# Reverse nearest neighbor queries in fixed dimension\*

Otfried Cheong<sup>1</sup>

Antoine Vigneron<sup>2</sup>

Juyoung Yon<sup>1</sup>

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#### Abstract

Reverse nearest neighbor queries are defined as follows: Given an input point set P, and a query point q, find all the points p in P whose nearest point in  $P \cup \{q\} \setminus \{p\}$  is q. We give a data structure to answer reverse nearest neighbor queries in fixed-dimensional Euclidean space. Our data structure uses O(n) space, its preprocessing time is  $O(n \log n)$ , and its query time is  $O(\log n)$ .

#### **1** Introduction

Given a set P of n points in  $\mathbb{R}^d$ , a well-known problem in computational geometry is nearest neighbor searching: preprocess P such that, for any query point q, a point in P that is closest to q can be reported efficiently. This problem has been studied extensively; in this paper, we consider the related problem of *reverse nearest neighbor searching*, which has attracted some attention recently.

The reverse nearest neighbor searching problem is the following. Given a query point q, we want to report all the points in P that have q as one of their nearest neighbors. More formally, we want to find the points  $p \in P$  such that for all points  $p' \in P \setminus \{p\}$ , the distance |pp'| is larger or equal to the distance |pq|.

The earliest work on reverse nearest neighbor searching is by Korn and Muthukrishnan [11]. They motivate this problem by applications in databases. Their approach is based on R-Trees, so it is unlikely to give a good worst-case time bound. Subsequently, the reverse nearest neighbor searching problem has attracted some attention in the database community [3, 12, 13, 15, 16, 18, 19, 20].

The only previous work on reverse nearest neighbor searching where worst-case time bounds are given is the work by Maheshwari et al. [14]. They give a data structure for the two-dimensional case, using O(n) space, with  $O(n \log n)$  preprocessing time, and  $O(\log n)$  query time. Their approach is to show that the arrangement of the largest empty circles centered at data points has linear size, and then they answer queries by doing point location in this arrangement.

In this paper, we extend the result of Maheshwari et al. [14] to arbitrary fixed dimension. We give a data structure for reverse nearest neighbor searching in  $\mathbb{R}^d$ , where d = O(1), using the Euclidean distance. Our data structure has size O(n), with preprocessing time  $O(n \log n)$ , and with query time  $O(\log n)$ . It is perhaps surprising that we can match the bounds for the two-dimensional case in arbitrary fixed dimension. For *nearest neighbor queries*, this does not seem to be possible: The bounds for nearest neighbor searching in higher dimension depend on the complexity of the Voronoi diagram, which is  $\Theta(n^{\lceil d/2 \rceil})$  in d-dimensional space.

Our approach is similar to some previous work on approximate Voronoi diagrams [2, 9, 10]: the space is partitioned using a compressed quadtree, each cell of this quadtree containing a small set of

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<sup>&</sup>lt;sup>1</sup>Dept. of Computer Science, Korea Advanced Institute of Science and Technology, Gwahangno 335, Daejeon 305-701, South Korea. Email: otfried@kaist.edu, yeon-ju-young@kaist.ac.kr.

<sup>&</sup>lt;sup>2</sup>INRA, UR 341 Mathématiques et Informatique Appliquées, 78352 Jouy-en-Josas, France. Email: antoine.vigneron@jouy.inra.fr.

candidate points. Queries are answered by finding the cell containing the query point, and checking all the candidate points in this cell. Interestingly, this approach allows to answer reverse nearest neighbor queries efficiently and *exactly*, while it only seems to give approximations for nearest neighbor searching.

Our model of computation is the real-RAM model, with some additional operations that are common in quadtree algorithms, such as the floor function, the logarithm  $\log_2$ , and the bitwise XOR. In particular, we need to be able to find in constant time the first binary digit at which two numbers differ. This allows, for instance, to find in constant time the smallest quadtree box that contains two given points. For more details on this issue, we refer to the lecture notes of Har-Peled [9], and to previous work related to quadtrees [4, 8].

#### 2 Compressed quadtrees

In this section, we describe compressed quadtrees, a well known data structure in computational geometry. A more detailed presentation can be found in Har-Peled's lecture notes [9], in the article on skip quadtrees by Eppstein, Goodrich, and Sun [8], or in the article by Bern, Eppstein and Teng [4]. We first describe quadtrees, and then we describe their compressed version.

We consider quadtrees in  $\mathbb{R}^d$ , where d = O(1). We denote by  $H_r$  the hypercube  $[-1, 1]^d$ ; the leaves of a quadtree will form a partition of  $H_r$ .

A quadtree box is either  $H_r$ , or is obtained by partitioning a quadtree box H into  $2^d$  equal sized hypercubes—these hypercubes are called the quadrants of H. A quadtree is a data structure that stores quadtree boxes in a hierarchical manner. Each node  $\nu$  of a quadtree stores a quadtree box  $C(\nu)$ , and pointers to its parent and its children. We call  $C(\nu)$  the *cell* of node  $\nu$ . In this paper, the cell of the root of a quadtree is always the box  $H_r$ . Each node  $\nu$  either is a leaf, or has  $2^d$  children that store the  $2^d$ quadrants of  $C(\nu)$ . With this definition, the cells of the leaves of a quadtree form a partition of  $H_r$ .

Let S denote a set of m quadtree boxes. We can construct the smallest quadtree whose nodes store all boxes in S as follows. We start by constructing the root. If  $S \subset \{H_r\}$ , then we are done. Otherwise, we construct the  $2^d$  children of the root. We consider the subset  $S_1 \subset S$  (resp.  $S_2, S_3, \ldots$ ) of the boxes in S contained in the first quadrant (resp. second, third,...). We construct recursively the quadtree, by taking the first (resp. second, third,...) child as the root and using the set of boxes  $S_1$  (resp.  $S_2, S_3, \ldots$ ).

The above construction results in a quadtree that stores all the boxes in S. Even though it is the smallest such quadtree, its size can be arbitrarily large when S contains very small boxes. To remedy this, we use a *compressed quadtree*, which allows to bypass long chains of internal nodes.

In order to reduce the size of the data structure, we allow two different kinds of nodes in a compressed quadtree. An *ordinary node* stores a quadtree box as before. A *compressed node*  $\nu$ , however, stores the difference  $H \setminus H'$  of two quadtree boxes H and H'. We still call this difference the cell  $C(\nu)$ . Compressed nodes are always leaves of the compressed quadtree.

As in a quadtree, the cells of the children of a node  $\nu$  form a partition of  $C(\nu)$ . But two cases are now possible: either these cells are the quadrants of  $C(\nu)$ , or  $\nu$  has two children, one of them storing a quadtree box  $H \subset C(\nu)$ , and the other storing  $C(\nu) \setminus H$ .

The construction of a compressed quadtree that stores all the boxes in S is analogous to the construction of the ordinary quadtree, with the following difference. Assume we are processing an internal node  $\nu$ . Let H denote the smallest quadtree box containing the boxes in S that are strictly contained in  $\mathcal{C}(\nu)$ . If  $H = \mathcal{C}(\nu)$ , then we proceed exactly as we did for the ordinary quadtree: we construct  $2^d$  children corresponding to the quadrants of  $\mathcal{C}(\nu)$ . Otherwise,  $\nu$  has two children, one stores H, and the other is a compressed node that stores  $\mathcal{C}(\nu) \setminus H$ . Intuitively, this construction of a compressed node allows us to "zoom in" when all the boxes in  $\mathcal{C}(\nu)$  are within a small area, and avoids a long chain of internal nodes. (See Figure 1.)

A direct implementation of the quadtree construction that we just described would lead to  $O(m^2)$ 



Figure 1: (a) A set of quadtree boxes. (b) The quadtree storing these quadtree boxes. (c) The compressed quadtree storing the same set of quadtree boxes. The two cells on the left side correspond to compressed nodes.

construction time and O(m) query time, since a quadtree may have linear depth. However, it is possible to achieve  $O(m \log m)$  construction time and  $O(\log m)$  query time using different algorithms for constructing and querying a compressed quadtree. One such construction is presented in Har-Peled's lecture notes [9]. The idea is first to find a quadtree box  $C_0$  that contains a constant fraction of the input boxes S, which can be done in linear time by choosing the center among the vertices of an irregular, constant-size grid. Then one computes recursively a compressed quadtree for the set  $S_{in}$  of the boxes in S that are contained in  $C_0$ , and for  $S_{out} = S \setminus S_{in}$ . Finally, these two quadtrees are merged in linear time, essentially by hanging the quadtree of  $S_{in}$  at a leaf of  $S_{out}$ .

The quadtree can be queried in  $O(\log m)$  time by constructing a *finger tree* over the quadtree, which is an auxiliary data structure to allow faster search. (This approach is also presented in Har Peled's notes [9].) We first find a cell of the quadtree such that the subtree rooted at the corresponding node contains a constant fraction of the boxes in S. This node is called a *separator* and can be found in linear time by traversing the quadtree. This construction is repeated recursively on the forest obtained by removing this node. The construction allows to answer a query in  $O(\log m)$  time, as this tree has  $O(\log m)$  depth. So we have the following bounds for constructing and querying a quadtree:

**Lemma 1.** Let S be a set of m quadtree boxes contained in  $H_r$ . We can construct in time  $O(m \log m)$  a compressed quadtree  $\mathcal{T}$ , such that each box in S is the cell of a node of  $\mathcal{T}$ . This compressed quadtree  $\mathcal{T}$  has size O(m). After  $O(m \log m)$  preprocessing time, we can find for any query point  $q \in H_r$  the leaf of  $\mathcal{T}$  whose cell contains q in time  $O(\log m)$ .

Note that a query point might lie on the boundaries of several cells. In this case, we break the tie arbitrarily, and we return only one cell containing q.

#### **3** Data structure for reverse nearest neighbor queries

In this section, we describe the construction of our data structure and how we answer reverse nearest neighbor queries. This data structure is a compressed quadtree, with a set of candidate points stored at each node. To answer a query, we locate the leaf of the compressed quadtree whose cell contains the query point, and we check all the candidate points in this leaf; the reverse nearest-neighbors are among these points. We start with some notation.

Our input point set is denoted by  $P = \{p_1, \ldots, p_n\}$ , with  $n \ge 2$ . We still work in  $\mathbb{R}^d$ , where d = O(1), and so  $P \subset \mathbb{R}^d$ . The *empty ball*  $b_i$  is the largest ball centered at  $p_i \in P$  that does not contain any other point of P in its interior. In other words, the boundary of the empty ball centered at p goes



Figure 2: Notation.

through the nearest point to p in  $P \setminus \{p\}$ . In this paper, we only consider closed balls, so  $p_i$  is a reverse nearest neighbor of a query point q if and only if  $q \in b_i$ .

Let  $H_P$  be a smallest axis-aligned *d*-dimensional hypercube containing the input point set *P*. Without loss of generality, we assume that  $H_P = [-1/2\sqrt{d}, 1/2\sqrt{d}]^d$ ; then any empty ball is contained in  $H_r = [-1, 1]^d$ . When  $\nu$  is an ordinary node, we denote by  $s(\nu)$  the side length of the quadtree box  $C(\nu)$ .

We first compute the set of all the largest empty balls for P. This can be done in  $O(n \log n)$  time using Vaidya's all-nearest neighbors algorithm [17]. We denote by  $r_i$  the radius of  $b_i$ . For each  $b_i$ , we compute the quadtree boxes with side length in  $[2r_i, 4r_i)$  that overlap  $b_i$ . (See Figure 2.) Our model of computation allows us to do this in O(1) time. There are at most  $2^d$  such boxes; we denote them by  $h_i^j, j \in \{1, \ldots, 2^d\}$ .

Using Lemma 1, we construct in  $O(n \log n)$  time a compressed quadtree  $\mathcal{T}$  of size O(n) such that each box  $h_i^j$  appears in  $\mathcal{T}$ . For each node  $\nu$  of  $\mathcal{T}$ , if the corresponding cell  $\mathcal{C}(\nu)$  is  $h_i^j$ , we store  $p_i$  as a candidate point for  $\nu$ . Storing these candidate points can be done during the construction of the quadtree within the same time bound. Notice that we may store several candidate points for a given node  $\nu$ .

These sets of candidate points are not sufficient for our purpose, so we will add some other points. For each ordinary (non-compressed) node  $\nu$ , we store the points  $p_i$  such that  $r_i > s(\nu)/4$  and  $b_i$  overlaps  $C(\nu)$ ; this list of candidate points is denoted by  $\mathcal{L}(\nu)$ . In order to analyze our algorithm, we need the following lemma, which is proved in Section 4.

**Lemma 2.** For any ordinary node  $\nu$ , the cardinality of the set of candidate points  $\mathcal{L}(\nu)$  stored at  $\nu$  is O(1).

We construct the lists  $\mathcal{L}(\cdot)$  by traversing  $\mathcal{T}$  recursively, starting from the root. Assume that  $\nu$  is the current ordinary node. The points  $p_i$  such that  $h_i^j = \mathcal{C}(\nu)$  for some j have already been stored at  $\nu$ . By our construction, they are the points  $p_i$  in  $\mathcal{L}(\nu)$  such that  $s(\nu)/4 < r_i \leq s(\nu)/2$ . So we need the other candidate points  $p_k$ , such that  $r_k > s(\nu)/2$ . These points can be found in  $\mathcal{L}(\nu')$ , where  $\nu'$  is the parent of  $\nu$ . So we insert in  $\mathcal{L}(\nu)$  all the points  $p_k \in \mathcal{L}(\nu')$  such that  $b_k$  overlaps  $\mathcal{C}(\nu)$ , which completes the construction of  $\mathcal{L}(\nu)$ . By Lemma 2, this can be done in O(1) time per node, and thus overall, computing the lists of candidate points for ordinary nodes takes O(n) time.

If  $\nu$  is a compressed node, and  $\nu'$  is its parent, we just set  $\mathcal{L}(\nu) = \mathcal{L}(\nu')$ . We complete the construction of our data structure by handling all the compressed nodes.

Given a query point q, we answer reverse nearest-neighbor queries as follows. If  $q \notin H_r$ , then we return  $\emptyset$ , because we saw earlier that all empty balls are in  $H_r$ . Otherwise, we find the leaf  $\nu$  such that  $q \in C(\nu)$ , which can be done in  $O(\log n)$  time by Lemma 1. For each point  $p_i \in \mathcal{L}(\nu)$ , we check whether  $p_i$  is a reverse nearest neighbor, that is, we check whether  $q \in b_i$ . If this is the case, we report  $p_i$ .

We still need to argue that we answered the query correctly. Assume that  $p_k$  is a reverse nearest neighbor of q, and the leaf  $\nu$  containing q is an ordinary node. As  $q \in b_k$ , we have  $q \in h_k^j$  for some j,

and since the side length of  $h_k^j$  is less than  $4r_k$ , we have  $s(\nu) < 4r_k$ . Since  $b_k$  contains q, it overlaps  $\mathcal{C}(\nu)$ , so by definition of  $\mathcal{L}(\nu)$ , we have  $p_k \in \mathcal{L}(\nu)$ , and thus  $p_k$  was reported. If  $\nu$  is a compressed node, then the same proof works if we replace  $\nu$  by its parent  $\nu'$ , since  $\mathcal{L}(\nu) = \mathcal{L}(\nu')$ .

The discussion above proves the main result of this paper:

**Theorem 3.** Let P be a set of n points in  $\mathbb{R}^d$ . We assume that d = O(1). Then we can construct in time  $O(n \log n)$  a data structure of size O(n) that answers reverse nearest-neighbor queries in  $O(\log n)$  time. The number of reverse nearest neighbors is O(1).

The fact that the number of reverse nearest neighbors is O(1) was known before: In fixed dimension, the in-degree of the vertices of the nearest neighbor graph is bounded by a constant.

## 4 Proof of Lemma 2

In this section, we prove Lemma 2, which was needed to establish the time bounds in Theorem 3. We start with a packing lemma.

**Lemma 4.** Let b be a ball with radius r. Then b intersects at most  $2 \times 5^d$  empty balls with radius larger or equal to r.

*Proof.* When  $x, y \in \mathbb{R}^d$ , we denote by |xy| the Euclidean distance between x and y, and we denote by  $\overline{xy}$  the line segment connecting them.

We denote by c the center of b, and we denote by b' the ball with center c and radius 2r. We first bound the number of empty balls with radius  $\ge r$  whose center is contained in b'. Let B denote this set of balls, and let C denote the set of their centers. Any two points in C are at distance at least r from each other. Hence, the balls with radius r/2 and with centers in C are disjoint. As they are all contained in the ball b'' with center c and radius 5r/2, the sum of their volumes is at most the volume of b''. Hence, we have  $|C| \le 5^d$ , and thus  $|B| \le 5^d$ .

We now consider the empty balls with radius  $\geq r$  that intersect b, and whose centers are not in b'. We denote by B' the set of these balls, and we denote by C' the set of their centers. Let  $b_1$  (resp.  $b_2$ ) be a ball in B' with radius  $r_1$  (resp.  $r_2$ ) and center  $c_1$  (resp.  $c_2$ ). (See Figure 3.) Without loss of generality,



Figure 3: Proof of Lemma 4.

we assume that  $r_1 \leq r_2$ .

Let  $c'_2$  be the point of  $\overline{cc_2}$  such that  $|cc'_2| = |cc_1|$ . Let  $r'_2 = r_2 - |c_2c'_2|$ , and let  $b'_2$  denote the ball with center  $c'_2$  and radius  $r'_2$ . As  $b'_2 \subset b_2$ , we know that  $b'_2$  does not contain  $c_1$  in its interior, and thus

 $r'_2 \leq |c_1c'_2|$ . As  $b'_2$  intersects b, we have  $|cc'_2| \leq r + r'_2$ . It implies that  $|cc'_2| - r \leq |c_1c'_2|$ . Since  $|cc'_2| \geq 2r$ , it follows that  $|cc'_2| \leq 2|c_1c'_2|$ .

Let  $c_1''$  (resp.  $c_2''$ ) denote the projection of  $c_1$  (resp.  $c_2$ ) onto the unit sphere u centered at c. In other words,  $c_1'' = c + (1/|cc_1|)(c_1 - c)$ . Then it follows from the previous paragraph that  $|c_1''c_2''| \ge 1/2$ . Hence, the spheres with radius 1/4 and centered at the projections onto u of the points in C' are disjoint. As these spheres are contained in the sphere with radius 5/4 centered at c, we have  $|C'| \le 5^d$ , and thus  $|B'| \le 5^d$ .

Now we prove Lemma 2: For any ordinary node  $\nu$ , the number of candidate points stored in  $\mathcal{L}(\nu)$  is O(1). We assume that  $p_i \in \mathcal{L}(\nu)$ . By definition, we must have  $r_i > s(\nu)/4$ , and  $b_i$  overlaps  $\mathcal{C}(\nu)$ . As  $\mathcal{C}(\nu)$  can be covered by O(1) balls with radius  $s(\nu)/4$ , Lemma 4 implies that there can be only O(1) such candidate points.

#### 5 Concluding remarks

Our approach does not only give a data structure to answer reverse nearest neighbor queries, it also yields a *reverse Voronoi diagram*: a spatial subdivision with linear complexity such that, within each cell, the set of reverse nearest neighbors is fixed. To achieve this, we construct, within the cell of each leaf of our quadtree, the arrangement of the empty balls of the candidate points. As there is only a constant number of candidates per cell, each such arrangement has constant complexity, so overall we get a subdivision of linear size.

The time bounds of our data structure can be improved in the word RAM model of computation, when the coordinates of the input points are  $O(\log n)$ -bits integers. In this case, Chan showed that the all-nearest neighbors computation and the compressed quadtree construction can be done in linear time, so our data structure can be built in linear time as well. Then using the shuffle-and-sort approach of Chan [5], combined with van Emde Boas trees, the compressed quadtree in our data structure can be queried in  $O(\log \log n)$  time. So overall, we can construct in linear time a data structure for reverse nearest neighbors with query time  $O(\log \log n)$ .

The most natural extension to this problem would be to handle different metrics. Our approach applies directly to any norm of  $\mathbb{R}^d$ , with d = O(1), as its unit ball can be made *fat* after changing the coordinate system: we just need to apply an affine map such that the John ellipsoid of the unit ball of this norm becomes a Euclidean ball. The time bounds and space usage remain the same.

Another possible extension would be to make our algorithm dynamic. The main difficulty is that it seems that we would need to maintain the empty balls, which means maintaining all nearest neighbors. The result of Maheshwari et al. [14], combined with the data structure of Chan for dynamic nearest neighbors [7], gives polylogarithmic update time and query time in  $\mathbb{R}^2$ . In higher dimension, these bounds would be considerably worse, if one uses the best known data structures for dynamic nearest neighbors [1, 6].

Finally, it would be interesting to find the dependency of our time bounds on the dimension d. We did not deal with this issue, because one would first have to find this dependency for constructing compressed quadtrees, which is not the focus of this paper.

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