Maximizing a Voronoi Region: The Convex Case

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Abstract

Given a set S of s points in the plane, where do we place a new point, p, in order to maximize the area of its region in the Voronoi diagram of S and p? We study the case where the Voronoi neighbors of p are in convex position, and prove that there is at most one local maximum.

Keywords: Computational geometry, locational planning, optimization, Voronoi diagram.

1 Introduction

Suppose that we want to place a new supermarket where it wins over as many customers as possible from the competitors that already exist.

Let us assume that customers are equally distributed and that each customer shops at the market closest to her residence. Our task then amounts to finding a location, p, for the new market amidst the locations p_i of the existing markets, such that the Voronoi region of p, that is, the set of all points in the plane that are closer to p than to any p_i, has a maximum area.

Surprisingly, not much seems to be known about this problem. The area of Voronoi regions has been addressed in the context of games, where players can in turn move their existing sites, or insert new sites, such as to end up with a large total area of their Voronoi regions; see the Hotelling game described in Okabe et al. [8], and related work by Cheong et al. [4] and Ahn et al. [1]. But none of these papers gives an explicit method for maximizing the region of a new site. After the conference version of the present paper [5] appeared, the problem of maximizing the Voronoi region of a new site has been addressed by Cheong et al. [3]. They showed how to compute, in time $O(n\delta^{-4} + n \log n)$, a location for a new site whose Voronoi region approximates the maximum size, up to a $1 - \delta$ factor.

In this paper we describe the first nontrivial step towards an *exact* solution of the area maximization problem. We are given a finite set, S, of point sites $p_1, \ldots, p_s$, and we want to place a new site, p, at a location that maximizes the area of its Voronoi region $\mathrm{VR}(p, S \cup \{p\})$.

Two aspects of this problem statement need to be clarified. First, the Voronoi region of p is formally undefined in case $p = p_i$ holds for a point $p_i \in S$. In the context of our maximization problem, this can be fixed as follows. Suppose that p moves towards p_i along a straight line l. Then the bisector of p and p_i converges to the line through $p = p_i$

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perpendicular to l. If we suppose that p is free to choose its direction of attack against p_i, we can define, as p's Voronoi region, the largest part of $\mathrm{VR}(p_i, S)$ that can be cut off by a line through p_i. For all points p_i, these maximal region parts can be computed in total time $O(s)$. The maximum area p can win, in this way, is a candidate for the final solution. Consequently, we may now assume that $p \notin S$ holds.

Second, if p settled at some location outside of the convex hull of S its region would be unbounded. There are several ways of dealing with this fact, as will be discussed in Section 4. In the following we are assuming that the feasible locations for p are restricted to some closed domain F inside the convex hull of S. Then the Voronoi region of p in the Voronoi diagram $V(S \cup \{p\})$ is always of finite area.

Suppose that the Voronoi region p consist of parts of the former regions of certain sites $p_1, \ldots, p_n$ in $V(S)$; these sites form the set N of Voronoi neighbors of p in $V(S \cup \{p\})$. In general, this set N spans a polygon that is star-shaped as seen from p.[1] As our main result, we show that if the set N is in *convex* position then there can be at most one local maximum for the Voronoi area of p, in the interior of the locus of all positions that have N as their neighbor set. The proof is based on a delicate analysis of certain rational functions; it will be given in Section 3.

In Section 4 we analyze the loci of identical Voronoi neighbors, for a given set S of point sites. Moreover, we discuss how a possible extension of our result to the case of general star-shaped Voronoi neighborhoods could be used in an overall algorithm for determining exactly the location of p that attains a maximum Voronoi area. Finally, we mention some directions for future work in Section 5. Section 2 contains some preliminaries, among them tractable formulae for the area of a Voronoi region with convex neighbor set.

For general properties of Voronoi diagrams see the monograph by Okabe et al. [8] or the surveys by Fortune [6] and Aurenhammer and Klein [2].

2 The Area of a Voronoi Region

First, we restate some basic definitions and facts. Let S be a set of s point sites in the plane that are in general position, that is, no four of them are co-circular, no three of them co-linear. By $V(S)$ we denote the Voronoi diagram of the set S. It consists of Voronoi regions $\mathrm{VR}(q, S)$, one to each point q of S, containing all points in the plane that are closer to q than to any other site in S. The planar dual of $V(S)$ is the Delaunay triangulation, $\mathrm{DT}(S)$, of S. It consists of all triangles with vertices in S whose circumcircle does not contain a site of S in its interior. The circumcircle of a Delaunay triangle is also called a Delaunay circle. Both, $V(S)$ and $\mathrm{DT}(S)$, are of complexity $O(s)$ and can be constructed in optimal time $O(s \log s)$.

The set N of all Voronoi or Delaunay neighbors q of site p forms a polygon, $P(N)$, that is star-shaped as seen from p. In this section we derive some useful formulae for the area of the Voronoi region of a new site p with neighbor set N, assuming that $P(N)$ is convex. It is based on computing the *signed areas* of certain triangles. Let (v_0, v_1, v_2) be the vertices of a triangle D, where $v_i = (a_i, b_i)$ in Cartesian coordinates. Then,

$$\mathrm{SignedArea}(D) := \frac{1}{2} \sum_{i=0}^{2} (a_i b_{i+1} - a_{i+1} b_i)$$

[1] A set P is called *star-shaped* as seen from one of its points, p, if any line segment connecting p to a point in P is fully contained in P.

gives the positive area of D if (v_0, v_1, v_2) appear in counterclockwise order on the boundary of D; otherwise, we obtain the negative value. Here, indices are counted mod 3.

Now let p_i, p_{i+1} be two consecutive vertices on the boundary of $P(N)$, in counterclockwise order. Unless p is co-linear with p_i and p_{i+1}, these three point sites define a Voronoi vertex v_i that may or may not be contained in $P(N)$; see Figure 1.

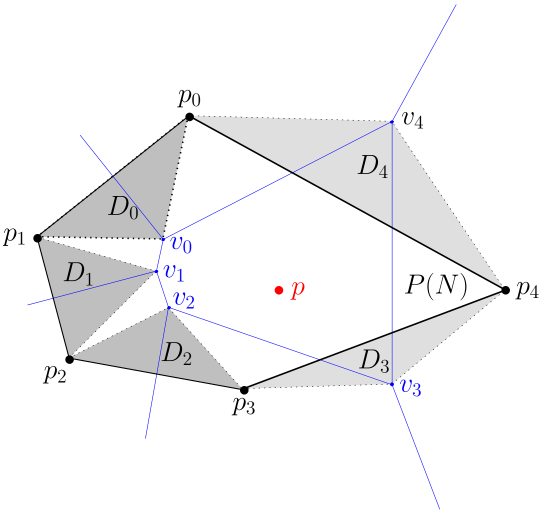

Figure 1: The triangles $D_i = (p_i, v_i, p_{i+1})$. Only D_0, D_1, D_2 are inside the convex neighbor polygon, $P(N)$. Their signed areas are negative, whereas D_3, and D_4 have a positive area.

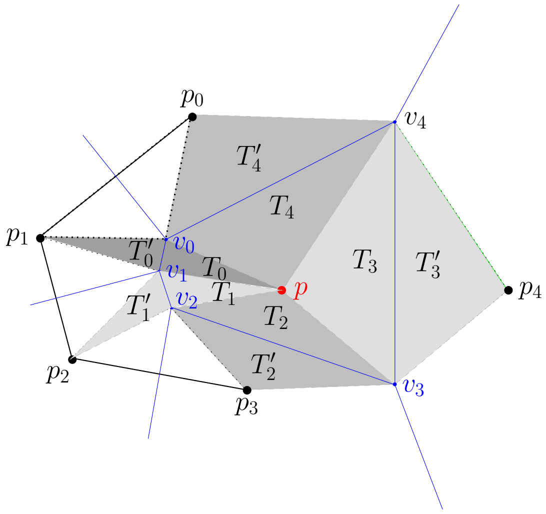

Figure 2: The triangles $T_i = (v_{i+1}, p, v_i)$ and their reflected images T'_i.

Let D_i denote the triangle (p_i, v_i, p_{i+1}), for $i = 0 \ldots n-1$. Its *signed area* is positive if and only if these vertices appear on D_i in counterclockwise order, that is, if and only if v_i lies outside the convex polygon $P(N)$.

Lemma 1 *With the notations from above we have the following identity.*

$$Area(VR(p, S \cup \{p\})) = \frac{1}{2}(Area(P(N)) + \sum_{i=1}^{n} SignedArea(D_i))$$

Proof. The area of $\mathrm{VR}(p, S \cup \{p\})$ equals the sum of the areas of the triangles $T_i := (v_{i+1}, p, v_i)$. Let T'_i be the result of reflecting triangle T_i about its edge $v_i v_{i+1}$. The union of all these triangles equals $P(N)$ minus those triangles D_j that are contained in $P(N)$, plus those D_i not contained in $P(N)$; see Figure 2. $\square$

Lemma 1 reduces the problem of maximizing the area of the Voronoi region of p to maximing the sum of the signed areas of the triangles D_i, assuming N is fixed. Two vertices of D_i are the given points p_i, p_{i+1}. Only the third vertex, v_i, depends on p, and its movement is constrained to the bisector of p_i, p_{i+1}.

Next, we express the signed area of D_i as a function of p in different ways. To this end, let $p_i = (s_i, t_i)$, and let $m_i = (\frac{s_i+s_{i+1}}{2}, \frac{t_i+t_{i+1}}{2})$ be the midpoint of $p_i p_{i+1}$. We put $b_i = |p_i m_i|$ and $l_i = |p m_i|$; see Figure 3 for an illustration.

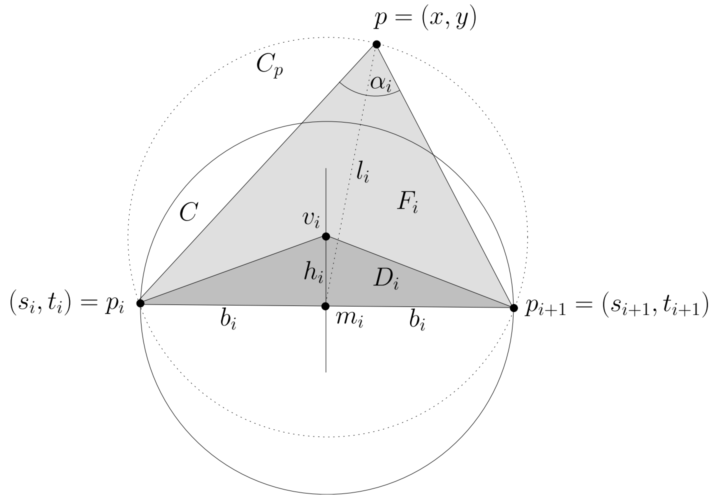

Figure 3: Computing the signed area of the triangle D_i as a function of p. In this case, the sign is negative.

Lemma 2 *Let $p = (x, y)$ be the new point site, different from p_i and p_{i+1}. Then the following identities hold.*

$$-SignedArea(D_i) = b_i^2 \frac{l_i^2 - b_i^2}{2SignedArea(F_i)} \tag{1}$$

$$= b_i^2 \frac{(x - \frac{s_i+s_{i+1}}{2})^2 + (y - \frac{t_i+t_{i+1}}{2})^2 - b_i^2}{x(t_i - t_{i+1}) + y(s_{i+1} - s_i) + s_i t_{i+1} - s_{i+1} t_i} \tag{2}$$

$$= b_i^2 \frac{(x - s_i)(x - s_{i+1}) + (y - t_i)(y - t_{i+1})}{(x - s_i)(t_i - t_{i+1}) + (y - t_i)(s_{i+1} - s_i)} \tag{3}$$

Proof. Let us first assume that p does not lie on the line through p_i and p_{i+1}. Let C denote the diametral circle of the line segment $p_i p_{i+1}$. By definition, $D_i = (p_i, v_i, p_{i+1})$. Clearly, the following equivalences hold.

$$\begin{aligned} p \text{ lies outside of } C \quad &\Leftrightarrow \quad l_i > b_i \\ &\Leftrightarrow \quad \text{SignedArea}(D_i) < 0 \\ &\Leftrightarrow \quad \text{Voronoi vertex } v_i \text{ is contained in } P(N) \end{aligned}$$

Let h_i denote the height of triangle D_i, so that $\text{Area}(D_i) = b_i h_i$. The Voronoi vertex v_i can be expressed as a vector sum

$$\mathbf{v_i} = \mathbf{m_i} + h_i \mathbf{e_i},$$

where $\mathbf{e_i}$ denotes the unit vector that runs from m_i towards v_i along the bisector of p_i, p_{i+1}. We have $\mathbf{e_i} = \frac{1}{2b_i}(t_i - t_{i+1}, s_{i+1} - s_i)$ if $\text{SignedArea}(D_i) < 0$; otherwise the direction of $\mathbf{e_i}$ is reversed. On the other hand, $p = (x, y)$ lies on a circle of radius $\sqrt{h_i^2 + b_i^2}$ centered at v_i. Plugging the cartesian coordinates of v_i into the equation of this circle, and solving for h_i, leads to formula (2), since the coefficient of h_i reduces to zero.

The numerators in formulae (1) and (2) are identical, and so are the denominators. Formula (3) follows directly from (2), using the identity

$$b_i^2 = (\frac{s_i - s_{i+1}}{2})^2 + (\frac{t_i - t_{i+1}}{2})^2.$$

□

It is interesting to observe that in the situation shown in Figure 3 the area of triangle D_i is also given by $b_i^2 \cot \alpha_i$, where α_i denotes the angle at vertex p of triangle F_i. Indeed, as p moves along circle C_p, the values of α_i and l_i do not change. When p is colinear with v_i and m_i we obtain $\text{Area}(F_i) = l_i b_i$. Moreover,

$$\begin{aligned}\cot \alpha_i &= \frac{\cos^2 \frac{\alpha_i}{2} - \sin^2 \frac{\alpha_i}{2}}{2 \sin \frac{\alpha_i}{2} \cos \frac{\alpha_i}{2}} = \frac{1}{2}(\cot \frac{\alpha_i}{2} - \tan \frac{\alpha_i}{2}) \\ &= \frac{1}{2}(\frac{l_i}{b_i} - \frac{b_i}{l_i}) = \frac{1}{2} \frac{l_i^2 - b_i^2}{l_i b_i}.\end{aligned}$$

If p lies on the line through, but differs from, p_i and p_{i+1}, then the denominator of (2), that is, the signed area of F_i, becomes 0, and the area of D_i is infinite since Voronoi vertex v_i is at infinity now. The numerator of formula (2) is the equation of the circle C. Therefore, the area of D_i vanishes whenever p is placed on $C \setminus \{p_i, p_{i+1}\}$, because v_i is then equal to m_i.

At the given points p_i and p_{i+1} the signed area of triangle D_i is undefined, and there is no continuous way of closing these gaps. However, when point p is restricted to move along a line $\{Y = eX + f\}$ through p_i, the area function can be continuously extended. If we substitute, in formula (3) of Lemma 2, $ex + f$ for y, where $f = t_i - es_i$, then the root $x - s_i$ cancels out, and we obtain a finite value at $x = s_i$ that depends on e.[2]

3 Uniqueness of the Local Maximum

In this section we assume that N, the set of Voronoi neighbors of the new site, p, consists of n points in convex position. Then the locus, L_N, of all placements of p that have N as their neighbor set is contained in the convex polygon $P(N)$.

Now we state our main result.

Theorem 3 *Let N be a convex neighbor set. Then the area of the Voronoi region of a new point p has at most one local maximum in the interior of $P(N) \cap L_N$.*

As usual, a function f is said to have a local maximum at point a if $f(a) \geq f(b)$ holds, for all b in a neighborhood of a.

Proof. By Lemma 1 it is sufficient to prove that the sum of the signed areas of the triangles D_i has at most one local maximum in the interior of $P(N)$. It is enough to show that this sum attains at most one maximum along each line through $P(N)$.

We substitute, in formula (2) of Lemma 2, the variable y by the coordinates $eX + f$ of a line G. By performing partial fraction decomposition, we obtain

$$-\text{SignedArea}(D_i(X)) = \frac{A_i}{X - a_i} + c_i X + d_i.$$

If G does not pass through p_i or p_{i+1} then there is a a proper pole at $X = a_i$, where G intersects the line G_i through p_i, p_{i+1}; compare the discussion at the end of Section 2. More precisely, if the point $G \cap G_i$ lies outside the line segment $p_i p_{i+1}$ then, in formula (1)

[2]The same holds in case line G passes through p_{i+1} because we can replace, in the denominator of formula (3), s_i with s_{i+1} and t_i with t_{i+1}.

of Lemma 2, we have $l_i > b_i$, while the sign of the area of F_i changes from $-$ to $+$. Consequently, the sign of $-D_i(X)$ changes from $-$ to $+$. But if G intersects the interior of $p_i p_{i+1}$ then $l_i < b_i$, so that $-D_i(X)$ changes from $+$ to $-$.

If line G does pass through the given point p_i or p_{i+1}, then there is no pole, and we have $A_i = 0$.

Let us assume that line G equals the X-axis, and let

$$a_1 \leq a_2 \leq \ldots \leq a_m \leq l < r \leq b_1 \leq \ldots \leq b_k$$

denote the n points that correspond to its intersections with the lines G_i. By the convexity of $P(N)$, the two intersections of the X-axis with the boundary of $P(N)$ must be consecutive in this sequence; they are denoted by l and r.

Figure 4 shows the behavior of

$$\begin{aligned} f(X) &:= -\sum_{i=1}^{n} \text{SignedArea}(D_i) = \\ &= \sum_{i=1}^{m} \frac{A_i}{X - a_i} - \frac{L}{X - l} + \frac{R}{X - r} - \sum_{i=1}^{k} \frac{B_i}{X - b_i} + cX + d \end{aligned}$$

as a function of X. By the above discussion, we have $A_i, B_i > 0$ and $L, R \geq 0$.

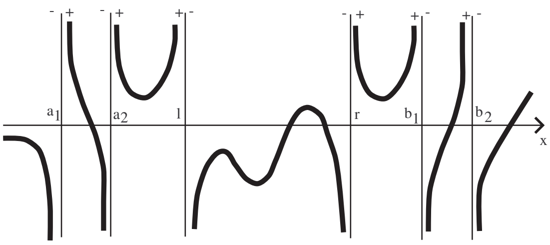

Figure 4: Between l and r, the function $f(X)$ can have at most one local minimum.

First, we assume that both L and R are strictly positive. We want to prove that $f(X)$ has at most one local minimum in the interval (l, r). Since f comes from $-\infty$ at l, and returns to $-\infty$ at r, it is sufficient to show that its second derivative

$$\frac{1}{2} f''(X) = \sum_{i=1}^{m} \frac{A_i}{(X - a_i)^3} - \frac{L}{(X - l)^3} + \frac{R}{(X - r)^3} - \sum_{i=1}^{k} \frac{B_i}{(X - b_i)^3}$$

has at most two zeros in (l, r). We split the function into two parts,

$$\begin{aligned} g(X) &:= \sum_{i=1}^{m} \frac{A_i}{(X - a_i)^3} - \frac{L}{(X - l)^3} \text{ and} \\ h(X) &:= \sum_{i=1}^{k} \frac{B_i}{(X - b_i)^3} - \frac{R}{(X - r)^3}, \end{aligned}$$

such that $f''/2 = g - h$ holds, and discuss g and h independently.

Lemma 4 *Each of the functions g and g'' has at most one zero in (l, ∞), and each of h, h'' has at most one zero in $(-\infty, r)$.*

Proof. Let $x_1 \neq x_0 \in (l, \infty)$ be such that x_0 is a zero of g. Then,

$$0 = g(x_0) \quad = \quad \sum_{i=1}^{m} \frac{A_i}{(x_0 - a_i)^3} - \frac{L}{(x_0 - l)^3}, \tag{4}$$

and by multiplying both sides by $\frac{(x_0-l)^3}{(x_1-l)^3}$ we obtain

$$\begin{aligned} g(x_0) &= \sum_{i=1}^{m} \frac{A_i}{(x_0 - a_i)^3} \frac{(x_0 - l)^3}{(x_1 - l)^3} - \frac{L}{(x_1 - l)^3} && (5) \\ &= \sum_{i=1}^{m} \frac{A_i}{(x_1 - a_i)^3} \left(\frac{(x_1 - a_i)^3}{(x_0 - a_i)^3} \frac{(x_0 - l)^3}{(x_1 - l)^3} \right) - \frac{L}{(x_1 - l)^3} && (6) \\ &< \sum_{i=1}^{m} \frac{A_i}{(x_1 - a_i)^3} - \frac{L}{(x_1 - l)^3} = g(x_1) \text{ , if } x_1 > x_0. && (7) \end{aligned}$$

Analogously, we have

$$0 = g(x_0) \quad > \quad g(x_1) \tag{8}$$

if $x_1 < x_0$ holds. The alternatives (7) or (8) follow from (6) because $a_i < l < x_0, x_1$ implies that

$$\frac{(x_1 - a_i)^3}{(x_0 - a_i)^3} \frac{(x_0 - l)^3}{(x_1 - l)^3}$$

is of value < 1 if $x_1 > x_0$ holds, and of value > 1, otherwise. Consequently, g has at most one zero in (l, ∞). The other claims are proven analogously. □

As a consequence of Lemma 4, the function g has at most one zero and at most one turning point to the right of l. Since g has a negative pole at l and tends to 0 for large values of X, its graph has one of the two possible shapes shown in Figure 5, together with the possible shapes of the graph of h.

Our next lemma implies that $f''/2 = g - h$ has at most two zeros in the interval (l, r).

Lemma 5 *The graphs of the functions g and h have at most two points of intersection over (l, r).*

Proof. If neither g nor h have a zero in (l, r) their graphs do not intersect; see Figure 5. Suppose that h has a zero in (l, r); then it has a unique minimum, m. Let us assume that p_1 and p_2 are, from left to right, the first points of intersection of the two graphs in (l, r).

We argue that p_2 must be situated to the right of minimum m of h. Indeed, to the left of m function h is decreasing, and runs below the X-axis. But below the X-axis, function g is increasing. Thus, at most one intersection, p_1, can be situated to the left of m.

If p_2 lies to the left of the maximum, M, of function g, or if g does not have a maximum, then, in (p_2, ∞), the two graphs are separated by the wedge between their tangents at p_2. To the right of M, function g is decreasing and runs above the X-axis, while h is increasing above the X-axis. In either case, there can be no further point of intersection to the right of p_2. □

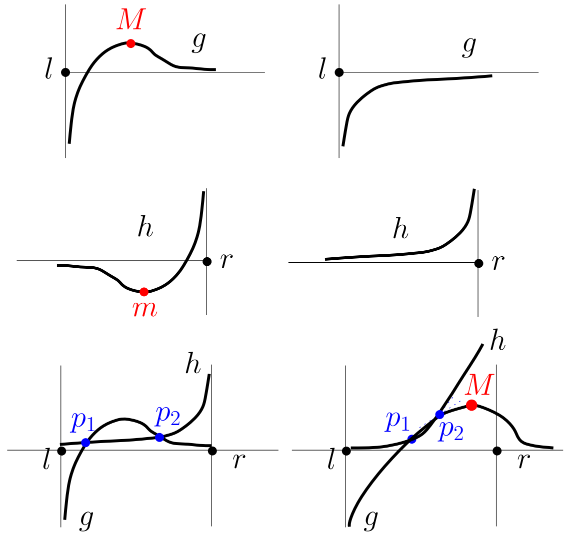

Figure 5: The possible shapes of the graphs of g and h. There can be at most two points of intersection between l and r.

So far we have shown that the function f takes on at most one local minimum along each line G that enters and leaves the convex polygon $P(N)$ through interior edge points.

It remains to generalize this statement to the case where the line G passes through one or two of the given points of N, upon entering and leaving $P(N)$. First, suppose G enters through p_i and leaves through an edge, so that $L = 0$ and $R > 0$ hold. We study the same functions f, g, h as before, but on the interval (a_m, r). Clearly, function g is strictly positive now. It comes from $+\infty$ and tends to 0 for large values of X. Hence, its graph can intersect the graph of h at most once. Consequently, function f has at most one turning point. Since f comes from $+\infty$ at $X = a_m$ and tends to $-\infty$ at $X = r$, it can have at most one minimum in between.

If both L and R vanish because line G enters and leaves $P(N)$ through vertices p_i, p_j then we consider the interval (a_m, b_1) between the innermost poles. Since both graphs of g and h are strictly positive now, they have at most one point in common. Function f comes from and returns to $+\infty$ at a_m and b_1. Because f has at most one turning point in between, it has exactly one minimum.

This completes the proof of Theorem 3. $\square$

To give an example, let us assume that n points are evenly placed on the boundary of the unit circle. For $n \leq 4$ there is no local maximum of the Voronoi area. In fact, there is a unique local minimum at the center for $n = 3$; for $n = 4$, the cross formed by the four point sites consists of minimal positions. But for $n \geq 5$ we have a unique local maximum at the center of the circle.

4 Global Considerations

In the preceding section we have studied the situation where the new site, p, moves only locally, so that the set N of its Voronoi neighbors does not change. During a global move of p, three events may happen. First, its set of Voronoi neighbors can change.

As before, let L_N be the locus of all placements of p that have exactly the points in N as their Voronoi neighbors. Figure 6 shows an example where the set L_N is not connected. In general, L_N consists of several maximal connected subsets C_N called the *neighborship*

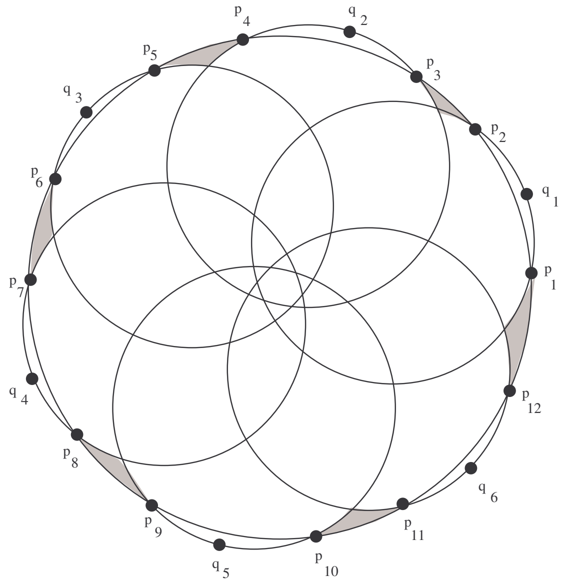

Figure 6: Each of the shaded cells has $p_1, \ldots, p_{12}$ as neighbor set.

cells of N, whose nature is determined by the following Lemma 6. Observe that for two neighboring sites, q and r, on the convex hull of S we can define, as their Delaunay triangle and circumcircle, the halfplane defined by the line through q, r that does not contain a site of S.

Lemma 6 *Let S be a set of s point sites in the plane. The neighborship cells with respect to S are the cells of the arrangement of the Delaunay circles of S. Each cell C has, as its neighbor set N, all sites that span a Delaunay circle containing C. The total complexity of all neighborship cells is in $O(s^2)$.*

Proof. The standard incremental algorithm for constructing the Delaunay triangulation is built on the following fact. On inserting a new site, p, into the Delaunay triangulation of S, there will be a Delaunay edge of $DT(S \cup \{p\})$ connecting p with $q \in S$ if and only if p lies in the circumcircle of a Delaunay triangle of $DT(S)$ that has q as a vertex. This shows that all points of the same cell have the same set of Voronoi neighbors, namely all sites that span a Delaunay circle containing the cell. Moreover, edge-adjacent cells have different sets of Voronoi neighbors. Indeed, if p leaves a Delaunay circle spanned by u, v, w through the arc between u and v then point w can no longer be a Delaunay neighbor of p because the edge pw would cross the edge uv of the Delaunay triangle (u, v, w) of $DT(S \cup \{p\})$. $\square$

The arrangement of $O(s)$ many circles can be constructed in time $O(s\lambda_4(s))$ by a deterministic[3] algorithm, or in expected time $O(s \log s + k)$, where k denotes the complexity of the arrangement; see Sharir and Agarwal [10].

[3]As usual, $\lambda_t(s)$ denotes the maximum length of a Davenport-Schinzel sequence of order t over s characters.

Another event happens when p hits the boundary of the convex hull of the site set S. At this point, the region of p becomes unbounded. There are several ways of dealing with this phenomenon. The most simple one we suggest here is to assume that a certain feasability domain, F, is given, that consists of neighborship cells contained in the interior of the convex hull of S, and that the placement of p is restricted to F ("far out of town there are no customers to win"). One could also think of allowing unbounded Voronoi regions, and measuring their area by the angle between the two unbounded Voronoi edges. Another approach could be to specify population densities, instead of the uniform distribution, with or without defining a feasibility domain F.

Finally, the position of the new site, p, could coincide with one of the existing sites, $p_i \in S$. At these points the area function fails to be continuous; in fact, the former region of p_i is split among p and p_i by a bisector through $p = p_i$ whose slope is perpendicular to the direction in which p has approached p_i, as we discussed in Section 1. But apart from the points p_i, the area function is smooth, as was shown independently by Okabe and Aoyagi [7] and by Piper [9] who generalized work by Sibson [11].

Let us assume the uniqueness of the local maximum proven in Section 3 for convex Voronoi neighbor sets were also true for the general star-shaped neighbor sets N. Then we could employ the following technique for finding the global maximum within the whole feasibility domain F. First, we compute how large an area p can obtain by moving close to an existing site from the right direction. This takes total time $O(s)$. Next, we compute the Delaunay triangulation of S, and the arrangement of all Delaunay circles in time $O(s\lambda_4(s))$. We inspect each cell C of F in turn, and compute the optimal placement of p within the closure of C. Within the interior of C we can simply follow the gradient which leads to the (unique!) maximum, or to the boundary of C. Finally, it would remain to check for maxima on the boundary of C, which consists of circular arcs, by Lemma 6.

5 Conclusions

In this paper we have shown that the Voronoi area of a new site has at most one local maximum in the interior of each neighborship cell, if the Voronoi neighbors are in convex position. This result gives rise to many further questions.

The obvious open problem is if the maximum is still unique if the neighbors are in star-shaped position. The main difference to the convex case is the following. The line G, along which the new site p was supposed to move in the proof of Theorem 3, can now intersect edge extensions of the neighbor polygon $P(N)$ *inside* $P(N)$, too. Consequently, the functions g and h in the proof of Lemma 4 become more complicated. We expect that considerably more (mathematical) effort will be necessary in order to settle this problem.

Other questions concern the customer model. Also, it would be interesting to study metrics different from the Euclidean, that are frequently used in location planning. From a theoretical point of view, it would also be interesting to investigate higher dimensions.

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