

# FOLDINGS, GRAPHS OF GROUPS AND THE MEMBERSHIP PROBLEM

ILYA KAPOVICH, RICHARD WEIDMANN, AND ALEXEI MYASNIKOV

ABSTRACT. We introduce a combinatorial version of Stallings-Bestvina-Feighn-Dunwoody folding sequences. We then show how they are useful in analyzing the solvability of the uniform subgroup membership problem for fundamental groups of graphs of groups. Applications include coherent right-angled Artin groups and coherent solvable groups.

## 1. INTRODUCTION

The idea of using foldings to study group actions on trees was introduced by Stallings in a seminal paper [45], where he applied foldings to investigate free groups. Free groups are exactly those groups that admit free actions on simplicial trees. Later Stallings [46] offered a way to extend these ideas to non-free actions of groups on graphs and trees. Bestvina and Feighn [5] gave a systematic treatment of Stallings' approach in the context of graphs of groups and applied this theory to prove a far-reaching generalization of Dunwoody's accessibility result. Later Dunwoody [20] refined the theory by introducing vertex morphism. Dunwoody [21] used foldings to construct a small unstable action on an  $\mathbb{R}$ -tree. Some other applications of foldings in the graph of groups context can be found in [39, 42, 43, 17, 19, 27, 28, 13, 12, 29].

In this paper we develop a combinatorial treatment of foldings geared towards more computational questions. In particular we are interested in the subgroup membership problem and in computing algorithmically the induced splitting for a subgroup of the fundamental group of a graph of groups. Recall that a finitely generated group

$$G = \langle x_1, \dots, x_k \mid r_1, r_2, \dots \rangle$$

is said to have *solvable membership problem* (or *solvable uniform membership problem*) if there is an algorithm which, for any finite family of words  $u, w_1, \dots, w_n$  in  $\{x_1, \dots, x_k\}^{\pm 1}$  decides whether or not the element of  $G$  represented by  $u$  belongs to the subgroup of  $G$  generated by the elements of  $G$  corresponding to  $w_1, \dots, w_n$  (it is easy to see that this definition does not depend on the choice of a finite generating set for  $G$ ). Similarly, if  $H \leq G$  is a specific subgroup, then  $H$  is said to have *solvable membership problem in  $G$*  if there is an algorithm deciding for any word  $u$  in  $\{x_1, \dots, x_k\}^{\pm 1}$  whether  $u$  represents an element of  $H$ .

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Amalgamated free products, HNN-extensions and more generally, fundamental groups of graphs of groups play a very important role in group theory. However, till now there has been relatively little understanding of how these fundamental constructions affect the subgroup membership problem. One of the first results in this direction is due to Mihailova, who proved [37, 38] that if  $A$  and  $B$  have solvable membership problem then so does their free product  $A * B$  (see also the subsequent work of Boydrón [14]). Mihailova [36] also produced some important counter-examples demonstrating the difficulty of the membership problem. Namely, she proved that the direct product  $G = F(a, b) \times F(x, y)$  of two free groups of rank two contains a finitely generated subgroup  $H$  with unsolvable membership problem in  $G$ . The group  $F(a, b) \times F(x, y)$  can be thought of as a double HNN-extension of  $F(a, b)$ :

$$G = \langle F(a, b), x, y \mid x^{-1}fx = f, y^{-1}fy = f \text{ for any } f \in F(a, b) \rangle.$$

It is well-known that a finitely generated free group has uniform membership problem solvable in quadratic time in terms of  $|u| + |w_1| + \dots + |w_n|$ . Thus even seemingly innocuous free constructions have the potential of greatly affecting the complexity of the membership problem. Another important example which to this date is not at all understood is that of the mapping torus of a free group automorphism.

Namely, let  $G$  be a group and let  $\phi : G \rightarrow G$  be an automorphism of  $G$ . Then the HNN-extension of  $G$  along  $\phi$

$$M_\phi := \langle G, t \mid t^{-1}gt = \phi(g), \text{ for every } g \in G \rangle = G \rtimes_\phi \mathbb{Z}.$$

is called the *mapping torus group* of  $\phi$ .

The case when  $G$  is a free group, or more generally, a surface group, is of particular importance in 3-dimensional topology. Yet, apart from a few obvious observations, nothing is known about the solvability of the membership problem for mapping tori of automorphisms of free groups and surface groups.

A substantial amount of work on the membership problem for amalgamated products and HNN-extension was done by Bezverkhii [7, 8, 9, 10]. However, he did not use the machinery of Bass-Serre theory of graphs of groups and groups acting on trees. Consequently, all of his results have to rely on Britton's lemma and the normal form theorem for amalgamated products, which makes his proofs extremely technical and statements of most results quite special.

Our goal is to present a more geometric and unified approach to this topic which relies on Bass-Serre theory [44, 4] as well as on combinatorial foldings methods. Because of our algorithmic goals, when approximating an induced splitting for a subgroup of the fundamental group of a graph of groups  $\mathbb{A}$ , we need to work primarily at the level of quotient graphs of groups rather than at the level of the Bass-Serre covering trees, as it is done in the Stallings-Bestvina-Feighn-Dunwoody treatment of foldings. We use finite combinatorial objects called  $\mathbb{A}$ -*graphs* (where  $\mathbb{A}$  is a given graph of groups) to provide such approximations. For algorithmic reasons,  $\mathbb{A}$ -graphs are labeled by elements and subgroups of the original vertex groups of  $\mathbb{A}$ , rather than by some abstractly defined groups and their homomorphisms into the original vertex groups.

The full list of conditions that guarantee that foldings of  $\mathbb{A}$ -graphs can be applied algorithmically and terminate yielding the induced splitting of an arbitrary finitely generated subgroup of  $\pi_1(\mathbb{A})$  turns out to be rather cumbersome (see Definition 5.6, Theorem 5.8 and

Theorem 5.13 below). The same is true for the conditions that guarantee that the membership problem is solvable (Theorem 5.13). Instead we formulate a corollary of the main results:

**Theorem 1.1.** *Let  $\mathbb{A}$  be a finite graph of groups such that:*

- (1) *For every vertex  $v$  of  $\mathbb{A}$  the vertex group  $A_v$  is either locally quasiconvex word-hyperbolic or polycyclic-by-finite.*
- (2) *Every edge group of  $\mathbb{A}$  is polycyclic-by-finite.*

*Then for any vertex  $v_0 \in VA$  the uniform membership problem for  $G = \pi_1(\mathbb{A}, v_0)$  is solvable. Moreover there is an algorithm which, given a finite subset  $S \subseteq G$ , constructs the induced splitting and a finite presentation for the subgroup  $U = \langle S \rangle \leq G$ .*

By the *induced splitting* of  $U$  in the theorem above we mean the decomposition of  $U$  as  $U = \pi_1(\mathbb{B}, v_0)$  where  $\mathbb{B}$  is the quotient graph of groups for the action of  $U$  on the minimal  $U$ -invariant subtree of the Bass-Serre covering tree  $X = \widetilde{(\mathbb{A}, v_0)}$  that contains the base-vertex of  $X$ .

The above theorem applies to a wide variety of situations. For example, it is applicable to a finite graph of groups where all vertex groups are virtually abelian or where all vertex groups are virtually free and edge groups are virtually cyclic. In particular, the mapping torus of an automorphism of a free abelian group of finite rank (or in fact of any virtually polycyclic group) falls into this category. While Theorem 1.1 does not say anything about the computational complexity of the algorithm solving the membership problem, we believe that in many specific cases this complexity can be analyzed and estimated explicitly. For example, in the case when all vertex groups are free and edge groups are cyclic, the folding algorithm provided by Theorem 1.1 appears to have polynomial complexity. Indeed, Paul Schupp [40] obtained more precise results with polynomial complexity estimates for multiple HNN-extensions of free groups with cyclic associated subgroups.

Not surprisingly, we also recover (see Corollary 5.15) a generalization of Mihailoiva's theorem regarding the membership problem for free products to graphs of groups with finite edge groups.

As an illustration of the usefulness of Theorem 1.1, we apply it to graph products and right-angled Artin groups. Recall that if  $\Gamma$  is a finite simple graph with a group  $G_v$  associated to each vertex of  $\Gamma$  then the *graph product* group  $G$  is defined as the free product  $*_{v \in V\Gamma} G_v$  modulo the relations  $[G_v, G_u] = 1$  whenever  $u$  and  $v$  are adjacent vertices in  $\Gamma$ . If each  $G_v$  is an infinite cyclic group, then  $G$  is called a *right-angled Artin group* or *graph group* and is denoted by  $G(\Gamma)$ .

**Corollary 1.2.** *Let  $T$  be a finite tree such that for every vertex  $v \in VT$  there is an associated finitely generated virtually abelian group  $G_v$ . Then the graph product group  $G$  has solvable uniform membership problem. Moreover, there is an algorithm which, given a finite subset  $S \subseteq G$ , constructs a finite presentation for the subgroup  $U = \langle S \rangle \leq G$ .*

*Proof.* Note that for any groups  $K, H$  we can write the direct product  $H \times K$  as an amalgam:

$$H \times K = H *_H (H \times K) *_K K.$$

Let  $v_1, \dots, v_n$  be the vertices of  $T$ . Let  $T'$  be the barycentric subdivision of  $T$ . We give  $T'$  the structure of a graph of groups as follows. For each vertex  $v_i$  of  $T$  assign the vertex

group  $T'_{v_i} := G_{v_i}$ . For each barycenter  $v$  of an edge  $[v_i, v_j]$  of  $T$  assign the vertex group  $T'_v := G_{v_i} \times G_{v_j}$ . Also, for  $e_i = [v_i, v] \in ET'$  and  $e_j = [v_j, v] \in ET'$  put  $T'_{e_i} := G_{v_i}$  and  $T'_{e_j} := G_{v_j}$ . Finally, we define the boundary monomorphisms  $T'_{e_i} \rightarrow T'_v$  and  $T'_{e_j} \rightarrow T'_v$  to be the inclusion map  $G_{v_i} \rightarrow G_{v_i} \times G_{v_j}$  and the identity map  $G_{v_i} \rightarrow G_{v_i}$  respectively. This defines a graph of groups  $\mathbb{T}'$  where all vertex groups are finitely generated virtually abelian. Moreover, we have  $G \cong \pi_1(\mathbb{T}', T')$ .

Corollary 1.2 now follows from Theorem 1.1.  $\square$

Theorem 1.1 also applies to many right-angled Artin groups:

**Corollary 1.3.** *Let  $G = G(\Gamma)$  be a coherent right-angled Artin group. Then  $G$  has solvable uniform membership problem. Moreover, there is an algorithm which, given a finite subset  $S \subseteq G$ , constructs a finite presentation for the subgroup  $U = \langle S \rangle \leq G$ .*

*Proof.* Recall that a simple graph is called *chordal* if it does not possess a chord-free simple circuit of length  $\geq 4$ , that is for every simple circuit of length  $\geq 4$  there are two non-neighboring vertices in the circuit which are adjacent in the graph. For example, every tree is a chordal graph. Chordal graphs are of particular importance in the theory of right-angled Artin groups since by a result of Droms [18] a right-angled Artin group  $G(\Gamma)$  is coherent if and only if  $\Gamma$  is chordal.

Let  $G = G(\Gamma)$  be a coherent right-angled Artin group based on a finite graph  $\Gamma$ . Hence  $\Gamma$  is chordal.

By the result of Mihailova about free products mentioned above we may assume that  $\Gamma$  is connected. We will think about the vertices of  $\Gamma$  as the generators of  $G$ .

Recall that a vertex  $v$  of a simple graph is called *simplicial* if any two vertices adjacent to  $v$  are joined by an edge. It is a well-known graph-theoretic fact that every chordal graph has a simplicial vertex (see, for example, Lemma 5.3.16 in [47]).

In order to establish the corollary we need the following:

**Claim.** Let  $\Gamma$  be a finite connected chordal graph. Then there exists a tree of free abelian groups  $\mathbb{T}$  with  $G = G(\Gamma) = \pi_1(\mathbb{T}, T)$  such that for every free Abelian subgroup  $A$  of  $G$  that corresponds to a complete subgraph of  $\Gamma$  there is a vertex group of  $\mathbb{T}$  containing  $A$ .

We will prove the Claim by induction on the number of vertices in  $\Gamma$ . When this number is 1 or 2, the statement is trivial. Suppose  $|\Gamma| = n > 2$  and the Claim has been verified for all graphs with fewer than  $n$  vertices.

Let  $v$  be a simplicial vertex of  $\Gamma$  and let  $\Gamma_0$  be the graph obtained from  $\Gamma$  by removing  $v$  and all edges adjacent to  $v$ . Then  $\Gamma_0$  is a chordal graph defining a right-angled Artin group  $G_0$  that is canonically embedded in  $G$ . Let  $S$  be the set of vertices of  $\Gamma$  adjacent to  $v$ . Since  $v$  is simplicial, the set  $S$  spans a complete subgraph of  $\Gamma$  (and of  $\Gamma_0$ ) and thus defines a free abelian subgroup  $A$  of  $G_0$  and of  $G$ . By the inductive hypothesis we may represent  $G_0$  as  $G_0 = \pi_1(\mathbb{T}_0, T_0)$  where  $\mathbb{T}_0$  is a tree of free abelian groups satisfying the requirements of the Claim for  $\Gamma_0$ . In particular, there is a vertex  $x$  of  $T_0$  with vertex group  $B$  such that  $A \leq B$ .

We now enlarge  $T_0$  to a tree  $T$  by attaching an extra edge  $e$  with origin  $x$ . We define the vertex group for the new vertex  $t(e)$  to be  $A \times \langle v \rangle$  and the edge group of  $e$  to be  $A$  (here  $\langle v \rangle$  is the infinite cyclic group  $G_v$ ). The boundary monomorphisms for  $e$  are defined as the obvious inclusions. This produces a tree of groups  $\mathbb{T}$ . By comparing the presentations for  $G_0 = \pi_1(\mathbb{T}_0, T_0)$  and for  $G$  we see that  $G = \pi_1(\mathbb{T}, T)$ .

Moreover, every complete subgraph of  $\Gamma$  is either contained in  $\Gamma_0$  or it is contained in the complete subgraph in  $\Gamma$  spanned by  $S$  and  $v$ . Hence  $\mathbb{T}$  satisfies the requirements of the Claim for  $\Gamma$ , and the Claim is established.

The statement of Corollary 1.3 now follows from Theorem 1.1.  $\square$

The simplest non-coherent right-angled Artin group is  $F(a, b) \times F(x, y)$ . This group is based on an “empty square”, that is a simple circuit of length four, which is also the simplest example of a non-chordal graph. By Mihailova’s theorem  $F(a, b) \times F(x, y)$  has unsolvable membership problem. Thus the statement of Corollary 1.3 need not hold for non-coherent right-angled Artin groups.

Another easy corollary of Theorem 1.1 is:

**Corollary 1.4.** *Let  $G$  be a finitely generated coherent solvable group. Then  $G$  has solvable uniform membership problem.*

*Proof.* By a result of Groves [24] and Bieri-Strebel [11] if  $G$  is a finitely generated coherent solvable group then either  $G$  is polycyclic or  $G$  is an ascending HNN-extension of a polycyclic group. Hence  $G$  has solvable uniform membership problem by Theorem 1.1.  $\square$

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## 2. GRAPHS OF GROUPS, SUBGROUPS AND INDUCED SPLITTINGS

We refer the reader to the book of Serre [44] as well as to [2, 4, 15, 41] for detailed background information regarding groups acting on trees and Bass-Serre theory.

**Convention 2.1** (Graph of groups notations). Following Serre, we say that a *graph*  $A$  consists of a vertex set  $VA$ , edge set  $EA$ , the inverse-edge function  $^{-1} : EA \rightarrow EA$  and two edge endpoint functions  $t : EA \rightarrow VA$ ,  $o : EA \rightarrow VA$  with the following properties:

- (1) The function  $^{-1}$  is a fixed-point free involution on  $EA$ ;
- (2) For any  $e \in EA$  we have  $o(e) = t(e^{-1})$ .

We call  $e^{-1}$  the *inverse edge* of  $e$ . For  $e \in EA$  we call  $o(e)$  the *initial vertex* of  $e$  and we call  $t(e)$  the *terminal vertex* of  $e$ .

An edge-path in  $A$  is *reduced* if it does not contain a subpath of the form  $e, e^{-1}$ , where  $e \in EA$ .

If  $T$  is a tree and  $v_0, v_1$  are vertices of  $T$ , we will denote by  $[v_0, v_1]_T$  the  *$T$ -geodesic path* from  $v_0$  to  $v_1$ , that is the unique reduced edge-path from  $v_0$  to  $v_1$  in  $T$ .

A *graph-of-groups*  $\mathbb{A}$  consists of an underlying graph  $A$  together with the following data. For each vertex  $v \in VA$  there is an associated *vertex group*  $A_v$  and for each edge  $e \in EA$  there is an associated *edge group*  $A_e$ . Every edge  $e \in EA$  comes equipped with two *boundary monomorphisms*  $\alpha_e : A_e \rightarrow A_{o(e)}$  and  $\omega_e : A_e \rightarrow A_{t(e)}$  for all  $e \in EA$ . If  $e^{-1}$  is the inverse edge of  $e$  then we assume that  $A_{e^{-1}} = A_e$ ,  $\alpha_{e^{-1}} = \omega_e$  and  $\omega_{e^{-1}} = \alpha_e$ .

**Definition 2.2** ( $\mathbb{A}$ -paths). Recall that in Bass-Serre theory if  $\mathbb{A}$  is a graph of groups, then an  $\mathbb{A}$ -*path of length*  $k \geq 0$  from  $v \in VA$  to  $v' \in VA$  is a sequence

$$p = a_0, e_1, a_1, \dots, e_k, a_k$$

where  $k \geq 0$  is an integer,  $e_1, \dots, e_k$  is an edge-path in  $A$  from  $v \in VA$  to  $v' \in VA$ , where  $a_0 \in A_v, a_k \in A_{v'}$  and  $a_i \in A_{t(e_i)} = A_{o(e_{i+1})}$  for  $0 < i < k$ . We will call  $k$  the *length* of  $p$  and denote it by  $|p|$ . Note that we allow  $k = |p|$  to be equal to zero, in which case  $v = v'$  and  $p = a_0 \in A_v$ .

If  $p$  is an  $\mathbb{A}$ -path from  $v$  to  $v'$  and  $q$  is an  $\mathbb{A}$ -path from  $v'$  to  $v''$ , then the *concatenation*  $pq$  of  $p$  and  $q$  is defined in the obvious way and is an  $\mathbb{A}$ -path from  $v$  to  $v''$  of length  $|p| + |q|$ .

The following notion plays an important role in Bass-Serre theory.

**Definition 2.3** (Fundamental group of a graph of groups). Let  $\mathbb{A}$  be a graph of groups. Let  $\sim$  be the equivalence relation on the set of all  $\mathbb{A}$ -paths generated (modulo concatenation) by:

$$a, e, \omega_e(c), e^{-1}, \bar{a} \sim a\alpha_e(c)\bar{a}, \quad \text{where } e \in EA, c \in A_e \text{ and } a, \bar{a} \in A_{o(e)}.$$

If  $p$  is an  $\mathbb{A}$ -path, we will denote the  $\sim$ -equivalence class of  $p$  by  $\bar{p}$ . Note that if  $p \sim p'$  then  $p, p'$  have the same initial vertex and the same terminal vertex in  $VA$ .

Let  $v_0 \in VA$  be a vertex of  $A$ . We define the *fundamental group*  $\pi_1(\mathbb{A}, v_0)$  as the set of  $\sim$ -equivalence classes of  $\mathbb{A}$ -paths from  $v_0$  to  $v_0$ . It can be shown that  $G$  is in fact a group with multiplication corresponding to concatenation of paths.

Suppose that an  $\mathbb{A}$ -path  $p$  has a subsequence of the form  $a, e, \omega_e(c), e^{-1}, \bar{a}$ . Replacing this subsequence in  $p$  by  $a\alpha_e(c)\bar{a}$  produces an  $\mathbb{A}$ -path  $q$ . In this situation we will say that  $q$  is obtained from  $p$  by an *elementary reduction*. Note that  $|q| = |p| - 2$  and that  $p \sim q$ . If no elementary reductions are applicable to  $p$ , we say that  $p$  is  $\mathbb{A}$ -*reduced* (or just *reduced*).

Any  $\mathbb{A}$ -path is equivalent to a reduced  $\mathbb{A}$ -path, and such a reduced  $\mathbb{A}$ -path can be obtained by applying elementary reductions as long as possible. The following proposition implies that the reduced  $\mathbb{A}$ -path obtained in this way is almost unique.

**Proposition 2.4** (Normal Form Theorem). *Let  $\mathbb{A}$  be a graph of groups. Then:*

- (1) *If  $a \in A_v, a \neq 1$  is a nontrivial vertex group element then the length zero path  $a$  from  $v$  to  $v$  is not  $\sim$ -equivalent to the trivial path  $1$  from  $v$  to  $v$ .*
- (2) *Suppose  $p = a_0, e_1, a_1, \dots, e_k, a_k$  is a reduced  $\mathbb{A}$ -path from  $v$  to  $v'$  with  $k > 0$ . Then  $p$  is not  $\sim$ -equivalent to a shorter path from  $v$  to  $v'$ . Moreover, if  $p$  is equivalent to a reduced  $\mathbb{A}$ -path  $p'$  from  $v$  to  $v'$  then  $p'$  has underlying edge-path  $e_1, e_2, \dots, e_k$ .*
- (3) *Suppose  $T$  is a maximal subtree of  $A$  and let  $v_0 \in VA$  be a vertex of  $V$ . Let  $G = \pi_1(\mathbb{A}, v_0)$ . For  $x, y \in VA$  we denote by  $[x, y]_T$  the  $T$ -geodesic edge-path in  $T$ . Then  $G$  is generated by the set  $\bar{S}$  where*

$$S = \bigcup_{e \in EA-ET} [v_0, o(e)]_T e [t(e), v_0]_T \bigcup_{v \in VA} \bigcup_{A_v} [v_0, v]_T A_v [v, v_0]_T$$

We also need to recall the explicit construction of the Bass-Serre universal covering tree for a graph of groups.

**Definition 2.5** (Bass-Serre covering tree). Let  $\mathbb{A}$  be a graph of groups with base-vertex  $v_0 \in VA$ . We define an equivalence relation  $\approx$  on the set of  $\mathbb{A}$ -paths originating at  $v_0$  by saying that  $p \approx p'$  if

- (1)  $p$  and  $p'$  are both  $\mathbb{A}$ -paths from  $v_0$  to  $v$  for some  $v \in VA$  and
- (2)  $p \approx p'a$  for some  $a \in A_v$ .

For a  $\mathbb{A}$ -path  $p$  from  $v_0$  to  $v$ , we shall denote the  $\approx$ -equivalence class of  $p$  by  $\overline{p}A_v$ .

We now define the Bass-Serre tree  $(\widetilde{\mathbb{A}}, v_0)$  as follows. The vertices of  $(\widetilde{\mathbb{A}}, v_0)$  are  $\approx$ -equivalence classes of  $\mathbb{A}$ -paths originating at  $v_0$ . Thus each vertex of  $(\widetilde{\mathbb{A}}, v_0)$  has the form  $\overline{p}A_v$ , where  $p$  is an  $\mathbb{A}$ -path from  $v_0$  to a vertex  $v \in VA$ . (Hence we can in fact choose  $p$  to be already  $\mathbb{A}$ -reduced and such that the last group-element in  $p$  is equal to 1.)

Two vertices  $x, x'$  of  $(\widetilde{\mathbb{A}}, v_0)$  are connected by an edge if and only if we can express  $x, x'$  as  $x = \overline{p}A_v, x' = \overline{pae}A_{v'}$ , where  $p$  is an  $\mathbb{A}$ -path from  $v_0$  to  $v$  and where  $a \in A_v, e \in EA$  with  $o(e) = v, t(e) = v'$ .

It follows from Proposition 2.4 that  $(\widetilde{\mathbb{A}}, v_0)$  is indeed a tree. This tree has a natural base-vertex, namely  $x_0 = \overline{1}A_{v_0}$  corresponding to the  $\approx$ -equivalence class of the trivial path 1 from  $v_0$  to  $v_0$ .

Moreover, the group  $G = \pi_1(\mathbb{A}, v_0)$  has a natural simplicial action on  $(\widetilde{\mathbb{A}}, v_0)$  defined as follows:

If  $g = \overline{q} \in G$  (where  $q$  is an  $\mathbb{A}$ -path from  $v_0$  to  $v_0$ ) and  $u = \overline{p}A_v$  (where  $p$  is an  $\mathbb{A}$ -path from  $v_0$  to  $v \in VA$ ), then  $g \cdot u := \overline{qp}A_v$ . It is not hard to check that the action is well-defined on the set of vertices of  $(\widetilde{\mathbb{A}}, v_0)$  and that it preserves the adjacency relation. Thus  $G$  in fact has a canonical simplicial action without inversions on  $(\widetilde{\mathbb{A}}, v_0)$ .

It follows from Proposition 2.4 that if  $p$  is an  $\mathbb{A}$ -path from  $v_0$  to  $v$  then the map  $A_v \rightarrow G, a \mapsto \overline{pap^{-1}}$  is an embedding. Moreover, in this case the  $G$ -stabilizer of the vertex  $\overline{p}A_v$  of  $(\widetilde{\mathbb{A}}, v_0)$  is equal to the image of the above map, that is to  $\overline{pA_v p^{-1}}$ . Similarly, the  $G$ -stabilizer of an edge in  $(\widetilde{\mathbb{A}}, v_0)$  connecting  $\overline{p}A_v$  to  $\overline{pae}A_{v'}$  is equal to  $\overline{p(a\alpha_e(A_e)a^{-1})p^{-1}}$ .

The following well-known statement is the heart of Bass-Serre theory and provides a duality between group actions on trees and fundamental groups of graphs of groups.

**Proposition 2.6.** *Let  $U$  be a group acting on a simplicial tree  $Y$  without inversions. Then the graph  $B = Y/U$  has a natural graph-of-groups structure  $\mathbb{B}$  such that  $U$  is canonically isomorphic to  $\pi_1(\mathbb{B}, v'_0)$  and  $Y$  is  $U$ -equivariantly isomorphic to the universal covering Bass-Serre tree of  $\mathbb{B}$  (here  $v'_0$  is the image of  $v_0$  in  $B$ ).*

**Remark 2.7.** We want to remind the reader of the explicit construction of  $\mathbb{B}$ . Let  $T_1 \subseteq Y$  and  $T_2 \subseteq Y$  be subtrees of  $Y$  such that the following hold:

- (1)  $T_1 \subseteq T_2$ .
- (2)  $T_1$  is the lift of a maximal subtree of  $Y/U$  to  $Y$ .
- (3)  $T_2$  is a fundamental domain for the action of  $U$  on  $Y$ , i.e.  $UT_2 = Y$  and no two distinct edges of  $T_2$  are  $U$ -equivalent.
- (4) Every vertex  $v \in VT_2 - VT_1$  is connected to a vertex of  $T_1$  by a single edge.

This clearly implies that no two vertices of  $T_1$  are  $U$ -equivalent, that  $U(VT_1) = VY$  and that for every vertex of  $v \in VT_2 - VT_1$  there is a unique vertex  $x(v) \in VT_1$  which is  $U$ -equivalent to  $v$ .

For each vertex  $v \in VT_2 - VT_1$  choose an element  $t_v \in U$  such that  $t_v v = x(v)$ . The graph of groups  $\mathbb{B}$  is then defined as follows.

- (1) The graph  $B = Y/U$  is obtained from  $T_2$  by identifying  $v$  with  $x(v)$  for each vertex  $v \in VT_2 - VT_1$ . Thus we can assume that  $T_1$  is a subgraph of  $B$  (in fact a spanning

- tree of  $B$ ) and that  $v'_0 = v_0$ . Similarly, we assume that  $EB = ET_2$ . For any edge  $e = [z, v]$  of  $T_2$  with  $z \in T_1$  and  $v \in VT_2 - VT_1$ , we set  $o_B(e) = z$  and  $t_B(e) = x(v)$ .
- (2) For each vertex  $v \in VT_1$  we set  $B_v := \text{Stab}_U(v)$ , where  $\text{Stab}_U(v)$  is the  $U$ -stabilizer of  $v \in X$ .
  - (3) For each edge  $e = [z, v] \in ET_2$  we set  $B_e := \text{Stab}_U(e)$ .
  - (4) For each edge  $e = [z, v] \in ET_1$  the boundary monomorphisms  $\alpha_e^B : B_e \rightarrow B_z$  and  $\omega_e^B : B_e \rightarrow B_v$  are defined as inclusions of  $\text{Stab}_U(e)$  in  $\text{Stab}_U(z)$  and  $\text{Stab}_U(v)$  accordingly.
  - (5) Suppose  $e = [z, v]$  is an edge of  $T_2$  with  $z \in T_1$ ,  $v \in VT_2 - VT_1$ . We set the boundary monomorphism  $\alpha_e^B : B_e \rightarrow B_z$  to be the inclusion of  $\text{Stab}_U(e)$  in  $\text{Stab}_U(z)$ . We set the boundary monomorphism  $\omega_e^B : B_e \rightarrow B_{x(v)}$  to be the map  $g \mapsto t_v g t_v^{-1}$ ,  $g \in B_e$ .

**Definition 2.8** (Induced splitting). Let  $\mathbb{A}$  be a graph of groups with a base-vertex  $v_0$ . Let  $G = \pi_1(\mathbb{A}, v_0)$  and let  $X = \widetilde{(\mathbb{A}, v_0)}$  be the universal Bass-Serre covering tree of the based graph-of-groups  $(\mathbb{A}, v_0)$ . Thus  $X$  has a base-vertex  $x_0$  mapping to  $v_0$  under the natural quotient map.

Suppose  $U \leq G$  is a subgroup of  $G$  and  $Y \subset X$  is a  $U$ -invariant subtree containing  $x_0$ . Then the graph-of-groups splitting  $\mathbb{B}$  of  $U$  obtained as in Proposition 2.6 on the quotient graph  $B = Y/U$  is said to be *an induced splitting of  $U \leq G$  with respect to  $Y$  corresponding to the splitting  $G = \pi_1(\mathbb{A}, v_0)$* .

If  $U$  acts on  $X$  without a global fixed point then there is a preferred choice of a  $U$ -invariant subtree of  $X$ , namely the smallest  $U$ -invariant subtree containing  $x_0$ , which will be denoted  $X_{U, x_0}$  (or by  $X_U$ , if no confusion is possible):

$$X_{U, x_0} = X_U := \cup_{u \in U} [x_0, ux_0]$$

Notice that because of the explicit construction of  $\mathbb{B}$  each vertex group of  $\mathbb{B}$  fixes a vertex of  $X$  and hence is conjugate to a subgroup of a vertex group of  $\mathbb{A}$ . Similarly, edge groups of  $\mathbb{B}$  are conjugate to subgroups of edge groups of  $\mathbb{A}$ . In practice, when talking about induced splittings, we will often choose  $Y$  to be  $X_{U, x_0}$ .

### 3. $\mathbb{A}$ -GRAPHS

In this section we introduce the combinatorial notion of an  $\mathbb{A}$ -graph. These  $\mathbb{A}$ -graphs will approximate induced splittings of subgroups of  $\pi_1(\mathbb{A}, v_0)$ . In good situations, namely when an  $\mathbb{A}$ -graph is “folded”, an induced splitting can be directly read off the  $\mathbb{A}$ -graph.

**Definition 3.1** ( $\mathbb{A}$ -graph). Let  $\mathbb{A}$  be a graph of groups. An  $\mathbb{A}$ -graph  $\mathcal{B}$  consists of an underlying graph  $B$  with the following additional data:

- (1) A graph-morphism  $[\cdot] : B \rightarrow \mathbb{A}$ .
- (2) Each vertex  $u \in VB$  has an associated group  $B_u$ , where  $B_u \leq A_{[u]}$ .
- (3) To each edge  $f \in EB$  there are two associated group elements  $f_\alpha \in A_{[o(f)]}$  and  $f_\omega \in A_{[t(f)]}$  such that  $(f^{-1})_\alpha = (f_\omega)^{-1}$  for all  $f \in EB$ .

**Convention 3.2.** If  $f \in EB$  and  $u \in VB$ , we shall refer to  $e = [f] \in EA$  and  $v = [u] \in VA$  as *the type of  $f$  and  $u$*  accordingly. Also, especially when representing  $\mathbb{A}$ -graphs by pictures, we will sometimes say that an edge  $f$  of an  $\mathbb{A}$ -graph  $\mathcal{B}$  has *label*  $(f_\alpha, [f], f_\omega)$ . Similarly, we will say that a vertex  $u \in VB$  has *label*  $(B_u, [u])$ .

We will visualize an  $\mathbb{A}$ -graph  $\mathcal{B}$  in the obvious way by drawing the underlying graph  $B$  with the appropriate labels next to its vertices and edges. For every geometric edge we choose the label of either edge of the corresponding edge-pair  $\{f, f^{-1}\}$ . For convenience we will further orient the edge by attaching an arrow such that for an edge with label  $(a, e, b)$  one travels from a vertex with label  $(B, o(e))$  to a vertex with label  $(B', t(e))$  if one follows the direction of the arrow. It follows that reversing the orientation of an edge and replacing the label  $(a, e, b)$  by  $(b^{-1}, e^{-1}, a^{-1})$  yields another diagram of the same  $\mathbb{A}$ -graph. An example is shown in Figure 1.

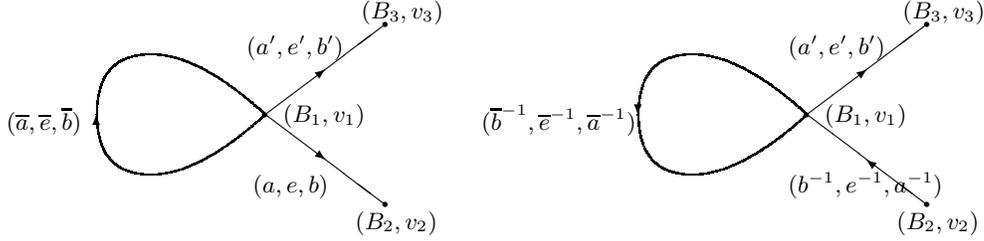


FIGURE 1. Two distinct diagrams associated to the same  $\mathbb{A}$ -graph

To any  $\mathbb{A}$ -graph we can then associate in a natural way a graph of groups:

**Definition 3.3** (Graph of groups defined by an  $\mathbb{A}$ -graph). Let  $\mathcal{B}$  be an  $\mathbb{A}$ -graph. The associated graph of groups  $\mathbb{B}$  is defined as follows:

- (1) The underlying graph of  $\mathbb{B}$  is the graph  $B$ .
- (2) For each  $u \in VB$  we put the vertex group of  $u$  to be  $B_u$ .
- (3) For each  $f \in EB$  we define the edge group of  $f$  in  $\mathbb{B}$  as

$$B_f := \alpha_{[f]}^{-1}(f_\alpha^{-1}B_{o(f)}f_\alpha) \cap \omega_{[f]}^{-1}(f_\omega B_{t(f)}f_\omega^{-1}) \leq A_{[f]}.$$

- (4) For each  $f \in EB$  we define the boundary monomorphism  $\alpha_f : B_f \rightarrow B_{o(f)}$  as  $\alpha_f(g) = f_\alpha(\alpha_{[f]}(g))f_\alpha^{-1}$ .

**Example 3.4.** Let  $\mathbb{A}$  be the “edge-of-groups” corresponding to an amalgamated product  $G = P *_C Q$ . Thus  $A$  consists of an edge  $e$  with two distinct endpoints  $v_0 = o(e)$  and  $v_1 = t(e)$ . The vertex and edge groups are:  $A_{v_0} = P$ ,  $A_{v_1} = Q$ ,  $A_e = A_{e^{-1}} = C$ . The boundary monomorphisms are the inclusions of  $C$  into  $P$  and  $Q$ .

Consider the  $\mathbb{A}$ -graph  $\mathcal{B}$ , shown in Figure 2, consisting of a single edge  $f$  of type  $e$  with  $o(f) = u_0$  of type  $v_0$  and  $t(f) = u_1$  of type  $v_1$ . The associated group of  $u_0$  is  $P_0 \leq P$  and the associated group of  $u_1$  is  $Q_0 \leq Q$ . Let  $a = f_\alpha \in P$  and  $b = f_\omega \in Q$ . Thus the label of  $f$  is  $(a, e, b)$ .

Then the graph of groups  $\mathbb{B}$  defined by  $\mathcal{B}$  looks as follows. The underlying graph of  $\mathbb{B}$  is still the single edge  $f$ . The vertex group of  $u_0$  is  $P_0$  and the vertex group of  $u_1$  is

$Q_0$ . The edge group of  $f$  is  $B_f = a^{-1}P_0a \cap bQ_0b^{-1} \leq C$ . The boundary monomorphisms corresponding to  $f$  are:  $\alpha_f(c) := aca^{-1}$  and  $\omega_f(c) := b^{-1}cb$  for  $c \in B_f$ .

$$\begin{array}{ccc} v_0 & \xrightarrow{e} & v_1 & & (P_0, v_0) & \xrightarrow{(a, e, b)} & (Q_0, v_1) \\ A_{v_0} = P & & A_{v_1} = Q & & & & \end{array}$$

FIGURE 2. Example of an amalgamated product and an  $\mathbb{A}$ -graph

**Convention 3.5.** Let  $\mathcal{B}$  be an  $\mathbb{A}$ -graph defining a graph-of-groups  $\mathbb{B}$ . Suppose  $u, u' \in VB$  and  $p$  is a  $\mathbb{B}$ -path from  $u$  to  $u'$ . Thus  $p$  has the form:

$$p = b_0, f_1, b_1, \dots, f_s, b_s$$

where  $s \geq 0$  is an integer,  $f_1, \dots, f_s$  is an edge path in  $B$  from  $u$  to  $u'$ , where  $b_0 \in B_u$ ,  $b_s \in B_{u'}$  and  $b_i \in B_{t(f_i)} = B_{o(f_{i+1})}$  for  $0 < i < s$ . Recall that each edge  $f_i$  has a label  $(g_i, e_i, k_i)$  in  $\mathcal{B}$ , where  $e_i = [f_i]$ ,  $g_i = (f_i)_\alpha$  and  $k_i = (f_i)_\omega$ .

Hence the  $\mathbb{B}$ -path  $p$  determines the  $\mathbb{A}$ -path  $\mu(p)$  from  $[u]$  to  $[u']$  in  $\mathbb{A}$  defined as follows:

$$\mu(p) = (b_0g_1), e_1, (k_1b_1g_2), e_2, \dots, (k_{s-1}b_{s-1}g_s), e_s, (k_sb_s)$$

Notice that  $|p| = |\mu(p)|$ .

We also want to think about an  $\mathbb{A}$ -graph as an ‘‘automaton’’ over  $\mathbb{A}$  which ‘‘accepts’’ a certain subgroup of the fundamental group of  $\mathbb{A}$ .

**Definition 3.6.** Let  $\mathcal{B}$  be an  $\mathbb{A}$ -graph with a base-vertex  $u_0 \in VB$ .

We define the *language*  $L(\mathcal{B}, u_0)$  as

$$L(\mathcal{B}, u_0) := \{\mu(p) \mid p \text{ is a reduced } \mathbb{B}\text{-path from } u_0 \text{ to } u_0 \text{ in } \mathbb{B}\}$$

Thus  $L(\mathcal{B}, u_0)$  consists of  $\mathbb{A}$ -paths from  $v_0 := [u_0]$  to  $v_0$ .

A simple but valuable observation states that the language of an  $\mathbb{A}$ -graph represents a subgroup in the fundamental group of  $\mathbb{A}$ .

**Proposition 3.7.** *Let  $\mathcal{B}$  be an  $\mathbb{A}$ -graph,  $u_0 \in VB$ ,  $v_0 = [u_0]$  and  $G = \pi_1(\mathbb{A}, v_0)$ .*

*Then:*

- (1) *If  $p, p'$  are  $\sim$ -equivalent  $\mathbb{B}$ -paths, then  $\mu(p) \sim \mu(p')$  as  $\mathbb{A}$ -paths.*
- (2) *The map  $\mu$  restricted to the set of  $\mathbb{B}$ -paths from  $u_0$  to  $u_0$  factors through to a homomorphism  $\nu : \pi_1(\mathbb{B}, u_0) \rightarrow G$ .*
- (3) *We have  $\overline{L(\mathcal{B}, u_0)} = \nu(\pi_1(\mathbb{B}, u_0))$ . In particular,  $\overline{L(\mathcal{B}, u_0)}$  is a subgroup of  $G$ .*
- (4) *There is a canonical  $\nu$ -equivariant simplicial map  $\phi : (\mathbb{B}, u_0) \rightarrow (\mathbb{A}, v_0)$  respecting the base-points.*

*Proof.* Part (1) follows directly from the definitions of  $\sim$  and  $\mathbb{B}$ . Part (1) immediately implies parts (2) and (3).

To establish (4) we will provide a direct construction of  $\nu$  which relies on the explicit definition of the Bass-Serre tree for a graph of groups given earlier. Denote  $X = \overline{(\mathbb{A}, v_0)}$

and  $Y = \widetilde{(\mathbb{B}, u_0)}$ . Let  $y = \overline{p}B_u$  be a vertex of  $Y$ , where  $p$  is a  $\mathbb{B}$ -path from  $u_0$  to  $u \in VB$ . Denote  $v = [u] \in VA$ . We put  $\phi(y) := \overline{\mu(p)}A_v \in VX$ . First, note that this definition does not depend on the choice of  $p$ . Indeed, suppose  $p'$  is another  $\mathbb{B}$ -path from  $u_0$  to  $u$  such that  $p \approx p'$ . Then by Definition 2.5  $\overline{p'} = \overline{pb}$  for some  $b \in B_u \leq A_v$ . Hence  $\overline{\mu(p)}A_v = \overline{\mu(p)b}A_v = \overline{\mu(pb)}A_v = \overline{\mu(p')}A_v$ . Thus  $\phi$  is well-defined on the vertex set of  $Y$ .

It remains to check that  $\phi$  preserves the adjacency relation. Let  $y = \overline{p}B_u \in VY$  be as above and let  $y' = \overline{pbf}B_{u'} \in VY$  be an adjacent vertex of  $Y$ , where  $b \in B_u \leq A_v$  and where  $f \in EB$  is an edge of type  $e \in EA$  with  $o(f) = u$ . Thus  $o(e) = v \in VA$ . We already know that  $\phi(y) = \overline{\mu(p)}A_v$ . Denote  $u' = t(f)$  and  $v' = t(e)$ , so that  $[u'] = v'$ . Also denote  $g = f_\alpha \in A_v$  and  $h = f_\omega \in A_{v'}$ . Then  $pbf$  is a  $\mathbb{B}$ -path from  $u_0$  to  $u'$ .

Therefore

$$\phi(y') = \overline{\mu(pbf)}A_{v'} = \overline{\mu(p)bgeh}A_{v'} = \overline{\mu(p)bge}A_{v'}$$

is an adjacent vertex of  $\phi(y) = \overline{\mu(p)}A_v$  since  $bg \in A_v$ . Thus indeed  $\phi$  is a well-defined simplicial map from  $Y$  to  $X$ . We leave checking the equivariance properties of  $\phi$  to the reader.  $\square$

We will see that every subgroup  $H$  of  $G = \pi_1(\mathbb{A}, v_0)$  arises in this fashion, i.e. for every  $H \leq G$  we have  $H = \nu(\pi_1(\mathbb{B}, u_0))$  where  $\mathbb{B}$  is the graph of groups associated to some  $\mathbb{A}$ -graph  $\mathcal{B}$ . Moreover, for an “efficient” choice of  $\mathcal{B}$  the homomorphism  $\nu : \pi_1(\mathbb{B}, u_0) \rightarrow H$  is an isomorphism and the graph of groups  $\mathbb{B}$  represents the induced splitting of the subgroup  $H \leq G$  with respect to the action of  $H$  on the Bass-Serre covering tree of  $\mathbb{A}$ .

**Remark 3.8.** Let  $\mathbb{A}$  and  $\mathcal{B}$  be as in Example 3.4. Then

$$\overline{L(\mathcal{B}, u_0)} = \nu(\pi_1(\mathbb{B}, u_0)) = \langle P_0, abQ_0b^{-1}a^{-1} \rangle \leq G = P *_C Q.$$

The following lemma is an immediate corollary of Proposition 2.4 and Proposition 3.7:

**Lemma 3.9.** *Let  $\mathcal{B}$  be an  $\mathbb{A}$ -graph with a base-vertex  $u_0$  of type  $v_0$ . Let  $T \subseteq B$  be a spanning tree. For any two vertices  $u, u' \in T$  denote by  $[u, u']_T$  the  $T$ -geodesic path from  $u$  to  $u'$ .*

*Then  $\pi_1(\mathbb{B}, u_0)$  is generated by  $\overline{S_T}$  where  $S_T$  is the following set:*

$$S_T := \{[u_0, u]_T B_u [u, u_0]_T \mid u \in VB\} \cup \{[u_0, o(e)]_T e [t(e), u_0]_T \mid e \in E(B - T)\}.$$

*In particular,  $\overline{L(\mathcal{B}, u_0)} \leq \pi_1(\mathbb{A}, v_0)$  is generated by  $\overline{\mu(S_T)} = \nu(\overline{S_T})$ .*

#### 4. FOLDING MOVES AND FOLDED $\mathbb{A}$ -GRAPHS

**Definition 4.1** (Folded  $\mathbb{A}$ -graph). Let  $\mathcal{B}$  be an  $\mathbb{A}$ -graph.

We will say that  $\mathcal{B}$  is *not folded* if at least one of the following applies:

- (1) There are two distinct edges  $f_1, f_2$  with  $o(f_1) = o(f_2) = z$  and labels  $(a_1, e, b_1), (a_2, e, b_2)$  accordingly, such that  $z$  has label  $(A', u)$  and  $a_2 = a' a_1 \alpha_e(c)$  for some  $c \in A_e$  and  $a' \in A'$ .
- (2) There is an edge  $f$  with label  $(a, e, b)$ , with  $o(f)$  labeled  $(A', u)$  and  $t(f)$  labeled  $(B', v)$  such that  $\alpha_e^{-1}(a^{-1}A'a) \neq \omega_e^{-1}(bB'b^{-1})$ .

Otherwise we will say that  $\mathcal{B}$  is *folded*.

It is easy to see that if  $\mathcal{B}$  is folded then any reduced  $\mathbb{B}$ -path translates into a reduced  $\mathbb{A}$ -path.

**Lemma 4.2.** *Let  $\mathcal{B}$  be a folded  $\mathbb{A}$ -graph defining the graph of groups  $\mathbb{B}$ . Suppose  $p$  is a reduced  $\mathbb{B}$ -path. Then the corresponding  $\mathbb{A}$ -path  $\mu(p)$  is  $\mathbb{A}$ -reduced.*

*Proof.* Suppose  $p$  is a  $\mathbb{B}$ -reduced  $\mathbb{B}$ -path and  $\mu(p)$  is the corresponding  $\mathbb{A}$ -path. Assume that  $\mu(p)$  is not reduced. Then  $p$  has a subsequence of the form  $f, a_1, f'$  where  $f^{-1}, f'$  are edges of  $B$  of the same type  $e \in EA$  such that the label of  $f^{-1}$  is  $aeb$ , the label of  $f'$  is  $a'eb'$ , where  $v \in VA$  is the type of  $o(f') = t(f) \in VB$ ,  $a, a' \in A_v$ ,  $a_1 \in B_{t(f)} \leq A_v$  and the  $\mathbb{A}$ -path  $b^{-1}, e^{-1}, a^{-1}a_1a', e, b'$  is not  $\mathbb{A}$ -reduced.

This means that for some  $c \in A_e$  we have  $a^{-1}a_1a' = \alpha_e(c)$ , that is  $a' = a_1^{-1}a\alpha_e(c)$ . If  $f^{-1}$  and  $f'$  are two distinct edges of  $B$ , this contradicts our assumption that  $\mathcal{B}$  is folded. Thus  $f^{-1} = f'$ , so that  $a = a', b = b'$ . Therefore  $a^{-1}a_1a = \alpha_e(c)$ . Recall that since  $\mathcal{B}$  is folded, part (2) of Definition 4.1 does not apply. Therefore the edge group in  $\mathbb{B}$  is  $B_{f'} = \alpha_e^{-1}(a^{-1}A_1a)$ , where  $A_1 = B_{t(f)} = B_{o(f')}$  and so  $c \in B_{f'}$ . Moreover, the boundary monomorphism of  $f'$  in  $\mathbb{B}$  was defined as  $\alpha_{f'}^B(c) = a\alpha_e(c)a^{-1}$ . Thus  $a_1 = a\alpha_e(c)a^{-1} \in \alpha_{f'}^B(B_{f'})$ . Hence  $f, a_1, f'$  is not  $\mathbb{B}$ -reduced, contrary to our assumptions.  $\square$

The above lemma immediately implies the following important fact:

**Proposition 4.3.** *Let  $\mathcal{B}$  be a folded  $\mathbb{A}$ -graph defining the graph of groups  $\mathbb{B}$ . Let  $u_0$  be a vertex of  $B$  of type  $v_0 \in VA$ . Denote  $G = \pi_1(\mathbb{A}, v_0)$  and  $U = \overline{L(\mathcal{B}, u_0)} = \nu(\pi_1(\mathbb{B}, u_0)) \leq G$ .*

*Then the epimorphism  $\nu : \pi_1(\mathbb{B}, u_0) \rightarrow U$  is an isomorphism and the graph map  $\phi$  between the Bass-Serre covering trees  $\phi : (\mathbb{B}, u_0) \rightarrow (\mathbb{A}, v_0)$  is injective.*

Proposition 4.3 essentially says that if  $\mathcal{B}$  is a folded  $\mathbb{A}$ -graph defining a subgroup  $U \leq G$ , then  $U = \pi_1(\mathbb{B}, u_0)$  is an induced splitting for  $U \leq G = \pi_1(\mathbb{A}, v_0)$ .

**Example 4.4.** Let  $\mathbb{A}$  and  $\mathcal{B}$  be as in Example 3.4. Recall that in this case  $G = P *_C Q$  and  $U = \langle P_0, abQ_0b^{-1}a^{-1} \rangle$ . Recall also that in the graph of groups  $\mathbb{B}$  the edge group of  $f$  is  $B_f = a^{-1}P_0a \cap bQ_0b^{-1} \leq C$ .

By Definition 4.1 the  $\mathbb{A}$ -graph  $\mathcal{B}$  is folded if and only if  $a^{-1}P_0a \cap C = bQ_0b^{-1} \cap C$  (in which case this last group is also equal to  $B_f$ ). It is easy to see that, as claimed by Proposition 4.3, if  $\mathcal{B}$  is folded then  $U = P_0 *_a B_f a^{-1} abQ_0b^{-1}a^{-1}$ .

We will now describe certain moves, called *folding moves* on  $\mathbb{A}$ -graphs, which preserve the corresponding subgroups of the fundamental group of  $\mathbb{A}$ . These folding moves are a more combinatorial version of the folding moves of Bestvina-Feighn [5] and Dunwoody [20]; implicitly they also contain Dunwoody's vertex morphisms.

Whenever we make changes to the label of an edge  $f$  of an  $\mathbb{A}$ -graph we assume that the corresponding changes are made to the label of  $f^{-1}$ .

**4.1. Auxiliary moves.** We will introduce three moves that can be applied to  $\mathbb{A}$ -graphs. These moves do not substantially change its structure and can be applied to any  $\mathbb{A}$ -graph.

**Definition 4.5** (Conjugation move A0). Let  $\mathcal{B}$  be an  $\mathbb{A}$ -graph. Suppose  $u$  is a vertex of  $\mathcal{B}$  and that  $g \in A_{[u]}$ .

Let  $\mathcal{B}'$  be the  $\mathbb{A}$ -graph obtained from  $\mathcal{B}$  as follows:

- (1) Replace  $B_u$  by  $gB_u g^{-1}$ .
- (2) For each non-loop edge  $f$  with  $o(f) = u$  replace  $f_\alpha$  with  $gf_\alpha$ .

(3) For each non-loop edge  $t(f) = u$  replace  $f_\omega$  with  $f_\omega g^{-1}$ .

(4) For each loop edge  $f$  with  $t(f) = o(f) = u$  replace  $f_\alpha$  with  $g f_\alpha$  and  $f_\omega$  with  $f_\omega g^{-1}$ .

In this case we will say that  $\mathcal{B}'$  is obtained from  $\mathcal{B}$  by a folding move of type A0.

If  $u' \in B$ ,  $u' \neq u$  is another vertex (whose vertex group is therefore not changed by the move), we will say that this A0-move is *admissible with respect to  $u'$* .

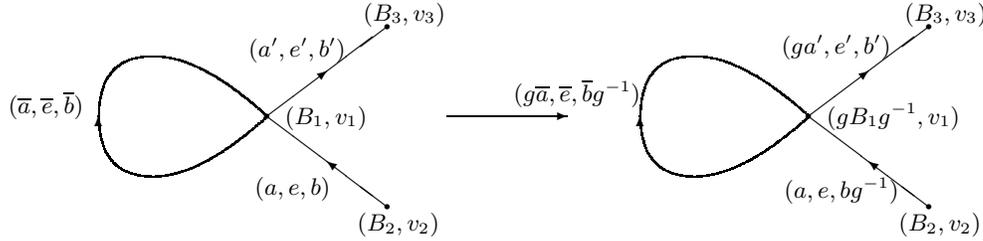


FIGURE 3. A move of type A0 with  $g \in A_{v_1}$

**Definition 4.6** (Bass-Serre move A1). Let  $\mathcal{B}$  be an  $\mathbb{A}$ -graph. Suppose  $f$  is an edge of  $\mathcal{B}$  and that  $c \in A_{[f]}$ .

Let  $\mathcal{B}'$  be the  $\mathbb{A}$ -graph obtained from  $\mathcal{B}$  by replacing  $f_\alpha$  with  $f_\alpha \alpha_{[e]}(c)^{-1}$  and  $f_\omega$  with  $\omega_{[e]}(c) f_\omega$ .

In this case we will say that  $\mathcal{B}'$  is obtained from  $\mathcal{B}$  by a folding move of type A1.



FIGURE 4. A move of type A1 with  $c \in A_e$

**Definition 4.7** (Simple adjustment A2). Let  $\mathcal{B}$  be an  $\mathbb{A}$ -graph. Suppose  $f$  is an edge of  $\mathcal{B}$  and that  $a' \in B_{o(f)}$ .

Let  $\mathcal{B}'$  be the  $\mathbb{A}$ -graph obtained from  $\mathcal{B}$  by replacing  $f_\alpha$  with  $a' f_\alpha$ .

In this case we will say that  $\mathcal{B}'$  is obtained from  $\mathcal{B}$  by a folding move of type A2.

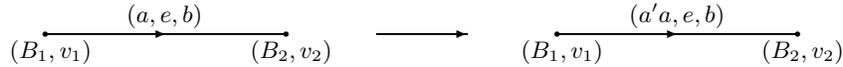


FIGURE 5. A move of type A2 with  $a' \in B_1$

**4.2. Main Stallings type folding moves.** In this section we introduce folding moves that change the structure of the underlying graph of an  $\mathbb{A}$ -graph. They can only be applied to  $\mathbb{A}$ -graphs that are not folded. On the level of underlying graphs these moves will correspond to the standard Stallings folds.

**Convention 4.8.** For the remainder of Section 4.2 let  $(\mathcal{B}, u_0)$  be an  $\mathbb{A}$ -graph with base vertex  $u_0$ . Suppose  $\mathcal{B}$  is not folded because case (1) of Definition 4.1 applies. Thus there exist distinct edges  $f_1$  and  $f_2$  with  $z = o(f_1) = o(f_2)$  and labels  $(a_1, e, a_2)$  and  $(a_2, e, b_2)$  such that  $a_2 = a' a_1 \alpha_e(c)$  for some  $c \in A_e$  and  $a' \in B_z$ . Suppose further that  $t(f_1) = x$  and  $t(f_2) = y$ . Clearly  $x$  and  $y$  are of the same type  $v \in VA$ . We also denote the type of  $z$  by  $w \in VA$ .

By applying a move of type A2 to the edge  $f_2$  we can change the label of  $f_2$  to  $(a'^{-1} a_2, e, b_2) = (a'^{-1} a' a_1 \alpha_e(c), e, b_2) = (a_1 \alpha_e(c), e, b_2)$ . A move of type A1 then yields the label  $(a_1, e, \omega_e(c) b_2)$  on  $f_2$ . We denote the resulting  $\mathbb{A}$ -graph by  $\mathcal{B}'$ .

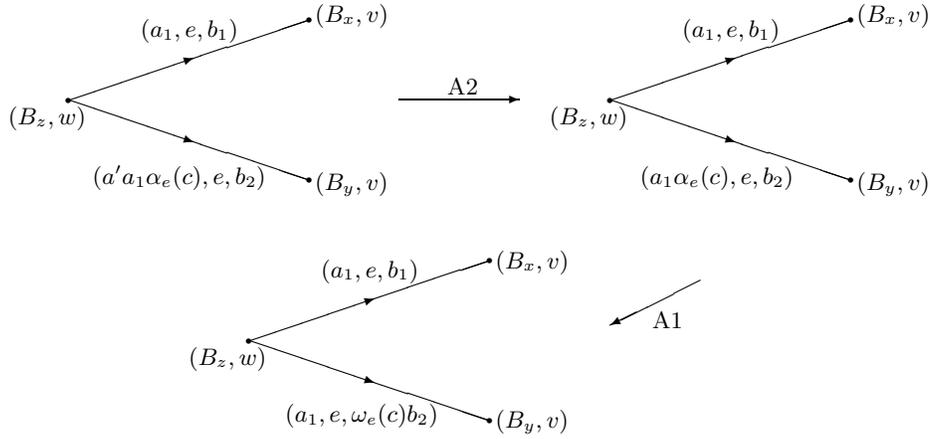


FIGURE 6. Constructing  $\mathcal{B}'$

We will use  $\mathcal{B}'$  as an intermediate object before defining the main folding moves on  $\mathcal{B}$ . Note that  $\mathcal{B}'$  is obtained from  $\mathcal{B}$  by moves that only alter labels of edges. Thus  $\mathcal{B}$  and  $\mathcal{B}'$  have the same underlying graphs as well as the same vertex groups.

It is possible that two or more of the vertices (that are drawn as distinct vertices) coincide. To simplify notations we put  $\bar{b}_2 := \omega_e(c) b_2$ . We now introduce four different types of folds,  $F1 - F4$ . They are distinguished by the topological type of the subgraph  $f_1 \cup f_2$  in  $B$ . Each of these moves will be defined as a sequence of several transformations, exactly one of which will correspond to performing a Stallings fold identifying the edges  $f_1$  and  $f_2$  in  $B$ . That particular portion of a move  $FN$ ,  $N = 1, \dots, 4$ , will be called an *elementary move of type  $\bar{F}N$* .

**Definition 4.9** (Simple fold  $F1$ ). Suppose  $f_1$  and  $f_2$  are two distinct non-loop edges and that  $t(f_1) \neq t(f_2)$ . Possibly after exchanging  $f_1$  and  $f_2$  we can assume that  $t(f_2)$  is not the base vertex  $u_0$  of  $\mathcal{B}$ .

We first perform a move of type  $A0$  on  $\mathcal{B}'$  at the vertex  $t(f_2) = y$  making the label of  $f_2$  to be  $(a_1, e, b_1)$  and the label of  $t(f_2)$  to be  $(b_1^{-1} \bar{b}_2 B_y \bar{b}_2^{-1} b_1, v)$ . Now both  $f_1$  and  $f_2$  have label  $(a_1, e, b_1)$ .

Next we identify the edges  $f_1$  and  $f_2$  into a single edge  $f$  with label  $(a_1, e, b_1)$ , as illustrated in Figure 7. The label of the vertex  $t(f)$  is set to be

$$\langle (B_x, b_1^{-1} \bar{b}_2 B_y \bar{b}_2^{-1} b_1), v \rangle.$$

The other labels do not change.

We call this last operation an *elementary move of type  $\bar{F}1$*  and say that the resulting  $\mathbb{A}$ -graph is obtained from the original  $\mathbb{A}$ -graph  $\mathcal{B}$  by a move of type  $F1$ .

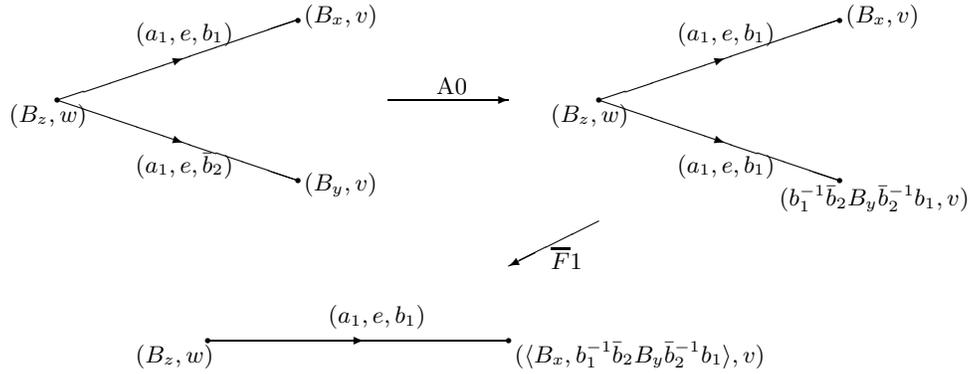


FIGURE 7. A move of type  $F1$

**Definition 4.10** (Mixed fold  $F2$ ). Suppose now that  $f_1$  is a loop edge and that  $f_2 \neq f_1$  is a non-loop edge. (The opposite situation is analogous).

This implies that  $e$  is a loop-edge in  $A$  based at the vertex  $v = w$ .

We first perform move  $A0$  on  $\mathcal{B}'$  making the label of  $f_2$  to be  $(a_1, e, b_1)$ . Next we fold the edges  $f_1$  and  $f_2$  into a single loop-edge  $f$  with label  $(a_1, e, b_1)$ , as shown in Figure 8. The label of  $o(f) = t(f)$  is set to be

$$\langle (B_z, b_1^{-1} \bar{b}_2 B_y \bar{b}_2^{-1} b_1), v \rangle.$$

We call this operation an *elementary move of type  $\bar{F}2$* .

If  $y = t(f_2) = u_0$  we then perform the auxiliary move  $A0$  corresponding to the element  $\bar{b}_2^{-1} b_1$ .

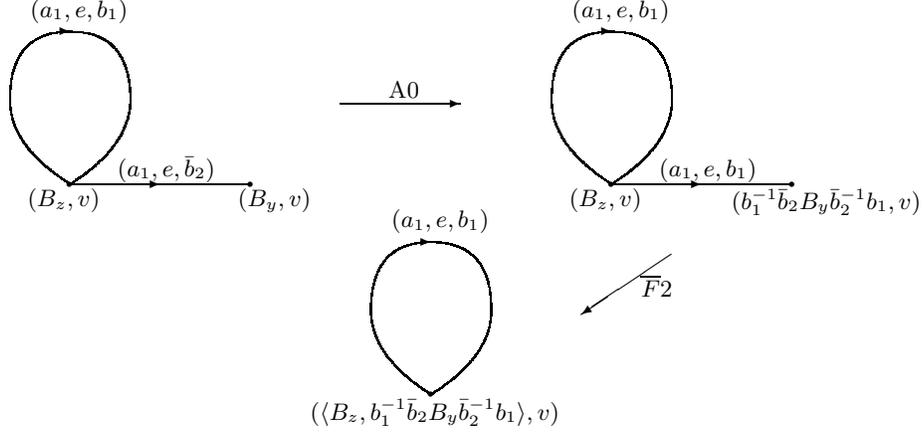
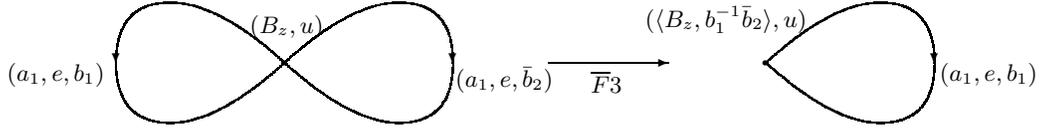
We will say that the resulting  $\mathbb{A}$ -graph is obtained from  $\mathcal{B}$  by a folding move of type  $F2$ .

**Definition 4.11** (Loop fold  $F3$ ). Suppose  $f_1$  and  $f_2$  are distinct loop-edges, so that  $x = y = z$ ,  $v = w$  and  $e$  is a loop-edge at  $v = w$  in  $A$ .

We identify the edges  $f_1$  and  $f_2$  in  $\mathcal{B}'$  into a single loop with label  $(a_1, e, b_1)$ , as shown in Figure 9. The new label of  $z$  is set to be

$$\langle (B_z, b_1^{-1} \bar{b}_2), v \rangle.$$

We call this last operation an *elementary move of type  $\bar{F}3$*  and say that the resulting  $\mathbb{A}$ -graph is obtained from  $\mathcal{B}$  by a folding move of type  $F3$ .

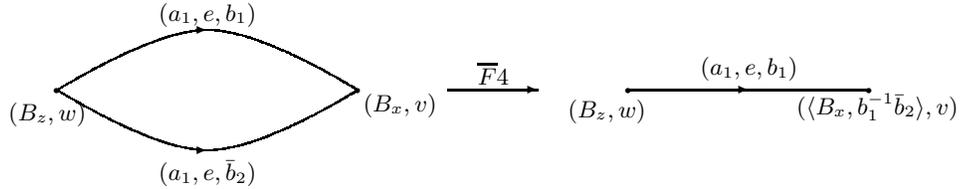
FIGURE 8. A move of type  $F2$ FIGURE 9. A move of type  $F3$ 

**Definition 4.12** (Double-edge fold  $F4$ ). Suppose that  $f_1$  are both non-loop edges such that  $x = t(f_1) = t(f_2) = y$ .

We identify the edges  $f_1$  and  $f_2$  of  $\mathcal{B}'$  into a single edge  $f$  with label  $(a_1, e, b_1)$ . We set the label of  $t(f)$  to be

$$((B_x, b_1^{-1} \bar{b}_2), v).$$

We call this last operation an *elementary move of type  $\bar{F}4$*  and say that the resulting  $\mathbb{A}$ -graph is obtained from  $\mathcal{B}$  by a *folding move of type  $F4$* .

FIGURE 10. A move of type  $F4$ 

**4.3. Edge-equalizing moves.** We will now introduce two folding moves that can be applied to an  $\mathbb{A}$ -graph that is not folded because of the second condition in Definition 4.1.

Thus suppose  $\mathcal{B}$  is an  $\mathbb{A}$ -graph with a base-vertex  $u_0$  and that there is an edge  $f \in EB$  with label  $(a, e, b)$ , with  $o(f)$  labeled  $(A', u)$  and  $t(f)$  labeled  $(B', v)$  such that  $\alpha_e^{-1}(a^{-1}A'a) \neq \omega_e^{-1}(bB'b^{-1})$ .

**Definition 4.13** (Equalizing an edge group  $F5$ ). Let  $\mathcal{B}$  be an  $\mathbb{A}$ -graph. Suppose  $f$  is a non-loop edge of  $\mathcal{B}$  with the label  $(a, e, b)$ . Let  $(A', w)$  be the label of  $z = o(f)$  and let  $(A'', v)$  be the label of  $t = t(f)$ .

Put  $C := \langle \alpha_e^{-1}(a^{-1}A'a), \omega_e^{-1}(bA''b^{-1}) \rangle \leq A_e$ .

Let  $\mathcal{B}'$  be the  $\mathbb{A}$ -graph obtained from  $\mathcal{B}$  by replacing the label of  $z$  with the label  $(\langle A', a\alpha_e(C)a^{-1} \rangle, w)$  and the label of  $t$  with  $(\langle A'', b^{-1}\omega_e(C)b \rangle, v)$ . In this case we will say that  $\mathcal{B}'$  is obtained from  $\mathcal{B}$  by a move of type  $F5$ .

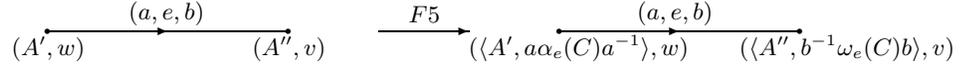


FIGURE 11. A move of type  $F6$  with  $C := \langle \alpha_e^{-1}(a^{-1}A'a), \omega_e^{-1}(bA''b^{-1}) \rangle$

**Definition 4.14** (Equalizing a loop-edge group  $F6$ ). Suppose  $f$  is a loop edge of  $\mathcal{B}$  with the label  $(a, e, b)$ . Let  $(A', v)$  be the label of  $z = o(f) = t(f)$ .

Put  $C := \langle \alpha_e^{-1}(a^{-1}A'a), \omega_e^{-1}(bA'b^{-1}) \rangle \leq A_e$ .

Let  $\mathcal{B}'$  be the  $\mathbb{A}$ -graph obtained from  $\mathcal{B}$  by replacing the label of  $z$  with

$$(\langle A', a\alpha_e(C)a^{-1}, b^{-1}\omega_e(C)b \rangle, v).$$

In this case we will say that  $\mathcal{B}'$  is obtained from  $\mathcal{B}$  by a move of type  $F6$ .

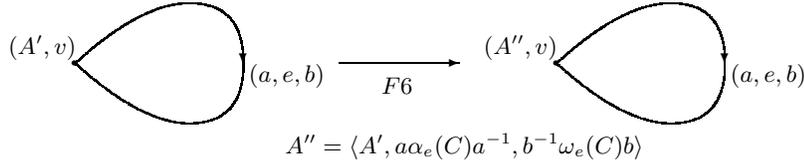


FIGURE 12. A move of type  $F6$  with  $C := \langle \alpha_e^{-1}(a^{-1}A'a), \omega_e^{-1}(bA'b^{-1}) \rangle$

Notice that each of the folding moves corresponds to a graph-morphism between the underlying graphs which preserves types of vertices and edges. In case of moves  $F1 - F4$  this morphism reduces the number of edge-pairs by one. For moves  $A0 - A3, F5 - F6$  the morphism is the identity map. Moreover, the moves  $F3, F4$  decrease the rank of the fundamental group of the underlying graph  $B$  by one, while  $F1$  and  $F2$  do not change it.

The following important proposition states that folding moves preserve the subgroup defined by an  $\mathbb{A}$ -graph.

**Proposition 4.15.** *Let  $\mathbb{A}$  be a graph of groups with a base-vertex  $v_0$ . Denote  $G = \pi_1(\mathbb{A}, v_0)$  and  $X = \widehat{(\mathbb{A}, v_0)}$ . Let  $\mathcal{B}'$  be an  $\mathbb{A}$ -graph obtained from  $\mathcal{B}$  by one of the folding moves  $A0 - A2, F1 - F6$  (where an  $A0$ -fold is  $u_0$ -admissible). Let  $u_0$  be a vertex of  $B$  and let  $u'_0$  be the image of  $u_0$  in  $B'$ . Suppose the type of the vertices  $u_0, u'_0$  is  $v_0 \in VA$ .*

Then there exists a canonical epimorphism  $\gamma : \pi_1(\mathbb{B}, u_0) \rightarrow \pi_1(\mathbb{B}', u'_0)$  and a  $\gamma$ -equivariant simplicial map  $\xi : \widetilde{(\mathbb{B}, u_0)} \rightarrow \widetilde{(\mathbb{B}', u'_0)}$  preserving the base-points such that the diagrams

$$\begin{array}{ccc} \pi_1(\mathbb{B}, u_0) & \xrightarrow{\gamma} & \pi_1(\mathbb{B}', u'_0) \\ & \searrow \nu & \downarrow \nu' \\ & & G \end{array} \quad \text{and} \quad \begin{array}{ccc} \widetilde{(\mathbb{B}, u_0)} & \xrightarrow{\xi} & \widetilde{(\mathbb{B}', u'_0)} \\ & \searrow \phi & \downarrow \phi' \\ & & X \end{array} \quad \text{commute.}$$

Hence the images of  $\phi : \widetilde{(\mathbb{B}, u_0)} \rightarrow \widetilde{(\mathbb{A}, v_0)}$  and  $\phi' : \widetilde{(\mathbb{B}', u'_0)} \rightarrow \widetilde{(\mathbb{A}, v_0)}$  coincide, and

$$\overline{L(\mathbb{B}, u_0)} = \overline{L(\mathbb{B}', u'_0)} \leq G = \pi_1(\mathbb{A}, v_0).$$

*Proof.* The proof of this proposition is a straightforward exercise in Bass-Serre theory. We will sketch a sample argument for the folding move  $F1$  and leave the other cases to the reader.

If we assume that auxiliary moves have already been shown to satisfy Proposition 4.15, we can assume that the move  $F1$  is actually an elementary move of type  $\overline{F1}$ , i.e. that both  $f_1$  and  $f_2$  have labels  $(a, e, b)$  and their terminal vertices  $t(f_1), t(f_2)$  have labels  $(B_x, v)$  and  $(B_y, v)$  accordingly. The folding move  $\overline{F1}$  identifies  $f_1$  and  $f_2$  into a single edge  $f$  with label  $(a, e, b)$  and with the label of  $t(f)$  equal  $(\langle B_x \cup B_y \rangle, v)$

Denote the folding graph-map in this move by  $P : B \rightarrow B'$  so that  $P(u_0) = u'_0$  and  $P(f_1) = P(f_2) = f$ . Note that by definition of  $\overline{F1}$  for any edge  $f' \in EB$  with  $f \notin \{f_1, f_2, f_1^{-1}, f_2^{-1}\}$  we have  $P(f') = f'$ . Also by construction for every vertex  $u_1 \in VB$  we have  $B_{u_1} \leq B'_{P(u_1)}$ . Thus the map  $P$  gives rise to the obvious map  $\gamma$  which takes a  $\mathbb{B}$ -path from  $u_1 \in VB$  to  $u_2 \in VB$  to a  $\mathbb{B}'$ -path from  $P(u_1)$  to  $P(u_2)$ . It is easy to see that  $\gamma$  respects the  $\sim$ -equivalence relation and therefore factors through to a group homomorphism, also denoted by  $\gamma$

$$\gamma : \pi_1(\mathbb{B}, u_0) \longrightarrow \pi_1(\mathbb{B}', u'_0).$$

The only nontrivial statement about the properties of  $\gamma$  is to check that  $\gamma$  is in fact ‘‘onto’’. It suffices to show that a generating set for  $\pi_1(\mathbb{B}', u'_0)$  provided by Lemma 3.9 lies in the image of  $\gamma$ . Since the edges  $f_1, f_2$  being folded by a move of type  $\overline{F1}$  are non-loops in  $B$ , we can choose a spanning tree  $T$  in  $B$  which contains both  $f_1$  and  $f_2$ . Then the graph  $P(T)$  obtained from  $T$  by identifying  $f_1$  and  $f_2$  is clearly a spanning tree for  $B'$ . Suppose  $f \in EB' - P(T)$ . Then  $f$  is in fact an edge of  $B$  which lies outside of  $T$ . Hence

$$\gamma([u_0, o(f)]_T f [t(f), u_0]_T) = [u'_0, o(f)]_{P(T)} f [t(f), u'_0]_{P(T)} =: s_f$$

and so the generator  $s_f$  of  $\pi_1(\mathbb{B}', u'_0)$  belongs to the image of  $\gamma$ .

Assume now that  $u' \in VB'$  is a vertex of  $B'$ . We need to show that the set  $[u'_0, u']_{P(T)} B'_w [u', u'_0]_{P(T)}$  is contained in the image of  $\gamma$ . If  $u' \neq t(f)$  then by construction  $u' = P(u') \in VB$  is a vertex of  $B$  with  $B_{u'} = B'_{u'}$ . In this case  $P([u_0, u']_T)$  is the  $P(T)$ -geodesic from  $u'_0 = P(u_0)$  to  $u'$  and so

$$[u'_0, u']_{P(T)} B'_w [u', u'_0]_{P(T)} = \gamma([u_0, u']_T B_w [u', u'_0]_T) \subseteq \text{im}(\gamma),$$

as required. Suppose next that  $u' = t(f) = P(t(f_1)) = P(t(f_2))$ . We will assume that  $f$  is contained in the  $P(T)$ -geodesic from  $u'_0$  to  $u'$  as the other case is similar.

Recall that by construction we have

$$B'_{u'} = \langle B_1, B_2 \rangle = \langle B_{t(f_1)}, B_{t(f_2)} \rangle.$$

Thus it suffices to show that for  $i = 1, 2$  the set

$$[u'_0, u']_{P(T)} B_i [u', u'_0]_{P(T)}$$

is contained in the image of  $\gamma$ . Since  $[u'_0, u']_{P(T)} = P([u_0, t(f_1)]_T) = P([u_0, t(f_2)]_T)$ , it follows that

$$[u'_0, u']_{P(T)} B_i [u', u'_0]_{P(T)} = \gamma([u_0, t(f_i)]_T B_i [t(f_i), u_0]_T) \subseteq \text{im}(\gamma),$$

as required. Thus indeed  $\widetilde{\gamma} : \pi_1(\mathbb{B}, u_0) \rightarrow \pi_1(\mathbb{B}', u'_0)$  is surjective.

We will now define  $\xi : (\mathbb{B}, u_0) \rightarrow (\mathbb{B}', u'_0)$ . Suppose  $x$  is a vertex of  $(\mathbb{B}, u_0)$ . Thus  $x$  has the form  $x = \overline{p} B_{u_1}$  for some vertex  $u_1 \in VB$  and some  $\mathbb{B}$ -path  $p$  from  $u_0$  to  $u_1$ . Then put  $\xi(x) := \overline{\gamma(p)} B_{P(u_1)}$ . It is not hard to see that  $\xi$  is well-defined on the vertex set of  $(\mathbb{B}, u_0)$  and that it preserves the adjacency relation for vertices. Thus indeed we have constructed a simplicial map  $\xi : (\mathbb{B}, u_0) \rightarrow (\mathbb{B}', u'_0)$ , as promised. The equivariant properties of  $\xi$  easily follow from the description of  $\xi$  and  $\gamma$  given above and from the explicit construction of the maps  $\nu$  and  $\phi$  given earlier in the proof of Proposition 3.7. We leave the details to the reader.  $\square$

**Lemma 4.16.** *Let  $\mathcal{B}$  be an  $\mathbb{A}$ -graph.*

- (1) *The  $\mathbb{A}$ -graph  $\mathcal{B}$  is folded if and only if none of the moves F1 – F6 apply. Moreover, if  $\mathcal{B}$  is folded then any application of moves of type A0 – A2 produces another folded graph.*
- (2) *Suppose that  $\mathcal{B}$  is not folded and case (1) of Definition 4.1 occurs. Then a move of type F1 – F4 can be applied to  $\mathcal{B}$ .*
- (3) *Suppose that  $\mathcal{B}$  is not folded and case (2) of Definition 4.1 occurs. Then a move of type F5 or F6 can be applied to  $\mathcal{B}$ .*

*Proof.* The statement of the lemma follows immediately from the definition of a folded graph and the section that introduces the folding moves.  $\square$

## 5. FINDING THE INDUCED SPLITTING ALGORITHMICALLY

In this section we describe an explicit procedure for finding an induced splitting for a subgroup and give a set of sufficient conditions which allow one to do this algorithmically.

The following notion allows us to easily construct a (usually non-folded)  $\mathbb{A}$ -graph for a subgroup  $U$  of  $G = \pi_1(\mathbb{A}, v_0)$  given by a generating set  $S \subset G$  of  $U$ .

**Definition 5.1** (Wedge). Let  $\mathbb{A}$  be a graph of groups with a base-vertex  $v_0$  and let  $S \subset G = \pi_1(\mathbb{A}, v_0)$ . For each  $s \in S$  we choose a reduced  $\mathbb{A}$ -path  $p_s$  from  $v_0$  to  $v_0$  such that  $\overline{p_s} = s$ . Put  $P_S = \{p_s | s \in S\}$ .

We construct an  $\mathbb{A}$ -graph  $\mathcal{B}_0$  as follows. The underlying graph  $B_0$  has base-vertex called  $u_0$  of type  $v_0$ . For each path  $p_s \in P_S$  of length at least 1 we write  $p_s$  as  $p_s = a_0, e_1, a_1, \dots, e_k, a_k$  and attach at the vertex  $u_0$  a circle subdivided into  $k$  edges. We give the first  $k - 1$  of these edges labels  $(a_0, e_1, 1), \dots, (a_{k-2}, e_{k-1}, 1)$  accordingly. We label the last edge of the circle by  $(a_{k-1}, e_k, a_k)$ . This describes the underlying graph  $B_0$  of  $\mathcal{B}_0$  with

the obvious assignment of types for vertices and edges (Note that  $B_0$  is either a single vertex or a wedge of circles). Every vertex  $u \in VB$  different from  $u_0$  and of type  $v \in VA$  is assigned the label  $(1, v)$  (so that the corresponding vertex group is trivial).

Note that for each  $p_s \in P_S$  of length zero we have  $s = \bar{p}_s \in A_{v_0}$ . We assign the vertex  $u_0$  of  $B_0$  label  $(K, v_0)$ , where

$$K = \langle \{s \in S \mid |p_s| = 0\} \rangle \leq A_{v_0}.$$

This completely describes the  $\mathbb{A}$ -graph  $\mathcal{B}_0$ . We call such an  $\mathbb{A}$ -graph an  $S$ -wedge.

**Example 5.2.** Suppose that  $\mathbb{A}$  is the edge-of groups with edge pair  $\{e, e^{-1}\}$  and  $o(e) = v_0$  and  $t(e) = v$  such that  $A_{v_0} = F(a, b)$ ,  $A_v = F(c, d)$ ,  $A_e = \langle a^2 = c^3 \rangle$  and that the boundary monomorphisms are the inclusion maps. In particular we have

$$G = \pi_1(\mathbb{A}, v_0) = F(a, b) *_{a^2=c^3} F(c, d).$$

Suppose that  $S = \{s_1 = a^4, s_2 = b^2, s_3 = c^3 d^{10}, s_4 = d^{10}\} \subset G$ . Clearly we have  $s_1, s_2 \in A_{v_0}$  and we can choose  $p_{s_3}$  and  $p_{s_4}$  as  $p_{s_3} = 1, e, c^3 d^{10}, e^{-1}, 1$  and  $p_{s_4} = 1, e, d^{10}, e^{-1}, 1$ , respectively. The diagram of the  $S$ -wedge then looks as follows:

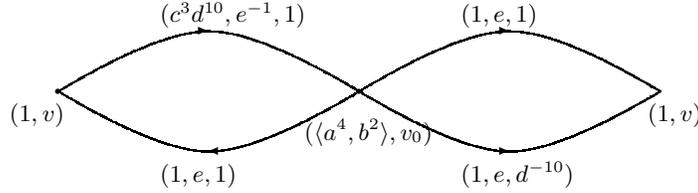


FIGURE 13. The  $S$ -wedge of  $S = \{a^4, b^2, c^3 d^{10}, d^{10}\}$

**Lemma 5.3.** *Let  $S, G, P_S, \mathbb{B}_0$  be as in the Definition 5.1 and  $U = \langle S \rangle$ . Then*

$$\overline{L(\mathbb{B}_0, u_0)} = U.$$

Moreover, the image of the map

$$\phi : (\widetilde{\mathbb{B}_0}, u_0) \rightarrow X = (\widetilde{\mathbb{A}}, v_0)$$

is equal to the tree  $X_{U, x_0}$  (where  $x_0$  is the base-vertex of  $X$ ).

*Proof.* It is clear from the definitions that  $S = \overline{P_S} \subseteq \overline{L(\mathbb{B}_0, u_0)} = \nu(\pi_1(\mathbb{B}, u_0)) \leq G$ . Thus  $U = \langle S \rangle \subset \overline{L(\mathbb{B}_0, u_0)}$ . On the other hand Lemma 3.9 implies that  $\nu(\pi_1(\mathbb{B}, u_0))$  is generated by  $S$ , and so  $\overline{L(\mathbb{B}_0, u_0)} = U$  as required. Denote  $H := \pi_1(\mathbb{B}, u_0)$ .

For each  $s \in S$  with  $|p_s| > 0$  denote by  $h_s$  the loop-path at  $u_0$  in the wedge  $B$  corresponding to  $s$ . For each  $s \in S$  with  $|p_s| = 0$  (so that  $s \in A_{v_0}$ ) put  $h_s = s$ . Then each  $h_s$  defines a  $\mathbb{B}$ -path from  $u_0$  to  $u_0$  and hence an element of  $H$  (if  $|p_s| > 0$  then  $h_s$  gives a  $\mathbb{B}$ -path with trivial group elements inserted between the consecutive edges). Then  $\nu(h_s) = s \in U \leq G$  and  $H = B_{u_0} * F(\{h_s \mid |p_s| > 0\})$ . It follows from the definition of the Bass-Serre covering tree that the action of  $H$  on  $(\widetilde{\mathbb{B}}, u_0)$  is minimal, that is, it has no proper  $H$ -invariant subtrees. Denote the base-vertex of  $(\widetilde{\mathbb{B}}, u_0)$  by  $y_0$ .

Since  $\pi_1(\mathbb{B}, u_0)$  is generated by the set  $\{h_s | s \in S\}$ , it follows that

$$\widetilde{(\mathbb{B}, u_0)} = \cup_{h \in H} [y_0, h y_0] = H(\cup_{s \in S} [y_0, h_s y_0])$$

and hence by Proposition 3.7

$$\phi(\widetilde{(\mathbb{B}_0, u_0)}) = U(\cup_{s \in S} \phi([y_0, h_s y_0]))$$

Since  $p_s$  is an  $\mathbb{A}$ -reduced path for each  $s \in S$ , we have  $\phi([y_0, h_s y_0]) = [x_0, s x_0]$ . Since  $U = \langle S \rangle \leq G$ , this implies

$$U(\cup_{s \in S} \phi([y_0, h_s y_0])) = U(\cup_{s \in S} [x_0, s x_0]) = \cup_{u \in U} [x_0, u x_0] = X_{U, x_0}$$

as required.  $\square$

The following statement is an immediate corollary of Proposition 4.3 and Proposition 4.15.

**Proposition 5.4** (Abstract Folding Algorithm). *Let  $\mathbb{A}$  be a graph of groups with a base-vertex  $v_0$  and let  $G = \pi_1(\mathbb{A}, v_0)$ . Suppose that  $S \subset G$  and that  $U = \langle S \rangle$ .*

*We first construct an  $S$ -wedge  $\mathcal{B}_0$  with base-vertex  $u_0$  as described in Definition 5.1. If this  $\mathbb{A}$ -graph is not folded, we start performing a sequence of folding moves F1 – F6 (in an arbitrarily chosen order) and construct a sequence of based  $\mathbb{A}$ -graphs  $(\mathcal{B}_n, u_n)$ , where each  $\mathcal{B}_{n+1}$  is obtained from  $\mathcal{B}_n$  by one of the folding moves F1 – F6.*

*If this sequence terminates in finitely many steps with a folded  $\mathbb{A}$ -graph  $\mathcal{B}_m$ , then we have  $\overline{L(\mathcal{B}_m, u_m)} = U$  and  $\mathbb{B}_m$  gives the induced splitting for  $U$  as described in Proposition 4.3.  $\square$*

**Convention 5.5.** When talking about actual algorithms related to graphs of groups we will not distinguish between an element of a vertex group and a word in the generators of that group. Thus, for example, when saying that we “construct an  $\mathbb{A}$ -path  $a_0, e_1, a_1, e_2, \dots$ ” we actually mean constructing a sequence  $w_0, e_1, w_1, e_2, \dots$  where  $w_i$  is a word in the generators of the corresponding vertex group representing the element  $a_i$ . Moreover, we will assume that vertex and edge groups are explicitly given by recursive presentations on finite generating sets and that boundary monomorphisms are explicitly given by specifying the images of the generators of vertex groups in the appropriate edge groups.

We will now describe a set of sufficient conditions which allows one to algorithmically carry out the abstract procedure described in Proposition 5.4.

**Definition 5.6.** We will say that a finite connected graph of finitely generated groups  $\mathbb{A}$  is *benign* if the following conditions are satisfied:

- (1) For each vertex  $v \in VA$  and an edge  $e \in EA$  with  $o(e) = v$  there is an algorithm with the following property. Given a finite set  $X \subseteq A_v$  and an element  $a \in A_v$  the algorithm decides whether  $I = \langle X \rangle \cap a\alpha_e(A_e)$  is empty. If  $I \neq \emptyset$ , the algorithm produces an element of  $I$ .
- (2) Every edge group  $A_e$  of  $\mathbb{A}$  is *Noetherian*, that is, it contains no infinite ascending sequence of subgroups. (Being Noetherian is equivalent to saying that all subgroups are finitely generated).
- (3) Every edge group  $A_e$  of  $\mathbb{A}$  has solvable uniform membership problem, i.e. there is an algorithm which, given a finite subset  $X \subseteq A_e$  and an element  $a \in A_e$  decides whether or not  $a \in \langle X \rangle$ .

- (4) For each vertex  $v \in VA$  and edge  $e \in EA$  with  $o(e) = v$  there is an algorithm with the following property. For any finite subset  $X \subseteq A_v$  the algorithm computes a finite generating set for the subgroup  $\alpha_e(A_e) \cap \langle X \rangle$ .

**Remark 5.7.** Notice that if  $\mathbb{A}$  is benign, then  $A_v$  has solvable membership problem with respect to  $\alpha_e(A_e)$  (where  $v = o(e), e \in EA$ ). Indeed, if  $a \in A_v$  then  $a \in \alpha_e(A_e)$  if and only if the intersection  $\{1\} \cap a\alpha_e(A_e)$  is nonempty.

**Theorem 5.8.** *Let  $\mathbb{A}$  be a benign graph of groups with base-vertex  $v_0$ . Then:*

- (a) *There is an algorithm which, given a finite set in  $G = \pi_1(\mathbb{A}, v_0)$  generating a subgroup  $U \leq G$ , constructs a folded  $\mathbb{A}$ -graph  $\mathcal{B}$  with base-vertex  $u_0$  such that  $L(\mathcal{B}, u_0) = U$ . In  $\mathcal{B}$ , each vertex group  $B_u$  is given by its finite generating set of words in the generators of  $A_{[v]}$ .*

*Moreover,  $\nu : \pi_1(\mathbb{B}, u_0) \rightarrow U \leq G$  is an isomorphism, the map*

$$\phi : \widetilde{(\mathcal{B}, u_0)} \rightarrow (X, x_0) = \widetilde{(\mathbb{A}, v_0)}$$

*is injective and the image of  $\phi$  is the tree  $X_U = \cup_{u \in U} [x_0, ux_0]$ .*

- (b) *Suppose, in addition, that for each  $v \in VA$  there is an algorithm which, given a finite subset  $Y$  of  $A_v$ , produces a finite presentation for the subgroup of  $A_v$  generated by  $Y$  (thus each  $A_v$  is coherent). Then there is an algorithm which, given a finite set  $S \subseteq G$ , constructs a finite presentation for the subgroup  $U = \langle S \rangle \leq G$ .*

**Remark 5.9.** Thus by Proposition 4.3 the identification of  $U$  with  $\pi_1(\mathbb{B}, u_0)$  via  $\nu$  gives the induced splitting for  $U \leq G = \pi_1(\mathbb{A}, v_0)$ . In particular, this identification gives us an explicit finite description of  $U$ , meaning that in  $\mathcal{B}$  for each vertex  $u \in VB$  of type  $v \in VA$  the group-label of  $u$  is given in the form  $\langle X \rangle$ , where  $X$  is a finite subset of  $A_v$ .

*Proof of Theorem 5.8.* Let  $S$  be a finite generating set of  $U$ . As  $\mathbb{A}$  is benign we can find for any  $s \in S$  a reduced  $\mathbb{A}$ -path  $p_s$ . Thus we can apply the abstract folding algorithm as described in Proposition 5.4.

We have to show that the process terminates in a finite number of steps and that each step can be performed effectively.

Recall that by construction since  $S$  is finite, the underlying graph  $B_0$  of  $\mathcal{B}_0$  is finite. Moreover the vertex groups in  $\mathcal{B}_0$  are trivial with the possible exception of the base-vertex  $w_0 \in VB_0$ . By construction, the vertex group at  $w_0$  is given by a finite generating set of cardinality at most  $\#S$ .

We can argue inductively that at each stage of the process for every vertex  $u \in VB_n$  of type  $v \in VA$  the group  $(B_n)_u \leq A_v$  is given by its finite generating set contained in  $A_v$ . At each stage it is easy to decide whether  $\mathcal{B}_n$  is folded. Namely, condition (1) of Definition 5.6 allows us to decide if Case (1) of Definition 4.1 occurs. Conditions (3) and (4) of Definition 5.6 allow us to decide if Case (2) of Definition 4.1 applies to  $\mathcal{B}_n$ . If  $\mathcal{B}_n$  turns out to be not folded, we perform one of the folded moves  $F1 - F6$ , whichever is appropriate.

By definition, performing folds of type  $F1 - F4$  allows us to effectively represent the vertex groups of  $\mathcal{B}_{n+1}$  by their finite generating sets. Suppose now that  $\mathcal{B}_{n+1}$  is obtained from  $B_n$  by a move of type  $F5$  or  $F6$ . Recall that edge groups of  $\mathbb{A}$  are Noetherian. Conditions (3) and (4) from the definition of a benign graph of groups and the definitions of folding moves  $F5 - F6$  allow us to effectively compute finite generating sets for the vertex groups of  $\mathcal{B}_{n+1}$ .

Suppose that the sequence  $(\mathcal{B}_n)$  is infinite. Each of the moves of type  $F1 - F4$  reduces the number of edges in  $\mathcal{B}_n$  and so can happen only finitely many types. Thus after a certain stage only the moves of type  $F5 - F6$  (which do not change the underlying finite graph) apply. Hence there is an edge to which moves of type  $F5 - F6$  apply infinitely often. Each such move increases the edge group of the corresponding edge in  $\mathbb{B}_n$ . This produces a strictly increasing infinite sequence of subgroups in an edge-group of  $\mathbb{A}$ , contradicting our assumption that edge-groups in  $\mathbb{A}$  are Noetherian.

Thus the sequence  $\mathcal{B}_n$  terminates in finitely many steps with a folded  $\mathbb{A}$ -graph  $\mathcal{B}_m$ , as required and part (a) of Theorem 5.8 is proved.

Once  $\mathbb{B}$  as in Theorem 5.8 is constructed, each vertex (edge) group of  $\mathbb{B}$  is given as a subgroup of some vertex (edge) group of  $\mathbb{A}$  generated by a given finite set of elements. If the additional assumptions on  $\mathbb{A}$  from part (b) of Theorem 5.8 hold, then we can recover finite presentations for each vertex group of  $\mathbb{B}$  and hence a finite presentation for  $U = \pi_1(\mathbb{B}, w_0)$ .  $\square$

**Example 5.10.** It is easy to produce an example of a non-benign graph of groups, where the folding algorithm described above does not necessarily terminate. For example, consider the HNN-extension of a free group  $F = F(a, b)$  along the endomorphism  $\phi : F \rightarrow F$ ,  $\phi(a) = ab^2a, \phi(b) = ba^2b$ :

$$G = \langle a, b, e \mid e^{-1}ae = ab^2a, e^{-1}be = ba^2b \rangle = \langle a, b, e \mid e^{-1}\alpha_e(f)e = \omega_e(f), f \in F \rangle$$

where  $\alpha_e = Id_F$  and  $\omega_e = \phi$ . Thus we may think of  $G$  as the fundamental group of the graph of groups  $\mathbb{A}$  consisting of a single vertex  $v$ , a single edge  $e$  with  $A_v = A_e = F$  and  $\alpha_e = Id_F$  and  $\omega_e = \phi$ . The group  $G$  is torsion-free and word-hyperbolic [31] by the Combination Theorem of Bestvina-Feighn [6]. Since  $[a, e] \neq 1$  in  $G$ , there is  $m > 0$  such that  $H = \langle e, a^m \rangle \leq G$  is free of rank two. It is not hard to see that in this case  $H \cap F$  is not finitely generated. In fact  $H \cap F$  is freely generated by the elements  $e^{-i}a^m e^i = \phi^i(a^m)$ . We can start the folding algorithm for  $H$  with an  $\mathbb{A}$ -graph  $\mathcal{B}$  consisting of a single vertex  $u$  of type  $v$ , a single edge  $f$  of type  $e$  with label  $(1, e, 1)$  and with  $B_u = \langle a^m \rangle$ . Then the folding algorithm results in a repeated application of an F6-move (no other moves are applicable) and produces an infinite sequence of  $\mathbb{A}$ -graphs  $\mathcal{B}_0 = \mathcal{B}, \mathcal{B}_1, \mathcal{B}_2, \dots$ . The only difference between  $\mathcal{B}_i$  and  $\mathcal{B}$  is that in  $\mathcal{B}_i$  the vertex group is  $\langle a^m, \phi(a^m), \dots, \phi^i(a^m) \rangle$ . This difficulty is caused by the fact that the edge-group in  $\mathbb{A}$  is not Noetherian.

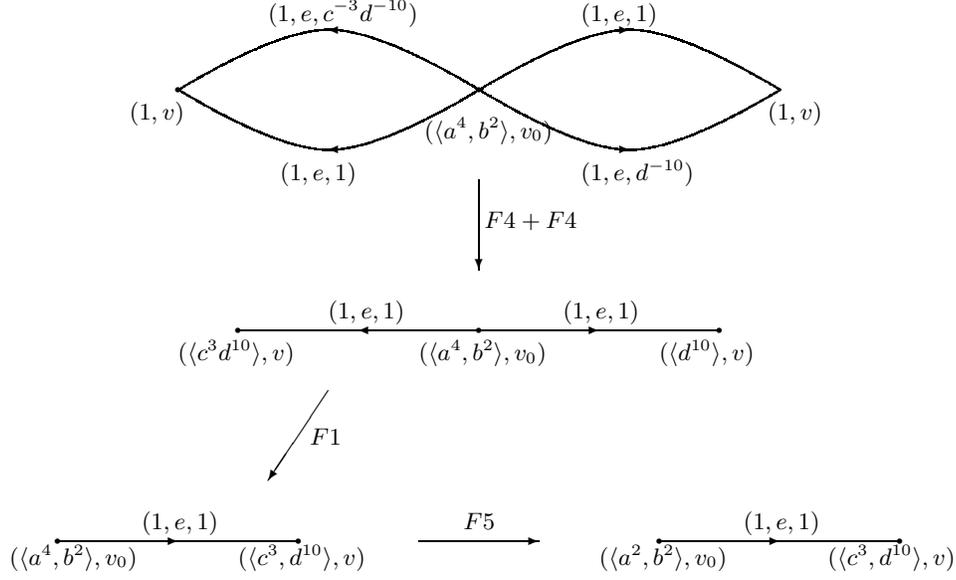
A similar effect occurs in the direct product  $F(a, b) \times \langle t \rangle$ , thought of as an HNN-extension of  $F(a, b)$  along the identity map, when we look at the subgroup  $H = \langle tb, a \rangle$ .

**Example 5.11.** We illustrate the folding algorithm in Figure 14. We start with the  $S$ -wedge discussed in Example 5.2. (Note that we have changed the orientation of two edges of the diagram.)

The F5-move corresponds to the fact that  $c^3 = a^2$  and hence  $a^2$  has to be “added” to the subgroup  $\langle a^4, b^2 \rangle$  yielding  $\langle a^2, a^4, b^2 \rangle = \langle a^2, b^2 \rangle$ . The final folded  $\mathbb{A}$ -graph corresponds to the induced splitting of  $H = \langle S \rangle$  as

$$H = \langle a^2, b^2 \rangle *_{a^2=c^3} \langle c^3, d^{10} \rangle.$$

We will need the following simple observation which says that reduced  $\mathbb{B}$ -paths of a folded  $\mathbb{A}$ -graph  $\mathcal{B}$  are the reduced  $\mathbb{A}$ -paths of the elements of  $U = \overline{L(\mathcal{B}, u_0)}$  up to the degree of freedom spelled out in the normal form theorem for fundamental groups of graphs of groups.

FIGURE 14. The folding algorithm applied to the  $S$ -wedge of Example 5.2

It provides a criterion to decide whether an element lies in a subgroup represented by a folded  $\mathbb{A}$ -graph.

**Lemma 5.12.** *Let  $\mathbb{A}$  be a graph of groups. Let  $\mathcal{B}$  be a folded  $\mathbb{A}$ -graph. Suppose that  $p = a_0, e_1, a_1, \dots, e_k, a_k$  is a reduced  $\mathbb{A}$ -path, where  $k \geq 0$ . Then  $\bar{p} = \mu(q)$  for some reduced  $\mathbb{B}$ -path  $q$  if and only if there exists a reduced  $\mathbb{B}$ -path*

$$q = b_0, f_1, b_1, \dots, b_{k-1}, f_k, b_k$$

and a sequence  $c_i \in A_{e_i}$ ,  $i = 1, \dots, k$  such that  $[f_i] = e_i$  and

$$\begin{aligned} a_0 &= b_0(f_1)_\alpha \alpha_{e_1}(c_1), \\ a_i &= \omega_{e_i}(c_i)^{-1}(f_i)_\omega b_i(f_{i+1})_\alpha \alpha_{e_{i+1}}(c_{i+1}) \text{ for } i = 1, \dots, k-1 \text{ and} \\ a_k &= \omega_{e_k}(c_k)^{-1}(f_k)_\omega b_k \end{aligned}$$

*Proof.* The existence of a path  $q$  and of  $(b_i)_i$ ,  $(c_i)_i$  with the required properties clearly implies that  $\bar{p} = \mu(q)$ .

If  $\bar{p} = \mu(q)$  for some reduced  $\mathbb{B}$ -path  $q$ , then the assertion follows from the normal form theorem applied to the product  $p\mu(q)^{-1}$  which is trivial in  $\pi_1(\mathbb{A}, o(e_1))$ .  $\square$

**Theorem 5.13.** *Let  $\mathbb{A}$  be a benign graph of groups. Suppose also that each vertex group of  $\mathbb{A}$  has solvable uniform membership problem. Let  $v_0 \in VA$  and denote  $G = \pi_1(\mathbb{A}, v_0)$ . Then the uniform membership problem for  $G$  is solvable. That is to say, there is an algorithm which, given finitely many elements  $h_1, \dots, h_k, g \in G$ , decides whether  $g$  belongs to the subgroup  $H = \langle h_1, \dots, h_k \rangle \leq G$ .*

*Proof.* Denote  $S = \{h_1, \dots, h_k\}$  and  $U = \langle S \rangle \leq G$ . First we apply Theorem 5.8 and construct a finite folded  $\mathbb{A}$ -graph  $\mathcal{B}$  with a base-vertex  $u_0$  such that  $U = \overline{L(\mathcal{B}, u_0)}$ . Every vertex group in  $\mathcal{B}$  is given by its finite generating set contained in the appropriate vertex group of  $\mathbb{A}$ .

Next we write  $g$  as a reduced  $\mathbb{A}$ -path  $p'$  from  $v_0$  to  $v_0$ . This is possible since by Remark 5.7 every vertex group in  $\mathbb{A}$  has solvable membership problem with respect to incident edge groups. Then  $g \in U$  if and only if there exists a reduced  $\mathbb{B}$ -path  $q'$  from  $u_0$  to  $u_0$  such that  $\overline{p'} = \mu(q')$ . The assertion of the theorem now immediately follows from:

**Claim.** There is an algorithm which, given a reduced  $\mathbb{A}$ -path

$$(*) \quad p = a_0, e_1, a_1, \dots, e_k, a_k$$

from some vertex  $v \in VA$  (possibly distinct from  $v_0$ ) to  $v_0$ , and given a vertex  $u \in VB$ , decides if there exists a reduced  $\mathbb{B}$ -path

$$q = b_0, f_1, b_1, \dots, b_{k-1}, f_k, b_k$$

from  $u \in VB$  to  $u_0$  such that  $\overline{\mu(q)} = \overline{p}$ .

We prove the Claim by induction on the length  $k$  of  $p$ . For  $k = 0$  the Claim is equivalent to deciding, given an element  $a_0 \in A_{v_0}$ , whether  $a_0 \in B_{u_0}$ . This is possible since  $B_{u_0} \subset A_{v_0}$  is a finitely generated subgroup and by assumption the group  $A_{v_0}$  has solvable uniform membership problem.

Suppose now that  $k > 0$  and the algorithm exists for reduced  $\mathbb{A}$ -paths of length  $k - 1$ .

Let  $p$  be a reduced  $\mathbb{A}$ -path of length  $k$  from  $v$  to  $v_0$  as in  $(*)$ . If a path  $q$  as in the Claim exists then it follows from Lemma 5.12 that there is such a path  $q$  with the property that  $[f_1] = e_1$  and  $a_0 = b_0(f_1)_\alpha \alpha_{e_1}(c_1)$  for some  $c_1 \in A_{e_1}$ .

Observe first that we can decide whether there exists an edge  $f$  with  $[f] = e_1$ ,  $b \in B_u$  and  $c \in A_{e_1}$  such that  $a_0 = bf_\alpha \alpha_{e_1}(c)$  and can find them if they do exist. (If there are no such  $f, b, c$  then by the previous remark the required  $q$  does not exist).

Since there are only finitely many edges in  $\mathcal{B}$  of type  $e_1$  emanating at  $u$ , we may assume that we are dealing with a fixed edge  $f$  and looking for  $b$  and  $c$  with the above properties. Recall that  $a_0, f_\alpha \in A_v$  are given. Thus we want to know if there are  $b \in B_u$  and  $c \in A_{e_1}$  such that  $a_0 = bf_\alpha \alpha_{e_1}(c)$ , i.e. such that  $f_\alpha^{-1}bf_\alpha = (f_\alpha^{-1}a_0)\alpha_{e_1}(c^{-1})$ . Thus the existence of such  $b$  and  $c$  is equivalent to  $f_\alpha^{-1}B_u f_\alpha \cap (f_\alpha^{-1}a_0)\alpha_{e_1}(A_{e_1}) \neq \emptyset$ . This can be checked by condition (1) of Definition 5.6 since  $\mathbb{A}$  is benign. Moreover, condition (1) of Definition 5.6 allows us to find such (not necessarily unique)  $b$  and  $c$  if they exist.

Suppose now that we have found  $f, b$  and  $c$  as above, so that  $a_0 = bf_\alpha \alpha_{e_1}(c)$ .

We now observe that if  $q$  as in the Claim exists, then there is such a  $q$  with  $b_0 = b$ . Indeed, if  $q$  is as in the Claim then by Lemma 5.12  $[f_1] = e_1$  and there exists an element  $c_1 \in \alpha_{e_1}(A_{e_1})$  such that  $a_0 = b_0(f_1)_\alpha \alpha_{e_1}(c_1) = bf_\alpha \alpha_{e_1}(c)$ . The assumption that  $\mathcal{B}$  is folded implies that  $f = f_1$ . Hence  $f_\alpha = (f_1)_\alpha$  and

$$b^{-1}b_0 = f_\alpha \alpha_{e_1}(cc_1^{-1})f_\alpha^{-1} \in f_\alpha \alpha_{e_1}(A_{e_1})f_\alpha^{-1}.$$

As  $\mathcal{B}$  is folded this implies that  $b^{-1}b_0 \in \alpha_f(B_f)$ . It follows that the  $\mathbb{B}$ -path  $q$  is equivalent to a  $\mathbb{B}$ -path starting with  $b$ , as required. We denote this new  $\mathbb{B}$ -path again by  $q$ .

As  $f_1 = f$ ,  $b_0 = b$  in  $q$  and  $\overline{\mu(q)} = \bar{p}$  it follows that

$$\overline{bf_\alpha e_1 f_\omega b_1 (f_2)_\alpha e_2 \dots} = \overline{a_0 e_1 a_1 e_2 \dots} = \overline{bf_\alpha \alpha_{e_1}(c) e_1 a_2 e_2 \dots} = \overline{bf_\alpha e_1 \omega_{e_1}(c) a_2 e_2 \dots}$$

and hence

$$\overline{b_1 (f_2)_\alpha e_2 \dots} = \overline{(f_\omega)^{-1} \omega_{e_1}(c) a_1 e_2 a_2 \dots}$$

Thus to decide if a desired  $q$  exists we need to determine if for the reduced  $\mathbb{A}$ -path

$$p' = (f_\omega)^{-1} \omega_{e_1}(c) a_1, e_2, a_2, \dots, a_{k-1}, e_k, a_k$$

from  $t(e_1)$  to  $v_0$  in  $\mathbb{A}$  there exists a path  $q'$  starting at  $t(f_1)$  as in the Claim. This is possible by the inductive hypothesis since  $|p'| = k - 1$ .  $\square$

We can now prove Theorem 1.1 from the Introduction:

**Theorem 5.14.** *Let  $\mathbb{A}$  be a finite graph of groups where each vertex group either is polycyclic-by-finite or is word-hyperbolic and locally quasiconvex, and where all edge groups are virtually polycyclic. Then for any  $v_0 \in VA$  the group  $G = \pi_1(\mathbb{A}, v_0)$  has solvable uniform membership problem. Moreover there is an algorithm which, given a finite subset  $S \subseteq G$ , constructs the induced splitting and a finite presentation for the subgroup  $U = \langle S \rangle \leq G$ .*

*Proof.* The uniform membership problem is solvable in polycyclic-by-finite groups [3] and in locally quasiconvex hyperbolic groups [30]. Thus by Theorem 5.13 to establish the solvability of the membership problem it suffices to check that the graph of groups  $\mathbb{A}$  is benign.

It is well-known (see for example [16]) that a polycyclic subgroup of a word-hyperbolic group is virtually cyclic. Hence all edge groups for edges incident to hyperbolic vertex groups are in fact virtually cyclic.

Suppose first that  $v \in VA$  is such that  $A_v$  is word-hyperbolic and locally quasiconvex. Let  $L$  be the regular language of all Short-Lex geodesic words in  $A_v$  over some fixed finite generating set of  $A_v$ . It is well known that  $L$  gives a bi-automatic structure with uniqueness for  $A_v$ . Since  $A_v$  is assumed to be locally quasiconvex, all finitely generated subgroups of  $A_v$  are  $L$ -rational. Therefore by the result of [30], there is a uniform algorithm which, given a finite set  $X \subseteq A_v$ , produces the pre-image  $L_X$  of the subgroup  $\langle X \rangle \leq A_v$  in  $L$ . For each edge  $e \in EA$  with  $o(e) = v$  denote by  $L_e$  the pre-image in  $L$  of the virtually cyclic subgroup  $\alpha_e(A_e)$ .

Suppose now that  $X \subseteq A_v$  is a finite set,  $a \in A_v$  and  $e \in EA$  is an edge with  $o(e) = v$ . We first construct the language  $L_X$ . Then using the biautomatic structure on  $A_v$  we construct the regular language  $L_{e,a}$  which is the pre-image in  $L$  of the set  $a\alpha_e(A_e)$ . Now to decide if  $\langle X \rangle \cap a\alpha_e(A_e)$  is empty we only need to check whether the intersection of the regular languages  $L_X \cap L_{e,a}$  is empty.

Moreover, we can also compute the intersection  $L_X \cap L_e$  which is the pre-image in  $L$  of the subgroup  $\langle X \rangle \cap \alpha_e(A_e)$ . Once the regular language  $L_X \cap L_e$  is known, it is easy to recover a finite generating set for  $\langle X \rangle \cap \alpha_e(A_e)$ . Thus we have verified that  $\mathbb{A}$  is benign at the vertex  $v$ .

Suppose now that  $A_v$  is virtually polycyclic. All virtually polycyclic groups are Noetherian and have solvable uniform membership problem (see for example [3]). Note that if  $H, K \leq A_v$  and  $a \in A_v$  then  $aH \cap K \neq \emptyset \iff a \in KH$ . Since  $A_v$  is virtually polycyclic,

by a result of [34] the set  $KH \subseteq A_v$  is closed in the profinite topology. Hence, given  $a \in A_v$  and finite generating sets for  $H, K$ , we can detect if  $a \notin KH$  in some finite quotient of  $A_v$ . On the other hand, we can enumerate the set  $KH$  and using the solvability of the word-problem in  $A_v$ , we can detect if  $a \in KH$ . Running this procedure parallel to enumerating all finite quotients of  $A_v$ , we can therefore decide whether or not  $a$  belongs to  $KH$ . This shows that condition (1) of Definition 5.6 holds at  $v$ . As proved in [3], there is an algorithm which, given two finitely generated subgroups of a virtually polycyclic group, computes the generating set of their intersection. Thus condition (4) of Definition 5.6 also holds at  $v$ .

We have verified that the graph of groups  $\mathbb{A}$  is benign. Hence Theorem 5.13 applies and  $G$  has solvable uniform membership problem.

By Theorem 5.8, given a finite subset  $S \subseteq G$  we can algorithmically construct a finite graph of group  $\mathbb{B}$  providing an induced splitting for  $U = \langle S \rangle \leq G = \pi_1(\mathbb{A}, v_0)$ . The vertex (and edge) groups of  $\mathbb{B}$  are given as subgroups of vertex groups of  $\mathbb{A}$  generated by some finite generating sets. By the result of Kapovich [30] if  $A_v$  is word-hyperbolic and locally quasiconvex, then there is an algorithm which, given a finite subset of  $A_v$ , produces a finite presentation for the subgroup generated by this set. The same is true for virtually polycyclic groups  $A_v$ , as proved in [3]. Hence we can recover a finite presentation of each vertex group of  $\mathbb{B}$  and thus produce a finite presentation of  $U$ , as claimed.  $\square$

Not surprisingly, we also recover (a generalization of) Mihailova's theorem regarding the membership problem for free products:

**Corollary 5.15.** *Let  $G = \pi_1(\mathbb{A}, v_0)$ , where  $\mathbb{A}$  is a finite graph of finitely generated groups such that all edge groups are finite and all vertex groups have solvable membership problem. Then  $G$  has solvable membership problem.*

*Suppose, in addition, that for each  $v \in VA$  there is an algorithm which, given a finite subset  $Y$  of  $A_v$ , produces a finite presentation for the subgroup of  $A_v$  generated by  $Y$  (thus each  $A_v$  is coherent). Then there is an algorithm which, given a finite set  $S \subseteq G$ , constructs a finite presentation for the subgroup  $U = \langle S \rangle \leq G$ .*

*Proof.* It is easy to see that  $\mathbb{A}$  is benign and hence Corollary 5.15 follows from Theorem 5.8 and Theorem 5.13.  $\square$

## 6. GRUSHKO'S THEOREM

As an application of our methods we can produce a quick proof of Grushko's Theorem [26]. Recall that for a finitely generated group  $G$  the rank  $rk(G)$  is defined as the smallest number of elements in a generating set of  $G$ . A classical result of Grushko states that rank behaves additively with respect to free products.

**Definition 6.1** (Complexity of an  $\mathbb{A}$ -graph). Let  $\mathcal{B}$  be a finite  $\mathbb{A}$ -graph.

We define the complexity of  $\mathcal{B}$  as

$$c(\mathcal{B}) := rk(\pi_1(B)) + \sum_{u \in VB} rk(B_u).$$

Recall that  $\pi_1(B)$  is a free group whose rank is equal to the number of edges in the complement of any maximal subtree of  $B$ .

**Theorem 6.2** (Grushko [26]). *Let  $G_1, G_2$  be nontrivial finitely generated groups. Then*

$$rk(G_1 * G_2) = rk(G_1) + rk(G_2)$$

*Proof.* It is obvious that  $rk(G_1 * G_2) \leq rk(G_1) + rk(G_2)$ . Thus it suffices to establish the opposite inequality.

Consider an edge of groups  $\mathbb{A}$  with a single edge  $e$ , two vertices  $v_0 = o(e), v_1 = t(e)$ , the trivial edge group  $A_e = 1$  and vertex groups  $A_{v_1} = G_1$  and  $A_{v_2} = G_2$ . Then

$$G := \pi_1(\mathbb{A}, v_0) = G_1 * G_2.$$

Let  $S$  be a generating set of  $G$  of minimal cardinality, given as a collection of  $\mathbb{A}$ -reduced paths from  $v_0$  to  $v_0$ . Thus  $\#S = rk(G)$ . Put  $(\mathcal{B}_0, u_0)$  to be the  $S$ -wedge. Notice that by construction  $c(\mathcal{B}_0) = \#S$ . We then start the abstract folding algorithm and construct a sequence of  $\mathbb{A}$ -graphs  $(\mathcal{B}_0, u_0), (\mathcal{B}_1, u_1), \dots$  by performing folding moves. Since the edge group in  $\mathbb{A}$  is trivial, moves of type  $F5 - F6$  will never occur. Each move of type  $F1 - F4$  reduces the number of edges, and hence this sequence will terminate with a folded graph  $(\mathcal{B}_n, u_n)$ . It is easy to see that moves  $F1 - F4$  do not increase the complexity and so  $c(\mathcal{B}_n) \leq c(\mathcal{B}_0) = \#S = rk(G)$ . On the other hand  $\mathcal{B}_n$  provides the induced splitting for the subgroup generated by  $S$ , that is for  $G$  itself. Thus  $\mathcal{B}_n$  recovers the original splitting  $\mathbb{A}$  of  $G$  which implies that  $c(\mathbb{B}_n) = rk(G_1) + rk(G_2)$ . Thus  $rk(G_1) + rk(G_2) \leq rk(G)$ .  $\square$

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DEPT. OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, 1409 WEST GREEN STREET, URBANA, IL 61801, USA

*E-mail address:* `kapovich@math.uiuc.edu`

FACHBEREICH MATHEMATIK, JOHANN WOLFGANG GOETHE-UNIVERSITÄT, ROBERT MAYER-STRASSE 6-8, 60325 FRANKFURT (MAIN), GERMANY

*E-mail address:* `rweidman@math.uni-frankfurt.de`

DEPARTMENT OF MATHEMATICS, CITY COLLEGE OF CUNY, NEW YORK, NY 10031, USA

*E-mail address:* `alexeim@att.net`