

Hurwitz Equivalence in Braid Group B_3

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ABSTRACT. In this paper we prove certain Hurwitz equivalence properties of B_n . In particular we prove that for $n = 3$ every two Artin's factorizations of Δ_3^2 of the form $H_{i_1} \cdots H_{i_6}, F_{j_1} \cdots F_{j_6}$ (with $i_k, j_k \in \{1, 2\}$) where $\{H_1, H_2\}, \{F_1, F_2\}$ are frames, are Hurwitz equivalent. The proof provided here is geometric, based on a newly defined frame type.

The results will be applied to the classification of algebraic surfaces up to deformation. It is already known that there exist surfaces that are diffeomorphic to each other but are not deformations of each other (Manetti example). We are constructing a new invariant based on Hurwitz equivalence class of factorization, to distinguish among diffeomorphic surfaces which are not deformation of each other. The precise definition of the new invariant can be found at [4] or [5]. The main result of this paper will help us to compute the new invariant.

1 Basic Definitions

In this section we recall some basic definitions and statements from [3]: Let D be a closed disk on \mathbb{R}^2 , $K \subset D$ finite set, $u \in \partial D$. Any diffeomorphism of D which fixes K and is the identity on ∂D acts naturally on $\Pi_1 = \Pi_1(D - K, u)$. We say that two such diffeomorphisms of D (which fix K and equal identity on ∂D) are equivalent if they define the same automorphism on $\Pi_1(D - K, u)$. This equivalence relation is compatible with composition of diffeomorphism and thus the equivalence classes form a group.

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Definition 1.1 *Braid Group $B_n[D, K]$.*

Let D, K be as above, $n = \#K$, and let \mathcal{B} be the group of all diffeomorphisms β of D such that $\beta(K) = K$, $\beta|_{\partial D} = Id_{\partial D}$. For $\beta_1, \beta_2 \in \mathcal{B}$ we say that β_1 is equivalent to β_2 if β_1 and β_2 define the same automorphism of $\Pi_1(D - K, u)$. The quotient of \mathcal{B} by this equivalence relation is called the Braid group $B_n[D, K]$.

Equivalently, if we take the canonical homomorphism $\psi : \mathcal{B} \rightarrow Aut(\Pi_1(D - K, u))$ then $B_n[D, K] = Im(\psi)$. The elements of $B_n[D, K]$ are called braids.

Lemma 1.2 *If $K' \subset D'$ in another pair as above with $\#K' = \#K = n$ then $B_n[D', K']$ is isomorphic to $B_n[D, K]$.*

This gives rise to the definition of B_n :

Definition 1.3 $B_n = B_n[D, K]$ for some D, K with $\#K = n$.

Definition 1.4 $H(\sigma)$, *half-twist defined by σ .*

Let D and K be defined as above. Let $a, b \in K$, $K_{a,b} = K \setminus \{a, b\}$ and σ be a simple path (without a self intersection) in $D \setminus \partial D$ connecting a with b such that $\sigma \cap K = \{a, b\}$. Choose a small regular neighborhood U of σ such that $K_{a,b} \cap U = \emptyset$, and an orientation preserving diffeomorphism $\psi : \mathbb{R}^2 \rightarrow \mathbb{C}^1$ such that $\psi(\sigma) = [-1, 1] = \{z \in \mathbb{C}^1 | Re(z) \in [-1, 1], Im(z) = 0\}$ and $\psi(U) = \{z \in \mathbb{C}^1 | |z| < 2\}$. Let $\alpha(r), r \geq 0$, be a real smooth monotone function such that $\alpha(r) = 1$ for $r \in [0, 3/2]$ and $\alpha(r) = 0$ for $r \geq 2$. Define a diffeomorphism $h : \mathbb{C}^1 \rightarrow \mathbb{C}^1$ as follows: for $z \in \mathbb{C}^1, z = re^{i\phi}$ let $h(z) = re^{i(\phi + \alpha(r)\pi)}$. It is clear that the restriction of h to $\{z \in \mathbb{C}^1 | |z| \leq 3/2\}$ coincides with the 180° positive rotation, and that the restriction to $\{z \in \mathbb{C}^1 | |z| \geq 2\}$ is the identity map. The diffeomorphism $\psi^{-1} \circ h \circ \psi$ is inducing a braid called half-twist and denoted by $H(\sigma)$

Definition 1.5 *Frame of $B_n[D, K]$*

Let $K = \{a_1, \dots, a_n\}$ and $\sigma_1, \dots, \sigma_{n-1}$ be a system of simple smooth paths in $D - \partial D$ such that σ_i connects a_i with a_{i+1} and $L = \bigcup \sigma_i$ is a simple smooth path. The ordered system of half-twists (H_1, \dots, H_{n-1}) defined by $\{\sigma_i\}_{i=1}^{n-1}$ is called a frame of $B_n[D, K]$

Theorem 1.6 *Let (H_1, \dots, H_{n-1}) be a frame of $B_n[D, K]$ then: $B_n[D, K]$ is generated by $\{H_i\}_{i=1}^{n-1}$ and the following is a complete list of relations:
 $H_i H_j = H_j H_i$ if $|i - j| > 1$ (Commutative relation)
 $H_i H_j H_i = H_j H_i H_j$ if $|i - j| = 1$ (Triple relation)*

Proof can be found for example in [3].

This theorem provides us with Artin's algebraic definition of Braid group.

2 Properties of Hurwitz Moves

In this section we define Hurwitz Move and Hurwitz Equivalence and prove a certain theorem on the Hurwitz Equivalence of words in the Braid group which is correct for any Braid group of arbitrary order.

Definition 2.1 *Hurwitz move on G^m (R_k, R_k^{-1})*

Let G be a group, $\vec{t} = (t_1, \dots, t_m) \in G^m$. We say that $\vec{s} = (s_1, \dots, s_m) \in G^m$ is obtained from \vec{t} by the Hurwitz move R_k (or \vec{t} is obtained from \vec{s} by the Hurwitz move R_k^{-1}) if

$$\begin{aligned} s_i &= t_i & \text{for } i \neq k, k+1, \\ s_k &= t_k t_{k+1} t_k^{-1}, & s_{k+1} = t_k. \end{aligned}$$

Definition 2.2 *Hurwitz move on factorization*

Let G be a group and $t \in G$. Let $t = t_1 \cdots t_m = s_1 \cdots s_m$ be two factorized expressions of t . We say that $s_1 \cdots s_m$ is obtained from $t_1 \cdots t_m$ by the Hurwitz move R_k if (s_1, \dots, s_m) is obtained from (t_1, \dots, t_m) by Hurwitz the move R_k .

Definition 2.3 *Hurwitz equivalence of factorization*

The factorizations $s_1 \cdots s_m, t_1 \cdots t_m$ are Hurwitz equivalent if they are obtained from each other by a finite sequence of Hurwitz moves. The notation is $t_1 \cdots t_m \overset{HE}{\sim} s_1 \cdots s_m$.

Definition 2.4 *Word in B_n*

A word in B_n is a representation of braid as a sequence of the frame elements and their inverses.

We will need a certain result of Garside:

Claim 2.5 (Garside): *Every two positive words (all generators with positive powers) which are equal are transformable into each other through a finite sequence of positive words, such that each word of the sequence is positive and obtained from the proceeding one by a direct application of the commutative relation or the triple relation.*

Proof: [1].

The following is obvious, never the less we give a short proof for clarification:

Claim 2.6 *G is a group $g_1, g_2 \in G$:*

1. If $g_1 g_2 = g_2 g_1$ then $g_1 \cdot g_2 \overset{HE}{\sim} g_2 \cdot g_1$
2. If $g_1 g_2 g_1 = g_2 g_1 g_2$ then $g_1 \cdot g_2 \cdot g_1 \overset{HE}{\sim} g_2 \cdot g_1 \cdot g_2$

Proof:

1. $g_1 \cdot g_2 \xrightarrow{R_1} g_1 g_2 g_1^{-1} \cdot g_1 = g_2 \cdot g_1$.
2. $g_1 \cdot g_2 \cdot g_1 \xrightarrow{R_2} g_1 \cdot g_2 g_1 g_2^{-1} \cdot g_2 \xrightarrow{R_1} g_1 g_2 g_1 g_2^{-1} g_1^{-1} \cdot g_1 \cdot g_2$ but $g_1 g_2 g_1 = g_2 g_1 g_2$ so $g_1 g_2 g_1 g_2^{-1} g_1^{-1} = g_2$ and we get $g_1 \cdot g_2 \cdot g_1 \overset{HE}{\sim} g_2 \cdot g_1 \cdot g_2$

From Garside and the above we get the following proposition:

Proposition 2.7 *Let H_1, \dots, H_{n-1} be a set of generators of B_n and $H_{i_1} \dots H_{i_p} = H_{j_1} \dots H_{j_p}$ two positive words (with $i_k, j_k \in \{1, \dots, n-1\}$) then $H_{i_1} \dots H_{i_p} \stackrel{HE}{\sim} H_{j_1} \dots H_{j_p}$.*

Proof:

Apply 2.5 on $H_{i_1} \dots H_{i_p} = H_{j_1} \dots H_{j_p}$ we get a finite sequence of positive words $\{W_r\}_{r=0}^q$ s.t. $W_0 = H_{i_1} \dots H_{i_p}$, $W_q = H_{j_1} \dots H_{j_p}$ and W_{r+1} is obtained from W_r by a single application of the commutative relation or the triple relation.

An application of the triple relation (as in 2.6) is equal to an application of 2 Hurwitz moves R_{t+1}, R_t . An application of the commutative relation is equal to an application of one Hurwitz move (on the commuting elements). Thus, $W_0 \stackrel{HE}{\sim} W_q$

Definition 2.8 $\Delta_n^2 \in B_n[D, K]$

$\Delta_n^2 = (H_1 \dots H_{n-1})^n$ where $\{H_i\}_{i=1}^{n-1}$ is a frame.

We apply 2.7 on Δ_n^2 :

Corollary 2.9 *All Δ_n^2 factorizations $H_{i_1} \dots H_{i_{n(n-1)}}$ $i_k \in \{1, \dots, n-1\}$ are Hurwitz Equivalent.*

3 B_3 Properties

This section concentrates on B_3 and we prove the following new result:

Theorem 3.1 *Let $\{H_1, H_2\}, \{F_1, F_2\}$ two frames of B_3 , then for every two words $\Delta_3^2 = H_{i_1} \dots H_{i_6} = F_{j_1} \dots F_{j_6}$ (with $i_k, j_k \in \{1, 2\}$) the factorizations $H_{i_1} \dots H_{i_6}, F_{j_1} \dots F_{j_6}$ are Hurwitz equivalent.*

The proof of the main theorem is at the end of the section following from propositions 3.7 and 3.9.

Let D be a closed disk on \mathbb{R}^2 and $K \subset D$, $K = \{a, b, c\}$. We fix two parallel lines L_1, L_2 in D which separate the points in K (see figure 1).

With the choice of L_1 and L_2 we define the length and the minimal paths related to a half-twist:

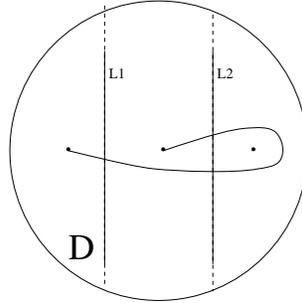


Figure 1: Simple path with $m(\sigma) = 3$

Definition 3.2 Let σ be a simple path in $D \setminus \partial D$ that defines the half-twist $H(\sigma)$. Then $m(\sigma) = \#\{\sigma \cap (L_1 \cup L_2)\}$ as demonstrated in figure 1.

Definition 3.3 Let H be a half-twist in $B_n[D, K]$. We define the length of H , $m(H) = \min\{m(\sigma) \mid H = H(\sigma)\}$ and we call such a path σ which gives the minimal value, the minimal path of H .

Notation 3.4 Any frame $\{H_1, H_2\}$ can be defined by the minimal paths σ_1, σ_2 which also satisfy the relations in 1.5.

Definition 3.5 We say that a frame $\{H_1, H_2\}$ is economic if there exist σ_1, σ_2 minimal paths of H_1, H_2 respectively s.t. for every small neighborhood U of $(\sigma_1 \cap K) \setminus \sigma_2$, $\sigma_1 \subset \sigma_2$ outside U (see 2)

Lemma 3.6 Let $\{H_1, H_2\}$ be economic frame then $m(H_1) \neq m(H_2)$

Proof: Trivial.

Proposition 3.7 Any frame $\{H_1, H_2\}$ which generates $B_3[D, K]$ s.t $\max(m(H_1), m(H_2)) > 1$ is economic.

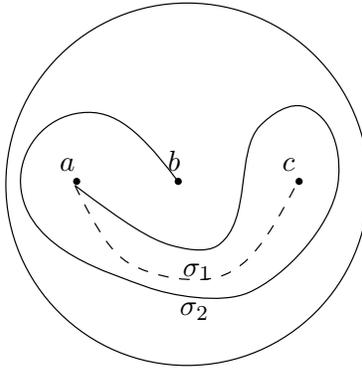


Figure 2: Economic frame $m(H(\sigma_1)) < m(H(\sigma_2))$.

Proof:

Suppose that there exists a frame $\{H_1, H_2\}$ as above which is not economic. From notation 3.4, there exist σ_1, σ_2 as defined in 1.5 s.t. σ_1, σ_2 are minimal paths of H_1, H_2 respectively and the two paths are 'separated' at some point. We will show that such a 'separation' is not possible. Starting from $s = \sigma_1 \cap \sigma_2$ we will examine the first 'separation' where the two paths are separated by one of the points in K .

The cases where $s = a$ and $s = c$ are symmetric. Proving the case where $s = b$ is similar.

We will assume that both paths start at a

Case 1: The two paths entering a neighborhood of the same point but with different orientation (one clockwise and the other counter-clockwise).

(1.a) Entering and turning back as shown in figure 3. The two paths create a cycle with one point inside, the internal path has no where to turn, and thus cannot end. This case is also relevant when one path is turning back and the other continues to the next point.

(1.b) Entering and continue to the next point:

(1.b.i) The two paths continue to the next point with the same orientation (figure 4). The two paths create a cycle that starts at a and contains b . One path has to end at b and will have to change it's orientation through c . This

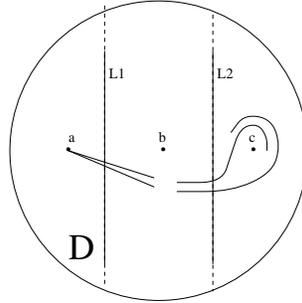


Figure 3: Same point different orientation and turning back.

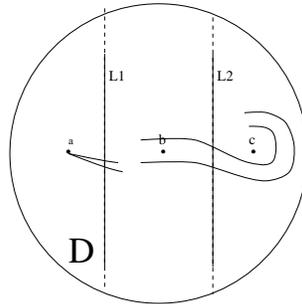


Figure 4: Same point different orientation and continue to next point with the same orientation.

will close a cycle with all the points inside, and the external path has no where to end.

(1.b.ii) The two paths continue to the next point with different orientation (figure 5). The two paths create a cycle with all points inside so the external path has no where to end.

Case 2: Entering to the same point with the same orientation and leaving to different points.

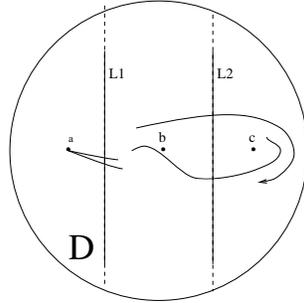


Figure 5: Same point different orientation and continue to next point with different orientation.

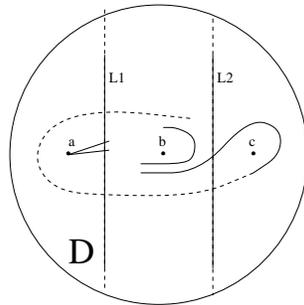


Figure 6: External path must create a cycle with the internal.

This can only happen in point b . The internal path must go back to a and the external will go to c .

(2.a) The external path goes to c by changing orientation (figure 6). This enforces the external path to go around a and create a cycle with the internal path. Thus, the two paths create a cycle with all points inside and the external path has no where to end.

(2.b) The external path goes to c with the same orientation (figure 7). The external path cannot go to the common part (this will create an empty cycle and the path has no where to end) of the paths so both paths continue to a .

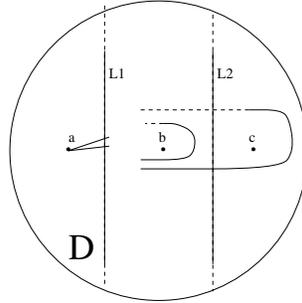


Figure 7: External path must create a cycle with the internal.

In this case we get an equivalent cycle as in 1.b.i.
This conclude the proof of proposition 3.7.

Lemma 3.8 *Let $\{H_1, H_2\}$ be a frame generates $B_3[D, K]$ and H_ζ half-twist, than $\{H_\zeta^{-1}H_1H_\zeta, H_\zeta^{-1}H_2H_\zeta\}$ is also a frame.*

Proof:

By III.1.0 of [2],

$$H_1^{-1}H_2H_1 = H(H_1(\sigma_2)) \text{ where } H_2 = H(\sigma_2).$$

$\{H_1 = H(\sigma_1), H_2 = H(\sigma_2)\}$ is a frame, therefore $\sigma_1 \cup \sigma_2$ is a simple smooth path. H_ζ is a diffeomorphism so $H_\zeta(\sigma_1) \cup H_\zeta(\sigma_2) = H_\zeta(\sigma_1 \cup \sigma_2)$ is also a simple smooth path.

$K \subset \sigma_1 \cup \sigma_2$, $H_\zeta(K) = K$ and therefore $H_\zeta(\sigma_1), H_\zeta(\sigma_2)$ connecting all points in K appropriately.

Proposition 3.9 *Let $\{H_1, H_2\}$ be a frame of $B_3[D, K]$ then there exist a frame $\{F_1, F_2\}$ s.t.*

$$H_1 \cdot H_2 \cdot H_1 \cdot H_2 \cdot H_1 \cdot H_2 \stackrel{HE}{\sim} F_1 \cdot F_2 \cdot F_1 \cdot F_2 \cdot F_1 \cdot F_2$$

$$\text{and } \max(m(H_1), m(H_2)) > \max(m(F_1), m(F_2)).$$

Proof:

From proposition 3.7 we get that $\{H_1, H_2\}$ is economic and from lemma 3.6 $m(H_1) \neq m(H_2)$ we will assume that $m(H_1) < m(H_2)$.

Since $\{H_1, H_2\}$ is economic and $m(H_1) < m(H_2)$, operating H_1 on σ_2 (conjugation) will cause σ_2 to start coming back from the starting point $(\sigma_1 \cap \sigma_2 \cap K)$ along the common part (σ_1) as shown in figure 8. Therefore, we get that either

$$\begin{aligned} m(H_1^{-1}H_2H_1) &= m(H(H_1(\sigma_2))) = m(H_2) - m(H_1) \text{ or} \\ m(H_1H_2H_1^{-1}) &= m(H(H_1^{-1}(\sigma_2))) = m(H_2) - m(H_1) \end{aligned}$$

In the case where $m(H(H_1(\sigma_2))) = m(H_2) - m(H_1)$:

By performing the Hurwitz moves $R_1R_0R_4R_3$ we get:

$$H_1 \cdot H_2 \cdot H_1 \cdot H_2 \cdot H_1 \cdot H_2 \stackrel{HE}{\rightsquigarrow} H_2 \cdot H_1 \cdot H_2 \cdot H_1 \cdot H_2 \cdot H_1$$

and by performing $R_0^{-1}R_2^{-1}R_4^{-1}$:

$$H_2 \cdot H_1 \cdot H_2 \cdot H_1 \cdot H_2 \cdot H_1 \stackrel{HE}{\rightsquigarrow} H_1 \cdot H_1^{-1}H_2H_1 \cdot H_1 \cdot H_1^{-1}H_2H_1 \cdot H_1 \cdot H_1^{-1}H_2H_1$$

Since $m(H_1^{-1}H_2H_1) = m(H(H_1(\sigma_2)))$ and $m(H_2) > m(H_1)$,

$$\max(m(H_1), m(H_2)) > \max(m(H_1), m(H_1^{-1}H_2H_1))$$

From Lemma 3.8, the set $\{H_1, H_1^{-1}H_2H_1\}$ is a frame, we get an equivalent factorization of the same form but with a reduced length.

In the case where $m(H(H_1^{-1}(\sigma_2))) = m(H_2) - m(H_1)$:

$$H_1 \cdot H_2 \cdot H_1 \cdot H_2 \cdot H_1 \cdot H_2 \stackrel{HE}{\rightsquigarrow} H_1H_2H_1^{-1} \cdot H_1 \cdot H_1H_2H_1^{-1} \cdot H_1 \cdot H_1H_2H_1^{-1} \cdot H_1$$

By performing the Hurwitz moves $R_0R_2R_4$. Once again, we get a frame where the maximum length is reduced.

Proof of the main theorem:

Let $H_{i_1} \cdot H_{i_2} \cdot H_{i_3} \cdot H_{i_4} \cdot H_{i_5} \cdot H_{i_6}$ and $F_{j_1} \cdot F_{j_2} \cdot F_{j_3} \cdot F_{j_4} \cdot F_{j_5} \cdot F_{j_6}$ be two factorizations as above.

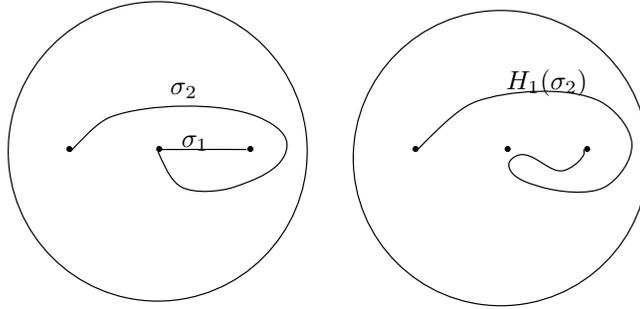


Figure 8: Half-twist conjugation.

From 2.7 we get that:

$$H_{i_1} \cdot H_{i_2} \cdot H_{i_3} \cdot H_{i_4} \cdot H_{i_5} \cdot H_{i_6} \stackrel{HE}{\sim} H_1 \cdot H_2 \cdot H_1 \cdot H_2 \cdot H_1 \cdot H_2 \quad \text{and}$$

$$F_{j_1} \cdot F_{j_2} \cdot F_{j_3} \cdot F_{j_4} \cdot F_{j_5} \cdot F_{j_6} \stackrel{HE}{\sim} F_1 \cdot F_2 \cdot F_1 \cdot F_2 \cdot F_1 \cdot F_2$$

By 3.9, both $H_1 \cdot H_2 \cdot H_1 \cdot H_2 \cdot H_1 \cdot H_2$ and $F_1 \cdot F_2 \cdot F_1 \cdot F_2 \cdot F_1 \cdot F_2$ are Hurwitz equivalent to $X_1 \cdot X_2 \cdot X_1 \cdot X_2 \cdot X_1 \cdot X_2$ where $\{X_1, X_2\}$ is a frame with $m(X_1) = m(X_2) = 1$. Since such a frame is unique we get that

$$H_{i_1} \cdot H_{i_2} \cdot H_{i_3} \cdot H_{i_4} \cdot H_{i_5} \cdot H_{i_6} \stackrel{HE}{\sim} F_{j_1} \cdot F_{j_2} \cdot F_{j_3} \cdot F_{j_4} \cdot F_{j_5} \cdot F_{j_6}$$

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