ON PRODUCTS OF QUASICONVEX SUBGROUPS IN HYPERBOLIC GROUPS

Ashot Minasyan

Abstract

An interesting question about quasiconvexity in a hyperbolic group concerns finding classes of quasiconvex subsets that are closed under finite intersections. A known example is the class of all quasiconvex subgroups [1]. However, not much is yet learned about the structure of arbitrary quasiconvex subsets. In this work we study the properties of products of quasiconvex subgroups; we show that such sets are quasiconvex, their finite intersections have a similar algebraic representation and, thus, are quasiconvex too.

0. Introduction

Let G be a hyperbolic group, $\Gamma(G, \mathcal{A})$ – its Cayley graph corresponding to a finite symmetrized generating set \mathcal{A} (i.e. for each element $a \in \mathcal{A}$, a^{-1} also belongs to this set). A subset $Q \subseteq G$ is said to be ε -quasiconvex, if any geodesic connecting two elements from Q belongs to a closed ε -neighborhood $\mathcal{O}_{\varepsilon}(Q)$ of Q in $\Gamma(G, \mathcal{A})$ for some $\varepsilon \geq 0$. Q will be called quasiconvex if there exists $\varepsilon > 0$ for which it is ε -quasiconvex.

In [4] Gromov proves that the notion of quasiconvexity in a hyperbolic group does not depend on the choice of a finite generating set (it is easy to show that this is not true in an arbitrary group).

If $A, B \subseteq G$ then their product is a subset of G defined by $A \cdot B = \{ab \mid a \in A, b \in B\}.$

Proposition 1. If the sets $A_1, \ldots, A_n \subset G$ are quasiconvex then their product set $A_1 A_2 \cdots A_n \stackrel{def}{=} \{a_1 a_2 \cdots a_n \mid a_i \in G_i\} \subset G$ is also quasiconvex.

Proposition 1 was proved by Zeph Grunschlag in 1999 in [11; Prop. 3.14] and, independently, by the author in his diploma paper in 2000.

If H is a subgroup of G and $x \in G$ then the subgroup conjugated to H by x will be denoted $H^x = xHx^{-1}$. The main result of the paper is

Theorem 1. Suppose $G_1, \ldots, G_n, H_1, \ldots, H_m$ are quasiconvex subgroups of the group $G, n, m \in \mathbb{N}$; $f, e \in G$. Then there exist numbers $r, t_l \in \mathbb{N} \cup \{0\}$ and $f_l, \alpha_{lk}, \beta_{lk} \in G, k = 1, 2, \ldots, t_l$ (for every fixed l), $l = 1, 2, \ldots, r$, such that

$$fG_1G_2\cdot\ldots\cdot G_n\cap eH_1H_2\cdot\ldots\cdot H_m = \bigcup_{l=1}^r f_lS_l$$

where for each $l, t = t_l$, there are indices $1 \le i_1 \le i_2 \le \ldots \le i_t \le n, 1 \le j_1 \le \le j_2 \le \ldots \le j_t \le m$:

$$S_{l} = (G_{i_{1}}^{\alpha_{l_{1}}} \cap H_{j_{1}}^{\beta_{l_{1}}}) \cdot \ldots \cdot (G_{i_{t}}^{\alpha_{l_{t}}} \cap H_{j_{t}}^{\beta_{l_{t}}}).$$

This claim does not hold if the group G is not hyperbolic : set $G_1 = \langle (1,0) \rangle$, $G_2 = \langle (0,1) \rangle$, $H = \langle (1,1) \rangle$ – cyclic subgroups of \mathbb{Z}^2 (they are quasiconvex in \mathbb{Z}^2 with generators $\{(1,0), (0,1), (1,1)\}$). $G_1 \cdot G_2 = \mathbb{Z}^2$, thus, $G_1G_2 \cap H = H$ but $G_1 \cap H = G_2 \cap H = \{(0,0)\}$ and, if the statement of the theorem held for \mathbb{Z}^2 then H would be finite – a contradiction.

The above example can also be used as another argument to prove the wellknown fact that \mathbb{Z}^2 can not be embedded into a hyperbolic group (because any cyclic subgroup is quasiconvex in a hyperbolic group).

The condition that the subgroups G_i , H_j are quasiconvex is also necessary: using Rips' Construction ([12]) one can achieve a group G satisfying the small cancellation condition C'(1/6) (and, therefore, hyperbolic) and its finitely generated normal subgroup K such that $G/K \cong \mathbb{Z}^2$. Let ϕ be the natural epimorphism from G to \mathbb{Z}^2 , $G_1 = \phi^{-1}(\langle (1,0) \rangle) \leq G$, $G_2 = \phi^{-1}(\langle (0,1) \rangle) \leq G$, $H = \phi^{-1}(\langle (1,1) \rangle) \leq G$. G_1, G_2 and H are finitely generated subgroups of G, $G_1 \cdot G_2 = G$ because $\langle (1,0) \rangle \cdot \langle (0,1) \rangle = \mathbb{Z}^2$ and $K \leq G_2$, thus $G_1 \cdot G_2 \cap H = H$. But for every $\alpha, \beta \in G \phi(G_i^{\alpha} \cap H^{\beta}) \subseteq \phi(G_i)^{\phi(\alpha)} \cap \phi(H)^{\phi(\beta)} = \{(0,0)\}, i = 1, 2$. Hence, it is impossible to obtain the infinite subgroup $\phi(H)$ from products of cosets to such sets, and we constructed the counterexample needed.

Definition : let G_1, G_2, \ldots, G_n be quasiconvex subgroups of $G, f_1, f_2, \ldots, f_n \in G, n \in \mathbb{N}$. Then the set

$$f_1G_1f_2G_2 \cdot \ldots \cdot f_nG_n = \{f_1g_1f_2g_2 \cdot \ldots \cdot f_ng_n \in G \mid g_i \in G_i, i = 1, \dots, n\}$$

will be called quasiconvex product.

Cor. 6].

Corollary 2. An intersection of finitely many quasiconvex products is a finite union of quasiconvex products.

Thus the class of finite unions of quasiconvex products is closed under taking finite intersections.

Recall that a group H is called *elementary* if it has a cyclic subgroup $\langle h \rangle$ of finite index. An elementary subgroup of a hyperbolic group is quasiconvex (see remark 5, Section 4). It is well known that any element x of infinite order in G is contained in a unique maximal elementary subgroup $E(x) \leq G$ [4], [5]. Every non-elementary hyperbolic group contains the free group of rank 2 [5,

Suppose $G_1, G_2, \ldots, G_n, H_1, H_2, \ldots, H_m$ are infinite maximal elementary subgroups of $G, f, e \in G$. And $G_i \neq G_{i+1}, H_j \neq H_{j+1}, i = 1, \ldots, n-1,$ $j = 1, \ldots, m-1$. Then we present the following uniqueness result for the products of such subgroups:

Theorem 2. The sets $fG_1 \cdot \ldots \cdot G_n$ and $eH_1 \cdot \ldots \cdot H_m$ are equal if and only if n = m, $G_n = H_n$, and there exist elements $z_j \in H_j$, $j = 1, \ldots, n$, such that $G_j = (z_n z_{n-1} \ldots z_{j+1}) \cdot H_j \cdot (z_n z_{n-1} \ldots z_{j+1})^{-1}$, $j = 1, 2, \ldots, n-1$, $f = ez_1^{-1}z_2^{-1} \ldots z_n^{-1}$.

Similarly to quasiconvex products one can define ME-products to be the products of cosets of maximal elementary subgroups in G (the full definition is

given in Section 4). The statement of Corollary 2 can be strengthened in this case :

Theorem 3. Intersection of any family (finite or infinite) of finite unions of ME-products is a finite union of ME-products.

An example which shows that an analogous property is not true for arbitrary quasiconvex products is constructed at the end of this paper.

Thus, all finite unions of ME-products constitute a topology \mathcal{T} (of closed sets) on the set of elements of a hyperbolic group. Taking an inverse, left and right shifts in G are continuous operations in \mathcal{T} . Also, by definition, any point is closed in \mathcal{T} , so \mathcal{T} is weakly separated (T_1) . However, if G is infinite elementary, then \mathcal{T} turns out to be the topology of finite complements which is not Hausdorff, also, in this case, the group multiplication is not continuous with respect to \mathcal{T} (since any product of two non-empty open sets contains the identity of G).

1. Preliminary information

Let $d(\cdot, \cdot)$ be the usual left-invariant metric on the Cayley graph of the group G with generating set \mathcal{A} . For any two points $x, y \in \Gamma(G, \mathcal{A})$ we fix a geodesic path between them and denote it by [x, y].

If $Q \subset \Gamma(G, \mathcal{A}), N \geq 0$, the closed N-neighborhood of Q will be denoted by

$$\mathcal{O}_N(Q) \stackrel{def}{=} \{ x \in \Gamma(G, \mathcal{A}) \mid d(x, Q) \le N \} .$$

If $x, y, w \in \Gamma(G, \mathcal{A})$, then the number

$$(x|y)_w \stackrel{def}{=} \frac{1}{2} \Big(d(x,w) + d(y,w) - d(x,y) \Big)$$

is called the *Gromov product* of x and y with respect to w.

Let abc be a geodesic triangle. There exist "special" points $O_a \in [b, c]$, $O_b \in [a, c], O_c \in [a, b]$ with the properties: $d(a, O_b) = d(a, O_c) = \alpha$, $d(b, O_a) = d(b, O_c) = \beta$, $d(c, O_a) = d(c, O_b) = \gamma$. From a corresponding system of linear equations one can find that $\alpha = (b|c)_a$, $\beta = (a|c)_b$, $\gamma = (a|b)_c$. Two points $O \in [a, b]$ and $O' \in [a, c]$ are called *a-equidistant* if $d(a, O) = d(a, O') \leq \alpha$. The triangle *abc* is said to be δ -*thin* if for any two points O, O' lying on its sides and equidistant from one of its vertices, $d(O, O') \leq \delta$ holds. *abc* is δ -*slim* if each of its sides belongs to a closed δ -neighborhood of the two others.

We assume the following equivalent definitions of hyperbolicity of $\Gamma(G, \mathcal{A})$ to be known to the reader (see [6], [2]):

1°. There exists $\delta \geq 0$ such that for any four points $x, y, z, w \in \Gamma(G, \mathcal{A})$ their Gromov products satisfy

$$(x|y)_{w} \ge \min\{(x|z)_{w}, (y|z)_{w}\} - \delta;$$

2°. All triangles in $\Gamma(G, \mathcal{A})$ are δ -thin for some $\delta \geq 0$;

3°. All triangles in $\Gamma(G, \mathcal{A})$ are δ -slim for some $\delta \geq 0$.

Now and below we suppose that G meets $1^{\circ}, 2^{\circ}$ and 3° for a fixed (sufficiently large) $\delta \geq 0$. 3° easily implies

<u>**Remark 0.**</u> Any side of a geodesic *n*-gon $(n \ge 3)$ in $\Gamma(G, \mathcal{A})$ belongs to a closed $(n-2)\delta$ -neighborhood of the union of the rest of its sides.

Let p be a path in the Cayley graph of G. Further on by p_- , p_+ we will denote the startpoint and the endpoint of p, by ||p|| - its length; lab(p), as usual, will mean the word in the alphabet \mathcal{A} written on p. $elem(p) \in G$ will denote the element of the group G represented by the word lab(p).

A path q is called (λ, c) -quasigeodesic if there exist $0 < \lambda \leq 1, c \geq 0$, such that for any subpath p of q the inequality $\lambda ||p|| - c \leq d(p_-, p_+)$ holds. In a hyperbolic space quasigeodesics and geodesics with same ends are mutually close :

Lemma 1.1. ([6; 5.6,5.11], [2; 3.3]) There is a constant $N = N(\delta, \lambda, c)$ such that for any (λ, c) -quasigeodesic path p in $\Gamma(G, \mathcal{A})$ and a geodesic q with $p_- = q_-, p_+ = q_+$, one has $p \in \mathcal{O}_N(q)$ and $q \in \mathcal{O}_N(p)$.

An important property of cyclic subgroups in a hyperbolic group states

Lemma 1.2. ([6; 8.21], [2; 3.2]) For any word w representing an element $g \in G$ of infinite order there exist constants $\lambda > 0$, $c \ge 0$, such that any path with a label w^m in the Cayley graph of G is (λ, c) -quasigeodesic for arbitrary integer m.

A broken line $p = [X_0, X_1, \ldots, X_n]$ is a path obtained as a consequent concatenation of geodesic segments $[X_{i-1}, X_i]$, $i = 1, 2, \ldots, n$. Later, in this paper, we will use the following fact concerning broken lines in a hyperbolic space:

Lemma 1.3. ([3, Lemma 21]) Let $p = [X_0, X_1, \ldots, X_n]$ be a broken line in $\Gamma(G, \mathcal{A})$ such that $||[X_{i-1}, X_i]|| > C_1 \forall i = 1, \ldots, n$, and $(X_{i-1}|X_{i+1})_{X_i} \leq C_0 \forall i = 1, \ldots, n-1$, where $C_0 \geq 14\delta$, $C_1 > 12(C_0 + \delta)$. Then p is contained in the closed $2C_0$ -neighborhood $\mathcal{O}_{2C_0}([X_0, X_n])$ of the geodesic segment $[X_0, X_n]$.

Suppose $H = \langle \mathcal{X} \rangle$ is a subgroup of G with a finite symmetrized generating set \mathcal{X} . If $h \in H$, then by $|h|_G$ and $|h|_H$ we will denote the lengths of the element h in \mathcal{A} and \mathcal{X} respectively. The distortion function $D_H : \mathbb{N} \to \mathbb{N}$ of H in G is defined by $D_H(n) = max\{|h|_H \mid h \in H, |h|_G \leq n\}.$

If $\alpha, \beta : \mathbb{N} \to \mathbb{N}$ are two functions then we write $\alpha \preceq \beta$ if $\exists K_1, K_2 > 0$: $\alpha(n) \leq K_1\beta(K_2n)$. $\alpha(n)$ and $\beta(n)$ are said to be equivalent if $\alpha \preceq \beta$ and $\beta \preceq \alpha$.

Evidently, the function D_H does not depend (up to this equivalence) on the choice of finite generating sets \mathcal{A} of G and \mathcal{X} of H. One can also notice that $D_H(n)$ is always at least linear (provided that H is infinite). If D_H is equivalent to linear, H is called *undistorted*.

<u>**Lemma 1.4.**</u> ([2; 3.8], [7; 10.4.2]) A quasiconvex subgroup H of a hyperbolic group G is finitely generated.

<u>**Remark 1.**</u> From the proof of this statement it also follows that D_H is equivalent to linear for a quasiconvex subgroup H.

Indeed, it was observed in [2] that if H is ε -quasiconvex, it is generated by

finitely many elements x_i , i = 1, ..., s, such that $|x_i|_G \leq 2\varepsilon + 1 \forall i$, and $\forall h \in H$, $h = a_1 \cdot \ldots \cdot a_r, a_i \in \mathcal{A}$, hence $\exists i_1, \ldots, i_r \in \{1, 2, \ldots, s\}$: $h = x_{i_1} x_{i_2} \cdot \ldots \cdot x_{i_r}$.

The proof of corollary 2 is based on

Lemma 1.5. ([1; Prop. 3]) Let G be a group generated by a finite set \mathcal{A} . Let A, B be subgroups of G quasiconvex with respect to A. Then $A \cap B$ is quasiconvex with respect to \mathcal{A} .

We will use the following notion in this paper :

Definition : let $H = \langle \mathcal{X} \rangle \leq G = \langle \mathcal{A} \rangle$, $card(\mathcal{X}) < \infty$, $card(\mathcal{A}) < \infty$. A path P in $\Gamma(G, \mathcal{A})$ will be called *H*-geodesic (or just *H*-path) if :

a) P is labelled by the word $a_{11} \ldots a_{1k_1} \ldots a_{s1} \ldots a_{sk_s}$ corresponding to an element $elem(P) = x \in H$, where $a_{ij} \in \mathcal{A}$;

b) $a_{j1} \dots a_{jk_i}$ is a shortest word for generator $x_j \in \mathcal{X}$ (i.e. $|x_j|_G = k_j$), $j=1,\ldots,s;$

c) $x = x_1 \dots x_s$ in $H, |x|_H = s$.

I.e. P is a broken line in $\Gamma(G, \mathcal{A})$ with segments corresponding to shortest representations of generators of H by means of \mathcal{A} .

Lemma 1.6. (see also [10; Lemma 2.4]) Let H be a (finitely generated) subgroup of a δ -hyperbolic group G. Then H is quasiconvex iff H is undistorted in G.

Proof. The necessity is given by remark 1.

To prove the sufficiency, suppose $H = \langle \mathcal{X} \rangle$, $card(\mathcal{X}) < \infty$, and $D_H(n) \leq cn$, $\forall n \in \mathbb{N}, c > 0$. For arbitrary two vertices $x, y \in H$ there is a H-path q connecting them in $\Gamma(G, \mathcal{A})$. Let p be any its subpath. By definition, there exists a subpath p' of q such that $p'_{-}, p'_{+} \in H$, subpaths of q from p_{-} to p'_{-} and from p_+ to p'_+ are geodesic, and $d(p_-, p'_-) \leq \varkappa/2$, $d(p_+, p'_+) \leq \varkappa/2$, where $\varkappa = max\{|h|_G \mid h \in \mathcal{X}\} < \infty$. In particular, p' is also H-geodesic.

Using the property c) from the definition of a H-path we obtain

$$||p'|| \le \varkappa \cdot |elem(p')|_H \le \varkappa \cdot c \cdot d(p'_-, p'_+) .$$

Therefore, $||p|| \leq ||p'|| + \varkappa \leq \varkappa \cdot c \cdot d(p'_{-}, p'_{+}) + \varkappa \leq \varkappa \cdot c \cdot d(p_{-}, p_{+}) + \varkappa^2 c + \varkappa$, which shows that q is $(\frac{1}{\varkappa c}, \varkappa + \frac{1}{c})$ -quasigeodesic. By lemma $1.1 \exists N = N(\varkappa, c)$ such that any geodesic path between x and y belongs to the closed N-neighborhood $\mathcal{O}_N(q)$ but $q \in \mathcal{O}_{\varkappa/2}(H)$ in the Cayley graph of G. Hence, H is quasiconvex with the constant $(N + \varkappa/2)$, and the lemma is proved. \Box

During this proof we showed

Remark 2. If H is a quasiconvex subgroup of a hyperbolic group G then any *H*-path is (λ, c) -quasigeodesic for some λ, c depending only on the subgroup H.

Let the words W_1, \ldots, W_l represent elements w_1, \ldots, w_l of infinite order in a hyperbolic group G. For a fixed constant K consider the set $S_M = S(W_1, \ldots, W_l; K, M)$ of words

$$W = X_0 W_1^{\alpha_1} X_1 W_2^{\alpha_2} X_2 \dots W_l^{\alpha_l} X_l$$

where $||X_i|| \leq K$ for $i = 0, 1, ..., l, |\alpha_2|, ..., |\alpha_{l-1}| \geq M$, and the element of G represented by $X_i^{-1}W_iX_i$ does not belong to the maximal elementary subgroup $E(w_{i+1}) \leq G$ containing w_{i+1} for i = 1, ..., l-1.

Lemma 1.7. ([5; Lemma 2.4]) There exist constants $\lambda > 0$, $c \ge 0$ and M > 0 (depending on K, W_1, \ldots, W_l) such that any path in $\Gamma(G, \mathcal{A})$ labelled by an arbitrary word $W \in S_M$ is (λ, c) -quasigeodesic.

Lemma 1.8. Suppose $l \in \mathbb{N}$, K > 0, and $w_1, \ldots, w_l \in G$ are elements of infinite order. Then there are $\lambda > 0$, $c \ge 0$ and M > 0 (depending on K, w_1, \ldots, w_l) such that for arbitrary $x_0, x_1, \ldots, x_l \in G$, $|x_i|_G \le K$, $i = 0, \ldots, l$, with conditions $w_i \notin x_i E(w_{i+1}) x_i^{-1} \forall i \in \{1, \ldots, l-1\}$, and any $\alpha_i \in \mathbb{Z}$, $|\alpha_i| \ge M, i = 2, \ldots, l-1$, the element

$$w = x_0 w_1^{\alpha_1} x_1 w_2^{\alpha_2} x_2 \cdot \ldots \cdot w_l^{\alpha_l} x_l \in G$$

satisfies $|w|_G \ge \lambda |\alpha_1| - c$.

Proof. As follows from Lemma 1.7 and the definition of a (λ, c) -quasigeodesic path, one has the following inequality:

$$|w|_G \ge \lambda \cdot \left(|x_0|_G + \sum_{i=1}^l (|\alpha_i||w_i|_G + |x_i|_G) \right) - c \ge \lambda \cdot |\alpha_1||w_1|_G - c \ge \lambda |\alpha_1| - c \,. \quad \Box$$

2. Quasiconvex sets and their products

<u>Remark 3.</u> Any finite subset of G is d-quasiconvex (where d is the diameter of this set).

Remark 4. Let $Q \subseteq G$ be ε -quasiconvex, $g \in G$. Then (a) the left shift $gQ = \{gx \mid x \in Q\}$ is quasiconvex with the same constant; (b) the right shift $Qg = \{xg \mid x \in Q\}$ is quasiconvex (possibly, with a different quasiconvexity constant).

(a) holds because the metric $d(\cdot, \cdot)$ is left-invariant. $x, y \in Q$ if and only if $xg, yg \in Qg$. By remark 0

$$[xg, yg] \subset \mathcal{O}_{2\delta}\Big([x, xg] \cup [x, y] \cup [y, yg]\Big) \subset \mathcal{O}_{2\delta+|g|_G}\Big([x, y]\Big) \subset \mathcal{O}_{2\delta+|g|_G+\varepsilon}(Q) \subset \mathcal{O}_{2\delta+2|g|_G+\varepsilon}(Qg)$$

therefore (b) is true.

Therefore, a left coset of a quasiconvex subgroup and a conjugate subgroup to it are quasiconvex (in a hyperbolic group).

Lemma 2.1. (see also [11; Prop. 3.14]) A finite union of quasiconvex sets in a hyperbolic group G is quasiconvex.

Proof. It is enough to prove that if $A, B \subset G$ are ε_i -quasiconvex, i = 1, 2, respectively, then $C = A \cup B$ is quasiconvex.



If both $x, y \in A$ or $x, y \in B$ then $[x, y] \subset \mathcal{O}_{max\{\varepsilon_1, \varepsilon_2\}}(C)$. So, assume that $x \in A, y \in B$. Fix $a \in A, b \in B$, and consider the geodesic quadrangle xyba (see Figure 1).

By remark 0 we have $[x, y] \subset \mathcal{O}_{2\delta}([x, a] \cup [a, b] \cup [b, y])$. After denoting $d(a, b) = 2\eta$ we obtain $[x, a] \subset \mathcal{O}_{\varepsilon_1}(A)$, $[b, y] \subset \mathcal{O}_{\varepsilon_2}(B)$, $[a, b] \subset \mathcal{O}_{\eta}(A \cup B)$. Hence $[x, a] \cup [a, b] \cup [b, y] \subset \mathcal{O}_{max\{\varepsilon_1, \varepsilon_2, \eta\}}(C)$, consequently, $[x, y] \subset \mathcal{O}_{max\{\varepsilon_1, \varepsilon_2, \eta\}+2\delta}(C)$, and the lemma is proved. \Box

Proof of Proposition 1. Assume n = 2 (for n > 2 the statement will follow by induction).

So, let A, B be ε_i -quasiconvex subsets of G respectively, i = 1, 2.

Consider arbitrary $a_1b_1, a_2b_2 \in AB$, $a_i \in A$, $b_i \in B_i$, i = 1, 2, and fix an element $b \in B$, $|b|_G = \eta$. Then, since the triangles are δ -slim,

$$[b_1, 1_G] \subset \mathcal{O}_{\delta}([b, 1_G] \cup [b, b_1]) \subset \mathcal{O}_{\delta+\eta}([b, b_1]) \subset \mathcal{O}_{\delta+\eta+\varepsilon_2}(B)$$
.

Denoting $\varepsilon_3 = \delta + \eta + \varepsilon_2$, one obtains $[b_1, 1_G] \subset \mathcal{O}_{\varepsilon_3}(B)$ and, similarly, $[b_2, 1_G] \subset \mathcal{O}_{\varepsilon_3}(B)$. Therefore, $[a_1b_1, a_1] \subset \mathcal{O}_{\varepsilon_3}(a_1B)$, $[a_2b_2, a_2] \subset \mathcal{O}_{\varepsilon_3}(a_2B)$. Also, observe that $\forall a \in A \quad d(a, ab) = |b|_G = \eta$, i.e. $A \subset \mathcal{O}_\eta(Ab) \subset \mathcal{O}_\eta(AB)$, hence $[a_1, a_2] \subset \mathcal{O}_{\varepsilon_1}(A) \subset \mathcal{O}_{\varepsilon_1+\eta}(AB)$. And using remark 0 we achieve

$$[a_1b_1, a_2b_2] \subset \mathcal{O}_{2\delta}\Big([a_1b_1, a_1] \cup [a_1, a_2] \cup [a_2b_2, a_2]\Big) \subset \mathcal{O}_{2\delta + max\{\varepsilon_1 + \eta, \varepsilon_3\}}(AB) ,$$

q.e.d. \Box

Corollary 1. In a hyperbolic group G every quasiconvex product is a quasiconvex set .

This follows directly from the proposition 1 and part (a) of remark 4.

3. Intersections of quasiconvex products

Set a partial order on \mathbb{Z}^2 : $(a_1, b_1) \leq (a_2, b_2)$ if $a_1 \leq a_2$ and $b_1 \leq b_2$. As usual, $(a_1, b_1) < (a_2, b_2)$ if $(a_1, b_1) \leq (a_2, b_2)$ and $(a_1, b_1) \neq (a_2, b_2)$.

Definition : a finite sequence $((i_1, j_1), (i_2, j_2), \ldots, (i_t, j_t))$ of pairs of positive integers will be called *increasing* if it is empty (t = 0) or (if t > 0) $(i_q, j_q) < (i_{q+1}, j_{q+1}) \forall q = 1, 2, \ldots, t - 1$. This sequence will also be called (n, m)-increasing $(n, m \in \mathbb{N})$ if $1 \le i_q \le n, 1 \le j_q \le m$ for all $q \in \{1, 2, \ldots, t\}$.

Note that the length of an (n, m)-increasing sequence never exceeds (n + m - 1).

Instead of proving theorem 1 we will prove

Theorem 1'. Suppose $G_1, \ldots, G_n, H_1, \ldots, H_m$ are quasiconvex subgroups of the group $G, n, m \in \mathbb{N}$; $f, e \in G$. Then there exist numbers $r, t_l \in \mathbb{N} \cup \{0\}$ and $f_l, \alpha_{lk}, \beta_{lk} \in G, k = 1, 2, \ldots, t_l$ (for every fixed l), $l = 1, 2, \ldots, r$, such that

(1)
$$fG_1G_2\cdot\ldots\cdot G_n\cap eH_1H_2\cdot\ldots\cdot H_m=\bigcup_{l=1}^r f_lS_l$$

where for each $l, t = t_l$, there are indices $1 \le i_1 \le i_2 \le \ldots \le i_t \le n, 1 \le j_1 \le \le j_2 \le \ldots \le j_t \le m$:

(2)
$$S_{l} = (G_{i_{1}}^{\alpha_{l_{1}}} \cap H_{j_{1}}^{\beta_{l_{1}}}) \cdot \ldots \cdot (G_{i_{t}}^{\alpha_{l_{t}}} \cap H_{j_{t}}^{\beta_{l_{t}}}),$$

and the sequence $((i_1, j_1), \ldots, (i_t, j_t))$ is (n, m)-increasing.

For our convenience, let us also introduce the following

Definition : the unions as in the right-hand side of (1) will be called *special*. S_l as in (2) will be called *increasing* (n,m)-products.

Lemma 3.1. Consider a geodesic polygon $X_0X_1 \ldots X_n$ in the Cayley graph $\Gamma(G, \mathcal{A}), n \geq 2$. Then there are points $\bar{X}_i \in [X_i; X_{i+1}], i = 1, 2, \ldots, n-1$, such that setting $\bar{X}_0 = X_0, \ \bar{X}_n = X_n$, we have $(\bar{X}_{i-1}|\bar{X}_{i+1})_{\bar{X}_i} \leq \delta$ and $d(\bar{X}_i, [\bar{X}_{i-1}; X_i]) \leq \delta$, for $1 \leq i \leq n-1$.

Proof of the lemma. First, we recursively construct the vertices \bar{X}_i . Let $\bar{X}_1 \in [X_1; X_2]$, $\bar{U}_1 \in [X_0; X_1]$ be the "special" points of the geodesic triangle $X_0X_1X_2$, i.e. $|X_1 - \bar{X}_1| = |X_1 - \bar{U}_1| = (X_0|X_2)_{X_1}$. Now, if \bar{X}_{i-1} is constructed, denote by $\bar{X}_i \in [X_i; X_{i+1}]$, $\bar{U}_i \in [\bar{X}_{i-1}; X_i]$ the special points of triangle $\bar{X}_{i-1}X_iX_{i+1}$ $(|X_i - \bar{X}_i| = |X_i - \bar{U}_i| = (\bar{X}_{i-1}|X_{i+1})_{X_i})$. (Figure 2) Then $d(\bar{X}_i, [\bar{X}_{i-1}; X_i]) \leq |\bar{X}_i - \bar{U}_i| \leq \delta, \forall i = 1, 2, \dots, n-1$.

For the other part of the claim we will use induction on n. n = 2, then

$$(X_0|X_2)_{\bar{X}_1} \stackrel{def}{=} \frac{1}{2}(|X_0 - \bar{X}_1| + |X_2 - \bar{X}_1| - |X_0 - X_2|) \le$$

$$\leq \frac{1}{2}(|X_0 - \bar{U}_1| + |\bar{U}_1 - \bar{X}_1| + |X_2 - \bar{X}_1| - |X_0 - X_2|) = \frac{1}{2}|\bar{U}_1 - \bar{X}_1| \leq \frac{\delta}{2} \leq \delta$$

Suppose, now, that $n \geq 3$. Let us evaluate the Gromov product $(\bar{X}_0 | \bar{X}_2)_{\bar{X}_1}$.

$$(\bar{X}_0|\bar{X}_2)_{\bar{X}_1} = \frac{1}{2}(|X_0 - \bar{X}_1| + |\bar{X}_2 - \bar{X}_1| - |X_0 - \bar{X}_2|),$$



 $|\bar{X}_2 - \bar{X}_1| \leq |\bar{X}_1 - \bar{U}_2| + |\bar{X}_2 - \bar{U}_2| \leq |\bar{X}_1 - \bar{U}_2| + \delta, |X_0 - \bar{X}_1| \leq |X_0 - \bar{U}_1| + \delta, |X_0 - \bar{U}_1| + |X_2 - \bar{X}_1| = |X_0 - X_2|$ - by the definition of special points of the triangle $X_0 X_1 X_2$. Therefore

$$|X_0 - \bar{X}_1| + |\bar{X}_2 - \bar{X}_1| \le |X_0 - \bar{U}_1| + |\bar{X}_1 - \bar{U}_2| + 2\delta =$$

 $=|X_0-\bar{U}_1|+(|X_2-\bar{X}_1|-|X_2-\bar{U}_2|)+2\delta=|X_0-X_2|-|X_2-\bar{X}_2|+2\delta .$ Now we notice that $|X_0-X_2|-|X_2-\bar{X}_2| \le |X_0-\bar{X}_2|$ and obtain:

$$(\bar{X}_0|\bar{X}_2)_{\bar{X}_1} \le \frac{1}{2}(|X_0 - \bar{X}_2| + 2\delta - |X_0 - \bar{X}_2|) = \delta$$
.

To the *n*-gon $\bar{X}_1 X_2 \dots X_n$ we can apply the induction hypothesis. The lemma is proved. \Box

Proof of theorem 1'. Define $T = fG_1G_2 \cdot \ldots \cdot G_n \cap eH_1H_2 \cdot \ldots \cdot H_m$. Fix some finite generating sets in every G_i, H_j and denote

 $K_1 = max\{1 ; |f|_G ; |generators of G_i|_G : i = 1, 2, ..., n\} < \infty$

 $K_2 = max\{1; |e|_G; |generators \ of \ H_j|_G : j = 1, 2, ..., m\} < \infty$.

Induction on (n+m).

If n = 0 or m = 0, then $card(T) \le 1$ and the statement is true.

Let $n \ge 1$ and $m \ge 1$, $n + m \ge 2$.

Choose an arbitrary $x \in T$, $x = fg_1g_2 \cdot \ldots \cdot g_n = eh_1h_2 \cdot \ldots \cdot h_m$ where $g_i \in G_i, h_j \in H_j, i = 1, \ldots, n, j = 1, \ldots, m$.

Consider a pair of (non-geodesic) polygons associated with x in $\Gamma(G, \mathcal{A})$:

 $P = X_0 p_1 X_1 p_2 \dots p_n X_n p_0$ and $Q = Y_0 q_1 Y_1 q_2 \dots q_m Y_m q_0$ with vertices

 $X_0 = Y_0 = 1_G, X_i = fg_1 \dots g_i \in G, Y_j = eh_1 \dots h_j \in G, i = 1, \dots, n, j = 1, \dots, m$, and edges $p_0, p_1, \dots, p_n, q_0, q_1, \dots, q_m$. Such that p_1 , starting at X_0 and ending at X_1 , is a union of a geodesic path corresponding to f and a G_1 -path corresponding to g_1, p_i is a G_i -path labelled by a word representing the element g_i in G from X_{i-1} to $X_i, i = 2, \dots, n; p_0$ is the geodesic path $[X_n, X_0]$ (Figure 3).



By construction, there are constants λ_i, c_i (not depending on $x \in T$) such that the segments $p_i, i = 1, ..., n$ are (λ_i, c_i) -quasigeodesic respectively.

Similarly one constructs the paths q_j , $j = 0, \ldots, m$.

Therefore the geodesic path $p_0 = [X_0; X_n] = [Y_0; Y_m] = q_0$ will be labelled by a word representing x in our Cayley graph.

We will also consider the geodesic polygons $X_0X_1...X_n$ and $Y_0Y_1...Y_m$ with same vertices as P and Q respectively.

Recalling the property of quasigeodesic paths, for each i = 1, ..., n

[j = 1, ..., m] we obtain a constant $N_i > 0$ $[M_j > 0]$ (not depending on the element $x \in T$) such that

(i)
$$[X_{i-1}, X_i] \subset \mathcal{O}_{N_i}(p_i) \quad \left[[Y_{j-1}, Y_j] \subset \mathcal{O}_{M_j}(q_j) \right].$$

Define $L = max\{N_1, ..., N_n, M_1, ..., M_m\}.$

a) Suppose $n, m \ge 2$ (after considering this case, we will see that the other cases, when n = 1 or m = 1 are easier).

Let's focus our attention on the polygons $X_0 \dots X_n$ and P since everything for the two others can be done analogously.

One can apply lemma 3.1 and obtain $\tilde{X}_i \in [X_i; X_{i+1}], i = 1, \ldots, n-1$, such that $(\tilde{X}_{i-1}|\tilde{X}_{i+1})_{\tilde{X}_i} \leq \delta, i = 1, \ldots, n-1, (\tilde{X}_0 = X_0 = 1_G, \tilde{X}_n = X_n = x)$, along with $\tilde{U}_i \in [\tilde{X}_{i-1}; X_i], |\tilde{X}_i - \tilde{U}_i| \leq \delta, i = 1, \ldots, n-1$. Now, using (i), we obtain points $\bar{X}_i \in p_{i+1}, i = 1, \ldots, n-1$, satisfying

Now, using (i), we obtain points $X_i \in p_{i+1}$, $i = 1, \ldots, n-1$, satisfying $d(\tilde{X}_i, \bar{X}_i) \leq L$ ($\bar{X}_0 = X_0 = 1_G$, $\bar{X}_n = X_n = x$) and $\bar{U}_1 \in p_1$ satisfying $d(\tilde{U}_1, \bar{U}_1) \leq L$. For each $i \in \{1, 2, \ldots, n-2\}$ the triangle $\tilde{X}_i \bar{X}_i X_{i+1}$ is δ -slim, hence $\exists \tilde{U}'_{i+1} \in [\bar{X}_i, X_{i+1}] : d(\tilde{U}'_{i+1}, \tilde{U}_{i+1}) \leq L+\delta$. The segment of p_{i+1} between \bar{X}_i and X_{i+1} is quasigeodesic with the same constants as p_{i+1} , therefore there is a point $\bar{U}_{i+1} \in p_{i+1}$ between \bar{X}_i and X_{i+1} such that $d(\tilde{U}'_{i+1}, \bar{U}_{i+1}) \leq L$, and, consequently, $d(\tilde{U}_{i+1}, \bar{U}_{i+1}) \leq 2L + \delta$ (see Figure 4).



Let α_t denote the segment of p_t from \bar{X}_{t-1} to X_t , $t = 2, \ldots, n$, and β_s – the subpath of p_s from \bar{X}_{s-1} to \bar{U}_s , $s = 1, \ldots, n-1$. Shifting the points \bar{X}_i, \bar{U}_i , $i = 1, \ldots, n-1$, along their sides of P (so that \bar{U}_i still stays between \bar{X}_{i-1} and X_i on p_i) by distances at most K_1 , we can achieve $elem(\beta_1) \in fG_1$ (i.e. $lab(\beta_1)$ represents an element of fG_1), $elem(\alpha_t) \in G_{t+1}$, $elem(\beta_s) \in G_s$, $t = 2, \ldots, n, s = 1, 2, \ldots, n-1$. And after this, setting, for brevity, $K = max\{K_1 + \frac{3}{2}L, K_2 + \frac{3}{2}L\}$, one obtains

$$(\bar{X}_{i-1}|\bar{X}_{i+1})_{\bar{X}_i} \le \delta + 3K_1 + 3L \le \delta + 3K \le 14\delta + 3K \stackrel{def}{=} C_0 ,$$

$$|\bar{X}_i - \bar{U}_i| \le \delta + 2K_1 + 3L + \delta \le 2\delta + 2K, \ i = 1, \dots, n-1.$$

Let $elem(\beta_1) = f\bar{g}_1$, $elem(\beta_i) = \bar{g}_i$, i = 1, ..., n-1, $elem(\alpha_n) = \bar{g}_n$, where $\bar{g}_k \in G_k$, k = 1, 2, ..., n. $elem([\bar{U}_i; \bar{X}_i]) = u_i \in G_i G_{i+1}$, i = 1, ..., n-1.

Then $|u_i|_G \leq 2\delta + 2K$, and there are only finitely many of possible u_i 's for every $i \in \{1, 2, ..., n-1\}$. Hence, we achieved the following representation for x:

(*)
$$x \stackrel{G}{=} f\bar{g}_1 u_1 \bar{g}_2 u_2 \cdot \ldots \cdot \bar{g}_{n-1} u_{n-1} \bar{g}_n$$

Similarly, one can obtain

(**)
$$x \stackrel{G}{=} e\bar{h}_1 v_1 \bar{h}_2 v_2 \cdot \ldots \cdot \bar{h}_{m-1} v_{m-1} \bar{h}_m \quad ,$$

where $\bar{h}_j \in H_j$, $j = 1, \ldots, m$; $v_j \in H_j H_{j+1}$ and $|v_j|_G \leq 2\delta + 2K$ for every $j = 1, 2, \ldots, m-1$ (see Figure 5).

 $\mathcal{U}_i \stackrel{def}{=} \{ u \in G_i G_{i+1} : |u|_G \le 2\delta + 2K \} \subset G, i = 1, \dots, n-1 . \ card(\mathcal{U}_i) < \infty,$

 $\forall i = 1, \dots, n-1$. For convenience, $\mathcal{U}_0 = \mathcal{U}_n = G_0 = G_{n+1} \stackrel{def}{=} \{1_G\}$. Analogously, define $\mathcal{V}_j \subset H_j H_{j+1}, j = 1, \dots, m-1$, and again,

 $\mathcal{V}_0 = \mathcal{V}_m = H_0 = H_{m+1} \stackrel{def}{=} \{1_G\}.$ Set $D = 14(\delta + C_0) + 3K = const$, and $\mathcal{L} = \{g \in G : |g|_G \leq D\}$. At last, we denote

$$\Delta_i = \mathcal{U}_{i-1} \cdot (\mathcal{L} \cap G_i) \cdot \mathcal{U}_i \subset G_{i-1}G_iG_{i+1} \subset G , \ i = 1, 2, \dots, n ,$$

$$\Theta_i = \mathcal{V}_{j-1} \cdot (\mathcal{L} \cap H_j) \cdot \mathcal{V}_j \subset H_{j-1}H_jH_{j+1} \subset G , \ j = 1, 2, \dots, m .$$

By construction, $card(\Delta_i) < \infty$, $card(\Theta_i) < \infty$, $\forall i, j$. Take any $i \in \{1, 2, ..., n\}$ and consider the intersection

$$T \supseteq fG_1G_2 \cdot \ldots \cdot G_{i-1}\Delta_iG_{i+1} \cdot \ldots \cdot G_n \cap eH_1 \cdot \ldots \cdot H_m =$$
$$= \bigcup_{g \in \Delta_i} [fG_1G_2 \cdot \ldots \cdot G_{i-1}gG_{i+1} \cdot \ldots \cdot G_n \cap eH_1 \cdot \ldots \cdot H_m] =$$
$$= \int_{G_1} [fG_1G_2 \cdot \ldots \cdot G_{i-1}gG_{i+1} \cdot \ldots \cdot G_n \cap eH_1 \cdot \ldots \cdot H_m] =$$

$$= \bigcup_{g \in \Delta_i} \left[fg(g^{-1}G_1g)(g^{-1}G_2g) \cdots (g^{-1}G_{i-1}g)G_{i+1} \cdots G_n \cap eH_1 \cdots H_m \right] .$$

Because of remark 4, one can apply the induction hypothesis to the last expression and conclude that it is a (finite) "special" union. Hence,

(3)
$$T_1 \stackrel{def}{=} \bigcup_{i=1}^n \left(fG_1G_2 \cdot \ldots \cdot G_{i-1}\Delta_i G_{i+1} \cdot \ldots \cdot G_n \cap eH_1 \cdot \ldots \cdot H_m \right)$$

is also a finite special union.

Because of the symmetry, we parallely showed that

(4)
$$T_2 \stackrel{def}{=} \bigcup_{j=1}^m \left(fG_1 \cdot \ldots \cdot G_n \cap eH_1H_2 \cdot \ldots \cdot H_{j-1}\Theta_jH_{j+1} \cdot \ldots \cdot H_m \right)$$

is a finite "special" union.

We have just proved that there exist $r_1 \in \mathbb{N} \cup \{0\}, f_l \in G$ and increasing (n, m)-products $S_l, l = 1, 2, \ldots, r_1$, such that

$$T_1 \cup T_2 = \bigcup_{l=1}^{r_1} f_l S_l \subseteq T$$

 $T = T_1 \cup T_2 \cup T_3$, where $T_3 \stackrel{def}{=} T \setminus (T_1 \cup T_2)$. Now, let's consider the case $x \in T_3$. It means that in representations (*) and (**) for $x, |\bar{g}_i|_G > D$, $|\bar{h}_{j}|_{G} > D$, for $D = 14(\delta + C_{0}) + 3K$ and $\forall i = 1, \dots, n, \forall j = 1, \dots, m$. Therefore, returning to the pair of polygons we constructed, one will have :



$$\begin{split} ||[\bar{X}_0;\bar{X}_1]|| &\geq |\bar{g}_1|_G - |f|_G - |u_1|_G > 14(\delta + C_0) + 3K - K - 2\delta - 2K = \\ &= 12(\delta + C_0) + 2C_0 \stackrel{def}{=} C_1, ||[\bar{X}_{i-1};\bar{X}_i]|| \geq |\bar{g}_i|_G - |u_i|_G > 14(\delta + C_0) + 3K - 2\delta - \\ &- 2K > C_1, \ i = 2, \dots, n-1, \ ||[\bar{X}_{n-1};\bar{X}_n]|| = |\bar{g}_n|_G > C_1. \\ \text{We also possess the following inequalities} : \ (\bar{X}_{i-1}|\bar{X}_{i+1})_{\bar{X}_i} < C_0, \ i = 1, \dots, n-1, \\ &C_0 \geq 14\delta, \ C_1 > 12(\delta + C_0). \end{split}$$

By lemma 1.3, the broken line $[\bar{X}_0; \bar{X}_1; \ldots; \bar{X}_n]$ is contained in the closed $C = 2C_0$ -neighborhood of the geodesic segment $[\bar{X}_0; \bar{X}_n]$. In particular,

(5)
$$d(\bar{X}_{n-1}, [\bar{X}_0; \bar{X}_n]) \le C$$

A similar argument shows that $d(\bar{Y}_{m-1}, [\bar{Y}_0; \bar{Y}_m]) \leq C$, and, since $[\bar{X}_0; \bar{X}_n] = [X_0; X_n] = [Y_0; Y_m] = [\bar{Y}_0; \bar{Y}_m]$, one has

(6)
$$d(\bar{Y}_{m-1}, [\bar{X}_0; \bar{X}_n]) \le C$$

b) In the previous case we assumed that $n, m \ge 2$ and we needed quite a long argument to prove (5) and (6). On the other hand, if, for example, n = 1, then $X_0 = \bar{X}_{n-1}$ and (5) is trivial.

Because of (5) and (6) one can choose $W, Z \in [X_0; X_n]$ with the properties $|W - \bar{X}_{n-1}| \leq C, |Z - \bar{Y}_{m-1}| \leq C.$

The first possibility is, when the point W on $[X_0; X_n]$ lies between Z and X_n , i.e. $W \in [Z; X_n]$.

Then, since triangles are δ -thin in the hyperbolic space $\Gamma(G, \mathcal{A})$,

 $d(W, [\bar{Y}_{m-1}; X_n]) \leq C + \delta$. Hence $d(\bar{X}_{n-1}, [\bar{Y}_{m-1}; X_n]) \leq 2C + \delta$. Consequently, because q_m is quasigeodesic, there exists a point R on the subpath γ of q_m from \bar{Y}_{m-1} to Y_m such that $d(\bar{X}_{n-1}, R) \leq 2C + \delta + K + M_m$ (M_m is the same as in (i)) and $elem([R; Y_m]) = elem(\gamma) = \hat{h}_m \in H_m$.

Define $\Omega = \{g \in G_n H_m : |g|_G \leq 2C + \delta + K + M_m\}$. Therefore $card(\Omega) < \infty$ and $elem([\bar{X}_{n-1}; R]) \in \Omega$.

For each element $g \in \Omega$ take a pair $g' \in G_n$, $h' \in H_m$ such that g = g'h'. By $G' \subset G_n$ denote the set of all elements g' which we have chosen, by $H' \subset H_m$ – the set of all h''s.

$$x = f\bar{g}_1u_1\dots u_{n-1}\bar{g}_n = e\bar{h}_1v_1\dots v_{m-1}\bar{h}_m .$$

From the triangle $\bar{X}_{n-1}X_nR$ we obtain $\bar{g}_n\hat{h}_m^{-1} = g'h' \in \Omega, g' \in G', h' \in H'$. Thus $(g')^{-1}\bar{g}_n = h'\hat{h}_m \in G_n \cap H_m$.

$$x \in fG_1G_2 \cdot \ldots \cdot G_{n-1} \cdot u_{n-1}g' \cdot ((g')^{-1}\overline{g}_n) \cap eH_1H_2 \cdot \ldots \cdot H_{m-1}H_m \subset$$

$$\subset fG_1G_2\cdot\ldots\cdot G_{n-1}\mathcal{U}_{n-1}G'\cdot (G_n\cap H_m)\cap eH_1H_2\cdot\ldots\cdot H_{m-1}H_m\subset T$$
.

Denote $I = \mathcal{U}_{n-1} \cdot G' \subset G_{n-1}G_n$ - a finite subset of G. Then

$$x \in fG_1G_2 \cdot \ldots \cdot G_{n-1}I \cdot (G_n \cap H_m) \cap eH_1H_2 \cdot \ldots \cdot H_{m-1}H_m =$$
$$= [fG_1G_2 \cdot \ldots \cdot G_{n-1}I \cap eH_1H_2 \cdot \ldots \cdot H_m] \cdot (G_n \cap H_m) \subset T.$$

The second possibility, when $Z \in [W; X_n]$ is considered analogously, and, in this case, one obtains a finite subset $J \subset H_{m-1}H_m$ such that

$$x \in [fG_1G_2 \cdot \ldots \cdot G_n \cap eH_1H_2 \cdot \ldots \cdot H_{m-1}J] \cdot (G_n \cap H_m) \subset T$$
.

Therefore, we showed that $T_3 \subseteq [T'_3 \cup T''_3] \cdot (G_n \cap H_m) \subset T$ where

(7)
$$T_3' \stackrel{def}{=} fG_1G_2 \cdot \ldots \cdot G_{n-1}I \cap eH_1H_2 \cdot \ldots \cdot H_m$$

(8)
$$T_3'' \stackrel{def}{=} fG_1G_2 \cdot \ldots \cdot G_n \cap eH_1H_2 \cdot \ldots \cdot H_{m-1}J$$

Combining the formulas (3),(4),(7),(8) and the property that if $H \leq G$ and $a \in H$ then aH = Ha = H, we obtain the following

Lemma 3.2. In notations of the theorem 1

$$fG_1G_2\cdot\ldots\cdot G_n\cap eH_1H_2\cdot\ldots\cdot H_m=T_1\cup T_2\cup [T'_3\cup T''_3]\cdot (G_n\cap H_m)$$

where

$$T_{1} = \bigcup_{i=1}^{n} \left(fG_{1}G_{2} \cdot \ldots \cdot G_{i-1}\overline{\Delta}_{i}G_{i+1} \cdot \ldots \cdot G_{n} \cap eH_{1} \cdot \ldots \cdot H_{m} \right) ,$$

$$T_{2} = \bigcup_{j=1}^{m} \left(fG_{1} \cdot \ldots \cdot G_{n} \cap eH_{1}H_{2} \cdot \ldots \cdot H_{j-1}\overline{\Theta}_{j}H_{j+1} \cdot \ldots \cdot H_{m} \right) ,$$

$$T_{3}' = fG_{1}G_{2} \cdot \ldots \cdot G_{n-1}\overline{I} \cap eH_{1}H_{2} \cdot \ldots \cdot H_{m} ,$$

$$T_{3}'' = fG_{1}G_{2} \cdot \ldots \cdot G_{n} \cap eH_{1}H_{2} \cdot \ldots \cdot H_{m-1}\overline{J}$$

for some finite subsets $\overline{\Delta}_i \subset G_i, \overline{\Theta}_j \subset H_j, \ \overline{I} \subset G_n, \overline{J} \subset H_m, \ 1 \leq i \leq n,$ $1 \leq j \leq m.$

Now, to finish the proof of the theorem, we apply the inductive hypothesis:

$$T_3 \subseteq \bigcup_{g \in I} \left[fG_1 G_2 \cdot \ldots \cdot G_{n-1}g \cap eH_1 H_2 \cdot \ldots \cdot H_m \right] \cdot \left(G_n \cap H_m \right) \cup$$

$$\bigcup_{h \in J} [fG_1G_2 \cdot \ldots \cdot G_n \cap eH_1H_2 \cdot \ldots \cdot H_{m-1}h] \cdot (G_n \cap H_m) =$$

$$= \bigcup_{g \in I} \left[fgG_1^{g^{-1}}G_2^{g^{-1}} \cdot \ldots \cdot G_{n-1}^{g^{-1}} \cap eH_1H_2 \cdot \ldots \cdot H_m \right] \cdot (G_n \cap H_m) \cup$$

$$\cup \bigcup_{h \in J} \left[fG_1G_2 \cdot \ldots \cdot G_n \cap ehH_1^{h^{-1}}H_2^{h^{-1}} \cdot \ldots \cdot H_{m-1}^{h^{-1}} \right] \cdot (G_n \cap H_m) =$$

$$= \left(\bigcup_{g \in I} \left[\bigcup_{k=1}^{\tilde{r}} \tilde{f}_k \tilde{S}_k \right] \cup \bigcup_{h \in J} \left[\bigcup_{q=1}^{\hat{r}} \hat{f}_q \hat{S}_q \right] \right) \cdot (G_n \cap H_m) =$$

$$= \bigcup_{l=r_1+1}^r f_l S_l \subset T .$$

Here $\tilde{r}, \hat{r}, r \in \mathbb{N} \cup \{0\}, r \geq r_1, \tilde{f}_k, \hat{f}_q, f_l \in G; \tilde{S}_k$ is an (n-1,m)-increasing product, \hat{S}_q is an (n,m-1)-increasing product and S_l is an (n,m)-increasing product; $k = 1, \ldots, \tilde{r}; q = 1 \ldots, \hat{r}; l = r_1 + 1, \ldots, r.$ Hence,

$$T = T_1 \cup T_2 \cup T_3 \subseteq \bigcup_{l=1}^r f_l S_l \subseteq T$$

and, thus

$$T = \bigcup_{l=1}^r f_l S_l \; .$$

So, the theorem is proved. \Box

Proof of Corollary 2. Observe that arbitrary quasiconvex product $f_1G_1f_2G_2$ f_nG_n is equal to a "transformed" product $fG'_1G'_2 \ldots G'_n$ where $G'_i = (f_{i+1} \ldots f_n)^{-1}G_i(f_{i+1} \ldots f_n), i = 1, \ldots, n-1, G'_n = G_n$, are quasiconvex subgroups of G by remark 4 and $f = f_1f_2 \ldots f_n \in G$. It remains to apply theorem 1 to the intersection of "transformed products" several times because a (n, m)-increasing product is also a quasiconvex product. \Box

4. Products of elementary subgroups

Recall that a group H is called *elementary* if it has a cyclic subgroup $\langle h \rangle$ of finite index.

<u>Remark 5.</u> An elementary subgroup H of a hyperbolic group G is quasiconvex .

Indeed, we have : $|H : \langle h \rangle| < \infty$. If the element *h* has a finite order, then *H* is finite and, thus, quasiconvex. In the case, when the order of *h* is infinite, by lemmas 1.2,1.1 $\langle h \rangle$ is a quasiconvex subgroup of *G*. By remark 4 and lemma 2.1 *H* is quasiconvex.

It is well known that any element x of infinite order in G is contained in a unique maximal elementary subgroup $E(x) \leq G$ (see [4]). And the intersection

of two distinct maximal elementary subgroups in a hyperbolic group is finite. Any infinite elementary subgroup contains an element of infinite order.

Obviously, a conjugate subgroup to a maximal elementary subgroup is also maximal elementary.

Proof of Theorem 2. The sufficiency is trivial.

Without loss of generality one can assume $n \ge m$. In this case theorem 2 immediately follows from

Theorem 2' Let $n \ge m$, $G_1, G_2, \ldots, G_n, H_1, H_2, \ldots, H_m$ be infinite maximal elementary subgroups of G, $f, e \in G$, and $g_i \in G_i$, $i = 1, 2, \ldots, n$, be elements of infinite order. Also, assume $G_i \ne G_{i+1}, H_j \ne H_{j+1}, i = 1, \ldots, n-1$, $j = 1, \ldots, m-1$. If there is a sequence of positive integers $(t_k)_{k=1}^{\infty}$ with the properties:

$$\lim_{k \to \infty} t_k = \infty \text{ and } fg_1^{t_k}g_2^{t_k} \cdot \ldots \cdot g_n^{t_k} \in eH_1H_2 \cdot \ldots \cdot H_m \text{ for all } k \in \mathbb{N},$$

then n = m, $G_n = H_n$, and there exist elements $z_i \in H_i$, i = 1, ..., n, such that $G_i = (z_n z_{n-1} \dots z_{i+1}) \cdot H_i \cdot (z_n z_{n-1} \dots z_{i+1})^{-1}$, i = 1, 2, ..., n-1, $f = ez_1^{-1} z_2^{-1} \dots z_n^{-1}$. Consequently, $fG_1 \cdot \dots \cdot G_n = eH_1 \cdot \dots \cdot H_m$.

In the conditions of theorem 2', let $h_j \in H_j$ be fixed elements of infinite order, j = 1, 2, ..., m. Then $G_i = E(g_i)$, $H_j = E(h_j)$ and $|G_i : \langle g_i \rangle| < \infty$, $|H_j : \langle h_j \rangle| < \infty$. Hence, there exists $T \in \mathbb{N}$ such that for all j and $\forall v \in H_j$ $\exists \beta \in \mathbb{Z}, y \in H_j : v = y \cdot h_j^\beta$ and $|y|_G \leq T$. Thus, every element $h \in eH_1 \cdot \ldots \cdot H_m$ can be presented in the form

(9)
$$h = ey_1 h_1^{\beta_1} y_2 h_2^{\beta_2} \cdot \ldots \cdot y_m h_m^{\beta_m}$$

where $\beta_j \in \mathbb{Z}, y_j \in H_j, |y_j|_G \leq T, j = 1, 2, \dots, m.$

Definition: the representation (9) for h will be called *reduced* if for any i, j, $1 \le i < j \le m$, such that $\beta_i, \beta_j \ne 0$, one has

$$(y_{i+1}h_{i+1}^{\beta_{i+1}}\dots h_{j-1}^{\beta_{j-1}}y_j)^{-1} \cdot h_i \cdot (y_{i+1}h_{i+1}^{\beta_{i+1}}\dots h_{j-1}^{\beta_{j-1}}y_j) \notin H_j = E(h_j) .$$

Observe that each element $h \in eH_1 \cdot \ldots \cdot H_m$ has a reduced representation. Indeed, if $(y_{i+1}h_{i+1}^{\beta_{i+1}} \ldots h_{j-1}^{\beta_{j-1}}y_j)^{-1} \cdot h_i \cdot (y_{i+1}h_{i+1}^{\beta_{i+1}} \ldots h_{j-1}^{\beta_{j-1}}y_j) \in H_j$ for some $1 \leq i < j \leq m$ then there are $\beta'_j \in \mathbb{Z}, \ y'_j \in H_j, \ |y'_j|_G \leq T$:

$$y_j \cdot (y_{i+1}h_{i+1}^{\beta_{i+1}} \dots h_{j-1}^{\beta_{j-1}}y_j)^{-1} \cdot h_i^{\beta_i} \cdot (y_{i+1}h_{i+1}^{\beta_{i+1}} \dots h_{j-1}^{\beta_{j-1}}y_j) \cdot h_j^{\beta_j} = y_j'h_j^{\beta_j'}.$$

Therefore,

$$h = ey_1 h_1^{\beta_1} \cdot \ldots \cdot y_{i-1} h_{i-1}^{\beta_{i-1}} y_i y_{i+1} h_{i+1}^{\beta_{i+1}} \cdot \ldots \cdot y_{j-1} h_{j-1}^{\beta_{j-1}} y_j' h_j^{\beta_j'} y_{j+1} h_{j+1}^{\beta_{j+1}} \cdot \ldots \cdot y_m h_m^{\beta_m}$$

and the number of non-zero β_k 's is decreased. Continuing this process, we will obtain a reduced representation for h after a finite number of steps .

Proof of Theorem 2'. Let $h_j \in H_j$, $1 \le j \le m, T$, be as above. Induction on n.

If n=1, then, evidently, m = 1, and $\forall k \in \mathbb{N}$ there is $y_{t_k} \in H_1$, $|y_{t_k}|_G \leq T$, and $\beta_{t_k} \in \mathbb{Z}$ such that $fg_1^{t_k} = ey_{t_k}h_1^{\beta_{t_k}}$. Because of having $\lim_{k\to\infty} t_k = \infty$, one can choose $p, q \in \mathbb{N}$ so that $t_p < t_q$ and $y_{t_p} = y_{t_q}$. Therefore,

$$fg_1^{t_p}h_1^{-\beta_{t_p}} = ey_{t_p} = fg_1^{t_q}h_1^{-\beta_{t_q}} ,$$

and, thus, $g_1^{t_p-t_q} = h_1^{\beta_{t_p}-\beta_{t_q}}$ – an element of infinite order in the intersection of G_1 and H_1 . Consequently, $G_1 = H_1$, because these subgroups are maximal elementary.

Assume, now, that n > 1. For every $k \in \mathbb{N}$ one has

(10)
$$fg_1^{t_k}g_2^{t_k}\cdot\ldots\cdot g_n^{t_k} = ey_{t_k1}h_1^{\beta_{t_k1}}y_{t_k2}h_2^{\beta_{t_k2}}\cdot\ldots\cdot y_{t_km}h_m^{\beta_{t_km}}$$

where the product in the right-hand side is reduced. Obviously, there exists a subsequence $(l_k)_{k=1}^{\infty}$ of (t_k) and $C \in \mathbb{N}$ such that for each $j \in \{1, 2, \ldots, m\}$ either $|\beta_{l_k j}| \leq C$ for all k or $\lim_{k\to\infty} |\beta_{l_k j}| = \infty$.

Therefore, since $|y_{l_kj}|_G \leq T \ \forall \ k \in \mathbb{N}, \ \forall \ j$, there is a subsequence $(s_k)_{k=1}^{\infty}$ of (l_k) such that $y_{s_kj} = y_j \in H_j \ \forall \ j$, and if for $j \in \{1, \ldots, m\}$ we had $|\beta_{l_kj}| \leq C$ $\forall \ k \in \mathbb{N}$ then $|\beta_{s_kj}| = \beta_j \in \mathbb{Z} \ \forall \ k \in \mathbb{N}$, and $\lim_{k \to \infty} |\beta_{s_kj}| = \infty$ for all other j's. Thus, $\{1, 2, \ldots, m\} = J_1 \cup J_2$ where if $j \in J_1$ then $|\beta_{s_kj}| = \beta_j$ for every k, and

if $j \in J_2$ then $\lim_{k\to\infty} |\beta_{s_k,j}| = \infty$. Let $J_2 = \{j_1, j_2, \dots, j_\varkappa\} \subset \{1, 2, \dots, m\}, j_1 < j_2 < \dots < j_\varkappa$, and denote

$$w_1 = y_1^{-1} \in H_1 \text{ if } j_1 = 1 \text{ , otherwise, if } j_1 > 1,$$

$$w_1 = y_{j_1}^{-1} h_{j_1-1}^{-\beta_{j_1-1}} y_{j_1-1}^{-1} \cdot \ldots \cdot h_1^{-\beta_1} y_1^{-1} \in H_{j_1} H_{j_1-1} \cdot \ldots \cdot H_1 \text{ ;}$$

$$\ldots \ldots$$

 $w_\varkappa = y_{j_\varkappa}^{-1} \in H_{j_\varkappa}$ if $j_\varkappa = j_{\varkappa-1}+1$, otherwise, if $j_\varkappa > j_{\varkappa-1}+1,$

$$w_{\varkappa} = y_{j_{\varkappa}}^{-1} h_{j_{\varkappa}-1}^{-\beta_{j_{\varkappa}-1}} y_{j_{\varkappa}-1}^{-1} \cdot \ldots \cdot h_{j_{\varkappa-1}+1}^{-\beta_{j_{\varkappa}-1}+1} y_{j_{\varkappa-1}+1}^{-1} \in H_{j_{\varkappa}} H_{j_{\varkappa-1}} \cdot \ldots \cdot H_{j_{\varkappa-1}+1} ;$$

$$w_{\varkappa+1} = 1_G \text{ if } j_{\varkappa} = m \text{ , otherwise, if } j_{\varkappa} < m,$$

$$w_{\varkappa+1} = h_m^{-\beta_m} y_m^{-1} \cdot \ldots \cdot h_{j_{\varkappa}+1}^{-\beta_{j_{\varkappa}+1}} y_{j_{\varkappa}+1}^{-1} \in H_m H_{m-1} \cdot \ldots \cdot H_{j_{\varkappa}+1} .$$

To simplify the formulas, denote $\delta_{k\nu} = -\beta_{s_k,j_\nu}$, $1 \le \nu \le \varkappa$. Then $\lim_{k\to\infty} |\delta_{k\nu}| = \infty$ for every $\nu = 1, 2, \ldots, \varkappa$. (10) is equivalent to

(11)
$$u_k \stackrel{def}{=} fg_1^{s_k}g_2^{s_k} \cdot \ldots \cdot g_{n-1}^{s_k}w_{\varkappa+1}h_{j_{\varkappa}}^{\delta_{k_{\varkappa}}}w_{\varkappa}h_{j_{\varkappa-1}}^{\delta_{k,\varkappa-1}} \cdot \ldots \cdot w_2h_{j_1}^{\delta_{k_1}}w_1e^{-1} = 1_G$$

So, $|u_k|_G = 0$ for all $k \in \mathbb{N}$. Denote $K = max\{|f|_G, |w_1e^{-1}|_G, |w_2|_G, \dots, |w_{\varkappa+1}|_G\}$, and assume that $g_n \notin w_{\varkappa+1}E(h_{j_\varkappa})w_{\varkappa+1}^{-1}$. The product in the right-hand side of (10) was reduced, therefore $h_{j_\nu} \notin w_\nu E(h_{j_{\nu-1}})w_{\nu-1}^{-1}, \nu = 2, 3, \dots, \varkappa$. Thus, we can apply Lemma 1.8 to (11) and obtain $\lambda > 0, c \geq 0$

and M > 0 (depending on K, $g_1, \ldots, g_n, h_{j_1}, \ldots, h_{j_\varkappa}$) such that if $s_k \ge M$ and $|\delta_{k\nu}| \ge M$, $\nu = 2, 3, \ldots, \varkappa$, then $|u_k|_G \ge \lambda \cdot s_k - c$. Now, by the choice of the sequence (s_k) , there exists $N \in \mathbb{N}$: $s_k > M$ and $|\delta_{k\nu}| > M \forall k \ge N$, $\nu = 2, 3, \ldots, \varkappa$. Thus, taking $k \ge max\{N, c/\lambda\} + 1$, we achieve a contradiction: $0 = |u_k|_G < \lambda \cdot s_k - c$.

Hence, $g_n \in w_{\varkappa+1} E(h_{j_{\varkappa}}) w_{\varkappa+1}^{-1}$ which implies

(12)
$$G_n = E(g_n) = w_{\varkappa+1} E(h_{j_{\varkappa}}) w_{\varkappa+1}^{-1} = E(w_{\varkappa+1} h_{j_{\varkappa}} w_{\varkappa+1}^{-1})$$

Consequently, for every $k \in \mathbb{N}$ $w_{\varkappa+1}^{-1}g_n^{s_k}w_{\varkappa+1}h_{j_\varkappa}^{\delta_{k,\varkappa}} = y'_{kj_\varkappa}h_{j_\varkappa}^{\gamma_k} \in H_{j_\varkappa}$ where $|y'_{kj_\varkappa}|_G \leq T$. By passing to a subsequence of (s_k) we can assume that $y'_{kj_\varkappa} = y'_{j_\varkappa} \in H_{j_\varkappa}$ for every k. Therefore

$$u_{k} = fg_{1}^{s_{k}}g_{2}^{s_{k}} \cdot \ldots \cdot g_{n-1}^{s_{k}}w_{\varkappa+1}y_{j_{\varkappa}}'h_{j_{\varkappa}}^{\gamma_{k}}w_{\varkappa}h_{j_{\varkappa-1}}^{\delta_{k,\varkappa-1}} \cdot \ldots \cdot w_{2}h_{j_{1}}^{\delta_{k1}}w_{1}e^{-1} = 1_{G}.$$

Suppose $\limsup_{k\to\infty} |\gamma_k| = \infty$. Since $E(g_{n-1}) = G_{n-1} \neq G_n = E(g_n)$, we have $g_{n-1} \notin w_{\varkappa+1} E(h_{j_{\varkappa}}) w_{\varkappa+1}^{-1} = w_{\varkappa+1} y'_{kj_{\varkappa}} E(h_{j_{\varkappa}}) (y'_{kj_{\varkappa}})^{-1} w_{\varkappa+1}^{-1}$ (because $y'_{kj_{\varkappa}} \in H_{j_{\varkappa}} = E(h_{j_{\varkappa}})$).

Then for $K' = max\{K, |w_{\varkappa+1}y'_{j_{\varkappa}}|_G\}$ by Lemma 1.8 there exist $\lambda > 0, c \ge 0$ and M > 0 (depending on $K', g_1, \ldots, g_{n-1}, h_{j_1}, \ldots, h_{j_{\varkappa}}$) such that if $s_k \ge M$, $|\delta_{k\nu}| \ge M, \nu = 2, 3, \ldots, \varkappa$, and $|\gamma_k| \ge M$ then $|u_k|_G \ge \lambda \cdot s_k - c$. Now, by the assumption on (s_k) and (γ_k) , there exists $N \in \mathbb{N}, N > c/\lambda$, such that $s_N > M$, $|\delta_{N\nu}| > M, \nu = 2, 3, \ldots, \varkappa$, and $|\gamma_N| > M$. Which leads us to a contradiction: $0 = |u_k|_G < \lambda \cdot s_k - c$.

Thus, $|\gamma_k| \leq C_1$ for some constant C_1 , so, by passing to a subsequence as above, we can assume that $\gamma_k = \gamma \ \forall \ k \in \mathbb{N}$. Hence, after setting $z_{\varkappa} = w_{\varkappa+1}y'_{i_{\varkappa}}h^{\gamma}_{i_{\varkappa}}w_{\varkappa}$, for every natural index k we will have

$$u_{k} = f g_{1}^{s_{k}} g_{2}^{s_{k}} \cdot \ldots \cdot g_{n-1}^{s_{k}} z_{\varkappa} h_{j_{\varkappa-1}}^{\delta_{k,\varkappa-1}} w_{j_{\varkappa-1}} \cdot \ldots \cdot w_{2} h_{j_{1}}^{\delta_{k1}} w_{1} e^{-1} = 1_{G}$$

Which implies $fg_1^{s_k}g_2^{s_k} \cdot \ldots \cdot g_{n-1}^{s_k} \in ew_1^{-1}H_{j_1}w_2^{-1}H_{j_2} \cdot \ldots \cdot w_{j_{\varkappa-1}}^{-1}H_{j_{\varkappa-1}}z_{\varkappa}^{-1} = uH_1^{v_2}H_2^{v_3} \cdot \ldots \cdot H_{j_{\varkappa-1}}^{v_{\varkappa}}$ where $v_{\nu} = z_{\varkappa}w_{\varkappa-1} \cdot \ldots \cdot w_{\nu+1}w_{\nu}, \ \nu = 2, 3, \ldots, \varkappa - 1,$ $v_{\varkappa} = z_{\varkappa}, \ u = ew_1^{-1}w_2^{-1} \cdot \ldots \cdot w_{\varkappa-1}^{-1}z_{\varkappa}^{-1}.$

 $n-1 \geq m-1 \geq \varkappa - 1$ and the other conditions of the theorem 2' are satisfied, therefore one can apply the induction hypothesis and obtain that $n-1 = \varkappa - 1$, hence, $\varkappa = m = n$, $j_{\nu} = \nu$, $1 \leq \nu \leq \varkappa$, and, by definition, $w_{\nu} = y_{j_{\nu}}^{-1} \in H_{\nu}$, $\nu = 1, 2, \ldots, n$, $w_{\varkappa+1} = 1_G$, $z_{\varkappa} = z_n \in H_n$. And also $G_{n-1} = H_{j_{\varkappa-1}}^{v_{\varkappa}} = H_{n-1}^{z_n}$, and there exist $\hat{z}_i \in H_i$, $1 \leq i \leq n-1$, such that

$$G_{i} = (\hat{z}_{n-1}^{v_{n}} \hat{z}_{n-2}^{v_{n-1}} \dots \hat{z}_{i+1}^{v_{i+2}}) \cdot H_{i}^{v_{i+1}} \cdot (\hat{z}_{n-1}^{v_{n}} \hat{z}_{n-2}^{v_{n-1}} \dots \hat{z}_{i+1}^{v_{i+2}})^{-1} =$$

 $= (z_n z_{n-1} \cdot \ldots \cdot z_{i+1}) \cdot H_i \cdot (z_n z_{n-1} \cdot \ldots \cdot z_{i+1})^{-1} , \ i = 1, 2, \dots, n-2 ,$

where $z_p = \hat{z}_p w_p \in H_p$, $1 \le p \le n-1$, $f = u (\hat{z}_1^{v_2})^{-1} (\hat{z}_2^{v_3})^{-1} \cdots (\hat{z}_{n-1}^{v_2})^{-1} = e z_1^{-1} z_2^{-1} \cdots z_n^{-1}$.

By (12) $G_n = E(h_n) = H_n$. The proof of the theorem 2' is finished. \Box

Suppose G_1, G_2, \ldots, G_n are infinite maximal elementary subgroups of G, $f_1, \ldots, f_n \in G$, $n \in \mathbb{N} \cup \{0\}$.

Definition : the set $P = f_1G_1f_2G_2 \cdot \ldots \cdot f_nG_n$ will be called *ME-product*. Thus, if n = 0, we have the empty set. For convenience, we will also consider every element $g \in G$ to be a ME-product. As in the proof of corollary 2, every such ME-product can be brought to a form (however, not unique)

$$P' = fG'_1G'_2 \cdot \ldots G'_k$$

where $0 \leq k \leq n, f \in G, G'_i$ are infinite maximal elementary subgroups, i = 1, 2, ..., k, and $G'_i \neq G'_{i+1}, 1 \leq i \leq k-1$. The number k in this case will be called *rank* of the ME-product P (thus, $rank(P) = rank(P') = k \leq n$). A set U which can be presented as a finite union of ME products has rank k.

A set U which can be presented as a finite union of ME-products has rank k, by definition, if $U = \bigcup_{i=1}^{t} P_i$, where P_i , $i = 1, \ldots, t$, are ME-products, and $k = max\{rank(P_i) \mid 1 \le i \le t\}$.

Note: an empty set is defined to have rank (-1); any element of the group G is a ME-product of rank 0; thus any finite non-empty subset of G is a finite union of ME-products of rank 0.

<u>**Remark 6.**</u> the rank of a ME-product is defined correctly by theorem 2. By theorem 2' the definition of the rank of a finite union of ME-products is correct.

Lemma 4.1. Suppose P,R are ME-products in a hyperbolic group G. Then the intersection $T \stackrel{def}{=} P \cap R$ is a finite union of ME-products and its rank is at most rank(P). If rank(T) = rank(P) then T = P.

Proof. Since a conjugate to an infinite maximal elementary subgroup is also infinite maximal elementary, it follows from theorem 1 that T is a finite union of ME-products P_i , $1 \le i \le t$ (for some $t \in \mathbb{N} \cup \{0\}$):

$$T = P \cap R = \bigcup_{i=1}^{t} P_i \; .$$

For each i = 1, ..., t, $P_i \subseteq P$, therefore by theorem 2', $rank(P_i) \leq rank(P)$ (otherwise we would get a contradiction), and $rank(P_i) = rank(P)$ if and only if $P_i = P$. Thus $rank(T) = max\{rank(P_i) \mid 1 \leq i \leq t\} \leq rank(P)$. If rank(T) = rank(P) then $rank(P_i) = rank(P)$ for some *i*, and so, $P_i = P = T$. Q.e.d. \Box

As an immediate consequence of lemma 4.1 one obtains

Corollary 3. let P be a ME-product of rank n and U be a finite union of ME-products. Then the set $P \cap U$ is a finite union of ME-products, $rank(P \cap U) \leq n$, and if $rank(P \cap U) = n$ then $P \cap U = P$.

Corollary 4. A non-elementary hyperbolic group G can not be equal to a finite union of its ME-products.

Proof. Suppose, by the contrary, that G is a finite union of ME-products: $G = P_1 \cup \ldots \cup P_l$ and rank(G) = m. Since G is not elementary, there exist two elements $x, y \in G$ of infinite order such that $E(x) \neq E(y)$. Hence, one can construct a ME-product $P = G_1 G_2 \cdot \ldots \cdot G_{m+1}$ in G where $G_i = E(x)$ if i is even, and $G_i = E(y)$ if i is odd. Consequently, rank(P) = m + 1, but $P \subset G$, thus

$$P \cap G = P = \bigcup_{j=1}^{l} (P_j \cap P)$$
.

By lemma 4.1, $rank(P_j \cap P) \leq rank(P_j) \leq m$ for every j = 1, 2, ..., l. Therefore, we achieve a contradiction with the definition of rank : $m + 1 = rank(P) = rank(P \cap G) \leq m$. \Box

A group H is called *bounded-generated* if it is a product of finitely many cyclic subgroups, i.e. there are elements $x_1, x_2, \ldots, x_k \in H$ such that every $h \in H$ is equal to $x_1^{s_1} x_2^{s_2} \cdot \ldots \cdot x_k^{s_k}$ for some $s_1, \ldots, s_k \in \mathbb{Z}$.

Corollary 5. Any bounded-generated hyperbolic group is elementary.

Proof. Indeed, any cyclic subgroup of a hyperbolic group either is finite or is contained in some infinite maximal elementary subgroup. Hence, their product is contained in a finite union of ME-products and we can apply corollary 4. \Box

Proof of Theorem 3. Since there exist at most countably many different ME-products in G, it is enough to consider only their countable intersections. Let P_{ji} , $1 \le i \le k_j$, k_j , $j \in \mathbb{N}$, be ME-products, and $U_j = \bigcup_{i=1}^{k_j} P_{ji}$ – their finite unions. Let

$$T = \bigcap_{j=1}^{\infty} U_j \; .$$

One has to show that there exist ME-products $R_1, \ldots, R_s, s \in \mathbb{N} \cup \{0\}$, such that $T = R_1 \cup \ldots \cup R_s$.

Induct on $n = rank(U_1)$.

$$T = \left(\bigcup_{i=1}^{k_1} P_{1i}\right) \cap \bigcap_{j=2}^{\infty} U_j = \bigcup_{i=1}^{k_1} \left(P_{1i} \cap \bigcap_{j=2}^{\infty} U_j\right)$$

So, it is enough to consider the case when $k_1 = 1$, $U_1 = P_{11} = P$. If n = 0 then P is finite and there is nothing to prove.

Assume that n > 0 and let $J \in \mathbb{N}$ be the smallest index such that $P \cap U_J \neq P$ (if there is no such J then T = P and the theorem is true). Therefore

$$T = P \cap \bigcap_{j=J}^{\infty} U_j = (P \cap U_J) \cap \bigcap_{j=J+1}^{\infty} U_j$$
.

By corollary 3, $P \cap U_J$ is a finite union of ME-products :

$$P \cap U_J = \bigcup_{l=1}^t R'_l , \ t \in \mathbb{N} \cup \{0\}$$

and $rank(P \cap U_J) < n$ because of the choice of J, therefore $rank(R'_l) < n$, $\forall \ l=1,2,\ldots,t.$

Hence, by the induction hypothesis,

$$T = \bigcup_{l=1}^{t} \left[R'_l \cap \bigcap_{j=J+1}^{\infty} U_j \right] = \bigcup_{l=1}^{t} \left[R_{l1} \cup \ldots \cup R_{ls_l} \right]$$

for some ME-products $R_{l1}, \ldots, R_{ls_l}, s_l \in \mathbb{N} \cup \{0\}, 1 \leq l \leq t$. \Box

The statement of the theorem 3 fails to be true if maximal elementary subgroups in the definition of ME-products one substitutes by arbitrary elementary subgroups. Below we construct an example to demonstrate that .

Let G = F(x, y) be the free group with two generators, $q_1 < q_2 < q_3 < \dots$ be an infinite sequence of prime numbers . Define $d_i = q_1 q_2 \cdot \dots \cdot q_i$, $c_i = q_1 q_2 \cdot \dots \cdot q_{i-1} q_i^2 = d_i \cdot q_i$, $i \in \mathbb{N}$, and the sets P_i , $i \in \mathbb{N}$, as follows : $P_1 = \langle x^{d_1} \rangle$ - cyclic subgroup of G generated by $x^{d_1} = x^{q_1}$, $P_2 = \langle y \rangle \cdot \langle y x^{c_1} y^{-1} \rangle \cdot \langle y^2 x^{d_2} y^{-2} \rangle \cdot \langle y \rangle$, $P_3 = \langle y \rangle \cdot \langle y x^{c_1} y^{-1} \rangle \cdot \langle y^2 x^{c_2} y^{-2} \rangle \cdot \langle y^3 x^{d_3} y^{-3} \rangle \cdot \langle y \rangle$, $\dots P_i = \langle y \rangle \cdot \langle y x^{c_1} y^{-1} \rangle \cdot \langle y^2 x^{c_2} y^{-2} \rangle \cdot \dots \cdot \langle y^{i-1} x^{c_{i-1}} y^{-(i-1)} \rangle \langle y^i x^{d_i} y^{-i} \rangle \cdot \langle y \rangle$, $\dots \dots$

Now consider the intersection $T = \bigcap_{i=1}^{\infty} P_i$. Let us observe that $P_1 \cap P_2 = \langle x^{c_1} \rangle \cup \langle x^{d_2} \rangle, \dots, \bigcap_{i=1}^k P_i = \langle x^{c_1} \rangle \cup \dots \cup \langle x^{c_{k-1}} \rangle \cup \langle x^{d_k} \rangle, \dots$. Indeed, $P_1 \cap P_2 = \langle x^{d_1} \rangle \cap (\langle x^{c_1} \rangle \cup \langle x^{d_2} \rangle) = \langle x^{c_1} \rangle \cup \langle x^{d_2} \rangle$. Inducting on k, we get

$$\bigcap_{i=1}^{k} P_{i} = \left(\bigcap_{i=1}^{k-1} P_{i}\right) \cap P_{k} = \langle x^{c_{1}} \rangle \cup \ldots \cup \langle x^{c_{k-2}} \rangle \cup \langle x^{d_{k-1}} \rangle \cap (\langle x^{c_{1}} \rangle \cup \ldots \cup \langle x^{c_{k-1}} \rangle \cup \cup \langle x^{d_{k}} \rangle) = \langle x^{d_{k}} \rangle = \langle x^{c_{1}} \rangle \cup \ldots \cup \langle x^{c_{k-2}} \rangle \cup \langle x^{d_{k-1}} \rangle \cap (\langle x^{c_{k-1}} \rangle \cup \langle x^{d_{k}} \rangle) = \langle x^{c_{1}} \rangle \cup \ldots \cup \langle x^{c_{k-2}} \rangle \cup \langle x^{d_{k-1}} \rangle \cup \langle x^{d_{k}} \rangle.$$

Since $\bigcap_{i=1}^{\infty} \langle x^{d_i} \rangle = \{1\}$, therefore $T = \bigcup_{i=1}^{\infty} \langle x^{c_i} \rangle$.

If $q_1 = 2, q_2 = 3, q_3 = 5, \ldots$, is chosen to be the enumeration of all primes, one can show directly that the set T can not be presented as a finite union of products $f_1G_1f_2G_2 \cdots f_nG_n$, where $f_1, \ldots, f_n \in G$ and G_1, \ldots, G_n are elementary (in this case cyclic) subgroups of G. We are not going to do that, instead we will use a set-theoretical argument : there are only countably many such finite unions, hence there is an infinite sequence of primes $q_1 < q_2 < q_3 < \ldots$ such that the corresponding set $\bigcap_{i=1}^{\infty} P_i$ is the example sought (because the sets $\bigcap_{i=1}^{\infty} P_i$ and $\bigcap_{i=1}^{\infty} P_i'$ corresponding to different increasing sequences of prime numbers $\alpha = \{q_1, q_2, q_3, \ldots\}$ and $\alpha' = \{q'_1, q'_2, q'_3, \ldots\}$ are distinct: if $q_l \in \alpha \setminus \alpha'$ then $x^{c_l} \in \bigcup_{i=1}^{\infty} \langle x^{c_i} \rangle \setminus \bigcup_{i=1}^{\infty} \langle x^{c'_j} \rangle$).

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