# Equivariant Cohomological Chern Characters 

Wolfgang Lück*<br>Fachbereich Mathematik<br>Universität Münster<br>Einsteinstr. 62<br>48149 Münster<br>Germany

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#### Abstract

We construct for an equivariant cohomology theory for proper equivariant $C W$-complexes an equivariant Chern character, provided that certain conditions about the coefficients are satisfied. These conditions are fulfilled if the coefficients of the equivariant cohomology theory possess a Mackey structure. Such a structure is present in many interesting examples.


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## 0. Introduction

The purpose of this paper is to construct an equivariant Chern character for a proper equivariant cohomology theory $\mathcal{H}_{?}^{*}$ with values in $R$-modules for a commutative associative ring $R$ with unit which satisfies $\mathbb{Q} \subseteq R$. It is a natural transformation of equivariant cohomology theories

$$
\operatorname{ch}_{?}^{*}: \mathcal{H}_{?}^{*} \rightarrow \mathcal{B} \mathcal{H}_{?}^{*}
$$

for a given equivariant cohomology theory $\mathcal{H}_{?}^{*}$. Here $\mathcal{B} \mathcal{H}_{\text {? }}^{*}$ is the associated equivariant cohomology theory which is defined by the Bredon cohomology with coefficients coming from the coefficients of $\mathcal{H}_{?}^{*}$. The notion of an equivariant cohomology theory and examples for it are presented in Section 1 and the associated Bredon cohomology is explained in Section 3 The point is that $\mathcal{B} \mathcal{H}_{\text {? }}^{*}$ is

[^0]much simpler and easier to compute than $\mathcal{H}_{?}^{*}$. If $\mathcal{H}_{?}^{*}$ satisfies the disjoint union axiom, then
$$
\operatorname{ch}_{G}^{n}(X, A): \mathcal{H}_{G}^{n}(X, A) \xrightarrow{\cong} \mathcal{B} \mathcal{H}_{G}^{n}(X, A)
$$
is bijective for every discrete group $G$, proper $G$ - $C W$-pair $(X, A)$ and $n \in \mathbb{Z}$.
The Chern character $\mathrm{ch}_{?}^{*}$ is only defined if the coefficients of $\mathcal{H}_{?}^{*}$ satisfy a certain injectivity condition (see Theorem4.6). This condition is fulfilled if the coefficients of $\mathcal{H}_{?}^{*}$ come with a Mackey structure (see Theorem 5.5) what is the case in many interesting examples.

The equivariant cohomological Chern character is a generalization to the equivariant setting of the classical non-equivariant Chern character for a (nonequivariant) cohomology theory $\mathcal{H}^{*}$ (see Example 4.1)

$$
\operatorname{ch}^{n}(X, A): \mathcal{H}^{n}(X, A) \stackrel{\cong}{\Longrightarrow} \prod_{p+q=n} H^{p}\left(X, A, \mathcal{H}^{q}(*)\right) .
$$

The equivariant cohomological Chern character has already been constructed in the special case, where $\mathcal{H}_{?}^{*}$ is equivariant topological $K$-theory $K_{?}^{*}$, in [12]. Its homological version has already been treated in [8] and plays an important role in the computation of the source of the assembly maps appearing in the Farrell-Jones Conjecture for $K_{n}(R G)$ and $L_{n}^{\langle-\infty\rangle}(R G)$ and the Baum-Connes Conjecture for $K_{n}\left(C_{r}^{*}(G)\right)$ (see also [9]).

The detailed formulation of the main result of this paper is presented in Theorem 5.5

The equivariant Chern character will play a key role in the proof of the following result which will be presented in [10].
Theorem 0.1 (Rational computation of the topological $K$-theory of $B G)$. Let $G$ be a discrete group. Suppose that there is a finite $G$ - $C W$-model for the classifying space $\underline{E} G$ for proper $G$-actions. Then there is a $\mathbb{Q}$-isomorphism, natural in $G$ and compatible with the multiplicative structures

$$
\begin{aligned}
& \overline{\operatorname{ch}}_{G}^{n}: K^{n}(B G) \otimes_{\mathbb{Z}} \mathbb{Q} \\
& \stackrel{\cong}{\Longrightarrow} \prod_{i \in \mathbb{Z}} H^{2 i+n}(B G ; \mathbb{Q}) \times \prod_{p \text { prime }} \prod_{(g) \in \operatorname{con}_{p}(G)} H^{2 i+n}\left(B C_{G}\langle g\rangle ; \mathbb{Q}_{p}\right) .
\end{aligned}
$$

Here $\operatorname{con}_{p}(G)$ is the set of conjugacy classes (g) of elements $g \in G$ of order $p^{m}$ for some integer $m \geq 1$ and $C_{G}\langle g\rangle$ is the centralizer of the cyclic subgroup generated by $g$ in $G$.

The assumption in Theorem 0.1 that there is a finite $G$ - $C W$-model for the classifying space $\underline{E} G$ for proper $G$-actions is satisfied for instance, if $G$ is wordhyperbolic in the sense of Gromov, if $G$ is a cocompact subgroup of a Lie group with finitely many path components, if $G$ is a finitely generated one-relator group, if $G$ is an arithmetic group, a mapping class group of a compact surface or the group of outer automorphisms of a finitely generated free group. For more information about $\underline{E} G$ we refer for instance to [1] and 11].

A group $G$ is always understood to be discrete and a ring $R$ is always understood to be associative with unit throughout this paper.

The paper is organized as follows:
1 Equivariant Cohomology Theories
2 Modules over a Category
3 The Associated Bredon Cohomology Theory
4 The Construction of the Equivariant Cohomological Chern Character
55 Mackey Functors
6 Multiplicative Structures References

## 1. Equivariant Cohomology Theories

In this section we describe the axioms of a (proper) equivariant cohomology theory. They are dual to the ones of a (proper) equivariant homology theory as described in [8, Section 1].

Fix a group $G$ and an commutative ring $R$. A $G$ - $C W$-pair $(X, A)$ is a pair of $G$ - $C W$-complexes. It is proper if all isotropy groups of $X$ are finite. It is relative finite if $X$ is obtained from $A$ by attaching finitely many equivariant cells, or, equivalently, if $G \backslash(X / A)$ is compact. Basic information about $G$ - $C W$-pairs can be found for instance in [7] Section 1 and 2]. A $G$-cohomology theory $\mathcal{H}_{G}^{*}$ with values in $R$-modules is a collection of covariant functors $\mathcal{H}_{G}^{n}$ from the category of $G$ - $C W$-pairs to the category of $R$-modules indexed by $n \in \mathbb{Z}$ together with natural transformations $\delta_{G}^{n}(X, A): \mathcal{H}_{G}^{n}(X, A) \rightarrow \mathcal{H}_{G}^{n+1}(A):=\mathcal{H}_{G}^{n+1}(A, \emptyset)$ for $n \in \mathbb{Z}$ such that the following axioms are satisfied:

- $G$-homotopy invariance

If $f_{0}$ and $f_{1}$ are $G$-homotopic maps $(X, A) \rightarrow(Y, B)$ of $G$-CW-pairs, then $\mathcal{H}_{G}^{n}\left(f_{0}\right)=\mathcal{H}_{G}^{n}\left(f_{1}\right)$ for $n \in \mathbb{Z}$;

- Long exact sequence of a pair

Given a pair $(X, A)$ of $G$ - $C W$-complexes, there is a long exact sequence

$$
\ldots \xrightarrow{\delta_{G}^{n-1}} \mathcal{H}_{G}^{n}(X, A) \xrightarrow{\mathcal{H}_{G}^{n}(j)} \mathcal{H}_{G}^{n}(X) \xrightarrow{\mathcal{H}_{G}^{n}(i)} \mathcal{H}_{G}^{n}(A) \xrightarrow{\delta_{G}^{n}} \ldots,
$$

where $i: A \rightarrow X$ and $j: X \rightarrow(X, A)$ are the inclusions;

- Excision

Let $(X, A)$ be a $G$ - $C W$-pair and let $f: A \rightarrow B$ be a cellular $G$-map of $G$ - $C W$-complexes. Equip $\left(X \cup_{f} B, B\right)$ with the induced structure of a $G$ $C W$-pair. Then the canonical map $(F, f):(X, A) \rightarrow\left(X \cup_{f} B, B\right)$ induces an isomorphism

$$
\mathcal{H}_{G}^{n}(F, f): \mathcal{H}_{G}^{n}(X, A) \stackrel{ }{\cong} \mathcal{H}_{G}^{n}\left(X \cup_{f} B, B\right) .
$$

Sometimes also the following axiom is required.

- Disjoint union axiom

Let $\left\{X_{i} \mid i \in I\right\}$ be a family of $G$ - $C W$-complexes. Denote by $j_{i}: X_{i} \rightarrow$ $\coprod_{i \in I} X_{i}$ the canonical inclusion. Then the map

$$
\prod_{i \in I} \mathcal{H}_{G}^{n}\left(j_{i}\right): \mathcal{H}_{G}^{n}\left(\coprod_{i \in I} X_{i}\right) \stackrel{\cong}{\leftrightarrows} \prod_{i \in I} \mathcal{H}_{G}^{n}\left(X_{i}\right)
$$

is bijective.
If $\mathcal{H}_{G}^{*}$ is defined or considered only for proper $G$ - $C W$-pairs $(X, A)$, we call it a proper $G$-cohomology theory $\mathcal{H}_{G}^{*}$ with values in $R$-modules.

The role of the disjoint union axiom is explained by the following result. Its proof for non-equivariant cohomology theories (see for instance [16, 7.66 and $7.67]$ ) carries over directly to $G$-cohomology theories.

Lemma 1.1. Let $\mathcal{H}_{G}^{*}$ and $\mathcal{K}_{G}^{*}$ be (proper) $G$-cohomology theories. Then
(a) Suppose that $\mathcal{H}_{G}^{*}$ satisfies the disjoint union axiom. Then there exists for every (proper) G-CW-pair $(X, A)$ a natural short exact sequence

$$
0 \rightarrow \lim _{n \rightarrow \infty}^{1} \mathcal{H}_{G}^{p-1}\left(X_{n} \cup A, A\right) \rightarrow \mathcal{H}^{p}(X, A) \rightarrow \lim _{n \rightarrow \infty} \mathcal{H}_{G}^{p}\left(X_{n} \cup A, A\right) \rightarrow 0
$$

(b) Let $T^{*}: \mathcal{H}_{G}^{*} \rightarrow \mathcal{K}_{G}^{*}$ be a transformation of (proper) $G$-cohomology theories, i.e. a collection of natural transformations $T^{n}: \mathcal{H}_{G}^{n} \rightarrow \mathcal{K}_{G}^{n}$ of contravariant functors from the category of (proper) G-CW-pairs to the category of $R$-modules indexed by $n \in \mathbb{Z}$ which is compatible with the boundary operator associated to (proper) G-CW-pairs. Suppose that $T^{n}(G / H)$ is bijective for every (proper) homogeneous space $G / H$ and $n \in \mathbb{Z}$.
Then $T^{n}(X, A): \mathcal{H}_{G}^{*}(X, A) \rightarrow \mathcal{K}_{G}^{*}(X, A)$ is bijective for all $n \in \mathbb{Z}$ provided that $(X, A)$ is relative finite or that both $\mathcal{H}^{*}$ and $\mathcal{K}^{*}$ satisfy the disjoint union axiom.

Remark 1.2 (The disjoint union axiom is not compatible with $-\otimes_{\mathbb{Z}} \mathbb{Q}$ ). Let $\mathcal{H}_{G}^{*}$ be a $G$-cohomology theory with values in $\mathbb{Z}$-modules. Then $\mathcal{H}_{G}^{*} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $G$-cohomology theory with values in $\mathbb{Q}$-modules since $\mathbb{Q}$ is flat as $\mathbb{Z}$-module. However, even if $\mathcal{H}^{*}$ satisfies the disjoint union axiom, $\mathcal{H}_{G}^{*} \otimes_{\mathbb{Z}} \mathbb{Q}$ does not satisfy the disjoint union axiom since $-\otimes_{\mathbb{Z}} \mathbb{Q}$ is not compatible with products over arbitrary index sets.

Example 1.3 (Rationalizing topological $K$-theory). Consider for instance the (non-equivariant) cohomology theory with values in $\mathbb{Z}$-modules satisfying the disjoint union axiom given by topological $K$-theory $K^{*}$. Let $K^{*}(-; \mathbb{Q})$ be the cohomology theory associated to the rationalization of the $K$-theory
spectrum. This is a (non-equivariant) cohomology theory with values in $\mathbb{Q}$ modules satisfying the disjoint union axiom. There is a natural transformation

$$
T^{*}(X): K^{*}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K^{*}(X ; \mathbb{Q})
$$

of (non-equivariant) cohomology theory with values in $\mathbb{Q}$-modules. The $\mathbb{Q}$-map $T^{n}(\{\mathrm{pt}\}$.$) is bijective for all n \in \mathbb{Z}$. Hence $T^{n}(X)$ is bijective for all finite $C W$-complexes by Lemma 1.1 (b). Notice that $K^{*}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ does not satisfy the disjoint union axiom in contrast to $K^{*}(X ; \mathbb{Q})$. Hence we cannot expect $T^{n}(X)$ to be bijective for all $C W$-complexes. Consider the case $X=B G$ for a finite group $G$. Since $H^{p}(B G ; \mathbb{Q}) \cong H^{p}(\{\mathrm{pt}.\} ; \mathbb{Q})$ for all $p \in \mathbb{Z}$, one obtains $K^{p}(B G ; \mathbb{Q}) \cong K^{p}(\{\mathrm{pt}.\} ; \mathbb{Q})$ for all $p \in \mathbb{Z}$. By the Atiyah-Segal Completion Theorem $K^{p}(B G) \otimes_{\mathbb{Z}} \mathbb{Q} \cong K^{p}(\{\mathrm{pt}\}.) \otimes_{\mathbb{Z}} \mathbb{Q}$ is only true if and only if the finite group $G$ is trivial.

Let $\alpha: H \rightarrow G$ be a group homomorphism. Given an $H$-space $X$, define the induction of $X$ with $\alpha$ to be the $G$-space $\operatorname{ind}_{\alpha} X$ which is the quotient of $G \times X$ by the right $H$-action $(g, x) \cdot h:=\left(g \alpha(h), h^{-1} x\right)$ for $h \in H$ and $(g, x) \in G \times X$. If $\alpha: H \rightarrow G$ is an inclusion, we also write $\operatorname{ind}_{H}^{G}$ instead of $\operatorname{ind}_{\alpha}$.

A (proper) equivariant cohomology theory $\mathcal{H}_{?}^{*}$ with values in $R$-modules consists of a collection of (proper) $G$-cohomology theory $\mathcal{H}_{G}^{*}$ with values in $R$ modules for each group $G$ together with the following so called induction structure: given a group homomorphism $\alpha: H \rightarrow G$ and a (proper) $H$ - $C W$-pair $(X, A)$ such that $\operatorname{ker}(\alpha)$ acts freely on $X$, there are for each $n \in \mathbb{Z}$ natural isomorphisms

$$
\begin{equation*}
\operatorname{ind}_{\alpha}: \mathcal{H}_{G}^{n}\left(\operatorname{ind}_{\alpha}(X, A)\right) \quad \cong \mathcal{H}_{H}^{n}(X, A) \tag{1.4}
\end{equation*}
$$

satisfying
(a) Compatibility with the boundary homomorphisms
$\delta_{H}^{n} \circ \operatorname{ind}_{\alpha}=\operatorname{ind}_{\alpha} \circ \delta_{G}^{n} ;$
(b) Functoriality

Let $\beta: G \rightarrow K$ be another group homomorphism such that $\operatorname{ker}(\beta \circ \alpha)$ acts freely on $X$. Then we have for $n \in \mathbb{Z}$

$$
\operatorname{ind}_{\beta \circ \alpha}=\operatorname{ind}_{\alpha} \circ \operatorname{ind}_{\beta} \circ \mathcal{H}_{K}^{n}\left(f_{1}\right): \mathcal{H}_{H}^{n}\left(\operatorname{ind}_{\beta \circ \alpha}(X, A)\right) \rightarrow \mathcal{H}_{K}^{n}(X, A),
$$

where $f_{1}: \operatorname{ind}_{\beta} \operatorname{ind}_{\alpha}(X, A) \xrightarrow{\cong} \operatorname{ind}_{\beta \circ \alpha}(X, A),(k, g, x) \mapsto(k \beta(g), x)$ is the natural $K$-homeomorphism;
(c) Compatibility with conjugation

For $n \in \mathbb{Z}, g \in G$ and a (proper) $G$ - $C W$-pair $(X, A)$ the homomorphism $\operatorname{ind}_{c(g): G \rightarrow G}: \mathcal{H}_{G}^{n}\left(\operatorname{ind}_{c(g)}: G \rightarrow G(X, A)\right) \rightarrow \mathcal{H}_{G}^{n}(X, A)$ agrees with $\mathcal{H}_{G}^{n}\left(f_{2}\right)$, where $f_{2}$ is the $G$-homeomorphism $f_{2}:(X, A) \rightarrow \operatorname{ind}_{c(g)}: G \rightarrow G(X, A), x \mapsto$ $\left(1, g^{-1} x\right)$ and $c(g)\left(g^{\prime}\right)=g g^{\prime} g^{-1}$.

This induction structure links the various $G$-cohomology theories for different groups $G$. It will play a key role in the construction of the equivariant Chern character even if we want to carry it out only for a fixed group $G$. In all of the relevant examples the induction homomorphism $\operatorname{ind}_{\alpha}$ of (1.4) exists for every group homomorphism $\alpha: H \rightarrow G$, the condition that $\operatorname{ker}(\alpha)$ acts freely on $X$ is only needed to ensure that $\operatorname{ind}_{\alpha}$ is bijective. If $\alpha$ is an inclusion, we sometimes write $\operatorname{ind}_{H}^{G}$ instead of $\operatorname{ind}_{\alpha}$.

We say that $\mathcal{H}_{?}^{*}$ satisfies the disjoint union axiom if for every group $G$ the $G$-cohomology theory $\mathcal{H}_{G}^{*}$ satisfies the disjoint union axiom.

We will later need the following lemma whose elementary proof is analogous to the one in [8, Lemma 1.2].

Lemma 1.5. Consider finite subgroups $H, K \subseteq G$ and an element $g \in G$ with $g H^{-1} \subseteq K$. Let $R_{g^{-1}}: G / H \rightarrow G / K$ be the $G$-map sending $g^{\prime} H$ to $g^{\prime} g^{-1} K$ and $\bar{c}(g): H \rightarrow K$ be the homomorphism sending $h$ to $\mathrm{ghg}^{-1}$. Let $\mathrm{pr}:\left(\operatorname{ind}_{c(g)}: H \rightarrow K\{\mathrm{pt}\}.\right) \rightarrow\{\mathrm{pt}$.$\} be the projection. Then the following diagram$ commutes

$$
\begin{array}{lll}
\mathcal{H}_{G}^{n}(G / K) & \xrightarrow{\mathcal{H}_{G}^{n}\left(R_{g-1}\right)} & \mathcal{H}_{G}^{n}(G / H) \\
\operatorname{ind}_{K}^{G} \downarrow & & \operatorname{ind}_{H}^{G} \downarrow \cong \\
\mathcal{H}_{K}^{n}(\{\mathrm{pt} .\}) & \xrightarrow{\operatorname{ind}_{c(g)} \circ \mathcal{H}_{K}^{n}(\mathrm{pr})} & \mathcal{H}_{H}^{n}(\{\mathrm{pt} .\})
\end{array}
$$

Example 1.6 (Borel cohomology). Let $\mathcal{K}^{*}$ be a cohomology theory for (nonequivariant) $C W$-pairs with values in $R$-modules. Examples are singular cohomology and topological $K$-theory. Then we obtain two equivariant cohomology theories with values in $R$-modules by the following constructions

$$
\begin{aligned}
\mathcal{H}_{G}^{n}(X, A) & =\mathcal{K}^{n}(G \backslash X, G \backslash A) \\
\mathcal{H}_{G}^{n}(X, A) & =\mathcal{K}^{n}\left(E G \times_{G}(X, A)\right)
\end{aligned}
$$

The second one is called the equivariant Borel cohomology associated to $\mathcal{K}$. In both cases $\mathcal{H}_{G}^{*}$ inherits the structure of a $G$-cohomology theory from the cohomology structure on $\mathcal{K}^{*}$.

The induction homomorphism associated to a group homomorphism $\alpha: H \rightarrow$ $G$ is defined as follows. Let $a: H \backslash X \stackrel{\cong}{\rightrightarrows} G \backslash\left(G \times_{\alpha} X\right)$ be the homeomorphism sending $H x$ to $G(1, x)$. Define $b: E H \times{ }_{H} X \rightarrow E G \times_{G} G \times_{\alpha} X$ by sending $(e, x)$ to $(E \alpha(e), 1, x)$ for $e \in E H, x \in X$ and $E \alpha: E H \rightarrow E G$ the $\alpha$-equivariant map induced by $\alpha$. The desired induction map $\operatorname{ind}_{\alpha}$ is given by $\mathcal{K}^{*}(a)$ and $\mathcal{K}^{*}(b)$. If the kernel $\operatorname{ker}(\alpha)$ acts freely on $X$, the map $b$ is a homotopy equivalence and hence in both cases $\operatorname{ind}_{\alpha}$ is bijective.

If $\mathcal{K}^{*}$ satisfies the disjoint union axiom, the same is true for the two equivariant cohomology theories constructed above.

Example 1.7 (Equivariant $K$-theory). In $12 G$-equivariant topological (complex) $K$-theory $K_{G}^{*}(X, A)$ is constructed for any proper $G$ - $C W$-pair ( $X, A$ ) and shown that $K_{G}^{*}$ defines a proper $G$-cohomology theory satisfying the disjoint
union axiom. Given a group homomorphism $\alpha: H \rightarrow G$, it induces an injective group homomorphism $\bar{\alpha}: H / \operatorname{ker}(\alpha) \rightarrow G$. Let

$$
\operatorname{Infl}_{H / \operatorname{ker}(\alpha)}^{H}: K_{H / \operatorname{ker}(\alpha)}^{*}(\operatorname{ker}(\alpha) \backslash X) \rightarrow K_{H}^{*}(X)
$$

be the inflation homomorphism of [12, Proposition 3.3] and

$$
\operatorname{ind}_{\bar{\alpha}}: K_{H / \operatorname{ker}(\alpha)}^{*}(\operatorname{ker}(\alpha) \backslash X) \xrightarrow{\cong} K_{G}^{*}\left(\operatorname{ind}_{\bar{\alpha}}(\operatorname{ker}(\alpha) \backslash X)\right)
$$

be the induction isomorphism of [12] Proposition 3.2 (b)]. Define the induction homomorphism

$$
\operatorname{ind}_{\alpha}: K_{G}^{*}\left(\operatorname{ind}_{\alpha} X\right) \rightarrow K_{H}^{*}(X)
$$

by $\operatorname{Infl}_{H / \operatorname{ker}(\alpha)}^{H} \circ\left(\operatorname{ind}_{\bar{\alpha}}\right)^{-1}$, where we identify $\operatorname{ind}_{\alpha} X=\operatorname{ind}_{\bar{\alpha}}(\operatorname{ker}(\alpha) \backslash X)$. On the level of complex finite-dimensional vector bundles the induction homomorphism $\operatorname{ind}_{\alpha}$ corresponds to considering for a $G$-vector bundle $E$ over $G \times_{\alpha} X$ the $H$ vector bundle obtained from $E$ by the pullback construction associated to the $\alpha$-equivariant map $X \rightarrow G \times{ }_{\alpha} X, x \mapsto(1, x)$.

Thus we obtain a proper equivariant cohomology theory $K_{?}^{*}$ with values in $\mathbb{Z}$-modules which satisfies the disjoint union axiom. There is also a real version $K O_{?}^{*}$.

Example 1.8 (Equivariant cohomology theories and spectra). Denote by GROUPOIDS the category of small groupoids. Let $\Omega$-SPECTRA be the category of $\Omega$-spectra, where a morphism $\mathbf{f}: \mathbf{E} \rightarrow \mathbf{F}$ is a sequence of maps $f_{n}: E_{n} \rightarrow F_{n}$ compatible with the structure maps and we work in the category of compactly generated spaces (see for instance [3 Section 1]). A contravariant GROUPOIDS- $\Omega$-spectrum is a contravariant functor $\mathbf{E}$ : GROUPOIDS $\rightarrow$ $\Omega$-SPECTRA.

Next we explain how we can associate to it an equivariant cohomology theory $H_{?}^{*}(-; \mathbf{E})$ satisfying the disjoint union axiom, provided that $\mathbf{E}$ respects equivalences, i.e. it sends an equivalence of groupoids to a weak equivalence of spectra. This construction is dual to the construction of an equivariant homology theory associated to a covariant GROUPOIDS-spectrum as explained in 13, Section 6.2], 14 Theorem 2.10 on page 21].

Fix a group $G$. We have to specify a $G$-cohomology theory $\mathcal{H}_{G}^{*}(-; \mathbf{E})$. Let $\operatorname{Or}(G)$ be the orbit category whose set of objects consists of homogeneous $G$ spaces $G / H$ and whose morphisms are $G$-maps. For a $G$-set $S$ we denote by $\mathcal{G}^{G}(S)$ its associated transport groupoid. Its objects are the elements of $S$. The set of morphisms from $s_{0}$ to $s_{1}$ consists of those elements $g \in G$ which satisfy $g s_{0}=s_{1}$. Composition in $\mathcal{G}^{G}(S)$ comes from the multiplication in $G$. Thus we obtain for a group $G$ a covariant functor

$$
\begin{equation*}
\mathcal{G}^{G}: \operatorname{Or}(G) \rightarrow \text { GROUPOIDS, } \quad G / H \mapsto \mathcal{G}^{G}(G / H) \tag{1.9}
\end{equation*}
$$

and a contravariant $\operatorname{Or}(G)-\Omega$-spectrum $\mathbf{E} \circ \mathcal{G}^{G}$. Given a $G$ - $C W$-pair $(X, A)$, we obtain a contravariant pair of $\operatorname{Or}(G)-C W$-complexes $\left(X^{?}, A^{?}\right)$ by sending $G / H$
to $\left(\operatorname{map}_{G}(G / H, X), \operatorname{map}_{G}(G / H, A)\right)=\left(X^{H}, A^{H}\right)$. The contravariant $\operatorname{Or}(G)$ spectrum $\mathbf{E} \circ \mathcal{G}^{G}$ defines a cohomology theory on the category of contravariant $\operatorname{Or}(G)-C W$-complexes as explained in [3] Section 4]. It value at ( $X^{?}, A^{?}$ ) is defined to be $H_{G}^{*}(X, A ; \mathbf{E})$. Explicitely, $H_{G}^{n}(X, A ; \mathbf{E})$ is the $(-n)$-th homotopy group of the spectrum $\operatorname{map}_{\operatorname{Or}(G)}\left(X_{+}^{?} \cup_{A_{+}^{?}} \operatorname{cone}\left(A_{+}^{?}\right), \mathbf{E} \circ \mathcal{G}^{G}\right)$. We need $\Omega$-spectra in order to ensure that the disjoint union axiom holds.

We briefly explain for a group homomorphism $\alpha: H \rightarrow G$ the definition of the induction homomorphism $\operatorname{ind}_{\alpha}: \mathcal{H}_{G}^{n}\left(\operatorname{ind}_{\alpha} X ; \mathbf{E}\right) \rightarrow \mathcal{H}_{H}^{n}(X ; \mathbf{E})$ in the special case $A=\emptyset$. The functor induced by $\alpha$ on the orbit categories is denoted in the same way

$$
\alpha: \operatorname{Or}(H) \rightarrow \operatorname{Or}(G), \quad H / L \mapsto \operatorname{ind}_{\alpha}(H / L)=G / \alpha(L)
$$

There is an obvious natural transformation of functors $\operatorname{Or}(H) \rightarrow$ GROUPOIDS

$$
T: \mathcal{G}^{H} \rightarrow \mathcal{G}^{G} \circ \alpha
$$

Its evaluation at $H / L$ is the functor of groupoids $\mathcal{G}^{H}(H / L) \rightarrow \mathcal{G}^{G}(G / \alpha(L))$ which sends an object $h L$ to the object $\alpha(h) \alpha(L)$ and a morphism given by $h \in H$ to the morphism $\alpha(h) \in G$. Notice that $T(H / L)$ is an equivalence if $\operatorname{ker}(\alpha)$ acts freely on $H / L$. The desired isomorphism

$$
\operatorname{ind}_{\alpha}: H_{G}^{n}\left(\operatorname{ind}_{\alpha} X ; \mathbf{E}\right) \rightarrow H_{H}^{n}(X ; \mathbf{E})
$$

is induced by the following map of spectra

$$
\begin{aligned}
\operatorname{map}_{\mathrm{Or}(G)}\left(\operatorname{map}_{G}(-\right. & \left.\left., \operatorname{ind}_{\alpha} X_{+}\right), \mathbf{E} \circ \mathcal{G}^{G}\right) \\
\stackrel{\cong}{\cong} & \operatorname{map}_{\mathrm{Or}(G)}\left(\alpha_{*}\left(\operatorname{map}_{H}\left(-, X_{+}\right)\right), \mathbf{E} \circ \mathcal{G}^{G}\right) \\
\cong & \operatorname{map}_{\mathrm{Or}(H)}\left(\operatorname{map}_{H}\left(-, X_{+}\right), \mathbf{E} \circ \mathcal{G}^{G} \circ \alpha\right) \\
& \xrightarrow{\operatorname{map}_{\mathrm{Or}(H)}(\operatorname{id}, \mathbf{E}(T))} \\
& \operatorname{map}_{\mathrm{Or}(H)}\left(\operatorname{map}_{H}\left(-, X_{+}\right), \mathbf{E} \circ \mathcal{G}^{H}\right) .
\end{aligned}
$$

Here $\alpha_{*} \operatorname{map}_{H}\left(-, X_{+}\right)$is the pointed $\operatorname{Or}(G)$-space which is obtained from the pointed $\operatorname{Or}(H)$-space $\operatorname{map}_{H}\left(-, X_{+}\right)$by induction, i.e. by taking the balanced product over $\operatorname{Or}(H)$ with the $\operatorname{Or}(H)-\mathrm{Or}(G)$ bimodule $\operatorname{mor}_{\mathrm{Or}(G)}(? ?, \alpha(?)$ ) 3 Definition 1.8]. The second map is given by the adjunction homeomorphism of induction $\alpha_{*}$ and restriction $\alpha^{*}$ (see [3, Lemma 1.9]). The first map comes from the homeomorphism of $\operatorname{Or}(G)$-spaces

$$
\alpha_{*} \operatorname{map}_{H}\left(-, X_{+}\right) \rightarrow \operatorname{map}_{G}\left(-, \operatorname{ind}_{\alpha} X_{+}\right)
$$

which is the adjoint of the obvious map of $\operatorname{Or}(H)$-spaces $\operatorname{map}_{H}\left(-, X_{+}\right) \rightarrow$ $\alpha^{*} \operatorname{map}_{G}\left(-, \operatorname{ind}_{\alpha} X_{+}\right)$whose evaluation at $H / L$ is given by $\operatorname{ind}_{\alpha}$.

## 2. Modules over a Category

In this section we give a brief summary about modules over a small category $\mathcal{C}$ as far as needed for this paper. They will appear in the definition of the equivariant Chern character.

Let $\mathcal{C}$ be a small category and let $R$ be a commutative ring. A contravariant $R \mathcal{C}$-module is a contravariant functor from $\mathcal{C}$ to the category $R$ - MOD of $R$ modules. Morphisms of contravariant $R \mathcal{C}$-modules are natural transformations. Given a group $G$, let $\widehat{G}$ be the category with one object whose set of morphisms is given by $G$. Then a contravariant $R \widehat{G}$-module is the same as a right $R G$ module. Therefore we can identify the abelian category MOD- $R \widehat{G}$ with the abelian category of right $R G$-modules MOD- $R G$ in the sequel. Many of the constructions, which we will introduce for $R \mathcal{C}$-modules below, reduce in the special case $\mathcal{C}=\widehat{G}$ to their classical versions for $R G$-modules. The reader should have this example in mind. There is also a covariant version. In the sequel $R \mathcal{C}$ module means contravariant $R \mathcal{C}$-module unless stated explicitly differently.

The category MOD- $R \mathcal{C}$ of $R \mathcal{C}$-modules inherits the structure of an abelian category from $R$ - MOD in the obvious way, namely objectwise. For instance a sequence $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$ of contravariant $R \mathcal{C}$-modules is called exact if its evaluation at each object in $\mathcal{C}$ is an exact sequence in $R$-MOD. The notion of an injective and of a projective $R \mathcal{C}$-module is now clear. For a set $S$ denote by $R S$ the free $R$-module with $S$ as basis. An $R \mathcal{C}$-module is free if it is isomorphic to $R \mathcal{C}$-module of the shape $\bigoplus_{i \in I} R \operatorname{mor}_{\mathcal{C}}\left(?, c_{i}\right)$ for some index set $I$ and objects $c_{i} \in \mathcal{C}$. Notice that by the Yoneda-Lemma there is for every $R \mathcal{C}$-module $N$ and every object $c$ a bijection of sets

$$
\operatorname{hom}_{R \mathcal{C}}\left(R \operatorname{mor}_{\mathcal{C}}(?, c), N\right) \xrightarrow{\cong} N(c), \quad \phi \mapsto \phi\left(\operatorname{id}_{x}\right) .
$$

This implies that every free $R \mathcal{C}$-module is projective and a $R \mathcal{C}$-module is projective if and only if it is a direct summand in a free $R \mathcal{C}$-module. The category of $R \mathcal{C}$-modules has enough projectives and injectives (see [7] Lemma 17.1] and [17. Example 2.3.13]).

Given a contravariant $R \mathcal{C}$-module $M$ and a covariant $R \mathcal{C}$-module $N$, their tensor product over $R \mathcal{C}$ is defined to be the following $R$-module $M \otimes_{R \mathcal{C}} N$. It is given by

$$
M \otimes_{R \mathcal{C}} N=\bigoplus_{c \in \mathrm{Ob}(\mathcal{C})} M(c) \otimes_{R} N(c) / \sim,
$$

where $\sim$ is the typical tensor relation $m f \otimes n=m \otimes f n$, i.e. for every morphism $f: c \rightarrow d$ in $\mathcal{C}, m \in M(d)$ and $n \in N(c)$ we introduce the relation $M(f)(m) \otimes$ $n-m \otimes N(f)(n)=0$. The main property of this construction is that it is adjoint to the $\operatorname{hom}_{R}$-functor in the sense that for any $R$-module $L$ there are natural isomorphisms of $R$-modules

$$
\begin{align*}
\operatorname{hom}_{R}\left(M \otimes_{R \mathcal{C}} N, L\right) & \cong \operatorname{hom}_{R \mathcal{C}}\left(M, \operatorname{hom}_{R}(N, L)\right) ;  \tag{2.1}\\
\operatorname{hom}_{R}\left(M \otimes_{R \mathcal{C}} N, L\right) & \cong \operatorname{hom}_{R \mathcal{C}}\left(N, \operatorname{hom}_{R}(M, L)\right) . \tag{2.2}
\end{align*}
$$

Consider a functor $F: \mathcal{C} \rightarrow \mathcal{D}$. Given a $R \mathcal{D}$-module $M$, define its restriction with $F$ to be $F^{*} M:=M \circ F$. Given a contravariant $R \mathcal{C}$-module $M$, its induction with $F$ is the contravariant $R \mathcal{D}$-module $F_{*} M$ given by

$$
\begin{equation*}
\left(F_{*} M\right)(? ?):=M(?) \otimes_{R \mathcal{C}} R \operatorname{mor}_{\mathcal{D}}(? ?, F(?)) \tag{2.3}
\end{equation*}
$$

and coinduction with $F$ is the contravariant $R \mathcal{D}$-module $F_{!} M$ given by

$$
\begin{equation*}
(F!M)(? ?):=\operatorname{hom}_{R \mathcal{C}}\left(R \operatorname{mor}_{\mathcal{D}}(F(?), ? ?), M(?)\right) \tag{2.4}
\end{equation*}
$$

Restriction with $F$ can be written as $F^{*} N(?)=\operatorname{hom}_{R \mathcal{D}}\left(R \operatorname{mor}_{\mathcal{D}}(? ?, F(?)), N(? ?)\right)$, the natural isomorphisms sends $n \in N(F(?))$ to the map

$$
R \operatorname{mor}_{\mathcal{D}}(? ?, F(?)) \rightarrow N(? ?), \quad \phi: ? ? \rightarrow F(?) \mapsto N(\phi)(n)
$$

Restriction with $F$ can also be written as $F^{*} N(?)=R \operatorname{mor}_{\mathcal{D}}(F(?), ? ?) \otimes_{R \mathcal{D}}$ $N(? ?)$, the natural isomorphisms sends $\phi \otimes_{R \mathcal{D}} n$ to $N(\phi)(n)$. We conclude from (2.2) that $\left(F_{*}, F^{*}\right)$ and $\left(F^{*}, F_{!}\right)$form adjoint pairs, i.e. for a $R \mathcal{C}$-module $M$ and a $R \mathcal{D}$-module $N$ there are natural isomorphisms of $R$-modules

$$
\begin{align*}
\operatorname{hom}_{R \mathcal{D}}\left(F_{*} M, N\right) & \cong \operatorname{hom}_{R \mathcal{C}}\left(M, F^{*} N\right) ;  \tag{2.5}\\
\operatorname{hom}_{R \mathcal{D}}\left(F^{*} N, M\right) & \cong \operatorname{hom}_{R \mathcal{C}}\left(N, F_{!} M\right) . \tag{2.6}
\end{align*}
$$

Consider an object $c$ in $\mathcal{C}$. Let aut $(c)$ be the group of automorphism of $c$. We can think of $\widehat{\operatorname{aut}(c)}$ as a subcategory of $\mathcal{C}$ in the obvious way. Denote by

$$
i(c): \widehat{\operatorname{aut}(c)} \rightarrow \mathcal{C}
$$

the inclusion of categories and abbreviate the group ring $R[\operatorname{aut}(c)]$ by $R[c]$ in the sequel. Thus we obtain functors

$$
\begin{align*}
i(c)^{*}: \mathrm{MOD}-R \mathcal{C} & \rightarrow \text { MOD }-R[c]  \tag{2.7}\\
i(c)_{*}: \mathrm{MOD}-R[c] & \rightarrow \text { MOD }-R \mathcal{C}  \tag{2.8}\\
i(c)_{!}: \operatorname{MOD}-R[c] & \rightarrow \text { MOD }-R \mathcal{C} \tag{2.9}
\end{align*}
$$

The projective splitting functor

$$
\begin{equation*}
S_{c}: \mathrm{MOD}-R \mathcal{C} \rightarrow \mathrm{MOD}-R[c] \tag{2.10}
\end{equation*}
$$

sends $M$ to the cokernel of the map


The injective splitting functor

$$
\begin{equation*}
T_{c}: \mathrm{MOD}-R \mathcal{C} \quad \rightarrow \mathrm{MOD}-R[c] \tag{2.11}
\end{equation*}
$$

sends $M$ to the kernel of the map

$$
\prod_{\substack{f: d \rightarrow c \\ f \text { not an isomorphism }}} M(f): M(c) \rightarrow \prod_{\substack{f: d \rightarrow c \\ f \text { not an isomorphism }}} M(d) .
$$

From now on suppose that $\mathcal{C}$ is an EI-category, i.e. a small category such that endomorphisms are isomorphisms. Then we can define the inclusion functor

$$
\begin{equation*}
I_{c}: \mathrm{MOD}-R[c] \quad \rightarrow \text { MOD }-R \mathcal{C} \tag{2.12}
\end{equation*}
$$

by $I_{c}(M)(?)=M \otimes_{R[c]} R \operatorname{mor}(?, c)$ if $c \cong ?$ in $\mathcal{C}$ and by $I_{c}(M)(?)=0$ otherwise. Let $B$ be the $R \mathcal{C}-R[c]$-bimodule, covariant over $\mathcal{C}$ and a right module over $R[c]$, given by

$$
B(c, ?)=\begin{array}{ll}
R \operatorname{mor}_{\mathcal{C}}(c, ?) & \text { if } c \cong ? ; \\
0 & \text { if } c \neq ?
\end{array}
$$

Let $C$ be the $R[c]-R \mathcal{C}$-bimodule, contravariant over $\mathcal{C}$ and a left module over $R[c]$, given by

$$
C(?, c)=\begin{array}{ll}
R \operatorname{mor}_{\mathcal{C}}(?, c) & \text { if } c \cong ? ; \\
0 & \text { if } c \not \approx ?
\end{array}
$$

One easily checks that there are natural isomorphisms

$$
\begin{aligned}
S_{c} M & \cong M \otimes_{R C} B \\
I_{c} N & \cong \operatorname{hom}_{R[c]}(B, N) \\
T_{c} M & \cong \operatorname{hom}_{R C}(C, M) \\
I_{c} N & \cong N \otimes_{R[c]} C
\end{aligned}
$$

Lemma 2.13. Let $\mathcal{C}$ be an EI-category and $c, d$ objects in $\mathcal{C}$.
(a) We obtain adjoint pairs $\left(i(c)_{*}, i(c)^{*}\right),\left(i(c)^{*}, i(c)!\right),\left(S_{c}, I_{c}\right)$ and $\left(I_{c}, T_{c}\right)$;
(b) There are natural equivalences of functors $S_{c} \circ i(c)_{*} \xrightarrow{\cong} \mathrm{id}$ and $T_{c} \circ i(c)!\xrightarrow{\cong}$ id of functors MOD $-R[c] \rightarrow \mathrm{MOD}-R[c]$. If $c \neq d$, then $S_{c} \circ i(d)_{*}=$ $T_{c} \circ i(d)!=0 ;$
(c) The functors $S_{c}$ and $i(c)_{*}$ send projective modules to projective modules. The functors $I_{c}$ and $i(c)$ ! send injective modules to injective modules.
Proof. (a) follows from (2.5), (2.6) and (2.1).
(b) This follows in the case $T_{c} \circ i(d)$ ! from the following chain of canonical isomorphisms

$$
\begin{aligned}
T_{c} \circ i(d)! & (M)=\operatorname{hom}_{R \mathcal{C}}\left(C(?, c), \operatorname{hom}_{R[d]}\left(R \operatorname{mor}_{\mathcal{C}}(d, ?), M\right)\right) \\
& \cong \operatorname{hom}_{R[d]}\left(C(?, c) \otimes_{R \mathcal{C}} R \operatorname{mor}_{\mathcal{C}}(d, ?), M\right)
\end{aligned} \stackrel{\cong}{\bigoplus} \operatorname{hom}_{R[c]}(C(c, d), M), ~ \$
$$

and analogously for $S_{d} \circ i(c)_{*}$.
(c) The functors $S_{c}$ and $i(c)_{*}$ are left adjoint to an exact functor and hence respect projective. The functors $T_{c}$ and $i(c)$ ! are right adjoint to an exact functor and hence respect injective.

The length $l(c) \in \mathbb{N} \cup\{\infty\}$ of an object $c$ is the supremum over all natural numbers $l$ for which there exists a sequence of morphisms $c_{0} \xrightarrow{f_{1}} c_{1} \xrightarrow{f_{2}} c_{2} \xrightarrow{f_{3}}$ $\ldots \xrightarrow{f_{l}} c_{l}$ such that no $f_{i}$ is an isomorphism and $c_{l}=c$. The colength $\operatorname{col}(c) \in$ $\mathbb{N} \cup\{\infty\}$ of an object $c$ is the supremum over all natural numbers $l$ for which there exists a sequence of morphisms $c_{0} \xrightarrow{f_{1}} c_{1} \xrightarrow{f_{2}} c_{2} \xrightarrow{f_{3}} \ldots \xrightarrow{f_{l}} c_{l}$ such that no $f_{i}$ is an isomorphism and $c_{0}=c$. If each object $c$ has length $l(c)<\infty$, we say that $\mathcal{C}$ has finite length. If each object $c$ has colength $\operatorname{col}(c)<\infty$, we say that $\mathcal{C}$ has finite colength.

Theorem 2.14. (Structure theorem for projective and injective $R \mathcal{C}$ modules). Let $\mathcal{C}$ be an EI-category. Then
(a) Suppose that $\mathcal{C}$ has finite colength. Let $M$ be a contravariant $R \mathcal{C}$-module such that the $R$ aut $(c)$-module $S_{c} M$ is projective for all objects $c$ in $\mathcal{C}$. Let $\sigma_{c}: S_{c} M \rightarrow M(c)$ be an $R$ aut $(c)$-section of the canonical projection $M(c) \rightarrow S_{c} M$. Consider the map of RC-modules

$$
\begin{aligned}
\mu(M): & \bigoplus_{(c) \in \operatorname{Is}(\mathcal{C})} i(c)_{*} S_{c} M \xrightarrow{\oplus_{(c) \in \operatorname{Is}(\mathcal{C})} i(c)_{*} \sigma_{c}} \bigoplus_{(c) \in \operatorname{Is}(\mathcal{C})} i(c)_{*} M(c) \\
& \xrightarrow[(c) \in \operatorname{Is}(\mathcal{C})]{ } \alpha(c)
\end{aligned} M,
$$

where $\alpha(c): i(c)_{*} M(c)=i(c)_{*} i(c)^{*} M \rightarrow M$ is the adjoint of the identity $i(c)^{*} M \rightarrow i(c)^{*} M$ under the adjunction (2.5). The map $\mu(M)$ is always surjective. It is bijective if and only if $M$ is a projective $R \mathcal{C}$-module;
(b) Suppose that $\mathcal{C}$ has finite length. Let $M$ be a contravariant $R \mathcal{C}$-module such that the $R$ aut(c)-module $T_{c} M$ is injective for all objects $c$ in $\mathcal{C}$. Let $\rho_{c}: M(c) \rightarrow I_{c} M$ be an $R$ aut $(c)$-retraction of the canonical injection $T_{c} M \rightarrow M(c)$. Consider the map of $R \mathcal{C}$-modules

$$
\begin{aligned}
& \nu(M): M \xrightarrow{\prod_{(c) \in \operatorname{Is}(\mathcal{C})} \beta(c)} \prod_{(c) \in \operatorname{Is}(\mathcal{C})} i(c)!M(c) \\
& \xrightarrow{\prod_{(c) \in \operatorname{Is}(\mathcal{C})} i(c)!\rho_{c}} \prod_{(c) \in \operatorname{Is}(\mathcal{C})} i(c)_{*} I_{c} M
\end{aligned}
$$

where $\beta(c): M \rightarrow i(c)!i(c)^{*} M=i(c)!M(c)$ is the adjoint of the identity $i(c)^{*} M \rightarrow i(c)^{*} M$ under the adjunction (2.6). The map $\nu(M)$ is always injective. It is bijective if and only if $M$ is an injective $R \mathcal{C}$-module.

Proof. (a) A contravariant $R \mathcal{C}$-module is the same as covariant $R \mathcal{C}^{\mathrm{op}}$-module, where $\mathcal{C}^{\mathrm{op}}$ is the opposite category of $\mathcal{C}$, just invert the direction of every morphisms. The corresponding covariant version of assertion (固) is proved in [8, Theorem 2.11].
(b) is the dual statement of assertion (目). We first show that $\nu(M)$ is always
injective. We show by induction over the length $l(x)$ of an object $x \in \mathcal{C}$ that $\nu(M)(x)$ is injective. Let $u$ be an element in the kernel of $\nu(M)(x)$. Consider a morphism $f: y \rightarrow x$ which is not an isomorphism. Then $l(y)<l(x)$ and by induction hypothesis $\nu(M)(y)$ is injective. Since the composite $\nu(M)(y) \circ M(f)$ factorizes through $\nu(M)(x)$, we have $u \in \operatorname{ker}(M(f))$. This implies $u \in I_{x} M$. Consider the composite

$$
I_{x} M \xrightarrow{i} M(x) \xrightarrow{\nu(M)(x)} \prod_{(c) \in \operatorname{Is}(\mathcal{C})} i(c)!I_{c} M(x) \xrightarrow{\mathrm{pr}_{x}} i(x)!I_{x} M(x) \xrightarrow{j} I_{x} M,
$$

where $i$ is the inclusion, $\mathrm{pr}_{x}$ is the projection onto the factor belonging to the isomorphism class of $x$ and $j$ is the isomorphism $\operatorname{hom}_{R[x]}\left(R \operatorname{mor}_{\mathcal{C}}(x, x), I_{x} M\right) \xrightarrow{\cong}$ $I_{x} M$ sending $\phi$ to $\phi\left(\mathrm{id}_{x}\right)$. Since this composite is the identity on $I_{x} M$ and $u$ lies in the kernel of $\nu(M)(x)$, we conclude $u=0$.

In particular we see that an injective $R \mathcal{C}$-module $M$ is trivial if and only if $i(d)!I_{d} M(x)$ is trivial for all objects $d \in \mathcal{C}$.

If $\nu(M)$ is bijective and each $I_{c} M$ is an injective $R[c]$-module, then $M$ is an injective $R \mathcal{C}$-module, since $i(c)$ ! sends injective $R[c]$-modules to injective $R \mathcal{C}$-modules by Lemma 2.13 (디) and the product of injective modules is again injective.

Now suppose that $M$ is injective. Let $N$ be the cokernel of $\nu(M)$. We have the exact sequence

$$
\begin{equation*}
0 \rightarrow M \xrightarrow{\nu(M)} \prod_{(c) \in \operatorname{Is}(\mathcal{C})} i(c)_{*} I_{c} M \xrightarrow{\mathrm{pr}} N \rightarrow 0 . \tag{2.15}
\end{equation*}
$$

Since $M$ is injective, this is a split exact sequence of injective $R \mathcal{C}$-modules. Fix an object $d$. The functors $i(d)$ ! and $I_{d}$ are left exact and hence send split exact sequences to split exact sequences. Therefore we obtain a split exact sequence if we apply $i(d)!I_{d}$ to (2.15). Using Lemma (b) the resulting exact sequence is isomorphic to the exact sequence

$$
0 \rightarrow i(d)!I_{d} M \xrightarrow{\text { id }} i(d)!I_{d} M \rightarrow i(d)!I_{d} N \rightarrow 0
$$

Hence $i(d)!I_{d} N$ vanishes for all objects $d$. This implies that $N$ is trivial and because of (2.15) that $\nu(M)$ is bijective.

For more details about modules over a category we refer to [7 Section 9A].

## 3. The Associated Bredon Cohomology Theory

Given a proper equivariant cohomology theory with values in $R$-modules, we can associate to it another proper equivariant cohomology theory with values in $R$-modules satisfying the disjoint union axiom called Bredon cohomology, which
is much simpler. The equivariant Chern character will identify this simpler proper equivariant cohomology theory with the given one.

The orbit category $\operatorname{Or}(G)$ has as objects homogeneous spaces $G / H$ and as morphisms $G$-maps. Let $\operatorname{Sub}(G)$ be the category whose objects are subgroups $H$ of $G$. For two subgroups $H$ and $K$ of $G$ denote by $\operatorname{conhom}_{G}(H, K)$ the set of group homomorphisms $f: H \rightarrow K$, for which there exists an element $g \in G$ with $g H^{-1} \subseteq K$ such that $f$ is given by conjugation with $g$, i.e. $f=$ $c(g): H \rightarrow K, \quad h \mapsto g h g^{-1}$. Notice that $f$ is injective and $c(g)=c\left(g^{\prime}\right)$ holds for two elements $g, g^{\prime} \in G$ with $g H g^{-1} \subseteq K$ and $g^{\prime} H\left(g^{\prime}\right)^{-1} \subseteq K$ if and only if $g^{-1} g^{\prime}$ lies in the centralizer $C_{G} H=\{g \in G \mid g h=h g$ for all $h \in H\}$ of $H$ in $G$. The group of inner automorphisms of $K$ acts on $\operatorname{conhom}_{G}(H, K)$ from the left by composition. Define the set of morphisms

$$
\operatorname{mor}_{\mathrm{Sub}(G)}(H, K):=\operatorname{Inn}(K) \backslash \operatorname{conhom}_{G}(H, K) .
$$

There is a natural projection $\operatorname{pr}: \operatorname{Or}(G) \rightarrow \operatorname{Sub}(G)$ which sends a homogeneous space $G / H$ to $H$. Given a $G$-map $f: G / H \rightarrow G / K$, we can choose an element $g \in G$ with $g H g^{-1} \subseteq K$ and $f\left(g^{\prime} H\right)=g^{\prime} g^{-1} K$. Then $\operatorname{pr}(f)$ is represented by $c(g): H \rightarrow K$. Notice that $\operatorname{mor}_{\mathbf{S u b}_{(G)}}(H, K)$ can be identified with the quotient $\operatorname{mor}_{\operatorname{Or}(G)}(G / H, G / K) / C_{G} H$, where $g \in C_{G} H$ acts on $\operatorname{mor}_{\mathrm{Or}(G)}(G / H, G / K)$ by composition with $R_{g^{-1}}: G / H \rightarrow G / H, g^{\prime} H \mapsto$ $g^{\prime} g^{-1} H$.

Denote by $\operatorname{Or}(G, \mathcal{F}) \subseteq \operatorname{Or}(G)$ and $\operatorname{Sub}(G, \mathcal{F}) \subseteq \operatorname{Sub}(G)$ the full subcategories, whose objects $G / H$ and $H$ are given by finite subgroups $H \subseteq G$. Both $\operatorname{Or}(G, \mathcal{F})$ and $\operatorname{Sub}(G, \mathcal{F})$ are EI-categories of finite length.

Given a proper $G$-cohomology theory $\mathcal{H}_{G}^{*}$ with values in $R$-modules we obtain for $n \in \mathbb{Z}$ a contravariant $\operatorname{ROr}(G, \mathcal{F})$-module

$$
\begin{equation*}
\mathcal{H}_{G}^{n}(G / ?): \operatorname{Or}(G, \mathcal{F}) \rightarrow R-\operatorname{MOD}, \quad G / H \mapsto \mathcal{H}_{G}^{n}(G / H) \tag{3.1}
\end{equation*}
$$

Let $(X, A)$ be a pair of proper $G$ - $C W$-complexes. Then there is a canonical identification $X^{H}=\operatorname{map}(G / H, X)^{G}$. Thus we obtain contravariant functors

$$
\begin{aligned}
\operatorname{Or}(G, \mathcal{F}) & \rightarrow C W \text {-PAIRS, } & & G / H \mapsto\left(X^{H}, A^{H}\right) ; \\
\operatorname{Sub}(G, \mathcal{F}) & \rightarrow C W \text {-PAIRS, } & & G / H \mapsto C_{G} H \backslash\left(X^{H}, A^{H}\right),
\end{aligned}
$$

where $C W$-PAIRS is the category of pairs of $C W$-complexes. If we compose them with the covariant functor $C W$-PAIRS $\rightarrow \mathbb{Z}$-CHCOM sending $(Z, B)$ to its cellular $\mathbb{Z}$-chain complex, then we obtain the contravariant $\mathbb{Z} O r(G, \mathcal{F})$ chain complex $C_{*}^{\mathrm{Or}(G, \mathcal{F})}(X, A)$ and the contravariant $\mathbb{Z} \operatorname{Sub}(G, \mathcal{F})$-chain complex $C_{*}^{\operatorname{Sub}(G, \mathcal{F})}(X, A)$. Both chain complexes are free in the sense that each chain module is a free $\mathbb{Z O r}(G, \mathcal{F})$-module resp. $\mathbb{Z S u b}(G, \mathcal{F})$-module. Namely, if $X_{n}$ is obtained from $X_{n-1} \cup A_{n}$ by attaching the equivariant cells $G / H_{i} \times D^{n}$ for $i \in I_{n}$, then

$$
\begin{align*}
C_{n}^{\mathrm{Or}(G, \mathcal{F})}(X, A) & \cong \bigoplus_{i \in I_{n}} \mathbb{Z} \operatorname{mor}_{\operatorname{Or}(G, \mathcal{F})}\left(G / ?, G / H_{i}\right)  \tag{3.2}\\
C_{n}^{\mathrm{Sub}(G, \mathcal{F})}(X, A) & \cong \bigoplus_{i \in I_{n}} \mathbb{Z} \operatorname{mor}_{\operatorname{Sub}(G, \mathcal{F})}\left(?, H_{i}\right) \tag{3.3}
\end{align*}
$$

Given a contravariant $\operatorname{ROr}(G, \mathcal{F})$-module $M$, the equivariant Bredon cohomology (see [2]) of a pair of proper $G$ - $C W$-complexes ( $X, A$ ) with coefficients in $M$ is defined by

$$
\begin{equation*}
H_{\mathrm{Or}(G, \mathcal{F})}^{n}(X, A ; M):=H^{n}\left(\operatorname{hom}_{\mathbb{Z O r}(G, \mathcal{F})}\left(C_{*}^{\mathrm{Or}(G, \mathcal{F})}(X, A), M\right)\right) . \tag{3.4}
\end{equation*}
$$

This is indeed a proper $G$-cohomology theory satisfying the disjoint union axiom. Hence we can assign to a proper $G$-homology theory $\mathcal{H}_{G}^{*}$ another proper $G$ cohomology theory which we call the associated Bredon cohomology

$$
\begin{equation*}
\mathcal{B H}_{G}^{n}(X, A):=\prod_{p+q=n} H_{\mathrm{Or}(G, \mathcal{F})}^{p}\left(X, A ; \mathcal{H}_{G}^{q}(G / ?)\right) \tag{3.5}
\end{equation*}
$$

There is an obvious $\mathbb{Z} \operatorname{Sub}(G ; \mathcal{F})$-chain map

$$
\operatorname{pr}_{*} C_{*}^{\mathrm{Or}(G, \mathcal{F})}(X, A) \xrightarrow{\cong} C_{*}^{\mathrm{Sub}(G, \mathcal{F})}(X, A)
$$

which is bijective because of (3.2), (3.3) and the canonical identification

$$
\operatorname{pr}_{*} \mathbb{Z} \operatorname{mor}_{\mathrm{Or}(G, \mathcal{F})}\left(G / ?, G / H_{i}\right)=\mathbb{Z} \operatorname{mor}_{\operatorname{Sub}(G, \mathcal{F})}\left(?, H_{i}\right)
$$

Given a covariant $\mathbb{Z S u b}(G, \mathcal{F})$-module $M$, we get from the adjunction $\left(\mathrm{pr}_{*}, \mathrm{pr}^{*}\right)$ (see Lemma 2.13 (国) natural isomorphisms

$$
\begin{align*}
H_{R \mathrm{Or}(G, \mathcal{F})}^{n}\left(X, A ; \operatorname{res}_{\mathrm{pr}} M\right) & \\
& \cong \tag{3.6}
\end{align*}
$$

This will allow us to work with modules over the category $\operatorname{Sub}(G ; \mathcal{F})$ which is smaller than the orbit category and has nicer properties from the homological algebra point of view. The main advantage of $\operatorname{Sub}(G ; \mathcal{F})$ is that the automorphism groups of every object is finite.

Suppose, we are given a proper equivariant cohomology theory $\mathcal{H}_{?}^{*}$ with values in $R$-modules. We get from (3.1) for each group $G$ and $n \in \mathbb{Z}$ a covariant $R S u b(G, \mathcal{F})$-module

$$
\begin{equation*}
\mathcal{H}_{G}^{n}(G / ?): \operatorname{Sub}(G, \mathcal{F}) \rightarrow R-\mathrm{MOD}, \quad H \mapsto \mathcal{H}_{G}^{n}(G / H) \tag{3.7}
\end{equation*}
$$

We have to show that for $g \in C_{G} H$ the $G$-map $R_{g^{-1}}: G / H \rightarrow G / H, \quad g^{\prime} H \rightarrow$ $g^{\prime} g^{-1} H$ induces the identity on $\mathcal{H}_{G}^{n}(G / H)$. This follows from Lemma 1.5, We will denote the covariant $\operatorname{ROr}(G, \mathcal{F})$-module obtained by restriction with pr: $\operatorname{Or}(G, \mathcal{F}) \rightarrow \operatorname{Sub}(G, \mathcal{F})$ from the $R \operatorname{Sub}(G, \mathcal{F})$-module $\mathcal{H}_{G}^{n}(G / ?)$ of (3.7) again by $\mathcal{H}_{G}^{n}(G / ?)$ as introduced already in (3.1).

It remains to show that the collection of $G$-cohomology theories $\mathcal{B H}_{G}^{*}(X, A)$ defined in (3.4) inherits the structure of a proper equivariant cohomology theory, i.e. we have to specify the induction structure. We leave it to the reader to carry out the obvious dualization of the construction for homology in [8, Section 3] and to check the disjoint union axiom.

## 4. The Construction of the Equivariant Cohomological Chern Character

We begin with explaining the cohomological version of the homological Chern character due to Dold [4].

Example 4.1 (The non-equivariant Chern character). Consider a (nonequivariant) cohomology theory $\mathcal{H}^{*}$ with values in $R$-modules. Suppose that $\mathbb{Q} \subseteq R$. For a space $X$ let $X_{+}$be the pointed space obtained from $X$ by adding a disjoint base point $*$. Since the stable homotopy groups $\pi_{p}^{s}\left(S^{0}\right)$ are finite for $p \geq 1$ by results of Serre [15], the condition $\mathbb{Q} \subseteq R$ imply that the Hurewicz homomorphism induces isomorphisms

$$
\operatorname{hur}_{R}: \pi_{p}^{s}\left(X_{+}\right) \otimes_{\mathbb{Z}} R \xrightarrow{\operatorname{hur} \otimes_{\mathbb{Z}} \operatorname{id}_{R}} H_{p}(X) \otimes_{\mathbb{Z}} R \xrightarrow{\cong} H_{p}(X ; R)
$$

and that the canonical map
$\alpha: H^{p}\left(X ; \mathcal{H}^{q}(\{\mathrm{pt}\}).\right) \xrightarrow{\cong} \operatorname{hom}_{\mathbb{Q}}\left(H_{p}(X ; \mathbb{Q}), \mathcal{H}^{q}(X)\right) \xrightarrow{\cong} \operatorname{hom}_{R}\left(H_{p}(X ; R), \mathcal{H}^{q}(X)\right)$ is bijective. Define a map

$$
\begin{equation*}
D^{p, q}: \mathcal{H}^{p+q}(X) \quad \rightarrow \quad \operatorname{hom}_{R}\left(\pi_{p}^{s}\left(X_{+}\right) \otimes_{\mathbb{Z}} R, \mathcal{H}^{q}(\{\mathrm{pt} .\})\right) \tag{4.2}
\end{equation*}
$$

as follows. Denote in the sequel by $\sigma^{k}$ the $k$-fold suspension isomorphism. Given $a \in \mathcal{H}^{p+q}(X)$ and an element in $\pi_{p}^{s}\left(X_{+}, *\right)$ represented by a map $f: S^{p+k} \rightarrow S^{k} \wedge$ $X_{+}$, we define $D^{p, q}(a)([f]) \in \mathcal{H}^{q}(\{\mathrm{pt}\}$.$) as the image of a$ under the composite

$$
\begin{aligned}
\mathcal{H}^{p+q}(X) \xrightarrow{\cong} \widetilde{\mathcal{H}}^{p+q}\left(X_{+}\right) \xrightarrow{\sigma^{k}} \widetilde{\mathcal{H}}^{p+q+k}\left(S^{k} \wedge X_{+}\right) \xrightarrow{\widetilde{\mathcal{H}}^{p+q+k}(f)} \widetilde{\mathcal{H}}^{p+q+k}\left(S^{p+k}\right) \\
\xrightarrow{\left(\sigma^{p+k}\right)^{-1}} \widetilde{\mathcal{H}}^{q}\left(S^{0}\right) \xrightarrow{\cong} \mathcal{H}^{q}(\{\mathrm{pt} .\}) .
\end{aligned}
$$

Then the (non-equivariant) Chern character for a $C W$-complex $X$ is given by the following composite

$$
\begin{aligned}
& \operatorname{ch}^{n}(X): \mathcal{H}^{n}(X) \xrightarrow{\prod_{p+q=n} D^{p, q}} \prod_{p+q=n} \operatorname{hom}_{R}\left(\pi_{p}^{s}\left(X_{+}, *\right) \otimes_{\mathbb{Z}} R, \mathcal{H}^{q}(*)\right) \\
& \xrightarrow{\prod_{p+q=n} \operatorname{hom}_{R}\left(\operatorname{hur}_{R}^{-1}, \mathrm{id}\right)} \prod_{p+q=n} \operatorname{hom}_{R}\left(H_{p}(X ; R), \mathcal{H}^{q}(*)\right) \\
& \xrightarrow{\prod_{p+q=n} \alpha^{-1}} \prod_{p+q=n} H^{p}\left(X, \mathcal{H}^{q}(*)\right) .
\end{aligned}
$$

There is an obvious version for a pair of $C W$-complexes

$$
\operatorname{ch}^{n}(X, A): \mathcal{H}^{n}(X, A) \cong \prod_{p+q=n} H^{p}\left(X, A, \mathcal{H}^{q}(*)\right)
$$

We get a natural transformation $\mathrm{ch}^{*}$ of cohomology theories with values in $R$ modules. One easily checks that it is an isomorphism in the case $X=\{\mathrm{pt}$.$\} .$ Hence $\operatorname{ch}^{n}(X, A)$ is bijective for all relative finite $C W$-pairs $(X, A)$ and $n \in \mathbb{Z}$ by Lemma 1.1 (b). If $\mathcal{H}^{*}$ satisfies the disjoint union axiom, then $\operatorname{ch}^{n}(X, A)$ is bijective for all $C W$-pairs ( $X, A$ ) and $n \in \mathbb{Z}$ by Lemma 1.1 (B).

Let $R$ be a commutative ring with $\mathbb{Q} \subseteq R$. Consider an equivariant cohomology theory $\mathcal{H}_{?}^{*}$ with values in $R$-modules. Let $G$ be a group and let $(X, A)$ be a proper $G$ - $C W$-pair. We want to construct an $R$-homomorphism

$$
\begin{align*}
\underline{\operatorname{ch}}_{G}^{p, q}(X, A)(H): \mathcal{H}_{G}^{p+q} & (X, A) \\
& \rightarrow \operatorname{hom}_{R}\left(H_{p}\left(C_{G} H \backslash\left(X^{H}, A^{H}\right) ; R\right), \mathcal{H}_{G}^{q}(G / H)\right) \tag{4.3}
\end{align*}
$$

We define it only in the case $A=\emptyset$, the general case is completely analogous.

$$
\begin{gathered}
\mathcal{H}_{G}^{p+q}(X) \\
\mathcal{H}_{G}^{p+q}\left(v_{H}\right) \downarrow \\
\mathcal{H}_{G}^{p+q}\left(\mathrm{ind}_{m_{H}} X^{H}\right) \\
\mathcal{H}_{G}^{p+q}\left(\mathrm{ind}_{m_{H}} \mathrm{pr}_{2}\right) \downarrow \\
\mathcal{H}_{G}^{p+q}\left(\mathrm{ind}_{m_{H}} E G \times X^{H}\right) \\
\operatorname{ind}_{m_{H}} \downarrow \cong \\
\mathcal{H}_{C_{G} H \times H}^{p+q}\left(E G \times X^{H}\right) \\
\left(\operatorname{ind}_{\mathrm{pr}: C_{G} H \times H \rightarrow H}\right)^{-1} \downarrow \cong \\
\mathcal{H}_{H}^{p+q}\left(E G \times_{C_{G} H} X^{H}\right) \\
D_{H}^{p, q}\left(E G \times_{C_{G} H} X^{H}\right) \\
\operatorname{hom}_{R}\left(\pi_{p}^{s}\left(\left(E G \times_{C_{G} H} X^{H}\right)_{+}\right) \otimes_{\mathbb{Z}} R, \mathcal{H}_{H}^{q}(\{\mathrm{pt} .\})\right) \\
\operatorname{hom}_{R}\left(\operatorname{hur}_{R}\left(E G \times_{C_{G} H} X^{H}\right), \mathrm{id}^{-1}\right. \\
\operatorname{hom}_{R}\left(H_{p}\left(E G \times_{C_{G} H} X^{H} ; R\right), \mathcal{H}_{H}^{q}(\{\mathrm{pt} .\})\right) \\
\operatorname{hom}_{R}\left(H_{p}\left(\mathrm{pr}_{1} ; R\right), \mathrm{id}\right)^{-1} \downarrow \\
\operatorname{hom}_{R}\left(H_{p}\left(C_{G} H \backslash X^{H} ; R\right), \mathcal{H}_{H}^{q}(\{\mathrm{pt} .\})\right) \\
\operatorname{hom}_{R}\left({\left.\mathrm{id} ;\left(\mathrm{ind}_{H}^{G}\right)^{-1}\right)} \downarrow\right. \\
\operatorname{hom}_{R}\left(H_{p}\left(C_{G} H \backslash X^{H} ; R\right), \mathcal{H}_{G}^{q}(G / H)\right)
\end{gathered}
$$

Here are some explanations, more details can be found in [8, Section 4].
We have a left free $C_{G} H$-action on $E G \times X^{H}$ by $g(e, x)=\left(e g^{-1}, g x\right)$ for $g \in$ $C_{G} H, e \in E G$ and $x \in X^{H}$. The map $\operatorname{pr}_{1}: E G \times_{C_{G} H} X^{H} \rightarrow C_{G} H \backslash X^{H}$ is the
canonical projection. Since the projection $B L \rightarrow\{\mathrm{pt}$.$\} induces isomorphisms$ $H_{p}(B L ; R) \xrightarrow{\cong} H_{p}(\{\mathrm{pt}\} ; R$.$) for all p \in \mathbb{Z}$ and finite groups $L$ because of $\mathbb{Q} \subseteq R$, we obtain for every $p \in \mathbb{Z}$ an isomorphism

$$
H_{p}\left(\mathrm{pr}_{1} ; R\right): H_{p}\left(E G \times_{C_{G} H} X^{H} ; R\right) \xrightarrow{\cong} H_{p}\left(C_{G} H \backslash X^{H} ; R\right) .
$$

The group homomorphism pr: $C_{G} H \times H \rightarrow H$ is the obvious projection and the group homomorphism $m_{H}: C_{G} H \times H \rightarrow G$ sends $(g, h)$ to $g h$. The $C_{G} H \times H$ action on $E G \times X^{H}$ comes from the obvious $C_{G} H$-action and the trivial $H$ action. In particular we equip $E G \times_{C_{G} H} X^{H}$ with the trivial $H$-action. The kernels of the two group homomorphisms pr and $m_{H}$ act freely on $E G \times X^{H}$. We denote by $\mathrm{pr}_{2}: E G \times X^{H} \rightarrow X^{H}$ the canonical projection. The $G$-map $v_{H}: \operatorname{ind}_{m_{H}} X^{H}=G \times_{m_{H}} X^{H} \rightarrow X$ sends $(g, x)$ to $g x$.

Since $H$ is a finite group, a $C W$-complex $Z$ equipped with the trivial $H$ action is a proper $H$ - $C W$-complex. Hence we can think of $\mathcal{H}_{H}^{*}$ as an (nonequivariant) homology theory if we apply it to a $C W$-pair $Z$ with respect to the trivial $H$-action. Define the map

$$
D_{H}^{p, q}(Z): \mathcal{H}_{H}^{p+q}(Z) \rightarrow \operatorname{hom}_{R}\left(\pi_{p}^{s}\left(Z_{+}\right) \otimes_{\mathbb{Z}} R, \mathcal{H}_{H}^{q}(\{\mathrm{pt} .\})\right)
$$

for a $C W$-complex $Z$ by the map $D^{p, q}$ of (4.2).
A calculation similar to the one in [8, Lemma 4.3] shows that the system of maps $\underline{\operatorname{ch}}_{G}^{p, q}(X, A)(H)$ (4.3) fit together to an in $X$ natural $R$-homomorphism

$$
\begin{align*}
{\underline{\operatorname{ch}_{G}^{p, q}}(X, A): \mathcal{H}_{G}^{p+q}(X, A)} & \\
& \rightarrow \operatorname{hom}_{\text {Sub }(G ; \mathcal{F})}\left(H_{p}\left(C_{G} ? \backslash X^{?} ; R\right), \mathcal{H}_{G}^{q}(G / ?)\right) \tag{4.4}
\end{align*}
$$

For any contravariant $R \operatorname{Sub}(G ; \mathcal{F})$-module $M$ and $p \in \mathbb{Z}$ there is an in $(X, A)$ natural $R$-homomorphism

$$
\begin{align*}
\alpha_{G}^{p}(X, A ; M): H_{R S u b(G ; \mathcal{F})}^{p}(X, & A ; M) \rightarrow \operatorname{hom}_{\mathbb{Q S u b}(G ; \mathcal{F})}\left(H_{p}\left(C_{G} ? \backslash X^{?} ; \mathbb{Q}\right), M\right) \\
& \cong \operatorname{hom}_{R S u b(G ; \mathcal{F})}\left(H_{p}\left(C_{G} ? \backslash X^{?} ; R\right), M\right) \tag{4.5}
\end{align*}
$$

which is bijective if $M$ is injective as $\mathbb{Q} \operatorname{Sub}(G ; \mathcal{F})$-module.
Theorem 4.6 (The equivariant Chern character). Let $R$ be a commutative ring $R$ with $\mathbb{Q} \subseteq R$. Let $\mathcal{H}_{?}^{*}$ be a proper equivariant cohomology theory with values in $R$-modules. Suppose that the $\operatorname{RSub}(G ; \mathcal{F})$-module $\mathcal{H}_{G}^{q}(G / ?)$ of (3.7), which sends $G / H$ to $\mathcal{H}_{G}^{q}(G / H)$, is injective as $\mathbb{Q} \operatorname{Sub}(G ; \mathcal{F})$-module for every group $G$ and every $q \in \mathbb{Z}$.

Then we obtain a transformation of proper equivariant cohomology theories with values in $R$-modules

$$
\operatorname{ch}_{?}^{*}: \mathcal{H}_{?}^{*} \quad \cong \quad \mathcal{B} \mathcal{H}_{?}^{*},
$$

if we define for a group $G$ and a proper $G-C W-\operatorname{pair}(X, A)$
$\operatorname{ch}_{G}^{n}(X, A): \mathcal{H}_{G}^{n}(X, A) \rightarrow \mathcal{B} \mathcal{H}_{G}^{n}(X, A):=\prod_{p+q=n} H_{R S u b(G ; \mathcal{F})}^{p}\left(X, A ; \mathcal{H}_{G}^{q}(G / ?)\right)$
by the composite

$$
\begin{aligned}
& \mathcal{H}_{G}^{n}(X, A) \xrightarrow{\prod_{p+q=n} \stackrel{\operatorname{ch}_{G}^{p, q}(X, A)}{\longrightarrow}} \prod_{p+q=n} \operatorname{hom}_{R S u b}(G ; \mathcal{F}) \\
& \xrightarrow{\prod_{p+q=n} \alpha_{G}^{p}\left(X, A ; \mathcal{H}_{G}^{q}(G / ?)\right)^{-1}} \\
& \prod_{p+q=n} H_{R S u b(G ; \mathcal{F})}^{p}\left(X, A ; \mathcal{H}_{G}^{q}(G / ?)\right)
\end{aligned}
$$

of the maps defined in (4.4) and (4.5).
The $R$-map $\operatorname{ch}_{G}^{n}(X, A)$ is bijective for all proper relative finite $G$ - $C W$-pairs $(X, A)$ and $n \in \mathbb{Z}$. If $\mathcal{H}_{\text {? }}^{*}$ satisfies the disjoint union axiom, then the $R$-map $\operatorname{ch}_{G}^{n}(X, A)$ is bijective for all proper $G$ - $C W$-pairs $(X, A)$ and $n \in \mathbb{Z}$.

Proof. First one checks that $\mathrm{ch}_{G}^{*}$ defines a natural transformation of proper $G$ cohomology theories. One checks for each finite subgroup $H \subseteq G$ and $n \in \mathbb{Z}$ that $\operatorname{ch}_{G}^{n}(G / H)$ is the identity if we identify for any $\operatorname{RSub}(G ; \mathcal{F})$-module $M$

$$
\begin{aligned}
& H_{R \mathrm{Sub}(G ; \mathcal{F})}^{p}(G / H ; M)=H^{p}\left(\operatorname{hom}_{R \operatorname{Sub}(G ; \mathcal{F})}\left(C_{*}^{R \operatorname{Sub}(G ; \mathcal{F})}(G / H), M\right)\right. \\
= & \begin{cases}\operatorname{hom}_{R \operatorname{Sub}(G ; \mathcal{F})}\left(R \operatorname{mor}_{\operatorname{Sub}(G ; \mathcal{F})}(?, G / H), M\right)=M(G / H) & \text { if } p=0 \\
0 & \text { if } p \neq 0\end{cases}
\end{aligned}
$$

Finally apply Lemma 1.1 (b).
Remark 4.7 (The Atiyah-Hirzebruch spectral sequence for equivariant cohomology). There exists a Atiyah-Hirzebruch spectral sequence for equivariant cohomology (see [3, Theroem $4.7(2)]$ ). It converges to $\mathcal{H}_{G}^{p+q}(X, A)$ and has as $E_{2}$-term the Bredon cohomology groups $H_{R S u b(G ; \mathcal{F})}^{p}\left(X, A ; \mathcal{H}_{G}^{q}(G / ?)\right)$. The conclusion of Theorem 4.6 is that the spectral sequences collapses.

Example 4.8 (Equivariant Chern character for $\mathcal{K}^{*}(G \backslash(X, A))$ ). Let $\mathcal{K}^{*}$ be a (non-equivariant) cohomology theory with values in $R$-modules for a commutative ring with $\mathbb{Q} \subseteq R$. In Example 1.6 we have assigned to it an equivariant cohomology theory by

$$
\mathcal{H}_{G}^{n}(X, A)=\mathcal{K}^{n}(G \backslash(X, A))
$$

We claim that the assumptions appearing in Theorem4.6 are satisfied We have to show that the constant functor

$$
\underline{\mathcal{K}^{q}(\{\mathrm{pt} .\})}: \operatorname{Sub}(G ; \mathcal{F}) \rightarrow \mathbb{Q}-\mathrm{MOD}, \quad H \mapsto \mathcal{K}^{q}(\{\mathrm{pt} .\})
$$

is injective. Let $i: \operatorname{Sub}(\{1\}) \rightarrow \operatorname{Sub}(G ; \mathcal{F})$ the obvious inclusion of categories. Since the object $\{1\}$ is an initial object in $\operatorname{Sub}(G ; \mathcal{F})$, the $\mathbb{Q} \operatorname{Sub}(G ; \mathcal{F})$-modules $i_{!}\left(\mathcal{K}^{q}(\{\mathrm{pt}\}).\right)$ and $\mathcal{K}^{q}(\{\mathrm{pt}\}$.$) are isomorphic. Since i_{!}$sends an injective $\mathbb{Q}$-module to an injective $R S \overline{\mathrm{ub}(G ; \mathcal{F})}$-module by Lemma2.13(c) and $\mathcal{H}^{q}(\{\mathrm{pt}\}$.$) is injective$
as $\mathbb{Q}$-module, $\mathcal{K}^{q}(\{$ pt. $\})$ is injective as $\mathbb{Q} \operatorname{Sub}(G ; \mathcal{F})$-module. From Theorem4.6 we get a transformation of equivariant cohomology theories

$$
\begin{aligned}
\operatorname{ch}_{G}^{n}(X, A): \mathcal{K}^{n}(G \backslash(X, A)) \stackrel{( }{\cong} \prod_{p+q=n} & H_{R S \mathfrak{u b}(G ; \mathcal{F})}^{p}\left(X, A ; \underline{\left.\mathcal{H}^{q}(\{\mathrm{pt} .\})\right)}\right) \\
& =\prod_{p+q=n} H^{p}\left(G \backslash(X, A) ; \mathcal{H}^{q}(\{\mathrm{pt} .\})\right) .
\end{aligned}
$$

One easily checks that this is precisely the Chern character of Example 4.1 applied to $\mathcal{K}^{*}$ and the $C W$-pair $G \backslash(X, A)$.

## 5. Mackey Functors

In Theorem4.6 the assumption appears that the contravariant $R \operatorname{Sub}(G ; \mathcal{F})$ module $\mathcal{H}_{G}^{q}(G /$ ?) is injective for each $q \in \mathbb{Z}$. We want to give a criterion which ensures that this assumption is satisfies and which turns out to apply to all cases of interest.

Let $R$ be a commutative ring. Let FGINJ be the category of finite groups with injective group homomorphisms as morphisms. Let $M$ : FGINJ $\rightarrow R$-MOD be a bifunctor, i.e. a pair $\left(M_{*}, M^{*}\right)$ consisting of a covariant functor $M_{*}$ and a contravariant functor $M^{*}$ from FGINJ to $R-$ MOD which agree on objects. We will often denote for an injective group homomorphism $f: H \rightarrow G$ the map $M_{*}(f): M(H) \rightarrow M(G)$ by $\operatorname{ind}_{f}$ and the map $M^{*}(f): M(G) \rightarrow M(H)$ by res ${ }_{f}$ and write $\operatorname{ind}_{H}^{G}=\operatorname{ind}_{f}$ and $\operatorname{res}_{G}^{H}=\operatorname{res}_{f}$ if $f$ is an inclusion of groups. We call such a bifunctor $M$ a Mackey functor with values in $R$-modules if
(a) For an inner automorphism $c(g): G \rightarrow G$ we have $M_{*}(c(g))=\mathrm{id}: M(G) \rightarrow$ $M(G)$;
(b) For an isomorphism of groups $f: G \xlongequal{\cong} H$ the composites $\operatorname{res}_{f} \circ \operatorname{ind}_{f}$ and $\operatorname{ind}_{f} \circ \operatorname{res}_{f}$ are the identity;
(c) Double coset formula

We have for two subgroups $H, K \subseteq G$

$$
\operatorname{res}_{G}^{K} \circ \operatorname{ind}_{H}^{G}=\sum_{K g H \in K \backslash G / H} \operatorname{ind}_{c(g): H \cap g^{-1} K g \rightarrow K} \circ \operatorname{res}_{H}^{H \cap g^{-1} K g},
$$

where $c(g)$ is conjugation with $g$, i.e. $c(g)(h)=g h g^{-1}$.
Let $G$ be a group. In the sequel we denote for a subgroup $H \subseteq G$ by $N_{G} H$ the normalizer and by $C_{G} H$ the centralizer of $H$ in $G$ and by $W_{G} H$ the quotient $N_{G} H / H \cdot C_{G} H$. Notice that $W_{G} H$ is finite if $H$ is finite. Let $R$ be a commutative ring. Let $M$ be a Mackey functor with values in $R$-modules. It induces a contravariant $R \operatorname{Sub}(G, \mathcal{F})$-module denoted in the same way

$$
M: \operatorname{Sub}(G, \mathcal{F}) \rightarrow R-\operatorname{MOD}, \quad(f: H \rightarrow K) \mapsto\left(M^{*}(f): M(H) \rightarrow M(K)\right) .
$$

We want to use Theorem [2.14 (B) to show that $M$ is injective and analyse its structure. The $R\left[W_{G} H\right]$-module $T_{H} M$ introduced in (2.11) is the same as the kernel of

$$
\prod_{K \subsetneq H} M\left(i_{K}\right): M(H) \rightarrow \prod_{K \subsetneq H} M(K)
$$

where for each subgroup $K \subsetneq H$ different from $H$ we denote by $i_{K}$ the inclusion. Suppose that $R\left[W_{G} H\right]$-module $T_{H} M$ is injective for every finite subgroup $H \subseteq$ $G$. For every finite subgroup $H \subseteq G$ choose a retraction $\rho_{H}: M(H) \rightarrow T_{H} M$ of the inclusion $T_{H} M \rightarrow M(H)$. Denote by $I=\operatorname{Is}(\operatorname{Sub}(G, \mathcal{F}))$ the set of isomorphism classes of objects in $\operatorname{Sub}(G ; \mathcal{F})$ which is the same as the set of conjugacy classes $(H)$ of finite subgroups $H$ of $G$. Let

$$
\begin{equation*}
\nu=\nu(M): M \rightarrow \prod_{(K) \in I} i(K)!\circ T_{K}(M) \tag{5.1}
\end{equation*}
$$

be the homomorphism of $R \operatorname{Sub}(G, \mathcal{F})$-modules uniquely determined by the property that for any $(K) \in I$ its composition with the projection onto the factor indexed by $(K)$ is the adjoint of $\rho_{K}: M(K) \rightarrow T_{K} M$ for the adjoint pair $\left(i(K)^{*}, i(K)!\right)$.

Theorem 5.2 (Injectivity and Mackey functors). Let $G$ be a group and let $R$ be a commutative ring such that the order of every finite subgroup of $G$ is invertible in $R$. Suppose that the $R\left[W_{G} H\right]$-module $T_{H} M$ is injective for each finite subgroup $H \subseteq G$. Then $M$ is injective as $R \operatorname{Sub}(G, \mathcal{F})$-module and the map $\nu$ of (5.1) is bijective.

Proof. The map $\nu$ of (5.1) is the map $\nu(M)$ appearing in Theorem 2.14 (b). Because of Theorem (2.14 (b) it suffices to show for each finite subgroup $H \subseteq G$ that $\nu(M)(H)$ is surjective.

Fix for any $(K) \in I$ a representative $K$. Then choose for any $W_{G} K \cdot f \in$ $W_{G} K \backslash \operatorname{mor}(K, H)$ an element $f \in \operatorname{conhom}(K, H)$ which represents a morphism $f: K \rightarrow H$ in $\operatorname{Sub}(G ; \mathcal{F})$ which belongs to $W_{G} K \cdot f \in W_{G} K \backslash \operatorname{mor}(K, H)$. Notice that $W_{G} K$ is the automorphism group of the object $K$ in $\operatorname{Sub}(G ; \mathcal{F})$ and $W_{G} K$, $\operatorname{mor}(K, H)$ and $W_{G} K \backslash \operatorname{mor}(K, H)$ are finite. With these choices we get for every object $H$ in $\operatorname{Sub}(G ; \mathcal{F})$ an identification

$$
i(K)!T_{K} M(H)=\operatorname{hom}_{R W_{G} K}\left(R \operatorname{mor}(K, H), T_{K} M\right)=\prod_{\substack{W_{G} K \cdot f \in \\ W_{G} K \backslash \operatorname{mor}(K, H)}} T_{K} M^{W_{G} K_{f}}
$$

where $W_{G} K_{f} \subseteq W_{G} K$ is the isotropy group of $f$ under the $W_{G} K$-action on $\operatorname{mor}(K, H)$. Under this identification $\nu(H)$ becomes the map

$$
\nu(H): M(H) \rightarrow \prod_{(K) \in I} \prod_{\substack{W_{G} K \cdot f \in \\ W_{G} K \backslash \operatorname{mor}(K, H)}} T_{K} M^{W_{G} K_{f}}
$$

for which the component of $\nu(H)(m)$, which belongs $(K) \in I$ and $W_{G} K \cdot f \in$ $W_{G} K \backslash \operatorname{mor}(K, H)$, is $\rho_{K} \circ \operatorname{res}_{f}(m)$ for $m \in M(H)$. Notice that the image of
$\operatorname{res}_{f}$ always is contained in $M(K)^{W_{G} K_{f}}$. Next we define a map

$$
\mu(H): \bigoplus_{\substack{(K) \in I,(K) \leq(H)}} \bigoplus_{\substack{W_{G} K \cdot f \in \in \\ W_{G} K \backslash \operatorname{mor}(K, H)}} T_{K} M^{W_{G} K_{f}} \rightarrow M(H)
$$

by requiring that its restriction to the summand, which belongs to $(K) \in I$ and $W_{G} K \cdot f \in W_{G} K \backslash \operatorname{mor}(K, H)$, is the composite of the inclusion $T_{K} M^{W_{G} K_{f}} \rightarrow$ $M(K)$ with $\operatorname{ind}_{f}: M(K) \rightarrow M(H)$. We want to show that the composite

$$
\begin{aligned}
& \nu(H) \circ \mu(H): \bigoplus_{\substack{(K) \in I,(K) \leq(H)}} \bigoplus_{\substack{W_{G} K \cdot f \in \\
W_{G} K \backslash \operatorname{mor}(K, H)}} T_{K} M^{W_{G} K_{f}} \\
& \rightarrow \prod_{(K) \in I} \prod_{\substack{W_{G} K \cdot f \in \\
W_{G} K \backslash \operatorname{mor}(K, H)}} T_{K} M^{W_{G} K_{f}}=\bigoplus_{\substack{(K) \in I,(K) \leq(H)}} \bigoplus_{\substack{W_{G} K \cdot f \in \\
W_{G} K \backslash \operatorname{mor}(K, H)}} T_{K} M^{W_{G} K_{f}}
\end{aligned}
$$

is bijective. If $K$ is subconjugated to $H$, we write $(K) \leq(H)$. Fix $(K),(L) \in$ $I$ with $(K) \leq(H)$ and $(L) \leq(H)$ and $W_{G} K \cdot f \in \operatorname{mor}(K, H)$ and $W_{G} L$. $g \in \operatorname{mor}(L, H)$. Then the homomorphism $T_{K} M^{W_{G} K_{f}} \rightarrow T_{L} M^{W_{G} L_{g}}$ given by $\nu(H) \circ \mu(H)$ and the summands corresponding to $(K, f)$ and $(L, g)$ is induced by the composite

$$
\begin{align*}
\alpha_{(K, f),(L, g)}: T_{K} M^{W_{G} K_{f}} \xrightarrow{i} M(K) \xrightarrow{\operatorname{ind}_{f: K \rightarrow \mathrm{im}(f)}} M(\operatorname{im}(f)) \xrightarrow{\operatorname{ind}_{\mathrm{im}(f)}^{H}} M(H) \\
\xrightarrow{\operatorname{res}_{H}^{\mathrm{im}(g)}} M(\operatorname{im}(g)) \xrightarrow{\operatorname{res}_{g: L \rightarrow \mathrm{im}(g)}} M(L) \xrightarrow{\rho_{L}} T_{L} M, \quad, \tag{5.3}
\end{align*}
$$

where $i$ is the inclusion. The double coset formula implies

$$
\begin{align*}
& \operatorname{res}_{H} \operatorname{im}_{H}(g) \\
& \circ \operatorname{ind}_{\mathrm{im}(f)}^{H}  \tag{5.4}\\
& \quad=\sum_{\substack{\operatorname{im}(g) h \operatorname{im}(f) \in \\
\operatorname{im}(g) \backslash H / \operatorname{im}(f)}} \operatorname{ind}_{c(h): \operatorname{im}(f) \cap h^{-1} \operatorname{im}(g) h \rightarrow \operatorname{im}(g)} \circ \operatorname{res}_{\mathrm{im}(f)}^{\operatorname{im}(f) \cap h^{-1} \operatorname{im}(g) h} .
\end{align*}
$$

The composite

$$
\begin{aligned}
T_{K} M^{W_{G} K_{f} \xrightarrow{i} M(K) \xrightarrow{\operatorname{ind}_{f: K \rightarrow \operatorname{im}(f)}}} \begin{aligned}
& M(\mathrm{im}(f)) \\
& \xrightarrow{\operatorname{res}_{\mathrm{im}(f)}^{\mathrm{im}(f)} \mathrm{h}^{-1} \mathrm{im}(g) h}
\end{aligned} M\left(\mathrm{im}(f) \cap h^{-1} \operatorname{im}(g) h\right)
\end{aligned}
$$

is trivial by the definition of $T_{K} M$ if $\operatorname{im}(f) \cap h^{-1} \operatorname{im}(g) h \neq \operatorname{im}(f)$ holds. Hence $\alpha_{(K, f),(L, g)} \neq 0$ is only possible if $\operatorname{im}(f) \cap h^{-1} \operatorname{im}(g) h=\operatorname{im}(f)$ for some $h \in H$ and hence $(K) \leq(L)$ hold.

Suppose that $(K)=(L)$. Then $K=L$ by our choice of representatives. Suppose that $\alpha_{(K, f),(K, g)} \neq 0$. We have already seen that this implies $\operatorname{im}(f) \cap$ $h^{-1} \operatorname{im}(g) h=\operatorname{im}(f)$ for some $h \in H$. Since $|\operatorname{im}(f)|=|K|=|\operatorname{im}(g)|$ we conclude
$h^{-1} \operatorname{im}(g) h=\operatorname{im}(f)$ and therefore $W_{G} K \cdot f=W_{G} K \cdot g$ in $W_{G} K \backslash \operatorname{mor}(K, H)$. This implies already $f=g$ as group homomorphism $K \rightarrow H$ by our choice of representatives. The double coset formula (5.4) implies that $\alpha_{(K, f),(K, f)}$ is $\left|H \cap N_{G} \operatorname{im}(f)\right| \cdot \operatorname{id}_{T_{K} M^{W_{G} K_{f}}}$ since for all $h \in N_{G} \operatorname{im}(f) \cap H$ the composite

$$
\begin{aligned}
T_{K} M^{W_{G} K_{f}} \xrightarrow{i} M(K) \xrightarrow{\operatorname{ind}_{f: K \rightarrow \mathrm{im}(f)}} M(\operatorname{im}(f)) \\
\xrightarrow{\operatorname{ind}_{c(h): \operatorname{im}(f) \rightarrow \mathrm{im}(f)}} M(\operatorname{im}(f))
\end{aligned}
$$

agrees with $T_{K} M^{W_{G}} K_{f} \xrightarrow{i} M(K) \xrightarrow{\operatorname{ind}_{f: K \rightarrow \mathrm{im}(f)}} M(\operatorname{im}(f))$. Since the order of $\left|H \cap N_{G} \operatorname{im}(f)\right|$ is invertible in $R$ by assumption, $\alpha_{(K, f),(K, f)}$ is bijective.

We conclude that $\nu(H) \circ \mu(H)$ can be written as a matrix of maps which has upper triangular form and isomorphisms on the diagonal. Therefore $\nu(H) \circ \mu(H)$ is surjective. This shows that $\nu(H)$ is surjective. This finishes the proof of Theorem 5.2

Theorem 5.5 (The equivariant Chern character and Mackey structures). Let $\mathcal{H}_{?}^{*}$ be a proper equivariant cohomology theory. Define a contravariant functor

$$
\mathcal{H}_{?}^{q}(\{\mathrm{pt} .\}): \text { FGINJ } \rightarrow R-\mathrm{MOD}
$$

by sending a homomorphism $\alpha: H \rightarrow K$ to the composite

$$
\mathcal{H}_{K}^{q}(\{\mathrm{pt} .\}) \xrightarrow{\mathcal{H}^{q}(\mathrm{pr})} \mathcal{H}_{K}^{q}(K / H) \xrightarrow{\operatorname{ind}_{\alpha}} \mathcal{H}_{H}^{q}(\{\mathrm{pt} .\})
$$

where $\operatorname{pr}: H / K=\operatorname{ind}_{\alpha}(\{\mathrm{pt}\}.) \rightarrow\{\mathrm{pt}$.$\} is the projection and \operatorname{ind}_{\alpha}$ comes from the induction structure of $\mathcal{H}_{?}^{*}$. Suppose that it extends to a Mackey functor for every $q \in \mathbb{Z}$. Then
(a) For every group $G$ the $\operatorname{RSub}(G ; \mathcal{F})$-module $\mathcal{H}_{G}^{q}(G /$ ?) of (3.7) is injective as $R \mathrm{Sub}(G ; \mathcal{F})$-module, provided that $R$ is semisimple;
(b) We obtain a natural transformation of proper equivariant cohomology theories with values in $R$-modules

$$
\operatorname{ch}_{?}^{*}(X, A): \mathcal{H}_{?}^{*} \rightarrow \mathcal{B} \mathcal{H}_{?}^{*}
$$

In particular we get for every proper $G$ - $C W$-pair $(X, A)$ and every $n \in \mathbb{Z}$ a natural $R$-homomorphism

$$
\begin{aligned}
\operatorname{ch}_{G}^{n}(X, A): & \mathcal{H}_{G}^{n}(X, A) \\
& \rightarrow \mathcal{B} \mathcal{H}_{G}^{n}(X, A):=\prod_{p+q=n} H_{R S u b(G ; \mathcal{F})}^{p}\left(X, A ; \mathcal{H}_{G}^{q}(G / ?)\right) .
\end{aligned}
$$

It is bijective for all proper relative finite $G$ - $C W$-pairs $(X, A)$ and $n \in \mathbb{Z}$. If $\mathcal{H}_{\text {? }}^{*}$ satisfies the disjoint union axiom, it is bijective for all proper $G$ $C W$-pairs $(X, A)$ and $n \in \mathbb{Z}$;
(c) Define for finite subgroup $H \subseteq G$ the $R\left[W_{G} H\right]$-module $T_{H}\left(\mathcal{H}_{H}^{q}(\{\mathrm{pt}\}).\right)$ by

$$
\operatorname{ker}\left(\prod_{L \subsetneq H} \operatorname{ind}_{L}^{K} \circ \mathcal{H}^{q}(\operatorname{pr}: H / L \rightarrow\{\mathrm{pt} .\}): \mathcal{H}_{H}^{q}(\{\mathrm{pt} .\}) \rightarrow \prod_{L \subsetneq H} \mathcal{H}_{L}^{q}(\{\mathrm{pt} .\})\right)
$$

Then the Bredon cohomology $\mathcal{B H}_{G}^{n}(X, A)$ of a proper $G$ - $C W$-pair $(X, A)$ is naturally $R$-isomorphic to

$$
\prod_{p+q=n} \prod_{(H), H \subseteq G \text { finite }} \operatorname{hom}_{R W_{G} H}\left(H_{p}\left(C_{G} H \backslash X^{H} ; R\right), T_{H}\left(\mathcal{H}_{H}^{q}(\{\mathrm{pt} .\})\right)\right.
$$

provided that $R$ is semisimple.
Proof. (a) This follows from Theorem 5.2 since for every finite subgroup $H \subseteq G$ the group $W_{G} H$ is finite and hence the ring $R\left[W_{G} H\right]$ is semisimple and every $R\left[W_{G} H\right]$-module is injective.
(b) This follows from assertion (a) applied in the case $R=\mathbb{Q}$ together with Theorem 4.6
(디) Since $R$ is semisimple, the ring $R\left[W_{G} H\right]$ is semisimple and every $R\left[W_{G} H\right]$ module is injective for every finite subgroup $H$. Because the map $\nu$ of (5.1) is an isomorphism by Theorem 5.2 it remains to show for a $C_{G} H$-module $N$
$\operatorname{hom}_{R S u b(G ; \mathcal{F})}\left(H_{p}\left(C_{G} ? \backslash X^{?} ; R\right), i(H)!N\right)=\operatorname{hom}_{R W_{G} H}\left(H_{p}\left(C_{G} H \backslash X^{H} ; R\right), N\right)$.
This follows from the adjunction $\left(i(H)^{*}, i(H)!\right.$ ) of Lemma 2.13 (回).
Example 5.6 (Mackey structures for Borel cohomology). Let $\mathcal{K}^{*}$ be a cohomology theory for (non-equivariant) $C W$-pairs with values in $R$-modules for a commutative ring $R$ such that $\mathbb{Q} \subseteq R$ and $R$ is semisimple. In Example 1.6 we have assigned to it an equivariant cohomology theory called equivariant Borel cohomology by

$$
\mathcal{H}_{G}^{n}(X, A)=\mathcal{K}^{n}\left(E G \times_{G}(X, A)\right)
$$

We claim that the assumptions appearing in Theorem5.5 are satisfied. Namely, the contravariant functor

$$
\text { FGINJ } \rightarrow R-\mathrm{MOD}, \quad H \mapsto \mathcal{K}^{n}(B H)
$$

extends to a Mackey functor, the necessary covariant functor comes from the Becker-Gottlieb transfer (see for instance [5] and [6] Corollary 6.4 on page 206]). Hence we get from Theorem 5.5 for every group $G$ and every proper $G$ - $C W$-pair ( $X, A$ ) natural $R$-maps

$$
\begin{aligned}
\operatorname{ch}_{G}^{n}(X, A): \mathcal{K}^{n}\left(E G \times_{G}(X, A)\right) & \xlongequal{\cong} \prod_{p+q=n} H_{R \mathrm{Sub}(G ; \mathcal{F})}^{p}\left(X, A ; \mathcal{K}^{q}(B ?)\right) \\
& \cong \prod_{p+q=n} \prod_{(H), H \subseteq G \text { finite }} \operatorname{hom}_{R W_{G} H}\left(H_{p}\left(C_{G} H \backslash X^{H} ; R\right), T_{H}\left(\mathcal{K}^{q}(B H)\right),\right.
\end{aligned}
$$

if we define
$T_{H}\left(\mathcal{K}^{q}(B H)\right):=\operatorname{ker}\left(\prod_{K \subsetneq H} \mathcal{K}^{q}(B K \rightarrow B H): \mathcal{K}^{q}(B H) \rightarrow \prod_{K \subsetneq H} \mathcal{K}^{q}(B K)\right)$.
If $(X, A)$ is relative finite or if $\mathcal{K}^{*}$ satisfies the disjoint union axiom, then these maps $\operatorname{ch}_{G}^{n}(X, A)$ are bijective.

Remark 5.7. We mention that this does not prove Theorem0.1 since we cannot apply it to $\mathcal{K}^{*}:=K^{*} \otimes_{\mathbb{Z}} \mathbb{Q}$. The problem is that $K^{*} \otimes_{\mathbb{Z}} \mathbb{Q}$ defines all axioms of a cohomology theory but not the disjoint union axiom. But this is needed if we want to deal with classifying spaces $B G$ of groups which are not finite $C W$-complexes, for instance of groups containing torsion (see also Remark 1.2 and Example 1.3).

A proof of Theorem 0.1 will be given in [10].
Example 5.8 (Equivariant $K$-theory and Mackey structures). In Example 1.7 we have introduced the equivariant cohomology theory $K_{?}^{*}$ given by topological $K$-theory. Recall that it takes values in $R$-modules for $R=\mathbb{Z}$. Notice that for a finite group $H$ we get an identification of $K_{H}^{0}(\{\mathrm{pt}\}$.$) with the$ complex representation ring $R(H)$ and the associated contravariant functor

$$
K_{?}^{q}: \text { FGINJ } \rightarrow R-\mathrm{MOD}, \quad H \mapsto K_{H}^{q}(\{\mathrm{pt} .\})=R(?)
$$

sends an injective group homomorphism $\alpha: H \rightarrow G$ of finite groups to the homomorphism of abelian groups $R(G) \rightarrow R(H)$ given by restriction with $\alpha$. Induction with $\alpha$ induces a covariant functor $H \mapsto R(H)$ and it turns out that this defines a Mackey structure on $K_{?}^{q}$.

For rationalized equivariant topological $K$-theory $K_{?}^{*} \otimes_{\mathbb{Z}} \mathbb{Q}$ the equivariant Chern character of Theorem 5.5 can be identified with the one constructed in [12] for proper relative finite $G$ - $C W$-pairs $(X, A)$.

## 6. Multiplicative Structures

Next we want to introduce a multiplicative structure on a proper equivariant cohomology theory $\mathcal{H}_{?}^{*}$ and show that it induces one on the associated Bredon cohomology $\mathcal{B H}$ ? such that the equivariant Chern character is compatible with it.

We begin with the non-equivariant case. Let $\mathcal{H}^{*}$ be a (non-equivariant) cohomology theory with values in $R$-modules. A multiplicative structure assigns to a $C W$-complex $X$ with $C W$-subcomplexes $A, B \subseteq X$ natural $R$-homomorphisms

$$
\begin{equation*}
\cup: \mathcal{H}^{n}(X, A) \otimes_{R} \mathcal{H}^{n^{\prime}}(X, B) \quad \rightarrow \quad \mathcal{H}^{n+n^{\prime}}(X, A \cup B) \tag{6.1}
\end{equation*}
$$

This product is required to be compatible with the boundary homomorphism of the long exact sequence of a pair, to be graded commutative, to be associative and to have a unit $1 \in \mathcal{H}^{0}(\{\mathrm{pt}\}$.$) .$

Given a multiplicative structure on $\mathcal{H}^{*}$, we obtain for every $p, q \in \mathbb{Z}$ a pairing

$$
\cup: \mathcal{H}^{q}(\{\mathrm{pt} .\}) \otimes_{R} \mathcal{H}^{q^{\prime}}(\{\mathrm{pt.} .\}) \rightarrow \mathcal{H}^{q+q^{\prime}}(\{\mathrm{pt} .\})
$$

It yields on singular (or equivalently cellular) cohomology a product
$H^{p}\left(X, A ; \mathcal{H}^{q}(\{\mathrm{pt}\}).\right) \otimes_{R} H^{p^{\prime}}\left(X, B ; \mathcal{H}^{q^{\prime}}(\{\mathrm{pt}\}).\right) \rightarrow H^{p+p^{\prime}}\left(X, A \cup B ; \mathcal{H}^{q+q^{\prime}}(\{\mathrm{pt}\}).\right)$.
The collection of these pairings induce a multiplicative structure on the cohomology theory given by $\prod_{p+q=n} H^{p}\left(X, A ; \mathcal{H}^{q}(\{\mathrm{pt}\}).\right)$. The straightforward proof of the next lemma is left to the reader.
Lemma 6.2. Let $R$ be a commutative ring with $\mathbb{Q} \subseteq R$. Let $\mathcal{H}^{*}$ be a (nonequivariant) cohomology theory satisfying the disjoint union axiom which comes with a multiplicative structure.

Then the (non-equivariant) Chern character of Example 4.1

$$
\operatorname{ch}^{n}(X, A): \mathcal{H}^{n}(X, A) \stackrel{\cong}{\rightrightarrows} \prod_{p+q=n} H^{p}\left(X, A, \mathcal{H}^{q}(*)\right)
$$

is compatible with the given multiplicative structure on $\mathcal{H}^{*}$ and the induced multiplicative structure on the target.

Next we deal with the equivariant version. We only deal with the proper case, the definitions below make also sense without this condition.

Let $\mathcal{H}_{G}^{*}$ be a proper $G$-cohomology theory. A multiplicative structure assigns to a proper $G$ - $C W$-complex $X$ with $G$ - $C W$-subcomplexes $A, B \subseteq X$ natural $R$ homomorphisms

$$
\begin{equation*}
\cup: \mathcal{H}_{G}^{n}(X, A) \otimes_{R} \mathcal{H}_{G}^{n^{\prime}}(X, B) \quad \rightarrow \quad \mathcal{H}_{G}^{n+n^{\prime}}(X, A \cup B) \tag{6.3}
\end{equation*}
$$

This product is required to be compatible with the boundary homomorphism of the long exact sequence of a $G$ - $C W$-pair, to be graded commutative, to be associative and to have a unit $1 \in \mathcal{H}_{G}^{0}(X)$ for every proper $G$ - $C W$-complex $X$

Let $\mathcal{H}_{?}^{*}$ be a proper equivariant cohomology theory. A multiplicative structure on it assigns a multiplicative structure to the associated proper $G$-cohomology theory $\mathcal{H}_{G}^{*}$ for every group $G$ such that for each group homomorphism $\alpha: H \rightarrow G$ the maps given by the induction structure of (1.4)

$$
\operatorname{ind}_{\alpha}: \mathcal{H}_{G}^{n}\left(\operatorname{ind}_{\alpha}(X, A)\right) \quad \cong \mathcal{H}_{H}^{n}(X, A)
$$

are in the obvious way compatible with the multiplicative structures on $\mathcal{H}_{G}^{*}$ and $\mathcal{H}_{H}^{*}$.

Next we explain how a given multiplicative structure on $\mathcal{H}_{?}^{*}$ induces one on $\mathcal{B} \mathcal{H}_{\text {? }}^{*}$. We have to specify for every group $G$ a multiplicative structure on
the $G$-cohomology theory $\mathcal{B} \mathcal{H}_{G}^{*}$. Consider a $G$ - $C W$-complex $X$ with $G$ - $C W$ subcomplexes $A, B \subseteq X$. For two contravariant $\operatorname{Ror}(G ; \mathcal{F})$-chain complexes $C_{*}$ and $D_{*}$ define the contravariant $R \operatorname{Or}(G ; \mathcal{F})$-chain complexes $C_{*} \otimes_{R} D_{*}$ by sending $G / H$ to the tensor product of $R$-chain complexes $C_{*}(G / H) \otimes_{R} D_{*}(G / H)$. Let

$$
a_{*}: C_{*}^{R \operatorname{Or}(G ; \mathcal{F})}(X, A) \otimes_{R} C_{*}^{R \mathrm{Or}(G ; \mathcal{F})}(X, B) \xrightarrow{\cong} C_{*}^{R \mathrm{Or}(G ; \mathcal{F})}((X, A) \times(X, B))
$$

be the isomorphism of $\operatorname{Ror}(G ; \mathcal{F})$-chain complexes which is given for an object $G / H$ by the natural isomorphism of cellular $R$-chain complexes

$$
C_{*}\left(X^{H}, A^{H}\right) \otimes_{R} C_{*}\left(X^{H}, B^{H}\right) \stackrel{\cong}{\longrightarrow} C_{*}\left(\left(X^{H}, A^{H}\right) \times\left(X^{H}, B^{H}\right)\right) .
$$

The multiplicative structure on $\mathcal{H}_{G}^{*}$ yields a map of contravariant $R \operatorname{Or}(G ; \mathcal{F})$ modules

$$
c: \mathcal{H}_{G}^{q}(G / ?) \otimes_{R} \mathcal{H}_{G}^{q}(G / ?) \rightarrow \mathcal{H}_{G}^{q+q}(G / ?) .
$$

Let

$$
\Delta:(X ; A \cup B) \rightarrow(X, A) \times(X, B), \quad x \mapsto(x, x)
$$

be the diagonal embedding. Define a $R$-cochain map by the composite

$$
\begin{aligned}
& \quad b^{*}: \operatorname{hom}_{R \operatorname{Or}(G ; \mathcal{F})}\left(C_{*}^{R \operatorname{Or}(G ; \mathcal{F})}(X, A), \mathcal{H}_{G}^{q}(G / ?)\right) \\
& \qquad \otimes_{R} \operatorname{hom}_{R \operatorname{Or}(G ; \mathcal{F})}\left(C_{*}^{R \operatorname{Or}(G ; \mathcal{F})}(X, B), \mathcal{H}_{G}^{q^{\prime}}(G / ?)\right) \xrightarrow{\otimes_{R}} \\
& \operatorname{hom}_{R \operatorname{Or}(G ; \mathcal{F})}\left(C_{*}^{R \operatorname{Or}(G ; \mathcal{F})}(X, A) \otimes_{R} C_{*}^{R \operatorname{Or}(G ; \mathcal{F})}(X, B), \mathcal{H}_{G}^{q}(G / ?) \otimes_{R} \mathcal{H}_{G}^{q^{\prime}}(G / ?)\right) \\
& \xrightarrow{\operatorname{hom}_{R \mathrm{Or}(G ; \mathcal{F})}\left(\left(a_{*}\right)^{-1}, c\right)} \operatorname{hom}_{R \mathrm{Or}(G ; \mathcal{F})}\left(C_{*}^{R \operatorname{Or}(G ; \mathcal{F})}((X, A) \times(X, B)), \mathcal{H}_{G}^{q+q^{\prime}}(G / ?)\right) \\
& \xrightarrow{\operatorname{hom}_{R \mathrm{Or}(G ; \mathcal{F})}\left(C_{*}^{R \mathrm{Or}(G ; \mathcal{F})}(\Delta), \operatorname{id)}\right.} \operatorname{hom}_{R \operatorname{Or}(G ; \mathcal{F})}\left(C_{*}^{R \operatorname{Or}(G ; \mathcal{F})}(X, A \cup B), \mathcal{H}_{G}^{q+q^{\prime}}(G / ?)\right) .
\end{aligned}
$$

There is a canonical $R$-map

$$
H^{*}\left(C^{*} \otimes_{R} D^{*}\right) \rightarrow H^{*}\left(C^{*} \otimes_{R} D^{*}\right)
$$

for two $R$-cochain complexes $C^{*}$ and $D^{*}$. This map together with the map induced by $b^{*}$ on cohomology yields an $R$-homomorphism

$$
\begin{aligned}
H_{R \operatorname{Or}(G ; \mathcal{F})}^{p}\left(X, A ; \mathcal{H}_{G}^{q}(G / ?)\right) \otimes_{R} H_{R \operatorname{Or}(G ; \mathcal{F})}^{p^{\prime}} & \left(X, B ; \mathcal{H}_{G}^{q^{\prime}}(G / ?)\right) \\
& \rightarrow H_{R \operatorname{Or}(G ; \mathcal{F})}^{p+p^{\prime}}\left(X, A \cup B ; \mathcal{H}_{G}^{q}(G / ?)\right)
\end{aligned}
$$

The collection of these $R$-homomorphisms yields the desired multiplicative structure on $\mathcal{B} \mathcal{H}_{G}^{*}$. We leave it to the reader to check that the axioms of a multiplicative structure on $\mathcal{B} \mathcal{H}_{G}^{*}$ are satisfied and that all these are compatible with the induction structure so that we obtain a multiplicative structure on the equivariant cohomology theory $\mathcal{B H}_{?}^{*}$.

We also omit the lengthy but straightforward proof of the following result which is based on Theorem 4.6 Lemma 6.2 and the compatibility of the multiplicative structure with the induction structure.

Theorem 6.4 (The equivariant Chern character and multiplicative structures). Let $R$ be a commutative ring such that $\mathbb{Q} \subseteq R$. Suppose that $\mathcal{H}_{?}^{*}$ is a proper cohomology theory with values in $R$-modules which comes with a multiplicative structure. Suppose that the $\operatorname{RSub}(G ; \mathcal{F})$-module $\mathcal{H}_{G}^{q}(G /$ ?) of (3.7), which sends $G / H$ to $\mathcal{H}_{G}^{q}(G / H)$, is injective for each $q \in \mathbb{Z}$.

Then the natural transformation of equivariant cohomology theories appearing in Theorem 4.6

$$
\mathrm{ch}_{?}^{*}: \mathcal{H}_{?}^{*} \rightarrow \mathcal{B} \mathcal{H}_{?}^{*}
$$

is compatible with the given multiplicative structure on $\mathcal{H}_{?}^{*}$ and the induced multiplicative structure on $\mathcal{B} \mathcal{H}_{\text {? }}^{*}$.

Remark 6.5 (External products and restriction structures). One can also define an external product for a proper equivariant cohomology theory $\mathcal{H}_{?}^{*}$ with values in $R$-modules. It assigns to every two groups $G$ and $H$, a proper $G$ $C W$-pair $(X, A)$, a proper $H-C W$-pair $(Y, B)$ and $p, q \in \mathbb{Z}$ an $R$-homomorphism

$$
\times: \mathcal{H}_{G}^{p}(X, A) \otimes_{R} \mathcal{H}_{H}^{q}(Y, B) \rightarrow \mathcal{H}_{G \times H}^{p+q}((X, A) \times(Y, B)) .
$$

One requires graded commutativity, associativity, the existence of a unit $1 \in$ $\mathcal{H}_{\{1\}}^{0}(\{\mathrm{pt}\}$.$) and compatibility with the induction structure and the boundary$ homomorphism associated to a pair. One can show that $\mathcal{B} \mathcal{H}_{?}^{*}$ inherits an external product and prove the analogon of Theorem 6.4 for external products.

One can also introduce the notion of a restriction structure on $\mathcal{H}_{?}^{*}$. It yields for every injective group homomorphism $\alpha: H \rightarrow G$, every proper $G$ - $C W$-pair $(X, A)$ and $p \in \mathbb{Z}$ an $R$-homomorphism

$$
\left.\operatorname{res}_{\alpha}: \mathcal{H}_{G}^{p}(X, A)\right) \rightarrow \mathcal{H}_{H}^{p}\left(\operatorname{res}_{\alpha}(X, A)\right)
$$

Again certain axioms are required such as compatibility with the boundary homomorphism associated to pair, compatibility with induction for group isomorphisms $\alpha: H \stackrel{\cong}{\cong} G$, compatibility with conjugation, the double coset formula and compatibility for projections onto quotients under free actions. One can show that $\mathcal{B H} \mathcal{H}_{\text {? }}^{*}$ inherits a restriction structure and prove the analogon of Theorem 6.4 for restriction structures.

An external product together with a restriction structure yields a multiplicative structure as follows. Consider $G$ - $C W$-pairs $(X, A)$ and $(X, B)$. Let $d: G \rightarrow G \times G$ and $D:(X, A \cup B) \rightarrow(X, A) \times(X, B)$ be the diagonal maps. Define

$$
\begin{equation*}
\cup: \mathcal{H}_{G}^{m}(X, A) \otimes_{R} \mathcal{H}_{G}^{n}(X, A) \quad \rightarrow \quad \mathcal{H}_{G}^{m+n}(X, A \cup B) \tag{6.6}
\end{equation*}
$$

to be the composite

$$
\begin{aligned}
\mathcal{H}_{G}^{m}(X, A) \otimes_{R} \mathcal{H}_{G}^{n}(X, A) & \stackrel{\times}{\longrightarrow} \mathcal{H}_{G \times G}^{m+n}((X, A) \times(X, B)) \\
& \xrightarrow{\operatorname{res}_{d}} \mathcal{H}_{G}^{m+n}((X, A) \times(X, B)) \xrightarrow{\mathcal{H}_{G}^{m+n}(D)} \mathcal{H}_{G}^{m+n}(X, A \cup B) .
\end{aligned}
$$

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[^0]:    *email: lueck@math.uni-muenster.de
    www: http://www.math.uni-muenster.de/u/lueck/
    FAX: 492518338370

