# Sketching 1-D Stable Manifolds of 2-D Maps without the inverse 

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#### Abstract

Saddle fixed points are the centerpieces of complicated dynamics in a system. The one-dimensional stable and unstable manifolds of these saddle-points are crucial to understanding the dynamics of such systems. While the problem of sketching the unstable manifold is simple, plotting the stable manifold is not as easy. Several algorithms exist to compute the stable manifold of saddle-points, but they have their limitations, especially when the system is not invertible. In this paper, we present a new algorithm to compute the stable manifold of 2-dimensional systems which can also be used for non-invertible systems. After outlining the logic of the algorithm, we demonstrate the output of the algorithm on several examples.


Keywords : Stable Manifold, Non-Invertible Map, Point-Iterative Algorithm

## 1 Introduction

Saddle fixed points play a vital role in determining the behaviour of dynamical systems. These are points that act as attractors in some directions and repel trajectories in other directions. This information is conveyed in the stable and the unstable manifolds of the saddle point. For a 2-D map, an unstable manifold is a one-dimensional subset with the property that (a) it is tangential to the unstable eigenvector of the concerned fixed point, (b) iterates starting from any point on that manifold always remain on the manifold, and (c) iterates progressively move away from the fixed point along this manifold. Similarly, the stable manifold is defined as a subset with the property that (a) it is tangential to the stable eigenvector of the concerned fixed point, (b) iterates starting from any point on that manifold always remain on the manifold, and (c) iterates progressively move closer to the fixed point along this manifold.

It is known that an unstable manifold has attractive property and all attractors in a system occur on an unstable manifold of a fixed point. A stable manifold, on the other hand, has repelling property and if a system has multiple attractors, a stable manifold forms the boundary between the basins of attraction of the attractors. The interplay between the stable and unstable manifolds is responsible for important dynamical phenomena like interior crises, boundary crises, homoclinic intersections, etc. Hence, computing the structure of the stable and unstable manifolds of saddle points assumes special importance in understanding the dynamics of systems.

It is generally not possible to obtain the onedimensional manifolds (higher dimensional manifolds are not considered here) of saddle fixed points analytically, and hence, must be computed numerically. It is simpler to obtain the unstable manifold: The technique generally starts with approximating an initial segment $s$ of the manifold close to the saddle point with the aid of the eigenvector corresponding to the unstable eigenvalue of
the local linearization of the system at the point. Different algorithms can then be applied to extend the manifold from this initial segment. Over the years, this "manifold extension algorithm" has seen a lot of variation, as techniques were developed to obtain sufficiently closely spaced points so that the manifold can be sketched Nusse \& Yorke (2012).

It may be noticed that the definition of the stable manifold becomes identical with that of the unstable manifold under the application of the inverse map. That is why the same method can be applied to compute the stable manifold by iterating backward using the inverse map. This is the method used in the software tool $D y$ namics Nusse \& Yorke (2012) to sketch the stable manifold. But that requires the map to be invertible, which may not always be the case. To overcome the problem, a few alternative techniques have been proposed.

One of the proposed algorithms Kostelich et al. (1996) computes the stable manifold through the use of local inverse(s) for computing a sequence of pre-images of points on the segments. Thus it avoids the use of a global definition of the inverse map. However, this algorithm cannot be applied in cases where local inverses cannot be found.

In an alternative method, the initial segment is extended by using a "prediction and correction" technique Li et al. (2012) that uses the inverse of the local linearization of the given system to first predict a preimage of the current point, and then 'correct it' so that it is close to the actual preimage. However, as this technique also uses the inverse system, it faces similar limitations as Kostelich et al. (1996). Alternatively, the algorithm described in Nien \& Wicklin (1998) may be used to compute the pre-images of points (even in case of non-invertible maps) on the segment $s$ to obtain a set of points that lie on the stable manifold.

The Search Circle Algorithm England et al. (2004) extends the stable set, starting with $s$, by finding points at a particular distance from the latest point in the set that map back to a region already included in the stable set.

The Search Circle algorithm was implemented on a software tool called DSTool Back et al. (1995). It is a powerful method that works well for most maps - invertible or non-invertible. But its performance may be limited in case there are sharp bends and folds in the structure of the stable manifold. Moreover, it is no longer convenient to use DSTool as it works only on 32-bit architecture.

Thus we see that, even though some algorithms have been developed to compute the stable manifolds of noninvertible maps, these have some limitations because of the requirement of computing the pre-iterates in roundabout ways, necessitating complicated implementations that need strong error corrections to maintain accuracy. Even then, most of these algorithms fail when a stable manifold has sharp bends and folds.

In this paper, we present a simple and effective algorithm that uses only forward iterates to compute the stable manifold and thus is free from the problems of computation of pre-iterates of non-invertible maps. We call it the 'Point-Iterative Algorithm', which uses the properties of the stable manifold to reduce the problem at hand to a computationally simpler problem. We demonstrate the performance of the algorithm on different systems and show that the errors occurring through this algorithm are minimal.

## 2 The Proposed Logic

For any study of a dynamical system, it is seldom necessary to plot the structure of the stable and unstable manifolds over the entire phase plane. Normally the necessity is to plot the manifolds in smaller regions of interest. Once this is achieved, a more complete sense of the dynamics of the system can be obtained by consolidating the behaviour observed in all these smaller regions. Hence, for all practical purposes, we need to sketch the stable manifold in a smaller region of the phase plane such that,

$$
\begin{align*}
& a_{1} \leq x \leq a_{2}  \tag{1}\\
& b_{1} \leq y \leq b_{2}, \tag{2}
\end{align*}
$$

where $x$ and $y$ are the state variables and the specification of the system (whether in the form of differential equations or maps) provides the rules for the evolution of these state variables.

In our algorithm, we first compute the points where the stable manifold intersects the boundary of this region. Then the system can be simply evolved forward starting from these points of intersection, and the plot so obtained would be the sketch of the stable manifold in that region.

So, the problem is now reduced to that of finding the points of intersection of the manifold with the boundary of a given region. For this, we use the fact that the stable manifold acts as a repeller of trajectories, i.e., trajectories starting at very small distances on opposite sides of the stable manifold end up with very different fates. Using this property, the points of intersection are found, which allows us to compute the stable manifold using only the forward iterates.

### 2.1 The Parameters

1. BISECTION_ERROR: This describes the maximum error that is allowed in the bisection. Usually, $1 \mathrm{e}-6$ is a good value for the bisection error.
2. X_LINEAR_ADVANCEMENT: This describes the step length along the horizontal axis. Usually, 3-7\% of the length of the horizontal boundary is a good measure to start with.
3. Y_LINEAR_ADVANCEMENT: This describes the step length along the vertical axis. Usually, 3-7\% of the length of the vertical boundary is a good measure to start with.
4. N_MAX: The utility of this parameter is described later. Usually, N_MAX $=5$ or 6 is a good starting point.
5. points[ ]: This is the set of the points of intersection of the stable manifold with the boundary of the region.

### 2.2 The Functions

1. evolve_system (initial, system): Given the system definition and the initial conditions, it calculates the N_MAX-th iterate of the given map. It returns the vector pointing towards the next point in the system.
2. check_for_opposite_sides (p1,p2, system): Given two points, p1 and p2 and the system description, this function checks whether p1 and p2 are on opposite sides of the stable manifold. It does so by evolving p 1 and p 2 under evolve_system, and checking whether the vectors corresponding to the two points are diverging at least along one of the horizontal and vertical directions. FSometimes, it may so happen that due to the particular choice of N_MAX, such points may be obtained, which are not actually on opposite sides of the stable manifold but the vectors happen to diverge for the current parameter values. It might be a good idea to check for two or more N_MAX values before applying the bisection procedure.
3. bisect ( $\mathrm{p} 1, \mathrm{p} 2$,mode,system): Given two points, p 1 and p 2 , it performs bisection between p 1 and p 2 and adds all points of intersections to the set points[ ]. The mode decides whether the bisection is performed along the horizontal or the vertical axis.
4. find_points_of_intersection (p1,p2,mode,system): Given the endpoints of the region boundary, this uses bisect to find all the points of intersection along with the boundary. The different values of mode correspond to the two directions of the boundary.

| Mode | Direction |
| :---: | :---: |
| 0 | Horizontal Axis |
| 1 | Vertical Axis |
| All other values | Invalid Mode |

The pseudocodes of all these functions are given in the Appendix.

### 2.3 Special Consideration for Maps

While the proposed logic of finding the points of intersection and evolving the system can be applied to differential equations as well as maps, some special considerations are necessary when it is applied to maps.

In case of a continuous time system, theoretically an infinite number of points lying on the stable manifold can be found through evolution of a point on the manifold. In order to sketch the manifold computationally, a finite set of discrete points is obtained using a numerical algorithm, which is a subset of the infinite number of points. The number of points in the set can be increased to any desirable extent so as to obtain an accurate plot of stable manifold, i.e., the dicretization is under the control of the programmer.

In case of maps, that is not the case. Due to the discrete nature of maps, iteration of the points of intersection of the stable manifold with the boundary of the region of interest results in a finite number of points lying on the stable manifold. The result is a discrete scatter of points from which it might be difficult to infer a pattern.

For this reason, if a reasonably accurate and (almost) continuous plot is to be obtained, we need to run the find_points_of_intersection for all distinct regions of size X_LINEAR_ADVANCEMENT $\times$ Y_LINEAR_ADVANCEMENT, present within our region of interest. Hence, the sketch_stable_manifold function takes a slightly different form in case of maps. sketch_stable_manifold function for maps is provided in This algorithm is a simple, yet powerful technique that can sketch complicated structures of the stable manifold, some of which are demonstrated in Section 3 ,

## 3 Examples

We now demonstrate the performance of the PointIterative algorithm with the plots obtained for 2dimensional maps. We verify the plots by sketching the basins of attraction alongside (since the stable manifold lies along the basin boundaries), and comparing their shapes.

### 3.1 Henon Map

For the parameter values we have considered, the Henon map is invertible but has relatively sharp bends and folds. The map is described by

$$
\begin{equation*}
\binom{x_{n+1}}{y_{n+1}}=\binom{a-x_{n}^{2}+b y_{n}}{x_{n}} \tag{3}
\end{equation*}
$$

where $a$ and $b$ are parameters which control the behaviour of the system. In Fig. 1, we have plotted the basins of attraction and the stable manifolds obtained by our algorithm for three different values of $(a, b)$. The obtained manifolds are found to lie on the basin boundaries, and additionally show intricate structures that are not visible in the basin plots because of the finite resolution of the basin images. The structure of the stable manifold for $a=1.4$ and $b=-0.3$ is also compared with the manifold obtained from DSTool Back et al. (1995) using the Search-Circle Algorithm England et al. (2004) (Fig. 2).

As can be seen from Fig. 11(b), the sharp bends and folds of the manifold are missing in Fig. 2.

### 3.2 Modified Ikeda Map

The Ikeda map is described by:

$$
\begin{equation*}
\binom{x_{n+1}}{y_{n+1}}=\binom{a+b x_{n} \cos (m)-e y_{n} \sin (m)}{b y_{n} \cos (m)+e x_{n} \sin (m)} \tag{4}
\end{equation*}
$$

where

$$
\begin{equation*}
m=\phi-\frac{q}{1+x_{n}^{2}+y_{n}^{2}} \tag{5}
\end{equation*}
$$

and $a, b, e, q, \phi$ are parameters that control the behaviour of the map. For the typical Ikeda map, $b=e$, and as mentioned in England et al. (2004), it is possible to find an explicit inverse. However, we consider slightly different values for $b$ and $e$ for which no closed analytic form of the inverse exists England et al. (2004); Kostelich et al. (1996).

Considering the values $a=1, b=0.9, \phi=0.4, q=6$ and $e=1$, we have obtained a plot of the stable manifold using the Point-Iterative Algorithm (Fig. 33). This is very similar to the plot in Kostelich et al. (1996) and in England et al. (2004) (generated by the Search Circle Algorithm).

### 3.3 Modified Gumowski-Mira Map

The Gumowski-Mira map is expressed as:

$$
\begin{equation*}
\binom{x_{n+1}}{y_{n+1}}=\binom{y_{n}}{a x_{n}+b x_{n}^{2}+y_{n}^{2}} \tag{6}
\end{equation*}
$$

where $a$ and $b$ are parameters which control the behaviour of the system. It is easy to notice that while finding the inverse of the above map, we obtain a quadratic equation of $x_{n}$ in terms of $x_{n+1}$ and $y_{n+1}$. Hence, the map is noninvertible as there might exist two distinct pre-images of a single point. Algorithms that make use of the inverse for sketching the stable manifold fail to work here.

Using $a=-0.8$ and $b=0.1$, we observe that there exists a saddle-fixed point at $(x, y)=(1.636,1.636)$. Fig. 4 shows the sketch of the stable manifold of this saddle point generated by the Point-Iterative Algorithm. The manifold has disjoint sets which is a result of the existence of two pre-images. Interestingly, the branches of the stable manifold on both sides of the saddle-point join to form a closed loop. The stable manifold plotted by our algorithm has a good correspondence to the plot in England et al. (2004) generated by the Search-Circle Algorithm.

### 3.4 Non-Smooth non-invertible Map

Non-smooth maps have been subject of intense investigation in recent times because of their applicability in many physical and engineering systems. The piecewise linear 'normal form' of the non-smooth map is given by

$$
\binom{x_{n+1}}{y_{n+1}}= \begin{cases}h_{1}\left(x_{n}, y_{n}\right) & \text { if } x<0  \tag{7}\\ h_{2}\left(x_{n}, y_{n}\right) & \text { if } x \geq 0\end{cases}
$$



Figure 1: (a) and (b): The basins of attraction and the stable manifold plotted by the Point-Iterative Algorithm for the Henon Map for $a=1.4, b=-0.3$. (c) and (d): The same for parameter values $a=0.5, b=0.9$. (e) and (f): The same for parameter values $a=1.42, b=0.3$.


Figure 2: Stable manifold of the Henon Map for $a=1.4$ and $b=-0.3$ obtained using the Search-Circle Algorithm


Figure 3: Sketch of the stable manifold for the modified Ikeda map for $a=1, b=0.9, \phi=0.4, q=6$ and $e=1$, obtained using the Point-Iterative algorithm.


Figure 4: Sketch of the stable manifold of the saddle-fixed point (marked in orange) of the modified Gumowski-Mira Map for $a=-0.8, b=0.1$ obtained using the Point-Iterative Algorithm


Figure 5: Sketch of the stable manifold for the non-smooth map for parameters $\delta_{L}=-0.3, \tau_{L}=-0.3, \delta_{R}=1.4, \mu=0.05$ and (a) $\tau_{R}=0.28$, (b) $\tau_{R}=0.53$
where

$$
\begin{aligned}
& h_{1}\left(x_{n}, y_{n}\right)=\binom{\tau_{L} x_{n}+y_{n}+\mu}{-\delta_{L} x_{n}} \\
& h_{2}\left(x_{n}, y_{n}\right)=\binom{\tau_{R} x_{n}+y_{n}+\mu}{-\delta_{R} x_{n}}
\end{aligned}
$$

and $\delta_{L}, \tau_{L}, \delta_{R}, \tau_{R}$ and $\mu$ are parameters that determine the behaviour of the map. The map becomes noninvertible if $\delta_{L}$ and $\delta_{R}$ are of opposite sign or if one of them is zero De et al. (2012). We take the parameter values $\delta_{L}=-0.3, \tau_{L}=-0.3, \delta_{R}=1.4, \tau_{R}=0.28$ and $\mu=0.05$, satisfying this condition. The phase portrait of the system is characterised by saddle-node connections of a pair of period- 4 cycles (a stable cycle and a saddle cycle). Fig. 5(a) shows the plot of the stable manifold of the non-smooth map for the above parameter values as generated by the Point-Iterative Algorithm. The plot largely resembles the sketch of the stable manifold in De et al. (2012) that were obtained using DSTool. Our algorithm could detect some additional sections (compare Fig. 5(a) with Fig. 7(a) of De et al. (2012)) which have been verified to belong to the stable manifold.

For parameter values $\delta_{L}=-0.3, \tau_{L}=-0.3, \delta_{R}=$ $1.4, \tau_{R}=0.53$ and $\mu=0.05$, the phase portrait consists of a chaotic attractor and a stable period-4 cycle with their basins of attraction separated by the stable manifold of a saddle period-4 cycle. Fig. 5(b) shows the stable and saddle period- 4 cycles and the stable manifold obtained through the Point-Iterative Algorithm. More details about the 'closed loop' shape of the stable manifold can be found in De et al. (2012). The plot has an exact correspondence to the basins of attraction shown in De et al. (2012). It must be noted here that the logic for the check_for_opposite_sides function used in the Appendix will produce several false-positives if there is a chaotic attractor present in the phase portrait. Hence, to obtain the plot in Fig. 5 (b), the implementation of check_for_opposite_sides $\left(p_{0}, p_{1}\right.$, system) was modified to return True if trajectories starting from $p_{0}$ were attracted to the stable cycle and those starting from $p_{1}$ were
not, and vice-versa. The function would return False otherwise.

### 3.5 Map obtained from a 3 dimensional continuous time system

In order to demonstrate the power of this algorithm, we use it to plot the stable manifold of the map obtained by placing a Poincare section on the phase space of the continuous time system which is expressed as:

$$
\left(\begin{array}{l}
\dot{x}  \tag{8}\\
\dot{y} \\
\dot{z}
\end{array}\right)=\left\{\begin{array}{l}
y \\
z \\
-y+0.1 x^{2}+1.1 x z+1.05
\end{array}\right.
$$

This continuous time system has attracted some research attention because it does not have any equilibrium point and yet has a stable and an unstable limit cycle and an attractor at infinity. Upon placing a Poincaré section $z=-2$ on the phase plane, we obtain a planar map with two fixed points. The map has a stable fixed point at $(0.2459,-2.4506)$ and a saddle fixed point at $(-0.2036,-2.93)$, the stable manifold of which separates the basins of attraction of the stable fixed point and the attractor at infinity. The basins of attraction and the stable manifold of the saddle fixed point obtained using the Point-Iterative Algorithm have been shown in Fig. 6 and the stable manifold is found to lie along the basin boundaries. This demonstrates the usefulness of the algorithm as it does not need a closed form expression of the 2-dimensional map to obtain the stable manifold.

## 4 Adjusting the parameters used by the Algorithm

Section 2.1 lists the parameters that are used by the algorithm to sketch the stable manifold. For some systems, the initial setting of parameters might produce the desired results. However, for others, some tuning may be


Figure 6: (a) Basins of attraction and (b) Stable Manifold of the saddle fixed point of the planar map obtained from the 3D continuous time system
necessary. We now list how changing the various parameters affects the plot obtained from the algorithm.

1. BISECTION_ERROR: As this parameter controls the accuracy to which a point of intersection is bisected, very small values $(<1 \mathrm{e}-7)$ might increase the computation time, and thereby, affect the performance of the algorithm. Whereas a larger value for this parameter might mean a larger error in the position of the point of intersection. Usually, values between 1e-4 and 1e-6 achieve an acceptable compromise between computation time and accuracy.
2. X and Y LINEAR_ADVANCEMENTS: These essentially control the size of the steps that are taken along the boundary, before carrying out bisection. While reducing their values might lead to a greater computation time, for large values of X and Y LINEAR_ADVANCEMENTS ( $>15-20 \%$ of the boundary length), the stable manifold for maps may be reduced to a discrete sequence of points. These large values may be used when it is required to find out the general outline of the stable manifold. However, if the actual shape manifold needs to be plotted, their values need to be reduced.
3. N_MAX: This is essentially used to identify whether two points are on opposite sides of the stable manifold. If trajectories starting at these two points are found to diverge along at least one of the directions, after N_MAX iterates, the points are said to be on opposite sides of the stable manifold. Hence, a small value $(<3)$ might see the stable manifold breaking off unexpectedly in the phase plane, i.e., a discontinuous manifold may be obtained. On the other hand, a very large value would demand a large computation time. We illustrate the issue with reference to the modified Gumowski-Mira map discussed in Sec. 3. Fig. 7 shows the plots of the stable manifold of the above map for the parameter values specified in Sec. 3.3 obtained using the PointIterative Algorithm for different values of $N \_M A X$. As can be seen, for $N_{-} M A X=3$ and $N_{-} M A X$ $=4$, the plots obtained break off in the middle, indicating that the value of $N_{-} M A X$ must be increased. However, at $N \_M A X=5$, the manifold does not break off in the middle and we obtain the structure of the stable manifold. In Fig. 7(d), the
stable manifold is computed for a higher value of $N \_M A X(=14)$. As this shows no additional structure in the manifold, the chosen value of $N_{-} M A X$ is 5 as this is the smallest value of $N_{-} M A X$ that computes the structure of the stable manifold in the minimum computation time. Hence, the value of $N \_M A X$ should be so chosen that it results in a continuous manifold for a continuous map, and a further increase in its value does not provide any additional information about the stable manifold. While $N_{-} M A X=5$ or 6 may be a good starting point, some maps may require an $N_{-} M A X$ value up to 15 or even in some cases, as high as 25-30.

It must be noted that the exact values of these parameters may vary from system to system. However, the general behaviour upon increasing/decreasing the values remains similar. Hence, these should be used as indicators for tuning the parameter values until the required plot is obtained.

### 4.1 Accuracy of the Plots Obtained

While it is necessary to estimate and through the use of necessary techniques, confine the errors within certain bounds with the use of certain approximations, it is also necessary to ensure that heavy computational resources are not lost in overcoming these errors.

The use of various approximate algorithms such as approximating the local inverse Kostelich et al. (1996) or the Search Circle England et al. (2004) to extend a preliminary section of the stable manifold are followed by strong error analysis to ensure that the plots obtained are accurate. However, the Point-Iterative algorithm does not need rigorous error analysis. We find the points of intersection at multiple steps, and the only error encountered here is the uncertainty in the position of the point, i.e., the BISECTION_ERROR. Since the BISECTION_ERROR usually lies between $1 \mathrm{e}-4$ and $1 \mathrm{e}-6$, the plots obtained are, for all practical purposes, quite accurate.

## 5 Conclusion

In this paper we have introduced a new algorithm to plot the stable manifolds of saddle points in 2-D maps. It does not require specification of the inverse, nor does


Figure 7: Plot of the stable manifold obtained using the algorithm for (a) $N_{-} M A X=3$ (b) $N \_M A X=4$ (c) $N_{-} M A X=5$ (d) $N_{-} M A X=14$.
it depend on computation of pre-iterates. The PointIterative algorithm identifies the points of intersection of the stable manifold with the boundary of the region of interest by utilizing the repelling property of the stable manifold. Forward iteration from these points give points on the stable manifold. We have described the implementation of the algorithm, and have shown that the algorithm can successfully plot the stable manifold in non-invertible maps even when the manifold has sharp folds and bends.

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## Appendix

Here, we provide the pseudo-code implementations of the functions necessary to sketch the stable manifold using the Point-Iterative Algorithm.

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## function evolve_system $\left(p_{0}\right.$, map $)$ :

$p_{1}=$ Final point after iterating $p_{0}$ under map $N_{\_} M A X-1$ times
$p_{2}=\operatorname{map}\left(p_{1}\right)$
return $\left(p_{2}-p_{1}\right)$
function check_for_opposite_sides $\left(p_{0}, p_{1}\right.$, system $)$ :
$v 1=$ evolve_system $\left(p_{0}\right.$, system $)$
$v 2=$ evolve_system $\left(p_{1}\right.$, system $)$
return $(v 1[0] \times v 2[0])<0$ or $(v 1[1] \times v 2[1])<0$
function bisect ( $p_{0}, p_{1}$, mode, system):
mid $=\left(p_{0}+p_{1}\right) / 2$
if $p_{1}[$ mode $]-p_{0}[$ mode $]<B I S E C T I O N_{-} E R R O R$ then points.append(mid)
return
else
if (check_for_opposite_sides $\left(p_{0}\right.$, mid, system $)$ then bisect( $p_{0}$, mid, mode, system)
end
if (check_for_opposite_sides(mid, $p_{1}$, system) then bisect(mid, $p_{1}$, mode, system)
end
end
function find_points_of_intersection ( $p_{0}, p_{1}$, mode, system $)$ :
if mode $!=0$ and mode $!=1$ then
Invalid mode
return
else
while $p_{0}[$ mode $]<p_{1}[$ mode $]$ do
initial $=p_{0}$
if mode $=0$ then
$n e x t=\left(p_{0}[0]+X_{-} L I N E A R_{-} A D V\right.$ ANCEMENT, $\left.p_{0}[1]\right)$ $p_{0}[0]+=X_{-} L I N E A R_{-} A D V A N C E M E N T$
else
next $=\left(p_{0}[0], p_{0}[1]+Y_{\_} L I N E A R_{-}-A D V A N C E M E N T\right)$
$p_{0}[1]+=Y_{-} L I N E A R \_A D V A N C E M E N T$
end
if (check_for_opposite_sides(initial, next, system) then
bisect(initial, next, mode, system)
end
end
end
function sketch_stable_manifold
(x_left, x_right, y_bottom, y_top, map) :
if mode $!=0$ and mode $!=1$ then
Invalid mode
return
else
if mode $=0$ then
while $y$ _bottom $<y_{\text {_top }}$ do
x_start $=\left(x \_l e f t, y \_b o t t o m\right)$
$x_{\_}$end $=\left(x \_r i g h t, y \_\right.$bottom $)$
find_points_of_intersection(x_start, x_end, 0, map $)$ $y \_$bottom $+=Y_{-} L I N E A R_{-} A D V A N C E M E N T$
end
else
while $x$ _left $<x$ _right do
$y \_$start $=\left(x \_l e f t, y \_\right.$bottom $)$
$y_{-}$end $=\left(x_{-} l e f t, y_{-} t o p\right)$
find_points_of_intersection( $y_{\text {_start }}, y_{\text {_end }}, 1$, map $)$ $x_{\_} l e f t+=X_{\_} L I N E A R_{-} A D V A N C E M E N T$
end
end
end

