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Rocard's 1941 Chaotic Relaxation Econometric Oscillator

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In the beginning of the Second World War, the French physicist, Yves Rocard, published a book entitled *Théorie des Oscillateurs* (Theory of Oscillators). In Chapter V, he designed a mathematical model consisting of a set of three nonlinear differential equations and allowing to account for economic cycles. Numerical integration of his model has highlighted a chaotic attractor. Its analysis with classical tools such as bifurcation diagram and Lyapunov Characteristic Exponents has confirmed the chaotic features of its solution. It follows that Rocard's 1941 chaotic econometric model has thus most likely preceded Lorenz' butterfly of twenty-two years. Moreover, apart from this historical discovery which upsets historiography, it is also established that this "new old" three-dimensional autonomous dynamical system is a new *jerk system* whose solution exhibits a chaotic attractor the topology of which varies, from a *double scroll* attractor to a *Möbius-strip* and then to a *toroidal attractor*, according to the values of a control parameter.

Keywords: Relaxation oscillations; chaotic attractor; bifurcations; econometrics oscillator; economic crises modeling.

1. Introduction

According to the historiography, it is generally considered that the very first chaotic attractor has been designed in 1963 by the late Edward Norton Lorenz [Lorenz, 1963]. But the winding road taken by the theory of nonlinear oscillations sometimes leads to surprises. Long and deep investigations performed in this domain [Ginoux, 2017] have led us to the "discovery" of a book entitled *Théorie des Oscillateurs* (*Theory of Oscillators*) and published by the French physicist Yves Rocard (1903–1992) in 1941. In chapter V: "Les oscillateurs des théories

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économiques" (Oscillators of economics theories), Rocard designed two mathematical models allowing to account for economic cycles and crises. In the first one, presented in Sec. 1.2, Rocard proves that economic crises can be modeled by using a Van der Pol's relaxation oscillator [Van der Pol, 1926] and obtained, to our knowledge, probably the very first *relaxation econometric oscillator*. Then, by considering that the frequency of oscillations may strongly depend on the amplitude, Rocard modified his first model and obtained a second one which is a *chaotic relaxation econometric oscillator* presented in Sec. 1.3. This latter model, which is the subject of this present work, will be analyzed in Secs. 2 and 3. Then, by using classical analysis tools such as bifurcation diagram and Lyapunov Characteristic Exponents, it will thus be established that this model is a new jerk system whose solution exhibits not only one but several chaotic attractors according to the value of the bifurcation parameter.

1.1. *Rocard's econometric oscillators*

Starting his analysis from an analogy with a nonholonom oscillator, Rocard [1941, p. 126] imagined an econometric model for studying economic cycles in a stable environment that he described as follows:

> "Suppose that y is the price of a commodity, that y_1 is the number of consumers of that commodity, or its total consumption, and assume that y_2 is the degree of tooling or mechanization, or rationalization, involved in the production process of this commodity and tending to decrease the price. We will reason less on the quantities themselves than on their deviations from an equilibrium position which will not be quantified."

Rocard proposed to study the market dynamic of a specific commodity using the standard assumption in economics which is to study the deviation from the equilibrium position. Variable y_2 refers to the part of physical capital in the production process, knowing that, in such a model, we generally consider two factors of production, the physical capital and the labor. If the part of physical capital in the production process increases, the productivity will increase, and then the production (namely the supply). Then, Rocard obtained the following three-dimensional dynamical model consisting of linear ordinary differential equations:

$$\begin{cases} \dfrac{dy_1}{dt} = -ay_1 + by, \\[2mm] \dfrac{dy_2}{dt} = K(y + y_1), \\[2mm] m\dfrac{dy}{dt} = -y_2, \end{cases} \tag{1}$$

where a is a positive parameter while b is negative and m and K are two unknown coefficients to be determined.

In this model, the first equation is a dynamic expression of the demand which includes part of Walras' concept of *encaisse désirée* (i.e. the increase in consumption will increase the demand for money, which will increase the price and therefore will decrease the consumption) and the classic law of demand (namely the quantity purchased varies inversely with price). Parameter a ($a > 0$) represents the *growth rate of the number of consumers* or the *growth rate of the total consumption*, b ($b < 0$) is the *growth rate of the commodity price*. The second equation, which is not common today, is a *dynamic expression of the supply*, and $(y + y_1)$ can be interpreted as the *nominal demand*, and K can be considered as the *growth rate of the nominal demand*. The third equation is a *dynamic expression of the price*, which depends only on the supply. By rewriting the third equation as $dy/dt = -y_2/m$, we observe that m corresponds to the *growth rate of the part of capital in the production process*. Indeed, when $m < 0$, we have a *disinvestment* in physical capital, and when $m > 0$ we have an *investment* in the physical capital. By taking the second time derivative of the last equation of (1) and by making a linear combination of two others, he obtained the following third-order linear ordinary differential equation:

$$m\dddot{y} + am\ddot{y} + K\dot{y} + K(a + b)y = 0. \tag{2}$$

1.2. *Relaxation econometric oscillator*

Then, Rocard explained that as long as b is negative, the dynamical system (1) or its corresponding linear differential equation (2) cannot exhibit any *self-oscillations*, i.e. *self sustained oscillations*. As a consequence, in order to obtain such kind of oscillations, he suggested to replace b by $b[1 - y^2/y_0^2]$ where y_0 is a constant in the above Eqs. (1) and (2).

Let us notice that Rocard introduced this *nonlinear oscillations characteristic* "by hand" in his model, i.e. without any economical justification. He obtained the following three-dimensional dynamical system consisting of nonlinear ordinary differential equations:

$$\begin{cases} \dfrac{dy_1}{dt} = -ay_1 + b\left(1 - \dfrac{y^2}{y_0^2}y\right), \\ \dfrac{dy_2}{dt} = K(y + y_1), \\ m\dfrac{dy}{dt} = -y_2, \end{cases} \tag{3}$$

that he transformed into:

$$m\dddot{y} + am\ddot{y} + K\dot{y} + K\left[a + b\left(1 - \frac{y^2}{y_0^2}\right)\right]y = 0. \tag{4}$$

According to Sprott [2003], the third-order nonlinear ordinary differential equation (4) is a *jerk equation* and the dynamical system (3) is *jerk system*. Then, Rocard [1941, p. 128] explained that:

> "The equations of the system (3) are no more linear, and their mathematical analysis becomes more difficult. However, we have the study of relaxation oscillations to guide us, and we will quickly see that we can conclude to the existence of self-sustaining oscillations of finite amplitude."

Thus, Rocard [1941, p. 130] performed a classical analysis of his third-order nonlinear ordinary differential equation, i.e. his *jerk equation* (4) and plotted its solution (see Fig. 1).

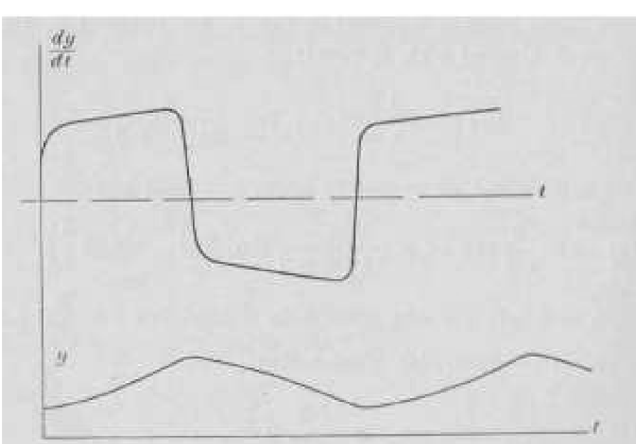


Fig. 1. Solution of Rocard's *jerk equation* (4).

Then, Rocard [1941, p. 130] explained that the curves represented in Fig. 1 are "very similar to those of relaxation oscillations". From Fig. 1, he deduced that "for the variation of prices y over time we obtain a fairly characteristic law of slow rise when prices are low, accelerated when they are high, then slowly fall, becoming a little faster when they are lower, etc ...". Finally, Rocard [1941, p. 131] stated mathematically that the frequency decreases as the amplitude increases and he considers that it would be interesting to analyze the case of an "oscillator whose frequency depends much on the amplitude." By posing:

$$\omega^2 = \frac{K}{m}; \quad a = \varepsilon\omega; \quad b = \eta\omega; \quad y = y_0 z,$$

he obtained the following dimensionless third-order nonlinear ordinary differential equation:

$$\dddot{z} + \varepsilon\omega\ddot{z} + \omega^2\dot{z} + \omega^3[\varepsilon + \eta(1 - z^2)]z = 0. \tag{5}$$

Although Le Corbeiller [1933] and Hamburger [1933a, 1933b] had suggested to apply Van der Pol's relaxation oscillations to economic cycles, they never developed a mathematical model. Only in 1951 did Goodwin [1951] propose a prototype nonlinear differential equation exhibiting maintained or self-sustained oscillations including relaxation oscillation. So, it appears that Rocard's *jerk equation* (5) which has preceded that of Goodwin of ten years can be considered as the *paradigm* of *relaxation oscillations* in Econometry and also upsets the historiography.

1.3. *Chaotic relaxation econometric oscillator*

In the second section of his chapter V, Rocard [1941, p. 133] designed a second model which is an "oscillator whose frequency depends much on the amplitude." To this aim, he modified the *nonlinear oscillations characteristic*, i.e. the last term of Eq. (5), by replacing $(1 - z^2)$ with $(1 - z^2 - \dot{z}^2/\omega^2)$. Thus, he obtains the following dimensionless third-order nonlinear ordinary differential equation:

$$\dddot{z} + \varepsilon\omega\ddot{z} + \omega^2\dot{z} + \omega^3\left[\varepsilon + \eta\left(1 - z^2 - \frac{\dot{z}^2}{\omega^2}\right)\right]z = 0. \tag{6}$$

Then, Rocard [1941, p. 133] explained that:

> "It would be interesting to provide a case study for which the frequency variation

according to the amplitude can even be totally abnormal."

This last model is, to our knowledge, the first *chaotic relaxation econometric oscillator*. It will be analyzed in the next sections. We will show that what he considered as "abnormal" is in fact the expression of the chaotic behavior of the solution of his *jerk equation* (6).

2. Rocard's 1941 Chaotic Relaxation Oscillator

First, let us notice that, according to D'Alembert [1748], the third-order nonlinear ordinary differential equation (6) can be cast in the form of a system of coupled first-order nonlinear differential equations as follows:

$$\begin{cases} \dfrac{dx}{dt} = -\omega[\varepsilon x + \omega y + \omega z], \\ \dfrac{dy}{dt} = \omega \left[\varepsilon + \eta\left(1 - z^2 - \dfrac{x^2}{\omega^2}\right)\right] z, \\ \dfrac{dz}{dt} = x. \end{cases} \tag{7}$$

Although in the second section of his chapter V, Rocard [1941, p. 133] did not assign any value to the parameter set of his model, we have performed many preliminary tests to determine the parameters range within which chaotic attractors may appear. This led us to use the following parameter set: $\varepsilon = 0.5$, $\omega = 2$ and $\eta \in [-1.34, 0.94]$ which will be used in the next sections. These parameter values correspond in the nondimensionless form to $a = \varepsilon\omega = 1$, $b \in [-2.68, 1.88]$ since $b = \eta\omega$ and $K = \omega^2 m = 4m$. Let us notice that they do have a real economic significance. Parameter $a = 1$ means that the population is considered as fixed, which is a standard hypothesis in the short run. b, which is the growth rate of the commodity price, can be negative (i.e. price decreases) or positive (i.e. price increases). It is consistent with the observations (see, for instance, the works of Jack and Stuermer [2020], Bakas and Triantafyllou [2020] and Baur and Dimpfl [2018a, 2018b]).

3. Stability Analysis

3.1. *Equilibrium points*

By using the classical nullclines method, we found that Rocard's system (7) has the following three equilibrium points:

$$\begin{aligned} &O(0,0,0); \\ &I_1\left(0, \sqrt{\frac{\varepsilon+\eta}{\eta}}, -\sqrt{\frac{\varepsilon+\eta}{\eta}}\right); \\ &I_2\left(0, -\sqrt{\frac{\varepsilon+\eta}{\eta}}, \sqrt{\frac{\varepsilon+\eta}{\eta}}\right). \end{aligned} \tag{8}$$

3.2. *Jacobian matrix*

The Jacobian matrix of Rocard's dynamical system (7) reads:

$$J = \begin{pmatrix} -\varepsilon\omega & -\omega^2 & -\omega^2 \\ -\dfrac{2\eta}{\omega}xz & 0 & -\dfrac{\eta x^2}{\omega} + \omega(\varepsilon + \eta - 3\eta z^2) \\ 1 & 0 & 0 \end{pmatrix}. \tag{9}$$

By replacing the coordinate of the equilibrium point O (8) in the Jacobian matrix (9) one obtains the following Cayley–Hamilton third degree eigenpolynomial:

$$\lambda^3 + \varepsilon\omega\lambda^2 + \omega^2\lambda - \omega^3(\varepsilon + \eta) = 0. \tag{10}$$

By using the Routh–Hurwitz criterion [Routh, 1877; Hurwitz, 1893] to state the stability of O, we obtain the following three determinants:

$$\begin{cases} \Delta_1 = \varepsilon\omega, \\ \Delta_2 = -\omega^3\eta, \\ \Delta_3 = -\omega^6(\eta + \varepsilon)\eta. \end{cases}$$

Since with our parameter set, $\varepsilon = 0.5$ and $\omega = 2$, it follows that all these three determinants are strictly positive provided that:

$$-\varepsilon < \eta < 0. \tag{11}$$

Thus, all the real parts of the eigenvalues of the eigenpolynomial (10) are negative and so, O is a *stable equilibrium point* provided that condition (11) is verified. Now, by replacing the coordinate of the equilibrium points I_1 or I_2 (8) in the Jacobian matrix (9) one obtains the following Cayley–Hamilton third degree eigenpolynomial:

$$\lambda^3 + \varepsilon\omega\lambda^2 + \omega^2\lambda + \omega^3(\varepsilon + \eta) = 0. \tag{12}$$

Still using the Routh–Hurwitz criterion [Routh, 1877; Hurwitz, 1893] to state the stability of $I_{1,2}$, we obtain the three determinants:

$$\begin{cases} \Delta_1 = \varepsilon\omega, \\ \Delta_2 = \omega^3(3\varepsilon + 2\eta), \\ \Delta_3 = -2\omega^6(3\varepsilon + 2\eta)(\varepsilon + \eta). \end{cases}$$

Since with our parameter set, $\varepsilon = 0.5$ and $\omega = 2$, it follows that all these three determinants are strictly positive provided that:

$$-\frac{3\varepsilon}{2} < \eta < -\varepsilon. \tag{13}$$

Thus, all the real parts of the eigenvalues of the eigenpolynomial (12) are negative and so, $I_{1,2}$ are *stable equilibrium points* provided that condition (13) is verified. Then, by taking into account both conditions (11) and (13), we find that if:

$$-\frac{3\varepsilon}{2} < \eta < 0$$

at least one of the three equilbrium points is stable, but this does not imply that other attractors can coexist with this local stable equilibrium, see for instance [Sprott, 2013].

The characteristic polynomial (10) becomes the polynomial $(\lambda - \kappa)(\lambda - \sigma i)(\lambda + \sigma i)$ with $\kappa\sigma \neq 0$, and consequently the equilibrium point at the origin of coordinates O has the possibility of exhibiting a Hopf bifurcation because then the eigenvalues of its linear part are κ and $\pm\sigma$ for $\omega > 0$ if and only if $\varepsilon = -\kappa/\sigma$, $\eta = 0$ and $\omega = \sigma$. We can expect to see a small-amplitude limit cycle biburcating from the equilibrium point O. In order to confirm that a Hopf bifurcation appears at O we must compute the first Lyapunov coefficient $\ell_1(O)$ of the differential system at O.

When $\ell_1(O) \neq 0$ the equilibrium point O is a weak focus of the differential system (7) restricted to the central manifold of O and the limit cycle that emerges from O is stable if $\ell_1(O) < 0$, and unstable if $\ell_1(O) > 0$. In the first case we say that the Hopf bifurcation is supercritical, and in the second case we say that the Hopf bifurcation is subcritical.

Here we use the following result presented in the pages 175–180 of the book [Kuznetsov, 2004] for computing $\ell_1(O)$.

Lemma 1. *Let $\dot{\mathbf{x}} = F(\mathbf{x})$ be a differential system with $\mathbf{x} \in \mathbb{R}^3$ having O as an equilibrium point. Consider the third order Taylor approximation of F around O given by $F(\mathbf{x}) = A\mathbf{x} + \frac{1}{2!}B(\mathbf{x},\mathbf{x}) + \frac{1}{3!}C(\mathbf{x},\mathbf{x},\mathbf{x}) + \mathcal{O}(|\mathbf{x}|^4)$, where A is a matrix, B is a bilinear function and C is a trilinear one. Assume that the matrix A has a pair of purely imaginary eigenvalues $\pm\sigma i$. Let q be the eigenvector of A corresponding to the eigenvalue σi, normalized so that $\bar{q} \cdot q = 1$, where $\bar{q}$ is the conjugate vector of q. Let p be the adjoint eigenvector such that $A^T p = -\sigma i p$ and $\bar{p} \cdot q = 1$. If I denotes the 3×3 identity matrix, then*

$$\begin{aligned} \ell_1(O) = \frac{1}{2\sigma} \operatorname{Re}(&\bar{p} \cdot C(q, q, \bar{q}) \\ &- 2\bar{p} \cdot B(q, A^{-1}B(q, \bar{q})) \\ &+ \bar{p} \cdot B(\bar{q}, (2\sigma i I - A)^{-1}B(q, q))). \end{aligned}$$

Some easy but tedious computations show that

$$A = \begin{pmatrix} \kappa & -\sigma^2 & -\sigma^2 \\ 0 & 0 & -\kappa \\ 1 & 0 & 0 \end{pmatrix}, \quad q = \left(\frac{i\sigma}{\sqrt{\frac{\kappa^2}{\sigma^2} + \sigma^2 + 1}}, \frac{i\kappa}{\sigma\sqrt{\frac{\kappa^2}{\sigma^2} + \sigma^2 + 1}}, \frac{1}{\sqrt{\frac{\kappa^2}{\sigma^2} + \sigma^2 + 1}} \right),$$

$$p = \left(-\frac{1}{(\kappa + i\sigma)\sqrt{\frac{\kappa^2}{\sigma^2} + \sigma^2 + 1}}, \frac{i\sigma}{(\kappa + i\sigma)\sqrt{\frac{\kappa^2}{\sigma^2} + \sigma^2 + 1}}, \frac{1}{\sqrt{\frac{\kappa^2}{\sigma^2} + \sigma^2 + 1}} \right),$$

$$B((x_1, y_1, z_1), (x_2, y_2, z_2)) = (0, 0, 0),$$

$$C((x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)) = \left(0, -\frac{6\eta(x_1x_2z_3 + x_1z_2x_3 + z_1x_2x_3)}{\omega} - 6\eta\omega z_1z_2z_3, 0 \right),$$

$$\ell_1(O) = -\frac{3\eta\sigma^3(\sigma^2 + \omega^2)}{2\omega(\kappa^2 + \sigma^2)(\kappa^2 + \sigma^4 + \sigma^2)}.$$

In summary, by Lemma 1, it follows that when $\ell_1(O) \neq 0$ the differential system (7) for $\varepsilon = -\kappa/\sigma$, $\eta = 0$ and $\omega = \sigma > 0$ exhibits a Hopf bifurcation at the equilibrium point O.

3.3. *Bifurcation diagram*

According to Rocard [1941, p. 133], the amplitude of his *jerk equation* (6) and so, of the *jerk system* (7) much depends on the parameter η. The same is true for the existence of chaotic attractors for the *jerk system* (7). Thus, in order to highlight how the changes of this control parameter impact the corresponding topology of the attractor, we have built a bifurcation diagram for $\eta \in [-1.34, -0.75]$ (see Fig. 2) and for $\eta \in [0, 0.94]$ (see Fig. 3) since in the interval $\eta \in [-0.75, 0]$ one at least of the three equilibrium points is stable. In Fig. 2, we observe a *reverse period doubling cascade* which confirms the existence of chaotic attractors for $-1.34 \leq \eta \leq -0.75$. As parameter η increases from -1.34 to -0.75, the *chaotic attractor* becomes a *limit cycle*. Let us notice in Fig. 2 the presence of several "windows" within which the *chaotic attractor* becomes a *limit cycle* whose period is determined by the number of branches. As an example, for $\eta \in [-1.319, -1.308]$, the attractor becomes a *limit cycle* of period 9. In Fig. 3, the bifurcation diagram highlights a *period doubling cascade* for $0 \leq \eta \leq 0.94$. Starting from $\eta = 0$ to $\eta \approx 0.72$ the attractor is a *limit cycle* and then becomes *chaotic*. There are also several windows within which the attractor topology changes and we observe *limit cycles* whose period is given by the number of branches. As an example, for $\eta \in [0.87, 0.92]$, the attractor becomes a *limit cycle* of period 5. The attractor topology changes according to the control parameter values of η may be represented as follows:

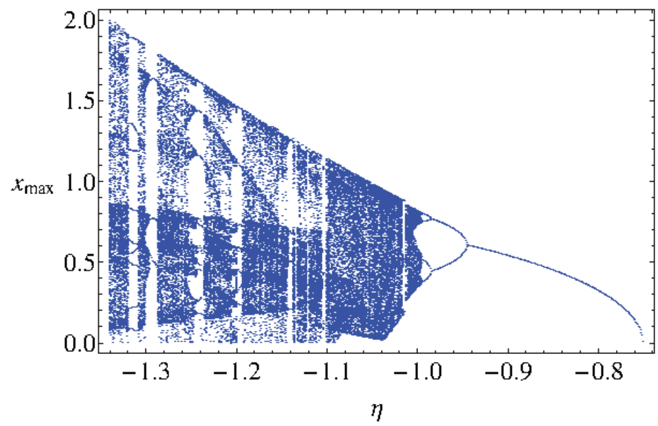


Fig. 2. Bifurcation diagram $x_{\max}$ as function of η for $\eta < 0$.

$$-1.34 \xrightarrow[\text{Chaos/LC}^n]{} \underset{HB}{-0.75} \xrightarrow[\text{SFP}]{} \underset{HB}{0} \xrightarrow[\text{LC}^1]{} 0.65 \xrightarrow[\text{Chaos/LC}^n]{} 0.94,$$

where LC^n means *limit cycle* of period n, SFP, Stable equilibrium Points and *HB*, Hopf Bifurcation.

For $-1.34 \leq \eta \leq -0.75$, a *reverse period doubling cascade* occurs (see Fig. 2) till the control parameter η reaches the value of the *Hopf bifurcation* $\eta_{\text{Hopf}} = -0.75$. Thus, we observe for $-1.34 \leq \eta \leq -1.15$ a *chaotic double-scroll* [see Figs. 4(a) and 4(c)]. Within this interval, several windows appear on the *bifurcation diagram* (see Fig. 2) and correspond to *limit cycles* of period n. As another example, for $\eta = -1.25$ the attractor becomes a *limit cycle* of period 5 [see Fig. 4(b)]. Then, starting with $-1.05 \leq \eta \leq -0.995$, the topology of the attractor changes and it becomes a *Möbius-strip* [see Figs. 4(d)–4(f)]. For $\eta \geq -0.75$, the attractor is a *limit cycle* of period 1.

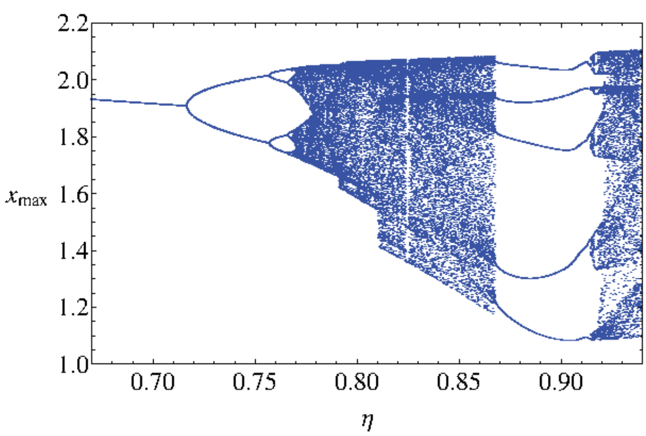
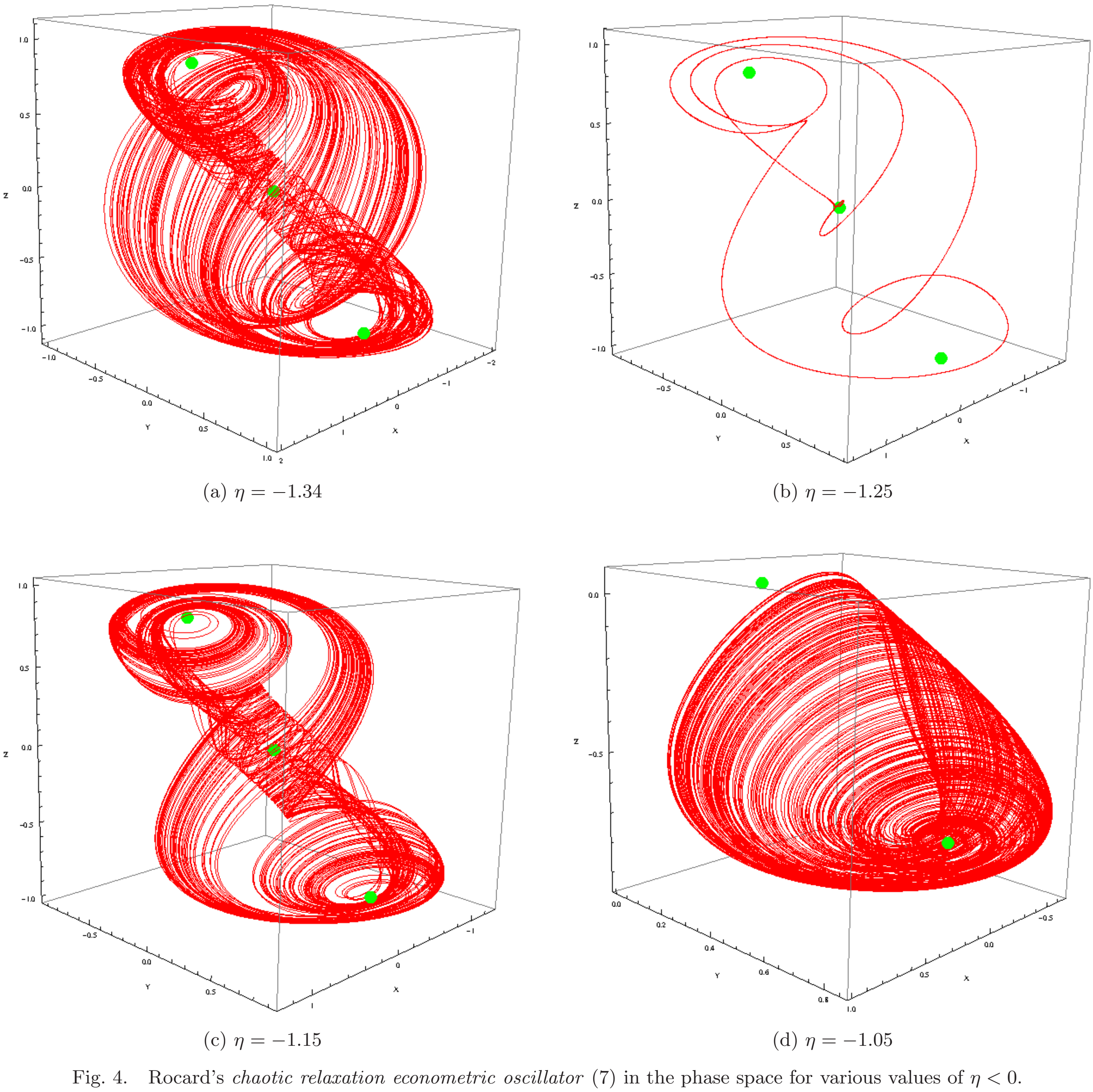


Fig. 3. Bifurcation diagram $x_{\max}$ as function of η for $\eta > 0$.

For $-0.75 \leq \eta \leq 0$, one of the three equilibrium points is *stable*. As highlighted in the *bifurcation diagram* (see Fig. 3), from $\eta = 0$ to $\eta \approx 0.72$ the attractor is a *limit cycle*. Then, a *period doubling cascade* occurs. From $\eta \approx 0.77$, we observe a *toroidal chaotic attractor* [see Figs. 5(a), 5(b), 5(d) and 5(e)]. Again, there are several windows within which some *limit cycles* of period n appear. As an example, Fig. 5(c) highlights a *limit cycle* of



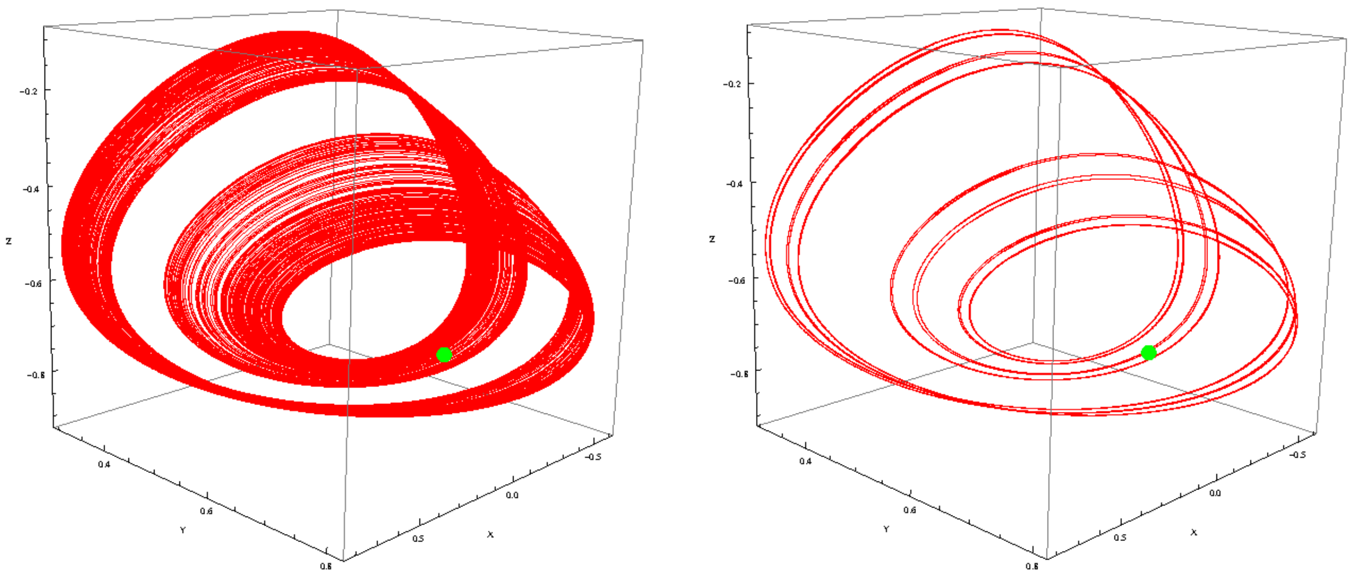
(a) $\eta = -1.34$

(b) $\eta = -1.25$

(c) $\eta = -1.15$

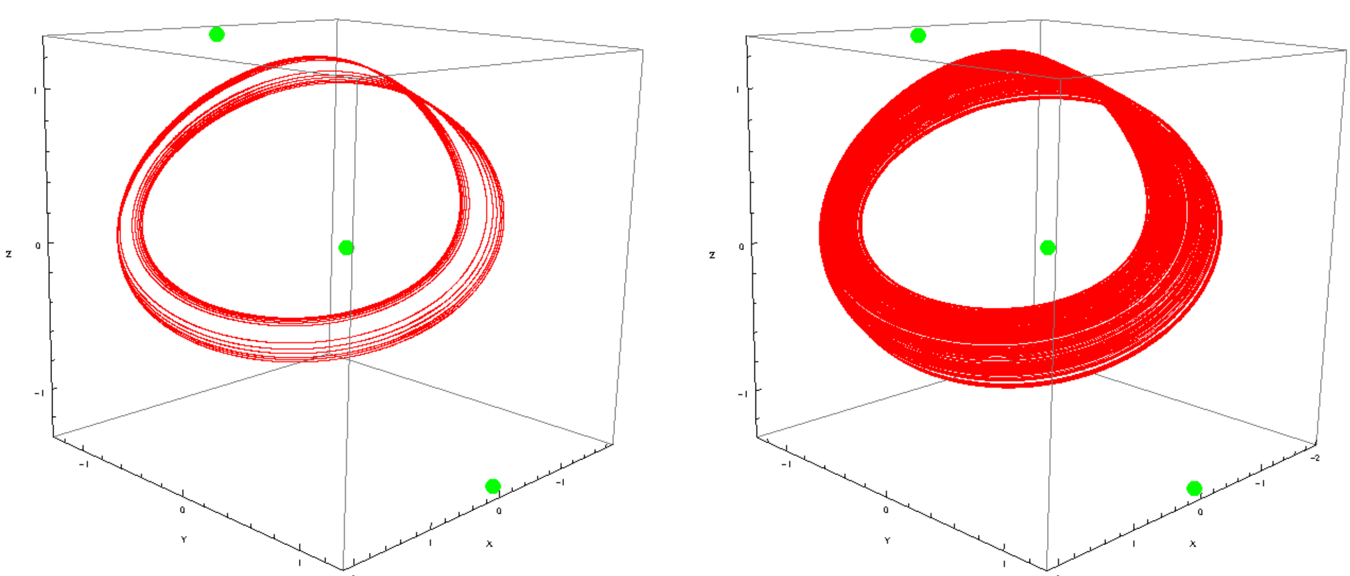
(d) $\eta = -1.05$

Fig. 4. Rocard's *chaotic relaxation econometric oscillator* (7) in the phase space for various values of $\eta < 0$.



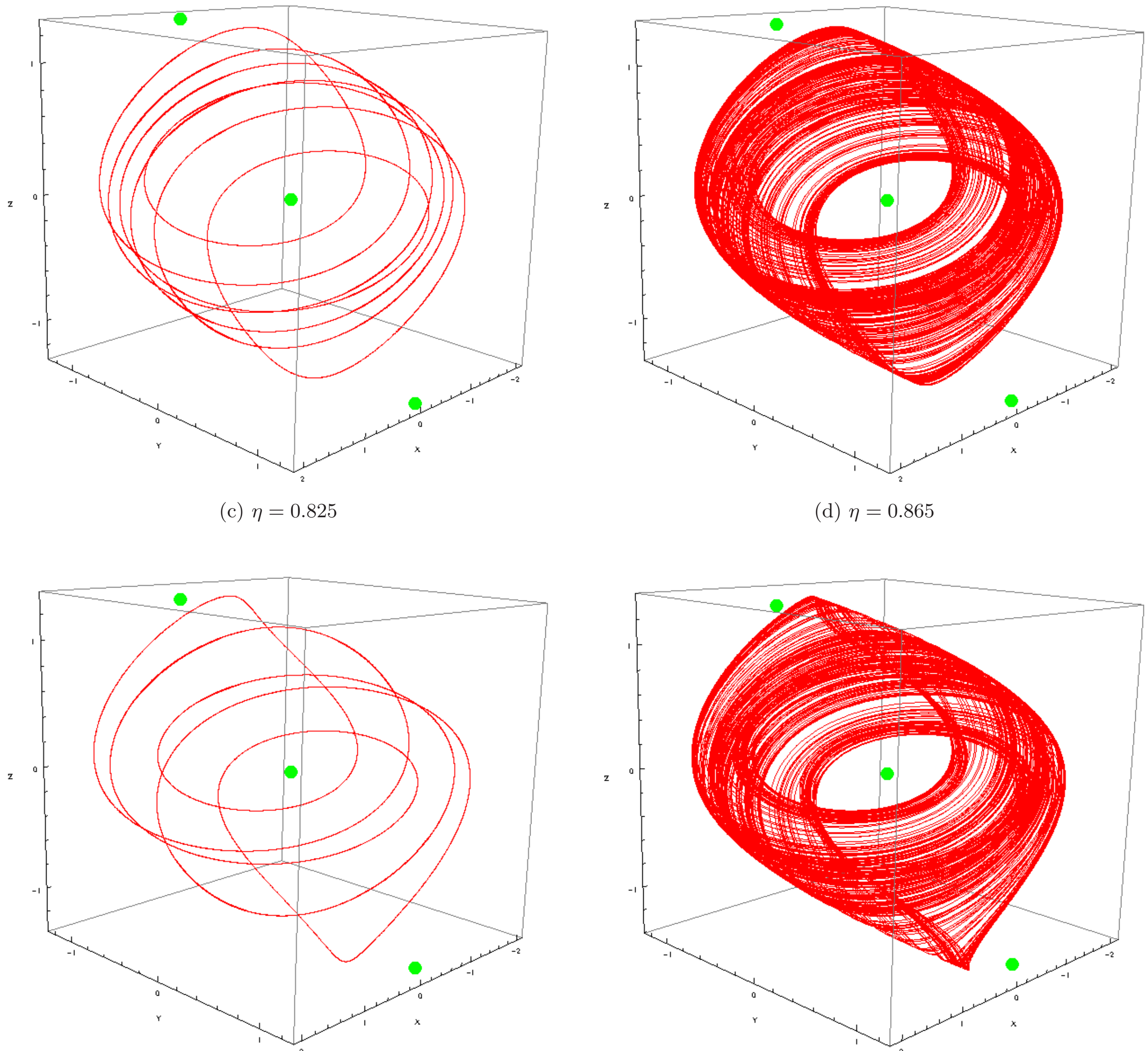
(e) $\eta = -1$ (f) $\eta = -0.995$

Fig. 4. (*Continued*)



(a) $\eta = 0.77$ (b) $\eta = 0.8$

Fig. 5. Rocard's *chaotic relaxation econometric oscillator* (7) in the phase space for various values of $\eta > 0$.



(c) $\eta = 0.825$

(d) $\eta = 0.865$

(e) $\eta = 0.9$

(f) $\eta = 0.925$

Fig. 5. (*Continued*)

Table 1. Lyapunov characteristics exponents of Rocard's dynamical system (7) for various values of η.

η	LCE Spectrum	Dynamics of the Attractor	Hausdorff Dimension
$-1.34 \leq \eta \leq -0.995$	$(+, 0, -)$	Chaos	$2.04 \leq D \leq 2.22$
$-0.95 \leq \eta \leq -0.75$	$(0, -, -)$	Limit cycle of period 1	$D = 1$
$0 \leq \eta \leq 0.72$	$(0, -, -)$	Limit cycle of period 1	$D = 1$
$0.72 \leq \eta \leq 0.94$	$(+, 0, -)$	Chaos	$2.04 \leq D \leq 2.17$

period 7 for $\eta = 0.825$ and a *limit cycle* of period 5 for $\eta = 0.9$ [see Fig. 5(e)].

In order to confirm the topology of these attractors, Lyapunov exponents have been computed in each case.

3.4. *Numerical computation of the Lyapunov exponents*

The algorithm developed by Sandri [1996] for Mathematica® has been used to perform the numerical calculation of the Lyapunov characteristics exponents (LCE) of the dynamical system (7) in each case. LCEs values have been computed within each considered interval ($\eta \in [-1.34, -0.75]$ and $[0, 0.94]$). Then, following the works of Klein and Baier [1991], a classification of (autonomous) continuous-time attractors of dynamical system (12) on the basis of their Lyapunov spectrum, together with their Hausdorff dimension is presented in Table 1. LCEs values have been also computed with the Lyapunov Exponents Toolbox (LET) developed by Siu for MatLab® and involving the two algorithms proposed by Wolf *et al.* [1985] and Eckmann and Ruelle [1985] (see https://fr.mathworks.com/matlabcentral/fileexchange/233-let). Results obtained by both algorithms are consistent.

We observe in Fig. 4 that the topology of the attractor of Rocard's *chaotic relaxation econometric oscillator* (7) varies. For $-1.34 \leq \eta \leq -1.15$ the attractor is a *double scroll* [see Figs. 4(a) and 4(c)] which may become for particular values of η, a *limit cycle* [see Fig. 4(b)], the period of which is given by the number of "branches" observed in the *bifurcation diagram* (see Fig. 2). Then, for $\eta \approx -1.09$, one of the two scrolls of the attractor disappears giving rise to the *Möbius-strip* [see Figs. 4(d)–4(f)]. The latter disappears on its turn to become a *limit cycle* according to the *reverse period doubling cascade* scenario presented in the *bifurcation diagram* (see Fig. 2). For $0 \leq \eta \leq 0.94$ the attractor slips from a *limit cycle* to a *Möbius-strip* [see Figs. 5(a) and 5(b)] via a *period doubling cascade* route to chaos as highlighted in the *bifurcation diagram* (see Fig. 3). Moreover, it may become for particular values of η, a *limit cycle* [see Figs. 5(c) and 5(e)], the period of which is given by the number of "branches" observed in the *bifurcation diagram* (see Fig. 3). Then, for $\eta \approx 0.825$, the attractor becomes *toroidal* [see Figs. 5(d) and 5(f)].

4. Conclusions

Deep investigations of the applications of *nonlinear oscillations theory* in the domain of *Econometric* induced by one of the authors (F. Jovanovic) has led us to the "discovery" of a book entitled *Théorie des Oscillateurs* (*Theory of Oscillators*) and published by the French physicist Yves Rocard (1903–1992) in 1941. In chapter V: "Les oscillateurs des théories économiques" (Oscillators of economics theories), Rocard designed two mathematical models allowing to account for economic crises. Each of them can be transformed into a third-order nonlinear ordinary differential equation, that is to say into a *jerk equation* according to Sprott [2003]. With the former model (3), Rocard proved that economic crises can be modeled by using a Van der Pol's relaxation oscillator [Van der Pol, 1926] and obtained, to our knowledge, probably the very first *relaxation econometric oscillator* (5). Till recently, the historiography [Velupillai, 1989] considered that this is only in 1951 that the American mathematician and economist, Richard Goodwin [1951], proposed a prototype nonlinear differential equation exhibiting maintained or self-sustained oscillations including *relaxation oscillation*. Then, Rocard modified his first equation (5) in order to have an "oscillator whose frequency depends much on the amplitude". Thus, he obtained the third-order nonlinear ordinary differential equation or *jerk equation* (6), the investigations of which led him to the conclusion that the "frequency variation according to the amplitude can even be totally abnormal". Nevertheless, he neither further analyzed this *jerk equation* (6) nor assigned any value to the parameter set. So, by considering a realistic range of parameter set, from the econometric point of view, we performed many preliminary tests and determined that for $\varepsilon = 0.5$, $\omega = 2$ and $\eta \in [-1.34, 0.94]$ several chaotic attractors may appear. Then, we transformed the third-order nonlinear ordinary differential equation or *jerk equation* (6) into a *dynamical system* that we have analyzed by using classical tools such as, equilibrium points stability, occurrence of Hopf bifurcations, bifurcation diagram and Lyapunov Characteristic Exponents. Such mathematical and numerical analysis has enabled to confirm that the solution of this "new old" three-dimensional autonomous dynamical system or new *jerk system* (7) exhibits a chaotic attractor the topology of which varies, from a *double scroll* attractor to a *Möbius-strip* and then to a *toroidal attractor*, according to the values of a control parameter η via a *reverse period doubling cascade* and *period doubling cascade*. Thus, it appears that Rocard [1941] has obtained in 1941, twenty-two years before Edward Norton Lorenz [1963], the very first *chaotic attractor*. This result upsets the historiography [Li & Yorke, 1975; Grassberger & Procaccia, 1983; Bergé *et al.*, 1984; Gleick, 1987] who considered till now that Lorenz [1963] had been the first to propose a *nonlinear dynamical system* the solution of which was exhibiting the famous *butterfly*. So, in this work, we have shown that contrary to what one thought, the very first chaotic attractor has not been designed for modeling atmospheric convection in the domain of *Meteorology* but for modeling great amplitude variations of relaxation oscillations in *Econometrics*.

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