Complex structures on indecomposable 6-dimensional nilpotent real Lie Algebras *

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December 8, 2018

Abstract

We compute all complex structures on indecomposable 6-dimensional real Lie algebras and their equivalence classes. We also give for each of them a global holomorphic chart on the connected simply connected Lie group associated to the real Lie algebra and write down the multiplication in that chart.

1 Introduction.

In the classification of nilmanifolds, a important question is to determine the set of all integrable left invariant complex structures on a given connected simply connected real nilpotent finite dimensional Lie group, or at the Lie algebra level the set $\mathfrak{X}_{\mathfrak{g}}$ of integrable complex structures on the nilpotent Lie algebra $\mathfrak{g}$, and its moduli space ([3],[2], [10]). In the case of 6-dimensional real nilpotent Lie groups, an upper bound has been given in [9] for the dimension of $\mathfrak{X}_{\mathfrak{g}}$, based on a subcomplex of the Dolbeault complex. These bounds are listed there, and Lie algebras which do not admit complex structures are specified. However, no detailed descriptions of the spaces $\mathfrak{X}_{\mathfrak{g}}$ are given. The aim of the present paper is to contribute in this area by supplying explicit computations of the various $\mathfrak{X}_{\mathfrak{g}}$ and their equivalence classes for any indecomposable 6-dimensional real Lie algebra $\mathfrak{g}$. We here are interested only in indecomposable Lie algebras, though direct products could be processed in the same way.

2 Preliminaries.

2.1 Labeling the algebras.

There are 22 indecomposable nonisomorphic nilpotent real 6-dimensional Lie algebras in the Morozov classification, labeled $M1$-$M22$ ([5]). Types $M14$ and $M18$ are splitted in $M14_{\pm 1}$ and $M18_{\pm 1}$. Over $\mathbb{C}$, types $M14$ and $M18$ are not splitted and types $M5$ and $M10$ do not appear. In [6], one is concerned with rank and weight systems over $\mathbb{C}$, and a different classification is used. The correspondance with Morozov types appears there on page 130. In the present paper, we label the algebras according to [6], except for $M5$, $M10$, $M14$ and $M18$. Note that $M5$ is the realification $\mathfrak{n}_{\mathbb{R}}$ of the 3-dimensional complex Heisenberg algebra $\mathfrak{n}$. Though $M10$ is not a realification, it appears as a subalgebra of the realification $(\mathfrak{g}_4)_{\mathbb{R}}$ of the complex 4-dimensional generic filiform Lie algebra $\mathfrak{g}_4$ in the isomorphic realisation $[a_1, a_2] = a_3, [a_1, a_3] = a_4, [a_2, a_3] = a_4$: just take $x_1 = a_1, x_2 = ia_2, x_3 = ia_3, x_4 = a_3, x_5 = ia_4, x_6 = a_4$. Let $\mathfrak{g}$ be any of the labeled 6-dimensional real Lie algebras, and let G_0 be the connected simply connected Lie group with Lie algebra $\mathfrak{g}$. From the commutation relations of the basis $(x_j)_{1\leqslant j\leqslant 6}$ of $\mathfrak{g}$ we use, the second kind canonical coordinates $(x \in G_0)$

$$x = \exp(x^1 x_1) \exp(y^1 x_2) \exp(x^2 x_3) \exp(y^2 x_4) \exp(x^3 x_5) \exp(y^3 x_6) \tag{1}$$

* *Math. Subj. Class. [2000]* : 17B30 (Primary), 53C15 (Secondary)

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yield a global chart for G_0 (see [11], Th. 3.18.11, p.243). We use this chart for G_0 in all cases but the case of $M5$ where the natural chart is used instead. For $1 \leqslant j \leqslant 6$, denote by X_j the left invariant vector field on G_0 associated to x_j, *i.e.*

$$(X_j f)(x) = \left[\frac{d}{dt} f(x \exp(tx_j))\right]_{t=0} \quad \forall\, f \in C^\infty(G_0).$$

Then due to the commutation relations, we have in each case except $M5$:

$$X_3 = \frac{\partial}{\partial x^2}\ ;\ X_4 = \frac{\partial}{\partial y^2}\ ;\ X_5 = \frac{\partial}{\partial x^3}\ ;\ X_6 = \frac{\partial}{\partial y^3}. \tag{2}$$

2.2 Complex structures.

Let $\mathfrak{g}$ any finite dimensional real Lie algebra, and let G_0 be the connected simply connected real Lie group with Lie algebra $\mathfrak{g}$. An almost complex structure on $\mathfrak{g}$ is a linear map $J : \mathfrak{g} \to \mathfrak{g}$ such that $J^2 = -1$. An almost complex structure on G_0 is a tensor field $x \mapsto J_x$ which at every point $x \in G_0$ is an endomorphism of $T_x(G_0)$ such that $J_x^2 = -1$. By definition, the almost complex structure on G_0 is left (resp. right) invariant if $J_{ax} = (\hat{L_a})_x J_x$ (resp. $J_{xa} = (\hat{R_a})_x J_x$) for all $a, x \in G_0$, where $(\hat{L_a})_x J_x$ (resp. $(\hat{R_a})_x J_x$) is the endomorphism $(L_a)_{*x} \circ J_x \circ (L_{a^{-1}})_{*ax}$ (resp. $(R_a)_{*x} \circ J_x \circ (R_{a^{-1}})_{*xa}$) of $T_{ax}(G_0)$ (resp. $T_{xa}(G_0)$), with L_a (resp. R_a) the left (resp. right) translation $x \mapsto ax$ (resp. $x \mapsto xa$) and $(.)_*$ the differential. For any almost complex structure J on $\mathfrak{g}$ there is a unique left invariant almost complex $\hat{J}$ structure on G_0 such that $\hat{J}_e = J$ (e is the identity of G_0), and one has $\hat{J}_a = (\hat{L_a})_{*e} J$ for all $a \in G_0$. It is easily seen that $\hat{J}$ is right invariant if and only if

$$J \circ ad\, X = ad\, X \circ J \quad \forall X \in \mathfrak{g},$$

that is $(\mathfrak{g}, J)$ is a *complex Lie algebra*. From the Newlander-Nirenberg theorem ([8]), $\hat{J}$ is *integrable*, that is G_0 can be given the structure of a complex manifold with the same underlying real structure and such that $\hat{J}$ is the canonical complex structure, if and only if the torsion tensor of $\hat{J}$ vanishes, i.e. :

$$[\hat{J}X, \hat{J}Y] - [X, Y] - \hat{J}[\hat{J}X, Y] - \hat{J}[X, \hat{J}Y] = 0$$

for all vector fields X, Y on G_0. By left invariance, this is equivalent to

$$[JX, JY] - [X, Y] - J[JX, Y] - J[X, JY] = 0 \quad \forall X, Y \in \mathfrak{g}. \tag{3}$$

By a complex structure on $\mathfrak{g}$, we'll mean an *integrable* almost complex structure on $\mathfrak{g}$, that is one satisfying (3).

Let J a complex structure on $\mathfrak{g}$ and denote by G the group G_0 endowed with the structure of complex manifold defined by $\hat{J}$. Then a smooth function $f : G_0 \to G_0$ is holomorphic if and only if its differential commutes with $\hat{J}$ ([4], Prop. 2.3 p. 123) : $\hat{J} \circ f_* = f_* \circ \hat{J}$. Hence left translations are holomorphic. Right translations are holomorphic, that is G is a complex Lie group, if and only if $\hat{J}$ is right invariant, i.e. $(\mathfrak{g}, J)$ is a complex Lie algebra. The complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ splits as $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{(1,0)} \oplus \mathfrak{g}^{(0,1)}$ where $\mathfrak{g}^{(1,0)} = \{X - iJX;\, X \in \mathfrak{g}\}$, $\mathfrak{g}^{(0,1)} = \{X + iJX;\, X \in \mathfrak{g}\}$. We will denote $\mathfrak{g}^{(1,0)}$ by $\mathfrak{m}$. The integrability of J amounts to $\mathfrak{m}$ being a complex subalgebra of $\mathfrak{g}_{\mathbb{C}}$. In that way the set of complex structures on $\mathfrak{g}$ can be identified with the set of all complex subalgebras $\mathfrak{m}$ of $\mathfrak{g}_{\mathbb{C}}$ such that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{m} \oplus \bar{\mathfrak{m}}$, bar denoting conjugation in $\mathfrak{g}_{\mathbb{C}}$. This is the algebraic approach. Our approach is more trivial since we simply fix a basis of $\mathfrak{g}$ and compute all possible matrices in that basis for a complex structure. From now on, we'll use the same notation J for J and $\hat{J}$ as well. For any $x \in G_0$, the complexification $T_x(G_0)_{\mathbb{C}}$ of the tangent space also splits as the direct sum of the holomorphic vectors $T_x(G_0)^{(1,0)} = \{X - iJX;\, X \in T_x(G_0)\}$ and the antiholomorphic vectors $T_x(G_0)^{(0,1)} = \{X + iJX;\, X \in T_x(G_0)\}$. Let $H_{\mathbb{C}}(G)$ be the space of complex valued holomorphic functions on G. Then $H_{\mathbb{C}}(G)$ is comprised of all complex smooth functions f on G_0 which are annihilated by any antiholomorphic vector field. This is equivalent to f being annihilated by all

$$\tilde{X}_j^- = X_j + iJX_j \quad 1 \leqslant j \leqslant n \tag{4}$$

whith $(X_j)_{1 \leqslant j \leqslant n}$ the left invariant vector fields associated to a basis $(x_j)_{1 \leqslant j \leqslant n}$ of $\mathfrak{g}$. Hence :

$$H_{\mathbb{C}}(G) = \{f \in C^\infty(G_0)\ ;\ \tilde{X}_j^- f = 0 \ \ \forall j \ \ 1 \leqslant j \leqslant n\}. \tag{5}$$

Finally, the automorphism group $\mathrm{Aut}\,\mathfrak{g}$ of $\mathfrak{g}$ acts on the set $\mathfrak{X}_{\mathfrak{g}}$ of all complex structures on $\mathfrak{g}$ by $J \mapsto \Phi^{-1} \circ J \circ \Phi \quad \forall \Phi \in \mathrm{Aut}\,\mathfrak{g}$. Two complex structures J_1, J_2 on $\mathfrak{g}$ are said to be *equivalent* if they are on the same $\mathrm{Aut}\,\mathfrak{g}$ orbit.

2.3 Presentation of results.

We consider here only indecomposable 6-dimensional nilpotent real Lie algebras which admit complex structures. For each such $\mathfrak{g}$, we first give the commutation relations of the basis $(x_j)_{1\leqslant j\leqslant 6}$ of $\mathfrak{g}$ we use, and the matrices $J = (J_j^k) = (\xi_j^k)$ in that basis of the elements of $\mathfrak{X}_\mathfrak{g}$. The parameters are $\boxed{'boxed'}$. These matrices have been obtained by developping specific programs with the computer algebra system *Reduce* by A. Hearn. The programs solve simultaneously the equation $J^2 = -1$ and the torsion equations $ij|k$ ($1 \leqslant i, j, k \leqslant 6$) obtained by projecting on x_k the equation $[Jx_i, Jx_j] - [x_i, x_j] - J[Jx_i, x_j] - J[x_i, Jx_j] = 0$. We do not enter computational technicalities here, referring instead to the technical report [7]. Let's simply say that the equations are *semilinear* in the sense that they can be solved in a succession of steps, each of which consists in solving some equation of degree 1 in some variable. For all the Lie algebras we consider, we prove that $\mathfrak{X}_\mathfrak{g}$ is a (smooth) submanifold of $\mathbb{R}^{36}$. The dimension of $\mathfrak{X}_\mathfrak{g}$ is equal to the upper bound given in [9] except in the case of $M10$. Then we give the automorphism group of $\mathfrak{g}$, representatives of the various equivalence classes, and the commutation relations of the corresponding algebra $\mathfrak{m} = \mathfrak{g}^{(1,0)}$ in terms of the basis $(\tilde{x}_j)_{1\leqslant j\leqslant 6}$ with $\tilde{x}_j = x_j - iJx_j$. As $\mathfrak{m}$ is a 3-dimensional complex Lie algebra, it is either abelian or isomorphic to the complex Heisenberg Lie algebra $\mathfrak{n}$. Hence, as a real Lie algebra, $\mathfrak{m}$ is either abelian or isomorphic to $M5$. Finally, we compute the left invariant vector fields X_1, X_2 on G_0 in terms of the second kind canonical coordinates (1) (except in the case of $M5$, the 4 others appear in (2)), a global holomorphic chart on G and the explicit look of the multiplication in G in terms of that chart. The fact that left translations are holomorphic, though the multiplication isn't except for the canonical structure on $M5$, appears clearly on these formulae. There we make use of the following formula which is easily checked from the Campbell-Hausdorff-Baker Formula :

$$e^X e^Y = e^{[X,Y]+\frac{1}{2}([X,[X,Y]]+[Y,[X,Y]])+\frac{1}{6}([X,[X,[X,Y]]]+[Y,[Y,[X,Y]]])+\frac{1}{4}[X,[Y,[X,Y]]] \bmod \mathcal{C}^5\mathfrak{g}}\, e^Y e^X \tag{6}$$

where $\mathcal{C}^5\mathfrak{g}$ denotes the 5th central derivative. As usual, for complex $z = x + iy$ ($x, y \in \mathbb{R}$),

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \;;\; \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

3 Lie Algebra $\mathcal{G}_{6,3}$ (isomorphic to $M3$).

Commutation relations for $\mathcal{G}_{6,3}$: $[x_1, x_2] = x_4$; $[x_1, x_3] = x_5$; $[x_2, x_3] = x_6$.

3.1 Case $\xi_6^1 \neq 0$.

$$J = \begin{pmatrix} \boxed{\xi_1^1} & \boxed{\xi_2^1} & \boxed{\xi_3^1} & \xi_6^3 & -\xi_6^2 & \boxed{\xi_6^1} \\ * & (-\xi_6^5\xi_6^2 - \xi_5^5\xi_6^1 + \xi_6^2\xi_2^1)/\xi_6^1 & (\xi_6^4\xi_6^2 + \xi_5^4\xi_6^1 + \xi_6^2\xi_3^1)/\xi_6^1 & (\xi_6^3\xi_6^2)/\xi_6^1 & (-{\xi_6^2}^2)/\xi_6^1 & \boxed{\xi_6^2} \\ * & * & (\xi_6^5\xi_6^2 + \xi_5^5\xi_6^1 + \xi_6^3\xi_3^1)/\xi_6^1 & {\xi_6^3}^2/\xi_6^1 & (-\xi_6^3\xi_6^2)/\xi_6^1 & \boxed{\xi_6^3} \\ * & (\xi_6^5\xi_3^1 - \xi_3^5\xi_6^1 + \xi_6^4\xi_2^1)/\xi_6^1 & * & (-\xi_6^5\xi_6^2 - \xi_5^5\xi_6^1 + \xi_6^4\xi_6^3)/\xi_6^1 & \boxed{\xi_5^4} & \boxed{\xi_6^4} \\ * & \boxed{\xi_2^5} & \boxed{\xi_3^5} & * & \boxed{\xi_5^5} & \boxed{\xi_6^5} \\ * & * & * & * & * & * \end{pmatrix} \tag{7}$$

where $J_1^2 = (\xi_6^5{\xi_6^2}^2 + \xi_5^5\xi_6^2\xi_6^1 - \xi_6^4\xi_6^3\xi_6^2 - \xi_5^4\xi_6^3\xi_6^1 + \xi_6^2\xi_6^1\xi_1^1)/{\xi_6^1}^2$;
$J_1^3 = ({\xi_6^5}^2{\xi_6^2}^3 + 2\xi_6^5\xi_5^5{\xi_6^2}^2\xi_6^1 - \xi_6^5\xi_6^4\xi_6^3{\xi_6^2}^2 - \xi_6^5\xi_5^4\xi_6^3\xi_6^2\xi_6^1 + {\xi_5^5}^2\xi_6^2{\xi_6^1}^2 - \xi_5^5\xi_6^4\xi_6^3\xi_6^2\xi_6^1 - \xi_5^5\xi_5^4\xi_6^3{\xi_6^1}^2 + \xi_6^4\xi_6^3\xi_6^2\xi_6^1\xi_1^1 + \xi_5^4\xi_6^3{\xi_6^1}^2\xi_1^1 + \xi_6^2{\xi_6^1}^2)/({\xi_6^1}^2(\xi_6^4\xi_6^2 + \xi_5^4\xi_6^1))$;
$J_2^3 = (-{\xi_6^5}^2{\xi_6^2}^2 - 2\xi_6^5\xi_5^5\xi_6^2\xi_6^1 - {\xi_5^5}^2{\xi_6^1}^2 + \xi_6^4\xi_6^3\xi_6^2\xi_2^1 + \xi_5^4\xi_6^3\xi_6^1\xi_2^1 - {\xi_6^1}^2)/(\xi_6^1(\xi_6^4\xi_6^2 + \xi_5^4\xi_6^1))$;
$J_1^4 = (-{\xi_6^5}^3\xi_5^4{\xi_6^2}^2\xi_6^1 - {\xi_6^5}^3{\xi_6^2}^3\xi_3^1 + {\xi_6^5}^2\xi_5^5\xi_6^4{\xi_6^2}^2\xi_6^1 - 2{\xi_6^5}^2\xi_5^5\xi_5^4\xi_6^2{\xi_6^1}^2 - 2{\xi_6^5}^2\xi_5^5{\xi_6^2}^2\xi_6^1\xi_3^1 + {\xi_6^5}^2\xi_3^5{\xi_6^2}^3\xi_6^1 + 2{\xi_6^5}^2\xi_6^4\xi_6^3{\xi_6^2}^2\xi_3^1 + {\xi_6^5}^2\xi_6^4{\xi_6^2}^2\xi_6^1\xi_1^1 + 2{\xi_6^5}^2\xi_5^4\xi_6^3\xi_6^2\xi_6^1\xi_3^1 + 2\xi_6^5{\xi_5^5}^2\xi_6^4\xi_6^2{\xi_6^1}^2 - \xi_6^5{\xi_5^5}^2\xi_5^4{\xi_6^1}^3 - \xi_6^5{\xi_5^5}^2\xi_6^2{\xi_6^1}^2\xi_3^1 + 2\xi_6^5\xi_5^5\xi_3^5{\xi_6^2}^2{\xi_6^1}^2 + 2\xi_6^5\xi_5^5\xi_6^4\xi_6^3\xi_6^2\xi_6^1\xi_3^1 + 2\xi_6^5\xi_5^5\xi_6^4\xi_6^2{\xi_6^1}^2\xi_1^1 + 2\xi_6^5\xi_5^5\xi_5^4\xi_6^3{\xi_6^1}^2\xi_3^1 - 2\xi_6^5\xi_3^5\xi_6^4\xi_6^3{\xi_6^2}^2\xi_6^1 - 2\xi_6^5\xi_3^5\xi_5^4\xi_6^3\xi_6^2{\xi_6^1}^2 + \xi_6^5{\xi_6^4}^2\xi_6^3{\xi_6^2}^2\xi_2^1 + 2\xi_6^5\xi_6^4\xi_5^4\xi_6^3\xi_6^2\xi_6^1\xi_2^1 + \xi_6^5{\xi_5^4}^2\xi_6^3{\xi_6^1}^2\xi_2^1 - \xi_6^5\xi_5^4{\xi_6^1}^3 - \xi_6^5\xi_6^2{\xi_6^1}^2\xi_3^1 + {\xi_5^5}^3\xi_6^4{\xi_6^1}^3 + {\xi_5^5}^2\xi_3^5\xi_6^2{\xi_6^1}^3 + {\xi_5^5}^2\xi_6^4{\xi_6^1}^3\xi_1^1 - 2\xi_5^5\xi_3^5\xi_6^4\xi_6^3\xi_6^2{\xi_6^1}^2 - 2\xi_5^5\xi_3^5\xi_5^4\xi_6^3{\xi_6^1}^3 + \xi_5^5\xi_6^4{\xi_6^1}^3 + \xi_3^5\xi_6^2{\xi_6^1}^3 - \xi_2^5{\xi_6^4}^2\xi_6^3{\xi_6^2}^2\xi_6^1 - 2\xi_2^5\xi_6^4\xi_5^4\xi_6^3\xi_6^2{\xi_6^1}^2 - \xi_2^5{\xi_5^4}^2\xi_6^3{\xi_6^1}^3 + \xi_6^4{\xi_6^1}^3\xi_1^1)/({\xi_6^1}^2({\xi_6^5}^2{\xi_6^2}^2 + 2\xi_6^5\xi_5^5\xi_6^2\xi_6^1 + {\xi_5^5}^2{\xi_6^1}^2 + {\xi_6^1}^2))$;
$J_3^4 = (-{\xi_6^5}^2\xi_6^4{\xi_6^2}^2\xi_3^1 - 2{\xi_6^5}^2\xi_5^4\xi_6^2\xi_6^1\xi_3^1 - 2\xi_6^5\xi_5^5\xi_5^4{\xi_6^1}^2\xi_3^1 + 2\xi_6^5\xi_3^5\xi_6^4{\xi_6^2}^2\xi_6^1 + 2\xi_6^5\xi_3^5\xi_5^4\xi_6^2{\xi_6^1}^2 - \xi_6^5{\xi_6^4}^2{\xi_6^2}^2\xi_2^1 - 2\xi_6^5\xi_6^4\xi_5^4\xi_6^2\xi_6^1\xi_2^1 - \xi_6^5{\xi_5^4}^2{\xi_6^1}^2\xi_2^1 + {\xi_5^5}^2\xi_6^4{\xi_6^1}^2\xi_3^1 + 2\xi_5^5\xi_3^5\xi_6^4\xi_6^2{\xi_6^1}^2 + 2\xi_5^5\xi_3^5\xi_5^4{\xi_6^1}^3 + \xi_2^5{\xi_6^4}^2{\xi_6^2}^2\xi_6^1 + 2\xi_2^5\xi_6^4\xi_5^4\xi_6^2{\xi_6^1}^2 + \xi_2^5{\xi_5^4}^2{\xi_6^1}^3 + \xi_6^4{\xi_6^1}^2\xi_3^1)/(\xi_6^1({\xi_6^5}^2{\xi_6^2}^2 + 2\xi_6^5\xi_5^5\xi_6^2\xi_6^1 + {\xi_5^5}^2{\xi_6^1}^2 + {\xi_6^1}^2))$;
$J_1^5 = (-{\xi_6^5}^2\xi_6^4\xi_5^4{\xi_6^2}^2\xi_6^1 - {\xi_6^5}^2{\xi_5^4}^2\xi_6^2{\xi_6^1}^2 + \xi_6^5\xi_5^5{\xi_6^4}^2{\xi_6^2}^2\xi_6^1 - \xi_6^5\xi_5^5{\xi_5^4}^2{\xi_6^1}^3 + \xi_6^5{\xi_6^4}^2\xi_6^3{\xi_6^2}^2\xi_3^1 + \xi_6^5{\xi_6^4}^2{\xi_6^2}^3\xi_2^1 + \xi_6^5{\xi_6^4}^2{\xi_6^2}^2\xi_6^1\xi_1^1 +$

$2\xi^5_6\xi^4_6\xi^4_5\xi^3_6\xi^2_6\xi^1_6\xi^1_3 + 2\xi^5_6\xi^4_6\xi^4_5{\xi^2_6}^2\xi^1_6\xi^1_2 + 2\xi^5_6\xi^4_6\xi^4_5\xi^2_6{\xi^1_6}^2\xi^1_1 + \xi^5_6{\xi^4_5}^2\xi^3_6{\xi^1_6}^2\xi^1_3 + \xi^5_6{\xi^4_5}^2\xi^2_6{\xi^1_6}^2\xi^1_2 + \xi^5_6{\xi^4_5}^2{\xi^1_6}^3\xi^1_1 + {\xi^5_5}^2{\xi^4_6}^2\xi^2_6{\xi^1_6}^2 + {\xi^5_5}^2\xi^4_6\xi^4_5{\xi^1_6}^3 -$
$\xi^5_3{\xi^4_6}^2\xi^3_6{\xi^2_6}^2\xi^1_6 - 2\xi^5_3\xi^4_6\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^2 - \xi^5_3{\xi^4_5}^2\xi^3_6{\xi^1_6}^3 - \xi^5_2{\xi^4_6}^2{\xi^2_6}^3\xi^1_6 - 2\xi^5_2\xi^4_6\xi^4_5{\xi^2_6}^2{\xi^1_6}^2 - \xi^5_2{\xi^4_5}^2\xi^2_6{\xi^1_6}^3 + {\xi^4_6}^2\xi^2_6{\xi^1_6}^2 + \xi^4_6\xi^4_5{\xi^1_6}^3)/({\xi^1_6}^2({\xi^4_6}^2{\xi^2_6}^2 +$
$2\xi^4_6\xi^4_5\xi^2_6\xi^1_6 + {\xi^4_5}^2{\xi^1_6}^2));$

$J^5_4 = (-{\xi^5_6}^2{\xi^2_6}^2 - 2\xi^5_6\xi^5_5\xi^2_6\xi^1_6 + \xi^5_6\xi^4_6\xi^3_6\xi^2_6 + \xi^5_6\xi^4_5\xi^3_6\xi^1_6 - {\xi^5_5}^2{\xi^1_6}^2 - {\xi^1_6}^2)/(\xi^1_6(\xi^4_6\xi^2_6 + \xi^4_5\xi^1_6));$

$J^6_1 = (-{\xi^5_6}^4\xi^4_5{\xi^2_6}^4\xi^1_6 - {\xi^5_6}^4{\xi^2_6}^5\xi^1_3 + {\xi^5_6}^3\xi^5_5\xi^4_6{\xi^2_6}^4\xi^1_6 - 3{\xi^5_6}^3\xi^5_5\xi^4_5{\xi^2_6}^3{\xi^1_6}^2 - 4{\xi^5_6}^3\xi^5_5{\xi^2_6}^4\xi^1_6\xi^1_3 + {\xi^5_6}^3\xi^4_6\xi^4_5\xi^3_6{\xi^2_6}^3\xi^1_6 + 3{\xi^5_6}^3\xi^4_6\xi^3_6{\xi^2_6}^4\xi^1_3 +$
${\xi^5_6}^3\xi^4_6{\xi^2_6}^4\xi^1_6\xi^1_1 + {\xi^5_6}^3{\xi^4_5}^2\xi^3_6{\xi^2_6}^2{\xi^1_6}^2 + 3{\xi^5_6}^3\xi^4_5\xi^3_6{\xi^2_6}^3\xi^1_6\xi^1_3 + {\xi^5_6}^3\xi^4_5{\xi^2_6}^3{\xi^1_6}^2\xi^1_1 + 3{\xi^5_6}^2{\xi^5_5}^2\xi^4_6{\xi^2_6}^3{\xi^1_6}^2 - 3{\xi^5_6}^2{\xi^5_5}^2\xi^4_5{\xi^2_6}^2{\xi^1_6}^3 - 6{\xi^5_6}^2{\xi^5_5}^2{\xi^2_6}^3{\xi^1_6}^2\xi^1_3 -$
${\xi^5_6}^2\xi^5_5{\xi^4_6}^2\xi^3_6{\xi^2_6}^3\xi^1_6 + {\xi^5_6}^2\xi^5_5\xi^4_6\xi^4_5\xi^3_6{\xi^2_6}^2{\xi^1_6}^2 + 7{\xi^5_6}^2\xi^5_5\xi^4_6\xi^3_6{\xi^2_6}^3\xi^1_6\xi^1_3 - {\xi^5_6}^2\xi^5_5\xi^4_6{\xi^2_6}^4\xi^1_6\xi^1_2 + 2{\xi^5_6}^2\xi^5_5\xi^4_6{\xi^2_6}^3{\xi^1_6}^2\xi^1_1 + 2{\xi^5_6}^2\xi^5_5{\xi^4_5}^2\xi^3_6\xi^2_6{\xi^1_6}^3 +$
$7{\xi^5_6}^2\xi^5_5\xi^4_5\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_3 - {\xi^5_6}^2\xi^5_5\xi^4_5{\xi^2_6}^3{\xi^1_6}^2\xi^1_2 + 2{\xi^5_6}^2\xi^5_5\xi^4_5{\xi^2_6}^2{\xi^1_6}^3\xi^1_1 - 2{\xi^5_6}^2\xi^5_3\xi^4_6\xi^3_6{\xi^2_6}^4\xi^1_6 - 2{\xi^5_6}^2\xi^5_3\xi^4_5\xi^3_6{\xi^2_6}^3{\xi^1_6}^2 - {\xi^5_6}^2\xi^5_2\xi^4_6{\xi^2_6}^5\xi^1_6 -$
${\xi^5_6}^2\xi^5_2\xi^4_5{\xi^2_6}^4{\xi^1_6}^2 - 2{\xi^5_6}^2{\xi^4_6}^2{\xi^3_6}^2{\xi^2_6}^3\xi^1_3 + {\xi^5_6}^2{\xi^4_6}^2\xi^3_6{\xi^2_6}^4\xi^1_2 - {\xi^5_6}^2{\xi^4_6}^2\xi^3_6{\xi^2_6}^3\xi^1_6\xi^1_1 - 4{\xi^5_6}^2\xi^4_6\xi^4_5{\xi^3_6}^2{\xi^2_6}^2\xi^1_6\xi^1_3 + 2{\xi^5_6}^2\xi^4_6\xi^4_5\xi^3_6{\xi^2_6}^3\xi^1_6\xi^1_2 -$
${\xi^5_6}^2\xi^4_6\xi^4_5\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_1 - {\xi^5_6}^2\xi^4_6\xi^3_6{\xi^2_6}^3\xi^1_6\xi^1_3\xi^1_1 - {\xi^5_6}^2\xi^4_6{\xi^2_6}^4\xi^1_6\xi^1_2\xi^1_1 - {\xi^5_6}^2\xi^4_6{\xi^2_6}^3{\xi^1_6}^2{\xi^1_1}^2 - 2{\xi^5_6}^2{\xi^4_5}^2{\xi^3_6}^2\xi^2_6{\xi^1_6}^2\xi^1_3 + {\xi^5_6}^2{\xi^4_5}^2\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_2 -$
${\xi^5_6}^2\xi^4_5\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_3\xi^1_1 - {\xi^5_6}^2\xi^4_5{\xi^2_6}^3{\xi^1_6}^2\xi^1_2\xi^1_1 - {\xi^5_6}^2\xi^4_5{\xi^2_6}^2{\xi^1_6}^3{\xi^1_1}^2 - 2{\xi^5_6}^2\xi^4_5{\xi^2_6}^2{\xi^1_6}^3 - 2{\xi^5_6}^2{\xi^2_6}^3{\xi^1_6}^2\xi^1_3 + 3\xi^5_6{\xi^5_5}^3\xi^4_6{\xi^2_6}^2{\xi^1_6}^3 - \xi^5_6{\xi^5_5}^3\xi^4_5\xi^2_6{\xi^1_6}^4 -$
$4\xi^5_6{\xi^5_5}^3{\xi^2_6}^2{\xi^1_6}^3\xi^1_3 - 2\xi^5_6{\xi^5_5}^2{\xi^4_6}^2\xi^3_6{\xi^2_6}^2{\xi^1_6}^2 - \xi^5_6{\xi^5_5}^2\xi^4_6\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^3 + 5\xi^5_6{\xi^5_5}^2\xi^4_6\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_3 - 2\xi^5_6{\xi^5_5}^2\xi^4_6{\xi^2_6}^3{\xi^1_6}^2\xi^1_2 + \xi^5_6{\xi^5_5}^2\xi^4_6{\xi^2_6}^2{\xi^1_6}^3\xi^1_1 +$
$\xi^5_6{\xi^5_5}^2{\xi^4_5}^2\xi^3_6{\xi^1_6}^4 + 5\xi^5_6{\xi^5_5}^2\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_3 - 2\xi^5_6{\xi^5_5}^2\xi^4_5{\xi^2_6}^2{\xi^1_6}^3\xi^1_2 + \xi^5_6{\xi^5_5}^2\xi^4_5\xi^2_6{\xi^1_6}^4\xi^1_1 - 4\xi^5_6\xi^5_5\xi^5_3\xi^4_6\xi^3_6{\xi^2_6}^3{\xi^1_6}^2 - 4\xi^5_6\xi^5_5\xi^5_3\xi^4_5\xi^3_6{\xi^2_6}^2{\xi^1_6}^3 -$
$2\xi^5_6\xi^5_5\xi^5_2\xi^4_6{\xi^2_6}^4{\xi^1_6}^2 - 2\xi^5_6\xi^5_5\xi^5_2\xi^4_5{\xi^2_6}^3{\xi^1_6}^3 - 2\xi^5_6\xi^5_5{\xi^4_6}^2{\xi^3_6}^2{\xi^2_6}^2\xi^1_6\xi^1_3 + 2\xi^5_6\xi^5_5{\xi^4_6}^2\xi^3_6{\xi^2_6}^3\xi^1_6\xi^1_2 - 2\xi^5_6\xi^5_5{\xi^4_6}^2\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_1 - 4\xi^5_6\xi^5_5\xi^4_6\xi^4_5{\xi^3_6}^2\xi^2_6{\xi^1_6}^2\xi^1_3 +$
$4\xi^5_6\xi^5_5\xi^4_6\xi^4_5\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_2 - 2\xi^5_6\xi^5_5\xi^4_6\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_1 - 2\xi^5_6\xi^5_5\xi^4_6\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_3\xi^1_1 - 2\xi^5_6\xi^5_5\xi^4_6{\xi^2_6}^3{\xi^1_6}^2\xi^1_2\xi^1_1 - 2\xi^5_6\xi^5_5\xi^4_6{\xi^2_6}^2{\xi^1_6}^3{\xi^1_1}^2 + \xi^5_6\xi^5_5\xi^4_6{\xi^2_6}^2{\xi^1_6}^3 -$
$2\xi^5_6\xi^5_5{\xi^4_5}^2{\xi^3_6}^2{\xi^1_6}^3\xi^1_3 + 2\xi^5_6\xi^5_5{\xi^4_5}^2\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_2 - 2\xi^5_6\xi^5_5\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_3\xi^1_1 - 2\xi^5_6\xi^5_5\xi^4_5{\xi^2_6}^2{\xi^1_6}^3\xi^1_2\xi^1_1 - 2\xi^5_6\xi^5_5\xi^4_5\xi^2_6{\xi^1_6}^4{\xi^1_1}^2 - 3\xi^5_6\xi^5_5\xi^4_5\xi^2_6{\xi^1_6}^4 -$
$4\xi^5_6\xi^5_5{\xi^2_6}^2{\xi^1_6}^3\xi^1_3 + 2\xi^5_6\xi^5_3{\xi^4_6}^2{\xi^3_6}^2{\xi^2_6}^3\xi^1_6 + 4\xi^5_6\xi^5_3\xi^4_6\xi^4_5{\xi^3_6}^2{\xi^2_6}^2{\xi^1_6}^2 + 2\xi^5_6\xi^5_3{\xi^4_5}^2{\xi^3_6}^2\xi^2_6{\xi^1_6}^3 - \xi^5_6{\xi^4_6}^3{\xi^3_6}^2{\xi^2_6}^3\xi^1_2 - 3\xi^5_6{\xi^4_6}^2\xi^4_5{\xi^3_6}^2{\xi^2_6}^2\xi^1_6\xi^1_2 -$
$3\xi^5_6\xi^4_6{\xi^4_5}^2{\xi^3_6}^2\xi^2_6{\xi^1_6}^2\xi^1_2 + \xi^5_6\xi^4_6\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^3 + 3\xi^5_6\xi^4_6\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_3 + \xi^5_6\xi^4_6{\xi^2_6}^2{\xi^1_6}^3\xi^1_1 - \xi^5_6{\xi^4_5}^3{\xi^3_6}^2{\xi^1_6}^3\xi^1_2 + \xi^5_6{\xi^4_5}^2\xi^3_6{\xi^1_6}^4 + 3\xi^5_6\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_3 +$
$\xi^5_6\xi^4_5\xi^2_6{\xi^1_6}^4\xi^1_1 + {\xi^5_5}^4\xi^4_6\xi^2_6{\xi^1_6}^4 - {\xi^5_5}^4\xi^2_6{\xi^1_6}^4\xi^1_3 - {\xi^5_5}^3{\xi^4_6}^2\xi^3_6\xi^2_6{\xi^1_6}^3 - {\xi^5_5}^3\xi^4_6\xi^4_5\xi^3_6{\xi^1_6}^4 + {\xi^5_5}^3\xi^4_6\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_3 - {\xi^5_5}^3\xi^4_6{\xi^2_6}^2{\xi^1_6}^3\xi^1_2 + {\xi^5_5}^3\xi^4_5\xi^3_6{\xi^1_6}^4\xi^1_3 -$
${\xi^5_5}^3\xi^4_5\xi^2_6{\xi^1_6}^4\xi^1_2 - 2{\xi^5_5}^2\xi^5_3\xi^4_6\xi^3_6{\xi^2_6}^2{\xi^1_6}^3 - 2{\xi^5_5}^2\xi^5_3\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^4 - {\xi^5_5}^2\xi^5_2\xi^4_6{\xi^2_6}^3{\xi^1_6}^3 - {\xi^5_5}^2\xi^5_2\xi^4_5{\xi^2_6}^2{\xi^1_6}^4 + {\xi^5_5}^2{\xi^4_6}^2\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_2 - {\xi^5_5}^2{\xi^4_6}^2\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_1 +$
$2{\xi^5_5}^2\xi^4_6\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_2 - {\xi^5_5}^2\xi^4_6\xi^4_5\xi^3_6{\xi^1_6}^4\xi^1_1 - {\xi^5_5}^2\xi^4_6\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_3\xi^1_1 - {\xi^5_5}^2\xi^4_6{\xi^2_6}^2{\xi^1_6}^3\xi^1_2\xi^1_1 - {\xi^5_5}^2\xi^4_6\xi^2_6{\xi^1_6}^4{\xi^1_1}^2 + {\xi^5_5}^2\xi^4_6\xi^2_6{\xi^1_6}^4 + {\xi^5_5}^2{\xi^4_5}^2\xi^3_6{\xi^1_6}^4\xi^1_2 -$
${\xi^5_5}^2\xi^4_5\xi^3_6{\xi^1_6}^4\xi^1_3\xi^1_1 - {\xi^5_5}^2\xi^4_5\xi^2_6{\xi^1_6}^4\xi^1_2\xi^1_1 - {\xi^5_5}^2\xi^4_5{\xi^1_6}^5{\xi^1_1}^2 - {\xi^5_5}^2\xi^4_5{\xi^1_6}^5 - 2{\xi^5_5}^2\xi^2_6{\xi^1_6}^4\xi^1_3 + 2\xi^5_5\xi^5_3{\xi^4_6}^2{\xi^3_6}^2{\xi^2_6}^2{\xi^1_6}^2 + 4\xi^5_5\xi^5_3\xi^4_6\xi^4_5{\xi^3_6}^2\xi^2_6{\xi^1_6}^3 +$
$2\xi^5_5\xi^5_3{\xi^4_5}^2{\xi^3_6}^2{\xi^1_6}^4 - \xi^5_5{\xi^4_6}^2\xi^3_6\xi^2_6{\xi^1_6}^3 - \xi^5_5\xi^4_6\xi^4_5\xi^3_6{\xi^1_6}^4 + \xi^5_5\xi^4_6\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_3 - \xi^5_5\xi^4_6{\xi^2_6}^2{\xi^1_6}^3\xi^1_2 + \xi^5_5\xi^4_5\xi^3_6{\xi^1_6}^4\xi^1_3 - \xi^5_5\xi^4_5\xi^2_6{\xi^1_6}^4\xi^1_2 - 2\xi^5_3\xi^4_6\xi^3_6{\xi^2_6}^2{\xi^1_6}^3 -$
$2\xi^5_3\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^4 + \xi^5_2{\xi^4_6}^3{\xi^3_6}^2{\xi^2_6}^3\xi^1_6 + 3\xi^5_2{\xi^4_6}^2\xi^4_5{\xi^3_6}^2{\xi^2_6}^2{\xi^1_6}^2 + 3\xi^5_2\xi^4_6{\xi^4_5}^2{\xi^3_6}^2\xi^2_6{\xi^1_6}^3 - \xi^5_2\xi^4_6{\xi^2_6}^3{\xi^1_6}^3 + \xi^5_2{\xi^4_5}^3{\xi^3_6}^2{\xi^1_6}^4 - \xi^5_2\xi^4_5{\xi^2_6}^2{\xi^1_6}^4 +$
${\xi^4_6}^2\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_2 - {\xi^4_6}^2\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_1 + 2\xi^4_6\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_2 - \xi^4_6\xi^4_5\xi^3_6{\xi^1_6}^4\xi^1_1 - \xi^4_6\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_3\xi^1_1 - \xi^4_6{\xi^2_6}^2{\xi^1_6}^3\xi^1_2\xi^1_1 - \xi^4_6\xi^2_6{\xi^1_6}^4{\xi^1_1}^2 + {\xi^4_5}^2\xi^3_6{\xi^1_6}^4\xi^1_2 -$
$\xi^4_5\xi^3_6{\xi^1_6}^4\xi^1_3\xi^1_1 - \xi^4_5\xi^2_6{\xi^1_6}^4\xi^1_2\xi^1_1 - \xi^4_5{\xi^1_6}^5{\xi^1_1}^2 - \xi^4_5{\xi^1_6}^5 - \xi^2_6{\xi^1_6}^4\xi^1_3)/({\xi^1_6}^3({\xi^5_6}^2\xi^4_6{\xi^2_6}^3 + {\xi^5_6}^2\xi^4_5{\xi^2_6}^2\xi^1_6 + 2\xi^5_6\xi^5_5\xi^4_6{\xi^2_6}^2\xi^1_6 + 2\xi^5_6\xi^5_5\xi^4_5\xi^2_6{\xi^1_6}^2 +$
${\xi^5_5}^2\xi^4_6\xi^2_6{\xi^1_6}^2 + {\xi^5_5}^2\xi^4_5{\xi^1_6}^3 + \xi^4_6\xi^2_6{\xi^1_6}^2 + \xi^4_5{\xi^1_6}^3));$

$J^6_2 = ({\xi^5_6}^2{\xi^2_6}^2\xi^1_3 + 2\xi^5_6\xi^5_5\xi^2_6\xi^1_6\xi^1_3 - \xi^5_6\xi^4_6\xi^3_6\xi^2_6\xi^1_3 + \xi^5_6\xi^4_6{\xi^2_6}^2\xi^1_2 - \xi^5_6\xi^4_5\xi^3_6\xi^1_6\xi^1_3 + \xi^5_6\xi^4_5\xi^2_6\xi^1_6\xi^1_2 + {\xi^5_5}^2{\xi^1_6}^2\xi^1_3 + \xi^5_5\xi^4_6\xi^2_6\xi^1_6\xi^1_2 + \xi^5_5\xi^4_5{\xi^1_6}^2\xi^1_2 +$
$\xi^5_3\xi^4_6\xi^3_6\xi^2_6\xi^1_6 + \xi^5_3\xi^4_5\xi^3_6{\xi^1_6}^2 + \xi^5_2\xi^4_6{\xi^2_6}^2\xi^1_6 + \xi^5_2\xi^4_5\xi^2_6{\xi^1_6}^2 - {\xi^4_6}^2\xi^3_6\xi^2_6\xi^1_2 - \xi^4_6\xi^4_5\xi^3_6\xi^1_6\xi^1_2 - \xi^4_6\xi^3_6\xi^2_6\xi^1_3\xi^1_2 - \xi^4_6{\xi^2_6}^2{\xi^1_2}^2 - \xi^4_6\xi^2_6\xi^1_6\xi^1_2\xi^1_1 - \xi^4_5\xi^3_6\xi^1_6\xi^1_3\xi^1_2 -$
$\xi^4_5\xi^2_6\xi^1_6{\xi^1_2}^2 - \xi^4_5{\xi^1_6}^2\xi^1_2\xi^1_1 + {\xi^1_6}^2\xi^1_3)/({\xi^1_6}^2(\xi^4_6\xi^2_6 + \xi^4_5\xi^1_6));$

$J^6_3 = (-{\xi^5_6}^3{\xi^2_6}^3\xi^1_3 - 3{\xi^5_6}^2\xi^5_5{\xi^2_6}^2\xi^1_6\xi^1_3 + {\xi^5_6}^2\xi^5_3{\xi^2_6}^3\xi^1_6 + {\xi^5_6}^2\xi^4_6\xi^3_6{\xi^2_6}^2\xi^1_3 - {\xi^5_6}^2\xi^4_6{\xi^2_6}^3\xi^1_2 + 2{\xi^5_6}^2\xi^4_5\xi^3_6\xi^2_6\xi^1_6\xi^1_3 - {\xi^5_6}^2\xi^4_5{\xi^2_6}^2\xi^1_6\xi^1_2 - {\xi^5_6}^2\xi^3_6{\xi^2_6}^2{\xi^1_3}^2 -$
${\xi^5_6}^2{\xi^2_6}^3\xi^1_3\xi^1_2 - {\xi^5_6}^2{\xi^2_6}^2\xi^1_6\xi^1_3\xi^1_1 - 3\xi^5_6{\xi^5_5}^2\xi^2_6{\xi^1_6}^2\xi^1_3 + 2\xi^5_6\xi^5_5\xi^5_3{\xi^2_6}^2{\xi^1_6}^2 - 2\xi^5_6\xi^5_5\xi^4_6{\xi^2_6}^2\xi^1_6\xi^1_2 + 2\xi^5_6\xi^5_5\xi^4_5\xi^3_6{\xi^1_6}^2\xi^1_3 - 2\xi^5_6\xi^5_5\xi^4_5\xi^2_6{\xi^1_6}^2\xi^1_2 -$
$2\xi^5_6\xi^5_5\xi^3_6\xi^2_6\xi^1_6{\xi^1_3}^2 - 2\xi^5_6\xi^5_5{\xi^2_6}^2\xi^1_6\xi^1_3\xi^1_2 - 2\xi^5_6\xi^5_5\xi^2_6{\xi^1_6}^2\xi^1_3\xi^1_1 - 2\xi^5_6\xi^5_3\xi^4_6\xi^3_6{\xi^2_6}^2\xi^1_6 - 2\xi^5_6\xi^5_3\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^2 + \xi^5_6{\xi^4_6}^2\xi^3_6{\xi^2_6}^2\xi^1_2 + 2\xi^5_6\xi^4_6\xi^4_5\xi^3_6\xi^2_6\xi^1_6\xi^1_2 +$
$\xi^5_6{\xi^4_5}^2\xi^3_6{\xi^1_6}^2\xi^1_2 - \xi^5_6\xi^2_6{\xi^1_6}^2\xi^1_3 - {\xi^5_5}^3{\xi^1_6}^3\xi^1_3 + {\xi^5_5}^2\xi^5_3\xi^2_6{\xi^1_6}^3 - {\xi^5_5}^2\xi^4_6\xi^3_6{\xi^1_6}^2\xi^1_3 - {\xi^5_5}^2\xi^4_6\xi^2_6{\xi^1_6}^2\xi^1_2 - {\xi^5_5}^2\xi^4_5{\xi^1_6}^3\xi^1_2 - {\xi^5_5}^2\xi^3_6{\xi^1_6}^2{\xi^1_3}^2 - {\xi^5_5}^2\xi^2_6{\xi^1_6}^2\xi^1_3\xi^1_2 -$
${\xi^5_5}^2{\xi^1_6}^3\xi^1_3\xi^1_1 - 2\xi^5_5\xi^5_3\xi^4_6\xi^3_6\xi^2_6{\xi^1_6}^2 - 2\xi^5_5\xi^5_3\xi^4_5\xi^3_6{\xi^1_6}^3 - \xi^5_5{\xi^1_6}^3\xi^1_3 + \xi^5_3\xi^2_6{\xi^1_6}^3 - \xi^5_2{\xi^4_6}^2\xi^3_6{\xi^2_6}^2\xi^1_6 - 2\xi^5_2\xi^4_6\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^2 - \xi^5_2{\xi^4_5}^2\xi^3_6{\xi^1_6}^3 - \xi^4_6\xi^3_6{\xi^1_6}^2\xi^1_3 -$
$\xi^4_6\xi^2_6{\xi^1_6}^2\xi^1_2 - \xi^4_5{\xi^1_6}^3\xi^1_2 - \xi^3_6{\xi^1_6}^2{\xi^1_3}^2 - \xi^2_6{\xi^1_6}^2\xi^1_3\xi^1_2 - {\xi^1_6}^3\xi^1_3\xi^1_1)/({\xi^1_6}^2({\xi^5_6}^2{\xi^2_6}^2 + 2\xi^5_6\xi^5_5\xi^2_6\xi^1_6 + {\xi^5_5}^2{\xi^1_6}^2 + {\xi^1_6}^2));$

$J^6_4 = (-{\xi^5_6}^2{\xi^2_6}^3 - 2\xi^5_6\xi^5_5{\xi^2_6}^2\xi^1_6 + 2\xi^5_6\xi^4_6\xi^3_6{\xi^2_6}^2 + 2\xi^5_6\xi^4_5\xi^3_6\xi^2_6\xi^1_6 - {\xi^5_5}^2\xi^2_6{\xi^1_6}^2 + \xi^5_5\xi^4_6\xi^3_6\xi^2_6\xi^1_6 + \xi^5_5\xi^4_5\xi^3_6{\xi^1_6}^2 - {\xi^4_6}^2{\xi^3_6}^2\xi^2_6 - \xi^4_6\xi^4_5{\xi^3_6}^2\xi^1_6 -$
$\xi^4_6{\xi^3_6}^2\xi^2_6\xi^1_3 - \xi^4_6\xi^3_6{\xi^2_6}^2\xi^1_2 - \xi^4_6\xi^3_6\xi^2_6\xi^1_6\xi^1_1 - \xi^4_5{\xi^3_6}^2\xi^1_6\xi^1_3 - \xi^4_5\xi^3_6\xi^2_6\xi^1_6\xi^1_2 - \xi^4_5\xi^3_6{\xi^1_6}^2\xi^1_1 - \xi^2_6{\xi^1_6}^2)/({\xi^1_6}^2(\xi^4_6\xi^2_6 + \xi^4_5\xi^1_6));$

$J^6_5 = (\xi^5_5\xi^2_6\xi^1_6 - \xi^4_5\xi^3_6\xi^1_6 + \xi^3_6\xi^2_6\xi^1_3 + {\xi^2_6}^2\xi^1_2 + \xi^2_6\xi^1_6\xi^1_1)/{\xi^1_6}^2;$

$J^6_6 = (\xi^5_6\xi^2_6 - \xi^4_6\xi^3_6 - \xi^3_6\xi^1_3 - \xi^2_6\xi^1_2 - \xi^1_6\xi^1_1)/\xi^1_6;$

and the parameters are subject to the condition

$$\xi^1_6(\xi^4_6\xi^2_6 + \xi^4_5\xi^1_6) \neq 0. \tag{8}$$

Now, equivalence by a suitable automorphism of the form

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ b^4_1 & b^4_2 & b^4_3 & 1 & 0 & 0 \\ b^5_1 & b^5_2 & b^5_3 & 0 & 1 & 0 \\ b^6_1 & b^6_2 & b^6_3 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

reduces to the case $\xi^1_1 = \xi^1_2 = \xi^1_3 = \xi^4_6 = \xi^5_2 = \xi^5_3 = \xi^5_6 = 0$. Then from (8), $\xi^4_5\xi^1_6 \neq 0$ and applying equivalence by

$$\Psi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \xi^2_6/\xi^1_6 & 0 & -\xi^4_5 c & 0 & 0 & 0 \\ \xi^3_6/\xi^1_6 & c & -\xi^5_5 c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\xi^4_5 c & 0 \\ 0 & 0 & 0 & c & -\xi^5_5 c & 0 \\ 0 & 0 & 0 & (\xi^2_6 c)/\xi^1_6 & (c(-\xi^5_5\xi^2_6 + \xi^4_5\xi^3_6))/\xi^1_6 & \xi^4_5 c^2 \end{pmatrix}$$

where $c = |\xi_5^4 \xi_6^1|^{-\frac{1}{2}}$, we get reduced according to the sign of $\xi_5^4 \xi_6^1$ to either

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (10)$$

or

$$J_1^- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now J_1^- is equivalent to J_1, hence any complex structure with $\xi_6^1 \neq 0$ is equivalent to J_1.

3.2 Case $\xi_6^1 = 0, \xi_5^2 \neq 0$.

$$J = \begin{pmatrix} -\xi_6^6 & * & * & 0 & 0 & 0 \\ \boxed{\xi_1^2} & * & \boxed{\xi_3^2} & \boxed{\xi_4^2} & \boxed{\xi_5^2} & 0 \\ \boxed{\xi_1^3} & * & (\xi_6^6\xi_5^2 - \xi_4^2\xi_3^2)/\xi_5^2 & (-{\xi_4^2}^2)/\xi_5^2 & -\xi_4^2 & 0 \\ (-\xi_5^6\xi_3^2 + \xi_3^6\xi_5^2 + \xi_5^4\xi_1^2)/\xi_5^2 & * & \boxed{\xi_3^4} & (-\xi_6^6\xi_5^2 + \xi_5^4\xi_4^2)/\xi_5^2 & \boxed{\xi_5^4} & (\xi_5^2({\xi_6^6}^2 + 1))/(\xi_1^3\xi_5^2 + \xi_4^2\xi_1^2) \\ * & * & * & (\xi_4^2(\xi_6^6 + \xi_5^5))/\xi_5^2 & \boxed{\xi_5^5} & -(\xi_4^2({\xi_6^6}^2 + 1))/(\xi_1^3\xi_5^2 + \xi_4^2\xi_1^2) \\ * & * & \boxed{\xi_3^6} & (\xi_5^6\xi_4^2 - \xi_1^3\xi_5^2 - \xi_4^2\xi_1^2)/\xi_5^2 & \boxed{\xi_5^6} & \boxed{\xi_6^6} \end{pmatrix} \qquad (11)$$

where $J_3^1 = -(\xi_5^2({\xi_6^6}^2 + 1))/(\xi_1^3\xi_5^2 + \xi_4^2\xi_1^2)$;
$J_2^2 = (-\xi_5^5\xi_5^2 - \xi_5^4\xi_4^2 + \xi_4^2\xi_3^2)/\xi_5^2$;
$J_2^3 = (\xi_4^2(\xi_6^6\xi_5^2 + \xi_5^5\xi_5^2 + \xi_5^4\xi_4^2 - \xi_4^2\xi_3^2))/{\xi_5^2}^2$;
$J_2^4 = (-{\xi_6^6}^2\xi_5^6{\xi_5^2}^2 + \xi_6^6\xi_5^4\xi_1^3{\xi_5^2}^2 + \xi_6^6\xi_5^4\xi_5^2\xi_4^2\xi_1^2 - \xi_5^6{\xi_5^2}^2 - \xi_5^5\xi_5^4\xi_1^3{\xi_5^2}^2 - \xi_5^5\xi_5^4\xi_5^2\xi_4^2\xi_1^2 - {\xi_5^4}^2\xi_1^3\xi_5^2\xi_4^2 - {\xi_5^4}^2{\xi_4^2}^2\xi_1^2 + \xi_3^4\xi_1^3{\xi_5^2}^2\xi_4^2 + \xi_3^4\xi_5^2{\xi_4^2}^2\xi_1^2)/({\xi_5^2}^2(\xi_1^3\xi_5^2 + \xi_4^2\xi_1^2))$;
$J_1^5 = (\xi_6^6\xi_5^2\xi_1^2 + \xi_5^6\xi_4^2\xi_3^2 - \xi_3^6\xi_5^2\xi_4^2 + \xi_5^5\xi_5^2\xi_1^2 - \xi_1^3\xi_5^2\xi_3^2 - \xi_4^2\xi_3^2\xi_1^2)/{\xi_5^2}^2$;
$J_2^5 = ({\xi_6^6}^2\xi_5^6{\xi_5^2}^2\xi_4^2 + {\xi_6^6}^2{\xi_5^2}^2\xi_4^2\xi_1^2 - \xi_6^6\xi_5^4\xi_1^3{\xi_5^2}^2\xi_4^2 - \xi_6^6\xi_5^4\xi_5^2{\xi_4^2}^2\xi_1^2 - \xi_6^6\xi_1^3{\xi_5^2}^2\xi_4^2\xi_3^2 - \xi_6^6\xi_5^2{\xi_4^2}^2\xi_3^2\xi_1^2 + \xi_5^6{\xi_5^2}^2\xi_4^2 - {\xi_5^5}^2\xi_1^3{\xi_5^2}^3 - {\xi_5^5}^2{\xi_5^2}^2\xi_4^2\xi_1^2 - \xi_5^5\xi_5^4\xi_1^3{\xi_5^2}^2\xi_4^2 - \xi_5^5\xi_5^4\xi_5^2{\xi_4^2}^2\xi_1^2 + \xi_5^5\xi_1^3{\xi_5^2}^2\xi_4^2\xi_3^2 + \xi_5^5\xi_5^2{\xi_4^2}^2\xi_3^2\xi_1^2 + \xi_5^4\xi_1^3\xi_5^2{\xi_4^2}^2\xi_3^2 + \xi_5^4{\xi_4^2}^3\xi_3^2\xi_1^2 - \xi_3^4\xi_1^3{\xi_5^2}^2{\xi_4^2}^2 - \xi_3^4\xi_5^2{\xi_4^2}^3\xi_1^2 - \xi_1^3{\xi_5^2}^3)/({\xi_5^2}^3(\xi_1^3\xi_5^2 + \xi_4^2\xi_1^2))$;
$J_3^5 = ({\xi_6^6}^2{\xi_5^2}^2\xi_1^2 - \xi_6^6\xi_1^3{\xi_5^2}^2\xi_3^2 - \xi_6^6\xi_5^2\xi_4^2\xi_3^2\xi_1^2 + \xi_5^5\xi_1^3{\xi_5^2}^2\xi_3^2 + \xi_5^5\xi_5^2\xi_4^2\xi_3^2\xi_1^2 + \xi_5^4\xi_1^3\xi_5^2\xi_4^2\xi_3^2 + \xi_5^4{\xi_4^2}^2\xi_3^2\xi_1^2 - \xi_3^4\xi_1^3{\xi_5^2}^2\xi_4^2 - \xi_3^4\xi_5^2{\xi_4^2}^2\xi_1^2 + {\xi_5^2}^2\xi_1^2)/({\xi_5^2}^2(\xi_1^3\xi_5^2 + \xi_4^2\xi_1^2))$;
$J_1^6 = ({\xi_6^6}^2\xi_5^6{\xi_5^2}^2\xi_1^2 - 2\xi_6^6\xi_5^6\xi_1^3{\xi_5^2}^2\xi_3^2 - 2\xi_6^6\xi_5^6\xi_5^2\xi_4^2\xi_3^2\xi_1^2 + 2\xi_6^6\xi_3^6\xi_1^3{\xi_5^2}^3 + 2\xi_6^6\xi_3^6{\xi_5^2}^2\xi_4^2\xi_1^2 + \xi_5^6{\xi_5^2}^2\xi_1^2 + \xi_5^4{\xi_1^3}^2{\xi_5^2}^2\xi_3^2 + 2\xi_5^4\xi_1^3\xi_5^2\xi_4^2\xi_3^2\xi_1^2 + \xi_5^4{\xi_4^2}^2\xi_3^2{\xi_1^2}^2 - \xi_3^4{\xi_1^3}^2{\xi_5^2}^3 - 2\xi_3^4\xi_1^3{\xi_5^2}^2\xi_4^2\xi_1^2 - \xi_3^4\xi_5^2{\xi_4^2}^2{\xi_1^2}^2)/({\xi_5^2}^3({\xi_6^6}^2 + 1))$;
$J_2^6 = (-\xi_6^6\xi_5^6\xi_5^2 - \xi_5^6\xi_5^5\xi_5^2 - \xi_5^6\xi_5^4\xi_4^2 + \xi_3^6\xi_5^2\xi_4^2 + \xi_5^4\xi_1^3\xi_5^2 + \xi_5^4\xi_4^2\xi_1^2)/{\xi_5^2}^2$;
and the parameters are subject to the condition

$$\xi_5^2(\xi_5^2\xi_1^3 + \xi_4^2\xi_1^2) \neq 0. \qquad (12)$$

Now, equivalence by a suitable automorphism of the form (9) reduces to the case $\xi_1^2 = \xi_3^2 = \xi_5^4 = \xi_5^5 = \xi_5^6 = \xi_3^4 = \xi_3^6 = 0$. Applying then equivalence by

$$\Psi = \begin{pmatrix} (-\xi_6^6)/\xi_5^2 & (-\xi_6^6\xi_4^2)/(\xi_1^3\xi_5^2) & (-1)/\xi_1^3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \xi_1^3/\xi_5^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (-\xi_6^6)/\xi_5^2 & 0 & 1/\xi_1^3 \\ 0 & 0 & 0 & (\xi_6^6\xi_4^2)/{\xi_5^2}^2 & 1/\xi_5^2 & 0 \\ 0 & 0 & 0 & (-\xi_1^3)/\xi_5^2 & 0 & 0 \end{pmatrix}$$

$det\Psi = 1/{\xi_5^2}^3$, $J2 = \Psi^{-1}J\Psi$ is the matrix

$$J2 = \begin{pmatrix} 0 & 0 & (-\xi_5^2)/\xi_1^3 & 0 & (-\xi_4^2)/\xi_1^3 & (-{\xi_4^2}^2)/{\xi_1^3}^2 \\ 0 & 0 & 0 & 0 & 1 & \xi_4^2/\xi_1^3 \\ \xi_1^3/\xi_5^2 & \xi_4^2/\xi_5^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi_5^2/\xi_1^3 \\ 0 & -1 & 0 & \xi_4^2/\xi_5^2 & 0 & 0 \\ 0 & 0 & 0 & (-\xi_1^3)/\xi_5^2 & 0 & 0 \end{pmatrix}$$

• Suppose first that $\xi_4^2 \neq 0$. Then this $J2$ is a complex structure belonging in the case $\xi_6^1 \neq 0$, *i.e.* $J2_6^1 = (-{\xi_4^2}^2)/{\xi_1^3}^2 \neq 0$. Note that $J2$ satifies the compatibility condition $\xi_6^4\xi_6^2 + \xi_5^4\xi_6^1$ of that type, since $J2_6^4 J2_6^2 + J2_5^4 J2_6^1 = (\xi_5^2\xi_4^2)/{\xi_1^3}^2 \neq 0$.
• Suppose now that $\xi_4^2 = 0$. Then

$$J2 = \begin{pmatrix} 0 & 0 & (-\xi_5^2)/\xi_1^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \xi_1^3/\xi_5^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi_5^2/\xi_1^3 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (-\xi_1^3)/\xi_5^2 & 0 & 0 \end{pmatrix}$$

Applying equivalence by the automorphism $\Lambda = \mathrm{diag}(1, \xi_1^3/\xi_5^2, \xi_1^3/\xi_5^2, \xi_1^3/\xi_5^2, \xi_1^3/\xi_5^2, {\xi_1^3}^2/{\xi_5^2}^2)$, $\Lambda^{-1}J2\Lambda$ is the matrix

$$J_2 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \tag{13}$$

We've thus got into the case of a complex structure (11) where $\xi_1^2 = \xi_3^2 = \xi_3^4 = \xi_5^4 = \xi_5^5 = \xi_3^6 = \xi_5^6 = \xi_6^6 = \xi_4^2 = 0$, and $\xi_1^3 = \xi_5^2 = 1$.
• Hence, from the result of the case $\xi_6^1 \neq 0$, we have that any complex structure with $\xi_6^1 = 0, \xi_5^2 \neq 0$ is equivalent to either J_1 in (10) or J_2 in (13). Observe now that J_2 is equivalent to J_1 by the automorphism

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

i.e. $M^{-1}J_2M = J_1$, since $MJ_1 - J_2M = 0$.

Hence, any complex structure with $\xi_6^1 = 0, \xi_5^2 \neq 0$ is equivalent to J_1.

3.3 Case $\xi_6^1 = 0, \xi_5^2 = 0$.

$$J = \begin{pmatrix} \boxed{\xi_1^1} & \boxed{\xi_2^1} & 0 & 0 & 0 & 0 \\ -\frac{{\xi_1^1}^2+1}{\xi_2^1} & -\xi_1^1 & 0 & 0 & 0 & 0 \\ \boxed{\xi_1^3} & \frac{\xi_2^1(\xi_1^4\xi_4^3+\xi_3^3\xi_1^3+\xi_1^3\xi_1^1)}{{\xi_1^1}^2+1} & \boxed{\xi_3^3} & \boxed{\xi_4^3} & 0 & 0 \\ \boxed{\xi_1^4} & \frac{\xi_2^1(-\xi_1^4\xi_4^3\xi_3^3+\xi_1^4\xi_4^3\xi_1^1-{\xi_3^3}^2\xi_1^3-\xi_1^3)}{\xi_4^3({\xi_1^1}^2+1)} & -\frac{{\xi_3^3}^2+1}{\xi_4^3} & -\xi_3^3 & 0 & 0 \\ \frac{\xi_4^6\xi_1^4\xi_2^1+\xi_3^6\xi_1^3\xi_2^1-\xi_2^6{\xi_1^1}^2-\xi_2^6}{{\xi_1^1}^2+1} & * & * & \frac{\xi_2^1(-\xi_4^6\xi_3^3-\xi_4^6\xi_1^1+\xi_3^6\xi_4^3)}{{\xi_1^1}^2+1} & \xi_1^1 & \xi_2^1 \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & -\frac{{\xi_1^1}^2+1}{\xi_2^1} & -\xi_1^1 \end{pmatrix} \tag{14}$$

where

$J_2^5 = (\xi_2^1(-\xi_4^6\xi_1^4\xi_4^3\xi_3^3\xi_2^1 + \xi_4^6\xi_1^4\xi_4^3\xi_2^1\xi_1^1 - \xi_4^6{\xi_3^3}^2\xi_1^3\xi_2^1 - \xi_4^6\xi_1^3\xi_2^1 + \xi_3^6\xi_1^4{\xi_4^3}^2\xi_2^1 + \xi_3^6\xi_4^3\xi_3^3\xi_1^3\xi_2^1 + \xi_3^6\xi_4^3\xi_1^3\xi_2^1\xi_1^1 - 2\xi_2^6\xi_4^3{\xi_1^1}^3 - 2\xi_2^6\xi_4^3\xi_1^1 + \xi_1^6\xi_4^3\xi_2^1{\xi_1^1}^2 + \xi_1^6\xi_4^3\xi_2^1))/(\xi_4^3({\xi_1^1}^4 + 2{\xi_1^1}^2 + 1));$

$J_3^5 = (\xi_2^1(-\xi_4^6{\xi_3^3}^2 - \xi_4^6 + \xi_3^6\xi_4^3\xi_3^3 - \xi_3^6\xi_4^3\xi_1^1))/(\xi_4^3({\xi_1^1}^2 + 1));$
and the parameters are subject to the condition

$$\xi_2^1\xi_4^3 \neq 0. \tag{15}$$

Now, equivalence by a suitable automorphism of the form (9) reduces to the case $\xi_1^3 = \xi_3^3 = \xi_1^4 = \xi_1^6 = \xi_2^6 = \xi_3^6 = \xi_4^6 = 0$. Applying then equivalence by the automorphism

$$\Psi = \begin{pmatrix} \xi_2^1 & 0 & 0 & 0 & 0 & 0 \\ -\xi_1^1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_4^3\xi_2^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_2^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi_4^3{\xi_2^1}^2 & 0 \\ 0 & 0 & 0 & 0 & -\xi_4^3\xi_2^1\xi_1^1 & \xi_4^3\xi_2^1 \end{pmatrix}$$

we get into the case of a complex structure J where $\xi_1^3 = \xi_3^3 = \xi_1^4 = \xi_1^6 = \xi_2^6 = \xi_3^6 = \xi_4^6 = \xi_1^1 = 0$ and $\xi_2^1 = \xi_4^3 = 1$. Hence, we have that any complex structure with $\xi_6^1 = 0, \xi_5^2 = 0$ is equivalent to

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{16}$$

which is equivalent to the matrix J_1 in (10). Now, with the automorphism $\mathrm{diag}(1, -1, 1, -1, 1, -1)$, (16) is equivalent to

$$J_0 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{17}$$

Hence, any complex structure with $\xi_6^1 = 0, \xi_5^2 = 0$ is equivalent to J_0 in (17). With the complex structure J_0, the commutation relations of $\mathfrak{m} = \mathfrak{g}^{(1,0)}$ are $[\tilde{x}_1, \tilde{x}_3] = \tilde{x}_5$; $[\tilde{x}_1, \tilde{x}_4] = \tilde{x}_6$; $[\tilde{x}_2, \tilde{x}_3] = \tilde{x}_6$; $[\tilde{x}_2, \tilde{x}_4] = -\tilde{x}_5$.

3.4 Conclusions.

Any $J \in \mathfrak{X}_{6,3}$ is equivalent to J_0 defined by (17). Hence $\mathfrak{X}_{6,3}$ is comprised of the single Aut $\mathcal{G}_{6,3}$ orbit of J_0. As $\mathfrak{X}_{6,3}$ is a closed subset of $\mathbb{R}^{36}$, so is the orbit. Now Aut $\mathcal{G}_{6,3}$ consists of the matrices

$$\Phi = \begin{pmatrix} b_1^1 & b_2^1 & b_3^1 & 0 & 0 & 0 \\ b_1^2 & b_2^2 & b_3^2 & 0 & 0 & 0 \\ b_1^3 & b_2^3 & b_3^3 & 0 & 0 & 0 \\ b_1^4 & b_2^4 & b_3^4 & b_2^2b_1^1 - b_1^2b_2^1 & b_3^2b_1^1 - b_1^2b_3^1 & b_3^2b_2^1 - b_2^2b_3^1 \\ b_1^5 & b_2^5 & b_3^5 & b_2^3b_1^1 - b_1^3b_2^1 & b_3^3b_1^1 - b_1^3b_3^1 & b_3^3b_2^1 - b_2^3b_3^1 \\ b_1^6 & b_2^6 & b_3^6 & b_2^3b_1^2 - b_1^3b_2^2 & b_3^3b_1^2 - b_1^3b_3^2 & b_3^3b_2^2 - b_2^3b_3^2 \end{pmatrix}$$

where the b_j^i's are arbitrary reals with the condition $\det \Phi \neq 0$, and the stabilizer of J_0 is 6-dimensional. Hence $\mathfrak{X}_{6,3}$ is a submanifold of dimension 12 of $\mathbb{R}^{36}$ ([1], Chap. 3, par. 1, Prop. 14). It can either be represented as the disjoint union $V_{12} \cup V_{11} \cup V_{10}$ of the submanifolds of respective dimensions 12,11,10 given by (7), (11), (14), or parametrized by the orbital map of J_0. We also remark that $\mathfrak{X}_{6,3}$ is the zero set of a polynomial map $F : \mathbb{R}^{36} \to \mathbb{R}^{81} \times \mathbb{R}^{36}$. However this map is not a subimmersion, that is its rank is not locally constant.

3.5

3.5.1

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x^1} - x^2\frac{\partial}{\partial x^3} - y^1\frac{\partial}{\partial y^2} \\ X_2 &= \frac{\partial}{\partial y^1} - x^2\frac{\partial}{\partial y^3}. \end{aligned}$$

3.5.2 Holomorphic functions for J_0.

Let G denote the group G_0 endowed with the left invariant structure of complex manifold defined by J_0 (17). Then $H_{\mathbb{C}}(G) = \{f \in C^\infty(G_0)\ ;\ \tilde{X}_j^- f = 0\ \forall j = 1, 3, 5\}$. One has

$$\tilde{X}_1^- = 2\,\frac{\partial}{\partial \overline{z^1}} - 2\,x^2\,\frac{\partial}{\partial \overline{z^3}} - y^1\,\frac{\partial}{\partial y^2}\ ;\ \tilde{X}_3^- = 2\,\frac{\partial}{\partial \overline{z^2}}\ ;\ \tilde{X}_5^- = 2\,\frac{\partial}{\partial \overline{z^3}}$$

where

$$z^j = x^j + iy^j\ (1 \leqslant j \leqslant 3).$$

Then $f \in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to z^2 and z^3 and satisfies

$$2\,\frac{\partial f}{\partial \overline{z^1}} = \frac{z^1 - \overline{z^1}}{2}\,\frac{\partial f}{\partial z^2}\,.$$

Hence the 3 functions $w^1 = z^1, w^2 = z^2 + \frac{|z^1|^2}{4} - \frac{(\overline{z^1})^2}{8}, w^3 = z^3$ are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by

$$F = (w^1, w^2, w^3). \tag{18}$$

F is a biholomorphic bijection, hence a global chart on G. We determine now how the multiplication of G looks like in the chart (18). Let $a, x \in G$:

$$\begin{aligned} x &= \exp{(x^1 x_1)}\exp{(y^1 x_2)}\exp{(x^2 x_3)}\exp{(y^2 x_4)}\exp{(x^3 x_5)}\exp{(y^3 x_6)} \\ a &= \exp{(\alpha^1 x_1)}\exp{(\beta^1 x_2)}\exp{(\alpha^2 x_3)}\exp{(\beta^2 x_4)}\exp{(\alpha^3 x_5)}\exp{(\beta^3 x_6)}. \end{aligned}$$

With obvious notations, $a = [w_a^1, w_a^2, w_a^3]$, $x = [w_x^1, w_x^2, w_x^3]$, $a\,x = [w_{ax}^1, w_{ax}^2, w_{ax}^3]$.
Computations yield :

$$\begin{aligned} w_{ax}^1 &= w_a^1 + w_x^1 \\ w_{ax}^2 &= w_a^2 + w_x^2 - \frac{w_a^1 - \overline{w_a^1}}{4}\,w_x^1 \\ w_{ax}^3 &= w_a^3 + w_x^3 + \frac{1}{2}(w_a^2 - \overline{w_a^2} + \frac{(\overline{w_a^1})^2 - (w_a^1)^2}{4} - |w_a^1|^2)\,w_x^1. \end{aligned}$$

4 Lie Algebra $\mathcal{G}_{6,7}$ (isomorphic to $M6$).

Commutation relations for $\mathcal{G}_{6,7}$: $[x_1, x_2] = x_4$; $[x_1, x_3] = x_5$; $[x_1, x_4] = x_6$; $[x_2, x_3] = -x_6$.

$$J = \begin{pmatrix} \boxed{\xi_1^1} & \boxed{\xi_2^1} & 0 & 0 & 0 & 0 \\ -\frac{{\xi_1^1}^2+1}{\xi_2^1} & -\xi_1^1 & 0 & 0 & 0 & 0 \\ * & \boxed{\xi_2^3} & c & \boxed{\xi_4^3} & 0 & 0 \\ * & \boxed{\xi_2^4} & -\frac{c^2+1}{\xi_4^3} & -c & 0 & 0 \\ * & * & * & \boxed{\xi_4^5} & \boxed{\xi_5^5} & -\frac{\xi_4^3\xi_2^1}{\xi_4^3-\xi_2^1} \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & * & \boxed{\xi_4^6} & \frac{({\xi_5^5}^2+1)(\xi_4^3-\xi_2^1)}{\xi_4^3\xi_2^1} & -\xi_5^5 \end{pmatrix} \tag{19}$$

where $J_1^3 = (\xi_5^5\xi_4^3\xi_2^3 - \xi_5^5\xi_2^3\xi_2^1 - \xi_2^4\xi_4^3\xi_2^1 - \xi_4^3\xi_2^3\xi_1^1 + \xi_2^3\xi_2^1\xi_1^1)/{\xi_2^1}^2$;
$c = J_3^3 = (-\xi_5^5\xi_4^3 + \xi_5^5\xi_2^1 + \xi_4^3\xi_1^1)/\xi_2^1$;
$J_1^4 = ({\xi_5^5}^2{\xi_4^3}^2\xi_2^3 - 2{\xi_5^5}^2\xi_4^3\xi_2^3\xi_2^1 + {\xi_5^5}^2\xi_2^3{\xi_2^1}^2 - \xi_5^5\xi_2^4{\xi_4^3}^2\xi_2^1 + \xi_5^5\xi_2^4\xi_4^3{\xi_2^1}^2 - 2\xi_5^5{\xi_4^3}^2\xi_2^3\xi_1^1 + 2\xi_5^5\xi_4^3\xi_2^3\xi_2^1\xi_1^1 + \xi_2^4{\xi_4^3}^2\xi_2^1\xi_1^1 + \xi_2^4\xi_4^3{\xi_2^1}^2\xi_1^1 +$
${\xi_4^3}^2\xi_2^3{\xi_1^1}^2 + \xi_2^3{\xi_2^1}^2)/(\xi_4^3{\xi_2^1}^3)$;
$J_1^5 = (-\xi_4^6{\xi_5^5}^2{\xi_4^3}^2\xi_2^3\xi_2^1 + \xi_4^6{\xi_5^5}^2\xi_4^3\xi_2^3{\xi_2^1}^2 + \xi_4^6\xi_5^5\xi_2^4{\xi_4^3}^2{\xi_2^1}^2 + 2\xi_4^6\xi_5^5{\xi_4^3}^2\xi_2^3\xi_2^1\xi_1^1 - 2\xi_4^6\xi_5^5\xi_4^3\xi_2^3{\xi_2^1}^2\xi_1^1 - \xi_4^6\xi_2^4{\xi_4^3}^2{\xi_2^1}^2\xi_1^1 - \xi_4^6{\xi_4^3}^2\xi_2^3\xi_2^1{\xi_1^1}^2 -$
$\xi_4^6\xi_4^3\xi_2^3{\xi_2^1}^2 + \xi_2^6{\xi_4^3}^2{\xi_2^1}^2{\xi_1^1}^2 + \xi_2^6{\xi_4^3}^2{\xi_2^1}^2 + \xi_1^6\xi_5^5{\xi_4^3}^2{\xi_2^1}^3 - \xi_1^6{\xi_4^3}^2{\xi_2^1}^3\xi_1^1 + {\xi_5^5}^3\xi_4^5{\xi_4^3}^2\xi_2^3 - 2{\xi_5^5}^3\xi_4^5\xi_4^3\xi_2^3\xi_2^1 + {\xi_5^5}^3\xi_4^5\xi_2^3{\xi_2^1}^2 - {\xi_5^5}^2\xi_4^5\xi_2^4{\xi_4^3}^2\xi_2^1 +$
${\xi_5^5}^2\xi_4^5\xi_2^4\xi_4^3{\xi_2^1}^2 - {\xi_5^5}^2\xi_4^5{\xi_4^3}^2\xi_2^3\xi_1^1 + 2{\xi_5^5}^2\xi_4^5\xi_4^3\xi_2^3\xi_2^1\xi_1^1 - {\xi_5^5}^2\xi_4^5\xi_2^3{\xi_2^1}^2\xi_1^1 + \xi_5^5\xi_4^5{\xi_4^3}^2\xi_2^3 - 2\xi_5^5\xi_4^5\xi_4^3\xi_2^3\xi_2^1 + \xi_5^5\xi_4^5\xi_2^3{\xi_2^1}^2 - \xi_4^5\xi_2^4{\xi_4^3}^2\xi_2^1 + \xi_4^5\xi_2^4\xi_4^3{\xi_2^1}^2 -$
$\xi_4^5{\xi_4^3}^2\xi_2^3\xi_1^1 + 2\xi_4^5\xi_4^3\xi_2^3\xi_2^1\xi_1^1 - \xi_4^5\xi_2^3{\xi_2^1}^2\xi_1^1)/(\xi_4^3{\xi_2^1}^2({\xi_5^5}^2\xi_4^3 - {\xi_5^5}^2\xi_2^1 + \xi_4^3 - \xi_2^1))$;
$J_2^5 = (\xi_4^6\xi_5^5{\xi_4^3}^2\xi_2^3 - 2\xi_4^6\xi_5^5\xi_4^3\xi_2^3\xi_2^1 - \xi_4^6\xi_2^4{\xi_4^3}^2\xi_2^1 - \xi_4^6{\xi_4^3}^2\xi_2^3\xi_1^1 + \xi_2^6\xi_5^5{\xi_4^3}^2\xi_2^1 + \xi_2^6{\xi_4^3}^2\xi_2^1\xi_1^1 - \xi_1^6{\xi_4^3}^2{\xi_2^1}^2 + {\xi_5^5}^2\xi_4^5\xi_4^3\xi_2^3 - {\xi_5^5}^2\xi_4^5\xi_2^3\xi_2^1 + \xi_4^5\xi_4^3\xi_2^3 -$
$\xi_4^5\xi_2^3\xi_2^1)/(\xi_4^3({\xi_5^5}^2\xi_4^3 - {\xi_5^5}^2\xi_2^1 + \xi_4^3 - \xi_2^1))$;
$J_3^5 = (\xi_4^6{\xi_2^1}^2 - \xi_5^5\xi_4^5\xi_4^3 + \xi_5^5\xi_4^5\xi_2^1 + \xi_4^5\xi_4^3\xi_1^1 - \xi_4^5\xi_2^1\xi_1^1)/(\xi_2^1(\xi_4^3 - \xi_2^1))$;

$J^6_3 = (-\xi^6_4\xi^5_5{\xi^3_4}^2 + 2\xi^6_4\xi^5_5\xi^3_4\xi^1_2 + \xi^6_4{\xi^3_4}^2\xi^1_1 - {\xi^5_5}^2\xi^5_4\xi^3_4 + {\xi^5_5}^2\xi^5_4\xi^1_2 - \xi^5_4\xi^3_4 + \xi^5_4\xi^1_2)/({\xi^3_4}^2\xi^1_2)$;
and the parameters are subject to the condition

$$\xi^1_2\xi^3_4(\xi^1_2 - \xi^3_4) \neq 0. \tag{20}$$

Now the automorphism group of $\mathcal{G}_{6,7}$ is comprised of the matrices

$$\Phi = \begin{pmatrix} b^1_1 & 0 & 0 & 0 & 0 & 0 \\ b^2_1 & b^2_2 & 0 & 0 & 0 & 0 \\ b^3_1 & b^3_2 & {b^1_1}^2 & 0 & 0 & 0 \\ b^4_1 & b^4_2 & b^4_3 & b^2_2 b^1_1 & 0 & 0 \\ b^5_1 & b^5_2 & b^5_3 & b^3_2 b^1_1 & {b^1_1}^3 & 0 \\ b^6_1 & b^6_2 & b^6_3 & b^4_2 b^1_1 - b^3_2 b^2_1 + b^3_1 b^2_2 & b^1_1(b^4_3 - b^2_1 b^1_1) & b^2_2 {b^1_1}^2 \end{pmatrix}$$

where $\det\Phi = {b^2_2}^3 {b^1_1}^9 \neq 0$. Taking suitable values for the b^i_j's, equivalence by Φ leads to the case where $\xi^1_1 = \xi^3_2 = \xi^4_2 = \xi^5_4 = \xi^5_5 = \xi^6_1 = \xi^6_2 = \xi^6_4 = 0$ and $\xi^1_2 = 1, \xi^3_4 = \alpha$, where $\alpha \neq 0, 1$:

$$J_\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & (-1)/\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha/(\alpha-1) \\ 0 & 0 & 0 & 0 & (\alpha-1)/\alpha & 0 \end{pmatrix} \quad (\alpha \neq 0, 1) \tag{21}$$

Hence, any complex structure is equivalent to J_α in (21). It is easily seen that the J_α's corresponding to different values of α are not equivalent. With the complex structure J_α, the commutation relations of $\mathfrak{m} = \mathfrak{g}^{(1,0)}$ are :

$$[\tilde{x}_1, \tilde{x}_3] = \tilde{x}_5 \, ; \, [\tilde{x}_1, \tilde{x}_4] = (1-\alpha)\tilde{x}_6 \, ; \, [\tilde{x}_2, \tilde{x}_3] = \frac{1-\alpha}{\alpha}\tilde{x}_6 \, ; \, [\tilde{x}_2, \tilde{x}_4] = -\alpha\tilde{x}_5.$$

4.1 Conclusions.

From (19), $\mathfrak{X}_{6,7}$ is a submanifold of dimension 10 in $\mathbb{R}^{36}$. It is the disjoint union of the continuously many orbits of the J_α's in (21).

4.2

4.2.1

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x^1} - y^1\frac{\partial}{\partial y^2} - x^2\frac{\partial}{\partial x^3} - y^2\frac{\partial}{\partial y^3} \\ X_2 &= \frac{\partial}{\partial y^1} + x^2\frac{\partial}{\partial y^3}. \end{aligned}$$

4.2.2 Holomorphic functions for J_α.

Let G denote the group G_0 endowed with the left invariant structure of complex manifold defined by J_α in (21). Then $H_{\mathbb{C}}(G) = \{f \in C^\infty(G_0) \; ; \; \tilde{X}^-_j f = 0 \; \forall j = 1, 3, 5\}$. One has

$$\begin{aligned} \tilde{X}^-_1 &= 2\frac{\partial}{\partial \overline{w^1}} - y^1\frac{\partial}{\partial y^2} - x^2\frac{\partial}{\partial x^3} - (y^2 + ix^2)\frac{\partial}{\partial y^3} \\ \tilde{X}^-_3 &= 2\frac{\partial}{\partial \overline{w^2}} \\ \tilde{X}^-_5 &= 2\frac{\partial}{\partial \overline{w^3}} \end{aligned}$$

where

$$\begin{aligned} w^1 &= x^1 - iy^1 \\ w^2 &= x^2 - i\alpha y^2 \\ w^3 &= x^2 + i\frac{\alpha}{\alpha-1}y^3. \end{aligned}$$

Then $f \in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to w^2 and w^3 and satisfies

$$2\,\frac{\partial f}{\partial \overline{w^1}} - \alpha\,\frac{w^1 - \overline{w^1}}{2}\,\frac{\partial f}{\partial w^2} + \frac{w^2}{\alpha - 1}\,\frac{\partial f}{\partial w^3} = 0.$$

The 3 functions

$$\begin{aligned}
\varphi^1 &= w^1 \\
\varphi^2 &= w^2 + \alpha\left(-\frac{(\overline{w^1})^2}{8} + \frac{|w^1|^2}{4}\right) \\
\varphi^3 &= w^3 + \frac{\alpha}{8(1-\alpha)}\,(\overline{w^1})^2\left(\frac{w^1}{2} - \frac{\overline{w^1}}{3}\right) + \frac{\overline{w^1}w^2}{2(1-\alpha)}
\end{aligned}$$

are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by

$$F = (\varphi^1, \varphi^2, \varphi^3). \tag{22}$$

F is a biholomorphic bijection, hence a global chart on G. We determine now how the multiplication of G looks like in the chart (22). Let $a, x \in G$:

$$\begin{aligned}
x &= \exp(x^1 x_1)\exp(y^1 x_2)\exp(x^2 x_3)\exp(y^2 x_4)\exp(x^3 x_5)\exp(y^3 x_6) \\
a &= \exp(\alpha^1 x_1)\exp(\beta^1 x_2)\exp(\alpha^2 x_3)\exp(\beta^2 x_4)\exp(\alpha^3 x_5)\exp(\beta^3 x_6).
\end{aligned}$$

With obvious notations, $a = [w_a^1, w_a^2, w_a^3]$, $x = [w_x^1, w_x^2, w_x^3]$, $a\,x = [w_{ax}^1, w_{ax}^2, w_{ax}^3]$, $a = [\varphi_a^1, \varphi_a^2, \varphi_a^3]$, $x = [\varphi_x^1, \varphi_x^2, \varphi_x^3]$, $a\,x = [\varphi_{ax}^1, \varphi_{ax}^2, \varphi_{ax}^3]$. Computations yield :

$$\begin{aligned}
w_{ax}^1 &= w_a^1 + w_x^1 \\
w_{ax}^2 &= w_a^2 + w_x^2 + i\alpha\beta^1 x^1 \\
w_{ax}^3 &= w_a^3 + w_x^3 - \alpha^2 x^1 + i\frac{\alpha}{\alpha - 1}\left(-\beta^1 x^1 + \alpha^2 y^1 + \frac{1}{2}\,\beta^1 (x^1)^2\right).
\end{aligned}$$

We then get

$$\begin{aligned}
\varphi_{ax}^1 &= \varphi_a^1 + \varphi_x^1 \\
\varphi_{ax}^2 &= \varphi_a^2 + \varphi_x^2 + \frac{\alpha}{4}\left(2\overline{\varphi_a^1} - \varphi_a^1\right)\varphi_x^1 \\
\varphi_{ax}^3 &= \varphi_a^3 + \varphi_x^3 + \chi(a, x)
\end{aligned}$$

where

$$\begin{aligned}
\chi(a, x) = \varphi_x^1 \Bigg(&-\frac{1}{2}\,\varphi_a^2 + \frac{\alpha}{2(1-\alpha)}\,\overline{\varphi_a^2} + \frac{\alpha(2+\alpha)}{16(1-\alpha)}\,(\overline{\varphi_a^1})^2 + \frac{\alpha^2}{16(1-\alpha)}\,(\varphi_a^1)^2 - \frac{\alpha^2}{4(1-\alpha)}\,|\varphi_a^1|^2\Bigg) \\
&+ \frac{\alpha}{16(1-\alpha)}\left(\varphi_a^1 - \overline{\varphi_a^1}\right)(\varphi_x^1)^2 + \frac{1}{2(1-\alpha)}\,\overline{\varphi_a^1}\,\varphi_x^2.
\end{aligned}$$

5 Lie Algebra $\mathcal{G}_{6,4}$ (isomorphic to $M7$).

Commutation relations for $\mathcal{G}_{6,4}$: $[x_1, x_2] = x_4$; $[x_1, x_3] = x_6$; $[x_2, x_4] = x_5$.

$$J = \begin{pmatrix}
\boxed{\xi_1^1} & \boxed{\xi_2^1} & 0 & 0 & 0 & 0 \\
-({\xi_1^1}^2 + 1)/\xi_2^1 & -\xi_1^1 & 0 & 0 & 0 & 0 \\
* & \boxed{\xi_2^3} & b & \boxed{\xi_4^3} & 0 & 0 \\
* & \boxed{\xi_2^4} & -\frac{b^2+1}{\xi_4^3} & -b & 0 & 0 \\
* & * & * & * & \boxed{\xi_5^5} & -\frac{{\xi_5^5}^2+1}{c} \\
\boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & c & -\xi_5^5
\end{pmatrix} \tag{23}$$

where $J_1^3 = (-\xi_5^5\xi_2^4\xi_4^3 + 2\xi_5^5\xi_2^3\xi_1^1 - \xi_2^4\xi_4^3\xi_1^1 + \xi_2^3{\xi_1^1}^2 - \xi_2^3)/(\xi_2^1(\xi_5^5 + \xi_1^1))$;
$b = J_3^3 = (-\xi_5^5\xi_1^1 + 1)/(\xi_5^5 + \xi_1^1)$;

$J_1^4 = ({\xi_5^5}^2\xi_2^3{\xi_1^1}^2 + {\xi_5^5}^2\xi_2^3 + \xi_5^5\xi_2^4\xi_4^3{\xi_1^1}^2 + \xi_5^5\xi_2^4\xi_4^3 + \xi_2^4\xi_4^3{\xi_1^1}^3 + \xi_2^4\xi_4^3\xi_1^1 + \xi_2^3{\xi_1^1}^2 + \xi_2^3)/(\xi_4^3\xi_2^1({\xi_5^5}^2 + 2\xi_5^5\xi_1^1 + {\xi_1^1}^2));$
$J_1^5 = (\xi_4^6{\xi_5^5}^2\xi_2^3{\xi_1^1}^4 + 2\xi_4^6{\xi_5^5}^2\xi_2^3{\xi_1^1}^2 + \xi_4^6{\xi_5^5}^2\xi_2^3 + \xi_4^6\xi_5^5\xi_2^4\xi_4^3{\xi_1^1}^4 + 2\xi_4^6\xi_5^5\xi_2^4\xi_4^3{\xi_1^1}^2 + \xi_4^6\xi_5^5\xi_2^4\xi_4^3 + \xi_4^6\xi_2^4\xi_4^3{\xi_1^1}^5 + 2\xi_4^6\xi_2^4\xi_4^3{\xi_1^1}^3 + \xi_4^6\xi_2^4\xi_4^3\xi_1^1 +$
$\xi_4^6\xi_2^3{\xi_1^1}^4 + 2\xi_4^6\xi_2^3{\xi_1^1}^2 + \xi_4^6\xi_2^3 - \xi_3^6{\xi_5^5}^2\xi_2^4{\xi_4^3}^2{\xi_1^1}^2 - \xi_3^6{\xi_5^5}^2\xi_2^4{\xi_4^3}^2 + 2\xi_3^6{\xi_5^5}^2\xi_4^3\xi_2^3{\xi_1^1}^3 + 2\xi_3^6{\xi_5^5}^2\xi_4^3\xi_2^3\xi_1^1 - 2\xi_3^6\xi_5^5\xi_2^4{\xi_4^3}^2{\xi_1^1}^3 - 2\xi_3^6\xi_5^5\xi_2^4{\xi_4^3}^2\xi_1^1 +$
$3\xi_3^6\xi_5^5\xi_4^3\xi_2^3{\xi_1^1}^4 + 2\xi_3^6\xi_5^5\xi_4^3\xi_2^3{\xi_1^1}^2 - \xi_3^6\xi_5^5\xi_4^3\xi_2^3 - \xi_3^6\xi_2^4{\xi_4^3}^2{\xi_1^1}^4 - \xi_3^6\xi_2^4{\xi_4^3}^2{\xi_1^1}^2 + \xi_3^6\xi_4^3\xi_2^3{\xi_1^1}^5 - \xi_3^6\xi_4^3\xi_2^3\xi_1^1 - \xi_2^6{\xi_5^5}^2\xi_4^3{\xi_1^1}^4 - 2\xi_2^6{\xi_5^5}^2\xi_4^3{\xi_1^1}^2 -$
$\xi_2^6{\xi_5^5}^2\xi_4^3 - 2\xi_2^6\xi_5^5\xi_4^3{\xi_1^1}^5 - 4\xi_2^6\xi_5^5\xi_4^3{\xi_1^1}^3 - 2\xi_2^6\xi_5^5\xi_4^3\xi_1^1 - \xi_2^6\xi_4^3{\xi_1^1}^6 - 2\xi_2^6\xi_4^3{\xi_1^1}^4 - \xi_2^6\xi_4^3{\xi_1^1}^2 - \xi_1^6{\xi_5^5}^3\xi_4^3\xi_2^1{\xi_1^1}^2 - \xi_1^6{\xi_5^5}^3\xi_4^3\xi_2^1 - \xi_1^6{\xi_5^5}^2\xi_4^3\xi_2^1{\xi_1^1}^3 -$
$\xi_1^6{\xi_5^5}^2\xi_4^3\xi_2^1\xi_1^1 + \xi_1^6\xi_5^5\xi_4^3\xi_2^1{\xi_1^1}^4 + \xi_1^6\xi_5^5\xi_4^3\xi_2^1{\xi_1^1}^2 + \xi_1^6\xi_4^3\xi_2^1{\xi_1^1}^5 + \xi_1^6\xi_4^3\xi_2^1{\xi_1^1}^3)/({\xi_4^3}^2{\xi_2^1}^2({\xi_5^5}^3 + 3{\xi_5^5}^2\xi_1^1 + 3\xi_5^5{\xi_1^1}^2 + {\xi_1^1}^3));$
$J_2^5 = (\xi_4^6\xi_2^4{\xi_1^1}^2 + \xi_4^6\xi_2^4 + \xi_3^6\xi_2^3{\xi_1^1}^2 + \xi_3^6\xi_2^3 - \xi_2^6\xi_5^5{\xi_1^1}^2 - \xi_2^6\xi_5^5 - \xi_2^6{\xi_1^1}^3 - \xi_2^6\xi_1^1 + \xi_1^6\xi_2^1{\xi_1^1}^2 + \xi_1^6\xi_2^1)/(\xi_4^3\xi_2^1(\xi_5^5 + \xi_1^1));$
$J_3^5 = (-\xi_4^6{\xi_5^5}^2{\xi_1^1}^4 - 2\xi_4^6{\xi_5^5}^2{\xi_1^1}^2 - \xi_4^6{\xi_5^5}^2 - \xi_4^6{\xi_1^1}^4 - 2\xi_4^6{\xi_1^1}^2 - \xi_4^6 - \xi_3^6{\xi_5^5}^3\xi_4^3{\xi_1^1}^2 - \xi_3^6{\xi_5^5}^3\xi_4^3 - 3\xi_3^6{\xi_5^5}^2\xi_4^3{\xi_1^1}^3 - 3\xi_3^6{\xi_5^5}^2\xi_4^3\xi_1^1 - 2\xi_3^6\xi_5^5\xi_4^3{\xi_1^1}^4 -$
$\xi_3^6\xi_5^5\xi_4^3{\xi_1^1}^2 + \xi_3^6\xi_5^5\xi_4^3 + \xi_3^6\xi_4^3{\xi_1^1}^3 + \xi_3^6\xi_4^3\xi_1^1)/({\xi_4^3}^2\xi_2^1({\xi_5^5}^3 + 3{\xi_5^5}^2\xi_1^1 + 3\xi_5^5{\xi_1^1}^2 + {\xi_1^1}^3));$
$J_4^5 = (-\xi_4^6{\xi_5^5}^2{\xi_1^1}^2 - \xi_4^6{\xi_5^5}^2 - \xi_4^6{\xi_1^1}^2 - \xi_4^6 + \xi_3^6\xi_5^5\xi_4^3{\xi_1^1}^2 + \xi_3^6\xi_5^5\xi_4^3 + \xi_3^6\xi_4^3{\xi_1^1}^3 + \xi_3^6\xi_4^3\xi_1^1)/(\xi_4^3\xi_2^1({\xi_5^5}^2 + 2\xi_5^5\xi_1^1 + {\xi_1^1}^2));$
$c = J_5^6 = -(\xi_4^3\xi_2^1(\xi_5^5 + \xi_1^1))/({\xi_1^1}^2 + 1);$
and the parameters are subject to the condition

$$\xi_2^1\xi_4^3(\xi_1^1 + \xi_5^5) \neq 0. \tag{24}$$

Now the automorphism group of $\mathcal{G}_{6,4}$ is comprised of the matrices

$$\Phi = \begin{pmatrix} b_1^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 & 0 \\ b_1^3 & b_2^3 & b_3^3 & 0 & 0 & 0 \\ b_1^4 & b_2^4 & 0 & b_2^2 b_1^1 & 0 & 0 \\ b_1^5 & b_2^5 & b_3^5 & -b_1^4 b_2^2 & {b_2^2}^2 b_1^1 & 0 \\ b_1^6 & b_2^6 & b_3^6 & b_2^3 b_1^1 & 0 & b_3^3 b_1^1 \end{pmatrix}$$

where $\det \Phi = {b_3^3}^2 {b_2^2}^4 {b_1^1}^4 \neq 0$. Taking suitable values for the b_j^i's, equivalence by Φ leads to the case where $\xi_2^3 = \xi_2^4 = \xi_2^6 = \xi_1^6 = \xi_3^6 = \xi_4^6 = 0$, $\xi_2^1 = \xi_4^3 = 1$ and $\xi_1^1 = \alpha, \xi_5^5 = \beta, \alpha \neq -\beta$.

$$J_{\alpha,\beta} = \begin{pmatrix} \alpha & 1 & 0 & 0 & 0 & 0 \\ -(\alpha^2+1) & -\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-\alpha\beta+1}{\alpha+\beta} & 1 & 0 & 0 \\ 0 & 0 & -\frac{\alpha^2\beta^2+\alpha^2+\beta^2+1}{\alpha^2+2\alpha\beta+\beta^2} & \frac{\alpha\beta-1}{\alpha+\beta} & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & \frac{\alpha^2\beta^2+\alpha^2+\beta^2+1}{\alpha+\beta} \\ 0 & 0 & 0 & 0 & -\frac{\alpha+\beta}{\alpha^2+1} & -\beta \end{pmatrix} \quad (\alpha \neq -\beta) \tag{25}$$

Hence, any complex structure is equivalent to $J_{\alpha,\beta}$ in (25). The $J_{\alpha,\beta}$'s corresponding to different couples (α, β) are not equivalent. With the complex structure $J_{\alpha,\beta}$, the commutation relations of $\mathfrak{m} = \mathfrak{g}^{(1,0)}$ are
$[\tilde{x}_1, \tilde{x}_3] = \tilde{x}_5(-\alpha^4\beta^2 - \alpha^4 - 2\alpha^2\beta^2 - 2\alpha^2 - \beta^2 - 1)/(\alpha^2 + 2\alpha\beta + \beta^2) + (\tilde{x}_6\beta(\alpha^3 + \alpha^2\beta + \alpha + \beta))/(\alpha^2 + 2\alpha\beta + \beta^2);$
$[\tilde{x}_1, \tilde{x}_4] = \tilde{x}_5(\alpha^3\beta - \alpha^2 + \alpha\beta - 1)/(\alpha + \beta) - \tilde{x}_6\alpha;$
$[\tilde{x}_2, \tilde{x}_3] = -(\tilde{x}_5\alpha(\alpha^2\beta^2 + \alpha^2 + \beta^2 + 1))/(\alpha^2 + 2\alpha\beta + \beta^2) + \tilde{x}_6(\alpha^2\beta + \alpha\beta^2 - \alpha - \beta)/(\alpha^2 + 2\alpha\beta + \beta^2);$
$[\tilde{x}_2, \tilde{x}_4] = (\tilde{x}_5\beta(\alpha^2 + 1))/(\alpha + \beta) - \tilde{x}_6.$

5.1 Conclusions.

From (23), $\mathfrak{X}_{6,4}$ is a submanifold of dimension 10 in $\mathbb{R}^{36}$. It is the disjoint union of the continuously many orbits of the $J_{\alpha,\beta}$'s in (25).

5.2

5.2.1

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x^1} + \frac{1}{2}(y^1)^2 \frac{\partial}{\partial x^3} - x^2 \frac{\partial}{\partial y^3} - y^1 \frac{\partial}{\partial y^2} \\ X_2 &= \frac{\partial}{\partial y^1} + y^2 \frac{\partial}{\partial x^3}. \end{aligned}$$

5.2.2 Holomorphic functions for $J_{\alpha,\beta}$.

Let G denote the group G_0 endowed with the left invariant structure of complex manifold defined by $J_{\alpha,\beta}$ (25). Then $H_{\mathbb{C}}(G) = \{f \in C^\infty(G_0)\ ;\ \tilde{X}_j^- f = 0\ \forall j = 1,3,5\}$. One has

$$\begin{aligned}
\tilde{X}_1^- &= (1+i\alpha)\left[2\,\frac{\partial}{\partial\overline{w^1}} + \left(\frac{1}{2}(y^1)^2 + iy^2(1-i\alpha)\right)\frac{\partial}{\partial x^3} - x^2\frac{\partial}{\partial x^3} - y^1\frac{\partial}{\partial y^2}\right]\\
\tilde{X}_3^- &= 2\,\frac{\partial}{\partial\overline{w^2}}\\
\tilde{X}_5^- &= 2\,\frac{\partial}{\partial\overline{w^3}}
\end{aligned}$$

where

$$\begin{aligned}
w^1 &= x^1 - i(\alpha x^1 + y^1)\\
w^2 &= x^2 + \frac{(1-\alpha\beta)(\alpha+\beta)}{(1+\alpha^2)(1+\beta^2)}y^2 - i\frac{(\alpha+\beta)^2}{(1+\alpha^2)(1+\beta^2)}y^2\\
w^3 &= x^3 + \frac{\beta(\alpha^2+1)}{\alpha+\beta}\,y^3 - i\frac{\alpha^2+1}{\alpha+\beta}\,y^3.
\end{aligned}$$

Then $f \in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to w^2 and w^3 and satisfies the equation

$$2\,\frac{\partial f}{\partial\overline{w^1}} + \left(\frac{1}{4}((\alpha-i)w^1 + (\alpha+i)\overline{w^1})^2 - \frac{(\alpha^2+1)(\beta-i)}{\alpha+\beta}w^2\right)\frac{\partial f}{\partial w^3} + \frac{A}{2}((\alpha-i)w^1+(\alpha+i)\overline{w^1})\,\frac{\partial f}{\partial w^2} = 0$$

where

$$A = \frac{(\alpha+\beta)(1-\alpha\beta-i(\alpha+\beta))}{(1+\alpha^2)(1+\beta^2)}.$$

The 3 functions

$$\begin{aligned}
\varphi^1 &= w^1\\
\varphi^2 &= w^2 - \frac{A}{4}\left(\frac{\alpha+i}{2}\,(\overline{w^1})^2 + (\alpha-i)\,|w^1|^2\right)
\end{aligned}$$

$$\begin{aligned}
\varphi^3 = w^3 &- \frac{1}{16}(\alpha-i)^2\,(w^1)^2\overline{w^1} - \frac{\alpha^2+1}{16}\left(1 + A\frac{(\beta-i)(\alpha-i)}{\alpha+\beta}\right)w^1\,(\overline{w^1})^2\\
&- \frac{\alpha+i}{48}\left(\alpha+i+2A\frac{(\beta-i)(\alpha^2+1)}{\alpha+\beta}\right)(\overline{w^1})^3 + \frac{(\alpha^2+1)(\beta-i)}{2(\alpha+\beta)}\,\overline{w^1}w^2\,.
\end{aligned}$$

are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by

$$F = (\varphi^1, \varphi^2, \varphi^3). \tag{26}$$

F is a biholomorphic bijection, hence a global chart on G. We determine now how the multiplication of G looks like in the chart (26). Let $a, x \in G$:

$$\begin{aligned}
x &= \exp(x^1x_1)\exp(y^1x_2)\exp(x^2x_3)\exp(y^2x_4)\exp(x^3x_5)\exp(y^3x_6)\\
a &= \exp(\alpha^1x_1)\exp(\beta^1x_2)\exp(\alpha^2x_3)\exp(\beta^2x_4)\exp(\alpha^3x_5)\exp(\beta^3x_6).
\end{aligned}$$

With obvious notations, $a = [w_a^1, w_a^2, w_a^3]$, $x = [w_x^1, w_x^2, w_x^3]$, $a\,x = [w_{ax}^1, w_{ax}^2, w_{ax}^3]$, $a = [\varphi_a^1, \varphi_a^2, \varphi_a^3]$, $x = [\varphi_x^1, \varphi_x^2, \varphi_x^3]$, $a\,x = [\varphi_{ax}^1, \varphi_{ax}^2, \varphi_{ax}^3]$. Computations yield :

$$\begin{aligned}
w_{ax}^1 &= w_a^1 + w_x^1\\
w_{ax}^2 &= w_a^2 + w_x^2 - \frac{\alpha+\beta}{(1+\alpha^2)(1+\beta^2)}(1-\alpha\beta-i(\alpha+\beta)\,\beta^1x^1\\
w_{ax}^3 &= w_a^3 + w_x^3 + \frac{1}{2}\,(\beta^1)^2x^1 - \beta^2y^1 + \beta^1x^1y^1 - \frac{(\alpha^2+1)(\beta-i)}{\alpha+\beta}\,\alpha^2x^1.
\end{aligned}$$

We then get

$$
\begin{aligned}
\varphi^1_{ax} &= \varphi^1_a + \varphi^1_x \\
\varphi^2_{ax} &= \varphi^2_a + \varphi^2_x + \frac{\alpha+\beta}{4(\alpha^2+1)(\beta-i)}\left(2(1-i\alpha)\overline{\varphi^1_a} - (\alpha^2+1)\varphi^1_a\right)\varphi^1_x \\
\varphi^3_{ax} &= \varphi^3_a + \varphi^3_x + \chi(a,x)
\end{aligned}
$$

where

$$
\chi(a,x) = \frac{1}{16(\alpha+\beta)^2}\Big(-(\alpha+\beta)(\alpha+i)^2(\beta+i)\,(\overline{\varphi^1_a})^2 + (\alpha+\beta)(\alpha+2\beta+i)(\alpha-i)^2\,(\varphi^1_a)^2
$$
$$
+4(\alpha+\beta)(\alpha+\beta+i(\alpha\beta-1))\,|\varphi^1_a|^2 - 8(1+\alpha^2)(1+\beta^2)\,\overline{\varphi^2_a} - 8(1+\alpha^2)(\alpha-i)(\beta-i)\,\varphi^2_a\Big)\,\varphi^1_x
$$
$$
+\frac{1}{16}\left(2(\alpha^2-1-2i\alpha)\,\varphi^1_a + (\alpha^2+3+2i)\,\overline{\varphi^1_a}\right)(\varphi^1_x)^2 + \frac{(\alpha^2+1)(\beta-i)}{2(\alpha+\beta)}\,\overline{\varphi^1_a}\,\varphi^2_x .
$$

6 Lie Algebra $\mathcal{G}_{6,1}$ (isomorphic to $M4$).

Commutation relations for $\mathcal{G}_{6,1}$: $[x_1,x_2]=x_5$; $[x_1,x_4]=x_6$; $[x_2,x_3]=x_6$.

6.1 Case $\xi^2_1 \neq 0, \xi^2_3 \neq 0, \xi^1_3 \neq \xi^2_4$.

$$
J = \begin{pmatrix}
\boxed{\xi^1_1} & * & \boxed{\xi^1_3} & * & 0 & 0 \\
\boxed{\xi^2_1} & \boxed{\xi^2_2} & \boxed{\xi^2_3} & \boxed{\xi^2_4} & 0 & 0 \\
* & * & * & * & 0 & 0 \\
* & * & * & * & 0 & 0 \\
* & * & * & * & \boxed{\xi^5_5} & (-({\xi^5_5}^2+1))/\xi^6_5 \\
\boxed{\xi^6_1} & \boxed{\xi^6_2} & \boxed{\xi^6_3} & \boxed{\xi^6_4} & \boxed{\xi^6_5} & -\xi^5_5
\end{pmatrix} \tag{27}
$$

where $J^1_2 = (-((\xi_4^2\xi_3^2\xi_1^1 - \xi_4^2\xi_1^2\xi_3^1 - \xi_3^2\xi_2^2\xi_3^1 - (\xi_4^2-\xi_3^1)\xi_5^5\xi_3^2)\xi_5^6 + ({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1)\xi_1^2))/(\xi_5^6{\xi_3^2}^2);$

$J^1_4 = (-(({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1) - \xi_5^6\xi_4^2\xi_3^1))/(\xi_5^6\xi_3^2);$

$J^3_1 = (-(\xi_5^6\xi_5^5{\xi_4^2}^2\xi_3^2\xi_1^2 - \xi_5^6\xi_5^5\xi_4^2\xi_3^2\xi_1^2\xi_3^1 - \xi_5^6{\xi_4^2}^2\xi_3^2\xi_1^2\xi_1^1 + \xi_5^6{\xi_4^2}^2{\xi_1^2}^2\xi_3^1 + \xi_5^6\xi_4^2{\xi_3^2}^2{\xi_1^1}^2 + \xi_5^6\xi_4^2{\xi_3^2}^2 - \xi_5^6\xi_4^2\xi_3^2\xi_1^2\xi_3^1\xi_1^1 - {\xi_5^5}^2{\xi_4^2}^2{\xi_1^2}^2 + {\xi_5^5}^2\xi_4^2\xi_3^2\xi_2^2\xi_1^2 + {\xi_5^5}^2\xi_4^2\xi_3^2\xi_1^2\xi_1^1 + {\xi_5^5}^2\xi_4^2{\xi_1^2}^2\xi_3^1 - {\xi_5^5}^2\xi_3^2\xi_2^2\xi_1^2\xi_3^1 - {\xi_5^5}^2\xi_3^2\xi_1^2\xi_3^1\xi_1^1 - {\xi_4^2}^2{\xi_1^2}^2 + \xi_4^2\xi_3^2\xi_2^2\xi_1^2 + \xi_4^2\xi_3^2\xi_1^2\xi_1^1 + \xi_4^2{\xi_1^2}^2\xi_3^1 - \xi_3^2\xi_2^2\xi_1^2\xi_3^1 - \xi_3^2\xi_1^2\xi_3^1\xi_1^1))/(({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1){\xi_3^2}^2);$

$J^3_2 = (-({\xi_5^6}^2\xi_5^5{\xi_4^2}^2{\xi_3^2}^2\xi_2^2 + {\xi_5^6}^2\xi_5^5{\xi_4^2}^2{\xi_3^2}^2\xi_1^1 - {\xi_5^6}^2\xi_5^5{\xi_4^2}^2\xi_3^2\xi_1^2\xi_3^1 - {\xi_5^6}^2\xi_5^5\xi_4^2{\xi_3^2}^2\xi_2^2\xi_3^1 - {\xi_5^6}^2\xi_5^5\xi_4^2{\xi_3^2}^2\xi_3^1\xi_1^1 + {\xi_5^6}^2\xi_5^5\xi_4^2\xi_3^2\xi_1^2{\xi_3^1}^2 - {\xi_5^6}^2{\xi_4^2}^2{\xi_3^2}^2\xi_2^2\xi_1^1 - {\xi_5^6}^2{\xi_4^2}^2{\xi_3^2}^2{\xi_1^1}^2 + {\xi_5^6}^2{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2\xi_3^1 + 2{\xi_5^6}^2{\xi_4^2}^2\xi_3^2\xi_1^2\xi_3^1\xi_1^1 - {\xi_5^6}^2{\xi_4^2}^2{\xi_1^2}^2{\xi_3^1}^2 + {\xi_5^6}^2\xi_4^2{\xi_3^2}^2\xi_2^2\xi_3^1\xi_1^1 - {\xi_5^6}^2\xi_4^2{\xi_3^2}^2\xi_3^1 - {\xi_5^6}^2\xi_4^2\xi_3^2\xi_2^2\xi_1^2{\xi_3^1}^2 + \xi_5^6{\xi_5^5}^3{\xi_4^2}^2\xi_3^2\xi_1^2 - 2\xi_5^6{\xi_5^5}^3\xi_4^2\xi_3^2\xi_1^2\xi_3^1 + \xi_5^6{\xi_5^5}^3\xi_3^2\xi_1^2{\xi_3^1}^2 - \xi_5^6{\xi_5^5}^2{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2 - 2\xi_5^6{\xi_5^5}^2{\xi_4^2}^2\xi_3^2\xi_1^2\xi_1^1 + 2\xi_5^6{\xi_5^5}^2{\xi_4^2}^2{\xi_1^2}^2\xi_3^1 + \xi_5^6{\xi_5^5}^2\xi_4^2{\xi_3^2}^2{\xi_2^2}^2 + \xi_5^6{\xi_5^5}^2\xi_4^2{\xi_3^2}^2 + 2\xi_5^6{\xi_5^5}^2\xi_4^2\xi_3^2\xi_2^2\xi_1^2\xi_3^1 + 2\xi_5^6{\xi_5^5}^2\xi_4^2\xi_3^2\xi_1^2\xi_3^1\xi_1^1 - 2\xi_5^6{\xi_5^5}^2\xi_4^2{\xi_1^2}^2{\xi_3^1}^2 - \xi_5^6{\xi_5^5}^2{\xi_3^2}^2{\xi_2^2}^2\xi_3^1 - \xi_5^6{\xi_5^5}^2{\xi_3^2}^2\xi_3^1 - \xi_5^6{\xi_5^5}^2\xi_3^2\xi_2^2\xi_1^2{\xi_3^1}^2 + \xi_5^6\xi_5^5{\xi_4^2}^2\xi_3^2\xi_1^2 - 2\xi_5^6\xi_5^5\xi_4^2\xi_3^2\xi_1^2\xi_3^1 + \xi_5^6\xi_5^5\xi_3^2\xi_1^2{\xi_3^1}^2 - \xi_5^6{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2 - 2\xi_5^6{\xi_4^2}^2\xi_3^2\xi_1^2\xi_1^1 + 2\xi_5^6{\xi_4^2}^2{\xi_1^2}^2\xi_3^1 + \xi_5^6\xi_4^2{\xi_3^2}^2{\xi_2^2}^2 + \xi_5^6\xi_4^2{\xi_3^2}^2 + 2\xi_5^6\xi_4^2\xi_3^2\xi_2^2\xi_1^2\xi_3^1 + 2\xi_5^6\xi_4^2\xi_3^2\xi_1^2\xi_3^1\xi_1^1 - 2\xi_5^6\xi_4^2{\xi_1^2}^2{\xi_3^1}^2 - \xi_5^6{\xi_3^2}^2{\xi_2^2}^2\xi_3^1 - \xi_5^6{\xi_3^2}^2\xi_3^1 - \xi_5^6\xi_3^2\xi_2^2\xi_1^2{\xi_3^1}^2 - {\xi_5^5}^4{\xi_4^2}^2{\xi_1^2}^2 + 2{\xi_5^5}^4\xi_4^2{\xi_1^2}^2\xi_3^1 - {\xi_5^5}^4{\xi_1^2}^2{\xi_3^1}^2 - 2{\xi_5^5}^2{\xi_4^2}^2{\xi_1^2}^2 + 4{\xi_5^5}^2\xi_4^2{\xi_1^2}^2\xi_3^1 - 2{\xi_5^5}^2{\xi_1^2}^2{\xi_3^1}^2 - {\xi_4^2}^2{\xi_1^2}^2 + 2\xi_4^2{\xi_1^2}^2\xi_3^1 - {\xi_1^2}^2{\xi_3^1}^2))/(({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1)\xi_5^6{\xi_3^2}^3);$

$J^3_3 = (-(\xi_5^6\xi_5^5\xi_4^2\xi_3^2 - \xi_5^6\xi_4^2\xi_3^2\xi_1^1 + \xi_5^6\xi_4^2\xi_1^2\xi_3^1 - {\xi_5^5}^2\xi_4^2\xi_1^2 + {\xi_5^5}^2\xi_3^2\xi_2^2 + {\xi_5^5}^2\xi_1^2\xi_3^1 - \xi_4^2\xi_1^2 + \xi_3^2\xi_2^2 + \xi_1^2\xi_3^1))/(({\xi_5^5}^2+1)\xi_3^2);$

$J^3_4 = (-({\xi_5^6}^2\xi_5^5{\xi_4^2}^2\xi_3^2 - {\xi_5^6}^2{\xi_4^2}^2\xi_3^2\xi_1^1 + {\xi_5^6}^2{\xi_4^2}^2\xi_1^2\xi_3^1 - \xi_5^6{\xi_5^5}^2{\xi_4^2}^2\xi_1^2 + \xi_5^6{\xi_5^5}^2\xi_4^2\xi_3^2\xi_2^2 - \xi_5^6{\xi_5^5}^2\xi_4^2\xi_3^2\xi_1^1 + 2\xi_5^6{\xi_5^5}^2\xi_4^2\xi_1^2\xi_3^1 - \xi_5^6{\xi_4^2}^2\xi_1^2 + \xi_5^6\xi_4^2\xi_3^2\xi_2^2 - \xi_5^6\xi_4^2\xi_3^2\xi_1^1 + 2\xi_5^6\xi_4^2\xi_1^2\xi_3^1 - {\xi_5^5}^4\xi_4^2\xi_1^2 + {\xi_5^5}^4\xi_1^2\xi_3^1 - 2{\xi_5^5}^2\xi_4^2\xi_1^2 + 2{\xi_5^5}^2\xi_1^2\xi_3^1 - \xi_4^2\xi_1^2 + \xi_1^2\xi_3^1))/(({\xi_5^5}^2+1)\xi_5^6{\xi_3^2}^2);$

$J^4_1 = (-((({\xi_5^5}^2+1)\xi_1^2 - \xi_5^6\xi_5^5\xi_3^2)(\xi_4^2-\xi_3^1)\xi_1^2 - ((\xi_3^2{\xi_1^1}^2 + \xi_3^2 - \xi_1^2\xi_3^1\xi_1^1)\xi_3^2 - (\xi_3^2\xi_1^1 - \xi_1^2\xi_3^1)\xi_4^2\xi_1^2)\xi_5^6))/(({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1)\xi_3^2);$

$J^4_2 = (-(((({\xi_5^5}^2+1)\xi_1^2 - \xi_5^6\xi_5^5\xi_3^2)(\xi_4^2-\xi_3^1)(\xi_3^2\xi_2^2 + \xi_3^2\xi_1^1 - \xi_1^2\xi_3^1) - ((\xi_3^2\xi_2^2\xi_1^1 - \xi_3^2 - \xi_2^2\xi_1^2\xi_3^1)\xi_3^2\xi_3^1 - (\xi_3^2\xi_2^2 + \xi_3^2\xi_1^1 - \xi_1^2\xi_3^1)(\xi_3^2\xi_1^1 - \xi_1^2\xi_3^1)\xi_4^2)\xi_5^6))/(({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1){\xi_3^2}^2);$

$J^4_3 = -((\xi_3^2\xi_1^1 - \xi_1^2\xi_3^1 - \xi_5^5\xi_3^2)\xi_5^6 + ({\xi_5^5}^2+1)\xi_1^2)/({\xi_5^5}^2+1);$

$J^4_4 = (-((\xi_3^2\xi_1^1 - \xi_1^2\xi_3^1 - \xi_5^5\xi_3^2)\xi_5^6\xi_4^2 + (\xi_3^2\xi_1^1 - \xi_1^2\xi_3^1 + \xi_4^2\xi_1^2)({\xi_5^5}^2+1)))/(({\xi_5^5}^2+1)\xi_3^2);$

$J^5_1 = ((((\xi_5^5 - \xi_1^1)\xi_1^6 - \xi_2^6\xi_1^2)({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1){\xi_3^2}^2 + (\xi_5^6\xi_5^5{\xi_4^2}^2\xi_3^2\xi_1^2 - 2\xi_5^6\xi_5^5\xi_4^2\xi_3^2\xi_1^2\xi_3^1 + \xi_5^6\xi_5^5\xi_3^2\xi_1^2{\xi_3^1}^2 - \xi_5^6{\xi_4^2}^2\xi_3^2\xi_1^2\xi_1^1 + \xi_5^6{\xi_4^2}^2{\xi_1^2}^2\xi_3^1 + \xi_5^6\xi_4^2{\xi_3^2}^2{\xi_1^1}^2 + \xi_5^6\xi_4^2{\xi_3^2}^2 - \xi_5^6\xi_4^2\xi_3^2\xi_1^2\xi_3^1\xi_1^1 - {\xi_5^5}^2{\xi_4^2}^2{\xi_1^2}^2 + {\xi_5^5}^2\xi_4^2\xi_3^2\xi_2^2\xi_1^2 + {\xi_5^5}^2\xi_4^2\xi_3^2\xi_1^2\xi_1^1 + 2{\xi_5^5}^2\xi_4^2{\xi_1^2}^2\xi_3^1 - {\xi_5^5}^2\xi_3^2\xi_2^2\xi_1^2\xi_3^1 - {\xi_5^5}^2\xi_3^2\xi_1^2\xi_3^1\xi_1^1 - {\xi_5^5}^2{\xi_1^2}^2{\xi_3^1}^2 - {\xi_4^2}^2{\xi_1^2}^2 + \xi_4^2\xi_3^2\xi_2^2\xi_1^2 + \xi_4^2\xi_3^2\xi_1^2\xi_1^1 + 2\xi_4^2{\xi_1^2}^2\xi_3^1 - \xi_3^2\xi_2^2\xi_1^2\xi_3^1 - \xi_3^2\xi_1^2\xi_3^1\xi_1^1 - {\xi_1^2}^2{\xi_3^1}^2)\xi_3^6 - ((\xi_3^2{\xi_1^1}^2 + \xi_3^2 - \xi_1^2\xi_3^1\xi_1^1)\xi_3^2 - (\xi_3^2\xi_1^1 - \xi_1^2\xi_3^1)\xi_4^2\xi_1^2)\xi_5^6\xi_4^6\xi_3^2 + (({\xi_5^5}^2+1)\xi_1^2 - \xi_5^6\xi_5^5\xi_3^2)(\xi_4^6\xi_3^2 - \xi_3^6\xi_3^1)(\xi_4^2-\xi_3^1)\xi_1^2)/(({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1)\xi_5^6{\xi_3^2}^2);$

$J^5_2 = (((({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1)\xi_1^6\xi_1^2 + \xi_5^6\xi_2^6\xi_5^5{\xi_3^2}^2 - \xi_5^6\xi_2^6{\xi_3^2}^2\xi_2^2)({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1) - (\xi_5^6\xi_5^5\xi_4^2{\xi_3^2}^2\xi_2^2 + \xi_5^6\xi_5^5\xi_4^2{\xi_3^2}^2\xi_1^1 - \xi_5^6\xi_5^5{\xi_3^2}^2\xi_2^2\xi_3^1 - \xi_5^6\xi_5^5{\xi_3^2}^2\xi_3^1\xi_1^1 - \xi_5^6\xi_4^2{\xi_3^2}^2\xi_2^2\xi_1^1 - \xi_5^6\xi_4^2{\xi_3^2}^2{\xi_1^1}^2 + \xi_5^6\xi_4^2\xi_3^2\xi_2^2\xi_1^2\xi_3^1 + 2\xi_5^6\xi_4^2\xi_3^2\xi_1^2\xi_3^1\xi_1^1 - \xi_5^6\xi_4^2{\xi_1^2}^2{\xi_3^1}^2 + \xi_5^6{\xi_3^2}^2\xi_2^2\xi_3^1\xi_1^1 - \xi_5^6{\xi_3^2}^2\xi_3^1 - \xi_5^6\xi_3^2\xi_2^2\xi_1^2{\xi_3^1}^2 - {\xi_5^5}^2\xi_4^2\xi_3^2\xi_2^2\xi_1^2 - {\xi_5^5}^2\xi_4^2\xi_3^2\xi_1^2\xi_1^1 + {\xi_5^5}^2\xi_3^2\xi_2^2\xi_1^2\xi_3^1 + {\xi_5^5}^2\xi_3^2\xi_1^2\xi_3^1\xi_1^1 - \xi_4^2\xi_3^2\xi_2^2\xi_1^2 - \xi_4^2\xi_3^2\xi_1^2\xi_1^1 + \xi_3^2\xi_2^2\xi_1^2\xi_3^1 + \xi_3^2\xi_1^2\xi_3^1\xi_1^1)\xi_5^6\xi_4^6)\xi_3^2 + (\xi_5^6{\xi_5^5}^2{\xi_4^2}^2{\xi_3^2}^2 - 2\xi_5^6{\xi_5^5}^2\xi_4^2{\xi_3^2}^2\xi_3^1 + \xi_5^6{\xi_5^5}^2{\xi_3^2}^2{\xi_3^1}^2 + \xi_5^6\xi_5^5{\xi_4^2}^2{\xi_3^2}^2\xi_2^2 - \xi_5^6\xi_5^5\xi_4^2{\xi_3^2}^2\xi_2^2\xi_3^1 + \xi_5^6\xi_5^5\xi_4^2{\xi_3^2}^2\xi_3^1\xi_1^1 - \xi_5^6\xi_5^5{\xi_3^2}^2{\xi_3^1}^2\xi_1^1 - \xi_5^6{\xi_4^2}^2{\xi_3^2}^2\xi_2^2\xi_1^1 - \xi_5^6{\xi_4^2}^2{\xi_3^2}^2{\xi_1^1}^2 + \xi_5^6{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2\xi_3^1 + 2\xi_5^6{\xi_4^2}^2\xi_3^2\xi_1^2\xi_3^1\xi_1^1 - \xi_5^6{\xi_4^2}^2{\xi_1^2}^2{\xi_3^1}^2 + \xi_5^6\xi_4^2{\xi_3^2}^2\xi_2^2\xi_3^1\xi_1^1 - \xi_5^6\xi_4^2{\xi_3^2}^2\xi_3^1 - \xi_5^6\xi_4^2\xi_3^2\xi_2^2\xi_1^2{\xi_3^1}^2 - {\xi_5^5}^3{\xi_4^2}^2\xi_3^2\xi_1^2 + 2{\xi_5^5}^3\xi_4^2\xi_3^2\xi_1^2\xi_3^1 -$

$\xi_5^{5\,3}\xi_3^2\xi_1^2{\xi_3^1}^2 - {\xi_5^5}^2{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2 - {\xi_5^5}^2{\xi_4^2}^2\xi_3^2\xi_1^2\xi_1^1 + {\xi_5^5}^2{\xi_4^2}^2{\xi_1^2}^2\xi_3^1 + {\xi_5^5}^2\xi_4^2{\xi_3^2}^2{\xi_2^2}^2 + {\xi_5^5}^2\xi_4^2{\xi_3^2}^2 + 2{\xi_5^5}^2\xi_4^2\xi_3^2\xi_2^2\xi_1^2\xi_3^1 - {\xi_5^5}^2\xi_4^2{\xi_1^2}^2{\xi_3^1}^2 -$
${\xi_5^5}^2{\xi_3^2}^2{\xi_2^2}^2\xi_3^1 - {\xi_5^5}^2{\xi_3^2}^2\xi_3^1 - {\xi_5^5}^2\xi_3^2\xi_2^2\xi_1^2{\xi_3^1}^2 + {\xi_5^5}^2\xi_3^2\xi_1^2{\xi_3^1}^2\xi_1^1 - \xi_5^5{\xi_4^2}^2\xi_3^2\xi_1^2 + 2\xi_5^5\xi_4^2\xi_3^2\xi_1^2\xi_3^1 - \xi_5^5\xi_3^2\xi_1^2{\xi_3^1}^2 - {\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2 - {\xi_4^2}^2\xi_3^2\xi_1^2\xi_1^1 +$
${\xi_4^2}^2{\xi_1^2}^2\xi_3^1 + \xi_4^2{\xi_3^2}^2{\xi_2^2}^2 + \xi_4^2{\xi_3^2}^2 + 2\xi_4^2\xi_3^2\xi_2^2\xi_1^2\xi_3^1 - \xi_4^2{\xi_1^2}^2{\xi_3^1}^2 - {\xi_3^2}^2{\xi_2^2}^2\xi_3^1 - {\xi_3^2}^2\xi_3^1 - \xi_3^2\xi_2^2\xi_1^2{\xi_3^1}^2 + \xi_3^2\xi_1^2{\xi_3^1}^2\xi_1^1)\xi_5^6\xi_3^6 + (\xi_4^2\xi_3^2\xi_1^1 - \xi_4^2\xi_1^2\xi_3^1 -$
$\xi_3^2\xi_2^2\xi_3^1 - (\xi_4^2 - \xi_3^1)\xi_5^5\xi_3^2)({\xi_5^5}^2 + 1)(\xi_4^2 - \xi_3^1)\xi_5^6\xi_1^6\xi_3^2 - (({\xi_5^5}^2 + 1)\xi_1^2 - \xi_5^6\xi_5^5\xi_3^2)(\xi_5^6\xi_4^6\xi_3^2\xi_1^2\xi_3^1 - \xi_5^6\xi_3^6\xi_5^5\xi_4^2\xi_3^2 + \xi_5^6\xi_3^6\xi_5^5\xi_3^2\xi_3^1 + \xi_5^6\xi_3^6\xi_4^2\xi_3^2\xi_1^1 -$
$\xi_5^6\xi_3^6\xi_4^2\xi_1^2\xi_3^1 - \xi_5^6\xi_3^6\xi_3^2\xi_3^1\xi_1^1 + \xi_3^6{\xi_5^5}^2\xi_4^2\xi_1^2 - \xi_3^6{\xi_5^5}^2\xi_1^2\xi_3^1 + \xi_3^6\xi_4^2\xi_1^2 - \xi_3^6\xi_1^2\xi_3^1)(\xi_4^2 - \xi_3^1))/(({\xi_5^5}^2 + 1)(\xi_4^2 - \xi_3^1){\xi_5^6}^2{\xi_3^2}^3);$
$J_3^5 = (-((((\xi_2^6\xi_3^2 + \xi_1^6\xi_3^1 - \xi_3^6\xi_5^5 - \xi_4^6\xi_1^2)({\xi_5^5}^2 + 1) - (\xi_3^2\xi_1^1 - \xi_1^2\xi_3^1 - \xi_5^5\xi_3^2)\xi_5^6\xi_4^6)\xi_3^2 + (\xi_4^2\xi_3^2\xi_1^1 - \xi_4^2\xi_1^2\xi_3^1 - \xi_3^2\xi_2^2\xi_3^1 - (\xi_4^2 - \xi_3^1)\xi_5^5\xi_3^2)\xi_5^6\xi_3^6 -$
$((({\xi_5^5}^2 + 1)\xi_2^2 + (\xi_5^5 - \xi_2^2)\xi_5^6\xi_3^1)\xi_3^2 - ({\xi_5^5}^2 + 1)(\xi_4^2 - \xi_3^1)\xi_1^2)\xi_3^6))/(({\xi_5^5}^2 + 1)\xi_5^6\xi_3^2);$
$J_4^5 = (-({\xi_5^6}^2\xi_4^6\xi_5^5\xi_4^2{\xi_3^2}^2 - {\xi_5^6}^2\xi_4^6\xi_4^2{\xi_3^2}^2\xi_1^1 + {\xi_5^6}^2\xi_4^6\xi_4^2\xi_3^2\xi_1^2\xi_3^1 - {\xi_5^6}^2\xi_3^6\xi_5^5{\xi_4^2}^2\xi_3^2 + {\xi_5^6}^2\xi_3^6{\xi_4^2}^2\xi_3^2\xi_1^1 - {\xi_5^6}^2\xi_3^6{\xi_4^2}^2\xi_1^2\xi_3^1 - \xi_5^6\xi_4^6{\xi_5^5}^3{\xi_3^2}^2 -$
$\xi_5^6\xi_4^6{\xi_5^5}^2\xi_4^2\xi_3^2\xi_1^2 - \xi_5^6\xi_4^6{\xi_5^5}^2{\xi_3^2}^2\xi_1^1 + \xi_5^6\xi_4^6{\xi_5^5}^2\xi_3^2\xi_1^2\xi_3^1 - \xi_5^6\xi_4^6\xi_5^5{\xi_3^2}^2 - \xi_5^6\xi_4^6\xi_4^2\xi_3^2\xi_1^2 - \xi_5^6\xi_4^6{\xi_3^2}^2\xi_1^1 + \xi_5^6\xi_4^6\xi_3^2\xi_1^2\xi_3^1 + \xi_5^6\xi_3^6{\xi_5^5}^2{\xi_4^2}^2\xi_1^2 -$
$\xi_5^6\xi_3^6{\xi_5^5}^2\xi_4^2\xi_3^2\xi_2^2 + \xi_5^6\xi_3^6{\xi_5^5}^2\xi_4^2\xi_3^2\xi_1^1 - 2\xi_5^6\xi_3^6{\xi_5^5}^2\xi_4^2\xi_1^2\xi_3^1 + \xi_5^6\xi_3^6{\xi_4^2}^2\xi_1^2 - \xi_5^6\xi_3^6\xi_4^2\xi_3^2\xi_2^2 + \xi_5^6\xi_3^6\xi_4^2\xi_3^2\xi_1^1 - 2\xi_5^6\xi_3^6\xi_4^2\xi_1^2\xi_3^1 + \xi_5^6\xi_2^6{\xi_5^5}^2\xi_4^2{\xi_3^2}^2 +$
$\xi_5^6\xi_2^6\xi_4^2{\xi_3^2}^2 + \xi_5^6\xi_1^6{\xi_5^5}^2\xi_4^2\xi_3^2\xi_3^1 + \xi_5^6\xi_1^6\xi_4^2\xi_3^2\xi_3^1 + \xi_3^6{\xi_5^5}^4\xi_4^2\xi_1^2 - \xi_3^6{\xi_5^5}^4\xi_1^2\xi_3^1 + 2\xi_3^6{\xi_5^5}^2\xi_4^2\xi_1^2 - 2\xi_3^6{\xi_5^5}^2\xi_1^2\xi_3^1 + \xi_3^6\xi_4^2\xi_1^2 - \xi_3^6\xi_1^2\xi_3^1 - \xi_1^6{\xi_5^5}^4\xi_4^2\xi_3^2 +$
$\xi_1^6{\xi_5^5}^4\xi_3^2\xi_3^1 - 2\xi_1^6{\xi_5^5}^2\xi_4^2\xi_3^2 + 2\xi_1^6{\xi_5^5}^2\xi_3^2\xi_3^1 - \xi_1^6\xi_4^2\xi_3^2 + \xi_1^6\xi_3^2\xi_3^1))/(({\xi_5^5}^2 + 1){\xi_5^6}^2{\xi_3^2}^2);$
and the parameters are subject to the condition

$$\xi_1^2\xi_3^2\xi_5^6(\xi_3^1 - \xi_4^2) \neq 0. \tag{28}$$

Now the automorphism group of $\mathcal{G}_{6,1}$ is comprised of the matrices

$$\Phi = \begin{pmatrix} b_1^1 & b_2^1 & 0 & 0 & 0 & 0 \\ b_1^2 & b_2^2 & 0 & 0 & 0 & 0 \\ b_1^3 & b_2^3 & b_1^1 u & -b_2^1 u & 0 & 0 \\ b_1^4 & b_2^4 & -b_1^2 u & b_2^2 u & 0 & 0 \\ b_1^5 & b_2^5 & b_3^5 & b_4^5 & b_2^2 b_1^1 - b_1^2 b_2^1 & 0 \\ b_1^6 & b_2^6 & b_3^6 & b_4^6 & b_2^3 b_1^2 - b_1^3 b_2^2 - b_1^4 b_2^1 + b_2^4 b_1^1 & (b_2^2 b_1^1 - b_1^2 b_2^1)u \end{pmatrix} \tag{29}$$

where $u \in \mathbb{R}$, $u \neq 0$ and $\det \Phi = (b_2^2 b_1^1 - b_1^2 b_2^1)^4 u^3 \neq 0$. Taking suitable values for u and the b_j^i's, equivalence by Φ leads to the case where $\xi_1^1 = \xi_3^1 = \xi_2^2 = \xi_5^5 = \xi_1^6 = \xi_2^6 = \xi_3^6 = \xi_4^6 = 0$ and $\xi_1^2 = \xi_3^2 = \xi_5^6 = 1$:

$$J_\alpha = \begin{pmatrix} 0 & -\alpha & 0 & -\alpha & 0 & 0 \\ 1 & 0 & 1 & \alpha & 0 & 0 \\ \alpha - 1 & \alpha - 1 & \alpha & (\alpha+1)\alpha & 0 & 0 \\ (-(\alpha-1))/\alpha & 0 & -1 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{30}$$

with $\alpha \neq 0$. The J_α's corresponding to distincts α's are not equivalent.

Commutation relations of $\mathfrak{m}$: $[\tilde{x}_1, \tilde{x}_2] = (1-\alpha)\tilde{x}_5$; $[\tilde{x}_1, \tilde{x}_3] = -\tilde{x}_6$; $[\tilde{x}_1, \tilde{x}_4] = -\alpha(\tilde{x}_5 + \tilde{x}_6)$; $[\tilde{x}_2, \tilde{x}_3] = \alpha\tilde{x}_5$; $[\tilde{x}_2, \tilde{x}_4] = \alpha(\alpha\tilde{x}_5 - \tilde{x}_6)$; $[\tilde{x}_3, \tilde{x}_4] = -\alpha\tilde{x}_5$.

6.2 Case $\xi_1^2 \neq 0, \xi_3^2 = 0, \xi_3^1 \neq \xi_4^2$.

$$J = \begin{pmatrix} -\frac{\xi_1^4\xi_4^2 + \xi_2^2\xi_1^2}{\xi_1^2} & -\frac{{\xi_2^2}^2 + 1 + \xi_2^4\xi_4^2}{\xi_1^2} & \boxed{\xi_3^1} & * & 0 & 0 \\ \boxed{\xi_1^2} & \boxed{\xi_2^2} & 0 & \boxed{\xi_4^2} & 0 & 0 \\ * & * & -\frac{\xi_5^5\xi_4^2 - \xi_5^5\xi_3^1 + \xi_2^2\xi_3^1}{\xi_4^2} & * & 0 & 0 \\ \boxed{\xi_1^4} & \boxed{\xi_2^4} & -\frac{\xi_1^2\xi_3^1}{\xi_4^2} & * & 0 & 0 \\ * & * & * & * & \boxed{\xi_5^5} & \frac{\xi_4^2\xi_3^1}{\xi_4^2 - \xi_3^1} \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & \frac{({\xi_5^5}^2 + 1)(\xi_4^2 - \xi_3^1)}{\xi_4^2\xi_3^1} & -\xi_5^5 \end{pmatrix} \tag{31}$$

where $J_4^1 = (-((\xi_4^2 + \xi_3^1)\xi_2^2\xi_1^2 + \xi_1^4{\xi_4^2}^2 + (\xi_4^2 - \xi_3^1)\xi_5^5\xi_1^2))/{\xi_1^2}^2;$
$J_1^3 = (-((\xi_4^2 - \xi_3^1)\xi_1^4\xi_2^2 - \xi_2^4\xi_4^2\xi_1^2 - (\xi_4^2 - \xi_3^1)\xi_5^5\xi_1^4))/(\xi_1^2\xi_3^1);$
$J_2^3 = ((\xi_4^2 + \xi_3^1)\xi_2^4\xi_2^2\xi_1^2 - ({\xi_2^2}^2 + 1)\xi_1^4\xi_4^2 + (\xi_4^2 - \xi_3^1)\xi_5^5\xi_2^4\xi_1^2)/({\xi_1^2}^2\xi_3^1);$
$J_4^3 = ((((\xi_4^2 - \xi_3^1){\xi_5^5}^2 - (\xi_4^2 - \xi_3^1)\xi_5^5\xi_2^2 + ({\xi_2^2}^2 + 1)\xi_4^2 + \xi_2^4{\xi_4^2}^2)(\xi_4^2 - \xi_3^1) + ({\xi_2^2}^2 + 1 + \xi_2^4\xi_4^2)\xi_4^2\xi_3^1)\xi_1^2 + ((\xi_4^2 + \xi_3^1)\xi_2^2\xi_1^2 + \xi_1^4{\xi_4^2}^2)(\xi_5^5 -$
$\xi_2^2)(\xi_4^2 - \xi_3^1))/(\xi_4^2{\xi_1^2}^2\xi_3^1);$
$J_4^4 = (\xi_1^4{\xi_4^2}^2 + \xi_2^2\xi_1^2\xi_3^1 + (\xi_4^2 - \xi_3^1)\xi_5^5\xi_1^2)/(\xi_4^2\xi_1^2);$
$J_1^5 = (-(\xi_4^6\xi_1^4\xi_1^2\xi_3^1 + \xi_3^6\xi_5^5\xi_1^4\xi_4^2 - \xi_3^6\xi_5^5\xi_1^4\xi_3^1 + \xi_3^6\xi_2^4\xi_4^2\xi_1^2 - \xi_3^6\xi_1^4\xi_4^2\xi_2^2 + \xi_3^6\xi_1^4\xi_2^2\xi_3^1 + \xi_2^6{\xi_1^2}^2\xi_3^1 - \xi_1^6\xi_5^5\xi_1^2\xi_3^1 - \xi_1^6\xi_1^4\xi_4^2\xi_3^1 - \xi_1^6\xi_2^2\xi_1^2\xi_3^1)\xi_4^2)/(({\xi_5^5}^2 +$
$1)(\xi_4^2 - \xi_3^1)\xi_1^2);$
$J_2^5 = (-(((\xi_4^6\xi_2^4\xi_1^2 - \xi_2^6\xi_5^5\xi_1^2 + \xi_2^6\xi_2^2\xi_1^2 - \xi_1^6\xi_2^4\xi_4^2 - \xi_1^6{\xi_2^2}^2 - \xi_1^6)\xi_1^2\xi_3^1 + (\xi_5^5\xi_2^4\xi_1^2 + \xi_2^4\xi_2^2\xi_1^2 - \xi_1^4{\xi_2^2}^2 - \xi_1^4)(\xi_4^2 - \xi_3^1)\xi_3^6 - (({\xi_2^2}^2 + 1)\xi_1^4 -$

$2\xi_2^4\xi_2^2\xi_1^2)\xi_3^6\xi_3^1)\xi_4^2)/((\xi_5^{5\,2}+1)(\xi_4^2-\xi_3^1)\xi_1^{2\,2})$;
$J_3^5=((\xi_4^6\xi_1^2\xi_3^1+2\xi_3^6\xi_5^5\xi_4^2-\xi_3^6\xi_5^5\xi_3^1+\xi_3^6\xi_2^2\xi_3^1-\xi_1^6\xi_4^2\xi_3^1)\xi_3^1)/((\xi_5^{5\,2}+1)(\xi_4^2-\xi_3^1))$;
$J_4^5=((((\xi_4^2-\xi_3^1)\xi_4^6\xi_1^2+\xi_1^6\xi_4^2\xi_3^1-(\xi_5^5-\xi_2^2)(\xi_4^2-\xi_3^1)\xi_3^6)((\xi_4^2+\xi_3^1)\xi_2^2\xi_1^2+\xi_1^4\xi_4^{2\,2})+((\xi_4^6\xi_5^5-\xi_2^6\xi_4^2)\xi_1^2+(\xi_4^2-\xi_3^1)\xi_1^6\xi_5^5)\xi_4^2\xi_1^2\xi_3^1-((\xi_1^4\xi_4^2+\xi_2^2\xi_1^2)(\xi_4^2-\xi_3^1)\xi_4^2+(\xi_1^4\xi_4^2+\xi_2^2\xi_1^2)\xi_4^2\xi_3^1+(\xi_4^2-\xi_3^1)\xi_5^5\xi_1^2\xi_3^1)\xi_4^6\xi_1^2-(((\xi_4^2-\xi_3^1)\xi_5^{5\,2}-(\xi_4^2-\xi_3^1)\xi_5^5\xi_2^2+(\xi_2^{2\,2}+1)\xi_4^2+\xi_2^4\xi_4^{2\,2})(\xi_4^2-\xi_3^1)+(\xi_2^{2\,2}+1+\xi_2^4\xi_4^2)\xi_4^2\xi_3^1)\xi_3^6\xi_1^2)/((\xi_5^{5\,2}+1)(\xi_4^2-\xi_3^1)\xi_1^{2\,2})$;

and the parameters are subject to the condition

$$\xi_1^2\xi_4^2\xi_3^1(\xi_3^1-\xi_4^2)\neq 0. \tag{32}$$

Taking $u=1$ and suitable values for the b_j^i's in (29), equivalence by Φ switches to the case 6.5 below $\xi_1^2=\xi_3^2=0$.

6.3 Case $\xi_1^2 \neq 0, \xi_3^1 = \xi_4^2$.

$$J=\begin{pmatrix} \boxed{\xi_1^1} & \boxed{\xi_2^1} & 0 & 0 & 0 & 0\\ -\frac{\xi_1^{1\,2}+1}{\xi_2^1} & -\xi_1^1 & 0 & 0 & 0 & 0\\ \xi_2^4 & \frac{(2\xi_2^4\xi_1^1-\xi_1^4\xi_2^1)\xi_2^1}{\xi_1^{1\,2}+1} & \xi_1^1 & -\xi_2^1 & 0 & 0\\ \boxed{\xi_1^4} & \boxed{\xi_2^4} & \frac{\xi_1^{1\,2}+1}{\xi_2^1} & -\xi_1^1 & 0 & 0\\ * & * & \frac{(\xi_5^5-\xi_1^1)\xi_3^6\xi_2^1-(\xi_1^{1\,2}+1)\xi_4^6}{\xi_5^6\xi_2^1} & \frac{\xi_4^6\xi_5^5+\xi_4^6\xi_1^1+\xi_3^6\xi_2^1}{\xi_5^6} & \boxed{\xi_5^5} & -\frac{\xi_5^{5\,2}+1}{\xi_5^6}\\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & \boxed{\xi_5^6} & -\xi_5^5 \end{pmatrix} \tag{33}$$

where $J_1^5=((\xi_5^5-\xi_1^1)\xi_1^6\xi_2^1+(\xi_1^{1\,2}+1)\xi_2^6-\xi_3^6\xi_2^4\xi_2^1-\xi_4^6\xi_1^4\xi_2^1)/(\xi_5^6\xi_2^1)$;
$J_2^5=(-((2\xi_2^4\xi_1^1-\xi_1^4\xi_2^1)\xi_3^6\xi_2^1+(\xi_1^{1\,2}+1)\xi_4^6\xi_2^4-(\xi_2^6\xi_5^5+\xi_2^6\xi_1^1-\xi_1^6\xi_2^1)(\xi_1^{1\,2}+1)))/((\xi_1^{1\,2}+1)\xi_5^6)$;
and the parameters are subject to the condition

$$\xi_2^1\xi_5^6\neq 0. \tag{34}$$

Taking $u=\frac{\xi_5^6}{1+\xi_5^{5\,2}}$ and suitable values for the b_j^i's in (29), equivalence by Φ leads to the case where $\xi_1^1=\xi_1^4=\xi_2^4=\xi_5^5=\xi_1^6=\xi_2^6=\xi_3^6=\xi_4^6=0$ and $\xi_2^1=\xi_5^6=1$:

$$J=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{35}$$

J is not equivalent to any J_α. $\mathfrak{m}$ is here an abelian algebra.

6.4 Case $\xi_1^2 = 0, \xi_3^2 \neq 0$.

$$J=\begin{pmatrix} \boxed{\xi_1^1} & * & * & * & 0 & 0\\ 0 & \boxed{\xi_2^2} & \boxed{\xi_3^2} & \boxed{\xi_4^2} & 0 & 0\\ -\frac{\xi_1^4\xi_4^2}{\xi_3^2} & * & -\frac{\xi_5^6\xi_5^5\xi_4^2-\xi_5^6\xi_4^2\xi_1^1+\xi_5^{5\,2}\xi_2^2+\xi_2^2}{\xi_5^{5\,2}+1} & * & 0 & 0\\ \boxed{\xi_1^4} & * & \frac{(\xi_5^5-\xi_1^1)\xi_5^6\xi_3^2}{\xi_5^{5\,2}+1} & -\frac{(\xi_5^{5\,2}+1)\xi_1^1-(\xi_5^5-\xi_1^1)\xi_5^6\xi_4^2}{\xi_5^{5\,2}+1} & 0 & 0\\ * & * & * & * & \boxed{\xi_5^5} & -\frac{\xi_5^{5\,2}+1}{\xi_5^6}\\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & \boxed{\xi_5^6} & -\xi_5^5 \end{pmatrix} \tag{36}$$

where $J_2^1=((\xi_5^{5\,2}+1)(\xi_2^2-\xi_1^1)\xi_1^4\xi_4^2+(\xi_5^5-\xi_2^2)(\xi_1^{1\,2}+1)\xi_5^6\xi_3^2)/((\xi_5^{5\,2}+1)\xi_1^4\xi_3^2)$;
$J_3^1=((\xi_5^{5\,2}+1)\xi_1^4\xi_4^2-(\xi_1^{1\,2}+1)\xi_5^6\xi_3^2)/((\xi_5^{5\,2}+1)\xi_1^4)$;
$J_4^1=(-((\xi_5^{5\,2}+1)(\xi_1^{1\,2}+1)\xi_3^2-(\xi_5^{5\,2}+1)\xi_1^4\xi_4^{2\,2}+(\xi_1^{1\,2}+1)\xi_5^6\xi_4^2\xi_3^2))/((\xi_5^{5\,2}+1)\xi_1^4\xi_3^2)$;
$J_2^3=(-(((\xi_2^{2\,2}+1)\xi_3^2-\xi_1^4\xi_4^{2\,2})(\xi_5^{5\,2}+1)-(\xi_2^2\xi_1^1-1-(\xi_2^2+\xi_1^1)\xi_5^5)\xi_5^6\xi_4^2\xi_3^2))/((\xi_5^{5\,2}+1)\xi_3^{2\,2})$;
$J_4^3=(-(\xi_5^6\xi_5^5\xi_4^2-\xi_5^6\xi_4^2\xi_1^1+\xi_5^{5\,2}\xi_2^2-\xi_5^{5\,2}\xi_1^1+\xi_2^2-\xi_1^1)\xi_4^2)/((\xi_5^{5\,2}+1)\xi_3^2)$;
$J_2^4=(-((\xi_2^2\xi_1^1-1-(\xi_2^2+\xi_1^1)\xi_5^5)\xi_5^6\xi_3^2+(\xi_5^{5\,2}+1)\xi_1^4\xi_4^2))/((\xi_5^{5\,2}+1)\xi_3^2)$;
$J_1^5=(-(\xi_4^6\xi_1^4\xi_3^2-\xi_3^6\xi_1^4\xi_4^2-\xi_1^6\xi_5^5\xi_3^2+\xi_1^6\xi_3^2\xi_1^1))/(\xi_5^6\xi_3^2)$;

$J_2^5 = ((({\xi_5^5}^2+1)(\xi_5^5-\xi_2^2)\xi_2^6\xi_1^4\xi_3^2-({\xi_5^5}^2+1)(\xi_2^2-\xi_1^1)\xi_1^6\xi_1^4\xi_4^2-(\xi_5^5-\xi_2^2)({\xi_1^1}^2+1)\xi_5^6\xi_1^6\xi_3^2-((\xi_2^2+\xi_1^1)\xi_5^5+{\xi_1^1}^2+1)\xi_5^6\xi_4^6\xi_1^4\xi_3^2-(\xi_2^2\xi_1^1-1-(\xi_2^2+\xi_1^1)\xi_5^5)\xi_5^6\xi_3^6\xi_1^4\xi_4^2)\xi_3^2+(({\xi_2^2}^2+1)\xi_3^2-\xi_1^4{\xi_4^2}^2)({\xi_5^5}^2+1)\xi_3^6\xi_1^4+(({\xi_5^5}^2+1)\xi_1^4\xi_4^2+(\xi_2^2+\xi_1^1)\xi_5^6\xi_3^2\xi_1^1)\xi_4^6\xi_1^4\xi_3^2)/(({\xi_5^5}^2+1)\xi_5^6\xi_1^4{\xi_3^2}^2)$;
$J_3^5 = (-((((\xi_5^5-\xi_1^1)\xi_4^6\xi_1^4-({\xi_1^1}^2+1)\xi_1^6)\xi_3^2-(\xi_2^2-\xi_1^1)\xi_3^6\xi_1^4\xi_4^2)\xi_5^6+(\xi_2^6\xi_3^2+\xi_1^6\xi_4^2-\xi_3^6\xi_5^5)({\xi_5^5}^2+1)\xi_1^4-(({\xi_5^5}^2+1)\xi_2^2+(\xi_5^5-\xi_2^2)\xi_5^6\xi_4^2)\xi_3^6\xi_1^4))/(({\xi_5^5}^2+1)\xi_5^6\xi_1^4)$;
$J_4^5 = (-((((\xi_5^5-\xi_1^1)\xi_4^6\xi_1^4-({\xi_1^1}^2+1)\xi_1^6)\xi_5^6\xi_4^2-({\xi_5^5}^2+1)({\xi_1^1}^2+1)\xi_1^6)\xi_3^2+((\xi_2^6\xi_3^2+\xi_1^6\xi_4^2)\xi_4^2-(\xi_5^5+\xi_1^1)\xi_4^6\xi_3^2)({\xi_5^5}^2+1)\xi_1^4+(({\xi_1^1}^2+1)\xi_3^2-\xi_1^4{\xi_4^2}^2)(\xi_5^5-\xi_2^2)\xi_5^6\xi_3^6-(({\xi_5^5}^2+1)(\xi_2^2-\xi_1^1)\xi_1^4\xi_4^2+(\xi_5^5-\xi_2^2)({\xi_1^1}^2+1)\xi_5^6\xi_3^2+(\xi_2^2-\xi_1^1)\xi_5^6\xi_1^4{\xi_4^2}^2)\xi_3^6))/(({\xi_5^5}^2+1)\xi_5^6\xi_1^4\xi_3^2)$;
and the parameters are subject to the condition

$$\xi_3^2\xi_1^4\xi_5^6 \neq 0. \tag{37}$$

Taking suitable values for u and the b_j^i's in (29), equivalence by Φ switches to the case 6.1, more precisely $\xi_1^2 = \xi_3^2 = \xi_4^2 = 1, \xi_3^1 = 0$. Hence J is equivalent to J_α in (30) with $\alpha = 1$.

6.5 Case $\xi_1^2 = 0, \xi_3^2 = 0$.

$$J = \begin{pmatrix} \frac{\xi_5^5\xi_4^2-\xi_5^5\xi_3^1+\xi_2^2\xi_3^1}{\xi_4^2} & \boxed{\xi_2^1} & \boxed{\xi_3^1} & \frac{(\xi_4^2+\xi_3^1)\xi_2^2\xi_2^1+\xi_2^3\xi_4^2\xi_3^1+(\xi_4^2-\xi_3^1)\xi_5^5\xi_2^1}{{\xi_2^2}^2+1} & 0 & 0 \\ 0 & \boxed{\xi_2^2} & 0 & \boxed{\xi_4^2} & 0 & 0 \\ * & \boxed{\xi_2^3} & -\frac{\xi_5^5\xi_4^2-\xi_5^5\xi_3^1+\xi_2^2\xi_3^1}{\xi_4^2} & * & 0 & 0 \\ 0 & -\frac{{\xi_2^2}^2+1}{\xi_4^2} & 0 & -\xi_2^2 & 0 & 0 \\ * & * & * & * & \boxed{\xi_5^5} & -\frac{\xi_4^2\xi_3^1}{\xi_4^2-\xi_3^1} \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & \frac{({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1)}{\xi_4^2\xi_3^1} & -\xi_5^5 \end{pmatrix} \tag{38}$$

where $J_1^3 = (-((\xi_4^2-\xi_3^1){\xi_5^5}^2\xi_4^2-(\xi_4^2-\xi_3^1){\xi_5^5}^2\xi_3^1+2(\xi_4^2-\xi_3^1)\xi_5^5\xi_2^2\xi_3^1+({\xi_2^2}^2+1)\xi_4^2\xi_3^1+(\xi_4^2-{\xi_2^2}^2\xi_3^1)(\xi_4^2-\xi_3^1)))/({\xi_4^2}^2\xi_3^1)$;
$J_4^3 = (-((((\xi_4^2+\xi_3^1)\xi_2^2\xi_2^1+\xi_2^3\xi_4^2\xi_3^1)(\xi_4^2-\xi_3^1)\xi_5^5-((\xi_4^2+\xi_3^1)\xi_2^2\xi_2^1+\xi_2^3\xi_4^2\xi_3^1)(\xi_4^2-\xi_3^1)\xi_2^2+((\xi_4^2-\xi_3^1)^2{\xi_5^5}^2-(\xi_4^2-\xi_3^1)^2\xi_5^5\xi_2^2+(\xi_4^2-\xi_3^1)({\xi_2^2}^2+1)\xi_4^2+({\xi_2^2}^2+1)\xi_4^2\xi_3^1)\xi_2^1))/(({\xi_2^2}^2+1)\xi_4^2\xi_3^1)$;
$J_1^5 = ((\xi_5^5-\xi_2^2)\xi_1^6\xi_4^2{\xi_3^1}^2+(\xi_4^2-{\xi_2^2}^2\xi_3^1)(\xi_4^2-\xi_3^1)\xi_3^6+((\xi_4^2-\xi_3^1){\xi_5^5}^2\xi_4^2-(\xi_4^2-\xi_3^1){\xi_5^5}^2\xi_3^1+2(\xi_4^2-\xi_3^1)\xi_5^5\xi_2^2\xi_3^1+({\xi_2^2}^2+1)\xi_4^2\xi_3^1)\xi_3^6)/(({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1)\xi_4^2)$;
$J_2^5 = ((((\xi_2^6\xi_5^5-\xi_2^6\xi_2^2-\xi_1^6\xi_2^1-\xi_3^6\xi_2^3)\xi_4^2+({\xi_2^2}^2+1)\xi_4^6)\xi_3^1)/(({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1))$;
$J_3^5 = ((2\xi_3^6\xi_5^5\xi_4^2-\xi_3^6\xi_5^5\xi_3^1+\xi_3^6\xi_2^2\xi_3^1-\xi_1^6\xi_4^2\xi_3^1)\xi_3^1)/(({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1))$;
$J_4^5 = ((((\xi_4^2-\xi_3^1)^2{\xi_5^5}^2-(\xi_4^2-\xi_3^1)^2\xi_5^5\xi_2^2+(\xi_4^2-\xi_3^1)({\xi_2^2}^2+1)\xi_4^2+({\xi_2^2}^2+1)\xi_4^2\xi_3^1)\xi_3^6\xi_2^1+(\xi_4^6\xi_5^5{\xi_2^2}^2+\xi_4^6\xi_5^5+\xi_4^6{\xi_2^2}^3+\xi_4^6\xi_2^2-\xi_2^6\xi_4^2{\xi_2^2}^2-\xi_2^6\xi_4^2-\xi_1^6\xi_5^5\xi_4^2\xi_2^1+\xi_1^6\xi_5^5\xi_3^1\xi_2^1)\xi_4^2\xi_3^1+((\xi_4^2+\xi_3^1)\xi_2^2\xi_2^1+\xi_2^3\xi_4^2\xi_3^1)((\xi_4^2-\xi_3^1)\xi_3^6\xi_5^5-(\xi_4^2-\xi_3^1)\xi_3^6\xi_2^2-\xi_1^6\xi_4^2\xi_3^1))/(({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1)({\xi_2^2}^2+1))$;
and the parameters are subject to the condition

$$\xi_3^1\xi_4^2(\xi_4^2-\xi_3^1) \neq 0. \tag{39}$$

• Suppose first $\xi_4^2 \neq -\xi_3^1$, taking suitable values for u and the b_j^i's in (29), equivalence by Φ leads to the case where $\xi_2^1 = \xi_2^2 = \xi_2^3 = \xi_5^5 = \xi_1^6 = \xi_2^6 = \xi_3^6 = \xi_4^6 = 0$ and $\xi_3^1 = 1, \xi_4^2 = \beta \neq 0, \pm 1$:

$$J'_\beta = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & (-1)/\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta/(\beta-1) \\ 0 & 0 & 0 & 0 & (\beta-1)/\beta & 0 \end{pmatrix} \tag{40}$$

where $\beta \neq 0, \pm 1$. The J's corresponding to different β, β' are not equivalent unless $\beta' = 1/\beta$. J'_β is equivalent to J_α in (30) if and only if $\alpha = \frac{(\beta-1)^2}{\beta}$. Note that the map $\beta \mapsto \alpha$ is invariant under the substitution $\beta \mapsto \frac{1}{\beta}$. This explains why the J_α are nonequivalent for different α though $J'_\beta, J'_{\frac{1}{\beta}}$ are equivalent. Note also that the image of $\mathbb{R} \setminus \{0, \pm 1\}$ under the map $\beta \mapsto \alpha$ is $\mathbb{R} \setminus [-4, 0]$.
• Suppose now $\xi_4^2 = -\xi_3^1$ in (38). Taking suitable values for u and the b_j^i's in (29), equivalence by Φ leads to the case where $\xi_2^1 = \xi_2^2 = \xi_5^5 = \xi_1^6 = \xi_2^6 = \xi_3^6 = \xi_4^6 = 0$ and $\xi_3^1 = 1, \xi_4^2 = -1$:

$$J''_\gamma = \begin{pmatrix} 0 & 0 & 1 & -\gamma & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & \gamma & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (-1)/2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} \tag{41}$$

Commutation relations of $\mathfrak{m}$: $[\tilde{x}_1, \tilde{x}_2] = \tilde{x}_5$; $[\tilde{x}_1, \tilde{x}_4] = 2\tilde{x}_6$; $[\tilde{x}_2, \tilde{x}_3] = 2\tilde{x}_6$; $[\tilde{x}_2, \tilde{x}_4] = -2\gamma\tilde{x}_6$; $[\tilde{x}_3, \tilde{x}_4] = \tilde{x}_5$.

J''_γ is not equivalent to any J_α in (30) nor to (35), and each $J''_\gamma\,(\gamma \neq 0)$ is equivalent to J''_1. J''_1 is not equivalent to J''_0.

6.6 Conclusions.

One has with obvious notations

$$\mathfrak{X}_{6,1} = \mathfrak{X}_{\xi^2_1 \neq 0} \cup \mathfrak{X}_{\xi^2_3 \neq 0} \cup \mathfrak{X}_{\xi^2_4 \neq 0} \tag{42}$$

and

$$\mathfrak{X}_{\xi^2_3 \neq 0} \subset \mathfrak{X}_{\xi^1_3 \neq \xi^2_4}.$$

It can be seen that the formula (27), which still makes sense for $\xi^6_5 \neq 0$ under the only assumption that $\xi^2_3 \neq 0$, yields all of $\mathfrak{X}_{\xi^2_3 \neq 0}$. Hence $\mathfrak{X}_{\xi^2_3 \neq 0}$ is is a 12-dimensional submanifold of $\mathbb{R}^{36}$ with a global chart. The automorphism

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

switches ξ^2_3 and ξ^2_4, hence $\mathfrak{X}_{\xi^2_4 \neq 0} = \Phi\, \mathfrak{X}_{\xi^2_3 \neq 0}\, \Phi^{-1}$ is also is a 12-dimensional submanifold of $\mathbb{R}^{36}$. Consider now $\mathfrak{X}_{\xi^2_1 \neq 0}$. There are 3 subcases :

$$\begin{aligned} \xi^2_3 &\neq 0 \qquad \text{(which implies } \xi^1_3 \neq \xi^2_4 \text{ from section 6.3)} && (43) \\ \xi^2_3 &= 0 \qquad \text{and} \qquad \xi^1_3 \neq \xi^2_4 && (44) \\ \xi^2_3 &= 0 \qquad \text{and} \qquad \xi^1_3 = \xi^2_4. && (45) \end{aligned}$$

To prove that $\mathfrak{X}_{\xi^2_1 \neq 0}$ is is a 12-dimensional submanifold of $\mathbb{R}^{36}$, it is sufficient to prove that it is a local submanifold in the neighborhood of any of its points. Take any $K \in \mathfrak{X}_{\xi^2_1 \neq 0} : K = (\xi^i_j(K))$. In case (43), $K \in \mathfrak{X}_{\xi^2_3 \neq 0}$ and then, from the section 6.1, $\mathfrak{X}_{\xi^2_1 \neq 0}$ is a local 12-dimensional submanifold of $\mathbb{R}^{36}$. Suppose now K belongs in case (44) or (45). To solve the initial system comprised of all the torsion equations and the equation $J^2 = -1$ in $\mathbb{R}^{36}$ in the neighborhood of K, one has to complete first a set of common steps, and then we are left with solving the system S of the remaining equations in the 15 variables $\xi^1_1, \xi^1_2, \xi^1_3, \xi^2_1, \xi^2_2, \xi^2_3, \xi^2_4, \xi^4_1, \xi^4_2, \xi^5_5, \xi^6_1, \xi^6_2, \xi^6_3, \xi^6_4, \xi^6_5$ in the open subset $\xi^6_5 \neq 0$ of $\mathbb{R}^{15}$. Among these equations, we single out the 3 following equations :

$$\begin{cases} f = 0 \\ g = 0 \\ h = 0 \end{cases} \tag{46}$$

where : $f = {J^2}^2_1 = (-\xi^6_5\xi^5_5\xi^2_3\xi^2_2 - \xi^6_5\xi^5_5\xi^2_3\xi^1_1 + \xi^6_5\xi^2_3\xi^2_2\xi^1_1 - \xi^6_5\xi^2_3\xi^2_1\xi^1_2 - \xi^6_5\xi^2_3 + {\xi^5_5}^2\xi^4_2\xi^2_3 + {\xi^5_5}^2\xi^4_1\xi^2_4 + {\xi^5_5}^2\xi^2_2\xi^2_1 + {\xi^5_5}^2\xi^2_1\xi^1_1 + \xi^4_2\xi^2_3 + \xi^4_1\xi^2_4 + \xi^2_2\xi^2_1 + \xi^2_1\xi^1_1)/({\xi^5_5}^2 + 1)$;
$g = {J^2}^2_2 = (-\xi^6_5\xi^5_5\xi^2_3{\xi^2_2}^2 - \xi^6_5\xi^5_5\xi^2_3\xi^2_1\xi^1_2 - \xi^6_5\xi^5_5\xi^2_3 + \xi^6_5\xi^2_3{\xi^2_2}^2\xi^1_1 - \xi^6_5\xi^2_3\xi^2_2\xi^2_1\xi^1_2 + \xi^6_5\xi^2_3\xi^1_1 + {\xi^5_5}^2\xi^4_2\xi^2_4\xi^2_1 + {\xi^5_5}^2\xi^4_2\xi^2_3\xi^2_2 - {\xi^5_5}^2\xi^4_2\xi^2_3\xi^1_1 + {\xi^5_5}^2\xi^4_1\xi^2_3\xi^1_2 + {\xi^5_5}^2{\xi^2_2}^2\xi^2_1 + {\xi^5_5}^2{\xi^2_1}^2\xi^1_2 + \xi^4_2\xi^2_4\xi^2_1 + \xi^4_2\xi^2_3\xi^2_2 - \xi^4_2\xi^2_3\xi^1_1 + \xi^4_1\xi^2_3\xi^1_2 + {\xi^2_2}^2\xi^2_1 + {\xi^2_1}^2\xi^1_2)/(\xi^2_1({\xi^5_5}^2 + 1))$;
$h = {J^2}^2_3 = (\xi^6_5\xi^5_5\xi^2_4\xi^2_3 - \xi^6_5\xi^5_5\xi^2_3\xi^1_3 - \xi^6_5\xi^2_4\xi^2_3\xi^1_1 + \xi^6_5\xi^2_4\xi^2_1\xi^1_3 - \xi^6_5{\xi^2_3}^2\xi^1_2 + \xi^6_5\xi^2_3\xi^2_2\xi^1_3 - {\xi^5_5}^2\xi^2_4\xi^2_1 + {\xi^5_5}^2\xi^2_1\xi^1_3 - \xi^2_4\xi^2_1 + \xi^2_1\xi^1_3)/({\xi^5_5}^2 + 1)$.

The solution J that's looked for is of type $\xi^2_1 \neq 0$ hence belongs in one of the 3 cases (43), (44), (45). If J belongs in case (44) or (45), the system S is equivalent to the 3 equations (46). If J belongs in the case (43), the system S is equivalent to the 3 equations (46) if and only if $c(J) \neq 0$ where

$$c(J) = ({\xi^5_5}^2 + 1)\xi^2_1 + \xi^2_1\xi^2_4\xi^6_5 - (\xi^5_5 - \xi^2_2)\xi^2_3\xi^6_5. \tag{47}$$

Now, if K belongs in case (44),

$$c(K) = \xi^2_1(K)({\xi^5_5(K)}^2 + 1) + \xi^2_4(K)\xi^6_5(K)) = \frac{1}{\xi^1_3(K)}\,\xi^2_1(K)\xi^2_4(K)({\xi^5_5(K)}^2 + 1) \neq 0$$

since in that case
$$\xi_5^6(K) = \frac{\xi_5^5(K)^2+1}{\xi_4^2(K)\xi_3^1(K)}\left(\xi_4^2(K)-\xi_3^1(K)\right)$$
(see (31)). If K belongs in case (45),
$$c(K) = \xi_1^2(K)(\xi_5^5(K)^2+1) \neq 0$$
(see (33)). Hence in both cases, one has $c(J)\neq 0$ in some neighborhood of K and the remaining system is equivalent in that neighborhood to the 3 equations (46). We will now show that the system (46) is of maximal rank 3 at K, that is some 3-jacobian doesn't vanish.

• Suppose K belongs in case (44). Then
$$\frac{D(f,g,h)}{D(\xi_1^1,\xi_2^1,\xi_5^6)}(K) = -\frac{\xi_4^2(K)\xi_1^2(K)^3\xi_3^1(K)}{\xi_5^5(K)^2+1} \neq 0.$$

• Suppose K belongs in case (45). Then
$$\frac{D(f,g,h)}{D(\xi_1^1,\xi_2^1,\xi_4^2)}(K) = -\xi_1^2(K)^3 \neq 0.$$

Hence the system (46) is of maximal rank 3 at K, and it follows that $\mathfrak{X}_{\xi_1^2\neq 0}$ is is a local submanifold in the neighborhood of K.

Hence $\mathfrak{X}_{\xi_1^2\neq 0}$ is is a 12-dimensional submanifold of $\mathbb{R}^{36}$, and so is $\mathfrak{X}_{6,1}$ from (42). Any element of $\mathfrak{X}_{6,1}$ is is equivalent to either J in (35), or $J_\alpha(\alpha\neq 0)$ in (30), or J_1'', or J_0'' in (41).

6.7

6.7.1

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x^1} - y^1\frac{\partial}{\partial x^3} - y^2\frac{\partial}{\partial y^3} \\ X_2 &= \frac{\partial}{\partial y^1} - x^2\frac{\partial}{\partial y^3}. \end{aligned}$$

6.7.2 Holomorphic functions for J_α.

Let G denote the group G_0 endowed with the left invariant structure of complex manifold defined by J_α (30). Then $H_{\mathbb{C}}(G) = \{f\in C^\infty(G_0)\ ;\ \tilde{X}_j^- f = 0\ \forall j\ 1\leqslant j\leqslant 6\}$. As $\tilde{X}_2^- = -i(\alpha\tilde{X}_1^-+(1-\alpha)\tilde{X}_3^-)$, $\tilde{X}_4^- = -i\alpha\tilde{X}_1^-+(1-i)\tilde{X}_3^-$, $\tilde{X}_6^- = -i\tilde{X}_5^-$, one has $H_{\mathbb{C}}(G) = \{f\in C^\infty(G_0)\ ;\ \tilde{X}_j^- f = 0\ \forall j=1,3,5\}$. Now
$$\begin{aligned} \tilde{X}_1^- &= 2\frac{\partial}{\partial\overline{z^1}} + i(\alpha-1)\left(\frac{\partial}{\partial x^2} - \frac{1}{\alpha}\frac{\partial}{\partial y^2}\right) - y^1\frac{\partial}{\partial x^3} - (y^2+ix^2)\frac{\partial}{\partial y^3} \\ \tilde{X}_3^- &= i\frac{\partial}{\partial y^1} + (1+i\alpha)\frac{\partial}{\partial x^2} - i\frac{\partial}{\partial y^2} - ix^2\frac{\partial}{\partial y^3} \\ \tilde{X}_5^- &= 2\frac{\partial}{\partial\overline{z^3}} \end{aligned}$$
where $z^1 = x^1+iy^1,\ z^3 = x^3+iy^3$. Then $f\in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to z^3 and satisfies the 2 equations
$$2\frac{\partial f}{\partial\overline{w^2}} + \frac{\partial f}{\partial\overline{z^1}} - \frac{\partial f}{\partial z^1} + \frac{1}{2}\left((1-i\alpha)w^2+(1+i\alpha)\overline{w^2}\right)\frac{\partial f}{\partial z^3} = 0 \qquad (48)$$
$$2\frac{\partial f}{\partial\overline{z^1}} + \frac{\alpha-1}{\alpha}\left(\frac{\partial f}{\partial\overline{w^2}} - \frac{\partial f}{\partial w^2}\right) + \left(\frac{\overline{z^1}-z^1}{2i} + \left(1-\frac{i\alpha}{2}\right)w^2 + \frac{i\alpha}{2}\overline{w^2}\right)\frac{\partial f}{\partial z^3} = 0 \qquad (49)$$
where $w^2 = x^2+\alpha y^2 - iy^2$. From now on we set $w^1 = z^1, w^3 = z^3$. The 3 functions
$$\begin{aligned} \varphi^1 &= 2w^1 + \overline{w^2} + w^2 \\ \varphi^2 &= 2w^2 + \frac{\alpha-1}{\alpha}(\overline{w^1}+w^1) \end{aligned}$$

$$\varphi^3 = w^3 + \frac{1}{32}\left(4i(\overline{w^1})^2 - 8i\overline{w^1 w^2} - 8i\overline{w^1}w^1 - 8(2-i)\overline{w^1}w^2 - (4+i)(\overline{w^2})^2 + 4i\overline{w^2}w^1 + 4i\overline{w^2}w^2\right)$$

if $\alpha = 1$ and if not

$$\varphi^3 = w^3 - \frac{\alpha i}{2(\alpha-1)}\,\overline{w^1}w^2 - \frac{1+i\alpha}{8}\,(\overline{w^2})^2 - \frac{1-\alpha+i\alpha(1+\alpha)}{4(\alpha-1)}\,\overline{w^2}w^2 + \frac{i\alpha}{2(\alpha-1)}w^1w^2 - \frac{3\alpha^2-2\alpha-1-i\alpha(\alpha+1)^2}{8(\alpha-1)^2}\,(w^2)^2$$

are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by

$$F = (\varphi^1, \varphi^2, \varphi^3). \tag{50}$$

F is a biholomorphic bijection, hence a global chart on G. We determine now how the multiplication of G looks like in the chart (50). Let $a, x \in G$:

$$\begin{array}{rcl} x &=& \exp(x^1x_1)\exp(y^1x_2)\exp(x^2x_3)\exp(y^2x_4)\exp(x^3x_5)\exp(y^3x_6) \\ a &=& \exp(a^1x_1)\exp(b^1x_2)\exp(a^2x_3)\exp(b^2x_4)\exp(a^3x_5)\exp(b^3x_6). \end{array}$$

With obvious notations, $a = [w_a^1, w_a^2, w_a^3]$, $x = [w_x^1, w_x^2, w_x^3]$, $a\,x = [w_{ax}^1, w_{ax}^2, w_{ax}^3]$, $a = [\varphi_a^1, \varphi_a^2, \varphi_a^3]$, $x = [\varphi_x^1, \varphi_x^2, \varphi_x^3]$, $a\,x = [\varphi_{ax}^1, \varphi_{ax}^2, \varphi_{ax}^3]$. Computations yield :

$$w_{ax}^1 = w_a^1 + w_x^1 \tag{51}$$

$$w_{ax}^2 = w_a^2 + w_x^2 \tag{52}$$

$$w_{ax}^3 = w_a^3 + w_x^3 - b^1x^1 - i(b^2x^1 + a^2y^1). \tag{53}$$

We then get

$$\begin{array}{rcl} \varphi_{ax}^1 &=& \varphi_a^1 + \varphi_x^1 \\ \varphi_{ax}^2 &=& \varphi_a^2 + \varphi_x^2 \\ \varphi_{ax}^3 &=& \varphi_a^3 + \varphi_x^3 + \chi(a,x) \end{array}$$

where for $\alpha \neq 1$

$$\chi(a,x) = \frac{1}{8}\,\varphi_x^1\left((\overline{\varphi_a^1} + \varphi_a^1)((1-i)\alpha - 1) - \alpha\,\overline{\varphi_a^2} + \frac{\alpha(1-\alpha+2i\alpha)}{\alpha-1}\,\varphi_a^2\right) + \frac{1}{8}\,\varphi_x^2\left(2\frac{i\alpha}{1-i}\,\overline{\varphi_a^1} + \alpha\overline{\varphi_a^2} + 2\frac{\alpha(1+i\alpha)}{(\alpha-1)(1-i)}\,\varphi_a^1 + \alpha\frac{(1-2i)\alpha^2-1}{(\alpha-1)^2}\,\varphi_a^2\right)$$

and for $\alpha = 1$

$$\chi(a,x) = \frac{1}{32}\,\varphi_x^1\left(-4i\overline{\varphi_a^1} + 2i\varphi_a^1 - 4\overline{\varphi_a^2} + 3i\varphi_a^2\right) + \frac{1}{64}\,\varphi_x^2\left(8(i-1)\,\overline{\varphi_a^1} + 8\overline{\varphi_a^2} - 2i\,\varphi_a^1 + (4-5i)\,\varphi_a^2\right).$$

6.7.3 Holomorphic functions for J.

Now J is defined in (35). Then $f \in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to z^2 and z^3 and satisfies the equation

$$2\,\frac{\partial f}{\partial \overline{w^1}} = (z^2 + y^1)\,\frac{\partial f}{\partial \overline{z^3}}$$

where $w^1 = x^1 - iy^1$, $z^2 = x^2 + iy^2$, $z^3 = x^3 + iy^3$. From now on we set $w^2 = z^2, w^3 = z^3$. The 3 functions

$$\begin{array}{rcl} \varphi^1 &=& w^1 \\ \varphi^2 &=& w^2 \\ \varphi^3 &=& w^3 + \dfrac{1}{2}w^2\overline{w^1} + \dfrac{i}{4}w^1\overline{w^1} - \dfrac{i}{8}\overline{w^1}^2 \end{array}$$

are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by

$$F = (\varphi^1, \varphi^2, \varphi^3). \tag{54}$$

F is a biholomorphic bijection, hence a global chart on G. We then get

$$\begin{aligned}
\varphi^1_{ax} &= \varphi^1_a + \varphi^1_x \\
\varphi^2_{ax} &= \varphi^2_a + \varphi^2_x \\
\varphi^3_{ax} &= \varphi^3_a + \varphi^3_x + \chi(a,x)
\end{aligned}$$

where from (53)

$$\chi(a,x) = \frac{1}{4}\,\varphi^1_x\left(2i\overline{\varphi^1_a} - i\varphi^1_a + 2\overline{\varphi^2_a}\right) + \frac{1}{2}\,\varphi^2_x\overline{\varphi^1_a}.$$

6.7.4 Holomorphic functions for J''_γ.

J''_γ is defined in (41) for any real γ. Here $\tilde{X}^-_2 = i\tilde{X}^-_1$, $\tilde{X}^-_4 = -i\tilde{X}^-_2 - i\gamma\tilde{X}^-_1$, $\tilde{X}^-_6 = -\frac{i}{2}\,\tilde{X}^-_5$, hence $H_{\mathbb{C}}(G) = \{f \in C^\infty(G_0)\ ;\ \tilde{X}^-_j\, f = 0\ \forall j = 1,2,5\}$. One has

$$\begin{aligned}
\tilde{X}^-_1 &= 2\,\frac{\partial}{\partial\overline{w^1}} - i\frac{\partial}{\partial x^2} - y^1\frac{\partial}{\partial x^3} - y^2\frac{\partial}{\partial y^3} \\
\tilde{X}^-_2 &= 2\,\frac{\partial}{\partial\overline{w^2}} + i\gamma\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial y^3} \\
\tilde{X}^-_5 &= 2\,\frac{\partial}{\partial\overline{w^3}}
\end{aligned}$$

where

$$\begin{aligned}
w^1 &= x^1 - ix^2 \\
w^1 &= y^1 + iy^2 \\
w^3 &= x^3 + \frac{i}{2}y^3.
\end{aligned}$$

Then $f \in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to w^3 and satisfies the 2 equations

$$2\,\frac{\partial f}{\partial\overline{w^2}} - \frac{\gamma}{2}\left(\frac{\partial f}{\partial\overline{w^1}} - \frac{\partial f}{\partial w^1}\right) + \frac{1}{4}\left(w^1 - \overline{w^1}\right)\frac{\partial f}{\partial w^3} = 0 \tag{55}$$

$$2\,\frac{\partial f}{\partial\overline{w^1}} - \frac{1}{4}\left(3w^2 + \overline{w^2}\right)\frac{\partial f}{\partial w^3} = 0 \tag{56}$$

The 3 functions

$$\begin{aligned}
\varphi^1 &= \gamma\overline{w^2} - 4w^1 \\
\varphi^2 &= w^2 \\
\varphi^3 &= w^3 + \frac{1}{32}\left(4\overline{w^1w^2} + 12\overline{w^1}w^2 + \gamma(\overline{w^2})^2 - 4\overline{w^2}w^1 + 12w^1w^2\right)
\end{aligned}$$

are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by

$$F = (\varphi^1, \varphi^2, \varphi^3). \tag{57}$$

F is a biholomorphic bijection, hence a global chart on G. Instead of (53), we here have

$$w^3_{ax} = w^3_a + w^3_x - b^1x^1 - \frac{i}{2}\,(b^2x^1 + a^2y^1),$$

whence

$$\begin{aligned}
\varphi^1_{ax} &= \varphi^1_a + \varphi^1_x \\
\varphi^2_{ax} &= \varphi^2_a + \varphi^2_x \\
\varphi^3_{ax} &= \varphi^3_a + \varphi^3_x + \chi(a,x)
\end{aligned}$$

with

$$\chi(a,x) = \frac{1}{16}\,\varphi^1_x\overline{\varphi^2_a} + \frac{1}{16}\,\varphi^2_x\left(-\overline{\varphi^1_a} - 2\varphi^1_a + 2\gamma\overline{\varphi^2_a} + \gamma\varphi^2_a\right).$$

7 Lie Algebra $\mathcal{G}_{6,6}$ (isomorphic to $M1$).

Commutation relations for $\mathcal{G}_{6,6}$: $[x_1, x_2] = x_4$; $[x_2, x_3] = x_6$; $[x_2, x_4] = x_5$.

7.1

$$J = \begin{pmatrix} \boxed{\xi_1^1} & -\frac{{\xi_1^1}^2+1}{\xi_1^2} & 0 & 0 & 0 & 0 \\ \boxed{\xi_1^2} & -\xi_1^1 & 0 & 0 & 0 & 0 \\ * & * & \boxed{\xi_3^3} & -\frac{{\xi_3^3}^2+1}{\xi_3^4} & 0 & 0 \\ \boxed{\xi_1^4} & \boxed{\xi_2^4} & \boxed{\xi_3^4} & -\xi_3^3 & 0 & 0 \\ \boxed{\xi_1^5} & * & \boxed{\xi_3^5} & \boxed{\xi_4^5} & -\xi_3^3 & \xi_3^4 \\ \boxed{\xi_1^6} & * & -\xi_4^5 & * & -\frac{{\xi_3^3}^2+1}{\xi_3^4} & \xi_3^3 \end{pmatrix} \tag{58}$$

where $J_1^3 = ((\xi_3^3 - \xi_1^1)\xi_1^4 - \xi_2^4\xi_1^2)/\xi_3^4$;
$J_2^3 = ((\xi_3^3 + \xi_1^1)\xi_2^4\xi_1^2 + ({\xi_1^1}^2 + 1)\xi_1^4)/(\xi_3^4\xi_1^2)$;
$J_2^5 = (-((\xi_3^3 - \xi_1^1)\xi_3^5\xi_1^4 - (\xi_3^3 - \xi_1^1)\xi_1^5\xi_3^4 - \xi_3^5\xi_2^4\xi_1^2 + \xi_4^5\xi_3^4\xi_1^4 + \xi_1^6{\xi_3^4}^2))/(\xi_3^4\xi_1^2)$;
$J_2^6 = (-(((\xi_3^3 + \xi_1^1)\xi_1^4 + \xi_2^4\xi_1^2)\xi_4^5\xi_3^4 + (\xi_3^5\xi_1^4 - \xi_1^5\xi_3^4)({\xi_3^3}^2 + 1) + (\xi_3^3 + \xi_1^1)\xi_1^6{\xi_3^4}^2))/({\xi_3^4}^2\xi_1^2)$;
$J_4^6 = (({\xi_3^3}^2 + 1)\xi_3^5 + 2\xi_4^5\xi_3^4\xi_3^3)/{\xi_3^4}^2$;
and the parameters are subject to the condition

$$\xi_1^2\xi_3^4 \neq 0. \tag{59}$$

Now the automorphism group of $\mathcal{G}_{6,6}$ is comprised of the matrices

$$\Phi = \begin{pmatrix} b_1^1 & b_2^1 & 0 & 0 & 0 & 0 \\ 0 & b_2^2 & b_3^2 & 0 & 0 & 0 \\ b_1^3 & b_2^3 & b_3^3 & 0 & 0 & 0 \\ b_1^4 & b_2^4 & b_3^4 & b_2^2b_1^1 & 0 & 0 \\ b_1^5 & b_2^5 & b_3^5 & -b_1^4b_2^2 & {b_2^2}^2b_1^1 & b_3^4b_2^2 \\ b_1^6 & b_2^6 & b_3^6 & -b_1^3b_2^2 & 0 & b_3^3b_2^2 \end{pmatrix}$$

where $b_1^1b_2^2b_3^3 \neq 0$. Taking

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -(\xi_4^5\xi_3^4)/({\xi_3^3}^2 + 1) & 0 & 1 & 0 \\ b_1^6 & b_2^6 & -\frac{({\xi_3^3}^2+1)\xi_3^5+2\xi_4^5\xi_3^4\xi_3^3}{({\xi_3^3}^2+1)\xi_3^4} & 0 & 0 & 1 \end{pmatrix}$$

with suitable values for b_1^6, b_2^6, equivalence by Φ leads to the case where $\xi_1^5 = \xi_3^5 = \xi_4^5 = \xi_1^6 = 0$. Then equivalence by

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & (-\xi_2^4\xi_1^2 + \xi_1^4\xi_3^3 - \xi_1^4\xi_1^1)/(\xi_3^4\xi_1^2) & 1 & 0 & 0 & 0 \\ 0 & \xi_1^4/\xi_1^2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

leads to the case where moreover $\xi_1^4 = \xi_2^4 = 0$. Finally equivalence by

$$\Phi = \begin{pmatrix} \xi_1^2 & -\xi_1^1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\xi_3^4\xi_1^2)/({\xi_3^3}^2 + 1) & 0 & 0 & 0 \\ 0 & 0 & -(\xi_3^4\xi_3^3\xi_1^2)/({\xi_3^3}^2 + 1) & \xi_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi_1^2 & -(\xi_3^4\xi_3^3\xi_1^2)/({\xi_3^3}^2 + 1) \\ 0 & 0 & 0 & 0 & 0 & (\xi_3^4\xi_1^2)/({\xi_3^3}^2 + 1) \end{pmatrix}$$

leads to the case where

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{60}$$

Commutation relations of $\mathfrak{m}$: $[\tilde{x}_1, \tilde{x}_3] = -\tilde{x}_5$; $[\tilde{x}_1, \tilde{x}_4] = \tilde{x}_6$; $[\tilde{x}_2, \tilde{x}_3] = \tilde{x}_6$; $[\tilde{x}_2, \tilde{x}_4] = \tilde{x}_5$.

7.2 Conclusions.

From (58), $\mathfrak{X}_{6,6}$ is a 10-dimensional submanifold of $\mathbb{R}^{36}$. There is only one Aut $\mathcal{G}_{6,6}$ orbit, and any element of $\mathfrak{X}_{6,6}$ is equivalent to J in (60).

7.3

7.3.1

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x^1} - y^1 \frac{\partial}{\partial y^2} + \frac{(y^1)^2}{2} \frac{\partial}{\partial x^3} \\ X_2 &= \frac{\partial}{\partial y^1} - y^2 \frac{\partial}{\partial x^3} - x^2 \frac{\partial}{\partial y^3}. \end{aligned}$$

7.3.2 Holomorphic functions for J.

Let G denote the group G_0 endowed with the left invariant structure of complex manifold defined by J in (60). Then $H_{\mathbb{C}}(G) = \{f \in C^\infty(G_0) \; ; \; \tilde{X}_j^- f = 0 \; \forall j = 1, 3, 5\}$. One has

$$\begin{aligned} \tilde{X}_1^- &= 2 \frac{\partial}{\partial \overline{w^1}} - y^1 \frac{\partial}{\partial y^2} + \left(\frac{(y^1)^2}{2} - i y^2 \right) \frac{\partial}{\partial x^3} - i x^2 \frac{\partial}{\partial y^3} \\ \tilde{X}_3^- &= 2 \frac{\partial}{\partial \overline{w^2}} \\ \tilde{X}_5^- &= 2 \frac{\partial}{\partial \overline{w^3}} \end{aligned}$$

where

$$\begin{aligned} w^1 &= x^1 + i y^1 \\ w^2 &= x^2 + i y^2 \\ w^3 &= x^3 - i y^3. \end{aligned}$$

Then $f \in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to w^2 and w^3 and satisfies the equation

$$2 \frac{\partial f}{\partial \overline{w^1}} - \frac{w^1 - \overline{w^1}}{2} \frac{\partial f}{\partial w^2} - \left(\frac{(w^1 - \overline{w^1})^2}{8} + w^2 \right) \frac{\partial f}{\partial w^3} = 0. \tag{61}$$

The 3 functions

$$\begin{aligned} \varphi^1 &= w^1 \\ \varphi^2 &= w^2 + \frac{1}{4} \left(w^1 \overline{w^1} - \frac{(\overline{w^1})^2}{2} \right) \\ \varphi^3 &= w^3 + \frac{1}{48} \left(-(\overline{w^1})^3 + 3 \overline{w^1} (w^1)^2 + 24 \overline{w^1} w^2 \right) \end{aligned}$$

are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by

$$F = (\varphi^1, \varphi^2, \varphi^3). \tag{62}$$

F is a biholomorphic bijection, hence a global chart on G. We determine now how the multiplication of G looks like in the chart (62). Let $a, x \in G$:

$$\begin{aligned} x &= \exp(x^1 x_1)\exp(y^1 x_2)\exp(x^2 x_3)\exp(y^2 x_4)\exp(x^3 x_5)\exp(y^3 x_6) \\ a &= \exp(a^1 x_1)\exp(b^1 x_2)\exp(a^2 x_3)\exp(b^2 x_4)\exp(a^3 x_5)\exp(b^3 x_6). \end{aligned}$$

With obvious notations, $a = [w_a^1, w_a^2, w_a^3]$, $x = [w_x^1, w_x^2, w_x^3]$, $a\,x = [w_{ax}^1, w_{ax}^2, w_{ax}^3]$, $a = [\varphi_a^1, \varphi_a^2, \varphi_a^3]$, $x = [\varphi_x^1, \varphi_x^2, \varphi_x^3]$, $a\,x = [\varphi_{ax}^1, \varphi_{ax}^2, \varphi_{ax}^3]$. Computations yield :

$$\begin{aligned} w_{ax}^1 &= w_a^1 + w_x^1 \\ w_{ax}^2 &= w_a^2 + w_x^2 - ib^1x^1 \\ w_{ax}^3 &= w_a^3 + w_x^3 + \frac{(b^1)^2}{2}x^1 - (b^2 - b^1x^1)y^1 + ia^2y^1. \end{aligned}$$

We then get

$$\begin{aligned} \varphi_{ax}^1 &= \varphi_a^1 + \varphi_x^1 \\ \varphi_{ax}^2 &= \varphi_a^2 + \varphi_x^2 + \frac{1}{4}\,\varphi_x^1(2\overline{\varphi_a^1} - \varphi_a^1) \\ \varphi_{ax}^3 &= \varphi_a^3 + \varphi_x^3 + \chi(a,x) \end{aligned}$$

where

$$\chi(a,x) = \frac{1}{16}\,(\varphi_x^1)^2\left(3\overline{\varphi_a^1} - 2\varphi_a^1\right) + \frac{1}{16}\,\varphi_x^1\left(2(\overline{\varphi_a^1})^2 - (\varphi_a^1)^2 + 8\overline{\varphi_a^2}\right) + \frac{1}{2}\,\varphi_x^2\overline{\varphi_a^1}.$$

8 Lie Algebra $\mathcal{G}_{6,5}$ (isomorphic to $M8$).

Commutation relations for $\mathcal{G}_{6,5}$: $[x_1, x_2] = x_4$; $[x_1, x_4] = x_5$; $[x_2, x_3] = x_6$; $[x_2, x_4] = x_6$.

8.1

$$\begin{pmatrix} a & -\frac{a^2+1}{\xi_1^2} & 0 & 0 & 0 & 0 \\ \boxed{\xi_1^2} & -a & 0 & 0 & 0 & 0 \\ \boxed{\xi_1^3} & * & b & -\frac{b^2+1}{\xi_3^4} & 0 & 0 \\ \boxed{\xi_1^4} & * & \boxed{\xi_3^4} & -b & 0 & 0 \\ \boxed{\xi_1^5} & * & \frac{((2\xi_5^5+\xi_3^4)\xi_3^6-\xi_4^6\xi_3^4)\xi_1^2-\xi_5^6\xi_3^6\xi_3^4}{\xi_5^6\xi_1^2} & * & \boxed{\xi_5^5} & -\frac{{\xi_5^5}^2+1}{\xi_5^6} \\ \boxed{\xi_1^6} & * & \boxed{\xi_3^6} & \boxed{\xi_4^6} & \boxed{\xi_5^6} & -\xi_5^5 \end{pmatrix} \qquad (63)$$

where $a = J_1^1 = (-(({\xi_5^5}^2 + 1)\xi_1^2 - \xi_5^6\xi_5^5\xi_3^4))/(\xi_5^6\xi_3^4)$;
$J_2^3 = ((((\xi_5^5 + 2\xi_3^4)\xi_5^5\xi_1^4 + {\xi_3^4}^2\xi_1^4 + {\xi_3^4}^2\xi_1^3 + \xi_1^4)\xi_5^6 + ({\xi_5^5}^2 + 1)\xi_1^3\xi_1^2){\xi_1^2}^2 + ((\xi_5^6\xi_3^4 - 2\xi_5^5\xi_1^2)\xi_1^4 - (2\xi_1^4 + \xi_1^3)\xi_3^4\xi_1^2){\xi_5^6}^2\xi_3^4)/(\xi_5^6\xi_3^4{\xi_1^2}^3)$;
$b = J_3^3 = (-((\xi_5^5 + \xi_3^4)\xi_1^2 - \xi_5^6\xi_3^4))/\xi_1^2$;
$J_2^4 = ((({\xi_5^5}^2 + 1){\xi_1^2}^2 + {\xi_5^6}^2{\xi_3^4}^2)\xi_1^4 - ((\xi_1^4 + \xi_1^3)\xi_3^4 + 2\xi_5^5\xi_1^4)\xi_5^6\xi_3^4\xi_1^2)/(\xi_5^6\xi_3^4{\xi_1^2}^2)$;
$J_2^5 = (-(((((\xi_5^5\xi_1^4 + 2\xi_3^4\xi_1^4 + 2\xi_3^4\xi_1^3)\xi_5^5 + {\xi_3^4}^2\xi_1^4 + {\xi_3^4}^2\xi_1^3 + \xi_1^4)\xi_3^6 - (\xi_1^6\xi_3^4 + \xi_1^5\xi_1^2)({\xi_5^5}^2 + 1) - (\xi_1^4 + \xi_1^3)\xi_4^6{\xi_3^4}^2){\xi_1^2}^2 - ((2(\xi_3^6\xi_1^4 - \xi_1^5\xi_1^2)\xi_5^5 + (2\xi_1^4 + \xi_1^3)\xi_3^6\xi_3^4)\xi_1^2 - (\xi_5^6\xi_3^6 + \xi_4^6\xi_1^2)\xi_3^4\xi_1^4)\xi_5^6\xi_3^4))/(\xi_5^6\xi_3^4{\xi_1^2}^3)$;
$J_4^5 = (-(((\xi_4^6{\xi_3^4}^2 - \xi_3^6)\xi_1^2 - \xi_5^6\xi_4^6{\xi_3^4}^2)\xi_1^2 - ((\xi_5^5 + \xi_3^4)\xi_1^2 - \xi_5^6\xi_3^4)^2\xi_3^6))/(\xi_5^6\xi_3^4{\xi_1^2}^2)$;
$J_2^6 = (-((\xi_5^6\xi_1^5 + \xi_4^6\xi_1^4 + \xi_3^6\xi_1^3)\xi_5^6\xi_3^4 - ({\xi_5^5}^2 + 1)\xi_1^6\xi_1^2))/(\xi_5^6\xi_3^4\xi_1^2)$;
and the parameters are subject to the condition

$$\xi_1^2\xi_3^4\xi_5^6 \neq 0. \qquad (64)$$

The automorphisms of $\mathcal{G}_{6,5}$ fall into 2 kinds (12-dimensional Lie group). The first kind is comprised of the matrices

$$\Phi = \begin{pmatrix} b_1^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 & 0 \\ b_1^3 & b_2^3 & b_2^2 b_1^1 & 0 & 0 & 0 \\ b_1^4 & b_2^4 & 0 & b_2^2 b_1^1 & 0 & 0 \\ b_1^5 & b_2^5 & b_3^5 & b_2^4 b_1^1 & b_2^2 {b_1^1}^2 & 0 \\ b_1^6 & b_2^6 & b_3^6 & -(b_1^4 + b_1^3) b_2^2 & 0 & {b_2^2}^2 b_1^1 \end{pmatrix} \qquad (65)$$

where $b^1_1 b^2_2 \neq 0$. The second kind is comprised of the matrices

$$\Phi = \begin{pmatrix} 0 & b^1_2 & 0 & 0 & 0 & 0 \\ b^2_1 & 0 & 0 & 0 & 0 & 0 \\ b^3_1 & b^3_2 & b^2_1 b^1_2 & 0 & 0 & 0 \\ b^4_1 & b^4_2 & -b^2_1 b^1_2 & -b^2_1 b^1_2 & 0 & 0 \\ b^5_1 & b^5_2 & b^5_3 & -b^4_1 b^1_2 & 0 & -b^2_1 {b^1_2}^2 \\ b^6_1 & b^6_2 & b^6_3 & b^2_1(b^4_2 + b^3_2) & -{b^2_1}^2 b^1_2 & 0 \end{pmatrix} \tag{66}$$

where $b^1_2 b^2_1 \neq 0$. Taking suitable values for the b^i_j's, equivalence by Φ in (65) leads to the case where $\xi^5_1 = \xi^5_5 = \xi^6_1 = \xi^6_3 = \xi^6_4 = \xi^6_5 = \xi^3_1 = \xi^4_1 = 0$ and moreover $\xi^2_1 = 1$:

$$J(\xi^4_3, \xi^5_5, \xi^6_5) = \begin{pmatrix} -\frac{{\xi^5_5}^2+1-\xi^6_5\xi^5_5\xi^4_3}{\xi^6_5\xi^4_3} & -\frac{({\xi^5_5}^2+1-\xi^6_5\xi^5_5\xi^4_3)^2}{(\xi^6_5\xi^4_3)^2} - 1 & 0 & 0 & 0 & 0 \\ 1 & \frac{{\xi^5_5}^2+1-\xi^6_5\xi^5_5\xi^4_3}{\xi^6_5\xi^4_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\xi^5_5 + \xi^4_3 - \xi^6_5\xi^4_3) & -\frac{(\xi^5_5+\xi^4_3-\xi^6_5\xi^4_3)^2+1}{\xi^4_3} & 0 & 0 \\ 0 & 0 & \xi^4_3 & \xi^5_5 + \xi^4_3 - \xi^6_5\xi^4_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi^5_5 & -\frac{{\xi^5_5}^2+1}{\xi^6_5} \\ 0 & 0 & 0 & 0 & \xi^6_5 & -\xi^5_5 \end{pmatrix} \tag{67}$$

where $\xi^4_3 \xi^6_5 \neq 0$.

Commutation relations of $\mathfrak{m}$:
$[\tilde{x}_1, \tilde{x}_3] = \frac{1}{\xi^6_5} \left((-\xi^5_5\xi^4_3 + {\xi^5_5}^2 + 1)\tilde{x}_5 + (-\xi^6_5\xi^4_3 + \xi^5_5)\tilde{x}_6 \right)$;
$[\tilde{x}_1, \tilde{x}_4] = \frac{1}{\xi^6_5\xi^4_3} \left(({\xi^5_5}^2+1)(\xi^5_5+\xi^4_3)+\xi^6_5\xi^5_5\xi^4_3(\xi^6_5\xi^4_3-2\xi^5_5-\xi^4_3)\right) \tilde{x}_5 + \frac{1}{\xi^4_3} \left((\xi^6_5\xi^4_3-\xi^5_5)^2+\xi^4_3(\xi^5_5-\xi^6_5\xi^4_3)+1\right) \tilde{x}_6$;
$[\tilde{x}_2, \tilde{x}_3] = \frac{(\xi^6_5\xi^4_3-\xi^5_5)^2+1}{{\xi^6_5}^2\xi^4_3} \left((1 + {\xi^5_5}^2)\tilde{x}_5 + \xi^6_5\xi^5_5\tilde{x}_6 \right)$;
$[\tilde{x}_2, \tilde{x}_4] = \frac{(\xi^6_5\xi^4_3-\xi^5_5)^2+1}{{\xi^6_5}^2{\xi^4_3}^2} \left(-({\xi^5_5}^2 + 1)(\xi^4_3\xi^6_5 - \xi^5_5 - \xi^4_3)\tilde{x}_5 + \xi^6_5(-\xi^6_5\xi^5_5\xi^4_3 + {\xi^5_5}^2 + 1 + \xi^5_5\xi^4_3)\tilde{x}_6 \right)$.

If $J(\xi^4_3, \xi^5_5, \xi^6_5), J(\eta^4_3, \eta^5_5, \eta^6_5)$ are as in (67), they are equivalent under some first kind automorphism if and only if $\eta^4_3 = \xi^4_3, \eta^5_5 = \xi^5_5, \eta^6_5 = \xi^5_5$. They are equivalent under some second kind automorphism if and only if $\eta^4_3 = (-({\xi^6_5}^2{\xi^4_3}^2 - 2\xi^6_5\xi^5_5\xi^4_3 + {\xi^5_5}^2 + 1))/\xi^4_3$, $\eta^5_5 = -\xi^5_5$, and $\eta^6_5 = (((\xi^6_5\xi^4_3 - 2\xi^5_5)\xi^6_5\xi^4_3 + {\xi^5_5}^2 + 1)\xi^6_5{\xi^4_3}^2)/({\xi^6_5}^2{\xi^4_3}^2 - 2\xi^6_5\xi^5_5\xi^4_3 + {\xi^5_5}^2 + 1)^2$.

8.2 Conclusions.

From (63), $\mathfrak{X}_{6,5}$ is a submanifold of dimension 10 in $\mathbb{R}^{36}$. It is the disjoint union of the continuously many orbits of the $J(\xi^4_3, \xi^5_5, \xi^6_5)$ in (67).

8.3

8.3.1

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x^1} - y^1 \frac{\partial}{\partial y^2} - y^2 \frac{\partial}{\partial x^3} + \frac{(y^1)^2}{2} \frac{\partial}{\partial y^3} \\ X_2 &= \frac{\partial}{\partial y^1} - (x^2 + y^2) \frac{\partial}{\partial y^3}. \end{aligned}$$

8.3.2 Holomorphic functions for $J(\xi^4_3, \xi^5_5, \xi^6_5)$.

Let G denote the group G_0 endowed with the left invariant structure of complex manifold defined by $J(\xi^4_3, \xi^5_5, \xi^6_5)$ in (67) where $\xi^4_3\xi^6_5 \neq 0$. Then $H_{\mathbb{C}}(G) = \{f \in C^\infty(G_0) \ ; \ \tilde{X}^-_j f = 0 \ \forall j = 1, 3, 5\}$. One has

$$\begin{aligned} \tilde{X}^-_1 &= 2\frac{\partial}{\partial \overline{w^1}} - y^1(1 + iA) \frac{\partial}{\partial y^2} - y^2(1 + iA) \frac{\partial}{\partial x^3} + \left(\frac{(y^1)^2}{2}(1 + iA) - i(x^2 + y^2) \right) \frac{\partial}{\partial y^3} \\ \tilde{X}^-_3 &= 2\frac{\partial}{\partial \overline{w^2}} \\ \tilde{X}^-_5 &= 2\frac{\partial}{\partial \overline{w^3}} \end{aligned}$$

where

$$
\begin{aligned}
w^1 &= x^1 - Ay^1 + iy^1 \\
w^2 &= x^2 + \frac{\xi_5^5 + \xi_3^4 - \xi_5^6\xi_3^4}{\xi_3^4}\, y^2 + \frac{i}{\xi_3^4}\, y^2 \\
w^3 &= x^3 - \frac{\xi_5^5}{\xi_5^6}\, y^3 + \frac{i}{\xi_5^6}\, y^3 \\
A &= -\frac{{\xi_5^5}^2 + 1 - \xi_5^6\xi_5^5\xi_3^4}{\xi_5^6\xi_3^4}.
\end{aligned}
$$

Then $f \in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to w^2 and w^3 and satisfies the equation

$$
\begin{aligned}
&2\,\frac{\partial f}{\partial \overline{w^1}} - \frac{w^1 - \overline{w^1}}{2i}\,\frac{(1+iA)(\xi_5^5 + \xi_3^4 - \xi_5^6\xi_3^4 + i)}{\xi_3^4}\,\frac{\partial f}{\partial w^2} \\
&-\left[(1+iA)\xi_3^4\,\frac{w^2 - \overline{w^2}}{2i} + \left(\frac{(w^1 - \overline{w^1})^2}{8}\,(1+iA) + i\left(\frac{w^2 + \overline{w^2}}{2} - (\xi_5^5 - \xi_5^6\xi_3^4)\,\frac{w^2 - \overline{w^2}}{2i}\right)\right)\frac{i - \xi_5^5}{\xi_5^6}\right]\frac{\partial f}{\partial w^3} = 0.
\end{aligned}
$$

The 3 functions

$$
\begin{aligned}
\varphi^1 &= w^1 \\
\varphi^2 &= w^2 + \frac{1+iA}{4i\xi_3^4}(\xi_5^5 + \xi_3^4 - \xi_5^6\xi_3^4 + i)\left(w^1\overline{w^1} - \frac{(\overline{w^1})^2}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\varphi^3 = w^3 &+ \frac{1}{48{\xi_5^6}^2{\xi_3^4}^2}\,\overline{w^1}^3(-2i{\xi_5^6}^2{\xi_5^5}^2{\xi_3^4}^2 - 4{\xi_5^6}^2\xi_5^5{\xi_3^4}^2 + 2i{\xi_5^6}^2{\xi_3^4}^2 + 4i\xi_5^6{\xi_5^5}^3\xi_3^4 + i\xi_5^6{\xi_5^5}^2{\xi_3^4}^2 + 4\xi_5^6{\xi_5^5}^2\xi_3^4 + 2\xi_5^6\xi_5^5{\xi_3^4}^2 \\
&+ 4i\xi_5^6\xi_5^5\xi_3^4 - i\xi_5^6{\xi_3^4}^2 + 4\xi_5^6\xi_3^4 - 2i{\xi_5^5}^4 - i{\xi_5^5}^3\xi_3^4 - {\xi_5^5}^2\xi_3^4 - 4i{\xi_5^5}^2 - i\xi_5^5\xi_3^4 - \xi_3^4 - 2i) \\
&+ \frac{1}{16{\xi_5^6}^2{\xi_3^4}^2}\,\overline{w^1}^2 w^1(i{\xi_5^6}^2{\xi_5^5}^2{\xi_3^4}^2 + 2{\xi_5^6}^2\xi_5^5{\xi_3^4}^2 - i{\xi_5^6}^2{\xi_3^4}^2 - 2i\xi_5^6{\xi_5^5}^3\xi_3^4 - 2\xi_5^6{\xi_5^5}^2\xi_3^4 - 2i\xi_5^6\xi_5^5\xi_3^4 - 2\xi_5^6\xi_3^4 + i{\xi_5^5}^4 + 2i{\xi_5^5}^2 + i) \\
&+ \frac{1}{16{\xi_5^6}^2\xi_3^4}\,\overline{w^1}{w^1}^2(-i\xi_5^6{\xi_5^5}^2\xi_3^4 - 2\xi_5^6\xi_5^5\xi_3^4 + i\xi_5^6\xi_3^4 + i{\xi_5^5}^3 + {\xi_5^5}^2 + i\xi_5^5 + 1) - \frac{i\xi_5^5 + 1}{2\xi_5^6}\,\overline{w^1}w^2
\end{aligned}
$$

are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by

$$
F = (\varphi^1, \varphi^2, \varphi^3). \tag{68}
$$

F is a biholomorphic bijection, hence a global chart on G. We determine now how the multiplication of G looks like in the chart (68). Let $a, x \in G$:

$$
\begin{aligned}
x &= \exp(x^1x_1)\exp(y^1x_2)\exp(x^2x_3)\exp(y^2x_4)\exp(x^3x_5)\exp(y^3x_6) \\
a &= \exp(a^1x_1)\exp(b^1x_2)\exp(a^2x_3)\exp(b^2x_4)\exp(a^3x_5)\exp(b^3x_6).
\end{aligned}
$$

With obvious notations, $a = [w_a^1, w_a^2, w_a^3]$, $x = [w_x^1, w_x^2, w_x^3]$, $a\,x = [w_{ax}^1, w_{ax}^2, w_{ax}^3]$, $a = [\varphi_a^1, \varphi_a^2, \varphi_a^3]$, $x = [\varphi_x^1, \varphi_x^2, \varphi_x^3]$, $a\,x = [\varphi_{ax}^1, \varphi_{ax}^2, \varphi_{ax}^3]$.
Computations yield :

$$
\begin{aligned}
w_{ax}^1 &= w_a^1 + w_x^1 \\
w_{ax}^2 &= w_a^2 + w_x^2 - \frac{b^1x^1}{\xi_3^4}\,(\xi_5^5 + \xi_3^4 - \xi_5^6\xi_3^4 + i) \\
w_{ax}^3 &= w_a^3 + w_x^3 - b^2x^1 + \frac{1}{2}b^1(x^1)^2 + \frac{i - \xi_5^5}{\xi_5^6}\,\left(\frac{1}{2}\,(b^1)^2x^1 - b^2y^1 + b^1x^1y^1 - a^2y^1\right).
\end{aligned}
$$

We then get

$$
\begin{aligned}
\varphi_{ax}^1 &= \varphi_a^1 + \varphi_x^1 \\
\varphi_{ax}^2 &= \varphi_a^2 + \varphi_x^2 + C\varphi_x^1 \\
\varphi_{ax}^3 &= \varphi_a^3 + \varphi_x^3 + D_1(\varphi_x^1)^2 + D_2\varphi_x^1 + D_3\varphi_x^2
\end{aligned}
$$

where $C = (\overline{\varphi_a^1}(i\xi_5^6\xi_3^4 - i\xi_5^5 - i\xi_3^4 + 1))/(2\xi_3^4) + (\varphi_a^1(-{\xi_5^6}^2\xi_5^5{\xi_3^4}^2 - i{\xi_5^6}^2{\xi_3^4}^2 + 2\xi_5^6{\xi_5^5}^2\xi_3^4 + \xi_5^6\xi_5^5{\xi_3^4}^2 + 2i\xi_5^6\xi_5^5\xi_3^4 + i\xi_5^6{\xi_3^4}^2 - {\xi_5^5}^3 - {\xi_5^5}^2\xi_3^4 - i{\xi_5^5}^2 - \xi_5^5 - \xi_3^4 - i))/(4\xi_5^6{\xi_3^4}^2)$;
$D_1 = (\overline{\varphi_a^1}(-i{\xi_5^6}^2{\xi_5^5}^2{\xi_3^4}^2 + 2{\xi_5^6}^2\xi_5^5{\xi_3^4}^2 + i{\xi_5^6}^2{\xi_3^4}^2 + 2i\xi_5^6{\xi_5^5}^3\xi_3^4 + i\xi_5^6{\xi_5^5}^2{\xi_3^4}^2 - 2\xi_5^6{\xi_5^5}^2\xi_3^4 - 2\xi_5^6\xi_5^5{\xi_3^4}^2 + 2i\xi_5^6\xi_5^5\xi_3^4 + 3i\xi_5^6{\xi_3^4}^2 - 2\xi_5^6\xi_3^4 - i{\xi_5^5}^4 - i{\xi_5^5}^3\xi_3^4 - {\xi_5^5}^2\xi_3^4 - 2i{\xi_5^5}^2 - i\xi_5^5\xi_3^4 - \xi_3^4 - i))/(16{\xi_5^6}^2{\xi_3^4}^2) + (\varphi_a^1(i{\xi_5^6}^2{\xi_5^5}^2{\xi_3^4}^2 - 2{\xi_5^6}^2\xi_5^5{\xi_3^4}^2 - i{\xi_5^6}^2{\xi_3^4}^2 - 2i\xi_5^6{\xi_5^5}^3\xi_3^4 - 2i\xi_5^6{\xi_5^5}^2{\xi_3^4}^2 + 2\xi_5^6{\xi_5^5}^2\xi_3^4 - 2i\xi_5^6\xi_5^5\xi_3^4 - 2i\xi_5^6{\xi_3^4}^2 + 2\xi_5^6\xi_3^4 + i{\xi_5^5}^4 + 2i{\xi_5^5}^3\xi_3^4 + 2{\xi_5^5}^2\xi_3^4 + 2i{\xi_5^5}^2 + 2i\xi_5^5\xi_3^4 + 2\xi_3^4 + i))/(16{\xi_5^6}^2{\xi_3^4}^2)$;
$D_2 = ({\overline{\varphi_a^1}}^2(-i{\xi_5^6}^2\xi_5^5{\xi_3^4}^2 - {\xi_5^6}^2{\xi_3^4}^2 + 2i\xi_5^6{\xi_5^5}^2\xi_3^4 + i\xi_5^6\xi_5^5{\xi_3^4}^2 + 4\xi_5^6\xi_5^5\xi_3^4 + \xi_5^6{\xi_3^4}^2 - 2i\xi_5^6\xi_3^4 - i{\xi_5^5}^3 - i{\xi_5^5}^2\xi_3^4 - 3{\xi_5^5}^2 - 2\xi_5^5\xi_3^4 - i\xi_5^5 + i\xi_3^4 - 3))/(16\xi_5^6\xi_3^4) + (\overline{\varphi_a^1}\varphi_a^1(\xi_5^6\xi_3^4 - \xi_5^5 - \xi_3^4 + i))/4 + (-i\overline{\varphi_a^2}\xi_3^4)/2 + ({\varphi_a^1}^2(i{\xi_5^6}^3\xi_5^5{\xi_3^4}^2 - {\xi_5^6}^3{\xi_3^4}^2 - 2i{\xi_5^6}^2{\xi_5^5}^2\xi_3^4 - i{\xi_5^6}^2\xi_5^5{\xi_3^4}^2 + {\xi_5^6}^2{\xi_3^4}^2 - 2i{\xi_5^6}^2\xi_3^4 + i\xi_5^6{\xi_5^5}^3 + \xi_5^6{\xi_5^5}^2 + i\xi_5^6\xi_5^5 + \xi_5^6 + i{\xi_5^5}^3 + {\xi_5^5}^2 + i\xi_5^5 + 1))/(16{\xi_5^6}^2\xi_3^4) + (\varphi_a^2(i\xi_5^6\xi_3^4 - i\xi_5^5 - 1))/(2\xi_5^6)$;
$D_3 = (\overline{\varphi_a^1}(-i\xi_5^5 - 1))/(2\xi_5^6)$.

9 Lie Algebra $\mathcal{G}_{6,8}$ (isomorphic to $M9$).

Commutation relations for $\mathcal{G}_{6,8}$: $[x_1, x_2] = x_4$; $[x_1, x_4] = x_5$; $[x_2, x_3] = x_5$; $[x_2, x_4] = x_6$.

9.1

$$J = \begin{pmatrix} \frac{\xi_5^5\xi_3^4+\xi_5^5\xi_1^2-\xi_3^3\xi_1^2}{\xi_3^4} & -\frac{(\xi_5^5\xi_3^4+\xi_5^5\xi_1^2-\xi_3^3\xi_1^2)^2+{\xi_3^4}^2}{{\xi_3^4}^2\xi_1^2} & 0 & 0 & 0 & 0 \\ \boxed{\xi_1^2} & -\frac{\xi_5^5\xi_3^4+\xi_5^5\xi_1^2-\xi_3^3\xi_1^2}{\xi_3^4} & 0 & 0 & 0 & 0 \\ \boxed{\xi_1^3} & \boxed{\xi_2^3} & \boxed{\xi_3^3} & -\frac{{\xi_3^3}^2+1}{\xi_3^4} & 0 & 0 \\ * & * & \boxed{\xi_3^4} & -\xi_3^3 & 0 & 0 \\ * & * & -\frac{(\xi_4^6\xi_3^4-\xi_3^6\xi_5^5+\xi_3^6\xi_3^3)(\xi_3^4+\xi_1^2)}{\xi_3^4\xi_1^2} & * & \boxed{\xi_5^5} & -\frac{({\xi_5^5}^2+1)(\xi_3^4+\xi_1^2)}{\xi_3^4\xi_1^2} \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & \frac{\xi_3^4\xi_1^2}{\xi_3^4+\xi_1^2} & -\xi_5^5 \end{pmatrix} \tag{69}$$

where $J_1^4 = (\xi_3^4\xi_3^3\xi_1^3 + \xi_3^4\xi_2^3\xi_1^2 - \xi_3^3\xi_1^3\xi_1^2 + (\xi_3^4 + \xi_1^2)\xi_5^5\xi_1^3)/({\xi_3^3}^2 + 1)$;
$J_2^4 = (-(((\xi_5^5 - \xi_3^3)(\xi_3^4\xi_3^3\xi_1^3 + \xi_3^4\xi_2^3\xi_1^2 - \xi_3^3\xi_1^3\xi_1^2) - (\xi_3^4 + \xi_1^2)\xi_5^5\xi_3^3\xi_1^3)(\xi_3^4 + \xi_1^2) + ((\xi_3^4 + \xi_1^2)^2{\xi_5^5}^2 + ({\xi_3^3}^2 + 1){\xi_3^4}^2)\xi_1^3))/(({\xi_3^3}^2 + 1)\xi_3^4\xi_1^2)$;
$J_1^5 = (-(\xi_4^6\xi_5^5{\xi_3^4}^2\xi_1^3 + \xi_4^6\xi_5^5\xi_3^4\xi_1^3\xi_1^2 + \xi_4^6{\xi_3^4}^2\xi_3^3\xi_1^3 + \xi_4^6{\xi_3^4}^2\xi_2^3\xi_1^2 - \xi_4^6\xi_3^4\xi_3^3\xi_1^3\xi_1^2 + \xi_3^6\xi_3^4{\xi_3^3}^2\xi_1^3 + \xi_3^6\xi_3^4\xi_1^3 + \xi_2^6\xi_3^4{\xi_3^3}^2\xi_1^2 + \xi_2^6\xi_3^4\xi_1^2 + \xi_1^6\xi_5^5{\xi_3^3}^2\xi_1^2 + \xi_1^6\xi_5^5\xi_1^2 - \xi_1^6{\xi_3^3}^3\xi_1^2 - \xi_1^6\xi_3^3\xi_1^2)(\xi_3^4 + \xi_1^2))/(({\xi_3^3}^2 + 1){\xi_3^4}^2\xi_1^2)$;
$J_2^5 = ((\xi_4^6{\xi_5^5}^2{\xi_3^4}^3\xi_1^3 + 2\xi_4^6{\xi_5^5}^2{\xi_3^4}^2\xi_1^3\xi_1^2 + \xi_4^6{\xi_5^5}^2\xi_3^4\xi_1^3{\xi_1^2}^2 + \xi_4^6\xi_5^5{\xi_3^4}^3\xi_2^3\xi_1^2 - 2\xi_4^6\xi_5^5{\xi_3^4}^2\xi_3^3\xi_1^3\xi_1^2 + \xi_4^6\xi_5^5{\xi_3^4}^2\xi_2^3{\xi_1^2}^2 - 2\xi_4^6\xi_5^5\xi_3^4\xi_3^3\xi_1^3{\xi_1^2}^2 - \xi_4^6{\xi_3^4}^3\xi_3^3\xi_2^3\xi_1^2 + \xi_4^6{\xi_3^4}^3\xi_1^3 - \xi_4^6{\xi_3^4}^2\xi_3^3\xi_2^3{\xi_1^2}^2 + \xi_4^6\xi_3^4{\xi_3^3}^2\xi_1^3{\xi_1^2}^2 - \xi_3^6{\xi_3^4}^2{\xi_3^3}^2\xi_2^3\xi_1^2 - \xi_3^6{\xi_3^4}^2\xi_2^3\xi_1^2 + 2\xi_2^6\xi_5^5{\xi_3^4}^2{\xi_3^3}^2\xi_1^2 + 2\xi_2^6\xi_5^5{\xi_3^4}^2\xi_1^2 + \xi_2^6\xi_5^5\xi_3^4{\xi_3^3}^2{\xi_1^2}^2 + \xi_2^6\xi_5^5\xi_3^4{\xi_1^2}^2 - \xi_2^6\xi_3^4{\xi_3^3}^3{\xi_1^2}^2 - \xi_2^6\xi_3^4\xi_3^3{\xi_1^2}^2 + \xi_1^6{\xi_5^5}^2{\xi_3^4}^2{\xi_3^3}^2 + \xi_1^6{\xi_5^5}^2{\xi_3^4}^2 + 2\xi_1^6{\xi_5^5}^2\xi_3^4{\xi_3^3}^2\xi_1^2 + 2\xi_1^6{\xi_5^5}^2\xi_3^4\xi_1^2 + \xi_1^6{\xi_5^5}^2{\xi_3^3}^2{\xi_1^2}^2 + \xi_1^6{\xi_5^5}^2{\xi_1^2}^2 - 2\xi_1^6\xi_5^5\xi_3^4{\xi_3^3}^3\xi_1^2 - 2\xi_1^6\xi_5^5\xi_3^4\xi_3^3\xi_1^2 - 2\xi_1^6\xi_5^5{\xi_3^3}^3{\xi_1^2}^2 - 2\xi_1^6\xi_5^5\xi_3^3{\xi_1^2}^2 + \xi_1^6{\xi_3^4}^2{\xi_3^3}^2 + \xi_1^6{\xi_3^4}^2 + \xi_1^6{\xi_3^3}^4{\xi_1^2}^2 + \xi_1^6{\xi_3^3}^2{\xi_1^2}^2)(\xi_3^4 + \xi_1^2))/(({\xi_3^3}^2 + 1){\xi_3^4}^3{\xi_1^2}^2)$;
$J_4^5 = ((\xi_4^6\xi_5^5\xi_3^4 + \xi_4^6\xi_3^4\xi_3^3 + \xi_3^6{\xi_3^3}^2 + \xi_3^6)(\xi_3^4 + \xi_1^2))/({\xi_3^4}^2\xi_1^2)$;
and the parameters are subject to the condition

$$\xi_1^2\xi_3^4(\xi_3^4 + \xi_1^2) \neq 0. \tag{70}$$

Now the automorphism group of $\mathcal{G}_{6,8}$ is comprised of the matrices

$$\Phi = \begin{pmatrix} b_1^1 & b_2^1 & 0 & 0 & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 & 0 \\ b_1^3 & b_2^3 & {b_1^1}^2 & 0 & 0 & 0 \\ b_1^4 & b_2^4 & 0 & b_2^2b_1^1 & 0 & 0 \\ b_1^5 & b_2^5 & b_3^5 & -(b_1^4b_2^1 + b_1^3b_2^2 - b_2^4b_1^1) & b_2^2{b_1^1}^2 & b_2^2b_2^1b_1^1 \\ b_1^6 & b_2^6 & b_3^6 & -b_1^4b_2^2 & 0 & {b_2^2}^2b_1^1 \end{pmatrix}$$

where $\det\Phi = {b_2^2}^5{b_1^1}^7 \neq 0$. Taking suitable values for the b_j^i's, we are led to the case where $\xi_1^2 = 1, \xi_1^3 = \xi_2^3 = \xi_5^5 = \xi_1^6 = \xi_3^6 = 0$ and moreover $\xi_2^6 = \xi_4^6 = 0$. Hence any J in (69) is equivalent to :

$$J(\xi_3^3,\xi_3^4)=\begin{pmatrix}(-\xi_3^3)/\xi_3^4 & (-({\xi_3^4}^2+{\xi_3^3}^2))/{\xi_3^4}^2 & 0 & 0 & 0 & 0\\ 1 & \xi_3^3/\xi_3^4 & 0 & 0 & 0 & 0\\ 0 & 0 & \xi_3^3 & (-({\xi_3^3}^2+1))/\xi_3^4 & 0 & 0\\ 0 & 0 & \xi_3^4 & -\xi_3^3 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & (-(\xi_3^4+1))/\xi_3^4\\ 0 & 0 & 0 & 0 & \xi_3^4/(\xi_3^4+1) & 0\end{pmatrix} \tag{71}$$

where $\xi_3^4\neq 0,-1$. $J(\xi_3^3,\xi_3^4)\cong J(\eta_3^3,\eta_3^4)$ if and only if $\xi_3^3=\eta_3^3$ and $\xi_3^4=\eta_3^4$.

Commutation relations of $\mathfrak{m}$: $[\tilde{x}_1,\tilde{x}_3]=-\tilde{x}_6\xi_3^4$; $[\tilde{x}_2,\tilde{x}_3]=\tilde{x}_5(\xi_3^4+1)-\tilde{x}_6\xi_3^3$; $[\tilde{x}_2,\tilde{x}_4]=(\tilde{x}_5\xi_3^3(-{\xi_3^4}^2+1))/{\xi_3^4}^2+(\tilde{x}_6(\xi_3^4+{\xi_3^3}^2))/\xi_3^4$.

9.2 Conclusions.

From (69), $\mathfrak{X}_{6,8}$ is a submanifold of dimension 10 in $\mathbb{R}^{36}$. It is the disjoint union of the continuously many orbits of the $J(\xi_3^3,\xi_3^4)$ defined in (71) where $\xi_3^4\neq 0,-1$.

9.3

9.3.1

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x^1}-y^1\frac{\partial}{\partial y^2}+y^2\frac{\partial}{\partial x^3}+\frac{(y^1)^2}{2}\frac{\partial}{\partial y^3}\\ X_2 &= \frac{\partial}{\partial y^1}-x^2\frac{\partial}{\partial x^3}-y^2\frac{\partial}{\partial y^3}. \end{aligned}$$

9.3.2 Holomorphic functions for $J(\xi_3^3,\xi_3^4)$.

Let G denote the group G_0 endowed with the left invariant structure of complex manifold defined by $J(\xi_3^3,\xi_3^4)$ defined in (71) where $\xi_3^4\neq 0,-1$. Let $H_{\mathbb{C}}(G)$ the space of complex valued holomorphic functions on G. Then $H_{\mathbb{C}}(G)=\{f\in C^\infty(G_0)\ ;\ \tilde{X}_j^-\,f=0\ \forall j=1,3,5\}$. One has

$$\begin{aligned} \tilde{X}_1^- &= 2\frac{\partial}{\partial\overline{w^1}}-y^1\left(1-\frac{i\xi_3^3}{\xi_3^4}\right)\frac{\partial}{\partial y^2}-\left(y^2\left(1-\frac{i\xi_3^3}{\xi_3^4}\right)+ix^2\right)\frac{\partial}{\partial x^3}+\left(\frac{(y^1)^2}{2}\left(1-\frac{i\xi_3^3}{\xi_3^4}\right)-iy^2\right)\frac{\partial}{\partial y^3}\\ \tilde{X}_3^- &= 2\frac{\partial}{\partial\overline{w^2}}\\ \tilde{X}_5^- &= 2\frac{\partial}{\partial\overline{w^3}} \end{aligned}$$

where

$$\begin{aligned} w^1 &= x^1+\frac{\xi_3^3}{\xi_3^4}y^1+iy^1\\ w^2 &= x^2-\frac{\xi_3^3}{\xi_3^4}y^2+\frac{i}{\xi_3^4}y^2\\ w^3 &= x^3+i\,\frac{\xi_3^4+1}{\xi_3^4}y^3. \end{aligned}$$

Then $f\in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to w^2 and w^3 and satisfies the equation

$$2\frac{\partial f}{\partial\overline{w^1}}-iA\frac{w^1-\overline{w^1}}{2}\frac{\partial f}{\partial w^2}-\left(iw^2+B\frac{(w^1-\overline{w^1})^2}{8}\right)\frac{\partial f}{\partial w^3} = 0$$

where

$$\begin{aligned} A &= \frac{1}{\xi_3^4}\left(\xi_3^3\left(1-\frac{1}{\xi_3^4}\right)-i\left(1+\frac{{\xi_3^3}^2}{\xi_3^4}\right)\right)\\ B &= \frac{\xi_3^4+1}{{\xi_3^4}^2}\left(\xi_3^3+i\xi_3^4\right). \end{aligned}$$

The 3 functions

$$\begin{aligned}
\varphi^1 &= w^1 \\
\varphi^2 &= w^2 + \frac{iA}{4}\left(w^1\overline{w^1} - \frac{(\overline{w^1})^2}{2}\right) \\
\varphi^3 &= w^3 + \frac{i}{2}\,\overline{w^1}w^2 - \frac{B}{48}\,(w^1 - \overline{w^1})^3 - \frac{A}{16}\,w^1(\overline{w^1})^2 + \frac{A}{24}\,(\overline{w^1})^3
\end{aligned}$$

are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by

$$F = (\varphi^1, \varphi^2, \varphi^3). \tag{72}$$

F is a biholomorphic bijection, hence a global chart on G. We determine now how the multiplication of G looks like in the chart (72). Let $a, x \in G$:

$$\begin{aligned}
x &= \exp(x^1x_1)\exp(y^1x_2)\exp(x^2x_3)\exp(y^2x_4)\exp(x^3x_5)\exp(y^3x_6) \\
a &= \exp(a^1x_1)\exp(b^1x_2)\exp(a^2x_3)\exp(b^2x_4)\exp(a^3x_5)\exp(b^3x_6).
\end{aligned}$$

With obvious notations, $a = [w_a^1, w_a^2, w_a^3]$, $x = [w_x^1, w_x^2, w_x^3]$, $a\,x = [w_{ax}^1, w_{ax}^2, w_{ax}^3]$, $a = [\varphi_a^1, \varphi_a^2, \varphi_a^3]$, $x = [\varphi_x^1, \varphi_x^2, \varphi_x^3]$, $a\,x = [\varphi_{ax}^1, \varphi_{ax}^2, \varphi_{ax}^3]$. Computations yield :

$$\begin{aligned}
w_{ax}^1 &= w_a^1 + w_x^1 \\
w_{ax}^2 &= w_a^2 + w_x^2 + b^1x^1\,\frac{\xi_3^3 - i}{\xi_3^4} \\
w_{ax}^3 &= w_a^3 + w_x^3 - b^2x^1 + b^1\,\frac{(x^1)^2}{2} - a^2y^1 + i\,\frac{\xi_3^4+1}{\xi_3^4}\left(\frac{(b^1)^2}{2}x^1 - (b^2 - b^1x^1)y^1\right).
\end{aligned}$$

We then get

$$\begin{aligned}
\varphi_{ax}^1 &= \varphi_a^1 + \varphi_x^1 \\
\varphi_{ax}^2 &= \varphi_a^2 + \varphi_x^2 + \chi^2(a,x) \\
\varphi_{ax}^3 &= \varphi_a^3 + \varphi_x^3 + \chi^3(a,x)
\end{aligned}$$

where

$$\chi^2(a,x) = \frac{\xi_3^3 - i}{4{\xi_3^4}^2}\,\varphi_x^1\left(2i\xi_3^4\overline{\varphi_a^1} + (\xi_3^3 - i\xi_3^4)\varphi_a^1\right)$$

$$\begin{aligned}
\chi^3(a,x) = \frac{i}{2}\,\varphi_x^2\overline{\varphi_a^1} &+ \frac{1}{16{\xi_3^4}^2}\,(\varphi_x^1)^2(4i{\xi_3^4}^2 - (\xi_3^3 - i)(3\xi_3^4 + i\xi_3^3))\left(\overline{\varphi_a^1} - \varphi_a^1\right) \\
&+ \varphi_x^1\left(\frac{1}{16\xi_3^4}\,(\overline{\varphi_a^1})^2(-\xi_3^3(\xi_3^4 - i\xi_3^3) - i\xi_3^4 - 3\xi_3^3 + 2i)\right. \\
&\quad + \frac{1}{16\xi_3^4}\,(\varphi_a^1)^2(-\xi_3^3(\xi_3^4 - i\xi_3^3) - i\xi_3^4 + \xi_3^3 - 2i) \\
&\quad\left. + \frac{\xi_3^3(\xi_3^4 - i\xi_3^3)}{4\xi_3^4}\,\overline{\varphi_a^1}\varphi_a^1 - \frac{i\xi_3^4}{2}\,\overline{\varphi_a^2} + \frac{i(\xi_3^4+1)}{2}\,\varphi_a^2\right).
\end{aligned}$$

10 Lie Algebra $M10$.

Commutation relations for $M10$: $[x_1, x_2] = x_3$; $[x_1, x_3] = x_5$; $[x_1, x_4] = x_6$; $[x_2, x_3] = -x_6$; $[x_2, x_4] = x_5$.

10.1 Case $\xi_3^4 \neq \xi_1^2$.

$$J = \begin{pmatrix}
\boxed{\xi_1^1} & -\frac{{\xi_1^1}^2+1}{\xi_1^2} & 0 & 0 & 0 & 0 \\
\boxed{\xi_1^2} & -\xi_1^1 & 0 & 0 & 0 & 0 \\
\boxed{\xi_1^3} & \frac{({\xi_3^3}^2+1)\xi_1^4 - (\xi_3^3+\xi_1^1)\xi_3^4\xi_1^3}{\xi_3^4\xi_1^2} & \boxed{\xi_3^3} & -\frac{{\xi_3^3}^2+1}{\xi_3^4} & 0 & 0 \\
\boxed{\xi_1^4} & \frac{(\xi_3^3-\xi_1^1)\xi_1^4 - \xi_3^4\xi_1^3}{\xi_1^2} & \boxed{\xi_3^4} & -\xi_3^3 & 0 & 0 \\
* & * & \boxed{\xi_3^5} & * & r & -\frac{r^2+1}{b} \\
\boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & * & b & -r
\end{pmatrix} \tag{73}$$

where $J_1^5 = (((2\xi_3^4\xi_1^1 - \xi_3^3\xi_1^2)\xi_1^2 + ({\xi_1^1}^2 + 1)\xi_3^3 - ({\xi_1^1}^2 + 1 + {\xi_1^2}^2)\xi_1^1)\xi_1^6{\xi_3^4}^2 + ((\xi_3^4{\xi_1^1}^2 + \xi_3^4 - 2\xi_3^3\xi_1^2\xi_1^1 - 2\xi_1^2)\xi_3^4 + ({\xi_3^3}^2 + 1){\xi_1^2}^2)\xi_3^5\xi_1^4 + ({\xi_3^4}^2\xi_1^2 - \xi_3^4{\xi_1^2}^2 - \xi_3^4{\xi_1^1}^2 - \xi_3^4 + {\xi_3^3}^2\xi_1^2 + \xi_1^2)\xi_2^6\xi_3^4\xi_1^2 - ((((\xi_3^3 - \xi_1^1)\xi_1^4 - \xi_3^4\xi_1^3)({\xi_3^3}^2 + 1) - ({\xi_3^4}^2\xi_1^3 - \xi_3^4\xi_1^4\xi_3^3 - \xi_3^4\xi_1^4\xi_1^1 + 2\xi_1^4\xi_3^3\xi_1^2)\xi_3^4)\xi_1^2 + ({\xi_1^1}^2 + 1 + {\xi_1^2}^2){\xi_3^4}^2\xi_1^3)\xi_3^6)/(({\xi_3^4}^2{\xi_1^1}^2 + {\xi_3^4}^2 - 2\xi_3^4\xi_3^3\xi_1^2\xi_1^1 - 2\xi_3^4\xi_1^2 + {\xi_3^3}^2{\xi_1^2}^2 + {\xi_1^2}^2)\xi_3^4)$;
$J_2^5 = (\xi_3^6{\xi_3^4}^2\xi_1^4\xi_1^2{\xi_1^1}^2 + \xi_3^6{\xi_3^4}^2\xi_1^4\xi_1^2 - \xi_3^6{\xi_3^4}^2\xi_3^3\xi_1^3{\xi_1^2}^2 + \xi_3^6{\xi_3^4}^2\xi_3^3\xi_1^3{\xi_1^1}^2 + \xi_3^6{\xi_3^4}^2\xi_3^3\xi_1^3 + \xi_3^6{\xi_3^4}^2\xi_1^3{\xi_1^2}^2\xi_1^1 + \xi_3^6{\xi_3^4}^2\xi_1^3{\xi_1^1}^3 + \xi_3^6{\xi_3^4}^2\xi_1^3\xi_1^1 + \xi_3^6\xi_3^4\xi_1^4{\xi_3^3}^2{\xi_1^2}^2 - \xi_3^6\xi_3^4\xi_1^4{\xi_3^3}^2{\xi_1^1}^2 - \xi_3^6\xi_3^4\xi_1^4{\xi_3^3}^2 - 2\xi_3^6\xi_3^4\xi_1^4\xi_3^3{\xi_1^2}^2\xi_1^1 - \xi_3^6\xi_3^4\xi_1^4{\xi_1^2}^2 - \xi_3^6\xi_3^4\xi_1^4{\xi_1^1}^2 - \xi_3^6\xi_3^4\xi_1^4 - 2\xi_3^6\xi_3^4{\xi_3^3}^2\xi_1^3\xi_1^2\xi_1^1 - 2\xi_3^6\xi_3^4\xi_1^3\xi_1^2\xi_1^1 + 2\xi_3^6\xi_1^4{\xi_3^3}^3\xi_1^2\xi_1^1 - \xi_3^6\xi_1^4{\xi_3^3}^2\xi_1^2{\xi_1^1}^2 + \xi_3^6\xi_1^4{\xi_3^3}^2\xi_1^2 + 2\xi_3^6\xi_1^4\xi_3^3\xi_1^2\xi_1^1 - \xi_3^6\xi_1^4\xi_1^2{\xi_1^1}^2 + \xi_3^6\xi_1^4\xi_1^2 - \xi_2^6{\xi_3^4}^2\xi_3^3{\xi_1^2}^3 + \xi_2^6{\xi_3^4}^2\xi_3^3\xi_1^2{\xi_1^1}^2 + \xi_2^6{\xi_3^4}^2\xi_3^3\xi_1^2 + \xi_2^6{\xi_3^4}^2{\xi_1^2}^3\xi_1^1 + \xi_2^6{\xi_3^4}^2\xi_1^2{\xi_1^1}^3 + \xi_2^6{\xi_3^4}^2\xi_1^2\xi_1^1 - 2\xi_2^6\xi_3^4{\xi_3^3}^2{\xi_1^2}^2\xi_1^1 - 2\xi_2^6\xi_3^4{\xi_1^2}^2\xi_1^1 - \xi_1^6{\xi_3^4}^3\xi_1^2{\xi_1^1}^2 - \xi_1^6{\xi_3^4}^3\xi_1^2 + \xi_1^6{\xi_3^4}^2{\xi_1^2}^2{\xi_1^1}^2 + \xi_1^6{\xi_3^4}^2{\xi_1^2}^2 + \xi_1^6{\xi_3^4}^2{\xi_1^1}^4 + 2\xi_1^6{\xi_3^4}^2{\xi_1^1}^2 + \xi_1^6{\xi_3^4}^2 - \xi_1^6\xi_3^4{\xi_3^3}^2\xi_1^2{\xi_1^1}^2 - \xi_1^6\xi_3^4{\xi_3^3}^2\xi_1^2 - \xi_1^6\xi_3^4\xi_1^2{\xi_1^1}^2 - \xi_1^6\xi_3^4\xi_1^2 - \xi_3^5{\xi_3^4}^3\xi_1^3{\xi_1^1}^2 - \xi_3^5{\xi_3^4}^3\xi_1^3 + \xi_3^5{\xi_3^4}^2\xi_1^4\xi_3^3{\xi_1^1}^2 + \xi_3^5{\xi_3^4}^2\xi_1^4\xi_3^3 - \xi_3^5{\xi_3^4}^2\xi_1^4{\xi_1^1}^3 - \xi_3^5{\xi_3^4}^2\xi_1^4\xi_1^1 + 2\xi_3^5{\xi_3^4}^2\xi_3^3\xi_1^3\xi_1^2\xi_1^1 + 2\xi_3^5{\xi_3^4}^2\xi_1^3\xi_1^2 - 2\xi_3^5\xi_3^4\xi_1^4{\xi_3^3}^2\xi_1^2\xi_1^1 + 2\xi_3^5\xi_3^4\xi_1^4\xi_3^3\xi_1^2{\xi_1^1}^2 - 2\xi_3^5\xi_3^4\xi_1^4\xi_3^3\xi_1^2 + 2\xi_3^5\xi_3^4\xi_1^4\xi_1^2\xi_1^1 - \xi_3^5\xi_3^4{\xi_3^3}^2\xi_1^3{\xi_1^2}^2 - \xi_3^5\xi_3^4\xi_1^3{\xi_1^2}^2 + \xi_3^5\xi_1^4{\xi_3^3}^3{\xi_1^2}^2 - \xi_3^5\xi_1^4{\xi_3^3}^2{\xi_1^2}^2\xi_1^1 + \xi_3^5\xi_1^4\xi_3^3{\xi_1^2}^2 - \xi_3^5\xi_1^4{\xi_1^2}^2\xi_1^1)/(({\xi_3^4}^2{\xi_1^1}^2 + {\xi_3^4}^2 - 2\xi_3^4\xi_3^3\xi_1^2\xi_1^1 - 2\xi_3^4\xi_1^2 + {\xi_3^3}^2{\xi_1^2}^2 + {\xi_1^2}^2)\xi_3^4\xi_1^2)$;
$J_4^5 = (-(\xi_3^6{\xi_3^4}^2{\xi_1^2}^2 + 2\xi_3^6\xi_3^4\xi_3^3\xi_1^2\xi_1^1 - 2\xi_3^6\xi_3^4\xi_1^2 + \xi_3^6{\xi_3^3}^2{\xi_1^1}^2 + \xi_3^6{\xi_3^3}^2 + \xi_3^6{\xi_1^1}^2 + \xi_3^6 + \xi_3^5{\xi_3^4}^2\xi_3^3\xi_1^2 - \xi_3^5{\xi_3^4}^2\xi_1^2\xi_1^1 - 2\xi_3^5\xi_3^4\xi_3^3{\xi_1^1}^2 - 2\xi_3^5\xi_3^4\xi_3^3 + \xi_3^5{\xi_3^3}^3\xi_1^2 + \xi_3^5{\xi_3^3}^2\xi_1^2\xi_1^1 + \xi_3^5\xi_3^3\xi_1^2 + \xi_3^5\xi_1^2\xi_1^1))/(({\xi_3^4}^2\xi_1^2 - \xi_3^4{\xi_1^2}^2 - \xi_3^4{\xi_1^1}^2 - \xi_3^4 + {\xi_3^3}^2\xi_1^2 + \xi_1^2)\xi_3^4)$;
$r = J_5^5 = -(((\xi_3^4\xi_1^1 - \xi_3^3\xi_1^2)\xi_1^2 + ({\xi_1^1}^2 + 1)\xi_3^3)\xi_3^4 - ({\xi_3^3}^2 + 1)\xi_1^2\xi_1^1)/(((\xi_3^4 - \xi_1^2)\xi_1^2 - ({\xi_1^1}^2 + 1))\xi_3^4 + ({\xi_3^3}^2 + 1)\xi_1^2)$;
$J_4^6 = (-(((\xi_3^4\xi_3^3 + \xi_3^4\xi_1^1 - 2\xi_3^3\xi_1^2)\xi_3^4 + ({\xi_3^3}^2 + 1)(\xi_3^3 - \xi_1^1))\xi_3^6\xi_1^2 - ((\xi_3^4{\xi_1^1}^2 + \xi_3^4 - 2\xi_3^3\xi_1^2\xi_1^1 - 2\xi_1^2)\xi_3^4 + ({\xi_3^3}^2 + 1){\xi_1^2}^2)\xi_3^5))/((({\xi_3^4}^2 - \xi_3^4\xi_1^2 + {\xi_3^3}^2 + 1)\xi_1^2 - ({\xi_1^1}^2 + 1)\xi_3^4)\xi_3^4)$;
$b = J_5^6 = -((\xi_3^4{\xi_1^1}^2 + \xi_3^4 - 2\xi_3^3\xi_1^2\xi_1^1 - 2\xi_1^2)\xi_3^4 + ({\xi_3^3}^2 + 1){\xi_1^2}^2)/(((\xi_3^4 - \xi_1^2)\xi_1^2 - ({\xi_1^1}^2 + 1))\xi_3^4 + ({\xi_3^3}^2 + 1)\xi_1^2)$;
and the parameters are subject to the condition

$$\begin{aligned} \xi_1^2\xi_3^4(\xi_1^2 - \xi_3^4) &\neq 0 && (74)\\ ((\xi_3^4 - \xi_1^2)\xi_1^2 - ({\xi_1^1}^2 + 1))\xi_3^4 + ({\xi_3^3}^2 + 1)\xi_1^2 &\neq 0. && (75)\end{aligned}$$

Note that $b \neq 0$ since $\mathrm{Num}(b) = -\|Y - Z\|^2$ where $Y = \xi_3^4 \begin{pmatrix} \xi_1^1 \\ 1 \end{pmatrix}$, $Z = \xi_1^2 \begin{pmatrix} \xi_3^3 \\ 1 \end{pmatrix}$.

Now the automorphism group of $M10$ is comprised of the matrices

$$\Phi = \begin{pmatrix} b_1^1 & b_2^1 & 0 & 0 & 0 & 0 \\ b_1^2 & b_2^2 & 0 & 0 & 0 & 0 \\ b_1^3 & b_2^3 & b_2^2b_1^1 - b_1^2b_2^1 & -(b_2^2b_2^1 - b_1^2b_1^1) & 0 & 0 \\ b_1^4 & b_2^4 & 0 & {b_2^1}^2 + {b_1^1}^2 & 0 & 0 \\ b_1^5 & b_2^5 & b_2^3b_1^1 - b_1^3b_2^1 - b_1^4b_2^2u + b_2^4b_1^2 & b_4^5 & (b_2^2b_1^1 - b_1^2b_2^1)b_1^1 & -(b_2^2b_1^1 - b_1^2b_2^1)b_2^1 \\ b_1^6 & b_2^6 & -(b_2^3b_1^2 + b_1^3b_2^2 + b_1^4b_2^1 - b_2^4b_1^1) & b_4^6 & -(b_2^2b_1^1 - b_1^2b_2^1)b_1^2 & ({b_1^2}^2 + {b_1^1}^2 + {b_2^1}^2)b_1^1 + b_2^2b_1^2b_2^1 \end{pmatrix}$$

where $\det\Phi = (b_2^2b_1^1 - b_1^2b_2^1)^3({b_1^2}^2 + {b_1^1}^2)({b_2^1}^2 + {b_1^1}^2)^2 \neq 0$ and $C_1 = {b_1^2}^2b_1^1 + 2b_1^2b_2^1b_2^2 + ({b_2^1}^2 + {b_1^1}^2 - {b_2^2}^2)b_1^1$ and $C_2 = -{b_1^2}^2b_2^1 + 2b_1^2b_1^1b_2^2 + ({b_2^2}^2 + {b_1^1}^2 + {b_2^1}^2)b_2^1$ must vanish. This requires the vanishing of the resultant $R = Res(C_1, C_2, b_1^2)$, hence $b_2^2 = b_1^1u$ with $u^2 = 1$ since $R = -4({b_2^2}^2 + {b_2^1}^2)({b_1^1}^2 + {b_2^1}^2)({b_2^2}^2 - {b_2^1}^2)$. Then one gets $C_1 = (b_1^2 + ub_2^1)^2b_1^1$. If $b_1^1 \neq 0$, then $b_1^2 = -ub_2^1$ is the only common root for C_1 and C_2. If $b_1^1 = 0$, then $b_2^1 \neq 0$ and $C_2 = b_2^1({b_2^1}^2 - {b_1^2}^2)$ vanishes for $b_1^2 = \pm b_2^1$. In both cases, the vanishing of C_1 and C_2 holds if and only if $b_2^2 = ub_1^1, b_1^2 = -ub_2^1$ with $u^2 = 1$. Hence the automorphism group of $M10$ is comprised of the matrices

$$\Phi = \begin{pmatrix} b_1^1 & b_1^2u & 0 & 0 & 0 & 0 \\ b_1^2 & -b_1^1u & 0 & 0 & 0 & 0 \\ b_1^3 & b_2^3 & -({b_1^2}^2 + {b_1^1}^2)u & 0 & 0 & 0 \\ b_1^4 & b_2^4 & 0 & {b_1^2}^2 + {b_1^1}^2 & 0 & 0 \\ b_1^5 & b_2^5 & b_2^3b_1^1 - b_1^3b_1^2u + b_1^4b_1^1u + b_2^4b_1^2 & b_4^5 & -({b_1^2}^2 + {b_1^1}^2)b_1^1u & ({b_1^2}^2 + {b_1^1}^2)b_1^2 \\ b_1^6 & b_2^6 & -(b_2^3b_1^2 + b_1^3b_1^1u + b_1^4b_1^2u - b_2^4b_1^1) & b_4^6 & ({b_1^2}^2 + {b_1^1}^2)b_1^2u & ({b_1^2}^2 + {b_1^1}^2)b_1^1 \end{pmatrix} \qquad (76)$$

where $u^2 = 1$ and $det\Phi = -({b_1^2}^2 + {b_1^1}^2)^6u \neq 0$. Equivalence by a suitable block-diagonal automorphism (76) leads to the case $\xi_1^1 = 0$ and we then can suppose $0 < \xi_1^2 \leqslant 1$. Now, equivalence by

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \xi_1^3/\xi_1^2 & 1 & 0 & 0 & 0 \\ 0 & \xi_1^4/\xi_1^2 & 0 & 1 & 0 & 0 \\ 0 & 0 & \xi_1^3/\xi_1^2 & 0 & 1 & 0 \\ 0 & 0 & \xi_1^4/\xi_1^2 & 0 & 0 & 1 \end{pmatrix} \qquad (77)$$

leads to the case where moreover $\xi_1^3 = 0, \xi_1^4 = 0$. Then, equivalence by

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ b_1^5 & b_2^5 & 0 & \xi_3^5/\xi_3^4 & 1 & 0 \\ 0 & 0 & 0 & \xi_3^6/\xi_3^4 & 0 & 1 \end{pmatrix}$$

with suitable b_1^5, b_2^5 leads to the case where moreover $\xi_1^3 = 0, \xi_1^4 = 0, \xi_3^5 = 0, \xi_1^6 = 0, \xi_2^6 = 0, \xi_3^6 = 0$:

$$J(\xi_1^2, \xi_3^3, \xi_3^4) = \begin{pmatrix} 0 & (-1)/\xi_1^2 & 0 & 0 & 0 & 0 \\ \xi_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_3^3 & (-({\xi_3^3}^2 + 1))/\xi_3^4 & 0 & 0 \\ 0 & 0 & \xi_3^4 & -\xi_3^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & r & -\frac{1+r^2}{b} \\ 0 & 0 & 0 & 0 & b & -r \end{pmatrix} \tag{78}$$

where

$$0 < \xi_1^2 \leqslant 1; \quad \xi_1^2 \xi_3^4 (\xi_1^2 - \xi_3^4) \neq 0; \quad (\xi_3^4 \xi_1^2 - {\xi_1^2}^2 - 1)\xi_3^4 + ({\xi_3^3}^2 + 1)\xi_1^2 \neq 0 \tag{79}$$

and $r = ((\xi_1^2 + 1)(\xi_1^2 - 1)\xi_3^4 \xi_3^3)/((\xi_3^4 \xi_1^2 - {\xi_1^2}^2 - 1)\xi_3^4 + ({\xi_3^3}^2 + 1)\xi_1^2)$,
$b = -((\xi_3^4 - 2\xi_1^2)\xi_3^4 + ({\xi_3^3}^2 + 1){\xi_1^2}^2)/((\xi_3^4 \xi_1^2 - {\xi_1^2}^2 - 1)\xi_3^4 + ({\xi_3^3}^2 + 1)\xi_1^2)$.
Now suppose $J(\eta_1^2, \eta_3^3, \eta_3^4) = \Phi^{-1} J(\xi_1^2, \xi_3^3, \xi_3^4)\Phi$ where Φ is given in (76) and the η's and ξ's satisfy (79). Computing the matrix $J2 = \Phi^{-1} J(\xi_1^2, \xi_3^3, \xi_3^4)\Phi$, one gets $J2_1^1 = (b_1^2 b_1^1({\xi_1^2}^2 - 1))/(\xi_1^2({b_1^2}^2 + {b_1^1}^2))$, $J2_1^2 = -({b_1^2}^2 + {b_1^1}^2 {\xi_1^2}^2)/(\xi_1^2 u({b_1^2}^2 + {b_1^1}^2))$, $J2_3^3 = \xi_3^3$, $J2_3^4 = -\xi_3^4 u$. From these formulae, we see that a necessary condition for equivalence is that $\eta_3^4 = -u\xi_3^4$ $(u = \pm 1)$ and $\eta_3^3 = \xi_3^3$. As $u = 1$ would change the sign of ξ_1^2, we conclude that $u = -1$. Now to keep $J2_1^1 = 0$, one must have either $\xi_1^2 = 1$ or $b_1^1 b_1^2 = 0$. If $\xi_1^2 = 1$, or if $\xi_1^2 < 1$ and $b_1^2 = 0$, then $\eta_1^2 = \xi_1^2$ and $\eta_3^3 = \xi_3^3, \eta_3^4 = \xi_3^4$. If $\xi_1^2 < 1$ and $b_1^1 = 0$, then $\eta_1^2 = 1/\xi_1^2 > 1$ which is contradictory. Hence $J(\eta_1^2, \eta_3^3, \eta_3^4)$ and $J(\xi_1^2, \xi_3^3, \xi_3^4)$ are not equivalent unless $\eta_1^2 = \xi_1^2, \eta_3^3 = \xi_3^3, \eta_3^4 = \xi_3^4$.

Commutation relations of $\mathfrak{m}$:
$[\tilde{x}_1, \tilde{x}_3] = \tilde{x}_5(-\xi_3^4 \xi_1^2 + 1) + \tilde{x}_6 \xi_3^3 \xi_1^2$; $[\tilde{x}_1, \tilde{x}_4] = \tilde{x}_5 \xi_3^3 \xi_1^2 + (\tilde{x}_6(\xi_3^4 - {\xi_3^3}^2 \xi_1^2 - \xi_1^2))/\xi_3^4$;
$[\tilde{x}_2, \tilde{x}_3] = (\tilde{x}_5 \xi_3^3)/\xi_1^2 + (\tilde{x}_6(\xi_3^4 - \xi_1^2))/\xi_1^2$; $[\tilde{x}_2, \tilde{x}_4] = (\tilde{x}_5(\xi_3^4 \xi_1^2 - {\xi_3^3}^2 - 1))/(\xi_3^4 \xi_1^2) + (-\tilde{x}_6 \xi_3^3)/\xi_1^2$.

10.2 Case $\xi_3^4 = \xi_1^2, \xi_3^3 = \xi_1^1$.

In that case one has necessarily $\xi_1^1 = 0$.

$$J = \begin{pmatrix} 0 & (-1)/\xi_1^2 & 0 & 0 & 0 & 0 \\ \boxed{\xi_1^2} & 0 & 0 & 0 & 0 & 0 \\ \boxed{\xi_1^3} & \xi_1^4 & 0 & (-1)/\xi_1^2 & 0 & 0 \\ \boxed{\xi_1^4} & -\xi_1^3 & \xi_1^2 & 0 & 0 & 0 \\ * & * & \boxed{\xi_3^5} & (({\xi_5^5}^2 + 1)\xi_3^6 - \xi_5^6 \xi_5^5 \xi_3^5)/(\xi_5^6 \xi_1^2) & \boxed{\xi_5^5} & (-({\xi_5^5}^2 + 1))/\xi_5^6 \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & (-(\xi_5^6 \xi_3^5 - \xi_3^6 \xi_5^5))/\xi_1^2 & \boxed{\xi_5^6} & -\xi_5^5 \end{pmatrix} \tag{80}$$

where $J_1^5 = (-(\xi_2^6 - \xi_1^6 \xi_5^5 \xi_1^2 + (\xi_5^5 \xi_1^4 + \xi_1^3 \xi_1^2)\xi_3^6 - \xi_5^6 \xi_3^5 \xi_1^4))/(\xi_5^6 \xi_1^2)$,
$J_2^5 = (\xi_2^6 \xi_5^5 + \xi_1^6 \xi_1^2 + (\xi_5^5 \xi_1^3 \xi_1^2 - \xi_1^4)\xi_3^6 - \xi_5^6 \xi_3^5 \xi_1^3 \xi_1^2)/\xi_5^6$, and the parameters are subject to the condition

$$\xi_1^2 = \pm 1, \quad \xi_5^6 \neq 0. \tag{81}$$

As in the preceding case, by equivalence by a suitable block-diagonal automorphism, we can suppose $\xi_5^5 = 0$, $0 < \xi_5^6 \leqslant 1$ and then equivalence by (77) leads to the case where moreover $\xi_1^3 = 0, \xi_1^4 = 0$. Then equivalence by

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ (\xi_1^2 - \xi_1^6)/\xi_5^6 & (\xi_2^6 \xi_1^2)/(\xi_5^6 \xi_1^2) & 0 & \xi_3^5/\xi_1^2 & 1 & 0 \\ 0 & 0 & 0 & \xi_3^6/\xi_1^2 & 0 & 1 \end{pmatrix}$$

leads to the case where moreover $\xi_3^5=0, \xi_1^6=0, \xi_2^6=0, \xi_3^6=0$:

$$J(\xi_1^2,\xi_5^6)=\begin{pmatrix} 0 & (-1)/\xi_1^2 & 0 & 0 & 0 & 0\\ \xi_1^2 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & (-1)/\xi_1^2 & 0 & 0\\ 0 & 0 & \xi_1^2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & (-1)/\xi_5^6\\ 0 & 0 & 0 & 0 & \xi_5^6 & 0 \end{pmatrix} \tag{82}$$

and the parameters are subject to the condition

$$\xi_1^2=\pm 1,\ 0<\xi_5^6\leqslant 1. \tag{83}$$

As in the preceding case, we can see that $J(\eta_1^2,\eta_5^6)$ and $J(\xi_1^2,\xi_5^4)$ are not equivalent unless $\eta_1^2=\xi_1^2, \eta_5^6=\xi_5^6$.

Commutation relations of $\mathfrak{m}$: $\mathfrak{m}$ is abelian.

Since $\mathfrak{m}$ is abelian, no $J(\xi_1^2,\xi_5^6)$ is equivalent to any $J(\xi_1^2,\xi_3^3,\xi_3^4)$ in (78).

10.3 Case $\xi_3^4=\xi_1^2, \xi_3^3\neq\xi_1^1$.

In that case one has necessarily $\xi_3^3\neq-\xi_1^1$ as well.

$$J=\begin{pmatrix} \boxed{\xi_1^1} & -\frac{{\xi_1^1}^2+1}{k} & 0 & 0 & 0 & 0\\ k & -\xi_1^1 & 0 & 0 & 0 & 0\\ \boxed{\xi_1^3} & \frac{(({\xi_3^3}^2+1)(\xi_3^3-\xi_1^1)\xi_1^4+(\xi_3^3+\xi_1^1)^2\xi_5^6\xi_1^3)(\xi_3^3-\xi_1^1)}{(\xi_3^3+\xi_1^1)^2{\xi_5^6}^2} & \boxed{\xi_3^3} & -\frac{{\xi_3^3}^2+1}{k} & 0 & 0\\ \boxed{\xi_1^4} & -\frac{(\xi_3^3+\xi_1^1)\xi_5^6\xi_1^3+(\xi_3^3-\xi_1^1)^2\xi_1^4}{(\xi_3^3+\xi_1^1)\xi_5^6} & k & -\xi_3^3 & 0 & 0\\ * & * & \boxed{\xi_3^5} & * & m & -\frac{m^2+1}{\xi_5^6}\\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & * & \boxed{\xi_5^6} & -m \end{pmatrix} \tag{84}$$

where

$k=J_1^2=-((\xi_3^3+\xi_1^1)\xi_5^6)/(\xi_3^3-\xi_1^1)$;

$J_1^5=(({\xi_5^6}^3\xi_1^6{\xi_3^3}^2+2{\xi_5^6}^3\xi_1^6\xi_3^3\xi_1^1+{\xi_5^6}^3\xi_1^6{\xi_1^1}^2+{\xi_5^6}^2\xi_3^6\xi_1^4{\xi_3^3}^2-{\xi_5^6}^2\xi_3^6\xi_1^4{\xi_1^1}^2+{\xi_5^6}^2\xi_2^6{\xi_3^3}^3+{\xi_5^6}^2\xi_2^6{\xi_3^3}^2\xi_1^1-{\xi_5^6}^2\xi_2^6\xi_3^3{\xi_1^1}^2-{\xi_5^6}^2\xi_2^6{\xi_1^1}^3-\xi_5^6\xi_3^6{\xi_3^3}^3\xi_1^3+\xi_5^6\xi_3^6{\xi_3^3}^2\xi_1^3\xi_1^1+\xi_5^6\xi_3^6\xi_3^3\xi_1^3{\xi_1^1}^2-\xi_5^6\xi_3^6\xi_1^3{\xi_1^1}^3-\xi_5^6\xi_1^6{\xi_3^3}^3\xi_1^1+\xi_5^6\xi_1^6{\xi_3^3}^2{\xi_1^1}^2+\xi_5^6\xi_1^6\xi_3^3{\xi_1^1}^3-\xi_5^6\xi_1^6{\xi_1^1}^4-\xi_5^6\xi_3^5\xi_1^4{\xi_3^3}^3+3\xi_5^6\xi_3^5\xi_1^4{\xi_3^3}^2\xi_1^1-3\xi_5^6\xi_3^5\xi_1^4\xi_3^3{\xi_1^1}^2+\xi_5^6\xi_3^5\xi_1^4{\xi_1^1}^3-\xi_3^6\xi_1^4{\xi_3^3}^4+3\xi_3^6\xi_1^4{\xi_3^3}^3\xi_1^1-3\xi_3^6\xi_1^4{\xi_3^3}^2{\xi_1^1}^2+\xi_3^6\xi_1^4\xi_3^3{\xi_1^1}^3)(\xi_3^3+\xi_1^1)+(\xi_5^6\xi_1^6\xi_3^3+\xi_5^6\xi_1^6\xi_1^1+\xi_3^6\xi_1^4\xi_3^3-\xi_3^6\xi_1^4\xi_1^1)(\xi_3^3\xi_1^1-1)(\xi_3^3-\xi_1^1)^2)/((\xi_3^3+\xi_1^1)^2(\xi_3^3-\xi_1^1)^2{\xi_5^6}^2)$;

$J_2^5=({\xi_5^6}^4\xi_2^6{\xi_3^3}^4+4{\xi_5^6}^4\xi_2^6{\xi_3^3}^3\xi_1^1+6{\xi_5^6}^4\xi_2^6{\xi_3^3}^2{\xi_1^1}^2+4{\xi_5^6}^4\xi_2^6\xi_3^3{\xi_1^1}^3+{\xi_5^6}^4\xi_2^6{\xi_1^1}^4-{\xi_5^6}^3\xi_3^6{\xi_3^3}^4\xi_1^3-2{\xi_5^6}^3\xi_3^6{\xi_3^3}^3\xi_1^3\xi_1^1+2{\xi_5^6}^3\xi_3^6\xi_3^3\xi_1^3{\xi_1^1}^3+{\xi_5^6}^3\xi_3^6\xi_1^3{\xi_1^1}^4-{\xi_5^6}^2\xi_3^6\xi_1^4{\xi_3^3}^5+{\xi_5^6}^2\xi_3^6\xi_1^4{\xi_3^3}^4\xi_1^1+2{\xi_5^6}^2\xi_3^6\xi_1^4{\xi_3^3}^3{\xi_1^1}^2-2{\xi_5^6}^2\xi_3^6\xi_1^4{\xi_3^3}^2{\xi_1^1}^3-{\xi_5^6}^2\xi_3^6\xi_1^4\xi_3^3{\xi_1^1}^4+{\xi_5^6}^2\xi_3^6\xi_1^4{\xi_1^1}^5+2{\xi_5^6}^2\xi_2^6{\xi_3^3}^5\xi_1^1+{\xi_5^6}^2\xi_2^6{\xi_3^3}^4{\xi_1^1}^2-{\xi_5^6}^2\xi_2^6{\xi_3^3}^4-4{\xi_5^6}^2\xi_2^6{\xi_3^3}^3{\xi_1^1}^3-2{\xi_5^6}^2\xi_2^6{\xi_3^3}^2{\xi_1^1}^4+2{\xi_5^6}^2\xi_2^6{\xi_3^3}^2{\xi_1^1}^2+2{\xi_5^6}^2\xi_2^6\xi_3^3{\xi_1^1}^5+{\xi_5^6}^2\xi_2^6{\xi_1^1}^6-{\xi_5^6}^2\xi_2^6{\xi_1^1}^4+{\xi_5^6}^2\xi_3^5{\xi_3^3}^5\xi_1^3-{\xi_5^6}^2\xi_3^5{\xi_3^3}^4\xi_1^3\xi_1^1-2{\xi_5^6}^2\xi_3^5{\xi_3^3}^3\xi_1^3{\xi_1^1}^2+2{\xi_5^6}^2\xi_3^5{\xi_3^3}^2\xi_1^3{\xi_1^1}^3+{\xi_5^6}^2\xi_3^5\xi_3^3\xi_1^3{\xi_1^1}^4-{\xi_5^6}^2\xi_3^5\xi_1^3{\xi_1^1}^5-2\xi_5^6\xi_3^6{\xi_3^3}^5\xi_1^3\xi_1^1+3\xi_5^6\xi_3^6{\xi_3^3}^4\xi_1^3{\xi_1^1}^2+\xi_5^6\xi_3^6{\xi_3^3}^4\xi_1^3+2\xi_5^6\xi_3^6{\xi_3^3}^3\xi_1^3{\xi_1^1}^3-2\xi_5^6\xi_3^6{\xi_3^3}^3\xi_1^3\xi_1^1-4\xi_5^6\xi_3^6{\xi_3^3}^2\xi_1^3{\xi_1^1}^4+2\xi_5^6\xi_3^6\xi_3^3\xi_1^3{\xi_1^1}^3+\xi_5^6\xi_3^6\xi_1^3{\xi_1^1}^6-\xi_5^6\xi_3^6\xi_1^3{\xi_1^1}^4-\xi_5^6\xi_1^6{\xi_3^3}^5{\xi_1^1}^2-\xi_5^6\xi_1^6{\xi_3^3}^5+\xi_5^6\xi_1^6{\xi_3^3}^4{\xi_1^1}^3+\xi_5^6\xi_1^6{\xi_3^3}^4\xi_1^1+2\xi_5^6\xi_1^6{\xi_3^3}^3{\xi_1^1}^4+2\xi_5^6\xi_1^6{\xi_3^3}^3{\xi_1^1}^2-2\xi_5^6\xi_1^6{\xi_3^3}^2{\xi_1^1}^5-2\xi_5^6\xi_1^6{\xi_3^3}^2{\xi_1^1}^3-\xi_5^6\xi_1^6\xi_3^3{\xi_1^1}^6-\xi_5^6\xi_1^6\xi_3^3{\xi_1^1}^4+\xi_5^6\xi_1^6{\xi_1^1}^7+\xi_5^6\xi_1^6{\xi_1^1}^5+\xi_5^6\xi_3^5\xi_1^4{\xi_3^3}^6-4\xi_5^6\xi_3^5\xi_1^4{\xi_3^3}^5\xi_1^1+5\xi_5^6\xi_3^5\xi_1^4{\xi_3^3}^4{\xi_1^1}^2-5\xi_5^6\xi_3^5\xi_1^4{\xi_3^3}^2{\xi_1^1}^4+4\xi_5^6\xi_3^5\xi_1^4\xi_3^3{\xi_1^1}^5-\xi_5^6\xi_3^5\xi_1^4{\xi_1^1}^6-2\xi_3^6\xi_1^4{\xi_3^3}^6\xi_1^1+8\xi_3^6\xi_1^4{\xi_3^3}^5{\xi_1^1}^2-12\xi_3^6\xi_1^4{\xi_3^3}^4{\xi_1^1}^3-2\xi_3^6\xi_1^4{\xi_3^3}^4\xi_1^1+8\xi_3^6\xi_1^4{\xi_3^3}^3{\xi_1^1}^4+8\xi_3^6\xi_1^4{\xi_3^3}^3{\xi_1^1}^2-2\xi_3^6\xi_1^4{\xi_3^3}^2{\xi_1^1}^5-12\xi_3^6\xi_1^4{\xi_3^3}^2{\xi_1^1}^3+8\xi_3^6\xi_1^4\xi_3^3{\xi_1^1}^4-2\xi_3^6\xi_1^4{\xi_1^1}^5)/((\xi_3^3+\xi_1^1)^3(\xi_3^3-\xi_1^1)^2{\xi_5^6}^3)$;

$J_4^5=((({\xi_5^6}^3\xi_3^5\xi_3^3+{\xi_5^6}^3\xi_3^5\xi_1^1+\xi_5^6\xi_3^5{\xi_3^3}^3-2\xi_5^6\xi_3^5{\xi_3^3}^2\xi_1^1+\xi_5^6\xi_3^5\xi_3^3{\xi_1^1}^2-\xi_3^6{\xi_3^3}^2+2\xi_3^6\xi_3^3\xi_1^1-\xi_3^6{\xi_1^1}^2)(\xi_3^3+\xi_1^1)+(\xi_3^3\xi_1^1-1)(\xi_3^3-\xi_1^1)^2\xi_5^6\xi_3^5)(\xi_3^3+\xi_1^1)(\xi_3^3-\xi_1^1)^2-((\xi_3^3\xi_1^1-1)(\xi_3^3-\xi_1^1)^2+(\xi_3^3+\xi_1^1)^2{\xi_5^6}^2)^2\xi_3^6)/((\xi_3^3+\xi_1^1)^3(\xi_3^3-\xi_1^1)^3{\xi_5^6}^2)$;

$m=J_5^5=((\xi_3^3\xi_1^1-1)(\xi_3^3-\xi_1^1)^2+(\xi_3^3+\xi_1^1)^2{\xi_5^6}^2)/((\xi_3^3+\xi_1^1)(\xi_3^3-\xi_1^1)^2)$;

$J_4^6=(-(({\xi_5^6}^2\xi_3^6\xi_3^3+{\xi_5^6}^2\xi_3^6\xi_1^1-\xi_5^6\xi_3^5{\xi_3^3}^2+2\xi_5^6\xi_3^5\xi_3^3\xi_1^1-\xi_5^6\xi_3^5{\xi_1^1}^2-\xi_3^6{\xi_3^3}^3+2\xi_3^6{\xi_3^3}^2\xi_1^1-\xi_3^6\xi_3^3{\xi_1^1}^2)(\xi_3^3+\xi_1^1)+(\xi_3^3\xi_1^1-1)(\xi_3^3-\xi_1^1)^2\xi_3^6))/((\xi_3^3+\xi_1^1)^2(\xi_3^3-\xi_1^1)\xi_5^6)$;

and the parameters are subject to the condition

$$\xi_3^3\neq\pm\xi_1^1,\ \xi_1^2\xi_4^3\xi_5^6\neq 0. \tag{85}$$

As in the preceding cases, equivalence by a suitable block-diagonal automorphism leads to the case $\xi_1^1=0$. Equivalence by

$$\Phi=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & (-\xi_1^3)/\xi_5^6 & 1 & 0 & 0 & 0\\ 0 & (-\xi_1^4)/\xi_5^6 & 0 & 1 & 0 & 0\\ 0 & 0 & (-\xi_1^3)/\xi_5^6 & 0 & 1 & 0\\ 0 & 0 & (-\xi_1^4)/\xi_5^6 & 0 & 0 & 1 \end{pmatrix}$$

leads to the case where moreover $\xi_1^3 = 0, \xi_1^4 = 0$. Then equivalence by

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ (-\xi_1^6\xi_3^3)/(\xi_5^6\xi_3^3) & (-\xi_5^6\xi_2^6\xi_3^3)/({\xi_5^6}^2\xi_3^3) & 0 & (-\xi_3^5)/\xi_5^6 & 1 & 0 \\ 0 & 0 & 0 & (-\xi_3^6)/\xi_5^6 & 0 & 1 \end{pmatrix}$$

leads to the case where moreover $\xi_3^5 = 0, \xi_1^6 = 0, \xi_2^6 = 0, \xi_3^6 = 0$:

$$J = \begin{pmatrix} 0 & 1/\xi_5^6 & 0 & 0 & 0 & 0 \\ -\xi_5^6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_3^3 & ({\xi_3^3}^2+1)/\xi_5^6 & 0 & 0 \\ 0 & 0 & -\xi_5^6 & -\xi_3^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & ((\xi_5^6+1)(\xi_5^6-1))/\xi_3^3 & (-(({\xi_5^6}^2-2){\xi_5^6}^2+{\xi_3^3}^2+1))/(\xi_5^6{\xi_3^3}^2) \\ 0 & 0 & 0 & 0 & \xi_5^6 & (-(\xi_5^6+1)(\xi_5^6-1))/\xi_3^3 \end{pmatrix} \tag{86}$$

Computing the matrix $J2 = \Phi^{-1}J\Phi$ where Φ is given in (76) and $b_1^2 = b_1^1 = 1$, one gets $J2_1^2 = \frac{1+{\xi_5^6}^2}{2\xi_5^6}u$, $J2_3^4 = \xi_5^6 u$. Hence if ${\xi_5^6}^2 \neq 1$ we are back to case 1 (10.1). Finally, if ${\xi_5^6}^2 = 1$, equivalence by $\Phi = \mathrm{diag}(1, -1, -1, 1, -1, 1)$ leads to the case where moreover $\xi_5^6 = 1$:

$$J(\xi_3^3) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_3^3 & {\xi_3^3}^2+1 & 0 & 0 \\ 0 & 0 & -1 & -\xi_3^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{87}$$

where $\xi_3^3 \neq 0$.
It si easily checked that $J(\xi_3^3)$ and $J(\eta_3^3)$ are non equivalent unless $\eta_3^3 = \xi_3^3$, and that $J(\xi_3^3)$ in (87) is equivalent neither to any $J(\xi_1^2, \xi_3^3, \xi_3^4)$ in (78) nor to any $J(\xi_1^2, \xi_5^6)$ in (82).

Commutation relations of $\mathfrak{m}$: $[\tilde{x}_1, \tilde{x}_3] = -\tilde{x}_6\xi_3^3$; $[\tilde{x}_1, \tilde{x}_4] = -\tilde{x}_5\xi_3^3 - \tilde{x}_6{\xi_3^3}^2$; $[\tilde{x}_2, \tilde{x}_3] = -\tilde{x}_5\xi_3^3$; $[\tilde{x}_2, \tilde{x}_4] = -\tilde{x}_5{\xi_3^3}^2 + \tilde{x}_6\xi_3^3$.

10.4 Conclusions.

To solve the initial system comprised of all the torsion equations and the equation $J^2 = -1$ in $\mathbb{R}^{36}$ in the 3 cases 1 (10.1), 2.1 (10.2), 2.2 (10.3), one has to complete first a set of common steps, and then we are left with solving the system S of the remaining equations in the 12 variables $\xi_1^1, \xi_1^2, \xi_1^3, \xi_3^3, \xi_1^4, \xi_3^4, \xi_3^5, \xi_5^5, \xi_1^6, \xi_2^6, \xi_3^6, \xi_5^6$ in the open subset $\xi_5^6\xi_3^4\xi_1^2 \neq 0$ of $\mathbb{R}^{12}$. Among these equations, we single out the 2 equations 13|6 and 14|6 which read:

$$\begin{cases} f = f_{13|6} = 0 \\ g = f_{14|6} = 0 \end{cases} \tag{88}$$

where :

$$f_{13|6} = \xi_5^6(\xi_3^3 + \xi_1^1) + \xi_5^5(\xi_1^2 - \xi_3^4) - \xi_3^4\xi_1^1 + \xi_3^3\xi_1^2$$

and

$$f_{14|6} = (\xi_5^6(\xi_3^4\xi_1^2 - {\xi_3^3}^2 - 1) + \xi_5^5\xi_3^4(\xi_3^3 - \xi_1^1) + \xi_3^4\xi_3^3\xi_1^1 + \xi_3^4 - ({\xi_3^3}^2 + 1)\xi_1^2)/\xi_3^4.$$

In each of the 3 cases, the remaining system is equivalent to the system (88). To conclude that $\mathfrak{X}_{M10}$ is a 10-dimensional submanifold of $\mathbb{R}^{36}$, it will be sufficient to prove that the preceding system is of maximal rank 2 at any point of $\mathfrak{X}_{M10}$, that is in each of the 3 cases some 2-jacobian doesn't vanish.

- In case 1, one has

$$\frac{D(f, g)}{D(\xi_5^5, \xi_5^6)} = -\frac{1}{\xi_3^4}(((\xi_3^4 - \xi_1^2)\xi_1^2 - ({\xi_1^1}^2 + 1))\xi_3^4 + ({\xi_3^3}^2 + 1)\xi_1^2) \neq 0.$$

• In case 2.1, one has $\frac{D(f,g)}{D(\xi_1^1,\xi_1^2)} = (\xi_5^6 - \xi_1^2)^2 + {\xi_5^5}^2 \neq 0$ if $\xi_5^5 \neq 0$. If $\xi_5^5 = 0$, $\frac{D(f,g)}{D(\xi_1^1,\xi_1^2)} = (\xi_5^6 - \xi_1^2)^2$ and $\frac{D(f,g)}{D(\xi_3^3,\xi_3^4)} = (\xi_5^6 + \xi_1^2)^2$. These 2-jacobians cannot simultaneously vanish.
• In case 2.2, one has $\frac{D(f,g)}{D(\xi_1^2,\xi_5^5)} = (\xi_3^3 - \xi_1^1)(\xi_5^5 + \xi_3^3) \neq 0$ if $\xi_5^5 \neq -\xi_3^3$. If $\xi_5^5 = -\xi_3^3$, $\frac{D(f,g)}{D(\xi_5^5,\xi_5^6)} = {\xi_1^1}^2 - {\xi_3^3}^2 \neq 0$.
Hence the system (88) is of maximal rank 2 at any point of $\mathfrak{X}_{M10}$, and $\mathfrak{X}_{M10}$ is a 10-dimensional submanifold of $\mathbb{R}^{36}$.

Any complex structure is equivalent to one and only one of the following : $J(\xi_1^2, \xi_3^3, \xi_3^4)$ in (78) or $J(\xi_1^2, \xi_5^6)$ in (82) or $J(\xi_3^3)$ in (87).

10.5

10.5.1

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x^1} - y^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3} - \left(y^2 + \frac{(y^1)^2}{2} \right) \frac{\partial}{\partial y^3} \\ X_2 &= \frac{\partial}{\partial y^1} - y^2 \frac{\partial}{\partial x^3} + x^2 \frac{\partial}{\partial y^3}. \end{aligned}$$

10.5.2 Holomorphic functions.

Let G denote the group G_0 endowed with the left invariant structure of complex manifold defined by $J(\xi_1^2, \xi_3^3, \xi_3^4)$ in (78), where $0 < \xi_1^2 \leqslant 1$, $\xi_1^2\xi_3^4 \neq 0$, $\xi_1^2 \neq \xi_3^4$, and $D = (\xi_3^4\xi_1^2 - {\xi_1^2}^2 - 1)\xi_3^4 + ({\xi_3^3}^2 + 1)\xi_1^2 \neq 0$. Then $H_{\mathbb{C}}(G) = \{f \in C^\infty(G_0)\ ;\ \tilde{X}_j^- f = 0\ \forall j = 1,3,5\}$. One has

$$\begin{aligned} \tilde{X}_1^- &= 2\frac{\partial}{\partial \overline{w^1}} - y^1 \frac{\partial}{\partial x^2} - \left(x^2 + i\xi_1^2 y^2\right) \frac{\partial}{\partial x^3} - \left(\frac{(y^1)^2}{2} + y^2 - i\xi_1^2 x^2 \right) \frac{\partial}{\partial y^3} \\ \tilde{X}_3^- &= 2\frac{\partial}{\partial \overline{w^2}} \\ \tilde{X}_5^- &= 2\frac{\partial}{\partial \overline{w^3}} \end{aligned}$$

where

$$\begin{aligned} w^1 &= x^1 + \frac{i}{\xi_1^2} y^1 \\ w^2 &= x^2 - \frac{\xi_3^3}{\xi_3^4} y^2 + \frac{i}{\xi_3^4} y^2 \\ w^3 &= x^3 - \frac{r}{b} y^3 + \frac{i}{b} y^3. \end{aligned}$$

Then $f \in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to w^2 and w^3 and satisfies the equation

$$\begin{aligned} 2\frac{\partial f}{\partial \overline{w^1}} + i\xi_1^2 \frac{w^1 - \overline{w^1}}{2i} \frac{\partial f}{\partial w^2} + \frac{1}{8b} \Big[&{\xi_1^2}^2 (r - i) \left(2\overline{w^1} w^1 - (\overline{w^1})^2 - {w^1}^2 \right) \\ &- 4\overline{w^2} \left(i(c + \xi_1^2 r) + \xi_1^2 + b \right) - 4w^2 \left(i(-c + \xi_1^2 r) + \xi_1^2 + b \right) \Big] \frac{\partial f}{\partial w^3} = 0. \end{aligned}$$

where

$$c = i\xi_3^4\xi_1^2 b - \xi_3^4 r + i\xi_3^4 + i\xi_3^3\xi_1^2 r + \xi_3^3\xi_1^2 + b\xi_3^3.$$

The 3 functions

$$\begin{aligned} \varphi^1 &= w^1 \\ \varphi^2 &= w^2 - \frac{i\xi_1^2}{4} \left(w^1\overline{w^1} - \frac{(\overline{w^1})^2}{2} \right) \end{aligned}$$

$$
\begin{aligned}
\varphi^3 = w^3 + \frac{1}{48b}\ \Big(& (\overline{w^1})^3 \xi_1^2 (c+ib) + 3(\overline{w^1})^2 w^1 {\xi_1^2}^2 (i-r) + 12\overline{w^1 w^2}(ic + i\xi_1^2 r + \xi_1^2 + b) \\
& + 3\overline{w^1}{w^1}^2 \xi_1^2(-c + 2\xi_1^2 r - 2i\xi_1^2 - ib) + 12(\overline{w^1} + w^1) w^2 (-ic + i\xi_1^2 r + \xi_1^2 + b)\Big)
\end{aligned}
$$

are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by

$$
F = (\varphi^1, \varphi^2, \varphi^3). \tag{89}
$$

F is a biholomorphic bijection, hence a global chart on G. We determine now how the multiplication of G looks like in the chart (89). Let $a, x \in G$:

$$
\begin{aligned}
x &= \exp(x^1 x_1) \exp(y^1 x_2) \exp(x^2 x_3) \exp(y^2 x_4) \exp(x^3 x_5) \exp(y^3 x_6) \\
a &= \exp(a^1 x_1) \exp(b^1 x_2) \exp(a^2 x_3) \exp(b^2 x_4) \exp(a^3 x_5) \exp(b^3 x_6).
\end{aligned}
$$

With obvious notations, $a = [w_a^1, w_a^2, w_a^3]$, $x = [w_x^1, w_x^2, w_x^3]$, $a\,x = [w_{ax}^1, w_{ax}^2, w_{ax}^3]$, $a = [\varphi_a^1, \varphi_a^2, \varphi_a^3]$, $x = [\varphi_x^1, \varphi_x^2, \varphi_x^3]$, $a\,x = [\varphi_{ax}^1, \varphi_{ax}^2, \varphi_{ax}^3]$. Computations yield :

$$
\begin{aligned}
w_{ax}^1 &= w_a^1 + w_x^1 \\
w_{ax}^2 &= w_a^2 + w_x^2 - b^1 x^1 \\
w_{ax}^3 &= w_a^3 + w_x^3 - a^2 x^1 + \frac{1}{2} b^1 (x^1)^2 - b^2 y^1 + \frac{i-r}{b}\left(-b^2 x^1 - \frac{1}{2}(b^1)^2 x^1 + y^1(a^2 - b^1 x^1)\right).
\end{aligned}
$$

We then get

$$
\begin{aligned}
\varphi_{ax}^1 &= \varphi_a^1 + \varphi_x^1 \\
\varphi_{ax}^2 &= \varphi_a^2 + \varphi_x^2 - \frac{i\xi_1^2}{4}(2\overline{\varphi_a^1} - \varphi_a^1)\,\varphi_x^1 \\
\varphi_{ax}^3 &= \varphi_a^3 + \varphi_x^3 + \chi^3(a,x)
\end{aligned}
$$

where

$$
\begin{aligned}
\chi^3(a,x) = {} & \frac{\xi_1^2}{16((\xi_3^4 - \xi_1^2)^2 + {\xi_3^3}^2 {\xi_1^2}^2)}\, D_1\,(\varphi_x^1)^2 \\
& + \frac{1}{16(\xi_3^4 - i\xi_3^3\xi_1^2 - \xi_1^2)}\ \left(D_2 \xi_1^2\, \varphi_x^1 + 8\xi_3^4 (1 - {\xi_1^2}^2)\ \left(\overline{\varphi_a^1} + \varphi_a^1\right)\ \varphi_x^2\right)
\end{aligned}
$$

with
$D_1 = i\overline{\varphi_a^1}\,(7{\xi_3^4}^2 {\xi_1^2}^2 - 3{\xi_3^4}^2 + 7i\xi_3^4\xi_3^3{\xi_1^2}^3 - 7i\xi_3^4\xi_3^3\xi_1^2 - 7\xi_3^4{\xi_1^2}^3 - \xi_3^4\xi_1^2 + 4{\xi_3^3}^2{\xi_1^2}^2 + 4{\xi_1^2}^2) + i\varphi_a^1\,(-4{\xi_3^4}^2{\xi_1^2}^2 + {\xi_3^4}^2 - 4i\xi_3^4\xi_3^3{\xi_1^2}^3 + 4i\xi_3^4\xi_3^3\xi_1^2 + 4\xi_3^4{\xi_1^2}^3 + 2\xi_3^4\xi_1^2 - 3{\xi_3^3}^2{\xi_1^2}^2 - 3{\xi_1^2}^2)$;
$D_2 = -i(\overline{\varphi_a^1})^2\,({\xi_3^4}^2\xi_1^2 - 2\xi_3^4{\xi_1^2}^2 + 2\xi_3^4 + {\xi_3^3}^2\xi_1^2 - 2i\xi_3^3\xi_1^2 - \xi_1^2) + 4i\overline{\varphi_a^1}\varphi_a^1\,({\xi_3^4}^2\xi_1^2 + \xi_3^4{\xi_1^2}^2 - \xi_3^4 + {\xi_3^3}^2\xi_1^2 - 2i\xi_3^3\xi_1^2 - \xi_1^2) - 8\overline{\varphi_a^2}\,({\xi_3^4}^2 + {\xi_3^3}^2 - 2i\xi_3^3 - 1) + 8\varphi_a^2\,({\xi_3^4}^2 + {\xi_3^3}^2 - 2\xi_3^3\xi_1^2 + 1) - i{\varphi_a^1}^2\,({\xi_3^4}^2\xi_1^2 + 3\xi_3^4{\xi_1^2}^2 - 2\xi_3^4 + {\xi_3^3}^2\xi_1^2 - 3i\xi_3^3\xi_1^2 - 2\xi_1^2)$.

In the case of $J(\xi_1^2, \xi_5^6)$ in (82) where $\xi_1^2 = \pm 1$, and $0 < \xi_5^6 \leqslant 1$, the preceding computations apply with $r = 0, b = \xi_5^6, \xi_3^3 = 0, \xi_3^4 = \xi_1^2$. The only difference is that we get now

$$
\chi^3(a,x) = \frac{1}{16\xi_5^6}\ \left(D_1\,(\varphi_x^1)^2 + D_2\,\varphi_x^1\right) + \frac{\xi_5^6 + \xi_1^2}{2\xi_5^6}\ \left(\overline{\varphi_a^1} + \varphi_a^1\right)\ \varphi_x^2
$$

with
$D_1 = -i\overline{\varphi_a^1}\,(3\xi_5^6\xi_1^2 + 7) + i\varphi_a^1\,(\xi_5^6\xi_1^2 + 4)$;
$D_2 = -i(\overline{\varphi_a^1})^2\,(3\xi_5^6\xi_1^2 + 1) - 8i\overline{\varphi_a^1}\varphi_a^1 - 8\overline{\varphi_a^2}\,(\xi_5^6 - \xi_1^2) + 8\varphi_a^2\,(\xi_5^6 + \xi_1^2) + i{\varphi_a^1}^2\,(\xi_5^6\xi_1^2 + 4)$

In the case of $J(\xi_3^3)$ in (87) where $\xi_3^3 \neq 0$, the general computations apply with $r = 0, b = 1, \xi_3^4 = \xi_1^2 = -1$. The only difference is that we get now

$$
\chi^3(a,x) = \frac{1}{16}\ \left(D_1\,(\varphi_x^1)^2 + D_2\,\varphi_x^1\right)
$$

with
$D_1 = -4i\overline{\varphi_a^1} + 3i\varphi_a^1$;
$D_2 = (2i - \xi_3^3)\,(\overline{\varphi_a^1})^2\ + 4(\xi_3^3 - 2i)\,\overline{\varphi_a^1}\varphi_a^1 - 8(i\xi_3^3 + 2)\,\overline{\varphi_a^2} + 8i\xi_3^3\,\varphi_a^2 + (3i - \xi_3^3)\,{\varphi_a^1}^2$.

11 Lie Algebra $M14_\gamma(\gamma = \pm 1)$.

Commutation relations for $M14_\gamma$: $[x_1, x_3] = x_4$; $[x_1, x_4] = x_6$; $[x_2, x_3] = x_5$; $[x_2, x_5] = \gamma x_6$.
$M14_{-1}$ has no complex structure.

11.1 Case $M14_1$.

$$J = \begin{pmatrix} 0 & -\frac{1}{\xi_1^2} & 0 & 0 & 0 & 0 \\ \boxed{\xi_1^2} & 0 & 0 & 0 & 0 & 0 \\ \boxed{\xi_1^3} & \boxed{\xi_2^3} & * & -(\xi_6^6\xi_5^3 - \xi_5^6\xi_6^3)\xi_1^2 & \boxed{\xi_5^3} & \boxed{\xi_6^3} \\ \boxed{\xi_1^4} & * & * & -\frac{(\xi_6^6\xi_5^3-\xi_5^6\xi_6^3)^2}{\xi_6^3} & -\frac{((\xi_5^6\xi_5^3+1)\xi_6^3-\xi_6^6{\xi_5^3}^2)\xi_1^2}{\xi_6^3} & (\xi_6^6\xi_5^3 - \xi_5^6\xi_6^3)\xi_1^2 \\ \boxed{\xi_1^5} & * & * & -\frac{((\xi_5^6\xi_5^3-1)\xi_6^3-\xi_6^6{\xi_5^3}^2)\xi_1^2}{\xi_6^3} & -\frac{{\xi_5^3}^2}{\xi_6^3} & -\xi_5^3 \\ * & * & * & -\frac{({\xi_6^6}^2\xi_5^3-\xi_6^6\xi_5^6\xi_6^3+\xi_5^3)\xi_1^2}{\xi_6^3} & \boxed{\xi_5^6} & \boxed{\xi_6^6} \end{pmatrix} \tag{90}$$

where $J_3^3 = ({\xi_5^6}^2{\xi_6^3}^2 + {\xi_5^3}^2 + {\xi_6^6}^2{\xi_5^3}^2 - (2\xi_5^6\xi_5^3 + 1)\xi_6^6\xi_6^3)/\xi_6^3$;
$J_2^4 = ((\xi_6^6\xi_5^3 - \xi_5^6\xi_6^3)\xi_2^3\xi_1^2 + \xi_1^5\xi_6^3 + \xi_5^3\xi_1^3)/\xi_6^3$;
$J_3^4 = ((({\xi_6^6}^2{\xi_5^3}^3 - 3\xi_6^6\xi_5^6\xi_6^3{\xi_5^3}^2 - \xi_6^6\xi_6^3\xi_5^3 + 3{\xi_5^6}^2{\xi_6^3}^2\xi_5^3 + \xi_5^6{\xi_6^3}^2 + {\xi_5^3}^3)\xi_6^6 - ({\xi_5^6}^3{\xi_6^3}^2 + \xi_5^6{\xi_5^3}^2 + \xi_5^3)\xi_6^3)\xi_1^2)/{\xi_6^3}^2$;
$J_2^5 = (-(\xi_1^4\xi_6^3 + \xi_5^3\xi_2^3 + \xi_5^6\xi_6^3\xi_1^3\xi_1^2 - \xi_6^6\xi_5^3\xi_1^3\xi_1^2))/\xi_6^3$;
$J_3^5 = (-(((\xi_6^6\xi_5^3 - 2\xi_5^6\xi_6^3)\xi_6^6{\xi_5^3}^2 + {\xi_5^6}^2{\xi_6^3}^2\xi_5^3 - \xi_5^6{\xi_6^3}^2 + {\xi_5^3}^3))/{\xi_6^3}^2$;
$J_1^6 = (-(\xi_6^3\xi_2^3\xi_1^2 + {\xi_5^3}^2\xi_1^3 + \xi_1^5\xi_6^3\xi_5^3 + (\xi_5^6\xi_1^3 + \xi_1^4\xi_1^2)\xi_5^6{\xi_6^3}^2 + ((\xi_6^6\xi_5^3 - 2\xi_5^6\xi_6^3)\xi_5^3\xi_1^3 - (\xi_1^4\xi_5^3\xi_1^2 + \xi_1^3)\xi_6^3)\xi_6^6))/{\xi_6^3}^2$;
$J_2^6 = (\xi_6^6\xi_1^5\xi_5^3 + \xi_6^6\xi_2^3\xi_1^2 - \xi_5^6\xi_1^5\xi_6^3 + \xi_1^4\xi_5^3\xi_1^2 + \xi_1^3)/(\xi_6^3\xi_1^2)$;
$J_3^6 = ({\xi_6^6}^3{\xi_5^3}^2 - 2{\xi_6^6}^2\xi_5^6\xi_6^3\xi_5^3 - {\xi_6^6}^2\xi_6^3 + \xi_6^6{\xi_5^6}^2{\xi_6^3}^2 + \xi_6^6{\xi_5^3}^2 - \xi_6^3)/{\xi_6^3}^2$;
and the parameters are subject to the condition

$$\xi_2^1 = \pm 1 \; ; \; \xi_6^3 \neq 0. \tag{91}$$

Now the automorphism group of $M14_1$ is comprised of the matrices

$$\Phi = \begin{pmatrix} b_1^1 & b_1^2 u & 0 & 0 & 0 & 0 \\ b_1^2 & -b_1^1 u & 0 & 0 & 0 & 0 \\ 0 & 0 & b_3^3 & 0 & 0 & 0 \\ b_1^4 & b_2^4 & b_3^4 & b_3^3 b_1^1 & b_3^3 b_1^2 u & 0 \\ -(b_2^4 - b_1^2 k)u & b_1^4 u - b_1^1 k & b_3^5 & b_3^3 b_1^2 & -b_3^3 b_1^1 u & 0 \\ b_1^6 & b_2^6 & b_3^6 & b_3^5 b_1^2 + b_3^4 b_1^1 & -(b_3^5 b_1^1 - b_3^4 b_1^2)u & ({b_1^2}^2 + {b_1^1}^2)b_3^3 \end{pmatrix}$$

where $\det\Phi = ({b_1^2}^2 + {b_1^1}^2)^3 {b_3^3}^4 \neq 0$ and $u = \pm 1$, $k \in \mathbb{R}$. Taking suitable values for the b_j^i's, equivalence by Φ leads to the case where $\xi_1^3 = \xi_2^3 = \xi_5^3 = \xi_1^4 = \xi_1^5 = \xi_6^6 = 0$. Then equivalence by

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ (-\xi_1^2)/2 & 0 & -\xi_5^6\xi_1^2 & 1 & 0 & 0 \\ 0 & (-1)/2 & 0 & 0 & \xi_1^2 & 0 \\ (\xi_5^6)/2 & 0 & 0 & -\xi_5^6\xi_1^2 & 0 & 1/|\xi_6^3| \end{pmatrix}$$

leads to the case where moreover $\xi_5^6 = 0, \xi_1^2 = 1, {\xi_6^3}^2 = 1$:

$$J(\xi_6^3) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi_6^3 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & (-1)/\xi_6^3 & 0 & 0 & 0 \end{pmatrix} \tag{92}$$

where $\xi_6^3 = \pm 1$. The 2 matrices corresponding to $\xi_6^3 = \pm 1$ are not equivalent.

Commutation relations of $\mathfrak{m}$: $[\tilde{x}_1, \tilde{x}_3] = \tilde{x}_4$; $[\tilde{x}_1, \tilde{x}_6] = -\tilde{x}_5\xi_6^3$; $[\tilde{x}_2, \tilde{x}_3] = \tilde{x}_5$; $[\tilde{x}_2, \tilde{x}_6] = \tilde{x}_4\xi_6^3$.

11.2 Conclusions.

From (90), $\mathfrak{X}_{M14_1}$ is a submanifold of dimension 8 in $\mathbb{R}^{36}$. There are only 2 orbits, and any complex structure on $M14_1$ is equivalent to one of the two non equivalent structures in (92).

11.3

11.3.1

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial x^1} - x^2\,\frac{\partial}{\partial y^2} - y^2\,\frac{\partial}{\partial y^3} \\
X_2 &= \frac{\partial}{\partial y^1} - x^2\,\frac{\partial}{\partial x^3} - x^3\,\frac{\partial}{\partial y^3}.
\end{aligned}
$$

11.3.2 Holomorphic functions for J.

Let G denote the group G_0 endowed with the left invariant structure of complex manifold defined by $J(\xi_6^3)$ in (92), where $\xi_6^3 = \pm 1$. Then $H_{\mathbb{C}}(G) = \{f \in C^\infty(G_0)\ ;\ \tilde{X}_j^- f = 0\ \forall j = 1,3,5\}$. One has

$$
\begin{aligned}
\tilde{X}_1^- &= 2\,\frac{\partial}{\partial \overline{w^1}} - x^2\,\frac{\partial}{\partial y^2} - ix^2\,\frac{\partial}{\partial x^3} - iw^3\,\frac{\partial}{\partial y^3} \\
\tilde{X}_3^- &= 2\frac{\partial}{\partial \overline{w^2}} \\
\tilde{X}_5^- &= 2\,\frac{\partial}{\partial \overline{w^3}}
\end{aligned}
$$

where

$$
\begin{aligned}
w^1 &= x^1 + iy^1 \\
w^2 &= x^2 - i\xi_6^3\,y^3 \\
w^3 &= x^3 - iy^2.
\end{aligned}
$$

Then $f \in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to w^2 and w^3 and satisfies the equation

$$
2\,\frac{\partial f}{\partial \overline{w^1}} - \xi_6^3 w^3\,\frac{\partial f}{\partial w^2} = 0.
$$

The 3 functions

$$
\begin{aligned}
\varphi^1 &= w^1 \\
\varphi^2 &= w^2 + \frac{\xi_6^3}{2}\,w^3\overline{w^1} \\
\varphi^3 &= w^3
\end{aligned}
$$

are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by

$$
F = (\varphi^1, \varphi^2, \varphi^3). \tag{93}
$$

F is a biholomorphic bijection, hence a global chart on G. We determine now how the multiplication of G looks like in the chart (93). Let $a, x \in G$:

$$
\begin{aligned}
x &= \exp{(x^1 x_1)}\exp{(y^1 x_2)}\exp{(x^2 x_3)}\exp{(y^2 x_4)}\exp{(x^3 x_5)}\exp{(y^3 x_6)} \\
a &= \exp{(a^1 x_1)}\exp{(b^1 x_2)}\exp{(a^2 x_3)}\exp{(b^2 x_4)}\exp{(a^3 x_5)}\exp{(b^3 x_6)}.
\end{aligned}
$$

With obvious notations, $a = [w_a^1, w_a^2, w_a^3]$, $x = [w_x^1, w_x^2, w_x^3]$, $a\,x = [w_{ax}^1, w_{ax}^2, w_{ax}^3]$, $a = [\varphi_a^1, \varphi_a^2, \varphi_a^3]$, $x = [\varphi_x^1, \varphi_x^2, \varphi_x^3]$, $a\,x = [\varphi_{ax}^1, \varphi_{ax}^2, \varphi_{ax}^3]$. Computations yield :

$$
\begin{aligned}
w_{ax}^1 &= w_a^1 + w_x^1 \\
w_{ax}^2 &= w_a^2 + w_x^2 - i\xi_6^3(-b^2x^1 + \frac{1}{2}\,a^2{x^1}^2 - a^3y^1 + \frac{1}{2}\,a^2{y^1}^2) \\
w_{ax}^3 &= w_a^3 + w_x^3 - a^2y^1 + ia^2x^1.
\end{aligned}
$$

We then get

$$
\begin{aligned}
\varphi_{ax}^1 &= \varphi_a^1 + \varphi_x^1 \\
\varphi_{ax}^2 &= \varphi_a^2 + \varphi_x^2 + \chi^2(a,x) \\
\varphi_{ax}^3 &= \varphi_a^3 + \varphi_x^3 + \chi^3(a,x)
\end{aligned}
$$

where

$$\begin{aligned}
\chi^2(a,x) &= \frac{1}{8}\,\varphi^1_x\left(2i\xi^3_6\overline{\varphi^1_a}(\overline{\varphi^2_a}+\varphi^2_a)+4\xi^3_6\overline{\varphi^3_a}-i(\overline{\varphi^1_a})^2\varphi^3_a-i\,\overline{\varphi^1_a\varphi^3_a}\varphi^1_a\right)+\frac{\xi^3_6}{2}\,\overline{\varphi^1_a}\varphi^3_x;\\
\chi^3(a,x) &= \frac{i}{4}\,\varphi^1_x\left(-3\xi^3_6(\overline{\varphi^1_a}\varphi^3_a+\varphi^1_a\overline{\varphi^3_a})+2(\varphi^2_a+\overline{\varphi^2_a})\right).
\end{aligned}$$

12 Lie Algebra $M18_\gamma(\gamma=\pm1)$.

Commutation relations for $M18_\gamma$:
$[x_1,x_2]=x_3$; $[x_1,x_3]=x_4$; $[x_1,x_4]=x_6$; $[x_2,x_3]=x_5$; $[x_2,x_5]=\gamma x_6$.
$M18_{-1}$ has no complex structure.

12.1 Case $M18_1$.

$$J = \text{same matrix as (90)} \tag{94}$$

and the parameters are subject to the condition

$$\text{same condition as (91)} \tag{95}$$

This comes as no surprise, since the commutations relations of $M18_\gamma$ are simply those of $M14_\gamma$ plus $[x_1,x_2]=x_3$, and any $J\in\mathfrak{X}_{M18_1}$ has $\xi^1_k=\xi^2_k=0$ for $3\leqslant k\leqslant 6$ and $\xi^1_1=\xi^2_2=0$.

Now the automorphism group of $M18_1$ is comprised of the matrices

$$\Phi=\begin{pmatrix}
b^1_1 & b^2_1u & 0 & 0 & 0 & 0\\
b^2_1 & -b^1_1u & 0 & 0 & 0 & 0\\
b^3_1 & b^3_2 & -({b^2_1}^2+{b^1_1}^2)u & 0 & 0 & 0\\
b^4_1 & b^4_2 & b^3_2b^1_1-b^3_1b^2_1u & -({b^2_1}^2+{b^1_1}^2)b^1_1u & -({b^2_1}^2+{b^1_1}^2)b^2_1 & 0\\
b^5_1 & b^5_2 & b^3_2b^2_1+b^3_1b^1_1u & -({b^2_1}^2+{b^1_1}^2)b^2_1u & ({b^2_1}^2+{b^1_1}^2)b^1_1 & 0\\
b^6_1 & b^6_2 & b^4_2b^1_1-b^4_1b^2_1u+b^5_1b^1_1u+b^5_2b^2_1 & ({b^2_1}^2+{b^1_1}^2)b^3_2 & -({b^2_1}^2+{b^1_1}^2)b^3_1 & -({b^2_1}^2+{b^1_1}^2)^2u
\end{pmatrix}$$

where $\det\Phi=({b^2_1}^2+{b^1_1}^2)^7\neq0$ and $u=\pm1$. Taking suitable values for the b^i_j's, equivalence by Φ leads to the case where $\xi^3_1=0,\xi^3_2=0,\xi^3_5=0,\xi^4_1=0,\xi^5_1=0,\xi^6_5=0,\xi^6_6=0$. Then equivalence by $\Phi=\mathrm{diag}(1,\xi^2_1,\xi^2_1/|\xi^3_6|,\xi^2_1/|\xi^3_6|,1/|\xi^3_6|,\xi^2_1/|\xi^3_6|^2)$ leads to the case where moreover $\xi^2_1=1,{\xi^3_6}^2=1$:

$$J(\xi^3_6)=\begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \xi^3_6\\
0 & 0 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & (-1)/\xi^3_6 & 0 & 0 & 0
\end{pmatrix} \tag{96}$$

where $\xi^3_6=\pm1$. The 2 matrices corresponding to $\xi^3_6=\pm1$ are not equivalent.

Commutation relations of $\mathfrak{m}$: $[\tilde{x}_1,\tilde{x}_3]=\tilde{x}_4$; $[\tilde{x}_1,\tilde{x}_6]=-\tilde{x}_5\xi^3_6$; $[\tilde{x}_2,\tilde{x}_3]=\tilde{x}_5$; $[\tilde{x}_2,\tilde{x}_6]=\tilde{x}_4\xi^3_6$.

12.2 Conclusions.

From (94), $\mathfrak{X}_{M18_1}$ is a submanifold of dimension 8 in $\mathbb{R}^{36}$. There are only 2 orbits, and any $J\in\mathfrak{X}_{M18_1}$ is equivalent to one of the two non equivalent structures in (96).

12.3

12.3.1

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial x^1}-y^1\,\frac{\partial}{\partial x^2}-x^2\,\frac{\partial}{\partial y^2}+\frac{1}{2}\,{y^1}^2\,\frac{\partial}{\partial x^3}-\left(y^2+\frac{1}{6}\,{y^1}^3\right)\frac{\partial}{\partial y^3}\\
X_2 &= \frac{\partial}{\partial y^1}-x^2\,\frac{\partial}{\partial x^3}-x^3\,\frac{\partial}{\partial y^3}.
\end{aligned}$$

12.3.2 Holomorphic functions for J.

Let G denote the group G_0 endowed with the left invariant structure of complex manifold defined by $J(\xi_6^3)$ in (96), where $\xi_6^3 = \pm 1$. Then $H_{\mathbb{C}}(G) = \{f \in C^\infty(G_0)\ ;\ \tilde{X}_j^- f = 0\ \forall j = 1,3,5\}$. One has

$$\begin{aligned}
\tilde{X}_1^- &= 2\,\frac{\partial}{\partial \overline{w^1}} - y^1\,\frac{\partial}{\partial x^2} - x^2\,\frac{\partial}{\partial y^2} + \left(\frac{1}{2}\,{y^1}^2 - ix^2\right)\frac{\partial}{\partial x^3} - \left(y^2 + \frac{1}{6}\,{y^1}^3 + ix^3\right)\frac{\partial}{\partial y^3} \\
\tilde{X}_3^- &= 2\frac{\partial}{\partial \overline{w^2}} \\
\tilde{X}_5^- &= 2\,\frac{\partial}{\partial \overline{w^3}}
\end{aligned}$$

where

$$\begin{aligned}
w^1 &= x^1 + iy^1 \\
w^2 &= x^2 - i\xi_6^3\, y^3 \\
w^3 &= x^3 - iy^2.
\end{aligned}$$

Then $f \in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to w^2 and w^3 and satisfies the equation

$$2\,\frac{\partial f}{\partial \overline{w^1}} + \left(-y^1 + \frac{i\xi_6^3}{6}\,{y^1}^3 - \xi_6^3 w^3\right)\frac{\partial f}{\partial w^2} + \frac{1}{2}\,{y^1}^2\,\frac{\partial f}{\partial w^3} = 0.$$

The 3 functions

$$\begin{aligned}
\varphi^1 &= w^1 \\
\varphi^2 &= w^2 + \frac{\xi_6^3}{2}\,w^3\overline{w^1} + \frac{1}{8}\,(w^1 - \overline{w^1})^2\left(i - \frac{\xi_6^3}{48}\,(w^1 - \overline{w^1})^2\right) + \frac{\xi_6^3}{384}\,(\overline{w^1})^2\left(6{w^1}^2 - 8w^1\overline{w^1} + 3(\overline{w^1})^2\right) \\
\varphi^3 &= w^3 + \frac{1}{48}\,\overline{w^1}\left(3{w^1}^2 - 3w^1\overline{w^1} + (\overline{w^1})^2\right)
\end{aligned}$$

are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by

$$F = (\varphi^1, \varphi^2, \varphi^3). \tag{97}$$

F is a biholomorphic bijection, hence a global chart on G. We determine now how the multiplication of G looks like in the chart (97). Let $a, x \in G$:

$$\begin{aligned}
x &= \exp(x^1 x_1)\exp(y^1 x_2)\exp(x^2 x_3)\exp(y^2 x_4)\exp(x^3 x_5)\exp(y^3 x_6) \\
a &= \exp(a^1 x_1)\exp(b^1 x_2)\exp(a^2 x_3)\exp(b^2 x_4)\exp(a^3 x_5)\exp(b^3 x_6).
\end{aligned}$$

With obvious notations, $a = [w_a^1, w_a^2, w_a^3]$, $x = [w_x^1, w_x^2, w_x^3]$, $a\,x = [w_{ax}^1, w_{ax}^2, w_{ax}^3]$, $a = [\varphi_a^1, \varphi_a^2, \varphi_a^3]$, $x = [\varphi_x^1, \varphi_x^2, \varphi_x^3]$, $a\,x = [\varphi_{ax}^1, \varphi_{ax}^2, \varphi_{ax}^3]$. Computations yield :

$$\begin{aligned}
w_{ax}^1 &= w_a^1 + w_x^1 \\
w_{ax}^2 &= w_a^2 + w_x^2 \\
&\quad -b^1x^1 + i\xi_6^3\left(a^3y^1 + b^2x^1 - \frac{1}{2}\,a^2{x^1}^2 + \frac{1}{6}\,{b^1}^3x^1 + \frac{1}{6}\,b^1{x^1}^3 + \frac{1}{2}\,{b^1}^2x^1y^1 - \frac{1}{2}\,{y^1}^2(a^2 - b^1x^1)\right) \\
w_{ax}^3 &= w_a^3 + w_x^3 + \frac{1}{2}\,{b^1}^2x^1 - y^1(a^2 - b^1x^1) - i\left(\frac{1}{2}\,b^1{x^1}^2 - a^2x^1\right).
\end{aligned}$$

We then get

$$\begin{aligned}
\varphi_{ax}^1 &= \varphi_a^1 + \varphi_x^1 \\
\varphi_{ax}^2 &= \varphi_a^2 + \varphi_x^2 + \chi^2(a,x) \\
\varphi_{ax}^3 &= \varphi_a^3 + \varphi_x^3 + \chi^3(a,x)
\end{aligned}$$

where

$$\chi^2(a,x) = \frac{\xi_6^3}{32}(\varphi_x^1)^3\left(\overline{\varphi_a^1}-\varphi_a^1\right)+\frac{\xi_6^3}{64}(\varphi_x^1)^2\left(4(\overline{\varphi_a^1})^2-3(\varphi_a^1)^2\right)+\frac{1}{512}D_2\,\varphi_x^1+\frac{\xi_6^3}{2}\varphi_x^3\overline{\varphi_a^1};$$

$$\chi^3(a,x) = \frac{1}{16}(\varphi_x^1)^2\left(4\overline{\varphi_a^1}-3\varphi_a^1\right)+\frac{1}{256}D_3\,\varphi_x^1;$$

where

$$\begin{aligned}D_2 = 16\xi_6^3\Big(&-(\overline{\varphi_a^1})^3+8i\overline{\varphi_a^1\varphi_a^2}+2\overline{\varphi_a^1}(\varphi_a^1)^2+8i\overline{\varphi_a^1}\varphi_a^2+16\overline{\varphi_a^3}-(\varphi_a^1)^3\Big)\\ &+i\Big((\overline{\varphi_a^1})^5-4(\overline{\varphi_a^1})^4\varphi_a^1+8(\overline{\varphi_a^1})^3(\varphi_a^1)^2-4(\overline{\varphi_a^1})^2(\varphi_a^1)^3\\ &-64(\overline{\varphi_a^1})^2\varphi_a^3-64\overline{\varphi_a^1\varphi_a^3}\varphi_a^1+\overline{\varphi_a^1}(\varphi_a^1)^4-256(\overline{\varphi_a^1}-\varphi_a^1)\Big)\end{aligned}$$

$$\begin{aligned}D_3 = i\xi_6^3\Big(&(\overline{\varphi_a^1})^4-4(\overline{\varphi_a^1})^3\varphi_a^1+8(\overline{\varphi_a^1})^2(\varphi_a^1)^2-4\overline{\varphi_a^1}(\varphi_a^1)^3-64\overline{\varphi_a^1}\varphi_a^3-64\overline{\varphi_a^3}\varphi_a^1+(\varphi_a^1)^4\Big)\\ &-32(\overline{\varphi_a^1})^2-16(\varphi_a^1)^2+64\overline{\varphi_a^1}\varphi_a^1+128i(\overline{\varphi_a^2}+\varphi_a^2).\end{aligned}$$

13 Lie Algebra $M5$.

Commutation relations for $M5 : [x_1, x_3] = x_5;\ [x_1, x_4] = x_6;\ [x_2, x_3] = -x_6;\ [x_2, x_4] = x_5$. This is the realification of the 3-dimensional complex Heisenberg Lie algebra $\mathfrak{n}\ \ [Z_1, Z_2] = Z_3$ we get by letting $x_1 = Z_1, x_2 = -iZ_1, x_3 = Z_2, x_4 = iZ_2, x_5 = Z_3, x_6 = iZ_3$.

13.1 Case ${\xi_4^2}^2 + {\xi_3^2}^2 \neq 0$.

$$J = \begin{pmatrix} \boxed{\xi_1^1} & * & \boxed{\xi_3^1} & \boxed{\xi_4^1} & 0 & 0\\ \boxed{\xi_1^2} & * & \boxed{\xi_3^2} & \boxed{\xi_4^2} & 0 & 0\\ * & * & * & * & 0 & 0\\ * & * & * & * & 0 & 0\\ * & * & * & * & \boxed{\xi_5^5} & -\frac{{\xi_5^5}^2+1}{\xi_5^6}\\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & \boxed{\xi_5^6} & -\xi_5^5 \end{pmatrix} \tag{98}$$

where $J_2^1 = (-(((\xi_5^6\xi_3^2-\xi_5^6\xi_4^1-\xi_5^5\xi_4^2)\xi_4^1+(\xi_3^2-2\xi_4^1)\xi_5^5\xi_3^1)\xi_5^6-({\xi_5^5}^2+1)(\xi_4^2+\xi_3^1)\xi_3^1-(({\xi_4^1}^2+{\xi_3^1}^2)\xi_1^2-\xi_3^2\xi_3^1\xi_1^1-\xi_4^2\xi_4^1\xi_1^1)\xi_5^6))/((\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)\xi_5^6)$;

$J_2^2 = (-(((\xi_5^6\xi_3^2-\xi_5^6\xi_4^1-\xi_5^5\xi_4^2-\xi_5^5\xi_3^1)\xi_4^2+(\xi_3^2-\xi_4^1)\xi_5^5\xi_3^2)\xi_5^6-({\xi_5^5}^2+1)(\xi_4^2+\xi_3^1)\xi_3^2+((\xi_4^2\xi_1^1-\xi_1^2\xi_4^1)\xi_4^2+(\xi_3^2\xi_1^1-\xi_1^2\xi_3^1)\xi_3^2)\xi_5^6))/((\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)\xi_5^6)$;

$J_1^3 = (-(((\xi_1^2\xi_3^1-\xi_4^1\xi_1^1-\xi_3^2\xi_1^1)\xi_1^2+({\xi_1^1}^2+1)\xi_4^2)\xi_5^6+(({\xi_5^5}^2+1)(\xi_4^2+\xi_3^1)-(\xi_3^2-\xi_4^1)\xi_5^6\xi_5^5)\xi_1^2))/((\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)\xi_5^6)$;

$J_2^3 = (-(((({\xi_5^6}^2{\xi_3^2}^2-2{\xi_5^6}^2\xi_3^2\xi_4^1+{\xi_5^6}^2{\xi_4^1}^2-2\xi_5^6\xi_5^5\xi_4^2\xi_3^2+2\xi_5^6\xi_5^5\xi_4^2\xi_4^1-2\xi_5^6\xi_5^5\xi_3^2\xi_3^1+2\xi_5^6\xi_5^5\xi_4^1\xi_3^1+\xi_5^6\xi_4^2\xi_1^2\xi_3^1-\xi_5^6\xi_3^2\xi_1^2\xi_4^1-2\xi_5^6\xi_3^2\xi_3^1\xi_1^1+{\xi_5^5}^2{\xi_4^2}^2+2{\xi_5^5}^2\xi_4^2\xi_3^1-2{\xi_5^5}^2{\xi_3^2}^2+2{\xi_5^5}^2\xi_3^2\xi_4^1+{\xi_5^5}^2{\xi_3^1}^2-\xi_5^5{\xi_4^2}^2\xi_1^1+\xi_5^5\xi_4^2\xi_1^2\xi_4^1-2\xi_5^5{\xi_3^2}^2\xi_1^1+2\xi_5^5\xi_3^2\xi_1^2\xi_3^1+\xi_5^5{\xi_3^1}^2\xi_1^1+{\xi_4^2}^2+2\xi_4^2\xi_3^1-2{\xi_3^2}^2+2\xi_3^2\xi_4^1+{\xi_3^1}^2+(\xi_1^2\xi_3^1+2\xi_4^1\xi_1^1)\xi_5^6\xi_3^1)\xi_4^2+((\xi_3^2-\xi_4^1)\xi_5^6\xi_5^5-2{\xi_5^5}^2\xi_3^1-2\xi_3^1)(\xi_3^2-\xi_4^1)\xi_3^2-(2{\xi_3^2}^2\xi_1^1-2\xi_3^2\xi_1^2\xi_3^1+\xi_1^2\xi_4^1\xi_3^1)\xi_5^5\xi_3^1+(2{\xi_3^2}^3\xi_1^1-2{\xi_3^2}^2\xi_1^2\xi_3^1-2{\xi_3^2}^2\xi_4^1\xi_1^1+3\xi_3^2\xi_1^2\xi_4^1\xi_3^1-2\xi_1^2{\xi_4^1}^2\xi_3^1)\xi_5^6)\xi_5^5-({\xi_4^2}^2\xi_1^1-\xi_4^2\xi_1^2\xi_4^1-\xi_4^2\xi_3^1\xi_1^1+2{\xi_3^2}^2\xi_1^1-2\xi_3^2\xi_1^2\xi_3^1+\xi_1^2\xi_4^1\xi_3^1)(\xi_4^2+\xi_3^1))\xi_5^6+({\xi_5^5}^2+1)^2(\xi_4^2+\xi_3^1)^2\xi_3^2+((\xi_3^2\xi_1^1-\xi_1^2\xi_3^1-\xi_4^1\xi_1^1)\xi_4^2+\xi_1^2{\xi_4^1}^2)(\xi_3^2-\xi_4^1){\xi_5^6}^3+(((\xi_4^2{\xi_1^1}^2-\xi_4^2-\xi_1^2\xi_4^1\xi_1^1-({\xi_1^1}^2+1)\xi_3^1)\xi_3^2-(\xi_1^2\xi_3^1\xi_1^1+\xi_4^1{\xi_1^1}^2-\xi_4^1)\xi_4^2+(\xi_1^2\xi_4^1\xi_3^1+2{\xi_4^1}^2\xi_1^1+{\xi_3^1}^2\xi_1^1)\xi_1^2)\xi_4^2+({\xi_3^2}^2{\xi_1^1}^2-2\xi_3^2\xi_1^2\xi_3^1\xi_1^1+{\xi_1^2}^2{\xi_3^1}^2+\xi_1^2\xi_4^1\xi_3^1\xi_1^1+{\xi_4^1}^2)\xi_3^2-({\xi_4^1}^2+{\xi_3^1}^2){\xi_1^2}^2\xi_4^1){\xi_5^6}^2))/((\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)^2{\xi_5^6}^2)$;

$J_3^3 = (-((({\xi_5^5}^2+1)(\xi_4^2+\xi_3^1)-(\xi_3^2-\xi_4^1)\xi_5^6\xi_5^5)\xi_3^2-({\xi_3^2}^2\xi_1^1-\xi_3^2\xi_1^2\xi_3^1+\xi_1^2\xi_4^1\xi_3^1-\xi_4^2\xi_3^1\xi_1^1)\xi_5^6))/((\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)\xi_5^6)$;

$J_4^3 = (-((({\xi_5^5}^2+1)(\xi_4^2+\xi_3^1)-(\xi_3^2-\xi_4^1)\xi_5^6\xi_5^5)\xi_4^2-(\xi_4^2\xi_3^2\xi_1^1-\xi_4^2\xi_1^2\xi_3^1-\xi_4^2\xi_4^1\xi_1^1+\xi_1^2{\xi_4^1}^2)\xi_5^6))/((\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)\xi_5^6)$;

$J_1^4 = -((\xi_1^2\xi_4^1+\xi_3^1\xi_1^1)\xi_1^2-({\xi_1^1}^2+1)\xi_3^2-\xi_4^2\xi_1^2\xi_1^1+(\xi_4^2+\xi_3^1)\xi_5^5\xi_1^2-(\xi_3^2-\xi_4^1)\xi_5^6\xi_1^2)/(\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)$;

$J_2^4 = (-(((((\xi_3^2-\xi_4^1)\xi_5^6-2\xi_5^5\xi_3^1)(\xi_3^2-\xi_4^1)\xi_5^5\xi_3^2+({\xi_5^6}^2{\xi_3^2}^2-2{\xi_5^6}^2\xi_3^2\xi_4^1+{\xi_5^6}^2{\xi_4^1}^2-2\xi_5^6\xi_5^5\xi_4^2\xi_3^2+2\xi_5^6\xi_5^5\xi_4^2\xi_4^1-2\xi_5^6\xi_5^5\xi_3^2\xi_3^1+2\xi_5^6\xi_5^5\xi_4^1\xi_3^1+{\xi_5^5}^2{\xi_4^2}^2+2{\xi_5^5}^2\xi_4^2\xi_3^1-2{\xi_5^5}^2{\xi_3^2}^2+2{\xi_5^5}^2\xi_3^2\xi_4^1+{\xi_5^5}^2{\xi_3^1}^2-2\xi_5^5{\xi_4^2}^2\xi_1^1+2\xi_5^5\xi_4^2\xi_1^2\xi_4^1-2\xi_5^5\xi_4^2\xi_3^1\xi_1^1+\xi_5^5\xi_3^2\xi_1^2\xi_3^1-2\xi_5^5\xi_3^2\xi_4^1\xi_1^1+3\xi_5^5\xi_1^2\xi_4^1\xi_3^1)\xi_4^2-({\xi_3^2}^2\xi_1^2-\xi_3^2\xi_1^2\xi_4^1+2\xi_3^2\xi_3^1\xi_1^1-2\xi_1^2{\xi_3^1}^2)\xi_5^5\xi_4^1)\xi_5^6+({\xi_5^5}^2+1)(\xi_5^5\xi_4^2\xi_3^2+\xi_5^5\xi_3^2\xi_3^1-\xi_4^2\xi_3^2\xi_1^1+\xi_3^2\xi_1^2\xi_4^1-\xi_3^2\xi_3^1\xi_1^1+\xi_1^2{\xi_3^1}^2)(\xi_4^2+\xi_3^1)+(2(\xi_4^2\xi_1^1-\xi_1^2\xi_4^1)\xi_4^2+(\xi_3^2\xi_1^1-\xi_1^2\xi_3^1)(\xi_3^2+\xi_4^1))(\xi_3^2-\xi_4^1){\xi_5^6}^2-(((\xi_3^2\xi_4^1\xi_1^1-\xi_1^2\xi_4^1\xi_3^1+{\xi_4^1}^2\xi_1^1+2{\xi_3^1}^2\xi_1^1)\xi_1^2-(\xi_1^1+1)(\xi_1^1-1)\xi_3^2\xi_3^1)\xi_3^2-({\xi_4^1}^2+{\xi_3^1}^2){\xi_1^2}^2\xi_3^1-({\xi_4^2}^2{\xi_1^1}^2-2\xi_4^2\xi_1^2\xi_4^1\xi_1^1+{\xi_3^2}^2{\xi_1^1}^2-{\xi_3^2}^2-\xi_3^2\xi_1^2\xi_3^1\xi_1^1+\xi_3^2\xi_4^1{\xi_1^1}^2+\xi_3^2\xi_4^1+{\xi_1^2}^2{\xi_4^1}^2-\xi_1^2\xi_4^1\xi_3^1\xi_1^1+{\xi_3^1}^2)\xi_4^2)\xi_5^6))/((\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)^2\xi_5^6)$;

$J_3^4 = -(\xi_3^2\xi_1^2\xi_4^1-\xi_3^2\xi_3^1\xi_1^1+\xi_1^2{\xi_3^1}^2-\xi_4^2\xi_3^2\xi_1^1+(\xi_4^2+\xi_3^1)\xi_5^5\xi_3^2-(\xi_3^2-\xi_4^1)\xi_5^6\xi_3^2)/(\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)$;

$J_4^4 = ((\xi_4^2\xi_1^1-\xi_1^2\xi_4^1)\xi_4^2+(\xi_3^2\xi_1^1-\xi_1^2\xi_3^1)\xi_4^1-(\xi_4^2+\xi_3^1)\xi_5^5\xi_4^2+(\xi_3^2-\xi_4^1)\xi_5^6\xi_4^2)/(\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)$;

$J_1^5 = ((((\xi_5^5-\xi_1^1)\xi_1^6-\xi_2^6\xi_1^2)(\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)-(\xi_3^2-\xi_4^1)\xi_3^6\xi_5^5\xi_1^2+((\xi_1^2\xi_3^1-\xi_4^1\xi_1^1-\xi_3^2\xi_1^1)\xi_1^2+({\xi_1^1}^2+1)\xi_4^2)\xi_3^6)\xi_5^6+((\xi_1^2\xi_4^1+\xi_3^1\xi_1^1)\xi_1^2-({\xi_1^1}^2+1)\xi_3^2-\xi_4^2\xi_1^2\xi_1^1)\xi_5^6\xi_4^6-((\xi_5^6\xi_3^2-\xi_5^6\xi_4^1-\xi_5^5\xi_4^2-\xi_5^5\xi_3^1)\xi_5^6\xi_4^6-({\xi_5^5}^2+1)(\xi_4^2+\xi_3^1)\xi_3^6)\xi_1^2)/((\xi_4^2\xi_3^1-\xi_3^2\xi_4^1){\xi_5^6}^2)$;

$J_2^5 = ((((\xi_3^2 - \xi_4^1)\xi_5^6 - 2\xi_5^5\xi_3^1)(\xi_3^2 - \xi_4^1)\xi_5^6\xi_5^5\xi_3^2 + {\xi_5^6}^3\xi_4^2{\xi_3^2}^2 - 2{\xi_5^6}^3\xi_4^2\xi_3^2\xi_4^1 + {\xi_5^6}^3\xi_4^2{\xi_4^1}^2 - 2{\xi_5^6}^2\xi_5^5{\xi_4^2}^2\xi_3^2 + 2{\xi_5^6}^2\xi_5^5{\xi_4^2}^2\xi_4^1 - 2{\xi_5^6}^2\xi_5^5\xi_4^2\xi_3^2\xi_3^1 + 2{\xi_5^6}^2\xi_5^5\xi_4^2\xi_4^1\xi_3^1 + \xi_5^6{\xi_5^5}^2{\xi_4^2}^3 + 2\xi_5^6{\xi_5^5}^2{\xi_4^2}^2\xi_3^1 - 2\xi_5^6{\xi_5^5}^2\xi_4^2{\xi_3^2}^2 + 2\xi_5^6{\xi_5^5}^2\xi_4^2\xi_3^2\xi_4^1 + \xi_5^6{\xi_5^5}^2\xi_4^2{\xi_3^1}^2 - 2\xi_5^6\xi_5^5{\xi_4^2}^3\xi_1^1 + 2\xi_5^6\xi_5^5{\xi_4^2}^2\xi_1^2\xi_4^1 - 2\xi_5^6\xi_5^5{\xi_4^2}^2\xi_3^1\xi_1^1 + \xi_5^6\xi_5^5\xi_4^2\xi_3^2\xi_1^2\xi_3^1 - 2\xi_5^6\xi_5^5\xi_4^2\xi_3^2\xi_4^1\xi_1^1 + 3\xi_5^6\xi_5^5\xi_4^2\xi_1^2\xi_4^1\xi_3^1 + {\xi_5^5}^3{\xi_4^2}^2\xi_3^2 + 2{\xi_5^5}^3\xi_4^2\xi_3^2\xi_3^1 + {\xi_5^5}^3\xi_3^2{\xi_3^1}^2 - {\xi_5^5}^2{\xi_4^2}^2\xi_3^2\xi_1^1 + {\xi_5^5}^2\xi_4^2\xi_3^2\xi_1^2\xi_4^1 - 2{\xi_5^5}^2\xi_4^2\xi_3^2\xi_3^1\xi_1^1 + {\xi_5^5}^2\xi_4^2\xi_1^2{\xi_3^1}^2 + \xi_5^5{\xi_4^2}^2\xi_3^2 + 2\xi_5^5\xi_4^2\xi_3^2\xi_3^1 + \xi_5^5\xi_3^2{\xi_3^1}^2 + (\xi_3^2\xi_1^2\xi_4^1 - \xi_3^2\xi_3^1\xi_1^1 + \xi_1^2{\xi_3^1}^2){\xi_5^5}^2\xi_3^1 - ({\xi_3^2}^2\xi_1^2 - \xi_3^2\xi_1^2\xi_4^1 + 2\xi_3^2\xi_3^1\xi_1^1 - 2\xi_1^2{\xi_3^1}^2)\xi_5^6\xi_5^5\xi_4^1 - (\xi_4^2\xi_3^2\xi_1^1 - \xi_3^2\xi_1^2\xi_4^1 + \xi_3^2\xi_3^1\xi_1^1 - \xi_1^2{\xi_3^1}^2)(\xi_4^2 + \xi_3^1))\xi_5^6\xi_4^6 + ({\xi_5^5}^2 + 1)^2(\xi_4^2 + \xi_3^1)^2\xi_3^6\xi_3^2 + (2(\xi_4^2\xi_1^1 - \xi_1^2\xi_4^1)\xi_4^2 + (\xi_3^2\xi_1^1 - \xi_1^2\xi_3^1)(\xi_3^2 + \xi_4^1))(\xi_3^2 - \xi_4^1){\xi_5^6}^3\xi_4^6 - ((\xi_2^6\xi_3^2 + \xi_1^6\xi_3^1)(\xi_4^2\xi_3^1 - \xi_3^2\xi_4^1) - (\xi_5^5{\xi_4^2}^2 + \xi_5^5\xi_4^2\xi_3^1 - 2\xi_5^5{\xi_3^2}^2 + 2\xi_5^5\xi_3^2\xi_4^1 - {\xi_4^2}^2\xi_1^1 + \xi_4^2\xi_1^2\xi_4^1 + \xi_4^2\xi_3^1\xi_1^1 - 2{\xi_3^2}^2\xi_1^1 + 2\xi_3^2\xi_1^2\xi_3^1 - \xi_1^2\xi_4^1\xi_3^1)\xi_3^6)({\xi_5^5}^2 + 1)(\xi_4^2 + \xi_3^1)\xi_5^6 + ((\xi_2^6\xi_4^2 + \xi_1^6\xi_4^1)(\xi_4^2\xi_3^1 - \xi_3^2\xi_4^1) + (\xi_3^2 - \xi_4^1)\xi_3^6\xi_5^5\xi_4^2 + ((\xi_3^2\xi_1^1 - \xi_1^2\xi_3^1 - \xi_4^1\xi_1^1)\xi_4^2 + \xi_1^2{\xi_4^1}^2)\xi_3^6)(\xi_3^2 - \xi_4^1){\xi_5^6}^3 - (((\xi_3^2\xi_4^1\xi_1^1 - \xi_1^2\xi_4^1\xi_3^1 + {\xi_4^1}^2\xi_1^1 + 2{\xi_3^1}^2\xi_1^1)\xi_1^2 - (\xi_1^1 + 1)(\xi_1^1 - 1)\xi_3^2\xi_3^1)\xi_3^2 - ({\xi_4^1}^2 + {\xi_3^1}^2){\xi_1^2}^2\xi_3^1 - ({\xi_4^2}^2{\xi_1^1}^2 - 2\xi_4^2\xi_1^2\xi_4^1\xi_1^1 + {\xi_3^2}^2{\xi_1^1}^2 - {\xi_3^2}^2 - \xi_3^2\xi_1^2\xi_3^1\xi_1^1 + \xi_3^2\xi_4^1{\xi_1^1}^2 + \xi_3^2\xi_4^1 + {\xi_1^2}^2{\xi_4^1}^2 - \xi_1^2\xi_4^1\xi_3^1\xi_1^1 + {\xi_3^1}^2)\xi_4^2){\xi_5^6}^2\xi_4^6 - (((2\xi_5^5\xi_4^2\xi_3^2 - 2\xi_5^5\xi_4^2\xi_4^1 + 2\xi_5^5\xi_3^2\xi_3^1 - 2\xi_5^5\xi_4^1\xi_3^1 - \xi_4^2\xi_1^2\xi_3^1 + \xi_3^2\xi_1^2\xi_4^1 + 2\xi_3^2\xi_3^1\xi_1^1 - (\xi_1^2\xi_3^1 + 2\xi_4^1\xi_1^1)\xi_3^1)\xi_4^2 - (\xi_3^2 - \xi_4^1)^2\xi_5^5\xi_3^2 - (2{\xi_3^2}^3\xi_1^1 - 2{\xi_3^2}^2\xi_1^2\xi_3^1 - 2{\xi_3^2}^2\xi_4^1\xi_1^1 + 3\xi_3^2\xi_1^2\xi_4^1\xi_3^1 - 2\xi_1^2{\xi_4^1}^2\xi_3^1))\xi_3^6\xi_5^5 - (({\xi_4^2}^2\xi_1^1 - \xi_4^2\xi_1^2\xi_4^1 + {\xi_3^2}^2\xi_1^1 - \xi_3^2\xi_1^2\xi_3^1 - ({\xi_4^2}^2 - {\xi_3^2}^2 + 2\xi_3^2\xi_4^1)\xi_5^5)\xi_2^6 + (\xi_4^2\xi_4^1\xi_1^1 + \xi_3^2\xi_3^1\xi_1^1 - \xi_1^2{\xi_4^1}^2 - \xi_1^2{\xi_3^1}^2 - (\xi_4^2\xi_4^1 - \xi_3^2\xi_3^1 + 2\xi_4^1\xi_3^1)\xi_5^5)\xi_1^6)(\xi_4^2\xi_3^1 - \xi_3^2\xi_4^1) - ((((\xi_4^2{\xi_1^1}^2 - \xi_4^2 - \xi_1^2\xi_4^1\xi_1^1 - ({\xi_1^1}^2 + 1)\xi_3^1)\xi_3^2 - (\xi_1^2\xi_3^1\xi_1^1 + \xi_4^1{\xi_1^1}^2 - \xi_4^1)\xi_4^2 + (\xi_1^2\xi_4^1\xi_3^1 + 2{\xi_4^1}^2\xi_1^1 + {\xi_3^1}^2\xi_1^1)\xi_1^2)\xi_4^2 + ({\xi_3^2}^2{\xi_1^1}^2 - 2\xi_3^2\xi_1^2\xi_3^1\xi_1^1 + {\xi_1^2}^2{\xi_3^1}^2 + \xi_1^2\xi_4^1\xi_3^1\xi_1^1 + {\xi_4^1}^2)\xi_3^2 - ({\xi_4^1}^2 + {\xi_3^1}^2){\xi_1^2}^2\xi_4^1)\xi_3^6){\xi_5^6}^2)/((\xi_4^2\xi_3^1 - \xi_3^2\xi_4^1)^2{\xi_5^6}^3);$

$J_3^5 = (-((((\xi_5^6\xi_3^2 - \xi_5^6\xi_4^1 - \xi_5^5\xi_4^2 - \xi_5^5\xi_3^1)\xi_5^6\xi_4^6 - ({\xi_5^5}^2 + 1)(\xi_4^2 + \xi_3^1)\xi_3^6)\xi_3^2 - (\xi_3^2\xi_1^2\xi_4^1 - \xi_3^2\xi_3^1\xi_1^1 + \xi_1^2{\xi_3^1}^2 - \xi_4^2\xi_3^2\xi_1^1)\xi_5^6\xi_4^6 + ((\xi_2^6\xi_3^2 + \xi_1^6\xi_3^1)(\xi_4^2\xi_3^1 - \xi_3^2\xi_4^1) - (\xi_4^2\xi_3^1 - {\xi_3^2}^2)\xi_3^6\xi_5^5 + ({\xi_3^2}^2\xi_1^1 - \xi_3^2\xi_1^2\xi_3^1 + \xi_1^2\xi_4^1\xi_3^1 - \xi_4^2\xi_3^1\xi_1^1)\xi_3^6)\xi_5^6))/((\xi_4^2\xi_3^1 - \xi_3^2\xi_4^1){\xi_5^6}^2);$

$J_4^5 = (-(((\xi_5^6\xi_4^2\xi_3^2 - \xi_5^6\xi_4^2\xi_4^1 - \xi_5^5{\xi_4^2}^2 - 2\xi_5^5\xi_4^2\xi_3^1 + \xi_5^5\xi_3^2\xi_4^1)\xi_5^6\xi_4^6 - ({\xi_5^5}^2 + 1)(\xi_4^2 + \xi_3^1)\xi_3^6\xi_4^2 + ((\xi_4^2\xi_1^1 - \xi_1^2\xi_4^1)\xi_4^2 + (\xi_3^2\xi_1^1 - \xi_1^2\xi_3^1)\xi_4^1)\xi_5^6\xi_4^6 + ((\xi_4^2\xi_3^2\xi_1^1 - \xi_4^2\xi_1^2\xi_3^1 - \xi_4^2\xi_4^1\xi_1^1 + \xi_1^2{\xi_4^1}^2 + (\xi_3^2 - \xi_4^1)\xi_5^5\xi_4^2)\xi_3^6 + (\xi_2^6\xi_4^2 + \xi_1^6\xi_4^1)(\xi_4^2\xi_3^1 - \xi_3^2\xi_4^1))\xi_5^6))/((\xi_4^2\xi_3^1 - \xi_3^2\xi_4^1){\xi_5^6}^2);$

and the parameters are subject to the condition

$$\xi_5^6({\xi_4^2}^2 + {\xi_3^2}^2)(\xi_4^2\xi_3^1 - \xi_3^2\xi_4^1) \neq 0. \tag{99}$$

13.2 Case $\xi_1^2 \neq 0$.

13.2.1 Case $\xi_1^2\xi_4^2 \neq 0$.

$$J = \begin{pmatrix} * & * & * & * & 0 & 0 \\ \boxed{\xi_1^2} & \boxed{\xi_2^2} & \boxed{\xi_3^2} & \boxed{\xi_4^2} & 0 & 0 \\ * & * & \boxed{\xi_3^3} & \boxed{\xi_4^3} & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & \boxed{\xi_5^5} & -\frac{{\xi_5^5}^2+1}{\xi_5^6} \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & \boxed{\xi_5^6} & -\xi_5^5 \end{pmatrix}$$

where $J_1^1 = (((\xi_5^6\xi_4^3\xi_4^2\xi_3^2 - \xi_5^6\xi_3^3{\xi_4^2}^2 + 2\xi_5^6\xi_4^2\xi_3^2\xi_1^2 + \xi_5^5\xi_4^3{\xi_3^2}^2 - \xi_5^5{\xi_4^2}^2\xi_1^2 + \xi_5^5\xi_4^2\xi_3^2\xi_2^2 + \xi_5^5{\xi_3^2}^2\xi_1^2)\xi_4^3 - (\xi_3^3\xi_4^2 - \xi_3^2\xi_1^2)\xi_5^6\xi_4^2\xi_1^2 - ((\xi_3^3 + \xi_2^2)\xi_3^3\xi_4^2 + (\xi_4^2\xi_1^2 - \xi_3^2\xi_2^2)\xi_1^2)\xi_5^5\xi_4^2)\xi_5^6 - ((\xi_3^3\xi_3^2 + \xi_4^2\xi_1^2)\xi_4^2 - \xi_4^3{\xi_3^2}^2)({\xi_5^5}^2 + 1)(\xi_3^3 + \xi_2^2) + ((\xi_4^3\xi_4^2\xi_3^2\xi_1^2 + \xi_4^3{\xi_3^2}^2\xi_2^2 - \xi_3^3{\xi_4^2}^2\xi_1^2 - 2\xi_3^3\xi_4^2\xi_3^2\xi_2^2 + \xi_3^3{\xi_3^2}^2\xi_1^2 + (\xi_4^2{\xi_1^2}^2 + \xi_4^2 + \xi_3^2\xi_2^2\xi_1^2)\xi_3^2)\xi_4^3 + ({\xi_3^3}^2\xi_4^2\xi_2^2 - {\xi_3^3}^2\xi_3^2\xi_1^2 - \xi_3^3\xi_4^2{\xi_1^2}^2 - \xi_3^3\xi_4^2 - \xi_3^3\xi_3^2\xi_2^2\xi_1^2 + \xi_3^2\xi_1^2)\xi_4^2)\xi_5^6)/((((\xi_5^6\xi_4^3\xi_4^2 + \xi_5^6\xi_4^2\xi_1^2 + \xi_5^5\xi_4^3\xi_3^2)\xi_5^6 + ({\xi_5^5}^2 + 1)(\xi_3^3 + \xi_2^2)\xi_3^2 + (\xi_4^2\xi_2^2 + \xi_3^2\xi_1^2 + \xi_3^3\xi_4^2)\xi_5^6\xi_5^5)\xi_4^2 - ((\xi_3^3\xi_4^2\xi_2^2 - \xi_3^3\xi_3^2\xi_1^2 - \xi_4^2)\xi_4^2 - (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_4^3\xi_3^2)\xi_5^6);$

$J_2^1 = -({\xi_5^6}^2{\xi_4^3}^2{\xi_3^2}^2 - {\xi_5^6}^2\xi_4^3\xi_3^3\xi_4^2\xi_3^2 - {\xi_5^6}^2\xi_4^3\xi_4^2\xi_3^2\xi_2^2 + {\xi_5^6}^2\xi_4^3{\xi_3^2}^2\xi_1^2 - {\xi_5^6}^2\xi_3^3\xi_4^2\xi_3^2\xi_1^2 - {\xi_5^6}^2\xi_4^2\xi_3^2\xi_2^2\xi_1^2 - \xi_5^6\xi_5^5{\xi_4^3}^2\xi_4^2\xi_3^2 + \xi_5^6\xi_5^5\xi_4^3\xi_3^3{\xi_4^2}^2 + \xi_5^6\xi_5^5\xi_4^3\xi_3^3{\xi_3^2}^2 + \xi_5^6\xi_5^5\xi_4^3{\xi_4^2}^2\xi_2^2 - \xi_5^6\xi_5^5\xi_4^3\xi_4^2\xi_3^2\xi_1^2 + \xi_5^6\xi_5^5\xi_4^3{\xi_3^2}^2\xi_2^2 - \xi_5^6\xi_5^5{\xi_3^3}^2\xi_4^2\xi_3^2 + \xi_5^6\xi_5^5\xi_3^3{\xi_4^2}^2\xi_1^2 - 2\xi_5^6\xi_5^5\xi_3^3\xi_4^2\xi_3^2\xi_2^2 + \xi_5^6\xi_5^5{\xi_4^2}^2\xi_2^2\xi_1^2 - \xi_5^6\xi_5^5\xi_4^2\xi_3^2{\xi_2^2}^2 - \xi_5^6{\xi_4^3}^2\xi_4^2\xi_3^2\xi_2^2 + \xi_5^6{\xi_4^3}^2{\xi_3^2}^2\xi_1^2 + \xi_5^6\xi_4^3\xi_3^3{\xi_4^2}^2\xi_2^2 - 2\xi_5^6\xi_4^3\xi_3^3\xi_4^2\xi_3^2\xi_1^2 - \xi_5^6\xi_4^3\xi_3^3{\xi_3^2}^2\xi_2^2 - \xi_5^6\xi_4^3{\xi_4^2}^2 - \xi_5^6\xi_4^3\xi_4^2\xi_3^2\xi_2^2\xi_1^2 - \xi_5^6\xi_4^3{\xi_3^2}^2{\xi_2^2}^2 + \xi_5^6{\xi_3^3}^2{\xi_4^2}^2\xi_1^2 + \xi_5^6{\xi_3^3}^2\xi_4^2\xi_3^2\xi_2^2 + \xi_5^6\xi_3^3{\xi_4^2}^2\xi_2^2\xi_1^2 + \xi_5^6\xi_3^3\xi_4^2\xi_3^2{\xi_2^2}^2 - \xi_5^6\xi_3^3\xi_4^2\xi_3^2 - \xi_5^6\xi_4^2\xi_3^2\xi_2^2 - {\xi_5^5}^2\xi_4^3\xi_3^3\xi_4^2\xi_3^2 - {\xi_5^5}^2\xi_4^3\xi_4^2\xi_3^2\xi_2^2 + {\xi_5^5}^2{\xi_3^3}^2{\xi_4^2}^2 + 2{\xi_5^5}^2\xi_3^3{\xi_4^2}^2\xi_2^2 + {\xi_5^5}^2{\xi_4^2}^2{\xi_2^2}^2 - \xi_4^3\xi_3^3\xi_4^2\xi_3^2 - \xi_4^3\xi_4^2\xi_3^2\xi_2^2 + {\xi_3^3}^2{\xi_4^2}^2 + 2\xi_3^3{\xi_4^2}^2\xi_2^2 + {\xi_4^2}^2{\xi_2^2}^2)/({\xi_5^6}^2\xi_4^3{\xi_4^2}^2 + {\xi_5^6}^2{\xi_4^2}^2\xi_1^2 + \xi_5^6\xi_5^5\xi_4^3\xi_4^2\xi_3^2 + \xi_5^6\xi_5^5\xi_3^3{\xi_4^2}^2 + \xi_5^6\xi_5^5{\xi_4^2}^2\xi_2^2 + \xi_5^6\xi_5^5\xi_4^2\xi_3^2\xi_1^2 + \xi_5^6\xi_4^3\xi_4^2\xi_3^2\xi_2^2 - \xi_5^6\xi_4^3{\xi_3^2}^2\xi_1^2 - \xi_5^6\xi_3^3{\xi_4^2}^2\xi_2^2 + \xi_5^6\xi_3^3\xi_4^2\xi_3^2\xi_1^2 + \xi_5^6{\xi_4^2}^2 + {\xi_5^5}^2\xi_3^3\xi_4^2\xi_3^2 + {\xi_5^5}^2\xi_4^2\xi_3^2\xi_2^2 + \xi_3^3\xi_4^2\xi_3^2 + \xi_4^2\xi_3^2\xi_2^2);$

$J_3^1 = (((\xi_5^6\xi_4^3\xi_3^2 + \xi_5^6\xi_3^2\xi_1^2 - \xi_5^5\xi_4^3\xi_4^2)\xi_5^6 - ({\xi_5^5}^2 + 1)(\xi_3^3 + \xi_2^2)\xi_4^2 - (\xi_4^2\xi_1^2 - \xi_3^2\xi_2^2 - \xi_3^3\xi_3^2)\xi_5^6\xi_5^5)\xi_4^2\xi_3^2 + ((({\xi_4^2}^2 + {\xi_3^2}^2)\xi_3^3 + (\xi_4^2\xi_1^2 + \xi_3^2\xi_2^2)\xi_3^2)\xi_4^3\xi_3^2 - ({\xi_3^3}^2{\xi_4^2}^2 + {\xi_3^3}^2{\xi_3^2}^2 + \xi_3^3\xi_4^2\xi_3^2\xi_1^2 + \xi_3^3{\xi_3^2}^2\xi_2^2 + {\xi_4^2}^2)\xi_4^2)\xi_5^6)/((((\xi_5^6\xi_4^3\xi_4^2 + \xi_5^6\xi_4^2\xi_1^2 + \xi_5^5\xi_4^3\xi_3^2)\xi_5^6 + ({\xi_5^5}^2 + 1)(\xi_3^3 + \xi_2^2)\xi_3^2 + (\xi_4^2\xi_2^2 + \xi_3^2\xi_1^2 + \xi_3^3\xi_4^2)\xi_5^6\xi_5^5)\xi_4^2 - ((\xi_3^3\xi_4^2\xi_2^2 - \xi_3^3\xi_3^2\xi_1^2 - \xi_4^2)\xi_4^2 - (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_4^3\xi_3^2)\xi_5^6);$

$J_4^1 = (((\xi_5^6\xi_4^3\xi_3^2 + \xi_5^6\xi_3^2\xi_1^2 - \xi_5^5\xi_4^3\xi_4^2)\xi_5^6 - ({\xi_5^5}^2 + 1)(\xi_3^3 + \xi_2^2)\xi_4^2 - (\xi_4^2\xi_1^2 - \xi_3^2\xi_2^2 - \xi_3^3\xi_3^2)\xi_5^6\xi_5^5){\xi_4^2}^2 + (((\xi_4^3\xi_3^2 - \xi_3^3\xi_4^2)({\xi_4^2}^2 + {\xi_3^2}^2) + (\xi_4^2\xi_1^2 + \xi_3^2\xi_2^2)\xi_4^2\xi_3^2)\xi_4^3 - (\xi_3^3\xi_4^2\xi_1^2 + \xi_3^3\xi_3^2\xi_2^2 - \xi_3^2){\xi_4^2}^2)\xi_5^6)/((((\xi_5^6\xi_4^3\xi_4^2 + \xi_5^6\xi_4^2\xi_1^2 + \xi_5^5\xi_4^3\xi_3^2)\xi_5^6 + ({\xi_5^5}^2 + 1)(\xi_3^3 + \xi_2^2)\xi_3^2 + (\xi_4^2\xi_2^2 + \xi_3^2\xi_1^2 + \xi_3^3\xi_4^2)\xi_5^6\xi_5^5)\xi_4^2 - ((\xi_3^3\xi_4^2\xi_2^2 - \xi_3^3\xi_3^2\xi_1^2 - \xi_4^2)\xi_4^2 - (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_4^3\xi_3^2)\xi_5^6);$

$J_1^3 = ((\xi_5^6\xi_4^3\xi_4^2 + \xi_5^6\xi_4^2\xi_1^2 + \xi_5^5\xi_4^3\xi_3^2)\xi_5^6 + ({\xi_5^5}^2 + 1)(\xi_3^3 + \xi_2^2)\xi_3^2 + (\xi_4^2\xi_2^2 + \xi_3^2\xi_1^2 + \xi_3^3\xi_4^2)\xi_5^6\xi_5^5 - (\xi_3^3\xi_4^2\xi_2^2 - \xi_3^3\xi_3^2\xi_1^2 - \xi_4^2 - (\xi_4^2\xi_1^2 + \xi_3^2\xi_2^2)\xi_4^3)\xi_5^6)/(({\xi_4^2}^2 + {\xi_3^2}^2)\xi_5^6);$

$J_2^3 = (-((\xi_5^6\xi_4^3\xi_3^2 + \xi_5^6\xi_3^2\xi_1^2 - \xi_5^5\xi_4^3\xi_4^2)\xi_5^6 - ({\xi_5^5}^2 + 1)(\xi_3^3 + \xi_2^2)\xi_4^2 - (\xi_4^2\xi_1^2 - \xi_3^2\xi_2^2 - \xi_3^3\xi_3^2)\xi_5^6\xi_5^5 - (\xi_3^3\xi_4^2\xi_1^2 + \xi_3^3\xi_3^2\xi_2^2 - \xi_3^2 + (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_4^3)\xi_5^6))/(({\xi_4^2}^2 + {\xi_3^2}^2)\xi_5^6);$

$J_1^4 = ((({\xi_5^6}^3{\xi_4^3}^2{\xi_4^2}^2 + 2{\xi_5^6}^3\xi_4^3{\xi_4^2}^2\xi_1^2 + {\xi_5^6}^3{\xi_4^2}^2{\xi_1^2}^2 + 2{\xi_5^6}^2\xi_5^5{\xi_4^3}^2\xi_4^2\xi_3^2 + 2{\xi_5^6}^2\xi_5^5\xi_4^3\xi_3^3{\xi_4^2}^2 + 2{\xi_5^6}^2\xi_5^5\xi_4^3{\xi_4^2}^2\xi_2^2 + 4{\xi_5^6}^2\xi_5^5\xi_4^3\xi_4^2\xi_3^2\xi_1^2 + \xi_5^6{\xi_5^5}^2{\xi_4^3}^2{\xi_3^2}^2 + 4\xi_5^6{\xi_5^5}^2\xi_4^3\xi_3^3\xi_4^2\xi_3^2 + 4\xi_5^6{\xi_5^5}^2\xi_4^3\xi_4^2\xi_3^2\xi_2^2 + 2\xi_5^6{\xi_5^5}^2\xi_4^3{\xi_3^2}^2\xi_1^2 + 2\xi_5^6\xi_5^5{\xi_4^3}^2\xi_4^2\xi_3^2\xi_1^2 + 2\xi_5^6\xi_5^5{\xi_4^3}^2{\xi_3^2}^2\xi_2^2 + \xi_5^6\xi_5^5\xi_4^3\xi_3^3{\xi_4^2}^2\xi_1^2 + \xi_5^6\xi_5^5\xi_4^3\xi_3^3{\xi_3^2}^2\xi_1^2 + 2{\xi_5^5}^3\xi_4^3\xi_3^3{\xi_3^2}^2 + 2{\xi_5^5}^3\xi_4^3{\xi_3^2}^2\xi_2^2 + 2{\xi_5^5}^2\xi_4^3\xi_3^3\xi_4^2\xi_3^2\xi_1^2 + 2{\xi_5^5}^2\xi_4^3\xi_3^3{\xi_3^2}^2\xi_2^2 + 2{\xi_5^5}^2\xi_4^3\xi_4^2\xi_3^2\xi_2^2\xi_1^2 + 2{\xi_5^5}^2\xi_4^3{\xi_3^2}^2{\xi_2^2}^2 + 2\xi_5^5\xi_4^3\xi_3^3{\xi_3^2}^2 + 2\xi_5^5\xi_4^3{\xi_3^2}^2\xi_2^2)\xi_3^2 - ({\xi_4^2}^3{\xi_1^2}^2 - 3{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2 - 2\xi_4^2{\xi_3^2}^2{\xi_2^2}^2 - \xi_4^2{\xi_3^2}^2{\xi_1^2}^2 - 2\xi_4^2{\xi_3^2}^2 - 3{\xi_3^2}^3\xi_2^2\xi_1^2)\xi_5^6\xi_5^5\xi_4^3 + 2({\xi_5^5}^2 + 1)(\xi_3^3\xi_4^2 + \xi_4^2\xi_2^2 + \xi_3^2\xi_1^2)(\xi_3^3 + \xi_2^2)\xi_5^5{\xi_3^2}^2 - (\xi_3^3{\xi_4^2}^2\xi_3^2\xi_1^2 + 2\xi_3^3\xi_4^2{\xi_3^2}^2\xi_2^2 - \xi_3^3{\xi_3^2}^3\xi_1^2 + {\xi_4^2}^3{\xi_1^2}^2 - {\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2 + \xi_4^2{\xi_3^2}^2{\xi_1^2}^2 - 2\xi_4^2{\xi_3^2}^2 - {\xi_3^2}^3\xi_2^2\xi_1^2)(\xi_3^3 + \xi_2^2){\xi_5^5}^2 + (2\xi_4^3{\xi_4^2}^2\xi_3^2\xi_1^2 + 2\xi_4^3\xi_4^2{\xi_3^2}^2\xi_2^2 - \xi_3^3{\xi_4^2}^3\xi_1^2 - 2\xi_3^3{\xi_4^2}^2\xi_3^2\xi_2^2 + \xi_3^3\xi_4^2{\xi_3^2}^2\xi_1^2 + {\xi_4^2}^3\xi_2^2\xi_1^2 + {\xi_4^2}^2\xi_3^2{\xi_1^2}^2 + 2{\xi_4^2}^2\xi_3^2 + \xi_4^2{\xi_3^2}^2\xi_2^2\xi_1^2 + {\xi_3^2}^3{\xi_1^2}^2)(\xi_4^3 + \xi_1^2){\xi_5^6}^2)\xi_5^6 + ({\xi_5^5}^2 + 1)^2(\xi_3^3 + \xi_2^2)^2{\xi_3^2}^3 +$

$2(\xi_4^2\xi_2^2+\xi_3^2\xi_1^2+\xi_3^3\xi_4^2){\xi_5^6}^3\xi_5^5\xi_4^2\xi_3^2\xi_1^2+((\xi_3^3\xi_4^2+2\xi_4^2\xi_2^2+4\xi_3^2\xi_1^2)\xi_3^3\xi_4^2+{\xi_4^2}^2{\xi_2^2}^2+4\xi_4^2\xi_3^2\xi_2^2\xi_1^2+{\xi_3^2}^2{\xi_1^2}^2){\xi_5^6}^2{\xi_5^5}^2\xi_3^2-(\xi_3^3{\xi_4^2}^2\xi_3^2\xi_1^2+2\xi_3^3\xi_4^2{\xi_3^2}^2\xi_2^2-$
$\xi_3^3{\xi_3^2}^3\xi_1^2+{\xi_4^2}^3{\xi_1^2}^2-{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2+\xi_4^2{\xi_3^2}^2{\xi_1^2}^2-2\xi_4^2{\xi_3^2}^2-{\xi_3^2}^3\xi_2^2\xi_1^2-2(\xi_4^2\xi_1^2+\xi_3^2\xi_2^2)\xi_4^3{\xi_3^2}^2)(\xi_3^3+\xi_2^2)\xi_5^6-((\xi_3^3{\xi_4^2}^3\xi_1^2+2\xi_3^3{\xi_4^2}^2\xi_3^2\xi_2^2-$
$\xi_3^3\xi_4^2{\xi_3^2}^2\xi_1^2+2{\xi_4^2}^2\xi_3^2{\xi_2^2}^2-2{\xi_4^2}^2\xi_3^2-2{\xi_3^2}^3{\xi_1^2}^2)\xi_3^3-({\xi_4^2}^3{\xi_2^2}^2\xi_1^2-{\xi_4^2}^3{\xi_1^2}^3+2{\xi_4^2}^2\xi_3^2\xi_2^2{\xi_1^2}^2+2{\xi_4^2}^2\xi_3^2\xi_2^2+\xi_4^2{\xi_3^2}^2{\xi_2^2}^2\xi_1^2-\xi_4^2{\xi_3^2}^2{\xi_1^2}^3+$
$2\xi_4^2{\xi_3^2}^2\xi_1^2+2{\xi_3^2}^3\xi_2^2{\xi_1^2}^2)){\xi_5^6}^2\xi_5^5+((\xi_3^3{\xi_4^2}^2\xi_2^2\xi_1^2+\xi_3^3\xi_4^2\xi_3^2{\xi_2^2}^2-\xi_3^3\xi_4^2\xi_3^2{\xi_1^2}^2-\xi_3^3{\xi_3^2}^2\xi_2^2\xi_1^2-{\xi_4^2}^2{\xi_2^2}^2\xi_1^2-{\xi_4^2}^2{\xi_1^2}^3-{\xi_4^2}^2\xi_1^2-2\xi_4^2\xi_3^2\xi_2^2-$
${\xi_3^2}^2{\xi_2^2}^2\xi_1^2-{\xi_3^2}^2{\xi_1^2}^3+3{\xi_3^2}^2\xi_1^2)\xi_3^3\xi_4^2+{\xi_4^2}^3\xi_2^2\xi_1^2+{\xi_4^2}^2\xi_3^2{\xi_1^2}^2+{\xi_4^2}^2\xi_3^2+3\xi_4^2{\xi_3^2}^2\xi_2^2\xi_1^2+{\xi_3^2}^3{\xi_1^2}^2+(\xi_4^3{\xi_4^2}^2\xi_3^2{\xi_1^2}^2+2\xi_4^3\xi_4^2{\xi_3^2}^2\xi_2^2\xi_1^2+\xi_4^3{\xi_3^2}^3{\xi_2^2}^2-$
$\xi_3^3{\xi_4^2}^3{\xi_1^2}^2-3\xi_3^3{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2-2\xi_3^3\xi_4^2{\xi_3^2}^2{\xi_2^2}^2+\xi_3^3\xi_4^2{\xi_3^2}^2{\xi_1^2}^2+2\xi_3^3\xi_4^2{\xi_3^2}^2+\xi_3^3{\xi_3^2}^3\xi_2^2\xi_1^2)\xi_4^3+(((\xi_4^2\xi_2^2\xi_1^2+4\xi_3^2)\xi_2^2+({\xi_1^2}^2+2)\xi_4^2\xi_1^2)\xi_4^2+({\xi_2^2}^2+$
${\xi_1^2}^2){\xi_3^2}^2\xi_1^2)\xi_4^3\xi_3^2){\xi_5^6}^2)/(((((\xi_4^2\xi_2^2-\xi_3^2\xi_1^2)\xi_3^3-\xi_4^2)\xi_4^2-(\xi_4^2\xi_2^2-\xi_3^2\xi_1^2)\xi_4^3\xi_3^2)\xi_5^6-({\xi_5^6}^2\xi_4^3\xi_4^2+{\xi_5^6}^2\xi_4^2\xi_1^2+\xi_5^6\xi_5^5\xi_4^3\xi_3^2+\xi_5^6\xi_5^5\xi_3^3\xi_4^2+\xi_5^6\xi_5^5\xi_4^2\xi_2^2+$
$\xi_5^6\xi_5^5\xi_3^2\xi_1^2+{\xi_5^5}^2\xi_3^3\xi_3^2+{\xi_5^5}^2\xi_3^2\xi_2^2+\xi_3^3\xi_3^2+\xi_3^2\xi_2^2)\xi_4^2)({\xi_4^2}^2+{\xi_3^2}^2)\xi_5^6);$

$J_2^4=(-(({\xi_5^6}^3{\xi_4^3}^2\xi_4^2{\xi_3^2}^2+2{\xi_5^6}^3\xi_4^3\xi_4^2{\xi_3^2}^2\xi_1^2+{\xi_5^6}^3\xi_4^2{\xi_3^2}^2{\xi_1^2}^2-{\xi_5^6}^2\xi_5^5{\xi_4^3}^2{\xi_4^2}^2\xi_3^2+{\xi_5^6}^2\xi_5^5{\xi_4^3}^2{\xi_3^2}^3+2{\xi_5^6}^2\xi_5^5\xi_4^3\xi_3^3\xi_4^2{\xi_3^2}^2-2{\xi_5^6}^2\xi_5^5\xi_4^3{\xi_4^2}^2\xi_3^2\xi_1^2+$
$2{\xi_5^6}^2\xi_5^5\xi_4^3\xi_4^2{\xi_3^2}^2\xi_2^2+2{\xi_5^6}^2\xi_5^5\xi_4^3{\xi_3^2}^3\xi_1^2-\xi_5^6{\xi_5^5}^2{\xi_4^3}^2\xi_4^2{\xi_3^2}^2-2\xi_5^6{\xi_5^5}^2\xi_4^3\xi_3^3{\xi_4^2}^2\xi_3^2+2\xi_5^6{\xi_5^5}^2\xi_4^3\xi_3^3{\xi_3^2}^3-2\xi_5^6{\xi_5^5}^2\xi_4^3{\xi_4^2}^2\xi_3^2\xi_2^2-2\xi_5^6{\xi_5^5}^2\xi_4^3\xi_4^2{\xi_3^2}^2\xi_1^2+$
$2\xi_5^6{\xi_5^5}^2\xi_4^3{\xi_3^2}^3\xi_2^2-\xi_5^6\xi_5^5{\xi_4^3}^2{\xi_4^2}^2\xi_3^2\xi_1^2-2\xi_5^6\xi_5^5{\xi_4^3}^2\xi_4^2{\xi_3^2}^2\xi_2^2+\xi_5^6\xi_5^5{\xi_4^3}^2{\xi_3^2}^3\xi_1^2+\xi_5^6\xi_5^5\xi_4^3\xi_3^3{\xi_4^2}^3\xi_1^2+\xi_5^6\xi_5^5\xi_4^3\xi_3^3\xi_4^2{\xi_3^2}^2\xi_1^2+\xi_5^6\xi_5^5\xi_4^3{\xi_4^2}^3\xi_2^2\xi_1^2-$
$2\xi_5^6\xi_5^5\xi_4^3{\xi_4^2}^2\xi_3^2{\xi_2^2}^2-\xi_5^6\xi_5^5\xi_4^3{\xi_4^2}^2\xi_3^2{\xi_1^2}^2-2\xi_5^6\xi_5^5\xi_4^3{\xi_4^2}^2\xi_3^2+\xi_5^6\xi_5^5\xi_4^3\xi_4^2{\xi_3^2}^2\xi_2^2\xi_1^2+\xi_5^6\xi_5^5\xi_4^3{\xi_3^2}^3{\xi_1^2}^2-2{\xi_5^5}^3\xi_4^3\xi_3^3\xi_4^2{\xi_3^2}^2-2{\xi_5^5}^3\xi_4^3\xi_4^2{\xi_3^2}^2\xi_2^2-$
${\xi_5^5}^2\xi_4^3\xi_3^3{\xi_4^2}^2\xi_3^2\xi_1^2-2{\xi_5^5}^2\xi_4^3\xi_3^3\xi_4^2{\xi_3^2}^2\xi_2^2+{\xi_5^5}^2\xi_4^3\xi_3^3{\xi_3^2}^3\xi_1^2-{\xi_5^5}^2\xi_4^3{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2-2{\xi_5^5}^2\xi_4^3\xi_4^2{\xi_3^2}^2{\xi_2^2}^2+{\xi_5^5}^2\xi_4^3{\xi_3^2}^3\xi_2^2\xi_1^2-2\xi_5^5\xi_4^3\xi_3^3\xi_4^2{\xi_3^2}^2-$
$2\xi_5^5\xi_4^3\xi_4^2{\xi_3^2}^2\xi_2^2-({\xi_5^5}^2+1)(\xi_3^3{\xi_4^2}^2-\xi_3^3{\xi_3^2}^2+{\xi_4^2}^2\xi_2^2+2\xi_4^2\xi_3^2\xi_1^2-{\xi_3^2}^2\xi_2^2)(\xi_3^3+\xi_2^2)\xi_5^5\xi_3^2+(\xi_3^3{\xi_4^2}^3\xi_1^2+\xi_3^3{\xi_4^2}^2\xi_3^2\xi_2^2-\xi_3^3\xi_4^2{\xi_3^2}^2\xi_1^2-\xi_3^3{\xi_3^2}^3\xi_2^2+$
${\xi_4^2}^3\xi_2^2\xi_1^2-{\xi_4^2}^2\xi_3^2{\xi_2^2}^2-2{\xi_4^2}^2\xi_3^2+\xi_4^2{\xi_3^2}^2\xi_2^2\xi_1^2-{\xi_3^2}^3{\xi_2^2}^2)(\xi_3^3+\xi_2^2){\xi_5^5}^2-(\xi_4^3{\xi_4^2}^2\xi_3^2\xi_2^2-2\xi_4^3\xi_4^2{\xi_3^2}^2\xi_1^2-\xi_4^3{\xi_3^2}^3\xi_2^2+2\xi_3^3{\xi_4^2}^2\xi_3^2\xi_1^2+2\xi_3^3\xi_4^2{\xi_3^2}^2\xi_2^2+$
${\xi_4^2}^3{\xi_2^2}^2+{\xi_4^2}^3+{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2+\xi_4^2{\xi_3^2}^2{\xi_2^2}^2-\xi_4^2{\xi_3^2}^2+{\xi_3^2}^3\xi_2^2\xi_1^2)(\xi_4^3+\xi_1^2){\xi_5^6}^2)\xi_5^6-({\xi_5^5}^2+1)^2(\xi_3^3+\xi_2^2)^2\xi_4^2{\xi_3^2}^2-({\xi_4^2}^2\xi_1^2-2\xi_4^2\xi_3^2\xi_2^2-{\xi_3^2}^2\xi_1^2-$
$2\xi_3^3\xi_4^2\xi_3^2){\xi_5^6}^3\xi_5^5\xi_3^2\xi_1^2+((\xi_3^3\xi_4^2\xi_3^2-2{\xi_4^2}^2\xi_1^2+2\xi_4^2\xi_3^2\xi_2^2+2{\xi_3^2}^2\xi_1^2)\xi_3^3-(2{\xi_4^2}^2\xi_2^2\xi_1^2-\xi_4^2\xi_3^2{\xi_2^2}^2+\xi_4^2\xi_3^2{\xi_1^2}^2-2{\xi_3^2}^2\xi_2^2\xi_1^2)){\xi_5^6}^2{\xi_5^5}^2\xi_3^2-((2\xi_3^3{\xi_4^2}^2\xi_3^2\xi_1^2+$
$2\xi_3^3\xi_4^2{\xi_3^2}^2\xi_2^2+{\xi_4^2}^3{\xi_2^2}^2-{\xi_4^2}^3{\xi_1^2}^2+{\xi_4^2}^3+2{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2+3\xi_4^2{\xi_3^2}^2{\xi_2^2}^2+\xi_4^2{\xi_3^2}^2{\xi_1^2}^2-\xi_4^2{\xi_3^2}^2+2{\xi_3^2}^3\xi_2^2\xi_1^2)\xi_3^3+{\xi_4^2}^3{\xi_2^2}^3-{\xi_4^2}^3\xi_2^2{\xi_1^2}^2+{\xi_4^2}^3\xi_2^2+$
$2{\xi_4^2}^2\xi_3^2{\xi_2^2}^2\xi_1^2+2{\xi_4^2}^2\xi_3^2\xi_1^2+\xi_4^2{\xi_3^2}^2{\xi_2^2}^3-\xi_4^2{\xi_3^2}^2\xi_2^2{\xi_1^2}^2-\xi_4^2{\xi_3^2}^2\xi_2^2+2{\xi_3^2}^3{\xi_2^2}^2\xi_1^2){\xi_5^6}^2\xi_5^5+(\xi_3^3{\xi_4^2}^3\xi_1^2+\xi_3^3{\xi_4^2}^2\xi_3^2\xi_2^2-\xi_3^3\xi_4^2{\xi_3^2}^2\xi_1^2-\xi_3^3{\xi_3^2}^3\xi_2^2+$
${\xi_4^2}^3\xi_2^2\xi_1^2-{\xi_4^2}^2\xi_3^2{\xi_2^2}^2-2{\xi_4^2}^2\xi_3^2+\xi_4^2{\xi_3^2}^2\xi_2^2\xi_1^2-{\xi_3^2}^3{\xi_2^2}^2-({\xi_4^2}^2\xi_1^2+2\xi_4^2\xi_3^2\xi_2^2-{\xi_3^2}^2\xi_1^2)\xi_4^3\xi_3^2)(\xi_3^3+\xi_2^2)\xi_5^6-((({\xi_4^2}^2\xi_1^2+\xi_4^2\xi_3^2{\xi_2^2}^3+\xi_4^2\xi_3^2\xi_2^2{\xi_1^2}^2+$
$3\xi_4^2\xi_3^2\xi_2^2-{\xi_3^2}^2\xi_1^2)\xi_4^2+({\xi_2^2}^2+{\xi_1^2}^2-1){\xi_3^2}^3\xi_2^2+\xi_4^3{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2+\xi_4^3\xi_4^2{\xi_3^2}^2{\xi_2^2}^2-\xi_4^3\xi_4^2{\xi_3^2}^2{\xi_1^2}^2-\xi_4^3{\xi_3^2}^3\xi_2^2\xi_1^2-\xi_3^3{\xi_4^2}^3\xi_2^2\xi_1^2-\xi_3^3{\xi_4^2}^2\xi_3^2{\xi_2^2}^2+$
$2\xi_3^3{\xi_4^2}^2\xi_3^2{\xi_1^2}^2+\xi_3^3{\xi_4^2}^2\xi_3^2+3\xi_3^3\xi_4^2{\xi_3^2}^2\xi_2^2\xi_1^2+\xi_3^3{\xi_3^2}^3{\xi_2^2}^2-\xi_3^3{\xi_3^2}^3)\xi_4^3-((\xi_3^3{\xi_4^2}^2{\xi_1^2}^2+2\xi_3^3\xi_4^2\xi_3^2\xi_2^2\xi_1^2+\xi_3^3{\xi_3^2}^2{\xi_2^2}^2+{\xi_4^2}^2{\xi_2^2}^3+{\xi_4^2}^2\xi_2^2{\xi_1^2}^2+{\xi_4^2}^2\xi_2^2-$
$4\xi_4^2\xi_3^2\xi_1^2+{\xi_3^2}^2{\xi_2^2}^3+{\xi_3^2}^2\xi_2^2{\xi_1^2}^2-{\xi_3^2}^2\xi_2^2)\xi_3^3-({\xi_4^2}^2{\xi_2^2}^2+{\xi_4^2}^2+2\xi_4^2\xi_3^2\xi_2^2\xi_1^2+{\xi_3^2}^2{\xi_2^2}^2))\xi_4^2){\xi_5^6}^2))/(((((\xi_4^2\xi_2^2-\xi_3^2\xi_1^2)\xi_3^3-\xi_4^2)\xi_4^2-(\xi_4^2\xi_2^2-$
$\xi_3^2\xi_1^2)\xi_4^3\xi_3^2)\xi_5^6-({\xi_5^6}^2\xi_4^3\xi_4^2+{\xi_5^6}^2\xi_4^2\xi_1^2+\xi_5^6\xi_5^5\xi_4^3\xi_3^2+\xi_5^6\xi_5^5\xi_3^3\xi_4^2+\xi_5^6\xi_5^5\xi_4^2\xi_2^2+\xi_5^6\xi_5^5\xi_3^2\xi_1^2+{\xi_5^5}^2\xi_3^3\xi_3^2+{\xi_5^5}^2\xi_3^2\xi_2^2+\xi_3^3\xi_3^2+\xi_3^2\xi_2^2)\xi_4^2)({\xi_4^2}^2+{\xi_3^2}^2)\xi_5^6);$

$J_3^4=-(((\xi_5^6\xi_3^3\xi_4^2+\xi_5^6\xi_4^2\xi_2^2+\xi_5^6\xi_3^2\xi_1^2+\xi_5^5\xi_3^3\xi_3^2-\xi_5^5\xi_4^2\xi_1^2+\xi_5^5\xi_3^2\xi_2^2)\xi_5^6\xi_4^3+({\xi_5^5}^2+1)(\xi_3^3\xi_3^2-\xi_4^2\xi_1^2+\xi_3^2\xi_2^2)(\xi_3^3+\xi_2^2)+(\xi_4^2\xi_2^2+\xi_3^2\xi_1^2+$
$\xi_3^3\xi_4^2){\xi_5^6}^2\xi_1^2+((\xi_3^3\xi_4^2+2\xi_4^2\xi_2^2+2\xi_3^2\xi_1^2)\xi_3^3+\xi_4^2{\xi_2^2}^2-\xi_4^2{\xi_1^2}^2+2\xi_3^2\xi_2^2\xi_1^2)\xi_5^6\xi_5^5)\xi_3^2-(((\xi_3^3\xi_4^2\xi_1^2+\xi_3^3\xi_3^2\xi_2^2+\xi_3^2{\xi_2^2}^2+\xi_3^2{\xi_1^2}^2-\xi_3^2)\xi_3^3+\xi_4^2\xi_1^2-$
$\xi_3^2\xi_2^2)\xi_4^2-((\xi_4^2\xi_1^2+\xi_3^2\xi_2^2)\xi_3^3+({\xi_2^2}^2+{\xi_1^2}^2)\xi_3^2)\xi_4^3\xi_3^2)\xi_5^6)/(((\xi_5^6\xi_4^3\xi_4^2+\xi_5^6\xi_4^2\xi_1^2+\xi_5^5\xi_4^3\xi_3^2)\xi_5^6+({\xi_5^5}^2+1)(\xi_3^3+\xi_2^2)\xi_3^2+(\xi_4^2\xi_2^2+\xi_3^2\xi_1^2+$
$\xi_3^3\xi_4^2)\xi_5^6\xi_5^5)\xi_4^2-((\xi_3^3\xi_4^2\xi_2^2-\xi_3^3\xi_3^2\xi_1^2-\xi_4^2)\xi_4^2-(\xi_4^2\xi_2^2-\xi_3^2\xi_1^2)\xi_4^3\xi_3^2)\xi_5^6);$

$J_4^4=-(((\xi_5^6\xi_4^3\xi_4^2\xi_3^2+\xi_5^6{\xi_4^2}^2\xi_2^2+2\xi_5^6\xi_4^2\xi_3^2\xi_1^2+\xi_5^5\xi_4^3{\xi_3^2}^2+\xi_5^5\xi_3^3\xi_4^2\xi_3^2)\xi_4^3+(\xi_4^2\xi_2^2+\xi_3^2\xi_1^2)\xi_5^6\xi_4^2\xi_1^2-({\xi_4^2}^2\xi_1^2-2\xi_4^2\xi_3^2\xi_2^2-{\xi_3^2}^2\xi_1^2)\xi_5^5\xi_4^3)\xi_5^6-$
$((\xi_4^2\xi_1^2-\xi_3^2\xi_2^2)\xi_4^2-\xi_4^3{\xi_3^2}^2)({\xi_5^5}^2+1)(\xi_3^3+\xi_2^2)+(\xi_4^2{\xi_2^2}^2-\xi_4^2{\xi_1^2}^2+2\xi_3^2\xi_2^2\xi_1^2+(\xi_4^2\xi_2^2+\xi_3^2\xi_1^2)\xi_3^3)\xi_5^6\xi_5^5\xi_4^2+((({\xi_1^2}^2+1+{\xi_2^2}^2)\xi_4^2\xi_3^2+(\xi_4^3\xi_3^2-$
$\xi_3^3\xi_4^2)(\xi_4^2\xi_1^2+\xi_3^2\xi_2^2))\xi_4^3+(\xi_4^2\xi_2^2+\xi_3^2\xi_1^2-({\xi_2^2}^2+{\xi_1^2}^2)\xi_3^3\xi_4^2)\xi_4^2)\xi_5^6)/(((\xi_5^6\xi_4^3\xi_4^2+\xi_5^6\xi_4^2\xi_1^2+\xi_5^5\xi_4^3\xi_3^2)\xi_5^6+({\xi_5^5}^2+1)(\xi_3^3+\xi_2^2)\xi_3^2+(\xi_4^2\xi_2^2+$
$\xi_3^2\xi_1^2+\xi_3^3\xi_4^2)\xi_5^6\xi_5^5)\xi_4^2-((\xi_3^3\xi_4^2\xi_2^2-\xi_3^3\xi_3^2\xi_1^2-\xi_4^2)\xi_4^2-(\xi_4^2\xi_2^2-\xi_3^2\xi_1^2)\xi_4^3\xi_3^2)\xi_5^6);$

$J_1^5=(-(((({\xi_5^6}^3{\xi_4^3}^2{\xi_4^2}^2+2{\xi_5^6}^3\xi_4^3{\xi_4^2}^2\xi_1^2+{\xi_5^6}^3{\xi_4^2}^2{\xi_1^2}^2+2{\xi_5^6}^2\xi_5^5{\xi_4^3}^2\xi_4^2\xi_3^2+2{\xi_5^6}^2\xi_5^5\xi_4^3\xi_3^3{\xi_4^2}^2+2{\xi_5^6}^2\xi_5^5\xi_4^3{\xi_4^2}^2\xi_2^2+4{\xi_5^6}^2\xi_5^5\xi_4^3\xi_4^2\xi_3^2\xi_1^2+$
$\xi_5^6{\xi_5^5}^2{\xi_4^3}^2{\xi_3^2}^2+4\xi_5^6{\xi_5^5}^2\xi_4^3\xi_3^3\xi_4^2\xi_3^2+4\xi_5^6{\xi_5^5}^2\xi_4^3\xi_4^2\xi_3^2\xi_2^2+2\xi_5^6{\xi_5^5}^2\xi_4^3{\xi_3^2}^2\xi_1^2+2\xi_5^6\xi_5^5{\xi_4^3}^2\xi_4^2\xi_3^2\xi_1^2+2\xi_5^6\xi_5^5{\xi_4^3}^2{\xi_3^2}^2\xi_2^2+\xi_5^6\xi_5^5\xi_4^3\xi_3^3{\xi_4^2}^2\xi_1^2+$
$\xi_5^6\xi_5^5\xi_4^3\xi_3^3{\xi_3^2}^2\xi_1^2+2{\xi_5^5}^3\xi_4^3\xi_3^3{\xi_3^2}^2+2{\xi_5^5}^3\xi_4^3{\xi_3^2}^2\xi_2^2+2{\xi_5^5}^2\xi_4^3\xi_3^3\xi_4^2\xi_3^2\xi_1^2+2{\xi_5^5}^2\xi_4^3\xi_3^3{\xi_3^2}^2\xi_2^2+2{\xi_5^5}^2\xi_4^3\xi_4^2\xi_3^2\xi_2^2\xi_1^2+2{\xi_5^5}^2\xi_4^3{\xi_3^2}^2{\xi_2^2}^2+2\xi_5^5\xi_4^3\xi_3^3{\xi_3^2}^2+$
$2\xi_5^5\xi_4^3{\xi_3^2}^2\xi_2^2)\xi_3^2-({\xi_4^2}^3{\xi_1^2}^2-3{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2-2\xi_4^2{\xi_3^2}^2{\xi_2^2}^2-\xi_4^2{\xi_3^2}^2{\xi_1^2}^2-2\xi_4^2{\xi_3^2}^2-3{\xi_3^2}^3\xi_2^2\xi_1^2)\xi_5^6\xi_5^5\xi_4^3+2({\xi_5^5}^2+1)(\xi_3^3\xi_4^2+\xi_4^2\xi_2^2+\xi_3^2\xi_1^2)(\xi_3^3+$
$\xi_2^2)\xi_5^5{\xi_3^2}^2-(\xi_3^3{\xi_4^2}^2\xi_3^2\xi_1^2+2\xi_3^3\xi_4^2{\xi_3^2}^2\xi_2^2-\xi_3^3{\xi_3^2}^3\xi_1^2+{\xi_4^2}^3{\xi_1^2}^2-{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2+\xi_4^2{\xi_3^2}^2{\xi_1^2}^2-2\xi_4^2{\xi_3^2}^2-{\xi_3^2}^3\xi_2^2\xi_1^2)(\xi_3^3+\xi_2^2){\xi_5^5}^2)\xi_4^6-(\xi_4^3+$
$\xi_1^2)^2{\xi_5^6}^3\xi_4^6{\xi_4^2}^3+(2\xi_3^3{\xi_4^2}^2\xi_3^2\xi_1^2+2\xi_3^3\xi_4^2{\xi_3^2}^2\xi_2^2-\xi_3^3{\xi_4^2}^3\xi_1^2-2\xi_3^3{\xi_3^2}^2\xi_2^2+\xi_3^3\xi_4^2{\xi_3^2}^2\xi_1^2+{\xi_4^2}^3\xi_2^2\xi_1^2+{\xi_4^2}^2\xi_3^2{\xi_1^2}^2+2{\xi_4^2}^2\xi_3^2+\xi_4^2{\xi_3^2}^2\xi_2^2\xi_1^2+$
${\xi_3^2}^3{\xi_1^2}^2)(\xi_4^3+\xi_1^2){\xi_5^6}^2\xi_4^6)\xi_5^6+(\xi_6^6\xi_3^2-\xi_3^6\xi_4^2)({\xi_5^5}^2+1)^2(\xi_3^3+\xi_2^2)^2{\xi_3^2}^2+2(\xi_4^2\xi_2^2+\xi_3^2\xi_1^2+\xi_3^3\xi_4^2){\xi_5^6}^3\xi_4^6\xi_5^5\xi_4^2\xi_3^2\xi_1^2+((\xi_3^3\xi_4^2+2\xi_4^2\xi_2^2+4\xi_3^2\xi_1^2)\xi_3^3\xi_4^2+$
${\xi_4^2}^2{\xi_2^2}^2+4\xi_4^2\xi_3^2\xi_2^2\xi_1^2+{\xi_3^2}^2{\xi_1^2}^2){\xi_5^6}^2\xi_4^6{\xi_5^5}^2\xi_3^2-(\xi_3^3{\xi_4^2}^2\xi_3^2\xi_1^2+2\xi_3^3\xi_4^2{\xi_3^2}^2\xi_2^2-\xi_3^3{\xi_3^2}^3\xi_1^2+{\xi_4^2}^3{\xi_1^2}^2-{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2+\xi_4^2{\xi_3^2}^2{\xi_1^2}^2-2\xi_4^2{\xi_3^2}^2-{\xi_3^2}^3\xi_2^2\xi_1^2-$
$2(\xi_4^2\xi_1^2+\xi_3^2\xi_2^2)\xi_4^3{\xi_3^2}^2)(\xi_3^3+\xi_2^2)\xi_5^6\xi_4^6-((\xi_3^3{\xi_4^2}^3\xi_1^2+2\xi_3^3{\xi_4^2}^2\xi_3^2\xi_2^2-\xi_3^3\xi_4^2{\xi_3^2}^2\xi_1^2+2{\xi_4^2}^2\xi_3^2{\xi_2^2}^2-2{\xi_4^2}^2\xi_3^2-2{\xi_3^2}^3{\xi_1^2}^2)\xi_3^3-({\xi_4^2}^3{\xi_2^2}^2\xi_1^2-{\xi_4^2}^3{\xi_1^2}^3+$
$2{\xi_4^2}^2\xi_3^2\xi_2^2{\xi_1^2}^2+2{\xi_4^2}^2\xi_3^2\xi_2^2+\xi_4^2{\xi_3^2}^2{\xi_2^2}^2\xi_1^2-\xi_4^2{\xi_3^2}^2{\xi_1^2}^3+2\xi_4^2{\xi_3^2}^2\xi_1^2+2{\xi_3^2}^3\xi_2^2{\xi_1^2}^2)){\xi_5^6}^2\xi_4^6\xi_5^5+(((\xi_5^5\xi_4^2\xi_3^2-\xi_4^3{\xi_3^2}^2+\xi_3^3\xi_4^2\xi_3^2+{\xi_4^2}^2\xi_1^2)\xi_4^6-$
$\xi_2^6\xi_4^2\xi_3^2\xi_1^2)({\xi_4^2}^2+{\xi_3^2}^2)-(2\xi_5^5\xi_4^3\xi_4^2\xi_3^2+2\xi_5^5\xi_3^3{\xi_4^2}^2+2\xi_5^5{\xi_4^2}^2\xi_2^2+2\xi_5^5\xi_4^2\xi_3^2\xi_1^2+\xi_4^3{\xi_4^2}^2\xi_1^2+2\xi_4^3\xi_4^2\xi_3^2\xi_2^2-\xi_4^3{\xi_3^2}^2\xi_1^2-2\xi_3^3{\xi_4^2}^2\xi_2^2+2\xi_3^3\xi_4^2\xi_3^2\xi_1^2+$
$2{\xi_4^2}^2)\xi_6^6\xi_3^2)({\xi_5^5}^2+1)(\xi_3^3+\xi_2^2)\xi_5^6+((\xi_3^3{\xi_4^2}^2\xi_2^2\xi_1^2+\xi_3^3\xi_4^2\xi_3^2{\xi_2^2}^2-\xi_3^3\xi_4^2\xi_3^2{\xi_1^2}^2-\xi_3^3{\xi_3^2}^2\xi_2^2\xi_1^2-{\xi_4^2}^2{\xi_2^2}^2\xi_1^2-{\xi_4^2}^2{\xi_1^2}^3-{\xi_4^2}^2\xi_1^2-2\xi_4^2\xi_3^2\xi_2^2-$
${\xi_3^2}^2{\xi_2^2}^2\xi_1^2-{\xi_3^2}^2{\xi_1^2}^3+3{\xi_3^2}^2\xi_1^2)\xi_3^3\xi_4^2+{\xi_4^2}^3\xi_2^2\xi_1^2+{\xi_4^2}^2\xi_3^2{\xi_1^2}^2+{\xi_4^2}^2\xi_3^2+3\xi_4^2{\xi_3^2}^2\xi_2^2\xi_1^2+{\xi_3^2}^3{\xi_1^2}^2+(\xi_4^3{\xi_4^2}^2\xi_3^2{\xi_1^2}^2+2\xi_4^3\xi_4^2{\xi_3^2}^2\xi_2^2\xi_1^2+\xi_4^3{\xi_3^2}^3{\xi_2^2}^2-$
$\xi_3^3{\xi_4^2}^3{\xi_1^2}^2-3\xi_3^3{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2-2\xi_3^3\xi_4^2{\xi_3^2}^2{\xi_2^2}^2+\xi_3^3\xi_4^2{\xi_3^2}^2{\xi_1^2}^2+2\xi_3^3\xi_4^2{\xi_3^2}^2+\xi_3^3{\xi_3^2}^3\xi_2^2\xi_1^2)\xi_4^3+(((\xi_4^2\xi_2^2\xi_1^2+4\xi_3^2)\xi_2^2+({\xi_1^2}^2+2)\xi_4^2\xi_1^2)\xi_4^2+({\xi_2^2}^2+$
${\xi_1^2}^2){\xi_3^2}^2\xi_1^2)\xi_4^3\xi_3^2){\xi_5^6}^2\xi_4^6-(((\xi_4^3\xi_3^2-\xi_3^3\xi_4^2+\xi_3^2\xi_1^2-\xi_5^5\xi_4^2)\xi_1^6+\xi_6^6\xi_4^2\xi_1^2)({\xi_4^2}^2+{\xi_3^2}^2)+(2\xi_5^5\xi_4^3\xi_4^2\xi_3^2+2\xi_5^5\xi_3^3{\xi_4^2}^2+2\xi_5^5{\xi_4^2}^2\xi_2^2+2\xi_5^5\xi_4^2\xi_3^2\xi_1^2+$
$\xi_4^3{\xi_4^2}^2\xi_1^2+2\xi_4^3\xi_4^2\xi_3^2\xi_2^2-\xi_4^3{\xi_3^2}^2\xi_1^2-2\xi_3^3{\xi_4^2}^2\xi_2^2+2\xi_3^3\xi_4^2\xi_3^2\xi_1^2+2{\xi_4^2}^2)\xi_6^6)(\xi_4^3+\xi_1^2){\xi_5^6}^3\xi_4^2-((((\xi_5^5\xi_4^3\xi_4^2{\xi_3^2}^2+4\xi_5^5\xi_3^3{\xi_4^2}^2\xi_3^2+\xi_4^3{\xi_4^2}^2\xi_3^2\xi_1^2+$
$2\xi_4^3\xi_4^2{\xi_3^2}^2\xi_2^2-\xi_4^3{\xi_3^2}^3\xi_1^2+\xi_3^3{\xi_4^2}^3\xi_1^2+\xi_3^3\xi_4^2{\xi_3^2}^2\xi_1^2+2(2\xi_4^2\xi_2^2+\xi_3^2\xi_1^2)\xi_5^5\xi_4^2\xi_3^2+{\xi_4^2}^3\xi_2^2\xi_1^2+2{\xi_4^2}^2\xi_3^2{\xi_2^2}^2+{\xi_4^2}^2\xi_3^2{\xi_1^2}^2+2{\xi_4^2}^2\xi_3^2+\xi_4^2{\xi_3^2}^2\xi_2^2\xi_1^2-$
${\xi_3^2}^3{\xi_1^2}^2)\xi_3^3-2(\xi_3^3\xi_4^2\xi_2^2-\xi_3^3\xi_3^2\xi_1^2-\xi_4^2)(\xi_3^3\xi_4^2+\xi_4^2\xi_2^2+\xi_3^2\xi_1^2)\xi_4^2+((\xi_3^3\xi_4^2+2\xi_4^2\xi_2^2+4\xi_3^2\xi_1^2)\xi_3^3\xi_4^2+{\xi_4^2}^2{\xi_2^2}^2+4\xi_4^2\xi_3^2\xi_2^2\xi_1^2+{\xi_3^2}^2{\xi_1^2}^2)\xi_5^5\xi_4^2)\xi_5^5+$
$(\xi_4^3{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2+\xi_4^3\xi_4^2{\xi_3^2}^2{\xi_2^2}^2-\xi_4^3\xi_4^2{\xi_3^2}^2{\xi_1^2}^2-\xi_4^3{\xi_3^2}^3\xi_2^2\xi_1^2-\xi_3^3{\xi_4^2}^3\xi_2^2\xi_1^2-2\xi_3^3{\xi_4^2}^2\xi_3^2{\xi_2^2}^2+\xi_3^3{\xi_4^2}^2\xi_3^2{\xi_1^2}^2+2\xi_3^3{\xi_4^2}^2\xi_3^2+3\xi_3^3\xi_4^2{\xi_3^2}^2\xi_2^2\xi_1^2-$
$\xi_3^3{\xi_3^2}^3{\xi_1^2}^2+({\xi_4^2}^2\xi_1^2+4\xi_4^2\xi_3^2\xi_2^2-{\xi_3^2}^2\xi_1^2)\xi_4^2)\xi_4^3+((\xi_3^3{\xi_4^2}^2{\xi_2^2}^2-2\xi_3^3\xi_4^2\xi_3^2\xi_2^2\xi_1^2+\xi_3^3{\xi_3^2}^2{\xi_1^2}^2-2{\xi_4^2}^2\xi_2^2+4\xi_4^2\xi_3^2\xi_1^2)\xi_3^3+(\xi_4^2+2\xi_3^2\xi_2^2\xi_1^2)\xi_4^2)\xi_4^2)\xi_4^6-$
$((((\xi_4^2\xi_2^2+\xi_3^2\xi_1^2+\xi_3^3\xi_4^2)\xi_5^5\xi_4^2+(\xi_5^5\xi_4^2\xi_3^2-\xi_4^3{\xi_3^2}^2+{\xi_4^2}^2\xi_1^2-2{\xi_3^2}^2\xi_1^2)\xi_4^3+((\xi_3^3\xi_4^2+\xi_3^2\xi_1^2)\xi_3^3+\xi_4^2{\xi_1^2}^2+\xi_4^2-\xi_3^2\xi_2^2\xi_1^2)\xi_4^2)\xi_5^5-((\xi_3^3\xi_4^2\xi_3^2\xi_1^2+$
$\xi_4^3{\xi_3^2}^2\xi_2^2-\xi_3^3{\xi_4^2}^2\xi_1^2-2\xi_3^3\xi_4^2\xi_3^2\xi_2^2+\xi_3^3{\xi_3^2}^2\xi_1^2+(\xi_4^2{\xi_1^2}^2+\xi_4^2+\xi_3^2\xi_2^2\xi_1^2)\xi_3^2)\xi_4^3+((\xi_3^3\xi_4^2\xi_2^2-\xi_3^3\xi_3^2\xi_1^2-\xi_4^2{\xi_1^2}^2-\xi_4^2-\xi_3^2\xi_2^2\xi_1^2)\xi_3^3+\xi_3^2\xi_1^2)\xi_4^2))\xi_1^6-$
$((\xi_4^2\xi_2^2+\xi_3^2\xi_1^2+\xi_3^3\xi_4^2+\xi_4^3\xi_3^2)\xi_5^5\xi_4^2-(((\xi_4^2\xi_2^2-\xi_3^2\xi_1^2)\xi_3^3-\xi_4^2)\xi_4^2-(\xi_4^2\xi_2^2-\xi_3^2\xi_1^2)\xi_4^3\xi_3^2))\xi_6^6\xi_1^2)({\xi_4^2}^2+{\xi_3^2}^2)){\xi_5^6}^2))/(((((\xi_4^2\xi_2^2-\xi_3^2\xi_1^2)\xi_3^3-$
$\xi_4^2)\xi_4^2-(\xi_4^2\xi_2^2-\xi_3^2\xi_1^2)\xi_4^3\xi_3^2)\xi_5^6-({\xi_5^6}^2\xi_4^3\xi_4^2+{\xi_5^6}^2\xi_4^2\xi_1^2+\xi_5^6\xi_5^5\xi_4^3\xi_3^2+\xi_5^6\xi_5^5\xi_3^3\xi_4^2+\xi_5^6\xi_5^5\xi_4^2\xi_2^2+\xi_5^6\xi_5^5\xi_3^2\xi_1^2+{\xi_5^5}^2\xi_3^3\xi_3^2+{\xi_5^5}^2\xi_3^2\xi_2^2+\xi_3^3\xi_3^2+$
$\xi_3^2\xi_2^2)\xi_4^2)({\xi_4^2}^2+{\xi_3^2}^2){\xi_5^6}^2);$

$J_2^5=((({\xi_5^6}^3{\xi_4^3}^2\xi_4^2{\xi_3^2}^2+2{\xi_5^6}^3\xi_4^3\xi_4^2{\xi_3^2}^2\xi_1^2+{\xi_5^6}^3\xi_4^2{\xi_3^2}^2{\xi_1^2}^2-{\xi_5^6}^2\xi_5^5{\xi_4^3}^2{\xi_4^2}^2\xi_3^2+{\xi_5^6}^2\xi_5^5{\xi_4^3}^2{\xi_3^2}^3+2{\xi_5^6}^2\xi_5^5\xi_4^3\xi_3^3\xi_4^2{\xi_3^2}^2-2{\xi_5^6}^2\xi_5^5\xi_4^3{\xi_4^2}^2\xi_3^2\xi_1^2+$
$2{\xi_5^6}^2\xi_5^5\xi_4^3\xi_4^2{\xi_3^2}^2\xi_2^2+2{\xi_5^6}^2\xi_5^5\xi_4^3{\xi_3^2}^3\xi_1^2-\xi_5^6{\xi_5^5}^2{\xi_4^3}^2\xi_4^2{\xi_3^2}^2-2\xi_5^6{\xi_5^5}^2\xi_4^3\xi_3^3{\xi_4^2}^2\xi_3^2+2\xi_5^6{\xi_5^5}^2\xi_4^3\xi_3^3{\xi_3^2}^3-2\xi_5^6{\xi_5^5}^2\xi_4^3{\xi_4^2}^2\xi_3^2\xi_2^2-2\xi_5^6{\xi_5^5}^2\xi_4^3\xi_4^2{\xi_3^2}^2\xi_1^2+$

$2\xi_5^6{\xi_5^5}^2\xi_4^3{\xi_3^2}^3\xi_2^2 - \xi_5^6\xi_5^5{\xi_4^3}^2{\xi_4^2}^2\xi_3^2\xi_1^2 - 2\xi_5^6\xi_5^5{\xi_4^3}^2\xi_4^2{\xi_3^2}^2\xi_2^2 + \xi_5^6\xi_5^5{\xi_4^3}^2{\xi_3^2}^3\xi_1^2 + \xi_5^6\xi_5^5\xi_4^3\xi_3^3{\xi_4^2}^3\xi_1^2 + \xi_5^6\xi_5^5\xi_4^3\xi_3^3\xi_4^2{\xi_3^2}^2\xi_1^2 + \xi_5^6\xi_5^5\xi_4^3{\xi_4^2}^3\xi_2^2\xi_1^2 -$
$2\xi_5^6\xi_5^5\xi_4^3{\xi_4^2}^2\xi_3^2{\xi_2^2}^2 - \xi_5^6\xi_5^5\xi_4^3{\xi_4^2}^2\xi_3^2{\xi_1^2}^2 - 2\xi_5^6\xi_5^5\xi_4^3{\xi_4^2}^2\xi_3^2 + \xi_5^6\xi_5^5\xi_4^3\xi_4^2{\xi_3^2}^2\xi_2^2\xi_1^2 + \xi_5^6\xi_5^5\xi_4^3{\xi_3^2}^3{\xi_1^2}^2 - 2{\xi_5^5}^3\xi_4^3\xi_3^3\xi_4^2{\xi_3^2}^2 - 2{\xi_5^5}^3\xi_4^3\xi_4^2{\xi_3^2}^2\xi_2^2 -$
${\xi_5^5}^2\xi_4^3\xi_3^3{\xi_4^2}^2\xi_3^2\xi_1^2 - 2{\xi_5^5}^2\xi_4^3\xi_3^3\xi_4^2{\xi_3^2}^2\xi_2^2 + {\xi_5^5}^2\xi_4^3\xi_3^3{\xi_3^2}^3\xi_1^2 - {\xi_5^5}^2\xi_4^3{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2 - 2{\xi_5^5}^2\xi_4^3\xi_4^2{\xi_3^2}^2{\xi_2^2}^2 + {\xi_5^5}^2\xi_4^3{\xi_3^2}^3\xi_2^2\xi_1^2 - 2\xi_5^5\xi_4^3\xi_3^3\xi_4^2{\xi_3^2}^2 -$
$2\xi_5^5\xi_4^3\xi_4^2{\xi_3^2}^2\xi_2^2 - ({\xi_5^5}^2+1)(\xi_3^3{\xi_4^2}^2 - \xi_3^3{\xi_3^2}^2 + {\xi_4^2}^2\xi_2^2 + 2\xi_4^2\xi_3^2\xi_1^2 - {\xi_3^2}^2\xi_2^2)(\xi_3^3+\xi_2^2)\xi_5^5\xi_3^2 + (\xi_3^3{\xi_4^2}^3\xi_1^2 + \xi_3^3{\xi_4^2}^2\xi_3^2\xi_2^2 - \xi_3^3\xi_4^2{\xi_3^2}^2\xi_1^2 - \xi_3^3{\xi_3^2}^3\xi_2^2 +$
${\xi_4^2}^3\xi_2^2\xi_1^2 - {\xi_4^2}^2\xi_3^2{\xi_2^2}^2 - 2{\xi_4^2}^2\xi_3^2 + \xi_4^2{\xi_3^2}^2\xi_2^2\xi_1^2 - {\xi_3^2}^3{\xi_2^2}^2)(\xi_3^3+\xi_2^2){\xi_5^5}^2)\xi_4^6 - (\xi_4^3+\xi_1^2)^2{\xi_5^6}^3\xi_3^6{\xi_4^2}^2\xi_3^2 - (\xi_4^3{\xi_4^2}^2\xi_3^2\xi_2^2 - 2\xi_4^3\xi_4^2{\xi_3^2}^2\xi_1^2 - \xi_4^3{\xi_3^2}^3\xi_2^2 +$
$2\xi_3^3{\xi_4^2}^2\xi_3^2\xi_1^2 + 2\xi_3^3\xi_4^2{\xi_3^2}^2\xi_2^2 + {\xi_4^2}^3{\xi_2^2}^2 + {\xi_4^2}^3 + {\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2 + \xi_4^2{\xi_3^2}^2{\xi_2^2}^2 - \xi_4^2{\xi_3^2}^2 + {\xi_3^2}^3\xi_2^2\xi_1^2)(\xi_4^3+\xi_1^2){\xi_5^6}^2\xi_4^6)\xi_5^6 - (\xi_4^6\xi_3^2 - \xi_3^6\xi_4^2)({\xi_5^5}^2 +$
$1)^2(\xi_3^3+\xi_2^2)^2\xi_4^2\xi_3^2 - ({\xi_4^2}^2\xi_1^2 - 2\xi_4^2\xi_3^2\xi_2^2 - {\xi_3^2}^2\xi_1^2 - 2\xi_3^3\xi_4^2\xi_3^2){\xi_5^6}^3\xi_4^6\xi_5^5\xi_3^2\xi_1^2 + ((\xi_3^3\xi_4^2\xi_3^2 - 2{\xi_4^2}^2\xi_1^2 + 2\xi_4^2\xi_3^2\xi_2^2 + 2{\xi_3^2}^2\xi_1^2)\xi_3^3 - (2{\xi_4^2}^2\xi_2^2\xi_1^2 -$
$\xi_4^2\xi_3^2{\xi_2^2}^2 + \xi_4^2\xi_3^2{\xi_1^2}^2 - 2{\xi_3^2}^2\xi_2^2\xi_1^2)){\xi_5^6}^2\xi_4^6{\xi_5^5}^2\xi_3^2 - ((2\xi_3^3{\xi_4^2}^2\xi_3^2\xi_1^2 + 2\xi_3^3\xi_4^2{\xi_3^2}^2\xi_2^2 + {\xi_4^2}^3{\xi_2^2}^2 - {\xi_4^2}^3{\xi_1^2}^2 + {\xi_4^2}^3 + 2{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2 + 3\xi_4^2{\xi_3^2}^2{\xi_2^2}^2 +$
$\xi_4^2{\xi_3^2}^2{\xi_1^2}^2 - \xi_4^2{\xi_3^2}^2 + 2{\xi_3^2}^3\xi_2^2\xi_1^2)\xi_3^3 + {\xi_4^2}^3{\xi_2^2}^3 - {\xi_4^2}^3\xi_2^2{\xi_1^2}^2 + {\xi_4^2}^3\xi_2^2 + 2{\xi_4^2}^2\xi_3^2{\xi_2^2}^2\xi_1^2 + 2{\xi_4^2}^2\xi_3^2\xi_1^2 + \xi_4^2{\xi_3^2}^2{\xi_2^2}^3 - \xi_4^2{\xi_3^2}^2\xi_2^2{\xi_1^2}^2 - \xi_4^2{\xi_3^2}^2\xi_2^2 +$
$2{\xi_3^2}^3{\xi_2^2}^2\xi_1^2){\xi_5^6}^2\xi_4^6\xi_5^5 + (\xi_3^3{\xi_4^2}^3\xi_1^2 + \xi_3^3{\xi_4^2}^2\xi_3^2\xi_2^2 - \xi_3^3\xi_4^2{\xi_3^2}^2\xi_1^2 - \xi_3^3{\xi_3^2}^3\xi_2^2 + {\xi_4^2}^3\xi_2^2\xi_1^2 - {\xi_4^2}^2\xi_3^2{\xi_2^2}^2 - 2{\xi_4^2}^2\xi_3^2 + \xi_4^2{\xi_3^2}^2\xi_2^2\xi_1^2 - {\xi_3^2}^3{\xi_2^2}^2 -$
$({\xi_4^2}^2\xi_1^2 + 2\xi_4^2\xi_3^2\xi_2^2 - {\xi_3^2}^2\xi_1^2)\xi_4^3\xi_3^2)(\xi_3^3+\xi_2^2)\xi_5^6\xi_4^6 - (((\xi_5^5 - \xi_2^2)\xi_2^6{\xi_4^2}^2 + (\xi_4^3\xi_3^2 - \xi_3^3\xi_4^2 - \xi_4^2\xi_2^2)\xi_1^6\xi_3^2)({\xi_4^2}^2 + {\xi_3^2}^2) - (\xi_5^5\xi_4^3{\xi_4^2}^3 - \xi_5^5\xi_4^3\xi_4^2{\xi_3^2}^2 -$
$2\xi_5^5\xi_3^3{\xi_4^2}^2\xi_3^2 + \xi_5^5{\xi_4^2}^3\xi_1^2 - 2\xi_5^5{\xi_4^2}^2\xi_3^2\xi_2^2 - \xi_5^5\xi_4^2{\xi_3^2}^2\xi_1^2 + \xi_4^3{\xi_4^2}^3\xi_2^2 - \xi_4^3{\xi_4^2}^2\xi_3^2\xi_1^2 - \xi_4^3\xi_4^2{\xi_3^2}^2\xi_2^2 + \xi_4^3{\xi_3^2}^3\xi_1^2 + \xi_3^3{\xi_4^2}^3\xi_1^2 + 2\xi_3^3{\xi_4^2}^2\xi_3^2\xi_2^2 - \xi_3^3\xi_4^2{\xi_3^2}^2\xi_1^2 -$
$2{\xi_4^2}^2\xi_3^2)\xi_3^6)(\xi_4^3+\xi_1^2){\xi_5^6}^3 - ((({\xi_4^2}^2\xi_1^2 + \xi_4^2\xi_3^2{\xi_2^2}^3 + \xi_4^2\xi_3^2\xi_2^2{\xi_1^2}^2 + 3\xi_4^2\xi_3^2\xi_2^2 - {\xi_3^2}^2\xi_1^2)\xi_4^2 + ({\xi_2^2}^2 + {\xi_1^2}^2 - 1){\xi_3^2}^3\xi_2^2 + \xi_4^3{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2 + \xi_4^3\xi_4^2{\xi_3^2}^2{\xi_2^2}^2 -$
$\xi_4^3\xi_4^2{\xi_3^2}^2{\xi_1^2}^2 - \xi_4^3{\xi_3^2}^3\xi_2^2\xi_1^2 - \xi_3^3{\xi_4^2}^3\xi_2^2\xi_1^2 - \xi_3^3{\xi_4^2}^2\xi_3^2{\xi_2^2}^2 + 2\xi_3^3{\xi_4^2}^2\xi_3^2{\xi_1^2}^2 + \xi_3^3{\xi_4^2}^2\xi_3^2 + 3\xi_3^3\xi_4^2{\xi_3^2}^2\xi_2^2\xi_1^2 + \xi_3^3{\xi_3^2}^3{\xi_2^2}^2 - \xi_3^3{\xi_3^2}^3)\xi_4^3 - ((\xi_3^3{\xi_4^2}^2{\xi_1^2}^2 +$
$2\xi_3^3\xi_4^2\xi_3^2\xi_2^2\xi_1^2 + \xi_3^3{\xi_3^2}^2{\xi_2^2}^2 + {\xi_4^2}^2{\xi_2^2}^3 + {\xi_4^2}^2\xi_2^2{\xi_1^2}^2 + {\xi_4^2}^2\xi_2^2 - 4\xi_4^2\xi_3^2\xi_1^2 + {\xi_3^2}^2{\xi_2^2}^3 + {\xi_3^2}^2\xi_2^2{\xi_1^2}^2 - {\xi_3^2}^2\xi_2^2)\xi_3^3 - ({\xi_4^2}^2{\xi_2^2}^2 + {\xi_4^2}^2 + 2\xi_4^2\xi_3^2\xi_2^2\xi_1^2 +$
${\xi_3^2}^2{\xi_2^2}^2))\xi_4^2){\xi_5^6}^2\xi_4^6 - ((((\xi_3^3+\xi_2^2)\xi_4^2 - \xi_4^3\xi_3^2)\xi_1^6 + (\xi_5^5 - \xi_2^2)\xi_2^6\xi_3^2)({\xi_4^2}^2 + {\xi_3^2}^2) - (2\xi_5^5\xi_4^3\xi_4^2\xi_3^2 + \xi_5^5\xi_3^3{\xi_4^2}^2 - \xi_5^5\xi_3^3{\xi_3^2}^2 + \xi_5^5{\xi_4^2}^2\xi_2^2 + 2\xi_5^5\xi_4^2\xi_3^2\xi_1^2 -$
$\xi_5^5{\xi_3^2}^2\xi_2^2 + 2\xi_4^3\xi_4^2\xi_3^2\xi_2^2 - 2\xi_4^3{\xi_3^2}^2\xi_1^2 - \xi_3^3{\xi_4^2}^2\xi_2^2 + 2\xi_3^3\xi_4^2\xi_3^2\xi_1^2 + \xi_3^3{\xi_3^2}^2\xi_2^2 + {\xi_4^2}^2 - {\xi_3^2}^2)\xi_3^6)({\xi_5^5}^2 + 1)(\xi_3^3 + \xi_2^2)\xi_5^6\xi_4^2 + ((((\xi_5^5\xi_4^3{\xi_4^2}^2\xi_3^2 +$
$2\xi_5^5\xi_3^3{\xi_4^2}^3 - 2\xi_5^5\xi_3^3\xi_4^2{\xi_3^2}^2 + 2\xi_4^3{\xi_4^2}^2\xi_3^2\xi_2^2 - 2\xi_4^3\xi_4^2{\xi_3^2}^2\xi_1^2 + \xi_3^3{\xi_4^2}^2\xi_3^2\xi_1^2 + \xi_3^3{\xi_3^2}^3\xi_1^2 + 2({\xi_4^2}^2\xi_2^2 + \xi_4^2\xi_3^2\xi_1^2 - {\xi_3^2}^2\xi_2^2)\xi_5^5\xi_4^2 + {\xi_4^2}^3{\xi_2^2}^2 + {\xi_4^2}^3 +$
${\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2 - \xi_4^2{\xi_3^2}^2{\xi_2^2}^2 - 2\xi_4^2{\xi_3^2}^2{\xi_1^2}^2 - \xi_4^2{\xi_3^2}^2 + {\xi_3^2}^3\xi_2^2\xi_1^2)\xi_4^3 - ((\xi_3^3\xi_4^2\xi_3^2 - 2{\xi_4^2}^2\xi_1^2 + 2\xi_4^2\xi_3^2\xi_2^2 + 2{\xi_3^2}^2\xi_1^2)\xi_3^3 - (2{\xi_4^2}^2\xi_2^2\xi_1^2 - \xi_4^2\xi_3^2{\xi_2^2}^2 +$
$\xi_4^2\xi_3^2{\xi_1^2}^2 - 2{\xi_3^2}^2\xi_2^2\xi_1^2))\xi_5^5\xi_4^2 + ((\xi_3^3{\xi_4^2}^2\xi_1^2 + 2\xi_3^3\xi_4^2\xi_3^2\xi_2^2 - \xi_3^3{\xi_3^2}^2\xi_1^2 + 2\xi_4^2\xi_3^2{\xi_2^2}^2 + 2\xi_4^2\xi_3^2{\xi_1^2}^2 - 2\xi_4^2\xi_3^2)\xi_3^3 + {\xi_4^2}^2\xi_1^2 - 2\xi_4^2\xi_3^2\xi_2^2 - {\xi_3^2}^2\xi_1^2)\xi_4^2)\xi_5^5 +$
$(\xi_3^3{\xi_4^2}^2\xi_3^2{\xi_2^2}^2 - 2\xi_3^3\xi_4^2{\xi_3^2}^2\xi_2^2\xi_1^2 + \xi_3^3{\xi_3^2}^3{\xi_1^2}^2 - \xi_3^3{\xi_3^2}^3{\xi_2^2}^2 + \xi_3^3{\xi_4^2}^3 + 3\xi_3^3{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2 + \xi_3^3\xi_4^2{\xi_3^2}^2{\xi_2^2}^2 - 2\xi_3^3\xi_4^2{\xi_3^2}^2{\xi_1^2}^2 - \xi_3^3\xi_4^2{\xi_3^2}^2 - \xi_3^3{\xi_3^2}^3\xi_2^2\xi_1^2 +$
$(2\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)(\xi_4^2+\xi_3^2)(\xi_4^2-\xi_3^2))\xi_4^3 - ((\xi_3^3{\xi_4^2}^2\xi_2^2\xi_1^2 + \xi_3^3\xi_4^2\xi_3^2{\xi_2^2}^2 - \xi_3^3\xi_4^2\xi_3^2{\xi_1^2}^2 - \xi_3^3{\xi_3^2}^2\xi_2^2\xi_1^2 - 2{\xi_4^2}^2\xi_1^2 - 2\xi_4^2\xi_3^2\xi_2^2 + 2{\xi_3^2}^2\xi_1^2)\xi_3^3 - ({\xi_4^2}^2\xi_2^2\xi_1^2 -$
$\xi_4^2\xi_3^2 - {\xi_3^2}^2\xi_2^2\xi_1^2))\xi_4^2)\xi_5^6 + (((\xi_4^3\xi_4^2\xi_3^2\xi_2^2 - \xi_4^3{\xi_3^2}^2\xi_1^2 - \xi_3^3{\xi_4^2}^2\xi_2^2 + 2\xi_3^3\xi_4^2\xi_3^2\xi_1^2 + \xi_3^3{\xi_3^2}^2\xi_2^2 + {\xi_4^2}^2 + \xi_4^2\xi_3^2\xi_2^2\xi_1^2 + {\xi_3^2}^2{\xi_2^2}^2)\xi_4^3 - (\xi_3^3\xi_4^2\xi_1^2 + \xi_3^3\xi_3^2\xi_2^2 -$
$\xi_3^2)(\xi_3^3+\xi_2^2)\xi_4^2 + (\xi_4^3\xi_4^2 - \xi_3^3\xi_3^2 + \xi_4^2\xi_1^2 - \xi_3^2\xi_2^2)(\xi_4^3\xi_3^2 - \xi_3^3\xi_4^2 - \xi_4^2\xi_2^2)\xi_5^5)\xi_1^6 - (\xi_5^5\xi_4^3\xi_4^2\xi_3^2 + \xi_5^5\xi_3^3{\xi_4^2}^2 + \xi_5^5{\xi_4^2}^2\xi_2^2 + \xi_5^5\xi_4^2\xi_3^2\xi_1^2 + \xi_4^3\xi_4^2\xi_3^2\xi_2^2 -$
$\xi_4^3{\xi_3^2}^2\xi_1^2 - \xi_3^3{\xi_4^2}^2\xi_2^2 + \xi_3^3\xi_4^2\xi_3^2\xi_1^2 + {\xi_4^2}^2)(\xi_5^5 - \xi_2^2)\xi_2^6)({\xi_4^2}^2 + {\xi_3^2}^2)){\xi_5^6}^2)/(((((\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_3^3 - \xi_4^2)\xi_4^2 - (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_4^3\xi_3^2)\xi_5^6 - ({\xi_5^6}^2\xi_4^3\xi_4^2 +$
${\xi_5^6}^2\xi_4^2\xi_1^2 + \xi_5^6\xi_5^5\xi_4^3\xi_3^2 + \xi_5^6\xi_5^5\xi_3^3\xi_4^2 + \xi_5^6\xi_5^5\xi_4^2\xi_2^2 + \xi_5^6\xi_5^5\xi_3^2\xi_1^2 + {\xi_5^5}^2\xi_3^3\xi_3^2 + {\xi_5^5}^2\xi_3^2\xi_2^2 + \xi_3^3\xi_3^2 + \xi_3^2\xi_2^2)\xi_4^2)({\xi_4^2}^2 + {\xi_3^2}^2){\xi_5^6}^2);$

$J_3^5 = (-((((\xi_4^2\xi_2^2 + \xi_3^2\xi_1^2 + \xi_3^3\xi_4^2)\xi_5^6\xi_1^6 + (\xi_5^6\xi_3^3\xi_4^2 + \xi_5^6\xi_4^2\xi_2^2 + \xi_5^6\xi_3^2\xi_1^2 + \xi_5^5\xi_3^3\xi_3^2 - \xi_5^5\xi_4^2\xi_1^2 + \xi_5^5\xi_3^2\xi_2^2)\xi_4^3 + ((\xi_3^3\xi_4^2 + 2\xi_4^2\xi_2^2 + 2\xi_3^2\xi_1^2)\xi_3^3 +$
$\xi_4^2{\xi_2^2}^2 - \xi_4^2{\xi_1^2}^2 + 2\xi_3^2\xi_2^2\xi_1^2)\xi_5^5)\xi_4^6\xi_3^2 - ((\xi_5^6\xi_4^2 + \xi_1^6\xi_3^2)\xi_3^2 - (\xi_5^5 - \xi_3^3)\xi_3^6\xi_4^2)(\xi_4^3 + \xi_1^2)\xi_5^6\xi_4^2 - (((\xi_3^3\xi_4^2\xi_1^2 + \xi_3^3\xi_3^2\xi_2^2 + \xi_3^2{\xi_2^2}^2 + \xi_3^2{\xi_1^2}^2 - \xi_3^2)\xi_3^3 +$
$\xi_4^2\xi_1^2 - \xi_3^2\xi_2^2)\xi_4^2 - ((\xi_4^2\xi_1^2 + \xi_3^2\xi_2^2)\xi_3^3 + ({\xi_2^2}^2 + {\xi_1^2}^2)\xi_3^2)\xi_4^3\xi_3^2)\xi_4^6)\xi_5^6 - ((\xi_5^6\xi_3^2 - \xi_1^6\xi_4^2 - (\xi_5^5 - \xi_3^3)\xi_3^6)\xi_4^2 - (\xi_3^3\xi_3^2 - \xi_4^2\xi_1^2 + \xi_3^2\xi_2^2)\xi_4^6)({\xi_5^5}^2 +$
$1)(\xi_3^3 + \xi_2^2)\xi_3^2 - (((({\xi_4^2}^2 + {\xi_3^2}^2)\xi_3^3 + (\xi_4^2\xi_1^2 + \xi_3^2\xi_2^2)\xi_3^2)\xi_4^3\xi_3^2 - ({\xi_3^3}^2{\xi_4^2}^2 + {\xi_3^3}^2{\xi_3^2}^2 + \xi_3^3\xi_4^2\xi_3^2\xi_1^2 + \xi_3^3{\xi_3^2}^2\xi_2^2 + {\xi_4^2}^2)\xi_4^2 - (\xi_4^2\xi_1^2 - \xi_3^2\xi_2^2 - \xi_3^3\xi_3^2 +$
$\xi_4^3\xi_4^2)\xi_5^5\xi_4^2\xi_3^2)\xi_1^6 + ((\xi_4^2\xi_2^2 + \xi_3^2\xi_1^2 + \xi_3^3\xi_4^2 + \xi_4^3\xi_3^2)\xi_5^5\xi_4^2 - ((\xi_3^3\xi_4^2\xi_2^2 - \xi_3^3\xi_3^2\xi_1^2 - \xi_4^2)\xi_4^2 - (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_4^3\xi_3^2))\xi_2^6\xi_3^2 - (\xi_5^5\xi_4^3\xi_4^2\xi_3^2 + \xi_5^5\xi_3^3{\xi_4^2}^2 +$
$\xi_5^5{\xi_4^2}^2\xi_2^2 + \xi_5^5\xi_4^2\xi_3^2\xi_1^2 + \xi_4^3\xi_4^2\xi_3^2\xi_2^2 - \xi_4^3{\xi_3^2}^2\xi_1^2 - \xi_3^3{\xi_4^2}^2\xi_2^2 + \xi_3^3\xi_4^2\xi_3^2\xi_1^2 + {\xi_4^2}^2)(\xi_5^5 - \xi_3^3)\xi_3^6)\xi_5^6))/((((\xi_3^3\xi_4^2\xi_2^2 - \xi_3^3\xi_3^2\xi_1^2 - \xi_4^2)\xi_4^2 - (\xi_4^2\xi_2^2 -$
$\xi_3^2\xi_1^2)\xi_4^3\xi_3^2)\xi_5^6 - ({\xi_5^6}^2\xi_4^3\xi_4^2 + {\xi_5^6}^2\xi_4^2\xi_1^2 + \xi_5^6\xi_5^5\xi_4^3\xi_3^2 + \xi_5^6\xi_5^5\xi_3^3\xi_4^2 + \xi_5^6\xi_5^5\xi_4^2\xi_2^2 + \xi_5^6\xi_5^5\xi_3^2\xi_1^2 + {\xi_5^5}^2\xi_3^3\xi_3^2 + {\xi_5^5}^2\xi_3^2\xi_2^2 + \xi_3^3\xi_3^2 + \xi_3^2\xi_2^2)\xi_4^2)\xi_5^6);$

$J_4^5 = (-((((\xi_5^6\xi_4^3{\xi_4^2}^2 + \xi_5^6{\xi_4^2}^2\xi_1^2 + \xi_5^5\xi_4^3\xi_4^2\xi_3^2 + {\xi_4^3}^2{\xi_3^2}^2 + \xi_4^3\xi_3^3\xi_4^2\xi_3^2 - (\xi_4^2\xi_1^2 - 3\xi_3^2\xi_2^2)\xi_4^3\xi_4^2)\xi_5^5 + (\xi_4^3\xi_3^2 + \xi_4^2\xi_2^2 + \xi_3^2\xi_1^2)(\xi_4^3 + \xi_1^2)\xi_5^6\xi_4^2 +$
$(\xi_4^2\xi_2^2 + \xi_3^2\xi_1^2 + \xi_3^3\xi_4^2){\xi_5^5}^2\xi_4^2 + (\xi_4^2{\xi_2^2}^2 - \xi_4^2{\xi_1^2}^2 + \xi_4^2 + 2\xi_3^2\xi_2^2\xi_1^2 + 2\xi_3^3\xi_3^2\xi_1^2)\xi_5^5\xi_4^2)\xi_4^6 - (\xi_5^6\xi_4^2 + \xi_1^6\xi_3^2 + \xi_3^6\xi_4^3)(\xi_4^3 + \xi_1^2)\xi_5^6{\xi_4^2}^2)\xi_5^6 - (((\xi_5^6\xi_3^2 -$
$\xi_1^6\xi_4^2)\xi_4^2 + \xi_3^6\xi_4^3\xi_3^2)\xi_4^2 - (\xi_5^5\xi_4^2\xi_3^2 + \xi_4^3{\xi_3^2}^2 - {\xi_4^2}^2\xi_1^2 + \xi_4^2\xi_3^2\xi_2^2)\xi_4^6)({\xi_5^5}^2 + 1)(\xi_3^3 + \xi_2^2) + ((({\xi_1^2}^2 + 1 + {\xi_2^2}^2)\xi_4^2\xi_3^2 + (\xi_4^3\xi_3^2 - \xi_3^3\xi_4^2)(\xi_4^2\xi_1^2 +$
$\xi_3^2\xi_2^2))\xi_4^3 + (\xi_4^2\xi_2^2 + \xi_3^2\xi_1^2 - ({\xi_2^2}^2 + {\xi_1^2}^2)\xi_3^3\xi_4^2)\xi_4^2)\xi_2^6\xi_4^6 - (((((\xi_4^3\xi_3^2 - \xi_3^3\xi_4^2)({\xi_4^2}^2 + {\xi_3^2}^2) + (\xi_4^2\xi_1^2 + \xi_3^2\xi_2^2)\xi_4^2\xi_3^2)\xi_4^3 - (\xi_3^3\xi_4^2\xi_1^2 + \xi_3^3\xi_3^2\xi_2^2 - \xi_3^2){\xi_4^2}^2 -$
$(\xi_4^2\xi_1^2 - \xi_3^2\xi_2^2 - \xi_3^3\xi_3^2 + \xi_4^3\xi_4^2)\xi_5^5{\xi_4^2}^2)\xi_1^6 + ((\xi_4^2\xi_2^2 + \xi_3^2\xi_1^2 + \xi_3^3\xi_4^2 + \xi_4^3\xi_3^2)\xi_5^5\xi_4^2 - ((\xi_3^3\xi_4^2\xi_2^2 - \xi_3^3\xi_3^2\xi_1^2 - \xi_4^2)\xi_4^2 - (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_4^3\xi_3^2))\xi_2^6\xi_4^2 +$
$((\xi_4^3\xi_3^2 + \xi_3^3\xi_4^2 + \xi_4^2\xi_2^2 + \xi_3^2\xi_1^2)\xi_5^5\xi_4^2 + \xi_4^3\xi_4^2\xi_3^2\xi_2^2 - \xi_4^3{\xi_3^2}^2\xi_1^2 - \xi_3^3{\xi_4^2}^2\xi_2^2 + \xi_3^3\xi_4^2\xi_3^2\xi_1^2 + {\xi_4^2}^2)\xi_3^6\xi_4^3)\xi_5^6))/((((\xi_3^3\xi_4^2\xi_2^2 - \xi_3^3\xi_3^2\xi_1^2 - \xi_4^2)\xi_4^2 - (\xi_4^2\xi_2^2 -$
$\xi_3^2\xi_1^2)\xi_4^3\xi_3^2)\xi_5^6 - ({\xi_5^6}^2\xi_4^3\xi_4^2 + {\xi_5^6}^2\xi_4^2\xi_1^2 + \xi_5^6\xi_5^5\xi_4^3\xi_3^2 + \xi_5^6\xi_5^5\xi_3^3\xi_4^2 + \xi_5^6\xi_5^5\xi_4^2\xi_2^2 + \xi_5^6\xi_5^5\xi_3^2\xi_1^2 + {\xi_5^5}^2\xi_3^3\xi_3^2 + {\xi_5^5}^2\xi_3^2\xi_2^2 + \xi_3^3\xi_3^2 + \xi_3^2\xi_2^2)\xi_4^2)\xi_5^6);$

and the parameters are subject to the conditions

$$\xi_1^2\xi_5^6\xi_4^2 C_2 \neq 0 \tag{100}$$

with $C_2 = ((\xi_5^6\xi_4^3\xi_4^2 + \xi_5^6\xi_4^2\xi_1^2 + \xi_5^5\xi_4^3\xi_3^2)\xi_5^6 + ({\xi_5^5}^2 + 1)(\xi_3^3 + \xi_2^2)\xi_3^2 + (\xi_4^2\xi_2^2 + \xi_3^2\xi_1^2 + \xi_3^3\xi_4^2)\xi_5^6\xi_5^5)\xi_4^2 - ((\xi_3^3\xi_4^2\xi_2^2 - \xi_3^3\xi_3^2\xi_1^2 - \xi_4^2)\xi_4^2 - (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_4^3\xi_3^2)\xi_5^6$.

13.2.2 Case $\xi_1^2\xi_3^2 \neq 0, \xi_4^2 = 0$.

$$J = \begin{pmatrix} * & * & -\frac{(\xi_3^3+\xi_2^2)\xi_3^2}{\xi_1^2} & -\frac{\xi_4^3\xi_3^2}{\xi_1^2} & 0 & 0 \\ \boxed{\xi_1^2} & \boxed{\xi_2^2} & \boxed{\xi_3^2} & 0 & 0 & 0 \\ * & * & \boxed{\xi_3^3} & \boxed{\xi_4^3} & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & \boxed{\xi_5^5} & -\frac{{\xi_5^5}^2+1}{\xi_5^6} \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & \boxed{\xi_5^6} & -\xi_5^5 \end{pmatrix}$$

where $J_1^1 = (-(({\xi_5^5}^2 + 1)(\xi_3^3 + \xi_2^2) + \xi_5^6\xi_2^2\xi_1^2 + (\xi_4^3\xi_2^2 + \xi_3^3\xi_1^2 + (\xi_4^3 + \xi_1^2)\xi_5^5)\xi_5^6))/(\xi_5^6\xi_1^2);$
$J_2^1 = (-((\xi_3^3 + \xi_2^2)\xi_2^2 - \xi_4^3\xi_1^2 - (\xi_3^3 + \xi_2^2)\xi_5^5 - (\xi_4^3 + \xi_1^2)\xi_5^6))/\xi_1^2;$

$J_1^3 = ((\xi_4^3\xi_2^2 + \xi_3^3\xi_1^2 + (\xi_4^3+\xi_1^2)\xi_5^5)\xi_5^6 + ({\xi_5^5}^2+1)(\xi_3^3+\xi_2^2))/(\xi_5^6\xi_3^2)$;
$J_2^3 = (-(\xi_5^6\xi_4^3 + \xi_5^6\xi_1^2 + \xi_5^5\xi_3^3 + \xi_5^5\xi_2^2 + \xi_4^3\xi_1^2 - \xi_3^3\xi_2^2 + 1))/\xi_3^2$;
$J_1^4 = ((((\xi_3^3+\xi_2^2)\xi_5^5 + 2\xi_5^6\xi_1^2 + 2{\xi_5^5}^2 + 2)(\xi_3^3+\xi_2^2)\xi_5^5\xi_1^2 + {\xi_5^6}^2\xi_4^3{\xi_1^2}^2 + {\xi_5^6}^2{\xi_1^2}^3 + \xi_5^6{\xi_5^5}^2{\xi_4^3}^2 + 2\xi_5^6{\xi_5^5}^2\xi_4^3\xi_1^2 + \xi_5^6{\xi_5^5}^2{\xi_1^2}^2 + 2\xi_5^6\xi_5^5{\xi_4^3}^2\xi_2^2 + \xi_5^6\xi_5^5\xi_4^3\xi_3^3\xi_1^2 + 3\xi_5^6\xi_5^5\xi_4^3\xi_2^2\xi_1^2 + 2{\xi_5^5}^3\xi_4^3\xi_3^3 + 2{\xi_5^5}^3\xi_4^3\xi_2^2 + 2{\xi_5^5}^2\xi_4^3\xi_3^3\xi_2^2 + 2{\xi_5^5}^2\xi_4^3{\xi_2^2}^2 + 2\xi_5^5\xi_4^3\xi_3^3 + 2\xi_5^5\xi_4^3\xi_2^2)\xi_5^6 + ({\xi_5^5}^2+1)^2(\xi_3^3+\xi_2^2)^2 + ((\xi_3^3+\xi_2^2)\xi_1^2 + 2\xi_4^3\xi_2^2)(\xi_3^3+\xi_2^2)\xi_5^6 + ({\xi_4^3}^2{\xi_2^2}^2 + \xi_4^3\xi_3^3\xi_2^2\xi_1^2 + {\xi_1^2}^2 + ({\xi_2^2}^2+{\xi_1^2}^2)\xi_4^3\xi_1^2){\xi_5^6}^2)/({\xi_5^6}^2\xi_4^3\xi_3^2\xi_1^2)$;
$J_2^4 = (-({\xi_5^6}^2\xi_5^5{\xi_4^3}^2 + 2{\xi_5^6}^2\xi_5^5\xi_4^3\xi_1^2 + {\xi_5^6}^2\xi_5^5{\xi_1^2}^2 + {\xi_5^6}^2{\xi_4^3}^2\xi_2^2 - {\xi_5^6}^2\xi_2^2{\xi_1^2}^2 + 2\xi_5^6{\xi_5^5}^2\xi_4^3\xi_3^3 + 2\xi_5^6{\xi_5^5}^2\xi_4^3\xi_2^2 + 2\xi_5^6{\xi_5^5}^2\xi_3^3\xi_1^2 + 2\xi_5^6{\xi_5^5}^2\xi_2^2\xi_1^2 + \xi_5^6\xi_5^5{\xi_4^3}^2\xi_1^2 + \xi_5^6\xi_5^5\xi_4^3{\xi_1^2}^2 - 2\xi_5^6\xi_5^5\xi_3^3\xi_2^2\xi_1^2 - 2\xi_5^6\xi_5^5{\xi_2^2}^2\xi_1^2 + \xi_5^6{\xi_4^3}^2\xi_2^2\xi_1^2 - \xi_5^6\xi_4^3\xi_3^3{\xi_2^2}^2 + \xi_5^6\xi_4^3\xi_3^3 - \xi_5^6\xi_4^3{\xi_2^2}^3 - \xi_5^6\xi_4^3\xi_2^2{\xi_1^2}^2 + \xi_5^6\xi_4^3\xi_2^2 + {\xi_5^5}^3{\xi_3^3}^2 + 2{\xi_5^5}^3\xi_3^3\xi_2^2 + {\xi_5^5}^3{\xi_2^2}^2 + {\xi_5^5}^2\xi_4^3\xi_3^3\xi_1^2 + {\xi_5^5}^2\xi_4^3\xi_2^2\xi_1^2 - {\xi_5^5}^2{\xi_3^3}^2\xi_2^2 - 2{\xi_5^5}^2\xi_3^3{\xi_2^2}^2 - {\xi_5^5}^2{\xi_2^2}^3 + \xi_5^5{\xi_3^3}^2 + 2\xi_5^5\xi_3^3\xi_2^2 + \xi_5^5{\xi_2^2}^2 + \xi_4^3\xi_3^3\xi_1^2 + \xi_4^3\xi_2^2\xi_1^2 - {\xi_3^3}^2\xi_2^2 - 2\xi_3^3{\xi_2^2}^2 - {\xi_2^2}^3))/(\xi_5^6\xi_4^3\xi_3^2\xi_1^2)$;
$J_3^4 = ((\xi_5^6\xi_4^3\xi_1^2 + \xi_5^6{\xi_1^2}^2 + \xi_5^5\xi_4^3\xi_3^3 + \xi_5^5\xi_4^3\xi_2^2 + 2(\xi_3^3+\xi_2^2)\xi_5^5\xi_1^2)\xi_5^6 + ({\xi_5^5}^2+1)(\xi_3^3+\xi_2^2)^2 + ({\xi_2^2}^2+{\xi_1^2}^2+\xi_3^3\xi_2^2)\xi_5^6\xi_4^3)/(\xi_5^6\xi_4^3\xi_1^2)$;
$J_4^4 = (({\xi_5^5}^2+1)(\xi_3^3+\xi_2^2) + (\xi_5^5\xi_4^3 + \xi_5^5\xi_1^2 + \xi_4^3\xi_2^2)\xi_5^6)/(\xi_5^6\xi_1^2)$;
$J_1^5 = (-(((((\xi_2^6\xi_1^2 - \xi_1^6\xi_5^5)\xi_5^6\xi_3^2 + ({\xi_5^5}^2+1)(\xi_3^3+\xi_2^2)\xi_3^6)\xi_1^2 + (\xi_4^3\xi_2^2 + \xi_3^3\xi_1^2 + (\xi_4^3+\xi_1^2)\xi_5^5)(\xi_3^6\xi_1^2 - \xi_1^6\xi_3^2)\xi_5^6)\xi_4^3 + ({\xi_4^3}^2{\xi_2^2}^2 + \xi_4^3\xi_3^3\xi_2^2\xi_1^2 + {\xi_1^2}^2 + ({\xi_2^2}^2+{\xi_1^2}^2)\xi_4^3\xi_1^2)\xi_5^6\xi_4^6 - (({\xi_5^5}^2+1)(\xi_3^3+\xi_2^2) + \xi_5^6\xi_2^2\xi_1^2)\xi_1^6\xi_4^3\xi_3^2)\xi_5^6 + ((((\xi_3^3+\xi_2^2)\xi_5^5 + 2\xi_5^6\xi_1^2 + 2{\xi_5^5}^2+2)(\xi_3^3+\xi_2^2)\xi_5^5\xi_1^2 + {\xi_5^6}^2\xi_4^3{\xi_1^2}^2 + {\xi_5^6}^2{\xi_1^2}^3 + \xi_5^6{\xi_5^5}^2{\xi_4^3}^2 + 2\xi_5^6{\xi_5^5}^2\xi_4^3\xi_1^2 + \xi_5^6{\xi_5^5}^2{\xi_1^2}^2 + 2\xi_5^6\xi_5^5{\xi_4^3}^2\xi_2^2 + \xi_5^6\xi_5^5\xi_4^3\xi_3^3\xi_1^2 + 3\xi_5^6\xi_5^5\xi_4^3\xi_2^2\xi_1^2 + 2{\xi_5^5}^3\xi_4^3\xi_3^3 + 2{\xi_5^5}^3\xi_4^3\xi_2^2 + 2{\xi_5^5}^2\xi_4^3\xi_3^3\xi_2^2 + 2{\xi_5^5}^2\xi_4^3{\xi_2^2}^2 + 2\xi_5^5\xi_4^3\xi_3^3 + 2\xi_5^5\xi_4^3\xi_2^2)\xi_5^6 + ({\xi_5^5}^2+1)^2(\xi_3^3+\xi_2^2)^2 + ((\xi_3^3+\xi_2^2)\xi_1^2 + 2\xi_4^3\xi_2^2)(\xi_3^3+\xi_2^2)\xi_5^6)\xi_4^6))/({\xi_5^6}^3\xi_4^3\xi_3^2\xi_1^2)$;
$J_2^5 = ((((\xi_5^6{\xi_4^3}^2 + 2\xi_5^6\xi_4^3\xi_1^2 + \xi_5^6{\xi_1^2}^2 + 2\xi_5^5\xi_4^3\xi_3^3 + 2\xi_5^5\xi_4^3\xi_2^2 + {\xi_4^3}^2\xi_1^2 + \xi_4^3{\xi_1^2}^2 + 2(\xi_5^5 - \xi_2^2)(\xi_3^3+\xi_2^2)\xi_1^2)\xi_5^5 + (\xi_4^3+\xi_1^2)(\xi_4^3-\xi_1^2)\xi_5^6\xi_2^2)\xi_5^6 + ({\xi_5^5}^2+1)(\xi_5^5\xi_3^3 + \xi_5^5\xi_2^2 + \xi_4^3\xi_1^2 - \xi_3^3\xi_2^2 - {\xi_2^2}^2)(\xi_3^3+\xi_2^2))\xi_4^6 + (\xi_3^6\xi_1^2 - \xi_1^6\xi_3^2)(\xi_4^3+\xi_1^2){\xi_5^6}^2\xi_4^3 + (\xi_4^3\xi_2^2\xi_1^2 - \xi_3^3{\xi_2^2}^2 + \xi_3^3 - ({\xi_2^2}^2 + {\xi_1^2}^2 - 1)\xi_2^2)\xi_5^6\xi_4^6\xi_4^3 - (((\xi_4^3\xi_1^2 - \xi_3^3\xi_2^2 - {\xi_2^2}^2 + (\xi_3^3+\xi_2^2)\xi_5^5)\xi_1^6 - (\xi_5^5 - \xi_2^2)\xi_2^6\xi_1^2)\xi_3^2 - (\xi_4^3\xi_1^2 - \xi_3^3\xi_2^2 + 1 + (\xi_3^3+\xi_2^2)\xi_5^5)\xi_3^6\xi_1^2)\xi_5^6\xi_4^3)/({\xi_5^6}^2\xi_4^3\xi_3^2\xi_1^2)$;
$J_3^5 = (-({\xi_5^6}^2\xi_4^6\xi_4^3\xi_1^2 + {\xi_5^6}^2\xi_4^6{\xi_1^2}^2 + \xi_5^6\xi_4^6\xi_5^5\xi_4^3\xi_3^3 + \xi_5^6\xi_4^6\xi_5^5\xi_4^3\xi_2^2 + 2\xi_5^6\xi_4^6\xi_5^5\xi_3^3\xi_1^2 + 2\xi_5^6\xi_4^6\xi_5^5\xi_2^2\xi_1^2 + \xi_5^6\xi_4^6\xi_4^3\xi_3^3\xi_2^2 + \xi_5^6\xi_4^6\xi_4^3{\xi_2^2}^2 + \xi_5^6\xi_4^6\xi_4^3{\xi_1^2}^2 - \xi_5^6\xi_3^6\xi_5^5\xi_4^3\xi_1^2 + \xi_5^6\xi_3^6\xi_4^3\xi_3^3\xi_1^2 + \xi_5^6\xi_2^6\xi_4^3\xi_3^2\xi_1^2 - \xi_5^6\xi_1^6\xi_4^3\xi_3^3\xi_3^2 - \xi_5^6\xi_1^6\xi_4^3\xi_3^2\xi_2^2 + \xi_4^6{\xi_5^5}^2{\xi_3^3}^2 + 2\xi_4^6{\xi_5^5}^2\xi_3^3\xi_2^2 + \xi_4^6{\xi_5^5}^2{\xi_2^2}^2 + \xi_4^6{\xi_3^3}^2 + 2\xi_4^6\xi_3^3\xi_2^2 + \xi_4^6{\xi_2^2}^2))/({\xi_5^6}^2\xi_4^3\xi_1^2)$;
$J_4^5 = (-(\xi_5^6\xi_4^6\xi_5^5\xi_4^3 + \xi_5^6\xi_4^6\xi_4^3\xi_2^2 + \xi_5^6\xi_3^6\xi_4^3\xi_1^2 - \xi_5^6\xi_1^6\xi_4^3\xi_3^2 + \xi_4^6{\xi_5^5}^2\xi_3^3 + \xi_4^6{\xi_5^5}^2\xi_2^2 + \xi_4^6\xi_3^3 + \xi_4^6\xi_2^2))/({\xi_5^6}^2\xi_1^2)$;

and the parameters are subject to the conditions

$$\xi_1^2\xi_5^6\xi_3^2\xi_4^3 \neq 0. \tag{101}$$

13.2.3 Case $\xi_1^2 \neq 0, \xi_3^2 = \xi_4^2 = 0, \xi_2^2 + \xi_3^3 \neq 0$.

$$J = \begin{pmatrix} -\xi_2^2 & -\frac{{\xi_2^2}^2+1}{\xi_1^2} & 0 & 0 & 0 & 0 \\ \boxed{\xi_1^2} & \boxed{\xi_2^2} & 0 & 0 & 0 & 0 \\ \boxed{\xi_1^3} & -\frac{\xi_1^4\xi_4^3 - \xi_3^3\xi_1^3 + \xi_1^3\xi_2^2}{\xi_1^2} & \boxed{\xi_3^3} & \boxed{\xi_4^3} & 0 & 0 \\ \boxed{\xi_1^4} & \frac{\xi_1^4\xi_4^3\xi_3^3 + \xi_1^4\xi_4^3\xi_2^2 + {\xi_3^3}^2\xi_1^3 + \xi_1^3}{\xi_4^3\xi_1^2} & -\frac{{\xi_3^3}^2+1}{\xi_4^3} & -\xi_3^3 & 0 & 0 \\ * & * & * & * & * & * \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & * & * \end{pmatrix}$$

where $J_1^5 = (\xi_4^6\xi_1^4{\xi_4^3}^2\xi_1^2 + \xi_4^6\xi_1^4\xi_4^3{\xi_2^2}^2 + \xi_4^6\xi_1^4\xi_4^3{\xi_1^2}^2 + \xi_4^6\xi_1^4\xi_4^3 + \xi_4^6\xi_1^4{\xi_3^3}^2\xi_1^2 + \xi_4^6\xi_1^4\xi_1^2 + \xi_3^6{\xi_4^3}^2\xi_1^3\xi_1^2 + \xi_3^6\xi_4^3\xi_1^3{\xi_2^2}^2 + \xi_3^6\xi_4^3\xi_1^3{\xi_1^2}^2 + \xi_3^6\xi_4^3\xi_1^3 + \xi_3^6{\xi_3^3}^2\xi_1^3\xi_1^2 + \xi_3^6\xi_1^3\xi_1^2 + \xi_2^6{\xi_4^3}^2{\xi_1^2}^2 + \xi_2^6\xi_4^3{\xi_2^2}^2\xi_1^2 + \xi_2^6\xi_4^3{\xi_1^2}^3 + \xi_2^6\xi_4^3\xi_1^2 + \xi_2^6{\xi_3^3}^2{\xi_1^2}^2 + \xi_2^6{\xi_1^2}^2 - \xi_1^6\xi_4^3\xi_3^3{\xi_2^2}^2 + \xi_1^6\xi_4^3\xi_3^3{\xi_1^2}^2 - \xi_1^6\xi_4^3\xi_3^3 - \xi_1^6\xi_4^3{\xi_2^2}^3 - \xi_1^6\xi_4^3\xi_2^2{\xi_1^2}^2 - \xi_1^6\xi_4^3\xi_2^2 - 2\xi_1^6{\xi_3^3}^2\xi_2^2\xi_1^2 - 2\xi_1^6\xi_2^2\xi_1^2)/({\xi_4^3}^2{\xi_1^2}^2 - 2\xi_4^3\xi_3^3\xi_2^2\xi_1^2 + 2\xi_4^3\xi_1^2 + {\xi_3^3}^2{\xi_2^2}^2 + {\xi_3^3}^2 + {\xi_2^2}^2 + 1)$;
$J_2^5 = (\xi_4^6\xi_1^4{\xi_4^3}^3\xi_3^3\xi_1^2 + \xi_4^6\xi_1^4{\xi_4^3}^3\xi_2^2\xi_1^2 + \xi_4^6\xi_1^4{\xi_4^3}^2\xi_3^3{\xi_2^2}^2 + \xi_4^6\xi_1^4{\xi_4^3}^2\xi_3^3{\xi_1^2}^2 + \xi_4^6\xi_1^4{\xi_4^3}^2\xi_3^3 + \xi_4^6\xi_1^4{\xi_4^3}^2{\xi_2^2}^3 + \xi_4^6\xi_1^4{\xi_4^3}^2\xi_2^2{\xi_1^2}^2 + \xi_4^6\xi_1^4{\xi_4^3}^2\xi_2^2 + \xi_4^6\xi_1^4\xi_4^3{\xi_3^3}^3\xi_1^2 + \xi_4^6\xi_1^4\xi_4^3{\xi_3^3}^2\xi_2^2\xi_1^2 + \xi_4^6\xi_1^4\xi_4^3\xi_3^3\xi_1^2 + \xi_4^6\xi_1^4\xi_4^3\xi_2^2\xi_1^2 + \xi_4^6{\xi_4^3}^2{\xi_3^3}^2\xi_1^3\xi_1^2 + \xi_4^6{\xi_4^3}^2\xi_1^3\xi_1^2 + \xi_4^6\xi_4^3{\xi_3^3}^2\xi_1^3{\xi_2^2}^2 + \xi_4^6\xi_4^3{\xi_3^3}^2\xi_1^3{\xi_1^2}^2 + \xi_4^6\xi_4^3{\xi_3^3}^2\xi_1^3 + \xi_4^6\xi_4^3\xi_1^3{\xi_2^2}^2 + \xi_4^6\xi_4^3\xi_1^3{\xi_1^2}^2 + \xi_4^6\xi_4^3\xi_1^3 + \xi_4^6{\xi_3^3}^4\xi_1^3\xi_1^2 + 2\xi_4^6{\xi_3^3}^2\xi_1^3\xi_1^2 + \xi_4^6\xi_1^3\xi_1^2 - \xi_3^6\xi_1^4{\xi_4^3}^4\xi_1^2 - \xi_3^6\xi_1^4{\xi_4^3}^3{\xi_2^2}^2 - \xi_3^6\xi_1^4{\xi_4^3}^3{\xi_1^2}^2 - \xi_3^6\xi_1^4{\xi_4^3}^3 - \xi_3^6\xi_1^4{\xi_4^3}^2{\xi_3^3}^2\xi_1^2 - \xi_3^6\xi_1^4{\xi_4^3}^2\xi_1^2 - \xi_3^6{\xi_4^3}^3\xi_3^3\xi_1^3\xi_1^2 + \xi_3^6{\xi_4^3}^3\xi_1^3\xi_2^2\xi_1^2 - \xi_3^6{\xi_4^3}^2\xi_3^3\xi_1^3{\xi_2^2}^2 - \xi_3^6{\xi_4^3}^2\xi_3^3\xi_1^3{\xi_1^2}^2 - \xi_3^6{\xi_4^3}^2\xi_3^3\xi_1^3 + \xi_3^6{\xi_4^3}^2\xi_1^3{\xi_2^2}^3 + \xi_3^6{\xi_4^3}^2\xi_1^3\xi_2^2{\xi_1^2}^2 + \xi_3^6{\xi_4^3}^2\xi_1^3\xi_2^2 - \xi_3^6\xi_4^3{\xi_3^3}^3\xi_1^3\xi_1^2 + \xi_3^6\xi_4^3{\xi_3^3}^2\xi_1^3\xi_2^2\xi_1^2 - \xi_3^6\xi_4^3\xi_3^3\xi_1^3\xi_1^2 + \xi_3^6\xi_4^3\xi_1^3\xi_2^2\xi_1^2 + 2\xi_2^6{\xi_4^3}^3\xi_2^2{\xi_1^2}^2 - \xi_2^6{\xi_4^3}^2\xi_3^3{\xi_2^2}^2\xi_1^2 + \xi_2^6{\xi_4^3}^2\xi_3^3{\xi_1^2}^3 - \xi_2^6{\xi_4^3}^2\xi_3^3\xi_1^2 + \xi_2^6{\xi_4^3}^2{\xi_2^2}^3\xi_1^2 + \xi_2^6{\xi_4^3}^2\xi_2^2{\xi_1^2}^3 + \xi_2^6{\xi_4^3}^2\xi_2^2\xi_1^2 - \xi_1^6{\xi_4^3}^3{\xi_2^2}^2\xi_1^2 - \xi_1^6{\xi_4^3}^3\xi_1^2 - \xi_1^6{\xi_4^3}^2{\xi_2^2}^4 - \xi_1^6{\xi_4^3}^2{\xi_2^2}^2{\xi_1^2}^2 - 2\xi_1^6{\xi_4^3}^2{\xi_2^2}^2 - \xi_1^6{\xi_4^3}^2{\xi_1^2}^2 - \xi_1^6{\xi_4^3}^2 - \xi_1^6\xi_4^3{\xi_3^3}^2{\xi_2^2}^2\xi_1^2 - \xi_1^6\xi_4^3{\xi_3^3}^2\xi_1^2 - \xi_1^6\xi_4^3{\xi_2^2}^2\xi_1^2 - \xi_1^6\xi_4^3\xi_1^2)/(\xi_4^3\xi_1^2({\xi_4^3}^2{\xi_1^2}^2 - 2\xi_4^3\xi_3^3\xi_2^2\xi_1^2 + 2\xi_4^3\xi_1^2 + {\xi_3^3}^2{\xi_2^2}^2 + {\xi_3^3}^2 + {\xi_2^2}^2 + 1))$;
$J_3^5 = (-\xi_4^6{\xi_4^3}^2{\xi_3^3}^2\xi_1^2 - \xi_4^6{\xi_4^3}^2\xi_1^2 - \xi_4^6\xi_4^3{\xi_3^3}^2{\xi_2^2}^2 - \xi_4^6\xi_4^3{\xi_3^3}^2{\xi_1^2}^2 - \xi_4^6\xi_4^3{\xi_3^3}^2 - \xi_4^6\xi_4^3{\xi_2^2}^2 - \xi_4^6\xi_4^3{\xi_1^2}^2 - \xi_4^6\xi_4^3 - \xi_4^6{\xi_3^3}^4\xi_1^2 - 2\xi_4^6{\xi_3^3}^2\xi_1^2 - \xi_4^6\xi_1^2 + \xi_3^6{\xi_4^3}^3\xi_3^3\xi_1^2 + \xi_3^6{\xi_4^3}^3\xi_2^2\xi_1^2 + 2\xi_3^6{\xi_4^3}^2\xi_3^3{\xi_1^2}^2 + \xi_3^6\xi_4^3{\xi_3^3}^3\xi_1^2 - \xi_3^6\xi_4^3{\xi_3^3}^2\xi_2^2\xi_1^2 + \xi_3^6\xi_4^3\xi_3^3\xi_1^2 - \xi_3^6\xi_4^3\xi_2^2\xi_1^2)/(\xi_4^3({\xi_4^3}^2{\xi_1^2}^2 - 2\xi_4^3\xi_3^3\xi_2^2\xi_1^2 + 2\xi_4^3\xi_1^2 + {\xi_3^3}^2{\xi_2^2}^2 + {\xi_3^3}^2 + {\xi_2^2}^2 + 1))$;
$J_4^5 = (-\xi_4^6{\xi_4^3}^2\xi_3^3\xi_1^2 + \xi_4^6{\xi_4^3}^2\xi_2^2\xi_1^2 - 2\xi_4^6\xi_4^3\xi_3^3{\xi_2^2}^2 - 2\xi_4^6\xi_4^3\xi_3^3 - \xi_4^6{\xi_3^3}^3\xi_1^2 - \xi_4^6{\xi_3^3}^2\xi_2^2\xi_1^2 - \xi_4^6\xi_3^3\xi_1^2 - \xi_4^6\xi_2^2\xi_1^2 + \xi_3^6{\xi_4^3}^3\xi_1^2 + \xi_3^6{\xi_4^3}^2{\xi_2^2}^2 + \xi_3^6{\xi_4^3}^2{\xi_1^2}^2 + \xi_3^6{\xi_4^3}^2 + \xi_3^6\xi_4^3{\xi_3^3}^2\xi_1^2 + \xi_3^6\xi_4^3\xi_1^2)/({\xi_4^3}^2{\xi_1^2}^2 - 2\xi_4^3\xi_3^3\xi_2^2\xi_1^2 + 2\xi_4^3\xi_1^2 + {\xi_3^3}^2{\xi_2^2}^2 + {\xi_3^3}^2 + {\xi_2^2}^2 + 1)$;
$J_5^5 = (-{\xi_4^3}^2\xi_2^2\xi_1^2 + \xi_4^3\xi_3^3{\xi_2^2}^2 - \xi_4^3\xi_3^3{\xi_1^2}^2 + \xi_4^3\xi_3^3 + {\xi_3^3}^2\xi_2^2\xi_1^2 + \xi_2^2\xi_1^2)/({\xi_4^3}^2\xi_1^2 + \xi_4^3{\xi_2^2}^2 + \xi_4^3{\xi_1^2}^2 + \xi_4^3 + {\xi_3^3}^2\xi_1^2 + \xi_1^2)$;
$J_6^5 = ({\xi_4^3}^2{\xi_2^2}^2 + {\xi_4^3}^2 + 2\xi_4^3\xi_3^3\xi_2^2\xi_1^2 + 2\xi_4^3\xi_1^2 + {\xi_3^3}^2{\xi_1^2}^2 + {\xi_1^2}^2)/({\xi_4^3}^2\xi_1^2 + \xi_4^3{\xi_2^2}^2 + \xi_4^3{\xi_1^2}^2 + \xi_4^3 + {\xi_3^3}^2\xi_1^2 + \xi_1^2)$;
$J_5^6 = (-{\xi_4^3}^2{\xi_1^2}^2 + 2\xi_4^3\xi_3^3\xi_2^2\xi_1^2 - 2\xi_4^3\xi_1^2 - {\xi_3^3}^2{\xi_2^2}^2 - {\xi_3^3}^2 - {\xi_2^2}^2 - 1)/({\xi_4^3}^2\xi_1^2 + \xi_4^3{\xi_2^2}^2 + \xi_4^3{\xi_1^2}^2 + \xi_4^3 + {\xi_3^3}^2\xi_1^2 + \xi_1^2)$;
$J_6^6 = ({\xi_4^3}^2\xi_2^2\xi_1^2 - \xi_4^3\xi_3^3{\xi_2^2}^2 + \xi_4^3\xi_3^3{\xi_1^2}^2 - \xi_4^3\xi_3^3 - {\xi_3^3}^2\xi_2^2\xi_1^2 - \xi_2^2\xi_1^2)/({\xi_4^3}^2\xi_1^2 + \xi_4^3{\xi_2^2}^2 + \xi_4^3{\xi_1^2}^2 + \xi_4^3 + {\xi_3^3}^2\xi_1^2 + \xi_1^2)$;

and the parameters are subject to the conditions

$$\xi_1^2\xi_4^3(\xi_3^3+\xi_2^2)({\xi_4^3}^2\xi_1^2 + \xi_4^3({\xi_2^2}^2 + {\xi_1^2}^2 + 1) + ({\xi_3^3}^2+1)\xi_1^2) \neq 0 : \tag{102}$$

(Note that ${\xi_4^3}^2{\xi_1^2}^2 - 2\xi_4^3\xi_1^2(\xi_3^3\xi_2^2 - 1) + ({\xi_2^2}^2+1)({\xi_3^3}^2+1) \neq 0$ is automatic from $\xi_3^3 + \xi_2^2 \neq 0$.)

13.2.4 Case $\xi^2_1 \neq 0, \xi^2_3 = \xi^2_4 = 0, \xi^3_3 = -\xi^2_2, \xi^3_4 = -\xi^2_1$.

$$J = \begin{pmatrix} 0 & -\frac{1}{\xi^2_1} & 0 & 0 & 0 & 0 \\ \xi^2_1 & 0 & 0 & 0 & 0 & 0 \\ \xi^3_1 & \xi^4_1 & 0 & -\xi^2_1 & 0 & 0 \\ \xi^4_1 & -\xi^3_1 & \frac{1}{\xi^2_1} & 0 & 0 & 0 \\ \frac{-\xi^6_4\xi^4_1-\xi^6_3\xi^3_1-\xi^6_2\xi^2_1+\xi^6_1\xi^5_5}{\xi^6_5} & \frac{\xi^6_4\xi^3_1\xi^2_1-\xi^6_3\xi^4_1\xi^2_1+\xi^6_2\xi^5_5\xi^2_1+\xi^6_1}{\xi^6_5\xi^2_1} & \frac{-\xi^6_4+\xi^6_3\xi^5_5\xi^2_1}{\xi^6_5\xi^2_1} & \frac{\xi^6_4\xi^5_5+\xi^6_3\xi^2_1}{\xi^6_5} & \xi^5_5 & -\frac{{\xi^5_5}^2+1}{\xi^6_5} \\ \xi^6_1 & \xi^6_2 & \xi^6_3 & \xi^6_4 & \xi^6_5 & -\xi^5_5 \end{pmatrix}$$

where

$$ {\xi^2_1}^2 = 1 \tag{103}$$

13.2.5 Case $\xi^2_1 \neq 0, \xi^2_3 = \xi^2_4 = 0, \xi^3_3 = -\xi^2_2, \xi^3_4 \neq -\xi^2_1$.

$$J = \begin{pmatrix} -\xi^2_2 & -\frac{{\xi^2_2}^2+1}{\xi^2_1} & 0 & 0 & 0 & 0 \\ \boxed{\xi^2_1} & \boxed{\xi^2_2} & 0 & 0 & 0 & 0 \\ \boxed{\xi^3_1} & -\frac{\xi^4_1\xi^3_4-2\xi^3_1\xi^2_2}{\xi^2_1} & -\xi^2_2 & \boxed{\xi^3_4} & 0 & 0 \\ \boxed{\xi^4_1} & \frac{\xi^3_1({\xi^2_2}^2+1)}{\xi^3_4\xi^2_1)} & -\frac{{\xi^2_2}^2+1}{\xi^3_4} & \xi^2_2 & 0 & 0 \\ * & * & * & * & \frac{\xi^2_2(-\xi^3_4+\xi^2_1)}{\xi^3_4+\xi^2_1} & \frac{{\xi^3_4}^2{\xi^2_2}^2+{\xi^3_4}^2-2\xi^3_4{\xi^2_2}^2\xi^2_1+2\xi^3_4\xi^2_1+{\xi^2_2}^2{\xi^2_1}^2+{\xi^2_1}^2}{{\xi^3_4}^2\xi^2_1+\xi^3_4{\xi^2_2}^2+\xi^3_4{\xi^2_1}^2+\xi^3_4+{\xi^2_2}^2\xi^2_1+\xi^2_1} \\ \boxed{\xi^6_1} & \boxed{\xi^6_2} & \boxed{\xi^6_3} & \boxed{\xi^6_4} & -\frac{\xi^3_4\xi^2_1+{\xi^2_2}^2+1}{\xi^3_4+\xi^2_1} & \frac{\xi^2_2(\xi^3_4-\xi^2_1)}{\xi^3_4+\xi^2_1} \end{pmatrix}$$

where $J^5_1 = (\xi^6_4\xi^4_1\xi^3_4 + \xi^6_4\xi^4_1\xi^2_1 + \xi^6_3\xi^3_4\xi^3_1 + \xi^6_3\xi^3_1\xi^2_1 + \xi^6_2\xi^3_4\xi^2_1 + \xi^6_2{\xi^2_1}^2 - 2\xi^6_1\xi^2_2\xi^2_1)/(\xi^3_4\xi^2_1 + {\xi^2_2}^2 + 1)$;
$J^5_2 = (\xi^6_4\xi^3_4\xi^3_1{\xi^2_2}^2 + \xi^6_4\xi^3_4\xi^3_1 + \xi^6_4\xi^3_1{\xi^2_2}^2\xi^2_1 + \xi^6_4\xi^3_1\xi^2_1 - \xi^6_3\xi^4_1{\xi^3_4}^3 - \xi^6_3\xi^4_1{\xi^3_4}^2\xi^2_1 + 2\xi^6_3{\xi^3_4}^2\xi^3_1\xi^2_2 + 2\xi^6_3\xi^3_4\xi^3_1\xi^2_2\xi^2_1 + 2\xi^6_2{\xi^3_4}^2\xi^2_2\xi^2_1 - \xi^6_1{\xi^3_4}^2{\xi^2_2}^2 - \xi^6_1{\xi^3_4}^2 - \xi^6_1\xi^3_4{\xi^2_2}^2\xi^2_1 - \xi^6_1\xi^3_4\xi^2_1)/(\xi^3_4\xi^2_1(\xi^3_4\xi^2_1 + {\xi^2_2}^2 + 1))$;
$J^5_3 = (-\xi^6_4\xi^3_4{\xi^2_2}^2 - \xi^6_4\xi^3_4 - \xi^6_4{\xi^2_2}^2\xi^2_1 - \xi^6_4\xi^2_1 - 2\xi^6_3\xi^3_4\xi^2_2\xi^2_1)/(\xi^3_4(\xi^3_4\xi^2_1 + {\xi^2_2}^2 + 1))$;
$J^5_4 = (\xi^3_4(2\xi^6_4\xi^2_2 + \xi^6_3\xi^3_4 + \xi^6_3\xi^2_1))/(\xi^3_4\xi^2_1 + {\xi^2_2}^2 + 1)$;
and the parameters are subject to the conditions

$$\xi^2_1\xi^3_4(\xi^3_4 + \xi^2_1)({\xi^2_2}^2 + \xi^3_4\xi^2_1 + 1) \neq 0 \tag{104}$$

13.2.6 Conclusions for the case $\xi^2_1 \neq 0$.

In each of the 5 subcases in the case $\xi^2_1 \neq 0$, after completing a set of common steps, one is left with solving the two equations 14|5 and 14|6 in the 14 variables

$$\xi^2_1, \xi^2_2, \xi^2_3, \xi^2_4, \xi^3_1, \xi^3_3, \xi^3_4, \xi^4_1, \xi^5_5, \xi^6_1, \xi^6_2, \xi^6_3, \xi^6_4, \xi^6_5$$

in the open subset $\xi^6_5(\xi^3_4\xi^2_1 - \xi^3_1\xi^2_4) \neq 0$ of $\mathbb{R}^{14}$. That is, the initial system comprised of all the torsion equations and the equation $J^2 = -1$ in $\mathbb{R}^{36}$ is reduced after the common steps to a system equivalent to the 2 mentioned equations, which reads

$$\begin{cases} f = 0 \\ g = 0 \end{cases} \tag{105}$$

where : $f = \xi^6_5\xi^5_5\xi^3_4{\xi^2_1}^2 + \xi^6_5\xi^5_5{\xi^2_1}^3 + \xi^6_5\xi^4_1\xi^3_4\xi^2_4\xi^2_1 - \xi^6_5\xi^4_1\xi^3_1{\xi^2_4}^2 + \xi^6_5\xi^3_4\xi^2_2{\xi^2_1}^2 + \xi^6_5\xi^3_3\xi^3_1\xi^2_4\xi^2_1 + \xi^6_5\xi^3_3{\xi^2_1}^3 - \xi^6_5{\xi^3_1}^2\xi^2_4\xi^2_3 - \xi^6_5\xi^3_1\xi^2_4\xi^2_2\xi^2_1 - \xi^6_5\xi^3_1\xi^2_3{\xi^2_1}^2 + {\xi^5_5}^2\xi^3_3{\xi^2_1}^2 + {\xi^5_5}^2\xi^2_2{\xi^2_1}^2 + \xi^3_3{\xi^2_1}^2 + \xi^2_2{\xi^2_1}^2$ and $g = \xi^6_5\xi^3_4{\xi^2_1}^2 + \xi^6_5{\xi^2_1}^3 + \xi^5_5\xi^3_3{\xi^2_1}^2 + \xi^5_5\xi^2_2{\xi^2_1}^2 - \xi^4_1\xi^3_4\xi^2_3\xi^2_1 + \xi^4_1\xi^3_1\xi^2_4\xi^2_3 + \xi^3_4{\xi^2_1}^3 - \xi^3_3\xi^3_1\xi^2_3\xi^2_1 - \xi^3_3\xi^2_2{\xi^2_1}^2 + {\xi^3_1}^2{\xi^2_3}^2 - \xi^3_1\xi^2_4{\xi^2_1}^2 + \xi^3_1\xi^2_3\xi^2_2\xi^2_1 + {\xi^2_1}^2$.
Hence, if $\mathfrak{X}_{\xi^2_1\neq 0}$ denotes the subset of $\mathfrak{X}_{M5}$ such that $\xi^2_1 \neq 0$, to conclude that $\mathfrak{X}_{\xi^2_1\neq 0}$ is a 12-dimensional submanifold of $\mathbb{R}^{36}$, it will be sufficient to prove that the preceding system is of maximal rank 2, that is in each of the subcases some 2-jacobian doesn't vanish.

- First, one has

$$\frac{D(f,g)}{D(\xi^4_1, \xi^3_1)} = -\frac{1}{{\xi^2_1}^2}\,(\xi^3_4\xi^2_1 - \xi^3_1\xi^2_4)({\xi^2_4}^2 + {\xi^2_3}^2)$$

hence this 2-jacobian doesn't vanish if $\xi^2_4 \neq 0$ or $\xi^2_3 \neq 0$.

- Suppose $\xi^2_4 = \xi^2_3 = 0$. Then

$$\frac{D(f,g)}{D(\xi^5_5, \xi^6_5)} = -\frac{1}{{\xi^6_5}^2}\,\left((\xi^6_5(\xi^3_4 + \xi^2_1) + \xi^5_5(\xi^3_3 + \xi^2_2))^2 + (\xi^3_3 + \xi^2_2)^2\right)$$

hence this 2-jacobian doesn't vanish if $\xi_3^3 + \xi_2^2 \neq 0$.
• Suppose $\xi_4^2 = \xi_3^2 = 0$ and $\xi_3^3 = -\xi_2^2$. Then

$$\frac{D(f,g)}{D(\xi_5^5, \xi_5^6)} = (\xi_4^3 + \xi_1^2)^2$$

hence this 2-jacobian doesn't vanish if $\xi_4^3 + \xi_1^2 \neq 0$.
• Suppose $\xi_4^2 = \xi_3^2 = 0$ and $\xi_3^3 = -\xi_2^2$ and $\xi_4^3 = -\xi_1^2$. Then the equation $f = 0$ reads $-2\xi_2^2\xi_1^2 = 0$, hence $\xi_2^2 = 0$. Then the equation $g = 0$ reads ${\xi_2^2}^2 - {\xi_1^2}^2 + 1 = 0$, hence $\xi_1^2 = \pm 1$. Then

$$\begin{aligned} \frac{D(f,g)}{D(\xi_2^2, \xi_3^3)} &= -2\xi_5^5\xi_1^2 \\ \frac{D(f,g)}{D(\xi_3^2, \xi_4^2)} &= {\xi_1^4}^2 + {\xi_1^3}^2 \end{aligned}$$

hence if ξ_5^5 or ξ_1^4 or ξ_1^3 is $\neq 0$, one of these 2-jacobians doesn't vanish. On the other hand, if $\xi_5^5 = \xi_1^4 = \xi_1^3 = 0$,

$$\frac{D(f,g)}{D(\xi_1^2, \xi_3^3)} = \xi_1^2(1 - {\xi_5^6}^2)$$

hence if $\xi_5^6 \neq \pm 1$, this 2-jacobian doesn't vanish. Suppose now that $\xi_5^6 = \pm 1$. Then

$$\frac{D(f,g)}{D(\xi_1^2, \xi_2^2)} = 2\xi_5^6(\xi_1^2 - \xi_5^6)$$

hence if $\xi_5^6 \neq \xi_1^2$, this 2-jacobian doesn't vanish. Suppose finally that $\xi_5^6 = \xi_1^2$. Then

$$\frac{D(f,g)}{D(\xi_3^3, \xi_4^3)} = 4.$$

This ends the proof that the system (105) is of maximal rank.

13.3

From 13.1, one has that the subset $\mathfrak{X}_{{\xi_4^2}^2 + {\xi_3^2}^2 \neq 0}$ of $\mathfrak{X}_{M5}$ such that ${\xi_4^2}^2 + {\xi_3^2}^2 \neq 0$, is a 12-dimensional submanifold of $\mathbb{R}^{36}$ with the global chart (98). Now if $\xi_4^2 = \xi_3^2 = 0$, then necessarily $\xi_1^2 \neq 0$. Hence

$$\mathfrak{X}_{M5} = \mathfrak{X}_{{\xi_4^2}^2 + {\xi_3^2}^2 \neq 0} \cup \mathfrak{X}_{\xi_1^2 \neq 0}$$

and we conclude from 13.2.6 that $\mathfrak{X}_{M5}$ is a 12-dimensional submanifold of $\mathbb{R}^{36}$. For example, one can take as a local chart in a neighborhood of the canonical complex structure J_0 :

$$J = \begin{pmatrix} * & * & * & * & 0 & 0 \\ \boxed{\xi_1^2} & \boxed{\xi_2^2} & \boxed{\xi_3^2} & \boxed{\xi_4^2} & 0 & 0 \\ \boxed{\xi_1^3} & * & * & * & 0 & 0 \\ \boxed{\xi_1^4} & * & * & * & 0 & 0 \\ * & * & * & * & \boxed{\xi_5^5} & -\frac{{\xi_5^5}^2+1}{\xi_5^6} \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & \boxed{\xi_5^6} & -\xi_5^5 \end{pmatrix}$$

where
$J_1^1 = (-(\xi_1^3\xi_3^2 + \xi_2^2\xi_1^2 + \xi_1^4\xi_4^2))/\xi_1^2;$
$J_2^1 = (-((((\xi_5^6\xi_1^4{\xi_4^2}^2 + \xi_5^6\xi_1^3\xi_4^2\xi_3^2 + \xi_5^5\xi_1^4\xi_4^2\xi_3^2 + \xi_5^5\xi_1^3{\xi_4^2}^2 + \xi_5^5\xi_1^3{\xi_3^2}^2)\xi_1^4 + ({\xi_2^2}^2 + 1)\xi_5^6{\xi_1^2}^2 - (\xi_4^2\xi_1^2 - 2\xi_3^2\xi_2^2)\xi_5^5\xi_1^4\xi_1^2 + (2\xi_4^2\xi_2^2 + \xi_3^2\xi_1^2)\xi_5^6\xi_1^4\xi_1^2 + (\xi_1^3\xi_4^2\xi_3^2 + 2\xi_4^2\xi_2^2\xi_1^2 + \xi_3^2{\xi_1^2}^2)\xi_5^5\xi_1^3)\xi_5^6 + ((\xi_1^3{\xi_3^2}^2 - \xi_4^2{\xi_1^2}^2 + 2\xi_3^2\xi_2^2\xi_1^2)\xi_1^3 + ({\xi_2^2}^2 + 1){\xi_1^2}^2 + \xi_1^4\xi_1^3\xi_4^2\xi_3^2)({\xi_5^5}^2 + 1))\xi_1^2 + (((\xi_1^3\xi_3^2\xi_2^2 + {\xi_2^2}^2\xi_1^2 - \xi_1^2)\xi_4^2\xi_1^2 + ({\xi_2^2}^2 + 1)\xi_1^3{\xi_3^2}^2)\xi_1^3 + ({\xi_2^2}^2 + {\xi_1^2}^2 + 1)({\xi_2^2}^2 + 1){\xi_1^2}^2 + (2{\xi_2^2}^2 + {\xi_1^2}^2 + 2)\xi_1^3\xi_3^2\xi_2^2\xi_1^2 + (((2{\xi_2^2}^2 + {\xi_1^2}^2 + 2)\xi_4^2\xi_2^2 - (\xi_2^2 + 1)(\xi_2^2 - 1)\xi_3^2\xi_1^2)\xi_1^2 + \xi_1^4{\xi_4^2}^2{\xi_2^2}^2 + \xi_1^4{\xi_4^2}^2 - \xi_1^4\xi_4^2\xi_3^2\xi_2^2\xi_1^2 + \xi_1^3{\xi_4^2}^2\xi_2^2\xi_1^2 + 2\xi_1^3\xi_4^2\xi_3^2{\xi_2^2}^2 + 2\xi_1^3\xi_4^2\xi_3^2 - \xi_1^3{\xi_3^2}^2\xi_2^2\xi_1^2)\xi_1^4)\xi_5^6))/((((({\xi_1^2}^2 + 1 + {\xi_2^2}^2)\xi_1^2 + (\xi_4^2\xi_1^2 + \xi_3^2\xi_2^2)\xi_1^3 + (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_1^4)\xi_5^6 + (\xi_5^6\xi_1^3\xi_4^2 + \xi_5^6{\xi_1^2}^2 - \xi_5^5\xi_1^4\xi_4^2 + \xi_5^5\xi_1^3\xi_3^2)\xi_5^6 - ({\xi_5^5}^2 + 1)(\xi_1^4\xi_3^2 - {\xi_1^2}^2))){\xi_1^2}^2);$
$J_3^1 = (-(((((\xi_5^6\xi_4^2 + \xi_5^5\xi_3^2)(\xi_1^4\xi_4^2 + 2\xi_1^3\xi_3^2) + 2(\xi_4^2\xi_2^2 + \xi_3^2\xi_1^2)\xi_5^6\xi_1^2)\xi_4^2 - (2{\xi_4^2}^2\xi_1^2 - 2\xi_4^2\xi_3^2\xi_2^2 - {\xi_3^2}^2\xi_1^2)\xi_5^5\xi_1^2)\xi_5^6\xi_1^4 - ({\xi_5^5}^2 + 1)(\xi_1^4\xi_3^2 - {\xi_1^2}^2)\xi_4^2{\xi_1^2}^2 + ((\xi_1^3\xi_4^2\xi_3^2 + 2\xi_4^2\xi_2^2\xi_1^2 + \xi_3^2{\xi_1^2}^2)\xi_1^3\xi_3^2 + (\xi_4^2{\xi_2^2}^2 + \xi_4^2 + 2\xi_3^2\xi_2^2\xi_1^2){\xi_1^2}^2){\xi_5^6}^2 + ((({\xi_2^2}^2 - {\xi_1^2}^2 + 1)\xi_3^2 - 2\xi_4^2\xi_2^2\xi_1^2){\xi_1^2}^2 + (\xi_1^3{\xi_3^2}^2 - \xi_4^2{\xi_1^2}^2 + 2\xi_3^2\xi_2^2\xi_1^2)\xi_1^3\xi_3^2)\xi_5^6\xi_5^5 + ((\xi_1^3\xi_4^2\xi_3^2\xi_1^2 + \xi_1^3{\xi_3^2}^2\xi_2^2 + \xi_4^2\xi_2^2{\xi_1^2}^2 + (2{\xi_2^2}^2 + {\xi_1^2}^2)\xi_3^2\xi_1^2)\xi_1^3\xi_3^2 + (({\xi_2^2}^2 + {\xi_1^2}^2 + 1)\xi_3^2\xi_2^2 + 2\xi_4^2\xi_1^2){\xi_1^2}^2 +$

$$(\xi_1^4{\xi_4^2}^2\xi_2^2 - \xi_1^4\xi_4^2\xi_3^2\xi_1^2 + \xi_1^3{\xi_4^2}^2\xi_1^2 + 2\xi_1^3\xi_4^2\xi_3^2\xi_2^2 - \xi_1^3{\xi_3^2}^2\xi_1^2 + (2\xi_4^2{\xi_2^2}^2 + \xi_4^2{\xi_1^2}^2 - \xi_3^2\xi_2^2\xi_1^2)\xi_1^2)\xi_1^4\xi_3^2)\xi_5^6))/(((({\xi_1^2}^2 + 1 + {\xi_2^2}^2)\xi_1^2 + (\xi_4^2\xi_1^2 + \xi_3^2\xi_2^2)\xi_1^3 + (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_1^4)\xi_5^6 + (\xi_5^6\xi_1^3\xi_4^2 + \xi_5^6{\xi_1^2}^2 - \xi_5^5\xi_1^4\xi_4^2 + \xi_5^5\xi_1^3\xi_3^2)\xi_5^6 - ({\xi_5^5}^2 + 1)(\xi_1^4\xi_3^2 - {\xi_1^2}^2)){\xi_1^2}^2);$$

$$J_4^1 = ((\xi_5^6\xi_1^3\xi_4^2\xi_3^2{\xi_1^2}^2 + \xi_5^6\xi_3^2{\xi_1^2}^4 + \xi_5^5{\xi_1^4}^2{\xi_4^2}^3 + 2\xi_5^5\xi_1^4\xi_1^3{\xi_4^2}^2\xi_3^2 + (2\xi_4^2\xi_2^2 + \xi_3^2\xi_1^2)\xi_5^5\xi_1^4\xi_4^2\xi_1^2 + ((\xi_1^3\xi_4^2{\xi_3^2}^2 - {\xi_4^2}^2{\xi_1^2}^2 + 2\xi_4^2\xi_3^2\xi_2^2\xi_1^2 + 2{\xi_3^2}^2{\xi_1^2}^2)\xi_1^3 + (\xi_4^2{\xi_2^2}^2 - \xi_4^2{\xi_1^2}^2 + \xi_4^2 + 2\xi_3^2\xi_2^2\xi_1^2){\xi_1^2}^2)\xi_5^5 - ((\xi_1^4{\xi_4^2}^2\xi_2^2 - \xi_1^4\xi_4^2\xi_3^2\xi_1^2 + \xi_1^3{\xi_4^2}^2\xi_1^2 + 2\xi_1^3\xi_4^2\xi_3^2\xi_2^2 - \xi_1^3{\xi_3^2}^2\xi_1^2 + (2\xi_4^2{\xi_2^2}^2 + \xi_4^2{\xi_1^2}^2 - \xi_3^2\xi_2^2\xi_1^2)\xi_1^2)\xi_1^4\xi_4^2 + (\xi_1^3\xi_4^2\xi_3^2\xi_1^2 + \xi_1^3{\xi_3^2}^2\xi_2^2 + \xi_4^2\xi_2^2{\xi_1^2}^2 + 2\xi_3^2{\xi_2^2}^2\xi_1^2 + \xi_3^2{\xi_1^2}^3)\xi_1^3\xi_4^2 + (\xi_4^2{\xi_2^2}^3 + \xi_4^2\xi_2^2{\xi_1^2}^2 + \xi_4^2\xi_2^2 - 2\xi_3^2\xi_1^2){\xi_1^2}^2))\xi_5^6 + ((({\xi_2^2}^2 + 1)\xi_3^2 - 2\xi_4^2\xi_2^2\xi_1^2){\xi_1^2}^2 + (\xi_1^3{\xi_3^2}^2 - 2\xi_4^2{\xi_1^2}^2 + 2\xi_3^2\xi_2^2\xi_1^2)\xi_1^3\xi_3^2 + ((\xi_1^4\xi_4^2 + 2\xi_1^3\xi_3^2)\xi_3^2 - (\xi_4^2\xi_1^2 - 2\xi_3^2\xi_2^2)\xi_1^2)\xi_1^4\xi_4^2)({\xi_5^5}^2 + 1))/(((({\xi_1^2}^2 + 1 + {\xi_2^2}^2)\xi_1^2 + (\xi_4^2\xi_1^2 + \xi_3^2\xi_2^2)\xi_1^3 + (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_1^4)\xi_5^6 + (\xi_5^6\xi_1^3\xi_4^2 + \xi_5^6{\xi_1^2}^2 - \xi_5^5\xi_1^4\xi_4^2 + \xi_5^5\xi_1^3\xi_3^2)\xi_5^6 - ({\xi_5^5}^2 + 1)(\xi_1^4\xi_3^2 - {\xi_1^2}^2)){\xi_1^2}^2);$$

$$J_2^3 = ((((\xi_5^6\xi_1^3\xi_4^2 + \xi_5^6{\xi_1^2}^2 + \xi_5^5\xi_1^3\xi_3^2 + 2\xi_5^5\xi_2^2\xi_1^2)\xi_1^4 + (\xi_1^3\xi_4^2 - {\xi_2^2}^2 + {\xi_1^2}^2 - 1)\xi_5^5\xi_1^3)\xi_5^6 + ((\xi_1^3\xi_3^2 + 2\xi_2^2\xi_1^2)\xi_1^3 + ({\xi_2^2}^2 + 1)\xi_1^4)({\xi_5^5}^2 + 1))\xi_1^2 + (((\xi_1^3\xi_4^2 + {\xi_2^2}^2 + {\xi_1^2}^2 + 1)\xi_2^2\xi_1^2 + ({\xi_2^2}^2 + 1)\xi_1^3\xi_3^2)\xi_1^3 + (\xi_1^3\xi_4^2{\xi_2^2}^2 + \xi_1^3\xi_4^2 - \xi_1^3\xi_3^2\xi_2^2\xi_1^2 + 2{\xi_1^2}^2)\xi_1^4)\xi_5^6)/(((({\xi_1^2}^2 + 1 + {\xi_2^2}^2)\xi_1^2 + (\xi_4^2\xi_1^2 + \xi_3^2\xi_2^2)\xi_1^3 + (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_1^4)\xi_5^6 + (\xi_5^6\xi_1^3\xi_4^2 + \xi_5^6{\xi_1^2}^2 - \xi_5^5\xi_1^4\xi_4^2 + \xi_5^5\xi_1^3\xi_3^2)\xi_5^6 - ({\xi_5^5}^2 + 1)(\xi_1^4\xi_3^2 - {\xi_1^2}^2))\xi_1^2);$$

$$J_3^3 = ((((\xi_5^6\xi_1^3\xi_4^2 + \xi_5^6{\xi_1^2}^2 + \xi_5^5\xi_1^3\xi_3^2)\xi_4^2 + (\xi_4^2\xi_2^2 + \xi_3^2\xi_1^2)\xi_5^5\xi_1^2)\xi_1^4 + (\xi_1^3\xi_4^2 + {\xi_1^2}^2)(\xi_1^3\xi_3^2 + \xi_2^2\xi_1^2)\xi_5^6)\xi_5^6 + ({\xi_5^5}^2 + 1)(\xi_1^4\xi_3^2 - {\xi_1^2}^2)\xi_2^2\xi_1^2 + ((\xi_1^3{\xi_3^2}^2 - \xi_4^2{\xi_1^2}^2 + \xi_3^2\xi_2^2\xi_1^2)\xi_1^3 + ({\xi_2^2}^2 - {\xi_1^2}^2 + 1){\xi_1^2}^2)\xi_5^6\xi_5^5 + ((\xi_1^3\xi_4^2\xi_3^2\xi_2^2 - \xi_1^3{\xi_3^2}^2\xi_1^2 + \xi_4^2\xi_1^2)\xi_1^4 + (\xi_1^3\xi_4^2\xi_1^2 + \xi_1^3\xi_3^2\xi_2^2 + {\xi_2^2}^2\xi_1^2 + {\xi_1^2}^3)\xi_1^3\xi_3^2)\xi_5^6)/(((({\xi_1^2}^2 + 1 + {\xi_2^2}^2)\xi_1^2 + (\xi_4^2\xi_1^2 + \xi_3^2\xi_2^2)\xi_1^3 + (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_1^4)\xi_5^6 + (\xi_5^6\xi_1^3\xi_4^2 + \xi_5^6{\xi_1^2}^2 - \xi_5^5\xi_1^4\xi_4^2 + \xi_5^5\xi_1^3\xi_3^2)\xi_5^6 - ({\xi_5^5}^2 + 1)(\xi_1^4\xi_3^2 - {\xi_1^2}^2))\xi_1^2);$$

$$J_4^3 = (({\xi_1^3}^2{\xi_4^2}^2\xi_1^2 + {\xi_1^3}^2\xi_4^2\xi_3^2\xi_2^2 + \xi_1^3\xi_4^2{\xi_2^2}^2\xi_1^2 + \xi_1^3\xi_4^2{\xi_1^2}^3 - \xi_1^3\xi_4^2\xi_1^2 - 2{\xi_1^2}^3 + (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_1^4\xi_1^3\xi_4^2 - (\xi_5^6\xi_1^3\xi_4^2{\xi_1^2}^2 + \xi_5^6{\xi_1^2}^4 + \xi_5^5\xi_1^4\xi_1^3{\xi_4^2}^2 + \xi_5^5{\xi_1^3}^2\xi_4^2\xi_3^2 + 2\xi_5^5\xi_1^3\xi_4^2\xi_2^2\xi_1^2 + 2\xi_5^5\xi_1^3\xi_3^2{\xi_1^2}^2 + 2\xi_5^5\xi_2^2{\xi_1^2}^3))\xi_5^6 - ((\xi_1^3{\xi_3^2}^2 - \xi_4^2{\xi_1^2}^2 + 2\xi_3^2\xi_2^2\xi_1^2)\xi_1^3 + ({\xi_2^2}^2 + 1){\xi_1^2}^2 + \xi_1^4\xi_1^3\xi_4^2\xi_3^2)({\xi_5^5}^2 + 1))/(((({\xi_1^2}^2 + 1 + {\xi_2^2}^2)\xi_1^2 + (\xi_4^2\xi_1^2 + \xi_3^2\xi_2^2)\xi_1^3 + (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_1^4)\xi_5^6 + (\xi_5^6\xi_1^3\xi_4^2 + \xi_5^6{\xi_1^2}^2 - \xi_5^5\xi_1^4\xi_4^2 + \xi_5^5\xi_1^3\xi_3^2)\xi_5^6 - ({\xi_5^5}^2 + 1)(\xi_1^4\xi_3^2 - {\xi_1^2}^2))\xi_1^2);$$

$$J_2^4 = (((\xi_5^6{\xi_1^4}^2\xi_4^2 + 2\xi_5^6\xi_1^4\xi_2^2\xi_1^2 - \xi_5^6\xi_1^3{\xi_2^2}^2 - \xi_5^6\xi_1^3 + \xi_5^5{\xi_1^4}^2\xi_3^2 + \xi_5^5\xi_1^4\xi_1^3\xi_4^2 + \xi_5^5\xi_1^4{\xi_2^2}^2 - \xi_5^5\xi_1^4{\xi_1^2}^2 + \xi_5^5\xi_1^4 + 2\xi_5^5\xi_1^3\xi_2^2\xi_1^2)\xi_5^6 + ({\xi_5^5}^2 + 1)(\xi_1^4\xi_3^2 - {\xi_1^2}^2)\xi_1^3)\xi_1^2 + ({\xi_1^4}^2\xi_4^2{\xi_2^2}^2 + {\xi_1^4}^2\xi_4^2 - {\xi_1^4}^2\xi_3^2\xi_2^2\xi_1^2 + \xi_1^4\xi_1^3\xi_4^2\xi_2^2\xi_1^2 + \xi_1^4\xi_1^3\xi_3^2{\xi_2^2}^2 + \xi_1^4\xi_1^3\xi_3^2 - 2\xi_1^3{\xi_1^2}^2 + ({\xi_2^2}^2 + {\xi_1^2}^2 + 1)\xi_1^4\xi_2^2\xi_1^2)\xi_5^6)/(((({\xi_1^2}^2 + 1 + {\xi_2^2}^2)\xi_1^2 + (\xi_4^2\xi_1^2 + \xi_3^2\xi_2^2)\xi_1^3 + (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_1^4)\xi_5^6 + (\xi_5^6\xi_1^3\xi_4^2 + \xi_5^6{\xi_1^2}^2 - \xi_5^5\xi_1^4\xi_4^2 + \xi_5^5\xi_1^3\xi_3^2)\xi_5^6 - ({\xi_5^5}^2 + 1)(\xi_1^4\xi_3^2 - {\xi_1^2}^2))\xi_1^2);$$

$$J_3^4 = ((\xi_5^6{\xi_1^4}^2{\xi_4^2}^2 + \xi_5^6\xi_1^4\xi_1^3\xi_4^2\xi_3^2 + 2\xi_5^6\xi_1^4\xi_4^2\xi_2^2\xi_1^2 + \xi_5^6\xi_1^4\xi_3^2{\xi_1^2}^2 + \xi_5^6{\xi_2^2}^2{\xi_1^2}^2 + \xi_5^6{\xi_1^2}^2 + \xi_5^5{\xi_1^4}^2\xi_4^2\xi_3^2 + \xi_5^5\xi_1^4\xi_1^3{\xi_3^2}^2 - 2\xi_5^5\xi_1^4\xi_4^2{\xi_1^2}^2 + 2\xi_5^5\xi_1^4\xi_3^2\xi_2^2\xi_1^2 - 2\xi_5^5\xi_2^2{\xi_1^2}^3)\xi_5^6 - ({\xi_5^5}^2 + 1)(\xi_1^4\xi_3^2 - {\xi_1^2}^2){\xi_1^2}^2 + ({\xi_1^4}^2\xi_4^2\xi_3^2\xi_2^2 - {\xi_1^4}^2{\xi_3^2}^2\xi_1^2 + \xi_1^4\xi_1^3\xi_4^2\xi_3^2\xi_1^2 + \xi_1^4\xi_1^3{\xi_3^2}^2\xi_2^2 + 2{\xi_1^2}^3 + ({\xi_2^2}^2 + {\xi_1^2}^2 - 1)\xi_1^4\xi_3^2\xi_1^2)\xi_5^6)/(((({\xi_1^2}^2 + 1 + {\xi_2^2}^2)\xi_1^2 + (\xi_4^2\xi_1^2 + \xi_3^2\xi_2^2)\xi_1^3 + (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_1^4)\xi_5^6 + (\xi_5^6\xi_1^3\xi_4^2 + \xi_5^6{\xi_1^2}^2 - \xi_5^5\xi_1^4\xi_4^2 + \xi_5^5\xi_1^3\xi_3^2)\xi_5^6 - ({\xi_5^5}^2 + 1)(\xi_1^4\xi_3^2 - {\xi_1^2}^2))\xi_1^2);$$

$$J_4^4 = (-((\xi_5^6\xi_1^3\xi_4^2\xi_2^2\xi_1^2 + \xi_5^6\xi_2^2{\xi_1^2}^3 + \xi_5^5{\xi_1^4}^2{\xi_4^2}^2 + \xi_5^5\xi_1^4\xi_1^3\xi_4^2\xi_3^2 + \xi_5^5\xi_1^4\xi_4^2\xi_2^2\xi_1^2 + \xi_5^5\xi_1^4\xi_3^2{\xi_1^2}^2)\xi_5^6 + ({\xi_5^5}^2 + 1)(\xi_1^4\xi_4^2 + \xi_1^3\xi_3^2 + \xi_2^2\xi_1^2)(\xi_1^4\xi_3^2 - {\xi_1^2}^2) - ((\xi_4^2\xi_1^2 - \xi_3^2\xi_2^2)\xi_1^3 - ({\xi_2^2}^2 - {\xi_1^2}^2 + 1)\xi_1^2)\xi_5^6\xi_5^5\xi_1^2 - ({\xi_1^4}^2{\xi_4^2}^2\xi_2^2 - {\xi_1^4}^2\xi_4^2\xi_3^2\xi_1^2 + \xi_1^4\xi_1^3{\xi_4^2}^2\xi_1^2 + \xi_1^4\xi_1^3\xi_4^2\xi_3^2\xi_2^2 + \xi_1^3\xi_3^2\xi_1^2 + ({\xi_2^2}^2 + {\xi_1^2}^2)\xi_1^4\xi_4^2\xi_1^2)\xi_5^6))/(((({\xi_1^2}^2 + 1 + {\xi_2^2}^2)\xi_1^2 + (\xi_4^2\xi_1^2 + \xi_3^2\xi_2^2)\xi_1^3 + (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_1^4)\xi_5^6 + (\xi_5^6\xi_1^3\xi_4^2 + \xi_5^6{\xi_1^2}^2 - \xi_5^5\xi_1^4\xi_4^2 + \xi_5^5\xi_1^3\xi_3^2)\xi_5^6 - ({\xi_5^5}^2 + 1)(\xi_1^4\xi_3^2 - {\xi_1^2}^2))\xi_1^2);$$

$$J_1^5 = ((\xi_1^3\xi_3^2 + \xi_2^2\xi_1^2 + \xi_1^4\xi_4^2 + \xi_5^5\xi_1^2)\xi_1^6 - \xi_2^6{\xi_1^2}^2 - \xi_3^6\xi_1^3\xi_1^2 - \xi_4^6\xi_1^4\xi_1^2)/(\xi_5^6\xi_1^2);$$

$$J_2^5 = (((((\xi_1^3\xi_3^2\xi_2^2 + {\xi_2^2}^2\xi_1^2 - \xi_1^2)\xi_4^2\xi_1^2 + ({\xi_2^2}^2 + 1)\xi_1^3{\xi_3^2}^2)\xi_1^3 + ({\xi_2^2}^2 + {\xi_1^2}^2 + 1)({\xi_2^2}^2 + 1){\xi_1^2}^2 + (2{\xi_2^2}^2 + {\xi_1^2}^2 + 2)\xi_1^3\xi_3^2\xi_2^2\xi_1^2 + (((2{\xi_2^2}^2 + {\xi_1^2}^2 + 2)\xi_4^2\xi_2^2 - (\xi_2^2 + 1)(\xi_2^2 - 1)\xi_3^2\xi_1^2)\xi_1^2 + \xi_1^4{\xi_4^2}^2{\xi_2^2}^2 + \xi_1^4{\xi_4^2}^2 - \xi_1^4\xi_4^2\xi_3^2\xi_2^2\xi_1^2 + \xi_1^3{\xi_4^2}^2\xi_2^2\xi_1^2 + 2\xi_1^3\xi_4^2\xi_3^2{\xi_2^2}^2 + 2\xi_1^3\xi_4^2\xi_3^2 - \xi_1^3{\xi_3^2}^2\xi_2^2\xi_1^2)\xi_1^4 + ((\xi_1^4\xi_4^2\xi_3^2 + \xi_1^3{\xi_4^2}^2 + \xi_1^3{\xi_3^2}^2 - (\xi_4^2\xi_1^2 - 2\xi_3^2\xi_2^2)\xi_1^2)\xi_1^4 + (\xi_1^3\xi_4^2\xi_3^2 + 2\xi_4^2\xi_2^2\xi_1^2 + \xi_3^2{\xi_1^2}^2)\xi_1^3)\xi_5^5\xi_1^2)\xi_1^6 - (\xi_5^5\xi_1^4\xi_4^2 - \xi_5^5\xi_1^3\xi_3^2 - \xi_1^4\xi_4^2\xi_2^2 + \xi_1^4\xi_3^2\xi_1^2 - \xi_1^3\xi_4^2\xi_1^2 - \xi_1^3\xi_3^2\xi_2^2 - {\xi_2^2}^2\xi_1^2 - {\xi_1^2}^3 - \xi_1^2)(\xi_5^5 - \xi_2^2)\xi_2^6{\xi_1^2}^2 - (((\xi_1^3\xi_4^2 + {\xi_2^2}^2 + {\xi_1^2}^2 + 1)\xi_2^2\xi_1^2 + ({\xi_2^2}^2 + 1)\xi_1^3\xi_3^2)\xi_1^3 + (\xi_1^3\xi_4^2{\xi_2^2}^2 + \xi_1^3\xi_4^2 - \xi_1^3\xi_3^2\xi_2^2\xi_1^2 + 2{\xi_1^2}^2)\xi_1^4 + ((\xi_1^3\xi_4^2 - {\xi_2^2}^2 + {\xi_1^2}^2 - 1)\xi_1^3 + (\xi_1^3\xi_3^2 + 2\xi_2^2\xi_1^2)\xi_1^4)\xi_5^5\xi_1^2)\xi_3^6\xi_1^2)\xi_5^6 + (((((\xi_1^4\xi_4^2 + \xi_1^3\xi_3^2)\xi_1^4\xi_4^2 + ({\xi_2^2}^2 + 1){\xi_1^2}^2 + (2\xi_4^2\xi_2^2 + \xi_3^2\xi_1^2)\xi_1^4\xi_1^2)\xi_1^6 + (\xi_5^5 - \xi_2^2)(\xi_1^3\xi_4^2 + {\xi_1^2}^2)\xi_2^6\xi_1^2 - (\xi_1^3\xi_4^2 + {\xi_1^2}^2)\xi_3^6\xi_1^4\xi_1^2)\xi_5^6 - ({\xi_1^4}^2\xi_4^2{\xi_2^2}^2 + {\xi_1^4}^2\xi_4^2 - {\xi_1^4}^2\xi_3^2\xi_2^2\xi_1^2 + \xi_1^4\xi_1^3\xi_4^2\xi_2^2\xi_1^2 + \xi_1^4\xi_1^3\xi_3^2{\xi_2^2}^2 + \xi_1^4\xi_1^3\xi_3^2 - 2\xi_1^3{\xi_1^2}^2 + ({\xi_2^2}^2 + {\xi_1^2}^2 + 1)\xi_1^4\xi_2^2\xi_1^2 + (\xi_5^6{\xi_1^4}^2\xi_4^2 + 2\xi_5^6\xi_1^4\xi_2^2\xi_1^2 - \xi_5^6\xi_1^3{\xi_2^2}^2 - \xi_5^6\xi_1^3 + \xi_5^5{\xi_1^4}^2\xi_3^2 + \xi_5^5\xi_1^4\xi_1^3\xi_4^2 + \xi_5^5\xi_1^4{\xi_2^2}^2 - \xi_5^5\xi_1^4{\xi_1^2}^2 + \xi_5^5\xi_1^4 + 2\xi_5^5\xi_1^3\xi_2^2\xi_1^2)\xi_1^2)\xi_4^6)\xi_5^6 + (((\xi_1^3{\xi_3^2}^2 - \xi_4^2{\xi_1^2}^2 + 2\xi_3^2\xi_2^2\xi_1^2)\xi_1^3 + ({\xi_2^2}^2 + 1){\xi_1^2}^2 + \xi_1^4\xi_1^3\xi_4^2\xi_3^2)\xi_1^6 - (\xi_5^5 - \xi_2^2)(\xi_1^4\xi_3^2 - {\xi_1^2}^2)\xi_2^6\xi_1^2 - ((\xi_1^3\xi_3^2 + 2\xi_2^2\xi_1^2)\xi_1^3 + ({\xi_2^2}^2 + 1)\xi_1^4)\xi_3^6\xi_1^2 - (\xi_1^4\xi_3^2 - {\xi_1^2}^2)\xi_4^6\xi_1^3\xi_1^2)({\xi_5^5}^2 + 1))\xi_1^2)/(((({\xi_1^2}^2 + 1 + {\xi_2^2}^2)\xi_1^2 + (\xi_4^2\xi_1^2 + \xi_3^2\xi_2^2)\xi_1^3 + (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_1^4)\xi_5^6 + {\xi_5^6}^2\xi_1^3\xi_4^2 + {\xi_5^6}^2{\xi_1^2}^2 - \xi_5^6\xi_5^5\xi_1^4\xi_4^2 + \xi_5^6\xi_5^5\xi_1^3\xi_3^2 - {\xi_5^5}^2\xi_1^4\xi_3^2 + {\xi_5^5}^2{\xi_1^2}^2 - (\xi_1^4\xi_3^2 - {\xi_1^2}^2))\xi_5^6{\xi_1^2}^2);$$

$$J_3^5 = ((((\xi_1^3\xi_4^2\xi_3^2\xi_1^2 + \xi_1^3{\xi_3^2}^2\xi_2^2 + \xi_4^2\xi_2^2{\xi_1^2}^2 + (2{\xi_2^2}^2 + {\xi_1^2}^2)\xi_3^2\xi_1^2)\xi_1^3\xi_3^2 + (({\xi_2^2}^2 + {\xi_1^2}^2 + 1)\xi_3^2\xi_2^2 + 2\xi_4^2\xi_1^2){\xi_1^2}^2 + (\xi_1^4{\xi_4^2}^2\xi_2^2 - \xi_1^4\xi_4^2\xi_3^2\xi_1^2 + \xi_1^3{\xi_4^2}^2\xi_1^2 + 2\xi_1^3\xi_4^2\xi_3^2\xi_2^2 - \xi_1^3{\xi_3^2}^2\xi_1^2 + (2\xi_4^2{\xi_2^2}^2 + \xi_4^2{\xi_1^2}^2 - \xi_3^2\xi_2^2\xi_1^2)\xi_1^2)\xi_1^4\xi_3^2 + ((({\xi_2^2}^2 - {\xi_1^2}^2 + 1)\xi_3^2 - 2\xi_4^2\xi_2^2\xi_1^2){\xi_1^2}^2 + (\xi_1^3{\xi_3^2}^2 - \xi_4^2{\xi_1^2}^2 + 2\xi_3^2\xi_2^2\xi_1^2)\xi_1^3\xi_3^2 + ((\xi_1^4\xi_4^2 + 2\xi_1^3\xi_3^2)\xi_4^2\xi_3^2 - (2{\xi_4^2}^2\xi_1^2 - 2\xi_4^2\xi_3^2\xi_2^2 - {\xi_3^2}^2\xi_1^2)\xi_1^2)\xi_1^4)\xi_5^5)\xi_1^6 - (({\xi_1^2}^2 + 1 + {\xi_2^2}^2)\xi_1^2 + (\xi_4^2\xi_1^2 + \xi_3^2\xi_2^2)\xi_1^3 + (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_1^4 - (\xi_1^4\xi_4^2 - \xi_1^3\xi_3^2)\xi_5^5)\xi_2^6\xi_3^2{\xi_1^2}^2 - ((\xi_5^5\xi_1^4\xi_4^2\xi_1^2 - \xi_5^5\xi_1^3\xi_3^2\xi_1^2 + \xi_1^4\xi_1^3\xi_4^2\xi_3^2 + 2\xi_1^4\xi_3^2{\xi_1^2}^2 + {\xi_1^3}^2{\xi_3^2}^2 - 2\xi_1^3\xi_4^2{\xi_1^2}^2 - 2{\xi_1^2}^4)\xi_5^5 + (\xi_1^3\xi_4^2\xi_3^2\xi_2^2 - \xi_1^3{\xi_3^2}^2\xi_1^2 + \xi_4^2\xi_1^2)\xi_1^4 + (\xi_1^3\xi_4^2\xi_1^2 + \xi_1^3\xi_3^2\xi_2^2 + {\xi_2^2}^2\xi_1^2 + {\xi_1^2}^3)\xi_1^3\xi_3^2)\xi_3^6\xi_1^2)\xi_5^6 + ((((\xi_1^4\xi_4^2 + 2\xi_1^3\xi_3^2)\xi_4^2 + 2(\xi_4^2\xi_2^2 + \xi_3^2\xi_1^2)\xi_1^2)\xi_1^4\xi_4^2 + (\xi_1^3\xi_4^2\xi_3^2 + 2\xi_4^2\xi_2^2\xi_1^2 + \xi_3^2{\xi_1^2}^2)\xi_1^3\xi_3^2 + (\xi_4^2{\xi_2^2}^2 + \xi_4^2 + 2\xi_3^2\xi_2^2\xi_1^2){\xi_1^2}^2)\xi_1^6 - (\xi_1^3\xi_4^2 + {\xi_1^2}^2)\xi_2^6\xi_3^2{\xi_1^2}^2 - (\xi_1^3\xi_3^2 + \xi_2^2\xi_1^2 + \xi_1^4\xi_4^2 - \xi_5^5\xi_1^2)(\xi_1^3\xi_4^2 + {\xi_1^2}^2)\xi_3^6\xi_1^2){\xi_5^6}^2 + ((\xi_2^6\xi_3^2 - \xi_1^6\xi_4^2 - (\xi_5^5 + \xi_2^2)\xi_3^6 + \xi_4^6\xi_1^2)({\xi_5^5}^2 + 1)(\xi_1^4\xi_3^2 - {\xi_1^2}^2)\xi_1^2 - ({\xi_1^4}^2\xi_4^2\xi_3^2\xi_2^2 - {\xi_1^4}^2{\xi_3^2}^2\xi_1^2 + \xi_1^4\xi_1^3\xi_4^2\xi_3^2\xi_1^2 + \xi_1^4\xi_1^3{\xi_3^2}^2\xi_2^2 + 2{\xi_1^2}^3 + ({\xi_2^2}^2 + {\xi_1^2}^2 - 1)\xi_1^4\xi_3^2\xi_1^2 + \xi_5^6{\xi_1^4}^2{\xi_4^2}^2 + \xi_5^6\xi_1^4\xi_1^3\xi_4^2\xi_3^2 + 2\xi_5^6\xi_1^4\xi_4^2\xi_2^2\xi_1^2 + \xi_5^6\xi_1^4\xi_3^2{\xi_1^2}^2 + \xi_5^6{\xi_2^2}^2{\xi_1^2}^2 + \xi_5^6{\xi_1^2}^2 + \xi_5^5{\xi_1^4}^2\xi_4^2\xi_3^2 + \xi_5^5\xi_1^4\xi_1^3{\xi_3^2}^2 - 2\xi_5^5\xi_1^4\xi_4^2{\xi_1^2}^2 + 2\xi_5^5\xi_1^4\xi_3^2\xi_2^2\xi_1^2 - 2\xi_5^5\xi_2^2{\xi_1^2}^3)\xi_5^6\xi_4^6)\xi_1^2)/(((({\xi_1^2}^2 + 1 + {\xi_2^2}^2)\xi_1^2 + (\xi_4^2\xi_1^2 + \xi_3^2\xi_2^2)\xi_1^3 + (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_1^4)\xi_5^6 + {\xi_5^6}^2\xi_1^3\xi_4^2 + {\xi_5^6}^2{\xi_1^2}^2 - \xi_5^6\xi_5^5\xi_1^4\xi_4^2 + \xi_5^6\xi_5^5\xi_1^3\xi_3^2 - {\xi_5^5}^2\xi_1^4\xi_3^2 + {\xi_5^5}^2{\xi_1^2}^2 - (\xi_1^4\xi_3^2 - {\xi_1^2}^2))\xi_5^6{\xi_1^2}^2);$$

$$J_4^5 = (-(((({\xi_1^2}^2 + 1 + {\xi_2^2}^2)\xi_1^2 + (\xi_4^2\xi_1^2 + \xi_3^2\xi_2^2)\xi_1^3 + (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_1^4 - (\xi_1^4\xi_4^2 - \xi_1^3\xi_3^2)\xi_5^5)\xi_2^6\xi_4^2{\xi_1^2}^2 - ((\xi_1^4{\xi_4^2}^2\xi_2^2 - \xi_1^4\xi_4^2\xi_3^2\xi_1^2 + \xi_1^3{\xi_4^2}^2\xi_1^2 + 2\xi_1^3\xi_4^2\xi_3^2\xi_2^2 - \xi_1^3{\xi_3^2}^2\xi_1^2 + (2\xi_4^2{\xi_2^2}^2 + \xi_4^2{\xi_1^2}^2 - \xi_3^2\xi_2^2\xi_1^2)\xi_1^2)\xi_1^4\xi_4^2 + (\xi_1^3\xi_4^2\xi_3^2\xi_1^2 + \xi_1^3{\xi_3^2}^2\xi_2^2 + \xi_4^2\xi_2^2{\xi_1^2}^2 + 2\xi_3^2{\xi_2^2}^2\xi_1^2 + \xi_3^2{\xi_1^2}^3)\xi_1^3\xi_4^2 + (\xi_4^2{\xi_2^2}^3 + \xi_4^2\xi_2^2{\xi_1^2}^2 + \xi_4^2\xi_2^2 - 2\xi_3^2\xi_1^2){\xi_1^2}^2 - (((\xi_1^4\xi_4^2 + 2\xi_1^3\xi_3^2)\xi_4^2 + (2\xi_4^2\xi_2^2 + \xi_3^2\xi_1^2)\xi_1^2)\xi_1^4\xi_4^2 + (\xi_1^3\xi_4^2{\xi_3^2}^2 - {\xi_4^2}^2{\xi_1^2}^2 + 2\xi_4^2\xi_3^2\xi_2^2\xi_1^2 + 2{\xi_3^2}^2{\xi_1^2}^2)\xi_1^3 + (\xi_4^2{\xi_2^2}^2 - \xi_4^2{\xi_1^2}^2 + \xi_4^2 + 2\xi_3^2\xi_2^2\xi_1^2){\xi_1^2}^2)\xi_5^5)\xi_1^6 + ({\xi_1^3}^2{\xi_4^2}^2\xi_1^2 + {\xi_1^3}^2\xi_4^2\xi_3^2\xi_2^2 + \xi_1^3\xi_4^2{\xi_2^2}^2\xi_1^2 + \xi_1^3\xi_4^2{\xi_1^2}^3 - \xi_1^3\xi_4^2\xi_1^2 - 2{\xi_1^2}^3 + (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_1^4\xi_1^3\xi_4^2 - ({\xi_1^3}^2\xi_4^2\xi_3^2 + 2\xi_1^3\xi_4^2\xi_2^2\xi_1^2 + 2\xi_1^3\xi_3^2{\xi_1^2}^2 + 2\xi_2^2{\xi_1^2}^3 + \xi_1^4\xi_1^3{\xi_4^2}^2)\xi_5^5)\xi_5^6\xi_3^2 - (((\xi_5^6\xi_1^3\xi_4^2\xi_1^2 + \xi_5^6{\xi_1^2}^3 - \xi_5^5\xi_1^4\xi_4^2\xi_1^2 + \xi_5^5\xi_1^3\xi_3^2\xi_1^2 + {\xi_1^4}^2{\xi_4^2}^2 + \xi_1^4\xi_1^3\xi_4^2\xi_3^2 + 2\xi_1^4\xi_4^2\xi_2^2\xi_1^2)\xi_5^5 + (\xi_1^3\xi_4^2 + {\xi_1^2}^2)\xi_5^6\xi_2^2\xi_1^2 + 2(({\xi_2^2}^2 + 1)\xi_1^2 + \xi_1^3\xi_3^2\xi_2^2)\xi_5^5\xi_1^2 - ({\xi_1^4}^2{\xi_4^2}^2\xi_2^2 - {\xi_1^4}^2\xi_4^2\xi_3^2\xi_1^2 + \xi_1^4\xi_1^3{\xi_4^2}^2\xi_1^2 + \xi_1^4\xi_1^3\xi_4^2\xi_3^2\xi_2^2 + \xi_1^3\xi_3^2\xi_1^2 + ({\xi_2^2}^2 + {\xi_1^2}^2)\xi_1^4\xi_4^2\xi_1^2))\xi_4^6 - (\xi_5^6\xi_4^2 + \xi_1^6\xi_3^2 - \xi_3^6\xi_1^2)(\xi_1^3\xi_4^2 + {\xi_1^2}^2)\xi_5^6\xi_1^2)\xi_1^2)\xi_5^6 + (((({\xi_2^2}^2 + 1)\xi_3^2 - 2\xi_4^2\xi_2^2\xi_1^2){\xi_1^2}^2 + (\xi_1^3{\xi_3^2}^2 - 2\xi_4^2{\xi_1^2}^2 + 2\xi_3^2\xi_2^2\xi_1^2)\xi_1^3\xi_3^2 + ((\xi_1^4\xi_4^2 + 2\xi_1^3\xi_3^2)\xi_3^2 - (\xi_4^2\xi_1^2 - 2\xi_3^2\xi_2^2)\xi_1^2)\xi_1^4\xi_4^2)\xi_1^6 - (\xi_1^4\xi_3^2 - {\xi_1^2}^2)\xi_2^6\xi_4^2{\xi_1^2}^2 - ((\xi_1^3{\xi_3^2}^2 - \xi_4^2{\xi_1^2}^2 + 2\xi_3^2\xi_2^2\xi_1^2)\xi_1^3 + ({\xi_2^2}^2 + 1){\xi_1^2}^2 + \xi_1^4\xi_1^3\xi_4^2\xi_3^2)\xi_3^6\xi_1^2 + (\xi_5^5\xi_1^2 - \xi_1^4\xi_4^2 - \xi_1^3\xi_3^2 - \xi_2^2\xi_1^2)(\xi_1^4\xi_3^2 - {\xi_1^2}^2)\xi_4^6\xi_1^2)({\xi_5^5}^2 + 1)))/(((({\xi_1^2}^2 + 1 + {\xi_2^2}^2)\xi_1^2 + (\xi_4^2\xi_1^2 + \xi_3^2\xi_2^2)\xi_1^3 + (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_1^4)\xi_5^6 + {\xi_5^6}^2\xi_1^3\xi_4^2 + {\xi_5^6}^2{\xi_1^2}^2 - \xi_5^6\xi_5^5\xi_1^4\xi_4^2 + \xi_5^6\xi_5^5\xi_1^3\xi_3^2 - {\xi_5^5}^2\xi_1^4\xi_3^2 + {\xi_5^5}^2{\xi_1^2}^2 - (\xi_1^4\xi_3^2 - {\xi_1^2}^2))\xi_5^6{\xi_1^2}^2);$$

13.4 Equivalence.

Due to the number of cases that had to be considered in the preceding computations, we'll tackle the equivalence problem in a slightly different way, mixing equations solving and reduction by equivalence. First we give explicit computation of all $\Phi \in \mathrm{Aut}(M5)$.

13.4.1

Lemma 1. *Aut($M5$) is comprised of all those (real) matrices of the following form :*

$$\Phi = \begin{pmatrix} b_1^1 & b_1^2 u & b_3^1 & -b_3^2 u & 0 & 0 \\ b_1^2 & -b_1^1 u & b_3^2 & b_3^1 u & 0 & 0 \\ b_1^3 & -b_1^4 u & b_3^3 & b_3^4 u & 0 & 0 \\ b_1^4 & b_1^3 u & b_3^4 & -b_3^3 u & 0 & 0 \\ b_1^5 & b_2^5 & b_3^5 & b_4^5 & H & Ku \\ b_1^6 & b_2^6 & b_3^6 & b_4^6 & K & -Hu \end{pmatrix} \tag{106}$$

where $u = \pm 1$ and $H + iK = \det(\mathbf{w}_1, \mathbf{w}_3) \neq 0$ with $\mathbf{w_1} = \begin{pmatrix} b_1^1 - ib_1^2 \\ b_1^3 + ib_1^4 \end{pmatrix}$ and $\mathbf{w_3} = \begin{pmatrix} b_3^1 - ib_3^2 \\ b_3^3 + ib_3^4 \end{pmatrix}$.

Proof. Let $\Phi = (b_j^i) \in \mathrm{Aut}(M5)$. Since Φ leaves the 2d central derivative $\mathcal{C}^2(M5)$ invariant, $b_5^i = b_6^i = 0$ for $1 \leqslant i \leqslant 4$. Denote by $ij|k$ the equation obtained by projecting on x_k the equation

$$[\Phi(x_i), \Phi(x_j)] - \Phi([x_i, x_j]) = 0.$$

Equations $13|5, 13|6, 14|5, 14|6$ yield $b_5^5 = b_3^4 b_1^2 - b_1^4 b_3^2 + b_3^3 b_1^1 - b_1^3 b_3^1$; $b_5^6 = b_3^4 b_1^1 - b_1^4 b_3^1 - b_3^3 b_1^2 + b_1^3 b_3^2$; $b_6^5 = b_4^4 b_1^2 - b_1^4 b_4^2 + b_4^3 b_1^1 - b_1^3 b_4^1$; $b_6^6 = b_4^4 b_1^1 - b_1^4 b_4^1 - b_4^3 b_1^2 + b_1^3 b_4^2$. Now equations $12|5$ and $12|6$ read respectively $\Delta^{1,3} + \Delta^{2,4} = 0$ and $\Delta^{1,4} - \Delta^{2,3} = 0$, with $\Delta^{i,j}$ the minors formed with the 1st and 2d columns and the lines indicated by the indices in the matrix

$$\Phi_\dagger = \begin{pmatrix} b_1^1 & b_2^1 & b_3^1 & b_4^1 \\ b_1^2 & b_2^2 & b_3^2 & b_4^2 \\ b_1^3 & b_2^3 & b_3^3 & b_4^3 \\ b_1^4 & b_2^4 & b_3^4 & b_4^4 \end{pmatrix}.$$

If we introduce for $1 \leqslant j \leqslant 4 \quad \mathbf{w}_j = \begin{pmatrix} b_j^1 - ib_j^2 \\ b_j^3 + ib_j^4 \end{pmatrix}$ then

$$\det(\mathbf{w}_1, \mathbf{w}_2) = \Delta^{1,3} + \Delta^{2,4} + i(\Delta^{1,4} - \Delta^{2,3})$$

hence equations $12|5$ and $12|6$ are equivalent to the single complex equation $\det(\mathbf{w}_1, \mathbf{w}_2) = 0$, i.e. to the existence of $z = \alpha + i\beta \in \mathbb{C}$ such that $\mathbf{w}_2 = z\,\mathbf{w}_1$. In the same way, equations $34|5$ and $34|6$ are equivalent to the existence of $w = \gamma + i\delta \in \mathbb{C}$ such that $\mathbf{w}_4 = w\,\mathbf{w}_3$. No, if we introduce $h = \alpha + \gamma, k = \beta + \delta$, the system $23|5, 23|6$ reads

$$\begin{cases} h\,H - k\,K = 0 \\ k\,H + h\,K = 0 \end{cases} \tag{107}$$

where $H = \Delta'^{2,4} + \Delta'^{1,3}$, $K = \Delta'^{1,4} - \Delta'^{2,3}$, $\Delta'^{i,j}$ the minors formed with the 1st and 3d columns and the lines indicated by the indices in the matrix $\Phi_\dagger$. Since $H + iK = \det(\mathbf{w}_1, \mathbf{w}_3)$, the case $H = K = 0$ would imply $\det(\mathbf{w}_1, \mathbf{w}_3) = 0$ which in turn leads to $\det \Phi = 0$. Hence (H, K) is a non trivial solution to the system (107). As its determinant is $h^2 + k^2$ we conclude that $h = k = 0$, i.e. $\gamma = -\alpha, \delta = -\beta$. Now we are left only with the system of equations $24|5, 24|6$. It reads

$$\begin{cases} (\alpha^2 - \beta^2 + 1)H - 2\alpha\beta K = 0 \\ 2\alpha\beta H + (\alpha^2 - \beta^2 + 1)K = 0. \end{cases} \tag{108}$$

Again, as (H, K) is a non trivial solution one has $\alpha\beta = 0, \alpha^2 - \beta^2 + 1 = 0$ i.e. $\alpha = 0, \beta = \pm 1$. Then we get (106) with $u = \beta$. ☐

The subgroup $\mathrm{Aut}(\mathfrak{n}) \subset \mathrm{Aut}(M5)$ of complex automorphisms of the complex Heisenberg Lie algebra $\mathfrak{n}$ is the subgroup comprised of all those matrices in (106) for which $u = -1$ and $b_2^6 = -b_1^5, b_2^5 = b_1^6, b_4^6 = b_3^5, b_4^5 = -b_3^6$.

13.4.2

Now, looking for complex structures J, after completing a set of general steps, one is left to find solutions of the torsion equations and $J^2 = -1$ of the following form:

$$J = \begin{pmatrix} \xi_1^1 & \xi_2^1 & \xi_3^1 & \xi_4^1 & 0 & 0 \\ \xi_1^2 & \xi_2^2 & \xi_3^2 & \xi_4^2 & 0 & 0 \\ \xi_1^3 & \xi_2^3 & \xi_3^3 & \xi_4^3 & 0 & 0 \\ \xi_1^4 & \xi_2^4 & \xi_3^4 & \xi_4^4 & 0 & 0 \\ \xi_1^5 & \xi_2^5 & \xi_3^5 & \xi_4^5 & \xi_5^5 & -\frac{{\xi_5^5}^2+1}{\xi_5^6} \\ \xi_1^6 & \xi_2^6 & \xi_3^6 & \xi_4^6 & \xi_5^6 & -\xi_5^5 \end{pmatrix}$$

where the ξ_j^5's are certain linear expressions in the ξ_k^6's ($1 \leqslant j,k \leqslant 4$) and $\xi_4^4 = -\sum_1^3 \xi_j^j$. Take the following $\Phi \in \mathrm{Aut}(M5)$:

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ b_1^5 & b_2^5 & b_3^5 & b_4^5 & 1 & 0 \\ b_1^6 & b_2^6 & b_3^6 & b_4^6 & 0 & 1 \end{pmatrix}$$

with $b_j^5 = \frac{1}{\xi_5^6}(-\xi_j^6 + \sum_{k=1}^4 b_k^6 \xi_j^k + \xi_5^5 b_j^6)$. Then equivalence by Φ leads to the case where $\xi_j^5 = \xi_j^6 = 0 \; \forall j \; 1 \leqslant j \leqslant 4$. Now consider the submatrix

$$J_\dagger = \begin{pmatrix} \xi_1^1 & \xi_2^1 & \xi_3^1 & \xi_4^1 \\ \xi_1^2 & \xi_2^2 & \xi_3^2 & \xi_4^2 \\ \xi_1^3 & \xi_2^3 & \xi_3^3 & \xi_4^3 \\ \xi_1^4 & \xi_2^4 & \xi_3^4 & \xi_4^4 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{109}$$

We get 2 cases 1 and 2 below. Before we proceed further, we record a lemma.

Lemma 2. *Let J be a complex structure on $M5$ of the following form :*

$$J = \begin{pmatrix} \begin{matrix} & & & \\ & J_\dagger & & \\ & & & \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{pmatrix} \xi_5^5 & -\frac{{\xi_5^5}^2+1}{\xi_5^6} \\ \xi_5^6 & -\xi_5^5 \end{pmatrix} \end{pmatrix}$$

where $J_\dagger$ is given in (109).
(i) Suppose that B and C are not simultaneously zero. Then J is equivalent to a structure of the same form for which $B \neq 0$.
(ii) Suppose $B = \begin{pmatrix} \xi_3^1 & \xi_4^1 \\ \xi_3^2 & \xi_4^2 \end{pmatrix} \neq 0$. Then J is equivalent to a structure of the same form for which $\xi_3^2 = 1, \xi_4^2 = 0$.
(iii) Suppose $B = \begin{pmatrix} \xi_3^1 & \xi_4^1 \\ 1 & 0 \end{pmatrix}$. Then J is equivalent to a structure of the same form having the same B and for which $\xi_1^2 = \xi_2^2 = 0$.
(iv) Suppose $B = C = 0$ and $\xi_2^2 = -\xi_1^1, \xi_4^4 = -\xi_3^3$. Then J is equivalent to a structure of the same form for which $B = C = 0$ and $\xi_1^1 = \xi_3^3 = 0$.

Proof. (i) The matrix J has the form

$$\begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & E \end{pmatrix}.$$

On the other hand, we know that the automorphism group of $M5$ is comprised of those matrices of the following form :

$$\Phi = \begin{pmatrix} & & & & 0 & 0 \\ & & \Phi_\dagger & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ * & * & * & * & H & Ku \\ * & * & * & * & K & -Hu \end{pmatrix}$$

with

$$\Phi_\dagger = \begin{pmatrix} b_1^1 & b_1^2 u & b_3^1 & -b_3^2 u \\ b_1^2 & -b_1^1 u & b_3^2 & b_3^1 u \\ b_1^3 & -b_1^4 u & b_3^3 & b_3^4 u \\ b_1^4 & b_1^3 u & b_3^4 & -b_3^3 u \end{pmatrix} \tag{110}$$

and

$$H + iK = \det(\mathbf{w}_1, \mathbf{w}_3) \neq 0 \tag{111}$$

where $\mathbf{w_1} = \begin{pmatrix} b_1^1 - ib_1^2 \\ b_1^3 + ib_1^4 \end{pmatrix}$ and $\mathbf{w_3} = \begin{pmatrix} b_3^1 - ib_3^2 \\ b_3^3 + ib_3^4 \end{pmatrix}$. Take

$$\Phi_\dagger = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

and $(u = 1)$

$$\Phi = \begin{pmatrix} \Phi_\dagger & 0 \\ 0 & \begin{pmatrix} H & Ku \\ K & -Hu \end{pmatrix} \end{pmatrix}.$$

Then $\mathbf{w_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\mathbf{w_3} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ so that $\Phi \in \mathrm{Aut}(M5)$. Now

$$(\Phi_\dagger)^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Phi_\dagger = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix}.$$

Hence if B and C are not simultaneously zero, one may suppose $B \neq 0$.
(ii) Suppose first $\xi_3^2 = \xi_4^2 = 0$. Then, since $B \neq 0$, the first line of B is not zero. Consider

$$\Phi_\dagger = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$$

and $(u = 1)$

$$\Phi = \begin{pmatrix} \Phi_\dagger & 0 \\ 0 & \begin{pmatrix} H & Ku \\ K & -Hu \end{pmatrix} \end{pmatrix}$$

with $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \neq 0$. Then $\mathbf{w_1} = \begin{pmatrix} -i \\ 0 \end{pmatrix}$ and $\mathbf{w_3} = \begin{pmatrix} 0 \\ i \end{pmatrix}$ so that $\Phi \in \mathrm{Aut}(M5)$. Now

$$(\Phi_\dagger)^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Phi_\dagger = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} UAU & UBU \\ UCU & UDU \end{pmatrix}.$$

where

$$UBU = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_3^1 & \xi_4^1 \\ \xi_3^2 & \xi_4^2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \xi_4^2 & \xi_3^2 \\ \xi_4^1 & \xi_3^1 \end{pmatrix}.$$

Hence we are led to the case where the second line of B doesn't vanish. Consider that case now. Introduce

$$\Phi_\dagger = \begin{pmatrix} I & 0 \\ 0 & V \end{pmatrix}$$

and $(u = -1, \alpha^2 + \beta^2 \neq 0)$

$$\Phi = \begin{pmatrix} \Phi_\dagger & 0 \\ 0 & \begin{pmatrix} H & Ku \\ K & -Hu \end{pmatrix} \end{pmatrix}$$

with $V = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \neq 0$. Then $\mathbf{w_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{w_3} = \begin{pmatrix} 0 \\ \alpha + i\beta \end{pmatrix}$ so that $\Phi \in \mathrm{Aut}(M5)$. Now

$$(\Phi_\dagger)^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Phi_\dagger = \begin{pmatrix} I & 0 \\ 0 & V^{-1} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} A & BV \\ V^{-1}C & V^{-1}DV \end{pmatrix}.$$

where

$$BV = \begin{pmatrix} \xi_3^1 & \xi_4^1 \\ \xi_3^2 & \xi_4^2 \end{pmatrix} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} * & * \\ {\xi'}_3^2 & {\xi'}_4^2 \end{pmatrix}$$

with ${\xi'}_3^2 = \alpha\xi_3^2 + \beta\xi_4^2$, ${\xi'}_4^2 = \alpha\xi_4^2 - \beta\xi_3^2$. We want ${\xi'}_3^2 = 1$ and ${\xi'}_4^2 = 0$. This is a Cramer system in α, β since $(\xi_3^2)^2 + (\xi_4^2)^2 \neq 0$, hence it has a nontrivial solution. Hence we are reduced to the case where $\xi_3^2 = 1$ and $\xi_4^2 = 0$.

(iii) Suppose $B = \begin{pmatrix} \xi_3^1 & \xi_4^1 \\ 1 & 0 \end{pmatrix}$. Consider

$$\Phi_\dagger = \begin{pmatrix} I & 0 \\ T & I \end{pmatrix}$$

and $(u = -1, \alpha^2 + \beta^2 \neq 0)$

$$\Phi = \begin{pmatrix} \Phi_\dagger & 0 \\ 0 & \begin{pmatrix} H & Ku \\ K & -Hu \end{pmatrix} \end{pmatrix}$$

with

$$T = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}.$$

Then $\mathbf{w_1} = \begin{pmatrix} 1 \\ \alpha + i\beta \end{pmatrix}$ and $\mathbf{w_3} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ so that $\Phi \in \mathrm{Aut}(M5)$. Now

$$\begin{aligned} (\Phi_\dagger)^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Phi_\dagger &= \begin{pmatrix} I & 0 \\ -T & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & 0 \\ T & I \end{pmatrix} \\ &= \begin{pmatrix} A & B \\ -TA + C & -TB + D \end{pmatrix} \begin{pmatrix} I & 0 \\ T & I \end{pmatrix} \\ &= \begin{pmatrix} A + BT & B \\ * & * \end{pmatrix}. \end{aligned}$$

We want

$$A + BT = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix},$$

that is $BT = \begin{pmatrix} * & * \\ -\xi_1^2 & -\xi_2^2 \end{pmatrix}$. As $BT = \begin{pmatrix} \xi_3^1 & \xi_4^1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} = \begin{pmatrix} * & * \\ \alpha & \beta \end{pmatrix}$, one has to let $\alpha = -\xi_1^2$ and $\beta = -\xi_2^2$, which is possible if one doesn't already has $\xi_1^2 = 0,\ \xi_2^2 = 0$. Hence we are reduced to the case where $\xi_1^2 = 0$ and $\xi_2^2 = 0$.

(iv) The matrix J has the form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & E \end{pmatrix}$$

with moreover $\xi_2^2 = -\xi_1^1$ and $\xi_4^4 = -\xi_3^3$. Consider

$$\Phi_\dagger = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$$

and $(u = -1,\ \alpha^2 + \beta^2 \neq 0,\ \gamma^2 + \delta^2 \neq 0)$

$$\Phi = \begin{pmatrix} \Phi_\dagger & 0 \\ 0 & \begin{pmatrix} H & Ku \\ K & -Hu \end{pmatrix} \end{pmatrix}$$

where

$$U = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},\ V = \begin{pmatrix} \gamma & -\delta \\ \delta & \gamma \end{pmatrix}.$$

Then $\mathbf{w_1} = \begin{pmatrix} \alpha - i\beta \\ 0 \end{pmatrix}$ and $\mathbf{w_3} = \begin{pmatrix} 0 \\ \gamma + i\delta \end{pmatrix}$ so that $\Phi \in \mathrm{Aut}(M5)$. Now

$$(\Phi_\dagger)^{-1} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \Phi_\dagger = \begin{pmatrix} U^{-1} & 0 \\ 0 & V^{-1} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} U^{-1}AU & 0 \\ 0 & V^{-1}D \end{pmatrix}.$$

One has

$$U^{-1}AU = \frac{1}{\alpha^2+\beta^2} \begin{pmatrix} {\xi'}^1_1 & {\xi'}^1_2 \\ {\xi'}^2_1 & {\xi'}^2_2 \end{pmatrix},$$

with

$$\begin{aligned} {\xi'}^1_1 &= \alpha^2 \xi^1_1 + \alpha\beta(\xi^1_2 + \xi^2_1) + \beta^2 \xi^2_2 \\ {\xi'}^2_2 &= \alpha^2 \xi^2_2 - \alpha\beta(\xi^1_2 + \xi^2_1) + \beta^2 \xi^1_1. \end{aligned}$$

Since $\xi^1_1 = -\xi^2_2$, one has ${\xi'}^1_1 = -{\xi'}^2_2$. Then the discriminant of the equation ${\xi'}^2_2 = 0$ is the sum of two squares, hence there exist $\alpha, \beta \in \mathbb{R}$ with $\beta = 1$ such that ${\xi'}^1_1 = 0$ and ${\xi'}^2_2 = 0$. Then

$$U^{-1}AU = \frac{1}{\alpha^2+\beta^2} \begin{pmatrix} 0 & {\xi'}^1_2 \\ {\xi'}^2_1 & 0 \end{pmatrix}.$$

Similarly, there exist $\gamma, \delta \in \mathbb{R}$ with $\delta = 1$ such that

$$V^{-1}DV = \frac{1}{\gamma^2+\delta^2} \begin{pmatrix} 0 & {\xi'}^3_4 \\ {\xi'}^4_3 & 0 \end{pmatrix}.$$

Hence we are reduced to the case where $\xi^1_1 = \xi^3_3 = 0$. $\square$

13.4.3 Case 1.

In the present case 1, we suppose that B and C in (109) are not both 0. Then by equivalence (Lemma 2 (i)), we may suppose $B \neq 0$. Since $B \neq 0$, by equivalence (Lemma 2 (ii)), we may assume $\xi^2_3 = 1$, $\xi^2_4 = 0$. Again by equivalence (Lemma 2 (iii)), we may assume without altering B that $\xi^2_1 = 0$, $\xi^2_2 = 0$. Then solving all equations, we finally get the matrix

$$J(\xi^1_3, \xi^1_4, \xi^5_5, \xi^6_5) = \begin{pmatrix} a & -(\xi^6_5\xi^1_4 - \xi^6_5 + \xi^5_5\xi^1_3) & \xi^1_3 & \xi^1_4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ b & c & d & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi^5_5 & (-({\xi^5_5}^2+1))/\xi^6_5 \\ 0 & 0 & 0 & 0 & \xi^6_5 & -\xi^5_5 \end{pmatrix} \tag{112}$$

where :
$a = J^1_1 = (({\xi^5_5}^2+1)\xi^1_3 + (\xi^1_4 - 1)\xi^6_5\xi^5_5)/\xi^6_5$,
$b = J^4_1 = -\frac{1+a^2}{\xi^1_4}$,
$c = J^4_2 = ((\xi^6_5\xi^5_5{\xi^1_4}^2 - 2\xi^6_5\xi^5_5\xi^1_4 + \xi^6_5\xi^5_5 + 2{\xi^5_5}^2\xi^1_4\xi^1_3 - 2{\xi^5_5}^2\xi^1_3 + \xi^1_4\xi^1_3)\xi^6_5 + ({\xi^5_5}^2+1)\xi^5_5{\xi^1_3}^2)/(\xi^6_5\xi^1_4)$
$d = J^4_3 = ((\xi^6_5\xi^1_4 - \xi^6_5 - \xi^5_5\xi^1_4\xi^1_3 + 2\xi^5_5\xi^1_3)\xi^6_5 - ({\xi^5_5}^2+1){\xi^1_3}^2)/(\xi^6_5\xi^1_4)$.
and the parameters are subject to the condition

$$\xi^1_4\xi^6_5 \neq 0. \tag{113}$$

Note also the following formulae from $J^2 = -1$:

$$\begin{aligned} d &= \frac{1}{\xi^1_4}\left(\xi^6_5(\xi^1_4 - 1) + \xi^1_3(\xi^5_5 - a)\right), && (114) \\ c &= ad + (1+a^2)\frac{\xi^1_3}{\xi^1_4}. && (115) \end{aligned}$$

Commutation relations of $\mathfrak{m}$:
$[\tilde{x}_1, \tilde{x}_2] = \frac{1}{\xi^6_5\xi^1_4}\left(\xi^6_5(\xi^6_5(\xi^1_4 - 1) + \xi^5_5\xi^1_3(2 - \xi^1_4)) - {\xi^1_3}^2({\xi^5_5}^2+1)\right)\tilde{x}_6 + \frac{1}{\xi^6_5}\left(\xi^5_5(\xi^1_4 - 1) + {\xi^5_5}^2 + 1\right)\tilde{x}_5$;
$[\tilde{x}_1, \tilde{x}_3] \quad = \quad -\frac{1}{{\xi^6_5}^2\xi^1_4}\left(\xi^5_5((\xi^6_5(\xi^1_4 - 1) + \xi^1_3\xi^5_5)^2 + {\xi^1_3}^2) + \xi^1_3{\xi^6_5}^2\xi^1_4\right)\tilde{x}_6 \quad -$

$\frac{1}{\xi_5^6\xi_4^1}(\xi_5^6{\xi_5^5}^2(\xi_4^1-1)^2+\xi_5^6+2\xi_5^5\xi_3^1(\xi_4^1-1)({\xi_5^5}^2+1)-{\xi_5^6}^2\xi_4^1)\,\tilde{x}_5$;
$[\tilde{x}_2,\tilde{x}_3]=\frac{1}{\xi_5^6\xi_4^1}\,({\xi_5^6}^2(\xi_4^1-1)^2+2\xi_5^6\xi_5^5\xi_3^1(\xi_4^1-1)+({\xi_5^5}^2+1){\xi_3^1}^2)\,(\xi_5^6\tilde{x}_6+\xi_5^5\tilde{x}_5)$;
$[\tilde{x}_2,\tilde{x}_4]=\xi_3^1\tilde{x}_6+(1-\xi_4^1)\tilde{x}_5$;

$[\tilde{x}_3,\tilde{x}_4]=(\xi_5^6(\xi_4^1-1)+\xi_5^5\xi_3^1)\,\tilde{x}_6+\frac{({\xi_5^5}^2+1)\xi_3^1+\xi_5^5\xi_5^6(\xi_4^1-1)}{\xi_5^6}\,\tilde{x}_5.$

$\mathfrak{m}$ is abelian if and only if

$$\xi_3^1=0 \quad , \quad \xi_4^1=1. \tag{116}$$

Suppose now that (116) holds, and denote $\alpha=\xi_5^5, \beta=\xi_5^6\neq 0$, $J(\alpha,\beta)=J(0,1,\xi_5^5,\xi_5^6)$. Then :

$$J(\alpha,\beta)=\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \alpha & -\frac{\alpha^2+1}{\beta}\\ 0 & 0 & 0 & 0 & \beta & -\alpha \end{pmatrix}$$

where $\beta\neq 0$. A direct study shows that any $J(\alpha,\beta)$ is equivalent to a unique $J(0,\gamma),\ 0<\gamma\leqslant 1$, and any two such $J(0,\gamma),\ 0<\gamma\leqslant 1$ are not equivalent unless $\gamma=\gamma'$.

$$J(0,\beta)=\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\beta}\\ 0 & 0 & 0 & 0 & \beta & 0 \end{pmatrix}. \tag{117}$$

13.4.4

Let $J_1=J(\xi_3^1,\xi_4^1,\xi_5^5,\xi_5^6)$ and $J_2=J(\eta_3^1,\eta_4^1,\eta_5^5,\eta_5^6)$ as in (112). It is clear that $J_1\cong J_2$ if and only if there exists a matrix Φ such that

$$\Phi=\begin{pmatrix} \Phi_\dagger & 0\\ 0 & \begin{pmatrix} H & Ku\\ K & -Hu\end{pmatrix}\end{pmatrix} \tag{118}$$

where $u=\pm 1$, $\Phi_\dagger, H, K$ are defined in (110), (111), $H^2+K^2\neq 0$ and $J_2=\Phi^{-1}J_1\Phi$. This equation implies that

$$\begin{pmatrix} \xi_5^5 & -\frac{{\xi_5^5}^2+1}{\xi_5^6}\\ \xi_5^6 & -\xi_5^5\end{pmatrix}\begin{pmatrix} H & Ku\\ K & -Hu\end{pmatrix}=\begin{pmatrix} H & Ku\\ K & -Hu\end{pmatrix}\begin{pmatrix} \eta_5^5 & -\frac{{\eta_5^5}^2+1}{\eta_5^6}\\ \eta_5^6 & -\eta_5^5\end{pmatrix}. \tag{119}$$

Equation (119) has non trivial solutions in H, K if and only if the following condition holds :

$$\frac{{\eta_5^5}^2+{\eta_5^6}^2+1}{\eta_5^6}=-u\,\frac{{\xi_5^5}^2+{\xi_5^6}^2+1}{\xi_5^6}. \tag{120}$$

However, though that condition is also sufficient for the existence of Φ in (118) in the abelian case where $\xi_3^1=\eta_3^1=0, \xi_4^1=\eta_4^1=1$, it is no longer sufficient in the nonabelian case. For example, take $J(1,1,1,1)$ and $J(1,1,1,\eta_5^6)$, with $\eta_5^6\neq 1$: then (120) holds if only if $u=-1, \eta_5^6=2$ or $u=1, \eta_5^6=-2,-1$, and in neither case does equivalence occur; hence $J(1,1,1,1)\ncong J(1,1,1,\eta_5^6)$ if $\eta_5^6\neq 1$. However $J(1,1,0,1)\cong J(1,1,0,-1)$. One also has $J(\xi_3^1,1,0,1))\cong J(\eta_3^1,\eta_4^1,0,1)\Leftrightarrow \eta_3^1=\pm\xi_3^1$; $\eta_4^1=\xi_4^1$. In general, if $\eta_4^1\neq\xi_4^1$, for $J(\xi_3^1,\xi_4^1,\xi_5^5,\xi_5^6)$ and $J(\eta_3^1,\eta_4^1,\eta_5^5,\eta_5^6)$ to be equivalent, it would be necessary that $P\xi_5^5+Q=0$ where P,Q are certain huge polynomials in the other variables. We simply conjecture here that equivalence implies $\xi_4^1=\eta_4^1$ and $\eta_3^1=\pm\xi_3^1$, and leave open the equivalence problem in the nonabelian case.

13.4.5 Case 2.

We now suppose $B=C=0$. Then necessarily $\xi_2^2=-\xi_1^1, \xi_4^4=-\xi_3^3$. By equivalence (Lemma 2 (iv)), we may suppose $\xi_1^1=0, \xi_3^3=0$.

Case 2.1. Suppose $\xi_5^5 = 0$. If $\xi_3^4\xi_1^2 \neq 1$ one gets then the matrix

$$J(\xi_1^2, \xi_3^4) = \begin{pmatrix} 0 & (-1)/\xi_1^2 & 0 & 0 & 0 & 0 \\ \xi_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (-1)/\xi_3^4 & 0 & 0 \\ 0 & 0 & \xi_3^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\xi_3^4\xi_1^2 - 1)/(\xi_3^4 - \xi_1^2) \\ 0 & 0 & 0 & 0 & (-\xi_3^4 + \xi_1^2)/(\xi_3^4\xi_1^2 - 1) & 0 \end{pmatrix} \qquad (121)$$

with the conditions

$$\xi_1^2, \xi_3^4 \neq 0 \; ; \; \xi_3^4 \neq \xi_1^2, \frac{1}{\xi_1^2}. \qquad (122)$$

If $\xi_3^4\xi_1^2 = 1$, we get

$$J = \begin{pmatrix} 0 & (-1)/\xi_1^2 & 0 & 0 & 0 & 0 \\ \xi_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\xi_1^2 & 0 & 0 \\ 0 & 0 & 1/\xi_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (-1)/\xi_5^6 \\ 0 & 0 & 0 & 0 & \xi_5^6 & 0 \end{pmatrix} \qquad (123)$$

with the conditions

$$\xi_1^2 = \pm 1 \; ; \; \xi_5^6 \neq 0. \qquad (124)$$

Case 2.2. Suppose $\xi_5^5 \neq 0$. Then we get:

$$J = \begin{pmatrix} 0 & (-1)/\xi_1^2 & 0 & 0 & 0 & 0 \\ \xi_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (-1)/\xi_1^2 & 0 & 0 \\ 0 & 0 & \xi_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi_5^5 & (-({\xi_5^5}^2 + 1))/\xi_5^6 \\ 0 & 0 & 0 & 0 & \xi_5^6 & -\xi_5^5 \end{pmatrix} \qquad (125)$$

with the conditions

$$\xi_1^2 = \pm 1 \; ; \; \xi_5^5\xi_5^6 \neq 0. \qquad (126)$$

13.4.6

Computing intertwining automorphisms, one can prove that for all $\xi_1^2, \xi_3^4, \eta_1^2, \eta_3^4 \in \mathbb{R}$ satisfying conditions (122), $J(\eta_1^2, \eta_3^4) \cong J(\xi_1^2, \xi_3^4)$ if and only if there exists $u = \pm 1$ such that one of the following is satisfied :

$$\left(\eta_1^2 = u\xi_1^2 \text{ or } \frac{u}{\xi_1^2}\right) \text{ and } \left(\eta_3^4 = u\xi_3^4 \text{ or } \frac{u}{\xi_3^4}\right)$$

or

$$\left(\eta_1^2 = u\xi_3^4 \text{ or } \frac{u}{\xi_3^4}\right) \text{ and } \left(\eta_3^4 = u\xi_1^2 \text{ or } \frac{u}{\xi_1^2}\right).$$

Conditions (122) are preserved by the transformations. For example, the canonical complex structure $J_0 = J(-1, 1)$ and its opposite $-J_0$ are equivalent. In fact, on has :

Lemma 3. *Let Ω denote the $Aut(M5)$-orbit of the canonical complex structure J_0. Then Ω is the 4-dimensional space comprised of the matrices*

$$J = \begin{pmatrix} 0 & (-1)/\xi_1^2 & 0 & 0 & 0 & 0 \\ \xi_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\xi_1^2 & 0 & 0 \\ 0 & 0 & -\xi_1^2 & 0 & 0 & 0 \\ \xi_2^6 & -\xi_1^6 & -\xi_4^6 & \xi_3^6 & 0 & 1/\xi_1^2 \\ \xi_1^6 & \xi_2^6 & \xi_3^6 & \xi_4^6 & -\xi_1^2 & 0 \end{pmatrix} \qquad (127)$$

where $\xi_1^2 = \pm 1, \xi_1^6, \xi_2^6, \xi_3^6, \xi_4^6 \in \mathbb{R}$.

On the other hand, the J in (123) appears simply as a limiting case when $\xi_5^5 \to 0$ of the structure $J(\xi_1^2, \xi_5^5, \xi_5^6)$ defined in (125) with $\xi_1^2 = \pm 1$; $\xi_5^5, \xi_5^6 \neq 0$.

13.4.7

$M5$ is a complex algebra for the structure $J(\xi_1^2,\xi_3^4)$ in (121) if and only if $\xi_1^2=-1,\xi_3^4=1$ or $\xi_1^2=1,\xi_3^4=-1$, *i.e.* $J(\xi_1^2,\xi_3^4)$ is the canonical complex structure $J_0=J(-1,1)$ or its opposite $-J_0$ respectively. Since $M5$ is not a complex algebra for the structure $J(\xi_3^1,\xi_4^1,\xi_5^5,\xi_5^6)$ in (112), the latter is not equivalent to J_0.

Lemma 4. *Suppose $J(\xi_1^2,\xi_3^4)\notin\{J_0,-J_0\}$. Then $J(\xi_1^2,\xi_3^4)$ is equivalent to some complex structure in case 1, i.e. there exist $\xi_3^1,\xi_4^1,\xi_5^5,\xi_5^6$ such that $J(\xi_1^2,\xi_3^4)\cong J(\xi_3^1,\xi_4^1,\xi_5^5,\xi_5^6)$.*

Proof. Take

$$\Phi_\dagger=\begin{pmatrix} I & S\\ 0 & I\end{pmatrix}$$

and

$$\Phi=\begin{pmatrix}\Phi_\dagger & 0\\ 0 & \begin{pmatrix}1&0\\0&1\end{pmatrix}\end{pmatrix} \tag{128}$$

where

$$S=\begin{pmatrix}\alpha & \beta\\ \beta & -\alpha\end{pmatrix}. \tag{129}$$

Then $\mathbf{w_1}=\begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{w_3}=\begin{pmatrix}\alpha-i\beta\\1\end{pmatrix}$ so that $\Phi\in\mathrm{Aut}(M5)$. Denote

$$J(\xi_1^2,\xi_3^4)_\dagger=\begin{pmatrix}A&0\\0&D\end{pmatrix}$$

where

$$A=\begin{pmatrix}0 & -\frac{1}{\xi_1^2}\\ \xi_1^2 & 0\end{pmatrix},\ D=\begin{pmatrix}0 & -\frac{1}{\xi_3^4}\\ \xi_3^4 & 0\end{pmatrix}.$$

Then

$$(\Phi_\dagger)^{-1}J(\xi_1^2,\xi_3^4)_\dagger\,\Phi_\dagger=\begin{pmatrix}I&-S\\0&I\end{pmatrix}\begin{pmatrix}A&0\\0&D\end{pmatrix}\begin{pmatrix}I&S\\0&I\end{pmatrix}=\begin{pmatrix}A&AS-SD\\0&D\end{pmatrix}.$$

Now

$$AS-SD=\begin{pmatrix}-\beta(\frac{1}{\xi_1^2}+\xi_3^4) & \alpha(\frac{1}{\xi_1^2}+\frac{1}{\xi_3^4})\\ \alpha(\xi_1^2+\xi_3^4) & \beta(\xi_1^2+\frac{1}{\xi_3^4})\end{pmatrix}.$$

The complex structure $\Phi^{-1}J(\xi_1^2,\xi_3^4)\,\Phi$ is of type 1 if $AS-SD\neq 0$. For this to hold, just choose $\alpha=1,\beta=0$ if $\xi_1^2+\xi_3^4\neq 0$ and $\alpha=0,\beta=1$ if $\xi_1^2+\xi_3^4=0$, noting in this latter case that $\xi_1^2+\frac{1}{\xi_3^4}=\frac{(\xi_1^2)^2-1}{\xi_1^2}\neq 0$ since $\xi_1^2\neq\pm1$ as $J(\xi_1^2,\xi_3^4)\notin\{J_0,-J_0\}$. □

Lemma 5. *Let J be either the complex structure defined in (125) with $\xi_1^2=\pm1$, $\xi_5^5,\xi_5^6\neq 0$ or the one in (123). Then $J\cong J(0,\beta)$, where $J(0,\beta)$ is defined in (117).*

Proof. In both cases, $\mathfrak{m}$ is abelian, and

$$J=\begin{pmatrix}A&0&0\\0&A&0\\0&0&E\end{pmatrix}$$

where

$$A=\begin{pmatrix}0 & -\frac{1}{\xi_1^2}\\ \xi_1^2 & 0\end{pmatrix}\ ,\ E=\begin{pmatrix}\xi_5^5 & -\frac{{\xi_5^5}^2+1}{\xi_5^6}\\ \xi_5^6 & \xi_5^5\end{pmatrix}$$

with $\xi_5^6\neq 0$, $\xi_1^2=\pm1$, $\xi_5^5=0$ in case (123) and $\xi_5^5\neq 0$ in case (125). Take Φ,S as in (128),(129). Denote

$$J_\dagger=\begin{pmatrix}A&0\\0&A\end{pmatrix}.$$

Then

$$(\Phi_\dagger)^{-1}J(\xi_1^2,\xi_3^4)_\dagger\,\Phi_\dagger=\begin{pmatrix}I&-S\\0&I\end{pmatrix}\begin{pmatrix}A&0\\0&A\end{pmatrix}\begin{pmatrix}I&S\\0&I\end{pmatrix}=\begin{pmatrix}A&AS-SA\\0&A\end{pmatrix}.$$

Now

$$AS - SA = \begin{pmatrix} -2\beta\xi_1^2 & 2\alpha\xi_1^2 \\ 2\alpha\xi_1^2 & 2\beta\xi_1^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

if we take $\alpha = \frac{\xi_1^2}{2}, \beta = 0$. Hence $\Phi^{-1} J\, \Phi$ is a complex structure of type 1, and we readily see that $J \cong J(0, 1, \xi_5^5, \xi_5^6)$, hence $J \cong J(0, \beta)$ for some β , $0 < \beta \leqslant 1$. $\square$

To summarize, we have shown the following :

Theorem 1. *Any complex structure on the Lie algebra $M5$ is equivalent to either the canonical complex structure J_0 or some complex structure $J = J(\xi_3^1, \xi_4^1, \xi_5^5, \xi_5^6)$ defined in (112).*

13.4.8

G_0 is here the complex 3-dimensional Heisenberg group, considered as a real Lie group, *i.e.* the real Lie group comprised of the matrices

$$x = \begin{pmatrix} 1 & x^1 + iy^1 & x^3 + iy^3 \\ 0 & 1 & x^2 + iy^2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{130}$$

We here depart from the second kind coordinates to use the natural coordinates defined by (130). Then the matrix x in (130) is

$$x = \exp\left(x^2 x_3 + y^2 x_4 + x^3 x_5 + y^3 x_6\right) \exp\left(x^1 x_1 - y^1 x_2\right), \tag{131}$$

and

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x^1} \quad , \quad X_2 = -\frac{\partial}{\partial y^1} \\ X_3 &= \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3} + y^1 \frac{\partial}{\partial y^3} \quad , \quad X_4 = \frac{\partial}{\partial y^2} - y^1 \frac{\partial}{\partial x^3} + x^1 \frac{\partial}{\partial y^3} \\ X_5 &= \frac{\partial}{\partial x^3} \quad , \quad X_6 = \frac{\partial}{\partial y^3}. \end{aligned}$$

13.4.9 Holomorphic functions for $J(\xi_3^1, \xi_4^1, \xi_5^5, \xi_5^6)$.

Let G denote the group G_0 endowed with the left invariant structure of complex manifold defined by $J = J(\xi_3^1, \xi_4^1, \xi_5^5, \xi_5^6)$ in (112). One easily checks (with formulae (114) and (115)) that

$$\tilde{X}_6^- = -i\, \frac{1 - i\xi_5^5}{\xi_5^6}\, \tilde{X}_5^- \ ; \ \tilde{X}_1^- = -i\, \frac{1 + ia}{\xi_4^1}\, \tilde{X}_4^- \ ; \ i\,\tilde{X}_2^- - \tilde{X}_3^- = -i\left(id + \frac{(1 + ia)\xi_3^1}{\xi_4^1}\right) \tilde{X}_4^-.$$

Hence $H_{\mathbb{C}}(G) = \{f \in C^\infty(G_0) \ ; \ \tilde{X}_j^- f = 0 \ \forall j = 1, 3, 5\}$. Consider first the equation

$$\tilde{X}_5^- f = 0. \tag{132}$$

One has

$$\tilde{X}_5^- = X_5 + i(\xi_5^5 X_5 + \xi_5^6 X_6) = \frac{\partial}{\partial x^3} + i\left(\xi_5^5 \frac{\partial}{\partial x^3} + \xi_5^6 \frac{\partial}{\partial y^3}\right) = \frac{\partial}{\partial u^3} + i\, \frac{\partial}{\partial v^3} = 2\, \frac{\partial}{\partial \overline{w^3}} \tag{133}$$

where

$$u^3 = x^3 - \frac{\xi_5^5}{\xi_5^6}\, y^3 \ ; \ v^3 = \frac{y^3}{\xi_5^6} \ ; \ w^3 = u^3 + iv^3.$$

Equation (132) simply means that f is holomorphic with respect to w^3. Consider now the equation

$$\tilde{X}_1^- f = 0. \tag{134}$$

One has

$$\tilde{X}_1^- = X_1 + i(aX_1 + bX_4) = \frac{\partial}{\partial x^1} + i\left(a\, \frac{\partial}{\partial x^1} + b\, \frac{\partial}{\partial y^2}\right) + ib\left(-y^1 \frac{\partial}{\partial x^3} + x^1 \frac{\partial}{\partial y^3}\right).$$

We suppose that f satisfies equation (132), *i.e.* f is holomorphic with respect to w^3. Hence

$$\frac{\partial f}{\partial x^3} = \frac{\partial f}{\partial u^3} = \frac{\partial f}{\partial w^3}$$

$$\frac{\partial f}{\partial y^3} = \frac{1}{\xi_5^6}\frac{\partial f}{\partial v^3} - \frac{\xi_5^5}{\xi_5^6}\frac{\partial f}{\partial u^3} = \frac{1}{\xi_5^6}(i-\xi_5^5)\frac{\partial f}{\partial w^3}.$$

Then equation (134) reads

$$\frac{\partial f}{\partial x^1} + i\left(a\frac{\partial f}{\partial x^1} + b\frac{\partial f}{\partial y^2}\right) - b\left(iy^1 + \frac{x^1}{\xi_5^6}(1+i\xi_5^5)\right)\frac{\partial f}{\partial w^3} = 0$$

that is

$$2\frac{\partial f}{\partial \overline{w^1}} - \frac{b}{2}\left(2iy^1 + \frac{1+i\xi_5^5}{\xi_5^6}((1-ia)w^1 + (1+ia)\overline{w^1})\right)\frac{\partial f}{\partial w^3} = 0 \tag{135}$$

where

$$u^1 = x^1 - \frac{a}{b}y^2\ ;\ v^1 = \frac{y^2}{b}\ ;\ w^1 = u^1 + iv^1.$$

Finally, we turn to the last equation

$$\tilde{X}_3^- f = 0. \tag{136}$$

One has

$$\begin{aligned}
\tilde{X}_3^- &= X_3 + i(\xi_3^1 X_1 + X_2 + dX_4)\\
&= i\xi_3^1 X_1 + iX_2 + X_3 + idX_4\\
&= i\xi_3^1\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial y^1} + \left(\frac{\partial}{\partial x^2} + x^1\frac{\partial}{\partial x^3} + y^1\frac{\partial}{\partial y^3}\right) + id\left(\frac{\partial}{\partial y^2} - y^1\frac{\partial}{\partial x^3} + x^1\frac{\partial}{\partial y^3}\right).
\end{aligned}$$

Since we suppose f holomorphic with respect to w^3, equation (136) then reads

$$\frac{\partial f}{\partial x^2} - i\frac{\partial f}{\partial y^1} + i\,\xi_3^1\frac{\partial f}{\partial x^1} + id\frac{\partial f}{\partial y^2} + \left(x^1 - idy^1 + (y^1 + idx^1)\frac{i-\xi_5^5}{\xi_5^6}\right)\frac{\partial f}{\partial w^3} = 0$$

that is

$$\begin{aligned}
2\frac{\partial f}{\partial \overline{w^2}} &+ \left(i\left(\xi_3^1 - \frac{ad}{b}\right) - \frac{d}{b}\right)\frac{\partial f}{\partial w^1} + \left(i\left(\xi_3^1 - \frac{ad}{b}\right) + \frac{d}{b}\right)\frac{\partial f}{\partial \overline{w^1}}\\
&+ \frac{1}{2}\left[\left((1-ia)w^1 + (1+ia)\overline{w^1}\right)\left(1 - \frac{d(1+i\xi_5^5)}{\xi_5^6}\right)\right.\\
&\left. -(w^2 - \overline{w^2})\left(-d + \frac{1+i\xi_5^5}{\xi_5^6}\right)\right]\frac{\partial f}{\partial w^3} = 0
\end{aligned} \tag{137}$$

where

$$w^2 = x^2 - iy^1.$$

Now equation (135) reads

$$2\frac{\partial f}{\partial \overline{w^1}} - \frac{b}{2}\left(-w^2 + \overline{w^2} + \frac{1+i\xi_5^5}{\xi_5^6}((1-ia)w^1 + (1+ia)\overline{w^1})\right)\frac{\partial f}{\partial w^3} = 0. \tag{138}$$

From equations (133), (138), (137), one readily sees that the functions φ^1 and φ^2 defined by

$$\varphi^1 = 2w^1 - \left(i\left(\xi_3^1 - \frac{ad}{b}\right) - \frac{d}{b}\right)\overline{w^2} \tag{139}$$

$$\varphi^2 = w^2 \tag{140}$$

are holomorphic on G. We look for a holomorphic function which depends on w^3. For any C^∞-function $\psi(w^1, w^2, \overline{w^2})$, *i.e.* ψ doesn't depend on $w^3, \overline{w^3}, \overline{w^1}$, the following function f_1 is a solution of equations (132) and (138) :

$$\begin{aligned}
f_1 = w^3 + \frac{b}{4}&\left[-w^2\,\overline{w^1} + \overline{w^2}\,\overline{w^1} + \frac{1+i\xi_5^5}{\xi_5^6}\left((1-ia)w^1\overline{w^1} + (1+ia)\frac{(\overline{w^1})^2}{2}\right)\right]\\
&+ \psi(w^1, w^2, \overline{w^2}).
\end{aligned} \tag{141}$$

We want to choose ψ such that f_1 is a solution of (137) as well. First, we have :

$$\begin{aligned}
\frac{\partial f_1}{\partial w^1} &= \frac{b}{4\xi_5^6}(1-ia)(1+i\xi_5^5)\,\overline{w^1} + \frac{\partial \psi}{\partial w^1}\\
\frac{\partial f_1}{\partial \overline{w^1}} &= \frac{b}{4}\left(-w^2+\overline{w^2}+\frac{1+i\xi_5^5}{\xi_5^6}((1-ia)w^1+(1+ia)\overline{w^1})\right)\\
\frac{\partial f_1}{\partial \overline{w^2}} &= \frac{b}{4}\,\overline{w^1}+\frac{\partial \psi}{\partial \overline{w^2}}\\
\frac{\partial f_1}{\partial w^3} &= 1\,.
\end{aligned}$$

Introducing these values in (137) we find that f_1 is a solution to (137) if and only if

$$N\,\overline{w^1} - M(w^2-\overline{w^2}) + \Lambda w^1 + \left(i\left(\xi_3^1-\frac{ad}{b}\right)-\frac{d}{b}\right)\frac{\partial \psi}{\partial w^1} + 2\,\frac{\partial \psi}{\partial \overline{w^2}} = 0$$

where

$$\begin{aligned}
N &= \frac{1}{\xi_5^6}\left(ib\xi_3^1 - b\xi_3^1\xi_5^5 + b\xi_5^6 + (1+ia)\xi_5^6 - d(1+ia)(1+i\xi_5^5)\right)\\
M &= \frac{1}{4}(ib\xi_3^1+(1-ia)d) + \frac{1}{2}\left(\frac{1}{\xi_5^6} - d + i\frac{\xi_5^5}{\xi_5^6}\right)\\
\Lambda &= \frac{(1+i\xi_5^5)(1-ia)}{4\xi_5^6}\left(ib\xi_3^1+(1-ia)d\right) + \frac{1-ia}{2}\left(1-\frac{d(1+i\xi_5^5)}{\xi_5^6}\right).
\end{aligned}$$

A computation shows that N is actually equal to 0. Hence f_1 is a solution to (137) if and only if

$$-M(w^2-\overline{w^2}) + \Lambda w^1 + \left(i\left(\xi_3^1-\frac{ad}{b}\right)-\frac{d}{b}\right)\frac{\partial \psi}{\partial w^1} + 2\,\frac{\partial \psi}{\partial \overline{w^2}} = 0. \tag{142}$$

Note that

$$i\left(\xi_3^1-\frac{ad}{b}\right)-\frac{d}{b} = 0 \Leftrightarrow \xi_3^1 = d = 0 \Leftrightarrow \xi_3^1 = 0, \xi_4^1 = 1.$$

Hence, in the nonabelian case where one doesn't have simulteanously $\xi_3^1 = 0, \xi_4^1 = 1$, for (142) to hold, it is sufficient to have

$$\begin{aligned}
\frac{\partial \psi}{\partial w^1} &= -\frac{\Lambda}{i\left(\xi_3^1-\frac{ad}{b}\right)-\frac{d}{b}}\,w^1\\
\frac{\partial \psi}{\partial \overline{w^2}} &= \frac{M}{2}(w^2-\overline{w^2})
\end{aligned}$$

which gives a solution

$$\psi = -\frac{\Lambda}{i\left(\xi_3^1-\frac{ad}{b}\right)-\frac{d}{b}}\,\frac{(w^1)^2}{2} + \frac{M}{2}\left(w^2\overline{w^2}-\frac{(\overline{w^2})^2}{2}\right).$$

In the abelian case, one can take

$$\psi = -\frac{\Lambda}{2}\,w^1\overline{w^2} + \frac{M}{2}\left(w^2\overline{w^2}-\frac{(\overline{w^2})^2}{2}\right).$$

We finally get the holomorphic function f_1 : in the nonabelian case,

$$\begin{aligned}
f_1 = w^3 + \frac{b}{4}\left[-w^2\,\overline{w^1} + \overline{w^2}\,\overline{w^1} + \frac{1+i\xi_5^5}{\xi_5^6}\left((1-ia)w^1\overline{w^1} + (1+ia)\frac{(\overline{w^1})^2}{2}\right)\right]\\
- \frac{\Lambda}{i\left(\xi_3^1-\frac{ad}{b}\right)-\frac{d}{b}}\,\frac{(w^1)^2}{2} + \frac{M}{2}\left(w^2\overline{w^2}-\frac{(\overline{w^2})^2}{2}\right).
\end{aligned} \tag{143}$$

In the abelian case,

$$f_1 = w^3 + \frac{b}{4}\left[-w^2\,\overline{w^1} + \overline{w^2}\,\overline{w^1} + \frac{1+i\xi_5^5}{\xi_5^6}\left((1-ia)w^1\overline{w^1} + (1+ia)\frac{(\overline{w^1})^2}{2}\right)\right]$$
$$- \frac{\Lambda}{2}\,w^1\overline{w^2} + \frac{M}{2}\left(w^2\overline{w^2} - \frac{(\overline{w^2})^2}{2}\right). \quad (144)$$

Note that in the abelian case, one can take $\xi_5^5 = 0, \xi_5^6 = \beta$, $0 < \beta \leqslant 1$, and then $a = c = d = 0$, $b = -1, M = \frac{1}{2\beta}, \Lambda = \frac{1}{2}$, hence

$$f_1 = w^3 - \frac{1}{4}\left[-w^2\,\overline{w^1} + \overline{w^2}\,\overline{w^1} + \frac{1}{\beta}\left(w^1\overline{w^1} + \frac{(\overline{w^1})^2}{2}\right)\right]$$
$$- \frac{1}{4}w^1\overline{w^2} + \frac{1}{4\beta}\left(w^2\overline{w^2} - \frac{(\overline{w^2})^2}{2}\right). \quad (145)$$

In both abelian and nonabelian cases, let $F : G \to \mathbb{C}^3$ defined by

$$F = (\varphi^1, \varphi^2, \varphi^3) \quad (146)$$

where φ^1, φ^2 are defined in (139), (140) and $\varphi^3 = f_1$. F is a biholomorphic bijection, hence a global chart on G. Then G is isomorphic as a complex variety to the complex Heisenberg group, though the complex structure $J(\xi_3^1, \xi_4^1, \xi_5^5, \xi_5^6)$ is not equivalent to the canonical complex structure. We determine now how the multiplication of G looks like in the chart (146). Recall first the formulae :

$$\begin{aligned} w^1 &= \left(x^1 - \frac{a}{b}y^2\right) + i\frac{y^2}{b} && (147)\\ w^2 &= x^2 - iy^1 && (148)\\ w^3 &= \left(x^3 - \frac{\xi_5^5}{\xi_5^6}\right) + i\,\frac{y^3}{\xi_5^6}. && (149)\end{aligned}$$

Let $\alpha, x \in G$:

$$x = \begin{pmatrix} 1 & x^1+iy^1 & x^3+iy^3 \\ 0 & 1 & x^2+iy^2 \\ 0 & 0 & 1\end{pmatrix} \; ; \; \alpha = \begin{pmatrix} 1 & \alpha^1+i\beta^1 & \alpha^3+i\beta^3 \\ 0 & 1 & \alpha^2+i\beta^2 \\ 0 & 0 & 1\end{pmatrix}.$$

With obvious notations, $\alpha = [w_\alpha^1, w_\alpha^2, w_\alpha^3]$, $x = [w_x^1, w_x^2, w_x^3]$, $\alpha\, x = [w_{\alpha x}^1, w_{\alpha x}^2, w_{\alpha x}^3]$, $\alpha = [\varphi_\alpha^1, \varphi_\alpha^2, \varphi_\alpha^3]$, $x = [\varphi_x^1, \varphi_x^2, \varphi_x^3]$, $\alpha\, x = [\varphi_{\alpha x}^1, \varphi_{\alpha x}^2, \varphi_{\alpha x}^3]$. We want to compute $\varphi_{\alpha x}^1, \varphi_{\alpha x}^2, \varphi_{\alpha x}^3$. First, compute $w_{\alpha x}^1, w_{\alpha x}^2, w_{\alpha x}^3$. By matrix multiplication and formulae (147)-(149) one gets

$$\begin{aligned} w_{\alpha x}^1 &= w_\alpha^1 + w_x^1 \\ w_{\alpha x}^2 &= w_\alpha^2 + w_x^2 \\ w_{\alpha x}^3 &= w_\alpha^3 + w_x^3 + \chi(\alpha, x)\end{aligned}$$

where $\chi(\alpha, x) = \alpha^1 x^2 - \beta^1 y^2 + \frac{i-\xi_5^5}{\xi_5^6}\,(\alpha^1 y^2 + \beta^1 x^2)$. Then from (139), (140) :

$$\begin{aligned} \varphi_{\alpha x}^1 &= \varphi_\alpha^1 + \varphi_x^1 \\ \varphi_{\alpha x}^2 &= \varphi_\alpha^2 + \varphi_x^2.\end{aligned}$$

To get $\varphi_{\alpha x}^3$, we just make the substitutions $w^1 \to w_\alpha^1 + w_x^1,\ w^2 \to w_\alpha^2 + w_x^2$, $w^3 \to w_\alpha^3 + w_x^3 + \chi(\alpha, x)$ in (143) (we consider here the nonabelian case). Now, let

$$\Delta = \varphi_{\alpha x}^3 - \varphi_\alpha^3 - \varphi_x^3.$$

Computations give the following result :

$$\Delta = \frac{1}{8\xi_5^6}\left(C_1\varphi_x^1 + C_2\xi_5^6\varphi_x^2\right) \quad (150)$$

where

$$C_1 = b(1+i\xi_5^5)\overline{\varphi_\alpha^1} + (b-1+ia)\xi_5^6\overline{\varphi_\alpha^2} + \frac{b(a+i)\xi_5^6}{b\xi_3^1 - d(a-i)}\varphi_\alpha^1 + \left((\xi_5^5 - i)(b\xi_3^1 - da - id) - b\xi_5^6\right)\varphi_\alpha^2 \quad (151)$$

$$C_2 = (1-b+ia)\overline{\varphi_\alpha^1} + \left(\frac{4(1+i\xi_5^5)}{\xi_5^6} + (a+i(b+1))\xi_3^1 - \frac{d}{b}(1+a^2) - d(1+ia)\right)\overline{\varphi_\alpha^2} + (1-ai)\varphi_\alpha^1 + \left(-\frac{2(1+i\xi_5^5)}{\xi_5^6} + (a+i(b-1))\xi_3^1 - \frac{d}{b}(1+a^2) + d(1-ia)\right)\varphi_\alpha^2. \quad (152)$$

For example, in the case of $J(1,1,1,1)$, the formula reads

$$\Delta = \left(-\frac{3-i}{4}\overline{\varphi_\alpha^2} - \frac{5+5i}{8}\overline{\varphi_\alpha^1} + \frac{3+4i}{8}\varphi_\alpha^2 + \frac{7+i}{16}\varphi_\alpha^1\right)\varphi_x^1 + \left(\frac{3+i}{4}\overline{\varphi_\alpha^2} + \frac{3+i}{4}\overline{\varphi_\alpha^1} - \frac{1+3i}{4}\varphi_\alpha^2 + \frac{1-2i}{8}\varphi_\alpha^1\right)\varphi_x^2 . \quad (153)$$

Finally, in the abelian case $J(0,1,0,\beta)$, one has to use (145) and one gets :

$$\Delta = -\left(\frac{1}{4}\overline{\varphi_\alpha^2} + \frac{1}{8\beta}\overline{\varphi_\alpha^1} - \frac{1}{8}\varphi_\alpha^2 + \frac{1}{16\beta}\varphi_\alpha^1\right)\varphi_x^1 + \left(\frac{1}{2\beta}\overline{\varphi_\alpha^2} + \frac{1}{4}\overline{\varphi_\alpha^1} - \frac{1}{4\beta}\varphi_\alpha^2 + \frac{1}{8}\varphi_\alpha^1\right)\varphi_x^2 .$$

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