# Functions on groups and computational complexity 

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#### Abstract

We give some connections between various functions defined on finitely presented groups (isoperimetric, isodiametric, Todd-Coxeter radius, filling length functions, etc.), and we study the relation between those functions and the computational complexity of the word problem (deterministic time, nondeterministic time, symmetric space). We show that the isoperimetric function can always be linearly decreased (unless it is the identity map). We present a new proof of the Double Exponential Inequality, based on context-free languages.


## 1 Introduction

The best studied functions on finitely presented group are the so-called "filling functions", in particular the isoperimetric functions and the isodiametric functions, and more recently, the filling length. The significance of the filling functions comes from the following:
(1) The connection between the filling functions of a group presentation and the computational complexity of the word problem of that presentation. By their very definition, filling functions express the difficulty or intricateness of certain aspects of a group presentation; hence they are a form of complexity by themselves. Moreover, there are strong connections between certain filling functions and certain computational complexity functions of the word problem. Actually, the computational complexity functions of the word problem could be considered as filling functions too.
(2) All the filling functions of a group (including the computational complexity functions of the word problem) are algebraic invariants of the group, in the sense that if one takes a different finite presentation of the same group the filling functions change only linearly (i.e., "up to bigO"). Some filling functions (e.g., the computational complexity functions of the word problem) are even more strongly invariant; they do not depend on the presentation, and are invariants of the group ("up to big-O") under change of finite set of generators. Filling functions can give structural information about a group, especially when the filling functions are small.

The term "filling function" is inspired from homotopy transformations, which "fill the space between" two objects that can be deformed into each other; this is Gromov's point of view, which has been very influential [18]. The same idea can be represented by different images in different contexts. From the point of view of computation, this consists of filling in the steps between the input and the output of a computation. Generally, any transformation that can

[^0]be decomposed into (or built up from) a set of smaller transformations has filling functions that count various steps in the transformation. Filling functions in this general sense have been known since antiquity (e.g., the most elementary filling functions are the length, area and volume of curves, surfaces, bodies; the number of primes in the prime decomposition of an integer is another old example of a filling function). One of the most interesting new developments is the connection between "static" fillings (like length and area) and "dynamic" fillings (e.g., space and time complexity of computations).

Notation and terminology: For an alphabet $A$ we let $A^{-1}$ be a disjoint copy of $A$. We write $A^{ \pm 1}$ for $A \cup A^{-1}$. We will usually assume that $|A| \geq 2$. The free monoid generated by $A^{ \pm 1}$ (i.e., the set of all finite sequences over the alphabet $A^{ \pm 1}$, including the empty sequence 1 ) is denoted by $\left(A^{ \pm 1}\right)^{*}$; its elements are called words. The length of a word $w \in\left(A^{ \pm 1}\right)^{*}$ is denoted by $|w|$. For a set $X$, we denote the cardinality of $X$ by $|X|$. For a set of words $R$, we denote the sum of the lengths of the words in $R$ by $\|R\|\left(=\sum_{r \in R}|r|\right)$. For a word $w=a_{1} \ldots a_{n}$ with $a_{i} \in A^{ \pm 1}$ for $i=1, \ldots, n$, we define $w^{-1}=a_{n}^{-1} \ldots a_{1}^{-1}$. For a set $R \subseteq\left(A^{ \pm 1}\right)^{*}$, we define $R^{-1}=\left\{r^{-1}: r \in R\right\}$, and we define $R^{ \pm 1}=R \cup R^{-1}$. We denote the free group over the generating set $A$ by $\mathrm{FG}(A)$. We denote the reduction in $\operatorname{FG}(A)$ by red; the word obtained from $w \in\left(A^{ \pm 1}\right)^{*}$ by reduction in $\operatorname{FG}(A)$ is red $(w)$. If $G$ is a group with generating set $A$ and if $x, y \in\left(A^{ \pm 1}\right)^{*}$ are words that represent the same element of $G$ we write $x={ }_{G} y$, and we say that $x$ and $y$ are equivalent in $G$ (or equivalent modulo $G$ ); we call the relation $=_{G}$ on $\left(A^{ \pm 1}\right)^{*}$ the congruence defining the group $G$. If a group $G$ has a presentation $\langle A ; R\rangle$ with set of generators $A$ and set of relators $R \subset\left(A^{ \pm 1}\right)^{*}$ then $=_{\langle A ; R\rangle}$ denotes the the congruence on $\left(A^{ \pm 1}\right)^{*}$ determined by this presentation. The congruence class of a word $w$ modulo $=_{G}$ (i.e., the set $\left\{x \in\left(A^{ \pm 1}\right)^{*}: x={ }_{G} w\right\}$ ) is denoted by $[w]_{G}$. If $x, y \in\left(A^{ \pm 1}\right)^{*}$ are the same word we write $x=y$ and we say that $x$ and $y$ are literally equal (the Russian literature calls this "graphically equal"). We carefully distinguish between words $\left(\in\left(A^{ \pm 1}\right)^{*}\right)$ and elements of the free group $\mathrm{FG}(A)$ (elements of a free group are equivalence classes of words); if $x, y \in\left(A^{ \pm 1}\right)^{*}$ are equivalent in the free group we write $x=_{\mathrm{FG}(A)} y$ or $x=_{\mathrm{FG}} y$. When used between words, " $=$ " denotes literal equality. By $g^{h}$ we denote the conjugate $\left(h^{-1} g h\right)$ of $g \in G$ by $h \in G$. If $x, y$ are words $\left(\in\left(A^{ \pm 1}\right)^{*}\right)$ we also use the notation $x^{y}$ for the word $y^{-1} x y$. For a finite presentations $\langle A ; R\rangle$ we always assume that the empty word 1 is not a relator $(1 \notin R)$.

## 2 Filling functions on groups

Definition 2.1 An isoperimetric function of a finite presentation $\langle A ; R\rangle$ of a group is any function $P: \mathbb{N} \rightarrow \mathbb{N}$ with the following property: For every $w \in\left(A^{ \pm 1}\right)^{*}$ such that $w=_{\langle A ; R\rangle} 1$ there exists a finite sequence $\left(r_{i}: i \in I\right)$ of relators in $R^{ \pm 1}$, and a finite sequence $\left(x_{i}: i \in I\right)$ of words in $\left(A^{ \pm 1}\right)^{*}$ such that

$$
w==_{\mathrm{FG}} \prod_{i \in I} r_{i}^{x_{i}} \quad \text { and } \quad|I| \leq P(|w|) .
$$

It is known that this definition is equivalent to stating that every word $w \in\left(A^{ \pm 1}\right)^{*}$ such that $w=_{\langle A ; R\rangle} 1$ has a van Kampen diagram with area $\leq P(|w|)$. This characterization motivates the term "isoperimetric".

It is also equivalent to saying that every word $w \in\left(A^{ \pm 1}\right)^{*}$ such that $w=_{\langle A ; R\rangle} 1$ can be rewritten to 1 in at most $P(|w|)$ "steps". To define a rewrite step we consider the finite rewrite system with alphabet $A^{ \pm 1}$ and set of rules $U_{R} \cup U_{\mathrm{FG}}$, where

$$
\begin{aligned}
& U_{R}=\left\{u \rightarrow v: u, v \in\left(A^{ \pm 1}\right)^{*}, u v^{-1} \in R^{ \pm 1}\right\}, \\
& U_{\mathrm{FG}}=\left\{a a^{-1} \rightarrow 1,1 \rightarrow a a^{-1}: a \in A^{ \pm 1}\right\}
\end{aligned}
$$

But in the isoperimetric function we do not count applications of the rules in $U_{\mathrm{FG}}$ (those are automatically part of any group and are taken for granted; moreover, they don't show up in the van Kampen diagrams). Note that this rewrite system is symmetric, i.e., if $x \rightarrow y$ is a rule then $y \rightarrow x$ is rule too. The rewriting characterization is the oldest explicit definition of the isoperimetric function, under the name "rewrite distance" (Madlener and Otto [23]). Gromov [17] introduced the geometric point of view on these functions; see also Gersten's explanations [14]. In [23] it is proved that the rewrite distances (i.e., the isoperimetric functions) are invariants of the group, up to big-O, in the following sense: If one takes a different finite presentation of the same group, the isoperimetric function $P(n)$ becomes $c_{1} P\left(c_{2} n\right)$, where $c_{1}, c_{2}>0$ are "constants" (i.e., they do not depend on $n$, but they depend on the two presentations). This fact was rediscovered a little later independently by several authors.

Yet another equivalent definition of the isoperimetric function can be obtained by using the Cayley 2-complex of the presentation. An isoperimetric function of the presentation is any function $P: \mathbb{N} \rightarrow \mathbb{N}$ with the following property: For every $w \in\left(A^{ \pm 1}\right)^{*}$ such that $w_{\langle A ; R\rangle} 1$ there exists a combinatorial homotopy in the Cayley 2-complex, starting with a loop labeled by $w$ (with base point 1 ) and ending with the trivial loop consisting of the base point 1 ; the area covered by the homotopy transformation is $\leq P(|w|)$ (the "area covered" consists of all the faces used, with multiplicities, i.e., a same face that is counted repeatedly if it is used repeatedly).

Definition $2.2 \quad A$ filling length function of a finite presentation $\langle A ; R\rangle$ of a group is any function $F: \mathbb{N} \rightarrow \mathbb{N}$ with the following property: Every word $w \in\left(A^{ \pm 1}\right)^{*}$ such that $w=_{\langle A ; R\rangle} 1$ can be rewritten to 1 (i.e., $w \rightarrow w_{N-1} \rightarrow \ldots \rightarrow w_{i} \rightarrow \ldots w_{1}=1$ ), using the rewrite system $\left(A^{ \pm 1}, U_{R} \cup U_{\mathrm{FG}}\right)$ defined above, in such a way that the intermediate words in the rewrite sequence are all of length $\left|w_{i}\right| \leq F(|w|)$.

Another characterization of the filling length is that every word $w \in\left(A^{ \pm 1}\right)^{*}$ such that $w=_{\langle A ; R\rangle} 1$ has a van Kampen diagram which can be contracted to one point, homotopically, in such a way that all intermediary van Kampen diagrams encountered during this contraction have perimeter $\leq F(|w|)$; see [15].

The concept of filling length was introduced by Gromov [18], and extensively studied by Gersten and Riley [15]. They proved that the filling length changes only up to "big-O" (just like the isoperimetric function) when the finite presentation is changed.

Definition 2.3 $A n$ isodiametric function of a finite presentation $\langle A ; R\rangle$ of a group $G$ is any function $D: \mathbb{N} \rightarrow \mathbb{N}$ with the following property: For every $w \in\left(A^{ \pm 1}\right)^{*}$ such that $w=_{\langle A ; R\rangle} 1$ there exists a finite sequence $\left(r_{i}: i \in I\right)$ of relators in $R^{ \pm 1}$, and a finite sequence $\left(x_{i}: i \in I\right)$ of words in $\left(A^{ \pm 1}\right)^{*}$ such that

$$
w=_{\mathrm{FG}} \prod_{i \in I} r_{i}^{x_{i}} \quad \text { and } \quad\left|x_{i}\right| \leq D(|w|) \quad(\text { for all } i \in I)
$$

This function was explicitly introduced by Gersten (see [14] for references). The above definition is equivalent to saying that every word $w \in\left(A^{ \pm 1}\right)^{*}$ such that $w=_{\langle A ; R\rangle} 1$ has a van Kampen diagram with diameter $\leq D(|w|)$ (the diameter being measured from the origin of $w$ on the perimeter of the diagram).

Another characterization of the isodiametric functions is by means of a loop complex. For any positive integer $j$ we define a labeled directed graph $\Lambda_{j}$ with labels in $A$, as follows:

- First, we create a new vertex $v_{0}$ (the "origin").
- Second, for every relator $r \in R$ and every reduced word $u \in\left(A^{ \pm 1}\right)^{*}$ such that $|u| \leq j$, we create a path labeled by $u$, starting from vertex $v_{0}$; at the non- $v_{0}$ end of this path we attach a loop labeled by $r$; in doing this, we create $|u|+|r|-1$ new vertices (for every pair $(u, r)$ ). All the paths (and all the loops) are disjoint, except that all the paths have the common vertex $v_{0}$. - In order to get by with only the alphabet $A$ we replace each edge $v_{1} \xrightarrow{a^{-1}} v_{2}$ by $v_{1} \stackrel{a}{\leftarrow} v_{2}$, where $v_{1}, v_{2}$ are vertices and $a \in A$.
- We can turn the graph $\Lambda_{j}$ into a 2 -complex by adding a face for each loop (the boundary of each face being the corresponding loop). We call this the loop-complex of radius $j$ of the presentation.
- We can turn the labeled graph $\Lambda_{j}$ into a nondeterministic finite automaton (an "NFA") over the alphabet $A^{ \pm 1}$. We take the vertex $v_{0}$ as both start and accept state, and we "symmetrize" each edge: for every edge $v_{1} \xrightarrow{a} v_{2}$ we also introduce the "inverse edge" $v_{2} \xrightarrow{a^{-1}} v_{1}$ into the NFA; no new vertices are created in this symmetrization. See [20] for definitions and basic facts about NFAs. The language accepted by an NFA $N$ is denoted by $L(N)$.

We call this NFA " $\Lambda_{j}$ " too. The context will always make it clear whether we refer to the graph, the 2-complex, or the NFA; the vertex set is the same in the three cases.

The number of vertices of $\Lambda_{j}$ is $\leq(\|R\|+|R| \cdot j)(2|A|)^{j} \leq c^{j}$, where $c>1$ is a constant that depends on the presentation $\langle A ; R\rangle$.

Characterization of the isodiametric functions: A function $D(\cdot)$ is an isodiametric function of $\langle A ; R\rangle$ iff for every word $w \in\left(A^{ \pm 1}\right)^{*}$ of length $\leq n$ we have: $w=_{\langle A ; R\rangle} 1$ iff there is a loop in $\Lambda_{D(n)}$, starting and ending at $v_{0}$, and labeled by a word equivalent (in $\operatorname{FG}(A)$ ) to $w$. This characterization of the isodiametric functions can be reformulated as follows:

Lemma 2.4 A function $D(\cdot)$ is an isodiametric function of $\langle A ; R\rangle$ iff for every word $w \in$ $\left(A^{ \pm 1}\right)^{*}$ of length $|w| \leq n$ we have:

$$
w=_{\langle A ; R\rangle} 1 \quad \text { iff } \quad[w]_{\mathrm{FG}} \cap L\left(\Lambda_{D(n)}\right) \neq \emptyset .
$$

The Lemma above can be refined to obtain an upper bound on the minimum isoperimetric function $P_{\min }(\cdot)$ of $\langle A ; R\rangle$. Lemma 2.5 is important because it connects $P_{\min }$ and $D_{\min }$; from this we will be able to re-prove the famous "double exponential inequality" in a later section.

Lemma 2.5 Let $\langle A ; R\rangle$ be a finite presentation with minimum isoperimetric function $P_{\min }$ and isodiametric function $D$. Suppose $\ell(n)$ has the property that for all words $w \in\left(A^{ \pm 1}\right)^{*}$ of length $\leq n$ such that $w=_{\langle A ; R\rangle} 1$, we have:

$$
[w]_{\mathrm{FG}} \cap L\left(\Lambda_{D(n)}\right) \text { contains a word of length } \leq \ell(n)
$$

Then $P_{\text {min }}(n) \leq \ell(n)$.

Proof: Since $w=_{\langle A ; R\rangle} 1$ and $|w| \leq 1$, Lemma 2.4 implies that $[w]_{\mathrm{FG}} \cap L\left(\Lambda_{D(n)}\right) \neq \emptyset$. Let $z \in[w]_{\mathrm{FG}} \cap L\left(\Lambda_{D(n)}\right)$ with $|z| \leq \ell(n)$. Then $z==_{\mathrm{FG}} w$ and $z=\prod_{i \in I} r_{i}^{x_{i}}$ (literal equality) for some sequence of relators $r_{i} \in R^{ \pm 1}$ and some sequence of words $x_{i}$ with $\left|x_{i}\right| \leq D(n)$. Since the equality $z=\prod_{i \in I} r_{i}^{x_{i}}$ is literal, and since all $r_{i}$ are non-empty, we conclude that $|I| \leq|z| \leq \ell(n)$. Hence, by the definition of the isoperimetric function, $P_{\min }(n) \leq|I|$.

For later use we introduce a slightly more compact NFA which can play the same role as $\Lambda_{D(n)}$ in Lemmas 2.4 and 2.5 (although it does not accept the same language). The tree NFA of radius $j$ for a presentation $\langle A ; R\rangle$ is defined as follows:

- First, we take the Cayley graph of $\operatorname{FG}(A)$, truncated to radius $j$ around the origin; this is a tree of depth $j$, with $1+\left|A^{ \pm 1}\right|\left(\left|A^{ \pm 1}\right|-1\right)^{j-1}$ vertices.
- Second, for every $r \in R$, at every vertex of the above graph we attach a loop labeled by $r$.
- We pick the root vertex of the tree as start and accept state.
- We symmetrize all edges, in the same way as we did for the NFA $\Lambda_{j}$.

We call the resulting NFA "tree $\Lambda_{j}$ ".
When $R \neq \emptyset$, the number of states of tree $\Lambda_{j}$ is $\leq\|R\| \cdot\left(1+\left|A^{ \pm 1}\right| \sum_{i=0}^{j-1}\left(\left|A^{ \pm 1}\right|-1\right)^{i}\right)$. Thus, when $R \neq \emptyset$, the number of states is $\leq\|R\|(2|A|)^{j}$.

Lemma 2.6 Let $\langle A ; R\rangle$ be a finite presentation.

- A function $D(\cdot)$ is an isodiametric function of $\langle A ; R\rangle$ iff for any word $w \in\left(A^{ \pm 1}\right)^{*}$ of length $|w| \leq n$ we have:

$$
w={ }_{\langle A ; R\rangle} 1 \quad \text { iff }[w]_{\mathrm{FG}} \cap L\left(\operatorname{tree} \Lambda_{D(n)}\right) \neq \emptyset .
$$

- Let the minimum isoperimetric function of the presentation be $P_{\min }$ and let $D$ be an isodiametric function. Suppose $\ell(n)$ has the property that for all words $w \in\left(A^{ \pm 1}\right)^{*}$ of length $\leq n$ such that $w=_{\langle A ; R\rangle} 1$, we have:

$$
[w]_{\mathrm{FG}} \cap L\left(\text { tree } \Lambda_{D(n)}\right) \quad \text { contains a word of length } \leq \ell(n) .
$$

Then $P_{\min }(n) \leq \ell(n)$.
Proof. The proof is very similar to the proofs of Lemmas 2.4 and 2.5 .
The loop complex $\Lambda_{D(n)}$ can be folded: Suppose in the graph we have $v_{1} \stackrel{a}{\leftarrow} v_{2} \xrightarrow{a} v_{3}$, (or we have $v_{1} \xrightarrow{a} v_{2} \stackrel{a}{\leftarrow} v_{3}$ ), where $a \in A$ and where $v_{1}, v_{2}, v_{3}$ are vertices. Then we "glue" $v_{1}$ and $v_{3}$ together; i.e., we replace $v_{1}, v_{3}$ by a new vertex $v_{1,3}$, and we replace the two edges $v_{1} \stackrel{a}{\leftarrow} v_{2}$ and $v_{2} \xrightarrow{a} v_{3}$ by the single edge $v_{1,3} \stackrel{a}{\leftarrow} v_{2}$ (respectively, replace $v_{1} \xrightarrow{a} v_{2}$ and $v_{2} \stackrel{a}{\leftarrow} v_{3}$ by $v_{1,3} \xrightarrow{a} v_{2}$ ). All in-edges (or out-edges) of $v_{1}$ and $v_{3}$ become in-edges (respectively out-edges) of $v_{1,3}$. This process continues as long as possible. Finally, to obtain a 2 -complex we attach faces on all loops labeled by relators.

Based on the folded graph one can obtain a "folded NFA", called $\mathrm{f} \Lambda_{D(n)}$, by symmetrizing all edges, and taking $v_{0}$ as start and accept state. The folded NFA is actually deterministic (it is a "DFA"), and it has the following property:

Lemma 2.7 Let $w \in\left(A^{ \pm 1}\right)^{*}$ be any word of length $\leq n \in\left(A^{ \pm 1}\right)^{*}$. Then we have $w=_{\langle A ; R\rangle} 1$ iff the reduced word $\operatorname{red}(w)$ is accepted by the folded DFA $\mathrm{f} \Lambda_{D(n)}$.

Proof: Let $\left(N_{i}: i=0,1, \ldots, m\right)$ be the sequence of NFAs obtained by successively folding edges; $N_{0}=\Lambda_{D(n)}$, and $N_{i+1}$ is obtained from $N_{i}$ by folding one pair of edges, for $i=0,1, \ldots, m$; the number of folding steps $m$ is less than the number of edges of $\Lambda_{D(n)}$, hence $m \leq c^{D(n)}$ for some constant $c>1$.

We want to show that for each NFA $N_{i}(i=0,1, \ldots, m)$, the language accepted satisfies $\operatorname{red}\left(L\left(N_{i}\right)\right)=\operatorname{red}\left(L\left(\Lambda_{D(n)}\right)\right)$. In other words, although $L\left(N_{i}\right)$ changes, the set of reductions of all words in $L\left(N_{i}\right)$ does not change.
(1) $L\left(\Lambda_{D(n)}\right) \subseteq L\left(N_{i}\right)$, hence we also have $\operatorname{red}\left(L\left(\Lambda_{D(n)}\right)\right) \subseteq \operatorname{red}\left(L\left(N_{i}\right)\right)$ for all $i$.

This is straightforward, since folding does not destroy any reachabilities, but adds additional reachabilities. So, $N_{i}$ contains all the accepting paths of $\Lambda_{D(n)}$ (up to changes of vertex names).
(2) $\operatorname{red}\left(L\left(N_{i}\right)\right) \subseteq \operatorname{red}\left(L\left(\Lambda_{D(n)}\right)\right)$ for all $i$.

To prove this we use induction on $i$. Inclusion (2) is obvious when $i=0$. Suppose red $\left(L\left(N_{i-1}\right)\right) \subseteq$ $\operatorname{red}\left(L\left(\Lambda_{D(n)}\right)\right)$. Let $p$ be an accepting path in $N_{i}$. If $p$ does not use an edge involved in the folding step that leads from $N_{i-1}$ to $N_{i}$, the path $p$ occurs in $N_{i-1}$ too; so, any word accepted by $N_{i}$ by means of $p$ is also accepted by $N_{i-1}$. If $p$ uses one edge involved in the folding step, the names of one of the vertices in $p$ changed when $N_{i-1}$ was transformed to $N_{i}$, but the edge labels in $p$ do not change; so, here too, any word accepted by $N_{i}$ by means of $p$ is also accepted by $N_{i-1}$. Finally, if two edges in $p$ are folded together then $p$ has the form $p_{1} p_{2}$, where

$$
p_{1}\left(v_{1} \stackrel{a}{\leftarrow} v_{2}\right)\left(v_{2} \xrightarrow{a} v_{3}\right) p_{2}
$$

is a path in $N_{i-1}, a \in A$ (we only consider one of the folding cases; the other is very similar). Let $x_{1}, x_{2} \in\left(A^{ \pm 1}\right)^{*}$ be the labels of $p_{1}$, respectively $p_{2}$. Along the path $p, N_{i}$ accepts $x_{1} x_{2}$, whereas $N_{i-1}$ accepts $x_{1} a a^{-1} x_{2}$; but red $\left(x_{1} x_{2}\right)=\operatorname{red}\left(x_{1} a a^{-1} x_{2}\right)$.

We will now define the "folded" versions of the above functions.
Definition $2.8 \quad A$ folded isoperimetric function of a finite presentation $\langle A ; R\rangle$ of a group is any function $p: \mathbb{N} \rightarrow \mathbb{N}$ with the following property: Every $w \in\left(A^{ \pm 1}\right)^{*}$ such that $w=_{\langle A ; R\rangle} 1$ has a van Kampen diagram whose folded area is $\leq p(|w|)$. The folded area of a van Kampen diagram is the number of faces in the 2-complex obtained by folding the van Kampen diagram. (Note: Faces that have the same boundary loop in the 2-complex are viewed as the same face.)

Definition $2.9 \quad A$ folded filling length function of a finite presentation $\langle A ; R\rangle$ of a group is any function $f: \mathbb{N} \rightarrow \mathbb{N}$ with the following property: Every $w \in\left(A^{ \pm 1}\right)^{*}$ such that $w=_{\langle A ; R\rangle} 1$ has a folded van Kampen diagram that admits a homotopy transformation which starts with the loop $w$, and ends with the origin point, and with all intermediate loops of length $\leq f(|w|)$.

As usual, "length" means length of a curve (or a path); it is not just the number of different edges; repetitions of edges are counted too.

Definition $2.10 \quad A$ folded isodiametric function of a finite presentation $\langle A ; R\rangle$ of a group is any function $d: \mathbb{N} \rightarrow \mathbb{N}$ with the following property: Every $w \in\left(A^{ \pm 1}\right)^{*}$ such that $w=_{\langle A ; R\rangle} 1$ has a folded van Kampen diagram of diameter (measured from the origin) $\leq d(|w|)$.

The folded isoperimetric function and the folded filling length function seem not to have appeared in the literature. The folded isodiametric function has been used a number of times; we will prove later that the minimum folded isodiametric function is equal to the minimum isodiametric function (and similarly for the filling length function).

Another function on finite presentations of groups can be defined by using the radius of the partial Cayley graphs; these partial Cayley graphs are constructed by the following version of the Todd-Coxeter process, used here for the word problem for words of length $\leq n$. We closely follow [10] (p. 110); see also [29] (which presents a somewhat different graphical version of Todd-Coxeter, however).

In the process below we use the following definition. In a graph with origin $v_{0}$, a hair is an edge $e$ such that one end vertex $v$ of $e$ has the following properties: $v$ has degree 1 , and $v \neq v_{0}$.

Process TC on input $\langle A ; R\rangle$ :
create a vertex $v_{0}$ (called "origin");
repeat

1. for every vertex $v$ of the graph constructed so far:
for every letter $a \in A$ such that $v$ does not have an out-edge with label $a$ : create a new vertex $(v, a)$ and a new edge $v \xrightarrow{a}(v, a)$;
for every letter $a \in A$ such that $v$ does not have an in-edge with label $a$ : create a new vertex $\left(v, a^{-1}\right)$ and a new edge $v \stackrel{a}{\leftarrow}\left(v, a^{-1}\right)$;
2. for every vertex $v$ of the graph constructed so far:
for every relator $r \in R$ which does not label a loop originating at $v$ : create a new loop labeled by $r$ and originating at $v$;
3. fold the graph obtained so far;
(* The folded graph constructed so far, with all hairs ignored, is called a "partial Cayley graph". *)

We call TC a "process" (as opposed to "algorithm") because it does not terminate.
Definition 2.11 $A$ Todd-Coxeter radius of a finite presentation $\langle A ; R\rangle$ of a group $G$ is any function $\rho_{\mathrm{TC}}: \mathbb{N} \rightarrow \mathbb{N}$ with the following property:

After some number of steps the Todd-Coxeter process TC constructs a partial Cayley graph, called $\mathrm{TC}_{n}$ with radius $\leq \rho_{\mathrm{TC}}(n)$ such that
$\left(\forall w \in\left(A^{ \pm 1}\right)^{*},|w| \leq n\right)\left[w=_{\langle A ; R\rangle} 1\right.$ iff $\operatorname{red}(w)$ labels a loop at the origin in $\left.\mathrm{TC}_{n}\right]$.
When $n$ is even, the condition $(\forall w,|w| \leq n)\left[w=_{\langle A ; R\rangle} 1\right.$ iff $\operatorname{red}(w)$ labels a loop at the origin] is equivalent to the following:

Within radius $n / 2$ from the origin, the graph $\mathrm{TC}_{n}$ is identical to the ball of radius $n / 2$ of the Cayley graph of the presentation.

The concept of the Todd-Coxeter radius function appears indirectly in [11] (in the case when it is linear). I learned about it from Stuart Margolis and John Meakin [24].

If there exists a computable function which is an upper bound on $\rho_{\text {TC }}(\cdot)$ then the process $T C$ can be used to decide the word problem of the presentation $\langle A ; R\rangle$.

We will view $\mathrm{TC}_{n}$ as a DFA (deterministic finite automaton), by taking the origin $v_{0}$ as start and accept state, and by symmetrizing the edges (as we did for $\Lambda_{D(n)}$ ). We will also view $\mathrm{TC}_{n}$ as a 2-complex (which agrees with the Cayley 2-complex within radius $n / 2$ when $n$ is even). The context will tell us which one of the three $\mathrm{TC}_{n}$ 's we are talking about.

Proposition 2.12 For any word $w \in\left(A^{ \pm 1}\right)^{*}$ of length $\leq n$ we have:

$$
w={ }_{\langle A ; R\rangle} 1 \text { iff the DFA } \mathrm{TC}_{n} \text { accepts } \operatorname{red}(w)
$$

Proof. This follows immediately from the fact that $w=_{\langle A ; R\rangle} 1 \mathrm{iff} \operatorname{red}(w)$ labels a loop in the Cayley graph. Moreover, the partial Cayley graph $\mathrm{TC}_{n}$ coincides with the Cayley graph within radius $n / 2$.

Another way to build a 2-complex in order to solve the word problem for words of length $\leq n$ is as follows: For each $w \in\left(A^{ \pm 1}\right)^{*}$ of length $\leq n$ we consider all the van Kampen diagrams of $w$ of minimum folded diameter. We create a new vertex $v_{0}$, and attach the origins of all these van Kampen diagrams to $v_{0}$; now we have a connected 2 -complex. Next, we fold this 2-complex. We call this the folded van Kampen 2-complex for words of length $\leq n$, and denote it by $\mathrm{fK}_{n}$.

A third way to build a 2-complex in order to solve the word problem for words of length $\leq n$ is as follows: We create a vertex $v_{0}$ (an origin). For every relator $r \in R$ and every word $x \in\left(A^{ \pm 1}\right)^{*}$ of length $\leq \lambda(n)$ (for a certain function $\lambda(n)$ to be determined soon), we create a loop with origin $v_{0}$, labeled by $r^{x}$; this loop bounds one face. We denote this 2 -complex by $\mathrm{LC}_{n}$. Next, we fold the 2-complex $\mathrm{LC}_{n}$, and denote the resulting 2-complex by $\mathrm{fLC}_{n}$. Finally, we choose $\lambda(n)$ large enough, but minimal, such that in $\mathrm{fLC}_{n}$ we have: For every word $w$ of length $\leq n, w=_{\langle A ; R\rangle} 1$ iff red $(w)$ labels a closed path through the origin.

The function $\lambda($.$) above is called "folded loop-complex function". One notes immediately$ that for the minimum function $\lambda(\cdot)$ of the presentation we have:

$$
\mathrm{LC}_{n}=\Lambda_{\lambda(n)}
$$

where $\Lambda_{j}$ is the loop complex introduced following Definition 2.3. We call $\mathrm{f} \Lambda_{\lambda(n)}$ the folded loop-complex for words of length $\leq n$.

Proposition 2.13 The Todd-Coxeter 2-complex $\mathrm{TC}_{n}$ is equal to the folded loop 2-complex $\mathrm{f} \Lambda_{\lambda(n)}$, and contains the folded van Kampen 2-complex $\mathrm{fK}_{n}$ as a subcomplex.

The minimum Todd-Coxeter radius function $\rho_{\mathrm{TC}}($.$) , the minimum folded isodiametric func-$ tion $d($.$) , the minimum folded loop-complex function \lambda($.$) , and the minimum isodiametric func-$ tion $D($.$) , are the same.$

Proof. (1) $\mathrm{TC}_{n}$ can be "pulled apart" into loops with labels $r^{x}$, with $r \in R,|x| \leq \rho_{\mathrm{TC}}(n)$ (one loop per face of the complex $\mathrm{TC}_{n}$ ). More precisely, the process of pulling a complex apart into loops goes as follows:

- For each face $f$ in the complex, choose a path $p_{f}$ of length $\leq \rho_{\text {TC }}(n)$ from the face to the origin of $\mathrm{TC}_{n}$. Let $x$ be the label of $p_{f}$ and let $r \in R$ be the label of the contour of $f$.
- Create a new origin for the loop complex to be constructed.
- Repeat the following, for each face $f$ of $\mathrm{TC}_{n}$ until all faces have been removed from $\mathrm{TC}_{n}$ :
- Create a new path with label $x$, attached at the new origin;
at the other end of this path, attach a face with contour label $r$
(so, viewed from the new origin, this path-and-loop has label $r^{x}$ ).
- Remove the face $f$ from its place in the TC complex.

The process of pulling $\mathrm{TC}_{n}$ apart can be viewed as the inverse of the folding process; it is reversible at each step. Therefore, if these loops are folded up again, we recover $\mathrm{TC}_{n}$. So we have:

$$
D(n) \leq \rho_{\mathrm{TC}}(n)
$$

where $D$ is the minimum isodiametric function.
(2) On the other hand, suppose we take all possible loops with label $r^{x}$, for every $r \in R$ and every reduced word $x$ of length $\leq \rho_{\mathrm{TC}}(n)$, and attach these loops to $\mathrm{TC}_{n}$ at the origin, and fold. We claim that the 2-complex obtained is again $\mathrm{TC}_{n}$. Indeed, in each added loop the path labeled by $x$ has length $\leq \rho_{\mathrm{TC}}(n)$. By the minimality of the Todd-Coxeter radius function $\rho_{\mathrm{TC}}($.$) , the process \mathrm{TC}$ glues on all relators within radius $\rho_{\mathrm{TC}}(n)$ anyway; hence, all $r^{x}$ (with $x$ reduced) occur already in $\mathrm{TC}_{n}$.

So, $\mathrm{TC}_{n}$ can be built by taking the folded loop-complex of radius $\rho_{\mathrm{TC}}(n)$. Since both $\rho_{\mathrm{TC}}($. and $\lambda($.$) are minimal, we conclude that \mathrm{TC}_{n}=\mathrm{f} \Lambda_{\lambda(n)}$, and $\rho_{\mathrm{TC}}(n)=\lambda(n)$.
(3) Since $\mathrm{TC}_{n}$ can be obtained by folding loops with labels $r^{x}$ with $|x| \leq \rho_{\mathrm{TC}}(n)$ (as seen at the beginning of the proof) we conclude that

$$
\rho_{\mathrm{TC}}(n) \leq D(n)
$$

Indeed, by Lemma 2.4, $D(n)$ is the radius of an unfolded loop-complex which can be used to decide the word problem for all words of length $\leq n$. Hence, the process $T C$ will decide the word problem after reaching radius $D(n)$.
(4) $\quad \mathrm{vK}_{n}$ can be pulled apart into loops with labels $r^{x}$, with $r \in R,|x| \leq d(n)$. This process of pulling $\mathrm{vK}_{n}$ apart is reversible at each step; therefore, if these loops are folded up again, we recover $\mathrm{vK}_{n}$.

Since $\mathrm{f} \Lambda_{\lambda(n)}$ has minimum radius, we conclude that $\lambda(n) \leq d(n)$.
(5) At the same time, each folded van Kampen diagram of a word of length $\leq n$ can be obtained by folding loops with labels $r^{x}$. If in a folded van Kampen diagram with minimal diameter, more loops are attached (at the origin) and folded in, this does not shrink the diameter (since the diameter is already minimum); hence all minimum-diameter folded van Kampen diagrams of words of length $\leq n$, as well as the folded van Kampen complex $\mathrm{fK}_{n}$ are subcomplexes of $\mathrm{f} \Lambda_{\lambda(n)}$. Hence we also have $d(n) \leq \lambda(n)$ (since subcomplexes of $\mathrm{f} \Lambda_{\lambda(n)}$ cannot have a larger radius than $\left.\mathrm{f} \Lambda_{\lambda(n)}\right)$.

Hence, combining this with (4) we obtain, $\lambda(n)=d(n)$.
(6) We saw in (1) that $D \leq \rho_{\mathrm{TC}}$, we saw in (2) that $\rho_{\mathrm{TC}}=\lambda$, and we saw in (3) that $\rho_{\mathrm{TC}} \leq D$. Hence, $D=\rho_{\text {тС }}=\lambda$. We saw in (5) that $\lambda=d$.

Since the four functions $\rho_{\mathrm{TC}}(),. \lambda(),. d(),. D($.$) are the same, we will use d($.$) to denote all$ of them. The fact that $d($.$) and D($.$) are the same appears implicitly in the literature (e.g., in$ [25] the definition of the folded isodiametric function is used for the "isodiametric function", without mention that this is not the usual definition).

Theorem 2.14 For any finite presentation, the minimum filling length function $F($.$) and the$ minimum folded filling length function $f($.$) , are the same.$

Proof. Recall the characterization of the filling length in terms of a rewriting system (see the definition of filling length and the subsequent characterizations). The same rewriting characterization applies to the folded filling length, based on the folded van Kampen diagram $\mathrm{fK}_{n}$. Both the minimum filling length and the minimum folded filling length for a word $w$ are equal to the length of the longest intermediate word derived in the rewrite process from $w$ to 1 . Hence the two functions $F$ and $f$ are equal.

Remark: Minimum-area van Kampen diagrams may have much larger area than their folded version. For example, consider a finite presentation $\langle A ;\{r\}\rangle$ where $r$ is cyclically reduced (relative to $F G(A)$ ) and $|r|>0$, and consider the word $w=r^{n}$ (for any $n>1$ ). Then the van Kampen diagram of $w$, consisting of $n$ positive (counter-clockwise) loops labeled by $r$, attached at the origin, has area $\geq n$. But the folded van Kampen diagram has only one face.

Theorem 2.15 Let $P_{\min }, f_{\min }, d_{\min }$ be the minimum isoperimetric function, respectively the minimum filling length function, respectively the minimum isodiametric function of a finite presentation $\langle A ; R\rangle$. Let $p_{\min }$ be the minimum folded isoperimetric function. These filling functions are related as follows (where $c>1$ is a constant that depends on the presentation; the constant may be different in different parts of the Theorem).

$$
\begin{array}{lr}
d_{\min }(n) \leq \frac{1}{2} f_{\min }(n) \leq c \cdot\left(P_{\min }(n)+n\right), \quad \text { and } \quad p_{\min }(n) \leq P_{\min }(n) . \\
P_{\min }(n) \leq n c^{c_{\min }(n)} & \quad \text { (Cohen's double exponential inequality) } \\
P_{\min }(n) \leq c^{f_{\min }(n)} & \text { (Gromov, Gersten) } \\
f_{\min }(n) \leq c^{d_{\min }(n)+n} & \text { (Gersten, Riley) } \\
p_{\min }(n) \leq c^{d_{\min }(n)} . & \tag{5}
\end{array}
$$

Proof (or references): For the proofs of the first two inequalities in (1) see [15]. The inequality $p \leq P$ is obvious. The double exponential inequality (2) is due to Daniel Cohen [8]; Steve Gersten [13] gave another proof, and Papasoglu [25] adapted Gersten's proof to more general 2-complexes. We will give another proof of the double exponential inequality in a later section. (3) is due to Gromov and Gersten ( 18$]$, pp. 100-101). (4) was proved by S.M. Gersten and T. Riley (Thm. 3 in [15]).
(5) The folded isoperimetric function $p(\cdot)$ is bounded by the number of faces in a folded van Kampen diagram with minimum diameter $d(n)$. Since every vertex in a folded van Kampen diagram has degree $\leq 2|A|$, it follows that a folded van Kampen diagram of diameter $d(n)$ has at most $a^{d(n)}$ edges (for some constant $a>1$ depending on $|A|$ ). Every face has a boundary of length $\leq m$, where $m=\max \{|r|: r \in R\}$ (so, $m$ depends on the presentation but not on $n$ ). Therefore the folded van Kampen diagram has $<\left(a^{d(n)}\right)^{m}$ different boundary edge-cycles, hence it has $<c^{d(n)}$ faces (for a constant $c$ ).

As a consequence of (4), (2) and (3) in the above Theorem we have the following break-up of the double exponential inequality (2) into two steps, when $d_{\min }(n) \geq a n$ (for a constant $a>0): \quad P_{\min }(n) \leq c^{f_{\min }(n)} \leq c^{C^{d_{\min }(n)}}$

In (15) Gersten and Riley use a slightly weaker form of the double exponential inequality, namely, $\quad P_{\min }(n) \leq c^{c^{d_{\min }(n)+n}}$. By using (2) above, we can improve (4) and the break-up of (2):

$$
f_{\min }(n) \leq c^{d_{\min }(n)}+d_{\min }(n) \log n
$$

When $d_{\text {min }}(n) \geq \log _{a} n$ (for a constant $a>1$ ):

$$
P_{\min }(n) \leq c^{f_{\min }(n)} \leq c^{C^{d_{\min }(n)}}
$$

Question: How are the minimum filling length function $f_{\min }$ and the minimum folded isoperimetric function $p_{\min }$ related? Do we have $f_{\min }(n) \leq c \cdot\left(p_{\min }(n)+n\right)$ (for some constant $c>0)$ ?

See [15], [16], and [21] for recent applications of isodiametric and other functions.
Earlier we discussed the folded van Kampen diagrams, and we used them to define the folded isoperimetric, isodiametric, and filling length functions (the latter two were later shown to be equal to their unfolded counterparts). We can define a further contraction of van Kampen diagrams by mapping van Kampen diagrams into the Cayley 2-complex; let's call the image of such a mapping of a van Kampen diagram the "Cayley image of the van Kampen diagram". We can then define new functions: The Cayley isoperimetric function (an upper bound on the number of faces in the Cayley image of van Kampen diagrams for words of length $n$ ), the Cayley isodiametric function (an upper bound on the diameter of the Cayley image), and the Cayley filling length function (an upper bound on the length of the homotopy loop within the Cayley complex, as word of length $n$ is contracted to a point).

This kind of mapping of van Kampen diagrams is different than folding; in the folding process we identify vertices (of the van Kampen diagram) that are equivalent in the free group; in the Cayley map, we identify vertices (of the van Kampen diagram) that are equivalent modulo the group $G$ under consideration.

## 3 Linear compression of the isoperimetric function

Computational complexity is usually studied up to big-O because of the linear speed-up theorem and the linear space compression theorem (see [20] for a reference). The filling functions are algebraic invariants up to big-O too; in addition, below we give an analogue of the linear speedup and compression theorems for the isoperimetric function. It is not clear whether such a compression is possible for the isodiametric function and the filling length function.

Theorem 3.1 Let $G$ be a group that has a finite presentation with isoperimetric function $\leq P(\cdot)$. Then $G$ also has a finite presentation with respect to which the isoperimetric function is $\leq P(n) / 2+n / 2$.

Proof. Let $\langle A ; R\rangle$ be a finite presentation of $G$ with respect to which the isoperimetric function is $\leq P(\cdot)$. Let $m=\max \{|r|: r \in R\}$ (length of the longest relator in $R$ ).

A new presentation of $G$ is obtained as follows. First, we symmetrize $R$, i.e., for each $r \in R$, we add $r^{-1}$ and all cyclic permutations of $r$ and of $r^{-1}$ as relators. Let $\left\langle A ; R_{s}\right\rangle$ be the symmetrized presentation obtained; this is still a presentation of the group $G$, with the same number $m$ defined above. Second, for any $i$-tuple of relators $\left(r_{1}, \ldots, r_{i}\right) \in\left(R_{s}\right)^{i}$ (with
$2 \leq i \leq m$ ), we introduce the new relator red $\left(r_{1} \ldots r_{i}\right)$. In terms of van Kampen diagrams this means that we glue $r_{1}, \ldots, r_{i}$ together along a part of their boundaries, starting at the origins of the relators. We call the set of newly created relators $R_{2}$. Obviously, $\left\langle A ; R_{s} \cup R_{2}\right\rangle$ is a finite presentation of $G$.

We claim that the isoperimetric function of $\left\langle A ; R_{s} \cup R_{2}\right\rangle$ is $\leq P(n) / 2+n / 2$.
For a word $w \in\left(A^{ \pm 1}\right)^{*}$ with $n=|w|$, if $w={ }_{G} 1$ then there is a van Kampen diagram $K$ (over the original presentation $\langle A ; R\rangle$ ) of area $\leq P(n)$. Let $K^{*}$ be the dual graph of $K$, and let $T^{*}$ be a spanning tree (a.k.a. maximal subtree) of $K^{*}$, whose root is chosen to be the outer (unbounded) face. Let us now remove the root of $T^{*}$; this yields a forest $F^{*}$, with $\leq P(n)$ vertices. For each member tree of $F^{*}$ we choose the child of the root of $T^{*}$ as the root. Since the root of $T^{*}$ has degree $n$ in $K^{*}$, there are $\leq n$ member trees in the forest $F^{*}$.

We will now use $F^{*}$ to transform the van Kampen diagram $K$ (over the presentation $\langle A ; R\rangle$ ) into a van Kampen diagram of area $\leq P(n) / 2+n / 2$ over the new presentation $\left\langle A ; R_{s} \cup R_{2}\right\rangle$. The main observation is that each vertex in $F^{*}$ has degree $\leq m$, and each tree root of $F^{*}$ has degree $\leq m-1$ in $F^{*}$.

1. Let $n_{1}(\leq n)$ be the the number of member trees in the forest $F^{*}$ that consist on only one vertex. We leave that part of $F^{*}$ alone.
2. For each member tree of $F^{*}$ that has at least two vertices we do the following. We consider a maximal set $S$ of sibling leaves at maximum depth (siblings are vertices with the same parent). We fuse all the siblings in $S$ and their parent, thus forming a new vertex. In the van Kampen diagram $K$, this corresponds to fusing $\leq m$ neighboring faces into one new face over the new presentation $\left\langle A ; R_{s} \cup R_{2}\right\rangle$. From now on we ignore this new vertex (remove it from the picture).
3. We repeat step 2 as often as possible. When we reach the root of a member tree of $F^{*}$, either it still has children (which are leaves now); then we fuse the root with these children into a new vertex. Or all the children were already removed. In the latter case, we fuse the root with any one of the new vertices that a child belongs to. Since every tree root in $F^{*}$ has degree $\leq m-1$, this creates a new vertex of $\leq m$ old vertices.

As a result, we obtain a van Kampen diagram over $\left\langle A ; R_{s} \cup R_{2}\right\rangle$ with the following upper bound on the number of vertices:

```
\leqn}\quad\mathrm{ (for the }\mp@subsup{n}{1}{}\mathrm{ one-vertex member trees of the forest F}\mp@subsup{F}{}{*}\mathrm{ )
+(P(n)-\mp@subsup{n}{1}{})/2 (for the multi-vertex trees of the forest F}\mp@subsup{F}{}{*}\mathrm{ , in which each vertex was
    fused with at least one other vertex; and at least one vertex was
    fused with more than one other vertex)
\leqn_/2+P(n)/2 \leq n/2 + P(n)/2.
```


## 4 A proof of the double exponential inequality, based on context-free languages

The double exponential inequality gives an upper bound on the minimum isoperimetric function $P(\cdot)$ in terms of the minimum isodiametric function $d(\cdot)$. It is surprising that $d$ should provide any bound at all on $P$.

There are many similarities between combinatorial group theory and the "low-complexity" theory of computation (see for example, [1], [2], [3], [4], [5], [6], [7], [9], [10], 23], [27], [28], [30], [31]). An interesting consequence of the following proof is that, from the point of view of the theory of computation, the double exponential inequality belongs into the theory of context-free languages.

Theorem 4.1 If $P(\cdot)$ is the minimum isoperimetric function of a finite presentation $\langle A ; R\rangle$ and $d(\cdot)$ is the minimum isodiametric function of that presentation then we have for all $n$ :

$$
P(n) \leq n 2^{C c^{d(n)}}
$$

where $C=2(2|A|+1)\|R\|^{2}$ and $c=(2|A|)^{2}$.
Proof. We will use Lemma 2.6. We fix a word $w \in\left(A^{ \pm 1}\right)^{*}$, of length $|w|=n>0$, and we assume $w=_{\langle A ; R\rangle} 1$. We consider the reduced word $\operatorname{red}(w)=v_{1} v_{2} \ldots v_{m}$ with $|\operatorname{red}(w)|=m \leq n$.
(1) The language $L\left(\operatorname{tree} \Lambda_{d(n)}\right)$ is of course a regular language, accepted by the NFA tree $\Lambda_{d(n)}$, which has $\leq\|R\|(2|A|)^{d(n)}$ states, as we saw just before Lemma 2.6. Notation: The set of next states of $\operatorname{tree} \Lambda_{d(n)}$, reached from state $q$ under input letter $a$, will be denoted by $q \cdot a$. The accept (and start) state of tree $\Lambda_{d(n)}$ will be denoted by $\mathbf{1}_{\Lambda}$.

It is well known that $[w]_{\mathrm{FG}}$ is a context-free language, accepted by a push-down automaton (a "pda") with $O(n)$ states (see 20, 19] for background on Dyck languages, and on context-free languages in general).

Here is a more detailed description of this pda, $\Pi_{w}$. The state set is $Q=\left\{f, s_{0}, s_{1}, \ldots, s_{m}\right\}$, where $s_{m}$ is the start state and $f$ is the accept state. Recall that $m=|\operatorname{red}(w)| \leq n$. The stack alphabet is $\Gamma=A^{ \pm 1} \cup\{\mathbf{z}\}$, where $\mathbf{z}$ is the bottom marker of the stack and is also the initial content of the stack. The input alphabet is $\Sigma=A^{ \pm 1}$. As before, we will denote the empty word by 1 (in 20 it is denoted by $\varepsilon$ ). The transition relation $\delta: Q \times(\Sigma \cup\{1\}) \times \Gamma \rightarrow Q \times \Gamma^{*}$ is defined as follows:
Phase 1: Pop the top letter off the stack if the next input letter is the inverse of the top of the stack; otherwise, push the input letter on top of the stack.

$$
\begin{aligned}
& \delta\left(s_{m}, a, a^{-1}\right)=\left(s_{m}, 1\right), \text { for all } a \in \Sigma ; \\
& \delta\left(s_{m}, a, b\right)=\left(s_{m}, a b\right), \text { for all } a \in \Sigma, b \in \Gamma \text { with } b \neq a^{-1} .
\end{aligned}
$$

Phase 2: Guess that the input is finished. Now, using "empty-input moves", pop the stack and check that its content is the fixed word $\operatorname{red}(w)$ (with the beginning of the word at the bottom of the stack):

$$
\begin{aligned}
& \delta\left(s_{i}, 1, v_{i}\right)=\left(s_{i-1}, 1\right) \text { for } i=m, \ldots, 1 \\
& \delta\left(s_{0}, 1, \mathbf{z}\right)=(f, 1)
\end{aligned}
$$

(2) The intersection of a regular language and a context-free language is a context-free language; a pda for the intersection can be obtained thanks to a cartesian product construction (see [20] Theorem 6.5, or [19] Theorem 6.4.1). Let $\Pi$ be this pda accepting $[w]_{\mathrm{FG}} \cap L\left(\operatorname{tree} \Lambda_{d(n)}\right)$, obtained by the cartesian product construction.

Let us describe the pda $\Pi$ in more detail. The stack alphabet of $\Pi$ is $\Gamma=A^{ \pm 1} \cup\{\mathbf{z}\}$, and the input alphabet is $\Sigma=A^{ \pm 1}$, as before. For the state set of $\Pi$ we could take the cartesian product of the state set of $\operatorname{tree} \Lambda_{d(n)}$ and the state set of $\Pi_{w}$, but we can leave out the states that will
not occur in any accepting computation. Hence, the states we keep form the set $Q=Q_{1} \cup Q_{2}$ where $Q_{1}=Q_{\Lambda} \times\left\{s_{m}\right\}$ and $Q_{2}=\left\{\mathbf{1}_{\Lambda}\right\} \times\left\{s_{m-1}, \ldots, s_{1}, s_{0}, f\right\}$.

The transitions of $\Pi$ form two groups (as in the case of $\Pi_{w}$ ), which we call phase 1 and phase 2. The first subset $Q_{1}$ corresponds to phase 1, and has $\left|Q_{1}\right|=\left|Q_{\Lambda}\right| \leq\|R\| \cdot(2|A|)^{d(n)}$ states. In phase 1 the transitions are

$$
\begin{aligned}
& \delta\left(\left(q, s_{m}\right), a, a^{-1}\right)=\left\{\left(\left(p, s_{m}\right), 1\right): p \in q \cdot a\right\}, \quad \text { for } a \in \Sigma, q \in Q_{\Lambda} \\
& \delta\left(\left(q, s_{m}\right), a, b\right)=\left\{\left(\left(p, s_{m}\right), a b\right): p \in q \cdot a\right\}, \quad \text { for } b \in \Gamma, a \in \Sigma, q \in Q_{\Lambda}, \text { with } b \neq a^{-1} .
\end{aligned}
$$

The second subset $Q_{2}$ is used in phase 2, and has $m+1$ states. In phase 2 the transitions are

$$
\begin{aligned}
& \delta\left(\left(\mathbf{1}_{\Lambda}, s_{i}\right), 1, v_{i}\right)=\left(\left(\mathbf{1}_{\Lambda}, s_{i-1}\right), 1\right), \quad i=m, \ldots, 1 \\
& \delta\left(\left(\mathbf{1}_{\Lambda}, s_{0}\right), 1, \mathbf{z}\right)=\left(\left(\mathbf{1}_{\Lambda}, f\right), 1\right)
\end{aligned}
$$

It is important to note that the $m+1$ states in $Q_{2}$ appear only in pop moves.
The start state is $\left(\mathbf{1}_{\Lambda}, s_{m}\right)$ and the accept state is $\left(\mathbf{1}_{\Lambda}, f\right)$. When the pda $\Pi$ reaches its accept state its stack will always become empty; so, $\Pi$ "accepts by empty stack".
(3) Next, from the pda $\Pi$ (accepting by empty stack) we construct a context-free grammar that generates the language $[w]_{\mathrm{FG}} \cap L\left(\right.$ tree $\left.\Lambda_{d(n)}\right)$, thanks to a construction of Chomsky, Evey, and Schützenberger (see [20] Section 5.3, or [19] Theorem 5.4.3).

The set of non-terminals corresponding to phase 1 are $S$ (the start symbol of the grammar), and all symbols of the form $[p, c, q] \in Q \times \Gamma \times Q$. The rules of phase 1 are of the form

$$
\begin{aligned}
& S \rightarrow\left[\left(\mathbf{1}_{\Lambda}, s_{m}\right), \mathbf{z},\left(\mathbf{1}_{\Lambda}, f\right)\right] ; \\
& {\left[\left(q, s_{m}\right), a,\left(p, s_{m}\right)\right] \rightarrow a^{-1}, \text { for any }\left(q, s_{m}\right) \in Q_{1}, \text { and } p \in q \cdot a ;} \\
& {\left[\left(q, s_{m}\right), b, r_{2}\right] \rightarrow a\left[\left(p, s_{m}\right), a, r_{1}\right]\left[r_{1}, b, r_{2}\right],} \\
& \quad \text { for any } r_{1}, r_{2} \in Q \text { and }\left(q, s_{m}\right),\left(p, s_{m}\right) \in Q_{1} \text { with }\left(\left(p, s_{m}\right), a b\right) \in \delta\left(\left(q, s_{m}\right), a, b\right) .
\end{aligned}
$$

The set of non-terminals corresponding to phase 2 is

$$
\left\{\left[\left(\mathbf{1}_{\Lambda}, s_{0}\right), \mathbf{z},\left(\mathbf{1}_{\Lambda}, f\right)\right]\right\} \cup\left\{\left[\left(\mathbf{1}_{\Lambda}, s_{i}\right), v_{i},\left(\mathbf{1}_{\Lambda}, s_{i-1}\right)\right]: i=1, \ldots, m\right\}
$$

These non-terminals belong to $Q_{2} \times \Gamma \times Q_{2}$, except for $\left[\left(\mathbf{1}_{\Lambda}, s_{m}\right), v_{m},\left(\mathbf{1}_{\Lambda}, s_{m-1}\right)\right]$, which belongs to $Q_{1} \times \Gamma \times Q_{2}$. The only rules that have these non-terminals on the left-side are "empty-word rules"

$$
\begin{aligned}
& {\left[\left(\mathbf{1}_{\Lambda}, s_{i}\right), v_{i},\left(\mathbf{1}_{\Lambda}, s_{i-1}\right)\right] \rightarrow 1, \text { for } i=1, \ldots, m} \\
& {\left[\left(\mathbf{1}_{\Lambda}, s_{0}\right), \mathbf{z},\left(\mathbf{1}_{\Lambda}, f\right)\right] \rightarrow 1}
\end{aligned}
$$

(4) We can simplify our grammar. First, we drop the non-terminals of phase 2 altogether, since they only generate the empty word; we directly replace them by the empty word wherever they occur in the grammar. Thus, we assume from now on that our grammar contains no non-terminals in $Q_{2} \times \Gamma \times Q_{2}$.

We can also drop $S$ and use $\left[\left(\mathbf{1}_{\Lambda}, s_{m}\right), \mathbf{z},\left(\mathbf{1}_{\Lambda}, f\right)\right]\left(\in Q_{1} \times \Gamma \times Q_{2}\right)$ asthe start symbol.
We can discard all non-terminals in $Q_{2} \times \Gamma \times Q_{1}$ because such non-terminals do not occur on the left side of any rule of the grammar. As a consequence, in every rule of the form
$\left[\left(q, s_{m}\right), b, r_{2}\right] \rightarrow a\left[\left(p, s_{m}\right), a, r_{1}\right]\left[r_{1}, b, r_{2}\right]$
we now have $r_{1} \in Q_{1}$. Hence, non-terminals in $Q_{1} \times \Gamma \times Q_{1}$ generate only non-terminals that are also in $Q_{1} \times \Gamma \times Q_{1}$. On the other hand, the words generated, in one step, by non-terminals in $Q_{1} \times \Gamma \times Q_{2}$ are in $\Sigma\left(Q_{1} \times \Gamma \times Q_{1}\right)\left(Q_{1} \times \Gamma \times Q_{2}\right)$. The main consequence of this is:

Fact: In a parse tree of a word in $\Sigma^{*}$, only the right-most path can contain non-terminals in $Q_{1} \times \Gamma \times Q_{2}$. All other non-terminals in the parse tree belong to $Q_{1} \times \Gamma \times Q_{1}$.

In the following we will need bounds on the number of non-terminals (recall that $\left|Q_{1}\right|=\left|Q_{\Lambda}\right| \leq$ $\|R\|(2|A|)^{d(n)}, \quad\left|Q_{2}\right|=m+1 \leq n+1$, and $\left.|\Gamma|=2|A|+1\right)$ :
$\left|Q_{1} \times \Gamma \times Q_{1}\right| \leq(2|A|+1)\|R\|^{2}(2|A|)^{2 d(n)}=C_{1} c^{d(n)}$,
$\left|Q_{1} \times \Gamma \times Q_{2}\right| \leq(n+1)(2|A|+1)\|R\|(2|A|)^{d(n)}\left(<n C_{1} c^{d(n)}\right)$,
where $C_{1}=(2|A|+1)\|R\|^{2}$, and $c=(2|A|)^{2}$.
(5) We now use the Pumping Lemma (due to Bar-Hillel, Perles, Shamir, see [20] Section 6.1, or [19] Theorem 6.2.1 and Corollary) which, among other things, states the following: If a language $L \neq \emptyset$ has a context-free grammar with $\nu$ non-terminals then $L$ contains a word of length $\leq \ell^{\nu}$, where $\ell$ is the maximum length of the right side of any rule. In our grammar, $\ell=3$ and $\nu \leq n c^{d(n)}$ for some constant $c>1$. Thus, we immediately get an upper bound $3^{n c^{d(n)}}$ on the length of the shortest word. However, we can obtain a smaller upper bound if we use the above Fact in our analysis of parse trees.

Recall that the Pumping Lemma is proved by looking at recurrences of non-terminals on any path of the parse tree. A shortest word in the language will have a parse tree with no recurrent non-terminals on any path from the root. Hence, the right-most path of the parse tree has length

$$
\begin{aligned}
& \leq\left|Q_{1} \times \Gamma \times Q_{2} \quad \cup Q_{1} \times \Gamma \times Q_{1}\right| \\
& \leq(n+1)(2|A|+1)\|R\|(2|A|)^{d(n)}+(2|A|+1)\|R\|^{2}(2|A|)^{2 d(n)} \\
& \leq n C_{1} c^{d(n)}, \quad \text { with } c=(2|A|)^{2} \text { and } C_{1}=(2|A|+1)\|R\|^{2} .
\end{aligned}
$$

By the Fact above, elsewhere the non-terminals in the parse tree that are not on the right-most path belong to $\in Q_{1} \times \Gamma \times Q_{2}$. In other words, every subtree of the parse tree, hanging at a vertex of the right-most path, has only non-terminals in $Q_{1} \times \Gamma \times Q_{1}$. Since there are no recurrent non-terminals, each one of these subtrees has depth

$$
\leq\left|Q_{1} \times \Gamma \times Q_{1}\right| \leq(2|A|+1)\|R\|^{2}(2|A|)^{2 d(n)}=C_{1} c^{d(n)}
$$

Hence, since each non-terminal has at most 2 non-terminal successors, the number of nonterminals in such a subtree is

$$
\leq 2^{C_{1} c^{d(n)}}
$$

Since the number of these subtrees is equal to the length of the right-most path, we find that the total number of non-terminals in the parse tree is

$$
\left.\leq n C_{1} c^{d(n)} \cdot 2^{C_{1} c^{d(n)}} \leq n 2^{2 C_{1} c^{d(n)}} \quad \text { (using the fact that } C_{1} c^{d(n)}<2^{C_{1} c^{d(n)}}\right)
$$

Every non-terminal vertex has at most one terminal descendant in the parse tree, so the above upper bound also bounds the length of the shortest word in the language. Thus we have proved that for all words $w$ of length $|w| \leq n$ :
$w={ }_{\langle A ; R\rangle} 1$ iff $[w]_{\mathrm{FG}} \cap L\left(\operatorname{tree} \Lambda_{d(n)}\right)$ contains a word of length $\leq n 2^{C c^{d(n)}}$, where $C=2 C_{1}$. Now the Theorem follows from Lemma 2.6.

## 5 Computational complexity

We know from [6] and [27] that if $P(\cdot)$ is an isoperimetric function for a finite presentation $\langle A ; R\rangle$, then the word problem of $\langle A ; R\rangle$ is in NTime $(P)$.

Proposition 5.1 Let $\langle A ; R\rangle$ be a finite presentation with isodiametric function $\leq d(\cdot)$. Then the word problem of $\langle A ; R\rangle$ is in $\operatorname{DTime}\left(c^{d(\cdot)}\right)$, where $c>1$ is a constant depending on the presentation.

Proof: Given a word $w$ of length $n$, we can decide whether $w{ }_{\langle A ; R\rangle} 1$ as follows. Construct the NFA $\Lambda_{d(n)}$; this can be done deterministically in time $\leq c^{d(n)}$ (for some constant $c>1$ ). Next, we fold this automaton as in Lemma 2.7; this can be done deterministically in time $\leq C^{d(n)}$ (for some constant $C>1$ ). Finally, check whether the folded DFA accepts red $(w)$.

Tim Riley [26] observed that if a finite presentation has a filling length function $f$ then the word problem of that presentation has nondeterministic space complexity $O(f)$.

The following proposition strengthens this fact, by using symmetric Turing machines. Those are nondeterministic Turing machines whose transition relation is symmetric (i.e., the reverse of any transition of the machine is also a transition of that machine); see [22] and [6]. One can define space complexity in relation to such machines: $\operatorname{SymSpace}(S)$ is the set of all languages accepted by symmetric Turing machines with space $\leq S(\cdot)$. For time-complexity it is known that $\operatorname{SymTime}(T)=\operatorname{NTime}(T)$ (proved by Lewis and Papadimitriou [22]). For space, DSpace $(S) \subseteq$ SymSpace $(S) \subseteq \operatorname{NSpace}(S)$; there are reasons to suspect that DSpace $(S) \neq \operatorname{SymSpace}(S) \neq$ NSpace $(S)$, but this remains an open problem.

Proposition 5.2 Let $\langle A ; R\rangle$ be a finite presentation of a group with filling length function $\leq f(\cdot)$. Then the word problem of $G$ is in SymSpace $(f(\cdot))$.

Proof: We use the rewriting system characterization of the filling length function. Note that this rewrite system (described after the definition of isoperimetric functions) is symmetric. A symmetric Turing machine can simulate this rewrite system. Since the longest words that occur in the rewrite process have length $\leq f(n)$, the space needed by the Turing machine is also $\leq f(n)$.

The relations between filling functions on groups and the complexity of the word problem of groups are summarized below. Following the standard notation for complexity classes, we introduce classes of finite presentations of groups, based on their filling functions.

Definition 5.3 Consider any function $h: \mathbb{N} \rightarrow \mathbb{N}$. We define $\operatorname{Isoper}(h)$ to be the set of all groups that have finite presentations $\langle A ; R\rangle$ whose minimum isoperimetric function $P_{\langle A ; R\rangle}$ satisfies

$$
P_{\langle A ; R\rangle}(n) \leq c_{1} h\left(c_{2} n\right), \quad \text { for all } n \geq c_{3}
$$

Here, $c_{1}, c_{2}, c_{3}$ are positive constants, depending on $\langle A ; R\rangle$, but not on $n$. We say, "the minimum isoperimetric function is $\leq h$ up to big- $O$ ".

In a similar way we define the sets of finite presentations $\operatorname{Isodiam}(h)$ for the isodiametric function, Filllen( $h$ ) for the filling length function, and $\mathbf{F I s o p e r}(h)$ for the folded isoperimetric function.

By NTime $(q)$ we denote all languages accepted by nondeterministic Turing machines with time complexity $O(q(O(n)))$. More precisely, for an accepted input of length $\leq n$ the Turing machine has at least one accepting computation whose time is $\leq c_{1} q\left(c_{2} n\right)$, for all $n \geq c_{3}$; here, $c_{1}, c_{2}, c_{3}$ are positive constants, depending on the Turing machine.

In a similar way we define DTime $(q)$ and $\operatorname{SymSpace}(q)$.
An inclusion between a class of groups and a class of languages (for example, $\operatorname{Isoper}(q) \subseteq$ NTime $(q))$, is defined to mean that every group in $\operatorname{Isoper}(q)$ has its word problem in NTime $(q)$.

In this notation, the inequalities in Theorem 2.15 and the inclusions in the above Propositions lead to the following (where $\subset$ and $\cap$ denote non-strict left-to-right or top-to-bottom inclusion).

Theorem 5.4 For any function $q: \mathbb{N} \rightarrow \mathbb{N}$ with $q(n) \geq \log \log n$ we have,

$\operatorname{NTime}(q) \subset \operatorname{SymSpace}(q) \subset \operatorname{DTime}\left(2^{O(q)}\right) \subset \operatorname{SymSpace}\left(2^{O(q)}\right) \subset \operatorname{NTime}\left(2^{2^{O(q)}}\right)$
Moreover, $\quad \operatorname{Isoper}(q) \subset \operatorname{FIsoper}(q), \quad$ and $\quad \operatorname{Isodiam}(q) \subset \operatorname{FIsoper}\left(2^{O(q)}\right)$.
Proof. To prove $\operatorname{Isoper}(q) \subset \operatorname{Filllen}(q)$, observe that if a group has a finite presentation $\langle A ; R\rangle$ with isoperimetric function $\leq q$ then $\langle A ; R\rangle$ has a filling length $\leq q$ too (since the minimum filling length is $\leq$ the minimum isoperimetric function up to big-O, by (1) of Theorem 2.15). Hence, every presentation in $\operatorname{Isoper}(q)$ is also in Filllen $(q)$.

The other inclusions follow from Theorem 2.15 in a similar way.
We do not know whether any of the inclusions in the above theorem are strict. For the complexity classes, this is a well known open problem. Along the lines of [6], (7] and [27] one could make the following conjecture.
Conjecture. A finitely generated group $G$ has its word problem in $\operatorname{SymSpace}(S)$ iff $G$ is embeddable in a finitely presented group $H$ whose filling length function is $O(S)$.

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## References

[1] J. Avenhaus, K. Madlener, "Subrekursive Komplexität bei Gruppen: I. Gruppen mit vorgeschriebener Komplexität", Acta Informatica 9 (1977) 87-104.
[2] J. Avenhaus, K. Madlener, "Subrekursive Komplexität bei Gruppen: II. Der Einbettungssatz von Higman für entscheidbare Gruppen", Acta Informatica 9 (1978) 183-193.
[3] J. Avenhaus, K. Madlener, "The Nielsen reduction and P-complete problems in free groups", Theoretical Computer Science 32 (1984) 61-76.
[4] J. Avenhaus, K. Madlener, "On the complexity of intersection and conjugacy problems in free groups", Theoretical Computer Science 32 (1984) 279-295.
[5] J. Avenhaus, K. Madlener, "An algorithm for the word problem in HNN extensions and dependence of its complexity on the group presentation", RAIRO Informatique Théorique 15 (1981) 355-371.
[6] J.C. Birget, "Time-complexity of the word problem for semigroups and the Higman Embedding Theorem", International J. of Algebra and Computation 8 (1998) 235-294.
[7] J.C. Birget, A. Ol'shanskii, E. Rips, M.V. Sapir, "Isoperimetric functions of groups and computational complexity of the word problem", Annals of Mathematics (accepted). Mathematics ArXiv, math.GR/9811106, http://front.math.ucdavis.edu
[8] D.E. Cohen, "Isodiametric and isoperimetric inequalities for group presentations", International J. of Algebra and Computation 1 (1991) 315-320.
[9] D.E. Cohen, K. Madlener, F. Otto, "Separating the intrinsic complexity and the derivational complexity of the word problem for finitely presented groups", Mathematical Logic Quarterly 39 (1993) 143-157.
[10] D.B.A. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson, W. Thurston, Word Processing in Groups, Jones and Bartlett (1992).
[11] W.J. Floyd, A.H.M. Hoare, R.C. Lyndon, "The word problem for geometrically finite groups", Geometriae Dedicata 20 (1986) 201-207.
[12] M. Garzon, Y. Zalcstein, "The complexity of Grigorchuk groups with application to cryptography", Theoretical Computer Science 88 (1991) 83-98.
[13] S.M. Gersten, "The double exponential theorem for isodiametric and isoperimetric functions", International J. of Algebra and Computation 1 (1991) 321-328.
[14] S.M. Gersten, "Isoperimetric and isodiametric functions", in Geometric Group Theory I. (G. Niblo, M. Roller, eds.), London Mathematical Society Lecture Notes Series 181, Cambridge Univ. Press (1993), pp. 79-96.
[15] S.M. Gersten, T. Riley, "Filling length in finitely presentable groups", Geometriae Dedicata, to appear.
[16] S.M. Gersten, T. Riley, "Filling radii of finitely presented groups", to appear in Quarterly Journal of Mathematics (Oxford).
[17] M. Gromov, "Hyperbolic groups", in Essays in Group Theory (S.M. Gersten, ed.), MSRI Series 8, Springer Verlag (1987).
[18] M. Gromov, "Asymptotic invariants of infinite groups", in Geometric Group Theory (G. Niblo, M. Roller, eds.), London Mathematical Society Lecture Notes Series 182, Cambridge Univ. Press (1993).
[19] M.A. Harrison, Introduction to Formal Language Theory, Addison-Wesley (1978).
[20] J. Hopcroft, J. Ullman, Introduction to Automata, Languages, and Computation, AddisonWesley (1979).
[21] Ilya Kapovich, "A note on the Poénaru condition", to appear in J. Group Theory.
[22] H.R. Lewis, Ch. Papadimitriou, "Symmetric space-bounded computation", Theoretical Computer Science 19 (1982) 161-187.
[23] K. Madlener, F. Otto, "Pseudo-natural algorithms for the word problem for finitely presented monoids and groups", J. of Symbolic Computation 1 (1985) 383-418.
[24] S. Margolis, J. Meakin, Personal communication on the Todd-Coxeter process (1991).
[25] P. Papasoglu, "Isodiametric and isoperimetric inequalities for complexes and groups", J. of the London Mathematical Society (2) 62 (2000) 97-106.
[26] T. Riley, Personal communication (Aug. 2001).
[27] M.V. Sapir, J.C. Birget, E. Rips, "Isoperimetric and isodiametric functions of groups", Annals of Mathematics (to appear). Mathematics ArXiv, math.GR/9811105, http://front.math.ucdavis.edu
[28] H.U. Simon, "Word problems for groups and context-free recognition", in Fundamentals of Computation Theory (ed., L. Budach), Akademie Verlag, Berlin (1979), pp. 417-422.
[29] J. Stallings, A.R. Wolf, "The Todd-Coxeter process, using graphs", in Combinatorial Group Theory and Topology (eds., S.M. Gersten, J. Stallings), Princeton Univ. Press (1987), pp. 157-161.
[30] M.K. Valiev, "On the complexity of the identity problem for finitely defined groups", Algebra i Logika 8 (1969) 5-43 (English translation, 2-21).
[31] S. Waack, "Tape complexity of word problems", in Fundamentals of Computation Theory (ed., F. Gecseg), Springer Lecture Notes in Computer Science 117 (1981) 467-471.

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