Bogdanov-Takens bifurcation of codimension 3 in the Gierer-Meinhardt model

Ranchao Wu*and Lingling Yang
Center for Pure Mathematics and School of Mathematical Sciences,
Anhui University, Hefei 230601, China

Abstract. Bifurcation of the local Gierer-Meinhardt model is analyzed in this paper. It is found that the degenerate Bogdanov-Takens bifurcation of codimension 3 happens in the model, except that teh saddle-node bifurcation and the Hopf bifurcation. That was not reported in the existing results about this model. The existence of equilibria, their stability, the bifurcation and the induced complicated and interesting dynamics are explored in detail, by using the stability analysis, the normal form method and bifurcation theory. Numerical results are also presented to validate theoretical results.

Key words: Gierer-Meinhardt model, Saddle-node, Hopf, Bogdanov-Takens bifurcation

1 Introduction

Early in [1], Turing discovered the common properties of the breakdown of spatial-temporal symmetry and the self-organization, selection, and stability of new spatial-temporal structures in systems, and proposed the idea of patterns as the results of diffusion driven instability. Since then more and more interests are focused on the Turing patterns and various models are put forward to describe the diffusion driven instability. One of the important models is the Gierer-Meinhardt model [2], which was proposed by Gierer and Meinhardt in 1972, and takes the following form

$$\begin{cases} \frac{\partial a}{\partial t} = \rho_0\rho + c\rho\frac{a^r}{h^s} - ua + D_a\frac{\partial^2 a}{\partial x^2}, \\ \frac{\partial h}{\partial t} = c'\rho'\frac{a^T}{h^u} - vh + D_h\frac{\partial^2 h}{\partial x^2}. \end{cases}$$

where $a(x,t)$ and $h(x,t)$ respectively represent the concentration of activators and inhibitors at spatial position x and time $t > 0$. $\rho_0\rho$ and ρ' are the source concentration of $a(x,t)$ and $h(x,t)$, respectively. The first-order kinetics of activator and inhibitor are represented by u_a and v_h, respectively. D_a and D_h represent the diffusion coefficients of activators and inhibitors, respectively. Generally, it is necessary to assume $\frac{sT}{u+1} > r - 1 > 0$, that is, $r \geq 2 (r \in \mathbb{Z})$.

*Corresponding author. E-mail address: rcwu@ahu.edu.cn (R. C. Wu)

In view of Turing's idea about pattern formation, to explore the patterns in such model, it is very necessary to carry out the stability and instability analysis. Instability will be accompanied by bifurcation in the model. Then spatiotemporal patterns will follow from the different bifurcation. Until not, various results about bifurcation and the resulting complex dynamics in the Gierer-Meinhardt model have been obtained. When $r = 2, s = 1, T = 2$, and $u = 0$, Song et al. [3] studied the Gierer-Meinhardt model with saturation terms and obtained the pattern formation in the certain parameter space. The Hopf bifurcation, the effect of diffusion on the stability and the subsequent Turing pattern were presented in [4]. For the delayed a delayed reaction-diffusion Gierer-Meinhardt system, the bifurcation analysis was also carried out in [5]. With the different sources for activators and inhibitors, Hopf bifurcation was treated in [6]. For the codimension-2 bifurcation, in [7] the Turing-Hopf bifurcation was considered, without the saturation term. The Turing-Turing bifurcation was given in [8], the coexistence of multi-stable patterns with different spatial responses and the superposition for patterns were demonstrated.

Recently, some results are obtained about the localized patterns in the Gray-Scott system and the bifurcation of the general Gierer-Meinhardt model in [9]. The local one-dimensional Gierer-Meinhardt model was given by

$$\begin{cases} \frac{\partial u}{\partial t} = a + \frac{u^2}{v} - u, \\ \frac{\partial v}{\partial t} = b + u^2 - dv. \end{cases} \tag{1}$$

where a, b and d are all positive constants. However, when $a = 0$, the system still has more complex dynamics and could be further explored. In this work, it is found that the model could admit the saddle-node, the Hopf and the degenerate Bogdanov-Takes bifurcations of codimension-3, which is not absent in the system in [9]. Note that highly degenerate bifurcations are more difficult to deal with and the resulting dynamical behaviors are richer and more interesting, so they are attracting increasing interests from mathematics and applied sciences. For example, degenerate bifurcations and the induced complicated dynamics were presented in[10–12], such as the nilpotent cusp singularity of order 3 and the degenerate Hopf bifurcation of codimension 3. In [13], Huang et al. discovered that there existed a degenerate Bogdanov-Takens singularity (focus case) of codimension 3 in the predator-prey model. In [14], the Bogdanov-Takens of codimension 3 and the Hopf bifurcation of codimension 2 were also found to happen.

In this paper, we will elaborate on these aspects for system (1) with $a = 0$. The existence and their stability of equilibrium points are introduced in Section 2. Bifurcations, such as, the saddle-node bifurcation, the Hopf bifurcation and the Bogdanov-Takes bifurcation of codimension-3 are presented in Section 4. Finally, a brief summary is made in Section 5.

2 Existence and stability of equilibria

Now consider the system in the following form

$$\begin{cases} \frac{du}{dt} = c\left(\frac{\beta u^2}{v} - u\right), \\ \frac{dv}{dt} = b + u^2 - dv. \end{cases} \tag{2}$$

Let $f(u,v) = c\left(\frac{\beta u^2}{v} - u\right)$, $g(u,v) = b + u^2 - dv$. Upon solving $f(u,v) = 0$, we obtain the solutions $u = 0$ or $v = u\beta$.

It is not difficult to get the boundary equilibrium $(0, \frac{b}{d})$ of system (2). Next, to find the existence of positive equilibria of system, substitute $v = u\beta$ into $g(u,v) = 0$, then we have

$$h(u) \triangleq u^2 - d\beta u + b = 0.$$

The discriminant of $h(u)$ is

$$\Delta = d^2\beta^2 - 4b.$$

It follows that

(i) if $d^2\beta^2 < 4b$, then $h(u) > 0$ for $u > 0$;

(ii) if $d^2\beta^2 = 4b$, then $h(u)$ has a real root $u_1 = \frac{d\beta}{2}$;

(iii) if $d^2\beta^2 > 4b$, then $h(u)$ has two distinct positive real roots,

$$u_2 = \frac{d\beta + \sqrt{\Delta}}{2}, \qquad u_3 = \frac{d\beta - \sqrt{\Delta}}{2}.$$

So we have the following result.

Theorem 1. *System (2) has only one boundary equilibrium* $E_0\,(0, v_0) = \left(0, \frac{b}{d}\right)$, *and*

(i) if $d^2\beta^2 < 4b$, *then there is no positive equilibria;*

(ii) If $d^2\beta^2 = 4b$, *then there is a positive equilibrium* $E_1\,(u_1, v_1) = \left(\frac{d\beta}{2}, \frac{d\beta^2}{2}\right)$;

(iii) If $d^2\beta^2 > 4b$,*then there are two positive equilibrium* $E_2\,(u_2, v_2) = \left(\frac{d\beta+\sqrt{\Delta}}{2}, \frac{d\beta^2+\beta\sqrt{\Delta}}{2}\right)$ *and* $E_3\,(u_3, v_3) = \left(\frac{d\beta-\sqrt{\Delta}}{2}, \frac{d\beta^2-\beta\sqrt{\Delta}}{2}\right)$.

Next the stability of the equilibria system (2) will be examined. First consider the boundary equilibrium $E_0\,(0, v_0)$. The Jacobian matrix of system (2) at equilibirum E_0 is

$$J_{E_0} = \begin{pmatrix} -1 & 0 \\ 0 & -d \end{pmatrix},$$

which has the eigenvalues $\lambda_1 = -1 < 0$ and $\lambda_2 = -d < 0$. Therefore, the equilibrium E_0 of system (2) is a stable node.

As for the stability of the equilibrium E_1, we have

Theorem 2. *(a) If* $d = c$, *then* E_1 *is a cusp of codimension three;*

(b) If $d > c$, *then* E_1 *is a saddle-node with an unstable parabolic sector;*

(b) If $d < c$, *then* E_1 *is a saddle-node with a stable parabolic sector.*

Proof. The Jacobian matrix of system (2) at equilibrium E_1 is

$$J_{E_1} = \begin{pmatrix} c & -\frac{c}{\beta} \\ d\beta & -d \end{pmatrix}.$$

It follows that

$$trJ_{E_1} = c - d, \;\; detJ_{E_1} = 0.$$

Now translate $E_1(u_1, v_1) = (\frac{d\beta}{2}, \frac{d\beta^2}{2})$ into the origin by the translation $(u, v) = (U + u_1, V + v_1)$, then system (2) is changed into

$$\begin{cases} \dot{U} = a_{10}U + a_{01}V + a_{20}U^2 + a_{11}UV + a_{02}V^2 + a_{21}U^2V \\ \qquad +a_{12}UV^2 + a_{03}V^3 + a_{22}U^2V^2 + a_{13}UV^3 + a_{04}V^4 + M\left(U, V\right), \\ \dot{V} = b_{10}U + b_{01}V + b_{20}U^2 + N\left(U, V\right), \end{cases} \tag{3}$$

where

$$\begin{array}{lllll} a_{10} = c, & a_{01} = -\dfrac{c}{\beta}, & a_{20} = \dfrac{2c}{d\beta}, & a_{11} = -\dfrac{4c}{d\beta^2}, & a_{02} = \dfrac{2c}{d\beta^3}, \\ a_{21} = -\dfrac{4c}{d^2\beta^3}, & a_{12} = \dfrac{8c}{d^2\beta^4}, & a_{03} = -\dfrac{4c}{d^2\beta^5}, & a_{22} = \dfrac{8c}{d^3\beta^5}, & a_{13} = -\dfrac{16c}{d^3\beta^6}, \\ a_{04} = \dfrac{8c}{d^3\beta 7}, & b_{10} = d\beta, & b_{01} = -d, & b_{20} = 1, & \end{array}$$

and $M(U, V)$, $N(U, V)$ are terms of at least order five in U and V.

First, assume $d = c$. Then both eigenvalues of J_{E_1} are zero. Applying the transformation $(U, V) = (x, \beta(x - \frac{y}{c}))$, we rewrite system (3) as

$$\begin{cases} \dot{x} = y + \frac{2y^2}{c^2\beta} - \frac{4xy^2}{c^3\beta^2} + \frac{4y^3}{c^4\beta^2} + \frac{8x^2y^2}{c^4\beta^3} - \frac{16xy^3}{c^5\beta^3} + \frac{8y^4}{c^6\beta^3} + M_2\left(x, y\right), \\ \dot{y} = -\frac{c}{\beta}x^2 + \frac{2y^2}{c\beta} - \frac{4xy^2}{c^2\beta^2} + \frac{4y^3}{c^3\beta^2} + \frac{8x^2y^2}{c^3\beta^2} - \frac{16xy^3}{c^4\beta^3} + \frac{8y^4}{c^5\beta^3} + N_2\left(x, y\right), \end{cases} \tag{4}$$

and $M_2(x, y)$, $N_2(x, y)$ are terms of at least order five in x and y. Further, let $(x, y) = (x_1, y_1 + x_1^2 + \frac{2}{c\beta}x_1y_1 - \frac{2}{c^2\beta}y_1^2)$, then (4) is transformed into the following form

$$\begin{cases} \dot{x}_1 = y_1 + x_1^2 + c_{11}x_1y_1 + c_{21}x_1^2y_1 + c_{12}x_1y_1^2 + c_{03}y_1^3 \\ \qquad +c_{40}x_1^4 + c_{22}x_1^2y_1^2 + c_{13}x_1y_1^3 + c_{04}y_1^4 + M_3(x_1, y_1), \\ \dot{y}_1 = d_{20}x_1^2 + d_{11}x_1y_1 + d_{30}x_1^3 + d_{21}x_1^2y_1 + d_{12}x_1y_1^2 + d_{03}y_1^3 \\ \qquad +d_{40}x_1^4 + d_{31}x_1^3y_1 + d_{22}x_1^2y_1^2 + d_{13}x_1y_1^3 + d_{04}y_1^4 + N_3(x_1, y_1), \end{cases} \tag{5}$$

where

$$\begin{array}{l} c_{11} = \dfrac{2}{c\beta}, \quad c_{21} = \dfrac{4}{c^2\beta}, \quad c_{12} = \dfrac{4}{c^3\beta^2}, \quad c_{03} = -\dfrac{4}{c^4\beta^2}, \quad c_{40} = \dfrac{2}{c^2\beta}, \quad c_{22} = \dfrac{4}{c^4\beta^2}, \\ c_{13} = \dfrac{8}{c^5\beta^3}, \quad c_{04} = -\dfrac{8}{c^6\beta^3}, \quad d_{20} = -\dfrac{c}{\beta}, \quad d_{11} = -2, \quad d_{30} = -2 + \dfrac{2}{\beta^2}, \\ d_{21} = -\dfrac{4}{c\beta^2} + \dfrac{2}{c\beta}, \quad d_{12} = -\dfrac{8}{c^2\beta}, \quad d_{03} = -\dfrac{4}{c^3\beta^2}, \quad d_{40} = -\dfrac{6}{c\beta} - \dfrac{4}{c\beta^3}, \\ d_{31} = \dfrac{4}{c^2\beta^2} + \dfrac{16}{c^2\beta^3} - \dfrac{16}{c^2\beta}, \quad d_{22} = \dfrac{12}{c^3\beta^2} - \dfrac{16}{c^3\beta^3}, \quad d_{13} = \dfrac{8}{c^4\beta^3} - \dfrac{24}{c^4\beta^2}, \quad d_{04} = -\dfrac{16}{c^5\beta^3}, \end{array}$$

and $M_3(x_1, y_1)$, $N_3(x_1, y_1)$ are terms of at least order five in x_1 and y_1.

Let $(x_2, y_2) = (x_1, y_1 + x_1^2 + \frac{2}{c\beta}x_1y_1 + M_4(x_2, y_2))$, then (5) takes the following form

$$\begin{cases} \dot{x}_2 = y_2, \\ \dot{y}_2 = e_{20}x_2^2 + e_{02}y_2^2 + e_{21}x_2^2y_2 + e_{12}x_2y_2^2 + e_{03}y_2^3 \\ \qquad +e_{40}x_2^4 + e_{31}x_2^3y_2 + e_{22}x_2^2y_2^2 + e_{13}x_2y_2^3 + e_{04}y_2^4 + N_4(x_2, y_2), \end{cases} \tag{6}$$

where

$$\begin{aligned} &e_{20} = -\frac{c}{\beta}, \quad e_{02} = \frac{2}{c\beta}, \quad e_{21} = -\frac{4}{c\beta^2}, \quad e_{12} = -\frac{4}{c^2\beta^2} - \frac{8}{c^2\beta}, \\ &e_{03} = -\frac{4}{c^3\beta^2}, \quad e_{40} = \frac{4}{c\beta^3}, \quad e_{31} = \frac{8}{c^2\beta^2} + \frac{16}{c^3\beta^3} + \frac{16}{c^2\beta^3}, \quad e_{22} = -\frac{8}{c^3\beta^3} + \frac{40}{c^3\beta^2} - \frac{16}{c^4\beta^4}, \\ &e_{13} = -\frac{24}{c^4\beta^2} + \frac{24}{c^4\beta^3}, \quad e_{04} = -\frac{16}{c^5\beta^3}, \end{aligned}$$

and $M_4(x_2, y_2)$, $N_4(x_2, y_2)$ are terms of at least order five in x_2 and y_2.

To eliminate the y_2- term in (6), change system (6) with the following transformation [14]

$$\begin{cases} x_3 = x_2 - \frac{e_{02}}{2}x_2^2 - \frac{e_{21}}{3e_{20}}x_2y_2 - \frac{e_{12}-e_{02}^2}{6}x_2^3 - \frac{e_{03}e_{20}-e_{02}e_{21}}{2e_{20}}x_2^2y_2 \\ \qquad -\frac{9e_{02}^3e_{20}-27e_{12}e_{02}e_{20}+18e_{20}e_{22}-32e_{21}^2}{216e_{20}}x_2^4 - \frac{7e_{02}^2e_{21}-12e_{02}e_{03}e_{20}-4e_{12}e_{21}+3e_{13}e_{20}}{18e_{20}}x_2^3y_2 \\ \qquad +\frac{e_{03}e_{21}-e_{04}e_{20}}{2e_{20}}x_2^2y_2^2, \\ y_3 = y_2 - e_{02}x_2y_2 - \frac{e_{21}}{3e_{20}}y_2^2 - \frac{e_{21}}{3}x_2^3 - \frac{e_{12}-e_{02}^2}{2}x_2^2y_2 - \frac{-2e_{02}e_{21}+3e_{03}e_{20}}{3e_{20}}x_2y_2^2 \\ \qquad -\frac{-3e_{02}e_{20}e_{21}+3e_{03}e_{20}^2+2e_{21}e_{30}}{6e_{20}}x_2^4 - \frac{9e_{02}^3e_{20}-27e_{02}e_{12}e_{20}+18e_{20}e_{22}-14e_{21}^2}{54e_{20}}x_2^3y_2 \\ \qquad -\frac{4e_{20}^2e_{21}-9e_{02}e_{03}e_{20}-2e_{12}e_{21}+3e_{13}e_{20}}{6e_{20}}x_2^2y_2^2 - \frac{-2e_{03}e_{21}+3e_{04}e_{20}}{3e_{20}}x_2y_2^3, \end{cases}$$

then we get

$$\begin{cases} \dot{x}_3 = y_3, \\ \dot{y}_3 = f_{20}x_3^2 + f_{40}x_3^4 + f_{31}x_3^3y_3 + N_5(x_3, y_3), \end{cases} \tag{7}$$

where

$$f_{20} = -\frac{c}{\beta}, \;\; f_{40} = \frac{11}{3c\beta^3} - \frac{4}{3c\beta^2}, \;\; f_{31} = -\frac{4}{c^2\beta^3} + \frac{16}{c^3\beta^3} + \frac{8}{c^2\beta^2},$$

and $M_5(x_3, y_3)$, $N_5(x_3, y_3)$ are terms of at least order five in x_3 and y_3.

Since $f_{20} = \frac{c}{\beta} \neq 0$, by the change of variables $(x_4, y_4) = \left(-x_3, -\frac{1}{\sqrt{-f_{20}}}y_3\right), \tau = \sqrt{-f_{20}}t$, we could turn system (7) into

$$\begin{cases} \frac{\mathrm{d}x_4}{\mathrm{d}\tau} = y_4, \\ \frac{\mathrm{d}y_4}{\mathrm{d}\tau} = x_4^2 + \left(\frac{4}{3c^2\beta} - \frac{11}{3c^2\beta^2}\right)x_4^3 - \frac{f_{31}}{\sqrt{-f_{20}}}x_4^3y_4 + N_6(x_4, y_4), \end{cases} \tag{8}$$

where $N_6(x_4, y_4)$ are terms of at least order five in x_4 and y_4.

From the proposition 5.3 in [15], we know that system (6) is equivalent to the system

$$\begin{cases} \frac{\mathrm{d}x_4}{\mathrm{d}\tau} = y_4, \\ \frac{\mathrm{d}y_4}{\mathrm{d}\tau} = x_4^2 + Ex_4^3y_4 + N_6(x_4, y_4), \end{cases}$$

where $E = -\frac{f_{31}}{\sqrt{-f_{20}}} = \frac{\frac{4}{c^2\beta^3} - \frac{16}{c^3\beta^3} - \frac{8}{c^2\beta^2}}{\sqrt{-f_{20}}} \neq 0$. Therefore, E_1 is a cusp of codimension three. This proves (a).

Next, assume $d \neq c$. The eigenvalues of J_{E_1} are $\lambda_3 = 0$ and $\lambda_4 = c - d$. Applying the transformation

$$U = \frac{u}{\beta} + \frac{v}{\beta d}, V = u + v, \tau = (d - c)\, t,$$

then system (3) becomes

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}\tau} = -\frac{3}{\beta^2(d-c)^2}u^2 - \frac{4d+2}{\beta^2(d-c)^2 d}uv + \frac{1-4d}{\beta^2 d^2(d-c)^2}v^2 + M_7(u,v), \\ \frac{\mathrm{d}v}{\mathrm{d}\tau} = v - \frac{2+d}{(d-c)^2\beta^2}u^2 - \frac{6}{\beta^2(d-c)^2}uv + \frac{2-5d}{\beta^2 d^2(d-c)^2}v^2 + N_7(u,v), \end{cases}$$

and $M_7(u,v), N_7(u,v)$ are terms of at least order three in u and v. The coefficient of u^2 is $-\frac{3}{\beta^2(d-c)^2} < 0$. From Theorem 7.1 in [16], the origin is a saddle-node. Considering the time vatiable τ, if $d - c < 0$, then E_1 is a saddle-node with a stable parabolic sector; if $d - c > 0$, then E_1 is a saddle-node with an unstable parabolic sector. □

If $\Delta > 0$, then $h(u)$ has two equilibria. Finally, we discuss the stability of the positive equilibria E_2 and E_3.

Theorem 3. *If $\Delta > 0$, then the positive equilibrium E_3 of system (2) is always a saddle point and the positive equilibrium E_2 is*

(a) a source if $d < c$;

(b) a center or a fine focus if $d = c$;

(c) a sink if $d > c$.

Proof. The Jacobian matrix of system (2) at equilibrium E_2 and E_3 are

$$J_{E_2} = \begin{pmatrix} c & -\frac{c}{\beta} \\ d\beta + \sqrt{\Delta} & -d \end{pmatrix}, \qquad J_{E_3} = \begin{pmatrix} c & -\frac{c}{\beta} \\ d\beta - \sqrt{\Delta} & -d \end{pmatrix}.$$

Then we could have

$$det J_{E_2} = \frac{c\sqrt{d^2\beta^2 - 4b}}{\beta} > 0$$

and

$$det J_{E_3} = -\frac{c\sqrt{d^2\beta^2 - 4b}}{\beta} < 0.$$

So E_3 is always a saddle point. The positive equilibrium E_2 is determined by the sign of the trace trJ_{E_2}. Specifically, when $trJ_{E_2} > 0$, i.e., $d < c$, E_2 is a source; When $trJ_{E_2} < 0$, i.e., $d > c$, E_2 is a sink. when $trJ_{E_2} = 0$,i.e., $d = c$, it is a center or a fine focus. □

3 Bifurcation

3.1 Saddle-node bifurcation

From Theorem 1 we note that the equilibrium points of system (2) vary as the parameter b changes. When $b > \frac{d^2\beta^2}{4}$, there is no positive equilibrium point. When $b = \frac{d^2\beta^2}{4}$, there is a positive equilibrium. When $b < \frac{d^2\beta^2}{4}$, there are two positive equilibria. This indicates the saddle-node bifurcation may occur around E_1.

Theorem 4. *When $b = b_{SN}$, the system (2) undergoes the saddle-node bifurcation around E_1, with the threshold value $b_{SN} = \frac{d^2\beta^2}{4}$.*

Proof. According to the Sotomayor's theorem [17], we need to verify the transversality condition for the occurrence of saddle-node bifurcation at $b \equiv b_{SN}$. The Jacobian matrix of system (2) at equilibrium E_1 is

$$J_{E_1} = \begin{pmatrix} c & -\frac{c}{\beta} \\ d\beta & -d \end{pmatrix}.$$

Because of $det(J_{E_1}) = \lambda_5\lambda_6 = 0$, J_{E_1} has a zero eigenvalue λ_5. Let V and W represent the eigenvectors of J_{E_1} and $J_{E_1}^T$ with respect to the eigenvalue λ_5, respectively.

Simple calculation gives

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \beta \end{pmatrix}$$

and

$$W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{c}{d\beta} \end{pmatrix}.$$

Further, we can obtain

$$F_b(E_1, b_{SN}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$D^2F(E_1, b_{SN})(V, V) = \begin{pmatrix} \frac{\partial^2 f}{\partial u^2}V_1^2 + 2\frac{\partial^2 f}{\partial u \partial v}V_1V_2 + \frac{\partial^2 f}{\partial v^2}V_2^2 \\ \frac{\partial^2 g}{\partial u^2}V_1^2 + 2\frac{\partial^2 g}{\partial u \partial v}V_1V_2 + \frac{\partial^2 g}{\partial v^2}V_2^2 \end{pmatrix}_{(E_1, b_{SN})} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Obviously, when $b \equiv b_{SN}$, V and W satisfy the transversality conditions

$$W^T F_b(E_1, b_{SN}) = -\frac{c}{d\beta} \neq 0$$

and

$$W^T \left[D^2F(E_1, b_{SN})(V, V)\right] = -\frac{2c}{d\beta} \neq 0.$$

Therefore, when the parameter b goes from one side of b_{SN} to the other, the system (2) experiences the saddle-node bifurcation at the equilibrium point E_1. □

3.2 Hopf bifurcation

From Theorem 3, it is found that the stability of the positive equilibrium of E_2 changes as the sign of $tr(E_2)$ varies, that will probably lead to the Hopf bifurcation around E_2.

According to the Hopf bifurcation theorem, we need to verify the transversal condition. Based on the fact that $\left.\frac{\mathrm{d}tr\left(J_{E_2}\right)}{\mathrm{d}d}\right|_{d=c} = -1 \neq 0$, system (2) undergoes the Hopf-bifurcation around E_2. Furthermore, we need to give the first Lyapunov coefficient and determine the stability of the limit cycle around E_2.

Now translate $E_2(u_2, v_2)$ to $(0,0)$ by $(\hat{u}, \hat{v}) = (u - u_2, v - v_2)$ and get the following form

$$\begin{cases} \dot{\hat{u}} = c\hat{u} - \frac{c}{\beta}\hat{v} + \frac{c}{u_2}\hat{u}^2 - \frac{2c}{\beta u_2}\hat{u}\hat{v} + \frac{c}{\beta^2 u_2}\hat{v}^2 + \hat{M}(\hat{u}, \hat{v}), \\ \dot{\hat{v}} = 2u_2\hat{u} - \hat{v} + \hat{u}^2 + \hat{N}(\hat{u}, \hat{v}), \end{cases} \tag{9}$$

where $\hat{M}(\hat{u}, \hat{v})$ and $\hat{N}(\hat{u}, \hat{v})$ are terms of at least order three in $\hat{u}$ and $\hat{v}$.

From another transformation $\hat{x} = -\hat{u}, \hat{y} = -\hat{u}, \hat{y} = \frac{1}{\sqrt{D}}(c\hat{u} - \frac{c}{\beta}\hat{v})$ and $\mathrm{d}\tau = \sqrt{D}\mathrm{d}t$, where $D = \frac{\sqrt{(d\beta)^2 - 4b}}{\beta}$, system (9) becomes

$$\begin{cases} \dot{\hat{x}} = -\dot{\hat{y}} + f(\hat{x}, \hat{y}), \\ \dot{\hat{y}} = \dot{\hat{x}} + g(\hat{x}, \hat{y}), \end{cases}$$

where

$$f(\hat{x}, \hat{y}) = -\frac{\sqrt{D}}{u_2}\hat{y}^2 + \hat{M}_1(\hat{u}, \hat{v}), \quad g(\hat{x}, \hat{y}) = -\frac{1}{D\beta}\hat{x}^2 + \frac{\hat{y}^2}{u_2} + \hat{N}_1(\hat{u}, \hat{v}),$$

where $\hat{M}_1(\hat{u}, \hat{v})$ and $\hat{N}_1(\hat{u}, \hat{v})$ are also terms of at least order three in $\hat{u}$ and $\hat{v}$.

Using the formal series method described in [16], we can calculate that the first-order Lyapunov number is

$$\sigma = \frac{\sqrt{D}}{4u_2^2} < 0.$$

Then the following theorem is available.

Theorem 5. *When $\Delta > 0$ and $d = c$, the system (2) at the equilibrium experiences the supercritical Hopf bifurcation with a stable limit cycle around E_2.*

3.3 Bogdanov-Takens bifurcation

When $u = \frac{v}{\beta}$ and $d = c$, it follows from Theorem 2(a) that the unique positive equilibrium E_1 of system (2) is a cusp of codimension three. The Bogdanov-Takens bifurcation may occur arround E_1. Now we select β, b and d as bifurcation parameters, and the Bogdanov-Takens bifurcation may occur under parameter perturbation.

Theorem 6. *When $u = \frac{v}{\beta}$ and $d = c$, the parameter (β, b, d) varies within the small neighborhood of $(\beta_{BT}, b_{BT}, d_{BT})$, where β_{BT}, b_{BT}, and d_{BT} are the Bogdanov-Takens bifurcation threshold values. Then system (2) undergoes the Bogdanov-Taken bifurcation of codimension 3 in the small neighborhood of E_1.*

Proof. When $u = \frac{v}{\beta}$ and $d = c$, it follows theorem 2(a) that E_1 is a cusp of codimension three of system (2). Perturb the parameters β, b and d at β_{BT}, b_{BT} and d_{BT} and denote $(\beta, b, d) = (\beta_{BT}+\epsilon_1, b_{BT}+\epsilon_2, d_{BT}+\epsilon_3)$, where $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$ is a vector of parameters in the small neighborhood of $(0, 0, 0)$. Then, the system (2) becomes

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = c(\frac{(\beta+\epsilon_1)u^2}{v} - u), \\ \frac{\mathrm{d}v}{\mathrm{d}t} = b + \epsilon_2 + u^2 - (d + \epsilon_3)v. \end{cases} \tag{10}$$

Then we translate $E_1(u_1, v_1) = (\frac{c\beta}{2}, \frac{c\beta^2}{2})$ into the origin by $(x, y) = (u - u_1, v - v_1)$. The system (10) is changed into

$$\begin{cases} \dot{x} = \bar{a}_{00} + \bar{a}_{10}x + \bar{a}_{01}y + \bar{a}_{20}x^2 + \bar{a}_{11}xy + \bar{a}_{02}y^2 + \bar{a}_{21}x^2y + \bar{a}_{12}xy^2 \\ \qquad + \bar{a}_{03}y^3 + \bar{a}_{22}x^2y^2 + \bar{a}_{13}xy^3 + \bar{a}_{04}y^4 + O(|x, y|^4), \\ \dot{y} = \bar{b}_{00} + \bar{b}_{10}x + \bar{b}_{01}y + \bar{b}_{20}x^2 + O(|x, y|^5), \end{cases} \tag{11}$$

where

$$\begin{aligned} &\bar{a}_{00} = \frac{c^2\epsilon_1}{2}, \quad \bar{a}_{10} = c(\frac{2(\beta+\epsilon_1)}{\beta} - 1), \quad \bar{a}_{01} = -\frac{c(\beta+\epsilon_1)}{\beta^2}, \quad \bar{a}_{20} = \frac{2(\beta+\epsilon_1)}{\beta^2}, \\ &\bar{a}_{11} = -\frac{4(\beta+\epsilon_1)}{\beta^3}, \quad \bar{a}_{02} = \frac{2(\beta+\epsilon_1)}{\beta^4}, \quad \bar{a}_{21} = -\frac{4(\beta+\epsilon_1)}{c\beta^4}, \quad \bar{a}_{12} = \frac{8(\beta+\epsilon_1)}{c\beta^5}, \\ &\bar{a}_{03} = -\frac{4(\beta+\epsilon_1)}{c\beta^6}, \quad \bar{a}_{22} = \frac{8(\beta+\epsilon_1)}{c^2\beta^6}, \quad \bar{a}_{13} = -\frac{16(\beta+\epsilon_1)}{c^2\beta^7}, \quad \bar{a}_{04} = \frac{8(\beta+\epsilon_1)}{c^2\beta^8}, \\ &\bar{b}_{00} = b + \epsilon_2 - \frac{1}{4}c^2\beta^2 - \frac{c\beta^2\epsilon_3}{2}, \quad \bar{b}_{10} = c\beta, \quad \bar{b}_{01} = -d - \epsilon_3, \qquad \bar{b}_{20} = 1. \end{aligned}$$

Then we rewrite system (11) with the transformation $(x, y) = (x_1 - \frac{2}{c\beta^2}x_1y_1, y_1)$ to

$$\begin{cases} \dot{x}_1 = \bar{c}_{00} + \bar{c}_{10}x_1 + \bar{c}_{01}y_1 + \bar{c}_{20}x_1^2 + \bar{c}_{11}x_1y_1 + O(|x, y|^4), \\ \dot{y}_1 = \bar{d}_{00} + \bar{d}_{10}x_1 + \bar{d}_{01}y_1 + \bar{d}_{20}x_1^2 + \bar{d}_{11}x_1y_1 + O(|x, y|^4), \end{cases}$$

where

$$\begin{aligned} &\bar{c}_{00} = \frac{c^2\epsilon_1}{2} \quad \bar{c}_{10} = \frac{2}{c\beta^2}\bar{b}_{00} + c(\frac{2(\beta+\epsilon_1)}{\beta} - 1), \quad \bar{c}_{01} = -\frac{c}{\beta}, \\ &\bar{c}_{20} = \frac{4}{\beta} + \frac{2\epsilon_1}{\beta^2}, \quad \bar{c}_{11} = -\frac{2(c+\epsilon_3)}{c\beta^2} - \frac{4(\beta+\epsilon_1)}{\beta^3}, \quad \bar{d}_{00} = \bar{b}_{00}, \\ &\bar{d}_{10} = c\beta, \quad \bar{d}_{01} = -c - \epsilon_3, \qquad \bar{d}_{20} = 1, \quad \bar{d}_{11} = -\frac{2}{\beta}. \end{aligned}$$

Further, we execute the transformation $(x_2, y_2) = (x_1, \dot{x}_1)$ and get

$$\begin{cases} \dot{x}_2 = y_2, \\ \dot{y}_2 = \bar{e}_{00} + \bar{e}_{10}x_2 + \bar{e}_{01}y_2 + \bar{e}_{20}x_2^2 + \bar{e}_{11}x_2y_2 + \bar{e}_{02}y_2^2 + \bar{e}_{30}x_2^3 + \bar{e}_{21}x_2^2y_2 + \bar{e}_{12}x_2y_2^2 \\ \qquad + \bar{e}_{03}y_2^3 + \bar{e}_{40}x_2^4 + \bar{e}_{31}x_2^3y_2 + \bar{e}_{22}x_2^2y_2^2 + \bar{e}_{13}x_2y_2^3 + \bar{e}_{04}y_2^4 + O(|x,y|^4), \end{cases} \tag{12}$$

where

$$\bar{e}_{00} = \bar{c}_{01}\bar{d}_{00} - \bar{c}_{00}\bar{d}_{01}, \quad \bar{e}_{10} = \bar{c}_{11}\bar{d}_{00} + \bar{c}_{01}\bar{d}_{10} - \bar{c}_{00}\bar{d}_{11} - \bar{c}_{10}\bar{d}_{01}, \quad \bar{e}_{01} = \bar{c}_{10} + \bar{d}_{01} - \frac{\bar{c}_{00}\bar{c}_{11}}{\bar{c}_{01}},$$
$$\bar{e}_{20} = \bar{c}_{11}\bar{d}_{10} + \bar{c}_{01}\bar{d}_{20} - \bar{c}_{10}\bar{d}_{11} - \bar{c}_{02}\bar{d}_{01}, \qquad \bar{e}_{11} = 2\bar{c}_{20} - \frac{\bar{c}_{10}\bar{c}_{11}}{\bar{c}_{01}} + \frac{\bar{c}_{00}\bar{c}_{11}^2}{\bar{c}_{01}^2} + \bar{d}_{11}, \quad \bar{e}_{02} = \frac{\bar{c}_{11}}{\bar{c}_{01}},$$
$$\bar{e}_{30} = \bar{c}_{11}\bar{d}_{20} - \bar{c}_{02}\bar{d}_{11}, \quad \bar{e}_{21} = -\frac{\bar{c}_{11}\bar{c}_{02}}{\bar{c}_{01}} + \frac{\bar{c}_{11}^2\bar{c}_{10}}{\bar{c}_{01}^2} - \frac{\bar{c}_{00}\bar{c}_{11}^3}{\bar{c}_{01}^3}, \quad \bar{e}_{12} = -\frac{\bar{c}_{11}^2}{\bar{c}_{01}^2}, \quad \bar{e}_{03} = 0,$$
$$\bar{e}_{40} = \frac{\bar{c}_{00}\bar{c}_{11}^4\bar{d}_{01}}{\bar{c}_{01}^4}, \quad \bar{e}_{31} = \frac{\bar{c}_{11}^2\bar{c}_{02}}{\bar{c}_{01}^2} - \frac{\bar{c}_{10}\bar{c}_{11}^3}{\bar{c}_{01}^3} + \frac{\bar{c}_{00}\bar{c}_{11}^4}{\bar{c}_{01}^4}, \quad \bar{e}_{22} = \frac{\bar{c}_{11}^3}{\bar{c}_{01}^3}, \quad \bar{e}_{13} = 0, \quad \bar{e}_{04} = 0.$$

To verify that the Bogdanov-Takens bifurcation occurs at equilibrium point E_1, we need to get the universal unfolding of system (9). So we need to eliminate y_2^2, x^3, x^2y, xy^2, y^3, and x^4 terms. Next, we transform system (12) by the procedure similar to that in [18].

(A) In order to eliminate the y_2^2 term, take the transformation $(x_2, y_2) = (u_1 + \frac{c_{02}}{2}u_1^2, v_1 + c_{02}u_1v_1)$, then system (12) takes the following form

$$\begin{cases} \frac{\mathrm{d}u_1}{\mathrm{d}t} = v_1, \\ \frac{\mathrm{d}v_1}{\mathrm{d}t} = \bar{f}_{00} + \bar{f}_{10}u_1 + \bar{f}_{01}v_1 + \bar{f}_{20}u_1^2 + \bar{f}_{11}u_1v_1 + \bar{f}_{30}u_1^3 + \bar{f}_{21}u_1^2v_1 + \bar{f}_{12}u_1v_1^2 \\ \qquad + \bar{f}_{40}u_1^4 + \bar{f}_{31}u_1^3v_1 + \bar{f}_{22}u_1^2v_1^2 + O(|u_1,v_1|^5), \end{cases} \tag{13}$$

where

$$\bar{f}_{00} = \bar{e}_{00}, \quad \bar{f}_{10} = \bar{e}_{10} - \bar{e}_{00}\bar{e}_{02}, \quad \bar{f}_{01} = \bar{e}_{01}, \quad \bar{f}_{20} = \bar{e}_{20} - \frac{\bar{e}_{10}\bar{e}_{02}}{2} + \bar{e}_{00}\bar{e}_{02}^2,$$
$$\bar{f}_{11} = \bar{e}_{11}, \quad \bar{f}_{30} = \bar{e}_{30} + \frac{\bar{e}_{10}\bar{e}_{02}^2}{2} - \bar{e}_{00}\bar{e}_{02}^3, \quad \bar{f}_{21} = \bar{e}_{21} + \frac{\bar{e}_{11}\bar{e}_{02}}{2},$$
$$\bar{f}_{40} = \frac{\bar{e}_{20}\bar{e}_{02}^2}{4} - \frac{\bar{e}_{10}\bar{e}_{02}^3}{2} + \bar{e}_{00}\bar{e}_{02}^4 + \frac{\bar{e}_{02}\bar{e}_{30}}{2}, \quad \bar{f}_{31} = \bar{e}_{31} + \bar{e}_{02}\bar{e}_{21}, \quad \bar{f}_{22} = \bar{e}_{22} + \frac{3\bar{e}_{02}\bar{e}_{12}}{2} - \bar{e}_{02}^3.$$

(B) In order to eliminate the $u_1v_1^2$ term, take the transformation $(u_1, v_1) = (u_2 + \frac{\bar{f}_{12}}{2}u_2^3, v_2 + \frac{d_{12}}{6}u_2^2v_2)$, then (13) is reduced to

$$\begin{cases} \frac{\mathrm{d}u_2}{\mathrm{d}t} = v_2, \\ \frac{\mathrm{d}v_2}{\mathrm{d}t} = \bar{g}_{00} + \bar{g}_{10}u_2 + \bar{g}_{01}v_2 + \bar{g}_{20}u_2^2 + \bar{g}_{11}u_2v_2 + \bar{g}_{30}u_2^3 + \bar{g}_{21}u_2^2v_2 \\ \qquad + \bar{g}_{40}u_2^4 + \bar{g}_{31}u_2^3v_2 + \bar{g}_{22}u_2^2v_2^2 + O(|u_2,v_2|^5), \end{cases} \tag{14}$$

where

$$\bar{g}_{00}=\bar{f}_{00},\quad \bar{g}_{10}=\bar{f}_{10},\quad \bar{g}_{01}=\bar{f}_{01},\quad \bar{g}_{20}=\bar{f}_{20}-\frac{\bar{f}_{00}\bar{f}_{12}}{2}+\frac{\bar{f}_{10}\bar{f}_{12}}{6},$$
$$\bar{g}_{11}=\bar{f}_{11},\quad \bar{g}_{30}=\bar{f}_{30}-\frac{\bar{f}_{10}\bar{f}_{12}}{2}+\frac{\bar{f}_{20}\bar{f}_{12}}{3},\quad \bar{g}_{21}=\bar{f}_{21}+\frac{\bar{f}_{11}\bar{f}_{12}}{6},$$
$$\bar{g}_{40}=\bar{f}_{40}-\frac{\bar{d}_{20}\bar{d}_{12}}{6}+\frac{\bar{f}_{00}\bar{f}_{12}^2}{4},\quad \bar{g}_{31}=\bar{f}_{31}+\frac{\bar{f}_{11}\bar{f}_{12}}{6},\quad \bar{g}_{22}=\bar{f}_{22}.$$

(C) Notice that $\bar{g}_{20}=-\frac{2b}{c\beta^3}-\frac{9c}{2\beta}+O(\epsilon)\neq 0$. To removing the u_2^3 and u_2^4 terms, we transform system (14) with $(u_2,v_2)=(u_3-\frac{\bar{g}_{30}}{4\bar{g}_{20}}u_3^2+\frac{15\bar{g}_{30}^2-16\bar{g}_{20}\bar{g}_{40}}{80\bar{g}_{20}^2}u_3^3,v_3)$ and scaling transformation $\mathrm{d}\tau=(1+\frac{\bar{g}_{30}}{2\bar{g}_{20}}u_3+\frac{48\bar{g}_{20}\bar{g}_{40}-25\bar{g}_{30}^2}{80\bar{g}_{20}}u_3^2+\frac{48\bar{g}_{20}\bar{g}_{30}\bar{g}_{40}-35\bar{g}_{30}^3}{80\bar{g}_{20}^3}u_3^3)\mathrm{d}t$ to obtain the following system

$$\begin{cases}\frac{\mathrm{d}u_3}{\mathrm{d}\tau}=v_3,\\ \frac{\mathrm{d}v_3}{\mathrm{d}\tau}=\bar{i}_{00}+\bar{i}_{10}u_3+\bar{i}_{01}v_3+\bar{i}_{20}u_3^2+\bar{i}_{11}u_3v_2+\bar{i}_{21}u_3^2v_3+\bar{i}_{12}u_3v_3^2\\ \qquad+\bar{i}_{40}u_3^4+\bar{i}_{31}u_3^3v_3+\bar{i}_{22}u_3^2v_3^2+R(u_3,v_3,\epsilon),\end{cases}\tag{15}$$

where

$$\bar{i}_{00}=\bar{g}_{00},\quad \bar{i}_{10}=\bar{g}_{10}-\frac{\bar{g}_{00}\bar{g}_{30}}{2\bar{g}_{20}},\quad \bar{i}_{01}=\bar{g}_{01},$$
$$\bar{i}_{20}=\bar{g}_{20}+\frac{45\bar{g}_{00}\bar{g}_{30}^2-60\bar{g}_{10}\bar{g}_{20}\bar{g}_{30}-48\bar{g}_{00}\bar{g}_{20}\bar{g}_{40}}{80\bar{g}_{20}^2},$$
$$\bar{i}_{11}=\bar{g}_{11}-\frac{\bar{g}_{01}\bar{g}_{30}}{2\bar{g}_{20}},\quad \bar{i}_{30}=\frac{\bar{g}_{10}(35\bar{g}_{30}^2-32\bar{g}_{20}\bar{g}_{40})}{4\bar{g}_{20}^2},$$
$$\bar{i}_{21}=\bar{g}_{21}-\frac{60\bar{g}_{11}\bar{g}_{20}\bar{g}_{30}-45\bar{g}_{01}\bar{g}_{30}^2+48\bar{g}_{01}\bar{g}_{20}\bar{g}_{40}}{80\bar{g}_{20}^2},$$
$$\bar{i}_{40}=\frac{g10g30g40}{4g20^2}-\frac{15g10g30^3}{64g20^3},\quad \bar{i}_{31}=\bar{g}_{31}+\frac{7\bar{g}_{11}\bar{g}_{30}^2}{8\bar{g}_{20}^2}-\frac{5\bar{g}_{30}\bar{g}_{21}+4\bar{g}_{11}\bar{g}_{40}}{5\bar{g}_{20}},$$
$$R(u_3,v_3,\epsilon)=v_3^2O(|u_3,v_3|^2)+O(|u_3,v_3|^5)+O(\epsilon)(O(v_3^2)+O(|u_3,v_3|^3))+O(\epsilon^2)O(|u_3,v_3|).$$

(D) Since

$$\begin{aligned}\bar{i}_{20}\ &=\ -\frac{9c}{2\beta}-\frac{2b}{c\beta^3}+\frac{2527200c^3}{\beta^4(\frac{c^3\beta}{4}-\frac{cb}{\beta})^2}+\frac{62104320b^2}{c\beta^7(\frac{c^3\beta}{4}-\frac{cb}{\beta})^2}\\ &\quad-\frac{23034240cb}{\beta^5(\frac{c^3\beta}{4}-\frac{cb}{\beta})^2}-\frac{416102406b^3}{c^3\beta^9(\frac{c^3\beta}{4}-\frac{cb}{\beta})^2}+O(\epsilon)\neq 0,\end{aligned}$$

by the transformation

$$(u_3,v_3)=(u_4,v_4+\frac{\bar{i}_{21}}{3\bar{i}_{20}}v_4^2+\frac{\bar{i}_{21}^2}{36\bar{i}_{20}^2}v_4^3),\mathrm{d}t=(1+\frac{\bar{i}_{21}}{3\bar{i}_{20}}v_4+\frac{\bar{i}_{21}^2}{36\bar{i}_{20}^2}v_4^2)\mathrm{d}\tau,$$

we can obtain the following form

$$\begin{cases}\frac{\mathrm{d}u_4}{\mathrm{d}\tau}=v_4,\\ \frac{\mathrm{d}v_4}{\mathrm{d}\tau}=\bar{j}_{00}+\bar{j}_{10}u_4+\bar{j}_{01}v_4+\bar{j}_{20}u_4^2+\bar{j}_{11}u_4v_4+\bar{j}_{31}u_4^3v_4+R(u_4,v_4,\epsilon),\end{cases}\tag{16}$$

where

$$\bar{j}_{00} = \bar{i}_{00}, \quad \bar{j}_{10} = \bar{i}_{10}, \quad \bar{j}_{01} = \bar{i}_{01} - \frac{\bar{i}_{00}\bar{i}_{21}}{\bar{i}_{20}},$$
$$\bar{j}_{20} = \bar{i}_{20}, \quad \bar{j}_{11} = \bar{i}_{11} - \frac{\bar{i}_{10}\bar{i}_{21}}{\bar{i}_{20}}, \quad \bar{j}_{31} = \bar{i}_{31} - \frac{\bar{i}_{21}\bar{i}_{30}}{\bar{i}_{20}}.$$

Additionally, $R(u_4, v_4, \epsilon)$ has the same properties as $R(u_3, v_3, \epsilon)$.

(E) We have $\bar{j}_{20}$ and $\bar{j}_{31}$ with the help of MAPLE

$$\begin{aligned}
\bar{j}_{20} =& -\frac{2b}{c\beta^3} - \frac{9c}{2\beta} + \frac{2527200c^2}{\beta^3(\frac{\beta c^3}{4} - \frac{cb}{\beta})^2} + \frac{62104320b^2}{c\beta^7(\frac{\beta c^3}{4} - \frac{cb}{\beta})^2} - \frac{23034240cb}{\beta^5(\frac{\beta c^3}{4} - \frac{cb}{\beta})^2} - \frac{41610240b^3}{c^3\beta^9(\frac{\beta c^3}{4} - \frac{cb}{\beta})^2} + O(\epsilon) \\
&\neq 0, \\
\bar{j}_{31} =& -\frac{151200b^3}{\beta^{11}c^6\left(\frac{2b\beta^3}{c} + \frac{9\beta c}{2}\right)^2} + \frac{128520b^2}{\beta^9c^4\left(\frac{2b\beta^3}{c} + \frac{9\beta c}{2}\right)^2} + \frac{71136b^2}{5\beta^8c^5\left(\frac{2b}{\beta^3c} + \frac{9c}{2\beta}\right)} - (\frac{57816b^3}{5\beta^{10}c^5\left(\frac{2b\beta^3}{c} + \frac{9\beta c}{2}\right)^2} \\
&- \frac{109278b^2}{5\beta^8c^3\left(\frac{2b\beta^3}{c} + \frac{9\beta c}{2}\right)^2} + \frac{36b}{\beta^4c^3} + \frac{44361b}{5\beta^6c\left(\frac{2b\beta^3}{c} + \frac{9\beta c}{2}\right)^2} - \frac{4131c}{4\beta^4\left(\frac{2b\beta^3}{c} + \frac{9\beta c}{2}\right)^2} + \frac{27}{\beta^2c}) \\
&\left(\frac{218976b^3}{\beta^{10}c^4\left(\frac{2b\beta^3}{c} + \frac{9\beta c}{2}\right)^2} - \frac{273048b^2}{\beta^8c^2\left(\frac{2b\beta^3}{c} + \frac{9\beta c}{2}\right)^2} + \frac{93456b}{\beta^6\left(\frac{2b\beta^3}{c} + \frac{9\beta c}{2}\right)^2} - \frac{9720c^2}{\beta^4\left(\frac{2b\beta^3}{c} + \frac{9\beta c}{2}\right)^2}\right) \\
&(-\frac{32508b^3}{5\beta^9c^3\left(\frac{\beta c^3}{4} - \frac{bc}{\beta}\right)^2} + \frac{48519b^2}{5\beta^7c\left(\frac{\beta c^3}{4} - \frac{bc}{\beta}\right)^2} + \frac{32508b}{5\beta^8\left(\frac{\beta c^3}{4} - \frac{bc}{\beta}\right)^3} - \frac{35991bc}{5\beta^5\left(\frac{\beta c^3}{4} - \frac{bc}{\beta}\right)^2} \\
&+ \frac{3159c^3}{8\beta^3\left(\frac{\beta c^3}{4} - \frac{bc}{\beta}\right)^2} - \frac{2b}{\beta^3c} - \frac{9c}{2\beta}) - \frac{72b}{\beta^5c^4} - \frac{36288b}{\beta^7c^2\left(\frac{2b\beta^3}{c} + \frac{9\beta c}{2}\right)^2} - \frac{13464b}{5\beta^6c^3\left(\frac{2b}{\beta^3c} + \frac{9c}{2\beta}\right)} \\
&+ \frac{3402}{\beta^5\left(\frac{2b\beta^3}{c} + \frac{9\beta c}{2}\right)^2} - \frac{432}{\beta^4c\left(\frac{2b}{\beta^3c} + \frac{9c}{2\beta}\right)} + \frac{18}{\beta^3c^2} + O(\epsilon) \neq 0.
\end{aligned}$$

Now, we want to turn $\bar{j}_{20}$ and $\bar{j}_{31}$ into 1 and notice that the signs of the coefficients of u_5^2 and $u_5^3v_5$ change as the signs of $\bar{j}_{20}$ and $\bar{j}_{31}$. The details are as follows.

(i) If $\bar{j}_{20} > 0, \bar{j}_{31} > 0$, then system (16) becomes the following form with the transformation $(u_4, v_4) = (\bar{j}_{20}^{\frac{1}{5}}\bar{j}_{31}^{-\frac{2}{5}}u_5, \bar{j}_{20}^{\frac{4}{5}}\bar{j}_{31}^{-\frac{3}{5}}v_5)$ and $\tau = \bar{j}_{20}^{-\frac{3}{5}}\bar{j}_{31}^{\frac{1}{5}}t$.

$$\begin{cases} \dot{u_5} = v_5, \\ \dot{v_5} = \bar{k}_{00} + \bar{k}_{10}u_5 + \bar{k}_{01}v_5 + u_5^2 + \bar{k}_{11}u_5v_5 + u_5^3v_5 + R(u_5, v_5, \epsilon), \end{cases}$$

where

$$\bar{k}_{00} = \bar{j}_{00}j_{20}^{-\frac{7}{5}}\bar{j}_{31}^{\frac{4}{5}}, \quad \bar{k}_{10} = \bar{j}_{10}\bar{j}_{20}^{-\frac{6}{5}}\bar{j}_{31}^{\frac{2}{5}}, \quad \bar{k}_{01} = \bar{j}_{01}\bar{j}_{20}^{-\frac{3}{5}}\bar{j}_{31}^{\frac{1}{5}}, \quad \bar{k}_{11} = \bar{j}_{11}\bar{j}_{20}^{-\frac{2}{5}}\bar{j}_{31}^{-\frac{1}{5}}.$$

Additionally, $R(u_5, v_5, \epsilon)$ has the same properties as $R(u_3, v_3, \epsilon)$.

(ii) If $\bar{j}_{20} < 0, \bar{j}_{31} > 0$ or $\bar{j}_{20} > 0, \bar{j}_{31} < 0$, then system (16) becomes the following form with the transformation $(u_4, v_4) = (\bar{j}_{20}^{\frac{1}{5}}\bar{j}_{31}^{-\frac{2}{5}}u_5, -\bar{j}_{20}^{\frac{4}{5}}\bar{j}_{31}^{-\frac{3}{5}}v_5)$ and $\tau = -\bar{j}_{20}^{-\frac{3}{5}}\bar{j}_{31}^{\frac{1}{5}}t$.

$$\begin{cases} \dot{u_5} = v_5, \\ \dot{v_5} = \bar{k}_{00} + \bar{k}_{10}u_5 + \bar{k}_{01}v_5 + u_5^2 + \bar{k}_{11}u_5v_5 - u_5^3v_5 + R(u_5, v_5, \epsilon), \end{cases}$$

where

$$\bar{k}_{00} = \bar{j}_{00}\bar{j}_{20}^{-\frac{7}{5}}\bar{j}_{31}^{\frac{4}{5}}, \quad \bar{k}_{10} = \bar{j}_{10}\bar{j}_{20}^{-\frac{6}{5}}\bar{j}_{31}^{\frac{2}{5}}, \quad \bar{k}_{01} = -\bar{j}_{01}\bar{j}_{20}^{-\frac{3}{5}}\bar{j}_{31}^{\frac{1}{5}}, \quad \bar{k}_{11} = -\bar{j}_{11}\bar{j}_{20}^{-\frac{2}{5}}\bar{j}_{31}^{-\frac{1}{5}}.$$

(iii) If $\bar{j}_{20} < 0, \bar{j}_{31} < 0$, then system (16) becomes the following form with the transformation $(u_4, v_4) = (-\bar{j}_{20}^{\frac{1}{5}}\bar{j}_{31}^{-\frac{2}{5}}u_5, -\bar{j}_{20}^{\frac{4}{5}}\bar{j}_{31}^{-\frac{3}{5}}v_5)$ and $\tau = \bar{j}_{20}^{-\frac{3}{5}}\bar{j}_{31}^{\frac{1}{5}}t$.

$$\begin{cases} \dot{u_5} = v_5, \\ \dot{v_5} = \bar{k}_{00} + \bar{k}_{10}u_5 + \bar{k}_{01}v_5 - u_5^2 + \bar{k}_{11}u_5v_5 - u_5^3v_5 + R(u_5, v_5, \epsilon), \end{cases}$$

where

$$\bar{k}_{00} = -\bar{j}_{00}\bar{j}_{20}^{-\frac{7}{5}}\bar{j}_{31}^{\frac{4}{5}}, \quad \bar{k}_{10} = \bar{j}_{10}\bar{j}_{20}^{-\frac{6}{5}}\bar{j}_{31}^{\frac{2}{5}}, \quad \bar{k}_{01} = \bar{j}_{01}\bar{j}_{20}^{-\frac{3}{5}}\bar{j}_{31}^{\frac{1}{5}}, \quad \bar{k}_{11} = -\bar{j}_{11}\bar{j}_{20}^{-\frac{2}{5}}\bar{j}_{31}^{-\frac{1}{5}}.$$

(F) Finally, we get the universal unfolding with the transformation $(u_6, v_6) = (u_5 - \frac{\bar{k}_{10}}{2}, v_5)$

$$\begin{cases} \dot{u_6} = v_6, \\ \dot{v_6} = \bar{l}_1 + \bar{l}_2v_6 + \bar{l}_3u_6v_6 + \bar{l}_4u_6^2 + \bar{l}_5u_6^3v_6 + R(u_6, v_6, \epsilon), \end{cases} \tag{17}$$

where $R(u_6, v_6, \epsilon)$ has the same properties as $R(u_3, v_3, \epsilon)$. There are three results corresponding to the three situations in (E).

(i) If $\bar{j}_{20} > 0, \bar{j}_{31} > 0$, then the cofficients of system (17) are

$$\bar{l}_1 = \bar{k}_{00} - \frac{1}{4}\bar{k}_{10}^2, \qquad \bar{l}_2 = \bar{k}_{01} - \frac{1}{8}\bar{k}_{10}^3 - \frac{1}{2}\bar{k}_{10}\bar{k}_{11}, \qquad \bar{l}_3 = \bar{k}_{11} + \frac{3}{4}\bar{k}_{10}^2, \qquad \bar{l}_4 = 1, \qquad \bar{l}_5 = 1.$$

(ii) If $\bar{j}_{20} < 0, \bar{j}_{31} > 0$ or $\bar{j}_{20} > 0, \bar{j}_{31} < 0$, then the cofficients of system (17) are

$$\bar{l}_1 = \bar{k}_{00} - \frac{1}{4}\bar{k}_{10}^2, \qquad \bar{l}_2 = \bar{k}_{01} + \frac{1}{8}\bar{k}_{10}^3 - \frac{1}{2}\bar{k}_{10}\bar{k}_{11}, \qquad \bar{l}_3 = \bar{k}_{11} - \frac{3}{4}\bar{k}_{10}^2, \qquad \bar{l}_4 = 1, \qquad \bar{l}_5 = -1.$$

(iii) If $\bar{j}_{20} < 0, \bar{j}_{31} < 0$, then the cofficients of system (17) are

$$\bar{l}_1 = \bar{k}_{00} + \frac{1}{4}\bar{k}_{10}^2, \qquad \bar{l}_2 = \bar{k}_{01} - \frac{1}{8}\bar{k}_{10}^3 + \frac{1}{2}\bar{k}_{10}\bar{k}_{11}, \qquad \bar{l}_3 = \bar{k}_{11} - \frac{3}{4}\bar{k}_{10}^2, \qquad \bar{l}_4 = -1, \qquad \bar{l}_5 = -1.$$

Then with the help of the MATLAB, we can obtain

$$\left|\frac{\partial(\bar{l}_1, \bar{l}_2, \bar{l}_3)}{\partial(\epsilon_1, \epsilon_2, \epsilon_3)}\right|_{\epsilon_1=\epsilon_2=\epsilon_3=0} \neq 0$$

for all three possible situations in (F). Therefore, according to the theory in [19, 20], system (2) undergoes the Bogdanov-Takens bifurcation of codimension 3 in a small neighborhood of E_1. □

The bifurcation diagram for system (17) can be described as follows. If $l_1 < 0$, there are no equilibria; if $l_1 = 0$, then there is a saddle-node bifurcation plane in a small neighborhood of the origin $(0, 0)$; if $l_1 > 0$, then the system has two equilibria, a saddle and an anti-saddle. The remaining surfaces of the bifurcation diagram in $\mathbb{R}^3$ have a conical structure, emanating from $(l_1, l_2, l_3) = (0, 0, 0)$, which can be demonstrated by drawing its intersection with the half sphere

$$S = \left\{(l_1, l_2, l_3) | l_1^2 + l_2^2 + l_3^2 = \rho^2, l_1 \geq 0, \rho > 0 \text{ sufficiently small}\right\}.$$

Now we project the traces onto the $l_2 l_3$-plane for clear visualization.

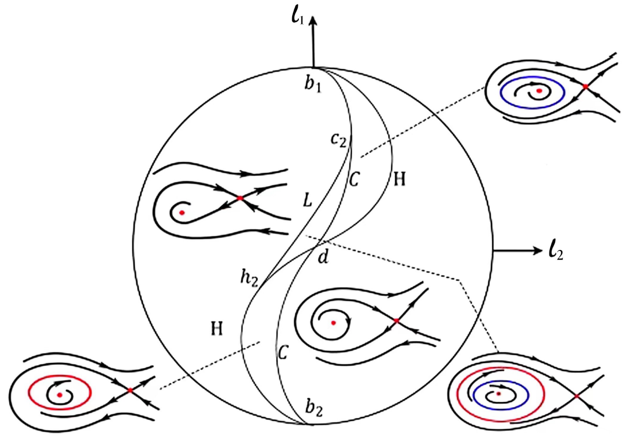

Figure 1: Bifurcation diagram of system (17) on S.

Here we summarize the bifurcation in system (17) based on the above discussion. Figure 1 contains the Hopf bifurcation curve, the homoclinic bifurcation curve and the saddle-node bifurcation curve, which are represented by H, C, and ∂S, respectively, where ∂S is the boundary of S. The curves H and C have the first order contact with the boundary of S at the points b_1 and b_2. The curve L is tangent to the curve C at point c_2 and tangent to the curve H at point h_2, which is the saddle-node bifurcation curve of double limit cycles. The system (17) is a cusp singularity unfolding of codimension 2 around b_1 and b_2.

Along the H, when crossing the arc $b_1 h_2$ of H from right to left, the curve H is a subercritical Hopf bifurcation with an unstable cycle curve of codimension one. And the curve H is a supercritical Hopf bifurcation with a stable cycle curve of codimension one when crossing the arc $h_2 b_2$ of H from left to right. The Hopf bifurcation of codimension 2 occurs at point h_2.

A homoclinic bifurcation of codimension 1 occurs along the curve C. When crossing the arc $b_1 c_2$ of C from left to right, the two parts of the saddle point coincide and an unstable limit cycle appears. And the two parts of the saddle point coincide and a stable limit cycle appears when crossing the arc $c_2 b_2$ of C from right to left. A homoclinic bifurcation of codimension 2 occurs at point c_2.

Then we give some numerical simulations about the system. In Figure 2, note that E_0 is a stable node when $c = 0.1, \beta = 0.12, b = 0.08, d = 0.08$. In Figure 3, there is a boundary equilibrium point

E_0 and a positive equilibrium point E_1. As $d = 0.4$, E_1 is a cusp when $c = 0.4, \beta = 0.5477, b = 0.012$; E_1 is a saddle-node with the stable parabolic sector when $c = 0.3, \beta = 0.5, b = 0.01$; E_1 is a saddle-node with the unstable parabolic sector when $c = 0.45, \beta = 0.5, b = 0.01$. There is a positive equilibrium E_2 in Figure 4. Choose $\beta = 0.6, b = 0.0125$, E_2 is a center when $c = 0.4, d = 0.4$; E_2 is a source when $c = 0.45, d = 0.38$; E_2 is a sink when $c = 0.3, d = 0.5$. E_3 is a saddle point when $c = 0.3, \beta = 0.5, b = 0.0075, d = 0.4$, which is shown in Figure 5.

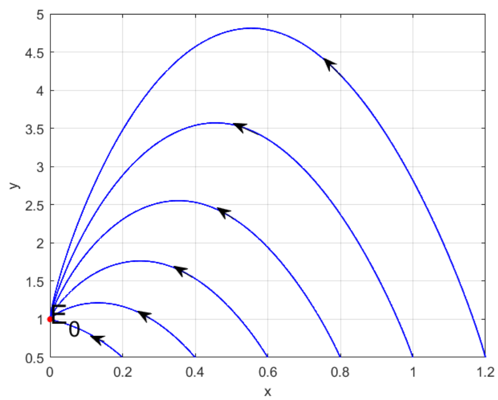

Figure 2: E_0 is a stable node when $c = 0.1, \beta = 0.12, b = 0.08, d = 0.08$.

4 Conclusion

Bifurcation analysis of the Gierer-Meinhardt model is carried out. Besides the saddle-node bifurcation and the Hopf bifurcation, it is found that the degenerate Bogdanov-Takens bifurcation of codimension-3 appears in the model. That was not reported in the previous results. By a series of transformation and based on the bifurcation theory, including the Sotomayor's theorem and the normal form method, the detailed bifurcation results are presented and more interesting dynamics are revealed. Theoretical findings are verified in numerical simulation. More further dynamics could be explored for the model.

Acknowledgments

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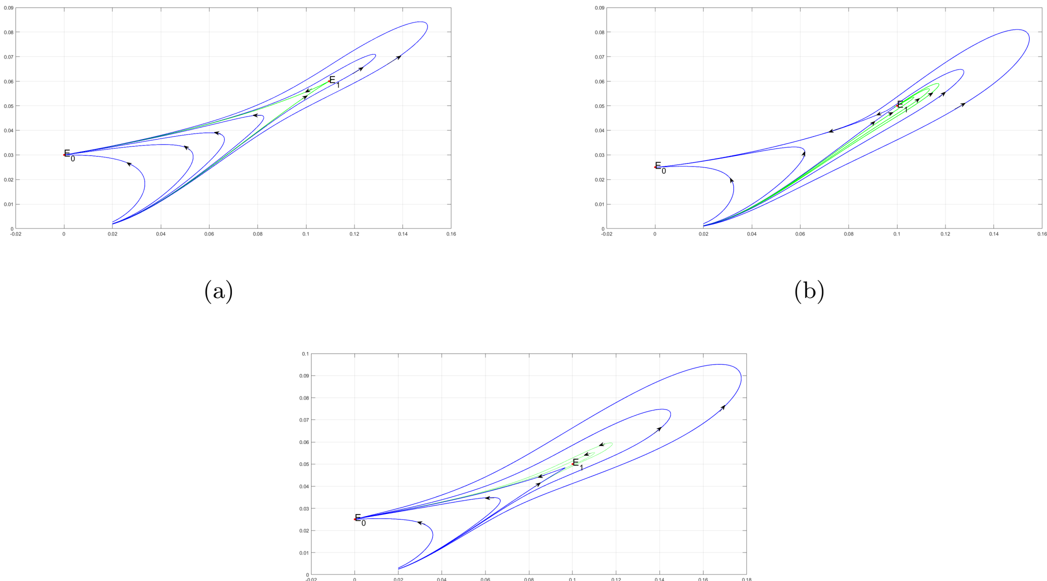
Conflict of Interest

The authors declare that there are no conflicts of interest.



(a) (b)

(c)

Figure 3: The stability of E_1 (a) E_1 is a cusp when $c = 0.4, \beta = 0.5477, b = 0.012, d = 0.4$ (b) E_1 is a saddle-node with stable parabolic sector when $c = 0.3, \beta = 0.5, b = 0.01, d = 0.4$. (c) E_1 is a saddle-node with unstable parabolic sector when $c = 0.45, \beta = 0.5, b = 0.01, d = 0.4$.

Data availability statement

All data generated or analysed during this study are included in this published article or available upon request.

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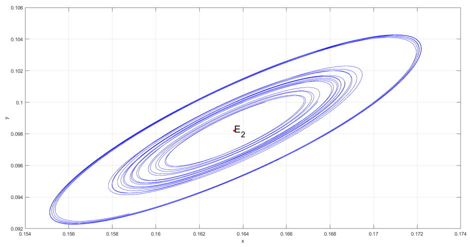
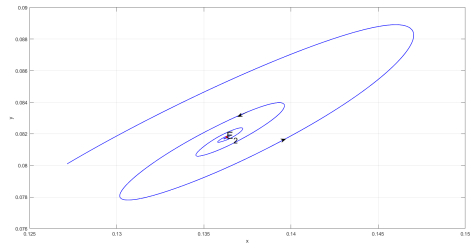
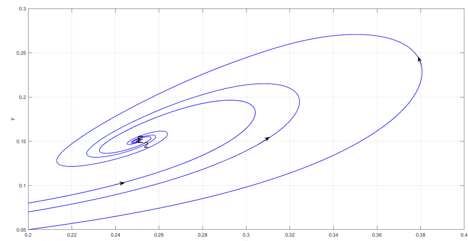


(a)



(b)



(c)

Figure 4: The stability of the E_2 (a) E_2 is a center when $c = 0.4, \beta = 0.6, b = 0.0125, d = 0.4$. (b) E_2 is a source when $c = 0.45, \beta = 0.6, b = 0.0125, d = 0.38$. (b) E_2 is a sink when $c = 0.3, \beta = 0.6, b = 0.0125, d = 0.5$.

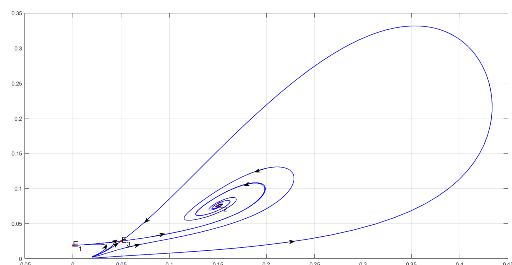


Figure 5: E_3 is a saddle point when $c = 0.3, \beta = 0.5, b = 0.0075, d = 0.4$.

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