Groups with quadratic-non-quadratic Dehn functions

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Abstract

We construct a finitely presented group G with non-quadratic Dehn function f majorizable by a quadratic function on arbitrary long intervals.

1 Introduction

Recall that the *Dehn function* of a finite presentation $\langle X \mid R \rangle$ of a group *G* is the smallest function $f : \mathbb{N} \to \mathbb{N}$ such that any word of length at most n in *X* that represents the identity of *G* is freely equal to a product of at most f(n) conjugates of elements of *R*. The Dehn functions f_1, f_2 of any two finite presentations of the same group *G* are *equivalent*, that is $f_2(n) \leq Cf_1(Cn) + Cn + C$, $f_1(n) < Cf_2(Cn) + Cn + C$ for some constant *C*. As usual, we do not distinguish equivalent functions. The Dehn function can also be defined as the smallest *isoperimetric* function of the presentation: that is the smallest function f(n) such that the area (i.e. the number of 2-cells) of a minimal van Kampen diagrams over $\langle X \mid R \rangle$ having perimeter (i.e. the combinatorial length of the contour) at most n does not exceed f(n). The connections of the properties of Dehn functions, on the one hand, to the asymptotic geometry of groups and spaces, and, on the other hand, to the computational complexity of the algorithmic word problem, are discussed in [6], [1] and [8].

The class of increasing functions which, up to equivalence, can be represented as Dehn functions of groups is vast (see [11], [4], [3], [9]), but there is one gap in the scale of their rates: if a Dehn function of a group G is subquadratic, then it is linear, and so G is a word hyperbolic group [5], [7], [2].

The goal of this paper is to give an example of a group whose Dehn function is not majorized on \mathbb{N} by a quadratic function but is smaller than a quadratic function on arbitrary long intervals of natural numbers.

Theorem 1.1. Let $\Psi(n) = n^2 \log' n / \log' \log' n$ where $n \ge 0$ and $\log' n = \max(\log_2 n, 1)$. There is a finitely generated group G whose Dehn function f(n) satisfies the following properties:

(1) $c_1 n^2 \leq f(n) \leq c_2 \Psi(n)$ for some positive constants c_1, c_2 and all sufficiently large n;

(2) there is a sequence $n_i \to \infty$ with $f(n_i) \ge c_3 \Psi(n_i)$ for a positive c_3 and every n_i ;

(3) there is a sequence $n'_i \to \infty$ with $f(n'_i) \leq c_4(n'_i)^2$ for a positive c_4 and every n'_i ;

Moreover:

(4) there are sequences of positive numbers $d_i \to \infty$ and $\lambda_i \to \infty$ such that $f(x) \leq c_4 x^2$ for arbitrary integer $x \in \bigcup_{i=1}^{\infty} [\frac{d_i}{\lambda_i}, \lambda_i d_i]$,

(5) there is a positive constant c_5 such that for every n_i defined in (2), and for every integer x with $x \leq c_5 n_i$, we have $f(x) \leq c_4 n_i^2$, in particular, $f([c_5 n_i])/f(n_i) \to 0$.

The group G is a multiple HNN extension of a free group.

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It will be clear from the proof that the sequences (n_i) , (n'_i) and (d_i) have double exponential growth. Notice that such sequences cannot grow as an ordinary exponential function (or slower), because any function f, up to equivalence, is determined by its values $f(a^i)$, i = 1, 2, ..., if a > 1.

The unusual almost quadratic behavior of the Dehn function, and especially the properties (4) and (5), lead in [10], to the solution of a well known problem about asymptotic cones of groups. Namely, property (4) guarantees that a cone defined by the scaling sequence (d_i) is simply connected, and property (5) implies that a cone defined by the sequence n_i is not simply connected. (See [10] for the details.)

To prove Theorem 1.1 we construct G as a special multiple HNN extension of a free groups, namely, an S-machine. Starting with [11], Sapir's S-machines are applied to a number of group theoretical tasks. In Section 3, we recall the definition and basic property of an auxiliary adding machine Z(A) introduced in [9]. In a sense, the main machine M defined in Section 4, is composed from various copies of adding machines.

It is seen from the definition of M that given number n, this machine can produce a computation $W_0 \to W_1 \to \cdots \to W_t$ such that the words W_0 and W_t are of length n, the maximal length of W_i is roughly $\exp n$, and t is roughly $\exp \exp n$. The corresponding diagram Δ for the conjugation of the words W_0 and W_t , has area roughly equal to $t \log t$. One can obtain a diagram of area $t^2 \log t / \log \log t$ and perimeter 4t when gluing together $t / \log \log t$ copies of Δ . This proofs the property (2) of Theorem 1.1.

To obtain the other inequalities, one has to strictly control what the non-deterministic machine M can do. (For example, the rules of ages (2) and (5) look useless for property (2) but we need them to prove the other properties.) The work of M is studied in Section 5. However one meets the primary difficulties when proceeding to the calculation of areas for *arbitrary* diagrams over the group G. In Section 6, we refine the technique of [9], and the exposition heavily depends on [9]. The quadratic upper bound for the dispersion of a bipartite chord diagram introduced in [9], plays a key role here as well.

2 Adding machine Z(A)

Following [9], we treat S-machines as HNN extensions of a free group F(Q, Y) generated by two sets of letters $Q = \bigcup_{i=1}^{N} Q_i$ and $Y = \bigcup_{i=1}^{N-1} Y_i$ where Q_i are disjoint and non-empty (below we always assume that $Y_N = Y_0 = \emptyset$). The set Q is called the set of q-letters, the set Y is called the set of a-letters.

Instead of the set of stable letters we have a collection Θ of N-tuples of θ -letters. Elements of Θ are called *rules*. The components of θ are called *brothers* $\theta_1, ..., \theta_N$.

With every $\theta \in \Theta$, we associate two sequences of elements in $F(Q \cup Y)$: $B(\theta) = [U_1, ..., U_N]$, $T(\theta) = [V_1, ..., V_N]$, and a subsets $Y_i(\theta) \subseteq Y_i$.

The generating set \mathcal{X} of the group S consists of all q-, a- and θ -letters. The relations, under condition $\theta_{N+1} = \theta_1$, are:

$$U_i\theta_{i+1} = \theta_i V_i, \quad i = 1, \dots, s, \qquad \theta_j a = a\theta_j \quad for \quad all \quad a \in Y_j(\theta)$$

$$(2.1)$$

Sometimes we will denote the rule θ by $[U_1 \to V_1, ..., U_N \to V_N]$. This notation contains all the necessary information about the rule except for the sets $Y_i(\theta)$. In most cases it will be clear what these sets are. By default $Y_i(\theta) = Y_i$.

In this section we recall the definition and some properties of an auxiliary adding machine Z(A) introduced in [9].

Let A be a finite set of letters. Let the set A_1 be a copy of A. It will be convenient to denote A by A_0 . For every letter $a \in A$ a_0 and a_1 denotes its copy in A_0 and A_1 , respectfully.

The set of state letters of Z(A) is $\{L\} \cup \{p(1), p(2), p(3)\} \cup \{R\}$, i.e., there are 3 states for the *p*-letter, and the letters *L* and *R* do not change their states. The set of tape letters is $Y_1 \cup Y_2$ where $Y_1 = A_0 \cup A_1$ and $Y_2 = A_0$.

The machine Z(A) has the following rules (there *a* is an arbitrary letter from *A*) and their inverses. The comments explain the meanings of these rules.

• $r_1(a) = [L \to L, p(1) \to a_1^{-1} p(1) a_0, R \to R].$

Comment. The state letter p(1) moves left searching for a letter from A_0 and replacing letters from A_1 by their copies in A_0 .

• $r_{12}(a) = [L \to L, p(1) \to a_0^{-1} a_1 p(2), R \to R].$

Comment. When the first letter a_0 of A_0 is found, it is replaced by a_1 , and p turns into p(2).

• $r_2(a) = [L \to L, p(2) \to a_0 p(2) a_0^{-1}, R \to R].$

Comment. The state letter p(2) moves toward R.

• $r_{21} = [L \to L, p(2) \xrightarrow{\ell} p(1), R \to R], Y_1(r_{21}) = Y_1, Y_2(r_{21}) = \emptyset.$

Comment. p(2) and R meet, the cycle starts again.

• $r_{13} = [L \xrightarrow{\ell} L, p(1) \to p(3), R \to R], Y_1(r_{13}) = \emptyset, Y_2(r_{13}) = A_0.$

Comment. If p(1) never finds a letter from A_0 , the cycle ends, p(1) turns into p(3); p and L must stay next to each other in order for this rule to be executable.

• $r_3(a) = [L \to L, p(3) \to a_0 p(3) a_0^{-1}, R \to R], Y_1(r_3(a)) = Y_2(r_3(a)) = A_0$

Comment. The letter r_3 returns to R.

Remark 2.1. If we replace every letter in A_i by its index *i*, then every word *u* in the alphabet $A_0 \cup A_1$ turns into a binary number b(u). If the machine starts with the word Lup(1)R where *u* is a positive word in A_0 , then b(u) = 0 and each 'regular' cycle of the machine adds 1 to b(u) modulo $2^{|u|}$. After $2^{|u|}$ we obtain Lup(3)R.

For every letter $a \in A$ we set $r_i(a^{-1}) = r_i(a)^{-1}$ (i = 1, 2, 3).

Remark 2.2. All the rules of machine Z(A) are transformed into relations by formulas 2.1, and so Z(A) can also be considered as a group. However, as in [9], we will use diagram and machine concepts in our proofs. All of them can be found in [9]: reduced diagrams, bands in diagrams, trapezia, their heights, bases and histories; admissible words, computation $W = W_0 \rightarrow_{\theta_1} W_1 \rightarrow_{\theta_2} \cdots \rightarrow_{\theta_t} W_t = f \cdot W$ with history $f = \theta_1 \theta_2 \dots \theta_t$ determined by a trapezia, the width and area of a computation.

There is an obvious mirror analog Z(A, mir) of the machine Z(A): $Y_1(mir) = A_0, Y_2(mir) = A_0 \cup A_1$, and, for example, the mirror analog of the rule $r_1(a)$ is $r_1(a, mir) = [L \to L, p(1) \to a_0 p(1)a_1^{-1}, R \to R]$. (The state letter moves right searching for a letter from A_0 and replacing letters from A_1 by their copies in A_0 .) There are obvious mirror analogs of lemmas 2.3 - 2.8, but we will not formulate these analogs. We often use p instead of some p(i) in subsequent formulations.

Lemma 2.3. Suppose (1) base(W) $\in \{LpR, p^{-1}pR\}$, both W and $f \cdot W$ contain p(1)R (resp. p(3)R) or (2) base(W) is $p^{-1}pR$ and both W and $f \cdot W$ contain p(1)R or p(3)R. Assume that all a-letters in W and in $f \cdot W$ are from A_0 in both cases. Then f is empty.

Proof. In case (1), the assertion is proved in [9], Lemma 2.27. It remains to consider case (2) where W contains p(1) but $f \cdot W$ contains p(3). But this is impossible since the *p*-letter cannot change state from p(1) to p(3) when its left neighbor in the base differs from L. (See the definition of the rule r_{13} .)

Lemma 2.4. Let W = LvpuR, base(W) = LpR. Suppose that $|\theta \cdot W| > |W|$. Then for every computation $W \rightarrow_{\theta} W_1 \rightarrow W_2 \rightarrow ... \rightarrow f \cdot W$, we have $|W_i| > |W|$ for every $i \ge 1$.

Proof. This assertion is proved in [9], Lemma 2.24.

Lemma 2.5. Suppose that an admissible word W has the form LupvR (resp. $p^{-1}upvR$ or $Lvpup^{-1}$) where u, v are words in $(A_0 \cup A_1)^{\pm 1}$. Let $\theta \cdot W = Lu'p'v'R$ (resp. $\theta \cdot W = (p')^{-1}u'p'v'R$ or $\theta \cdot W = Lv'p'u'p'^{-1}$). Then the projections of uv and u'v' (resp. $v^{-1}uv$ and $(v')^{-1}u'v'$, or vuv^{-1} and $v'u'(v')^{-1}$) onto A are freely equal.

Proof. This assertion is proved in [9], Lemma 2.18.

Lemma 2.6. Suppose that one of the following conditions for an admissible word W of Z(A) is satisfied (there $p = \{p(1), p(2), p(3)\}$): W does not contain a p-letter or base(W) = Lpp^{-1} , or base(W) = $pp^{-1}p$, or base(W) = $p^{-1}pR$, or base(W) = LpR. Then the width of any computation

$$W = W_0 \to_{\theta_0} W_1 \to_{\theta_1} \dots \to_{\theta_{t-1}} W_t$$

is at most $C \max(|W|, |W_t|)$ for some constant C.

Proof. This assertion is proved in [9], Lemma 2.29.

Lemma 2.7. Let base(W) = LpR. Then for every computation $W = W_0 \rightarrow W_1 \rightarrow \cdots \rightarrow W_t = f \cdot W$ of the S-machine Z(A):

- 1. $|W_i| \le \max(|W|, |f \cdot W|), i = 0, ..., t,$
- If W = LupR where p = p(1) (resp. p = p(3)), f ⋅ W contains p(3)R (resp. p(1)R) and all a-letters in W, f ⋅ W are from A₀^{±1}, then the length g(|u|) of f is between 2^{|u|} and 6 ⋅ 2^{|u|}, u is a positive word, and all words in the computation have the same length. Vice versa, for every positive word u, such a computation does exist.

Proof. This assertion is proved in [9], Lemma 2.25 and Remark 2.19.

Lemma 2.8. For every admissible word W with base(W) = LpR, every rule θ applicable to W, and every natural number t > 1, there is at most one computation $W \to_{\theta} W_1 \to ... \to W_t$ of length t where the lengths of the words are all the same.

Proof. This assertion is proved in [9], Lemma 2.21.

3 How machine \mathcal{M} works

In this section, we introduce the machine \mathcal{M} defining our group G.

We set N = 5 and $\mathcal{Q} = \bigcup_{i=1}^{5} \mathcal{Q}_i$. Here $\mathcal{Q}_6 = \mathcal{Q}_1 = \{k_0\}, \mathcal{Q}_3 = \{k_1\}, \mathcal{Q}_5 = \{k_2\}, \mathcal{Q}_2$ is of the form $\{q_1^*(*)\}, \mathcal{Q}_4$ is of the form $\{q_2^*(*)\}$, where the stars * and (*) will be replaced by particular indices below. The set of rules of machine \mathcal{M} will be partitioned in several *ages*.

Consider the machine $Z(\{a\})$ for a 1-letter alphabet $\{a\}$. Let Υ be the set of its rules. We introduce letters $a_{\tau} \in A(\Upsilon)$ for every $\tau \in \Upsilon$. Then we have two copies $A(\Upsilon)_0$ and $A(\Upsilon)_1$ of this alphabet. Let $Y = \bigcup_{i=1}^4 Y_i$ where $Y_1 = \{a_0\} \cup \{a_1\}, Y_2 = \{a_0\}, Y_3 = A(\Upsilon)_0, Y_4 = A(\Upsilon)_0 \cup A(\Upsilon)_1$.

Age(1) We correspond, to every rule τ of Z(A), a rule τ^1 of age (1) of the machine \mathcal{M} . For example, for $\tau = r_1(a)$, we have

$$r_1(a)^1 = [k_0 \to k_0, q_1(1)^1 \to a_1^{-1}q_1(1)^1 a_0, k_1 \to k_1, q_2^1 \to a_{r_1(a)}q_2^1, k_2 \to k_2]$$

with
$$Y_3(r_1(a)^1) = A(\Upsilon)_0, Y_4(r_1(a)^1) = \emptyset$$

Comment. Now the machine $Z(\{a\})$ works with letters L, p, R replaced by k_0, q_1, k_1 . At the same time it writes the history of its work in alphabet $A(\Upsilon)_0$ on the tape between the heads k_1 and q_2 . For example, it can start working with a word $k_0 a_0^n q_1(1)^1 k_1 q_2^1 k_2$ and finish 'adding' with $k_0 a_0^n q_1(3)^1 k_1 u q_2^1 k_2$ where u is the history f of such a computation copied in the alphabet $A(\Upsilon)$. The length of the positive word u is g(n) (see Lemma 2.7).

Age(12) The only *connecting* rule to the age (2) is

$$r^{12} = [k_0 \to k_0, q_1(3)^1 \to q_1(3)^2, k_1 \to k_1, q_2^1 \to q_2^2, k_2 \to k_2]$$

with $Y_1(r^{12}) = \{a_0\}, Y_2(r^{12}) = \emptyset, Y_3(r^{12}) = A(\Upsilon)_0, Y_4(r^{12}) = \emptyset$

Comment. This rule changes the states of the heads q_1, q_2 making possible the applications of rules of age (2). It is applicable under the restrictions imposed on the sets $Y_i(r^{12})$ above.

Age(2) Again, we correspond to every rule τ of machine $Z(\{a\})$ a rule τ^2 of age (2) of machine \mathcal{M} . For example, for $\tau = r_1(a)$, we define

$$\begin{aligned} r_1(a)^2 &= [k_0 \to k_0, q_1(1)^2 \to a_1^{-1} q_1(1)^2 a_0, k_1 \to k_1, q_2^2 \to a_{r_1(a)} q_2^2 a_{r_1(a)}^{-1}, k_2 \to k_2] \\ with \ Y_3(r_1(a)^2) &= Y_4(r_1(a)^2) = A(\Upsilon)_0 \end{aligned}$$

Comment. The work of \mathcal{M} is similar to that in age (1). But now the head q_2 runs to the left. For example, it can start working with the word $k_0 a_0^n q_1(3)^2 k_1 u q_2^2 k_2$ (see Comment to Age (1)), then it can simulate the computation of Z(A) with history f^{-1} and finish 'adding' with $k_0 a_0^n q_1(1)^2 k_1 q_2^2 k_2$. We show and use that such a smooth work between applications of rules of ages (12) and (23) is possible only when the word u has length g(n) for some n.

Age(23) The connecting rule to the age (3) is

$$r^{23} = [k_0 \to k_0, q_1(1)^2 \to q_1^3, k_1 \to k_1, q_2^2 \to q_2(1)^3, k_2 \to k_2]$$

with $Y_1(r^{23}) = \{a_0\}, Y_2(r^{23}) = \emptyset, \ , Y_3(r^{23}) = \emptyset, Y_4(r^{23}) = A(\Upsilon)_0,$

Comment. The role of this rule is similar to that of r^{12} .

Age(3) Here we use the machine $Z(A(\Upsilon), mir)$. To every rule τ of $Z(A(\Upsilon), mir)$, we correspond a rule τ^3 of age (3) for the machine \mathcal{M} . For example, for $\tau = r_1(a)$ $(a \in A(\Upsilon))$ we have

$$r_1(a)^3 = [k_0 \to k_0, q_1^3 \to q_1^3, k_1 \to k_1, q_2(1)^3 \to a_0 q_2(1)^3 a_1^{-1}, k_2 \to k_2]$$

with
$$Y_1(r_1(a)^3) = A(\Upsilon)_0, Y_2(r_1(a)^3) = \emptyset, Y_3(r_1(a)^3) = A(\Upsilon)_0$$

Comment. The machine $Z(A(\Upsilon), mir)$ works now with heads L, p, R replaced by k_1, q_2, k_2 . The head q_1 stays by k_2 , and the piece of tape between k_1 and q_1 is unchanged. For example, it can start working with $k_0 a_0^n q_1^3 k_1 q_2(1)^3 u k_2$ where u is a reduced word of length g(n) in the alphabet $A(\Upsilon)$, and finish 'adding' with $k_0 a_0^n q_1^3 k_1 q_2(3)^3 u k_2$ after application of g(g(n)) rules (double exponential in n time by Lemma 2.7).

Age(34) The connecting rule of age (34) is

$$r^{34} = [k_0 \to k_0, q_1^3 \to q_1^4, k_1 \to k_1, q_2(3)^3 \to q_2(3)^4, k_2 \to k_2]$$

with $Y_1(r^{34}) = \{a_0\}, Y_2(r^{34}) = \emptyset, \ , Y_3(r^{34}) = \emptyset, Y_4(r^{34}) = A(\Upsilon)_0,$

Ages (4), (45), (5), (56), (6). The rules of ages (4), (5), and (6) are similar to the rules of ages (3), (2) and (1), respectively, up to the superscripts at r- and q-letters: we replace 1 by 6, 2 by 5, and 3 by 4. The connecting rule of ages (45) and (56) are, respectively,

$$r^{45} = [k_0 \to k_0, q_1^4 \to q_1(1)^5, k_1 \to k_1, q_2(1)^4 \to q_2^5, k_2 \to k_2]$$

with
$$Y_1(r^{45}) = \{a_0\}, Y_2(r^{45}) = \emptyset, Y_3(r^{45}) = \emptyset, Y_4(r^{45}) = A(\Upsilon)_0,$$

and

$$r^{56} = [k_0 \to k_0, q_1(3)^5 \to q_1(3)^6, k_1 \to k_1, q_2^5 \to q_2^6, k_2 \to k_2]$$

with $Y_1(r^{56}) = \{a_0\}, Y_2(r^{56}) = \emptyset, Y_3(r^{56}) = A(\Upsilon)_0, Y_4(r^{56}) = \emptyset$

Comment. Let us start with the word $k_0 a_0^n q_1^3 k_1 q_2(3)^3 u k_2$. (See Comment to Age (3).) Then consecutive applications of rules of ages (34), (4), (45), (5), (56) and (6) can transform it as follows: $\rightarrow k_0 a_0^n q_1^4 k_1 q_2(3)^4 u k_2 \rightarrow \cdots \rightarrow k_0 a_0^n q_1^4 k_1 q_2(1)^4 u k_2 \rightarrow k_0 a_0^n q_1(1)^5 k_1 q_2^5 u k_2 \rightarrow \cdots \rightarrow k_0 a_0^n q_1(3)^5 k_1 u q_2^5 k_2 \rightarrow k_0 a_0^n q_1(3)^6 k_1 u q_2^6 k_2 \rightarrow \cdots \rightarrow k_0 a_0^n q_1(1)^6 k_1 q_2^6 k_2$. The computation $k_0 a_0^n q_1(1)^1 k_2 q_2^1 k_2 \rightarrow \cdots \rightarrow k_0 a_0^n q_1(1)^6 k_1 q_2^6 k_2$ we have considered as an example in the comments to the definition of machine \mathcal{M} , has an exponential width in n and double exponential length of the history.

As it was explained in the previous section, the machine \mathcal{M} defines the group $G = G(\mathcal{M})$. (See (2.1).)

If the history h of a computation is a product $h_1h_2...h_s$, where, for every subword h_i , each of its letter has the same age (j_i) $(j_i \in \{(1), (12), ..., (56), (6)\})$, and $j_i \neq j_{i+1}$ for i = 1, ..., s-1, then we say that this computation has brief history $(j_1)(j_2)...(j_s)$.

Since a history h is always a reduced word, it cannot contain subwords of the form $\tau\tau^{-1}$. If τ is a connecting rule, and the computation base has at least one q-letter, the h has no subwords τ^2 , since every connecting rule changes the states of q letters. For the same reason, a brief history of such a computation (with a q-letter in the base) cannot be of the form (1)(3) or (6)(1), or (3)(34)(3), etc.

We call a computation $W_0 \to W_1 \to \ldots W_t$ long if the brief history of this computation or of the inverse computation has a subword equal to $(1)(12)(2)\ldots(56)(6)$. Otherwise it is *short*.

4 *M*-computations with various bases and histories

Denote by \mathcal{A} the alphabet of all *a*-letters $\{a_0^{\pm 1}, a_1^{\pm 1}\} \cup A(\Upsilon)_0^{\pm 1} \cup A(\Upsilon)_1^{\pm 1}$. The length of a word W we denote by |W|, and the *a*-width $|W|_a$ of W is the number of *a*-letters in W. The width ||W|| of an admissible word of the form $q^{-1}uqvk$ and $kvquq^{-1}$ where u and v are words in \mathcal{A} , is defined as 3 + |u| + 2|v|, and ||W|| = |W| for all other words by definition.

We say that a reduced computation is *regular* if the applications of its rules do not change the width. An application of a rule $W \to W'$ increases the width of W if the word W' is longer than W. Then the application of the inverse rule to W' decreases the width.

Lemma 4.1. Let $W_0 \rightarrow_{\rho_1} \ldots \rightarrow_{\rho_l} W_l$ be a computation with base $k_1q_2k_2$. Assume that all the rules ρ_1, \ldots, ρ_l are of age (1) or (6) (of age (2), or (5)). Then there is an integer d, $0 \leq d \leq l$, such that the applications of ρ_1, \ldots, ρ_d decrease (do not increase) the widths of words W_0, \ldots, W_{d-1} , and the applications of $\rho_{d+1}, \ldots, \rho_l$ increase (do not decrease) the widths of W_d, \ldots, W_{l-1} .

Proof. Assume that an application of a rule ρ_i of age (1) increases the width of $W_{i-1} = k_1 w q_2 k_2$, and i < l. Then $W_i = k_1 w a q_2 k_2$ with a reduced word wa, $a \in A(\Upsilon)_0^{\pm 1}$. Since the history of a computation is reduced, we have $\rho_{i+1} \neq \rho_i^{-1}$, and so the application of ρ_{i+1} must also increase the width of W_i as this follows from the definition of the rules of age (1). The lemma statement follows from this observation. The proofs of the assertion for rules of ages (6), (2) and (5), are similar.

Lemma 4.2. Let the history of a computation be $\eta h \eta^{-1}$, where η is a connecting rule and h has no connecting rules. Assume that the base of this computation has one of the forms kqq^{-1} , $qq^{-1}q$, kqk, $q^{-1}qk$. Then the base is $q^{-1}qk$ or kqk, and if $q = q_1$, then rules of h have age (3) or (4), and if $q = q_2$, then rules of h have age (1) or (6).

Proof. No connecting rule is applicable to a word $quq^{-1}vq$ where u and v are a-words.

Now we assume that the base is kqk and the age of h is not (3) or (4). Then the equality $q = q_1$ is impossible by Lemma 2.3. Similarly, h cannot be of age (3) or (4) if $q = q_2$. The assumption that the history h is of age (2) or (5) and the base is $k_1q_2k_2$ leads to a contradiction since h is reduced, and a connecting rule ρ is applicable when $Y_4(\rho) = \emptyset$. Lemma 2.3 also works for bases kqq^{-1} , $q^{-1}qk$ if a connecting rule is applicable to a word having such a base. \Box

Lemma 4.3. Let the base of a computation $W_0 \to W_1 \to \ldots W_t$ have one of the forms kqq^{-1} , $k^{-1}k$, $qq^{-1}q$, kqk, $q^{-1}qk$, kk^{-1} , or k_2k_0 . Then

(1) all applications of the rules are regular if the first and the last rules are both connecting rules and there are no subwords (12)(1)(12) and (56)(6)(56) in the brief history; there are no such subwords if the base contains $q = q_1$;

(2) if there is an application $W_{i-1} \to W_i$ of a connecting rule in the computation and there are no letter q_2 in the base or there are no rules of ages (1) and (6) in the history, then $||W_i|| \leq ||W_s||$ for arbitrary $s \in \{0, 1, \ldots, t\}$. Proof. (1) We may exclude cases kk^{-1} , $k^{-1}k$ and k_2k_0 since they are trivial: $||W_0|| = ||W_1|| = \dots$ Then we may assume that the only connecting rules are the first one and the last one. The remaining rules must be of the same age, say (l) where $l \in \{2, 3, 4, 5\}$. (If, for example, l = 1, then there must be a subword (12)(1)(12) in the brief history, and $q = q_2$ by Lemma 4.2.) The *a*-letters of both W_0 and W_t must belong to $\{a_0^{\pm 1}\}$ or to $A(\Upsilon)_0^{\pm 1}$.

Assume the base is kqk. It is easy to see that the applications of rules do not change the projection of a word onto the subalphabet A_0 since neither of the rules are of age (1) or (6). It follows that $||W_0|| = ||W_t||$ and $||W_i|| \ge ||W_0||$ for $i = 1, \ldots, t - 1$. We notice now that no rule application is increasing by Lemma 2.4, if the age (l) is (2) or (5) and the base is $k_0q_2k_1$, or l = 3, 4 and the base is $k_1q_1k_2$ (i.e., a copy of machine $Z(\{a\})$ or machine $Z(A(\Upsilon), mir)$ works). In other cases, the assertion follows from Lemma 4.1 since $||W_0|| = ||W_t||$.

Let the base be kqq^{-1} . Then the connecting rules (12) and (56) are not applicable, and one may assume by the symmetry that l = 3. The mirror version of Lemma 2.3 makes this case impossible if $q = q_2$. Otherwise we just have $W_0 = W_1 = \ldots$

Similar arguments work for the bases $q^{-1}qk$ and $qq^{-1}q$.

(2) Let $W_s = k_0 u_s q_1 v_s k_1$. The reduced form of the projections of $u_s v_s$ on the alphabet A_0 do not depend on s by Lemma 2.5. But $u_i v_i$ is a reduced word in A_0 since one of the factors is empty (recall that W_i is the result of an application of a connecting rule). Hence $||W_s|| \ge ||W_i||$. The argument is similar if $q = q_2$ and there are no rules of ages (1) and (6) in the history.

Again, the base cannot be equal to $qq^{-1}q$, and the statement is obvious for bases kk^{-1} , $k^{-1}k$ and k_2k_0 . Then we may assume by part (1), that i = 1, the history is ηh where η is a connecting rule and h has no connecting rules.

Assume, for example, that the base is $q_1^{-1}q_1k_1$, the rule η is of age (12) or (23) ((45), or (56)) and h is of age (2) (of age (5)). Let $W_s = q_1^{-1}u_sq_1v_sk_1$. Then the reduced forms of the projections of $v_s^{-1}u_sv_s$ on the alphabet A_0 do not depend on s by Lemma 2.5. Since u_1 is a word in A_0 and v_1 is empty (recall that η is a connecting rule of age (12) or (23)), we have

$$||W_s|| = 3 + |u_s| + 2|v_s| \ge 3 + |v_s^{-1}u_sv_s| \ge 3 + |u_1| = ||W_1||$$

as desired. If h is of age (3) or (4), then obviously $||W_0|| = ||W_1|| = \dots$

Similar arguments work for $q = q_2$ and also for bases of the form kqq^{-1} . The lemma is proved.

Lemma 4.4. Let the base of a computation $W_0 \to W_1 \to \ldots \to W_t$ be one of the forms kqq^{-1} , $k^{-1}k$, $qq^{-1}q$, kqk, $q^{-1}qk$, kk^{-1} . Then

(1) if the history of the computation contains both connecting rules r^{12} and r^{23} (or their inverses), then the base has the form kqk;

(2) if $q = q_1$ or the computation is short, then $|W_i| \le c \max(|W_0|, |W_t|)$ for some constant c independent of the computation.

Proof. (1) The connecting rule r^{12} is not applicable whenever kqq^{-1} is the base.

We have $q = q_1$ if the base is $q^{-1}qk$, since otherwise r^{23} is not applicable. But after the application of r^{12} , the state of the q-letters is $q_1(3)^2$, and before the rule r^{23} is applied, the state must be $q_1(1)^2$. But the state $q_1(1)^2$ cannot be reached since the base has a subword $q_1^{-1}q_1$ (but not k_0q_1), a contradiction.

Similarly, the bases of forms $k^{-1}k$, $qq^{-1}q$, kqq^{-1} , and kk^{-1} can be eliminated.

(2) We can assume that there is a q-letter in the base since otherwise the assertion is obvious. Let $h = \rho_0 \dots \rho_t$ be the computation history, and $\rho_{i_1}, \dots, \rho_{i_l}$ all connecting rules in this computation.

For l = 0, the assertion follows from Lemma 2.6 if the age and the base are similar to those for machines $Z(\{a\})$ or $Z(\Upsilon)$, min). Otherwise it either obvious or follows from Lemma 4.1.

Let $l \ge 1$. Then denote by h_0, h_1, \ldots, h_l the subwords of h such that h_j starts with ρ_{i_j} (with ρ_0 for j = 0) and terminates with $\rho_{i_{j+1}}$ (with ρ_t for j = l).

First assume that either $q = q_1$ or h_0 has no rules of age (1) or (6). Then by Lemma 4.3(1), we have that $||W_{i_1-1}|| = ||W_{i_l}|| = ||W_s||$ for $i_1 - 1 \le s \le i_l$, and by Lemma 4.3(2), $||W_0|| \ge ||W_{i_1}|| = ||W_{i_l}||$. Therefore this case is reduced to the statement for the subcomputations $W_0 \to \ldots \to W_{i_1-1}$ and $W_{i_s} \to \ldots \to W_t$ having no connecting rules in the histories. The case where h_l has no rules of age (1) or (6) is similar.

Thus, it remains to eliminate the case: $l \ge 1$, $q = q_2$, h_0 contains rules of age (1) or (6), and similarly h_l does. Therefore l > 1, and we may assume that h_0 contains a rule of age (1). Then $\rho_{i_1} = (12)$, and it follows from Lemma 4.2 that $\rho_{i_2} = (23)$. Then the base is kq_2k by part (1) of the lemma. Now applying Lemma 4.2 several times, we have that $\rho_{i_3} = (34)$, $\rho_{i_4} = (45)$, $\rho_{i_5} = (56)$, l = 5, and the computation is long against the lemma condition.

Lemma 4.5. Let $W_0 \to W_1 \to \ldots \to W_t$ be a long computation with base $k_0q_1k_1q_2k_2$ or $k_1q_2k_2k_0q_1k_1$, or $k_2k_0q_1k_1q_2k_2$. Then for every W_i , where $1 \le i \le t$, we have $|W_i|_a \le \max(|W_1|_a, |W_t|_a, 2)$. If k_0uq_1 is a subword of W_0 with |u| = n, and a subcomputation $W_l \to W_{l+1} \to \ldots \to W_m$ starts (ends) with an application of the rule (12) (the rule (56)), then m - l = 5 + 2g(n) + 2g(g(n)) for some integer n, and the a-width of this subcomputation is n + g(n). The a-widths of the restrictions of this subcomputation to bases $k_0q_1k_2$ and $k_1q_2k_2$ are n and g(n), respectively.

Proof. We will assume that the base is $k_0q_1k_1q_2k_2$. By Lemma 4.3 (1), there are no subwords (12)(1)(12) and (56)(6)(56) in the brief history B of the computation, and so $B = (1)(12)\dots(56)(6)$.

Denote by W'_i and W''_i , respectively, the prefix (the suffix) of W_i ending (starting) with k_1 . Then $|W'_i| \leq \max(|W'_0|, |W'_l|)$ for $0 \leq i \leq l$, by Lemma 2.7, and $|W'_l| \leq |W'_0|$ by Lemma 4.3 (2). Also we have $|W''_i| \leq \max(|W''_0|, |W''_l|)$ by Lemma 4.1.

Then $|W_i| = |W_l| = |W_m|$ for $l \le i \le m$ by Lemma 4.3 (1), and $t \ge g(|v|_a) \ge 2^{|v|_a}$ by Lemma 2.7 for the subcomputation of age (3), where v is any of the words read between k_1 and k_2 in age (3). Similarly, when considering the maximal subcomputation of age (2), we have by lemmas 4.3 and 2.7, $|v|_a = g(|u|) = g(n) \ge 2^n$. Hence, for $l \le i \le m$, we have $|W_i|_a \le n + |v| < \log_2 \log_2 t + \log_2 t \le 2\log_2 t$. For i < l, $|W_i|_a = |W'_i|_a + |W''_i|_a \le$ $\max(|W_0|_a, n + |v|_a) \le \max(|W_0|_a, 2log_2 t)$. Since there is a similar estimate in case $i \ge m$, the desired upper bound for $|W_i|_a$ is obtained for all i.

The equality m - l = 5 + 2g(n) + 2g(g(n)) follows from lemmas 2.8 and 2.7 since there are 5 connecting rules in the subcomputation.

Lemma 4.6. Let $W_0 \to W_1 \to \ldots \to W_t$ be a computation with base $k_0q_1k_1q_2k_2$ or $k_1q_2k_2k_0q_1k_1$, or $k_2k_0q_1k_1q_2k_2$. Assume that $10g(g(n-1)) \leq t \leq g(g(n))$ for some integer n. Then the area of corresponding trapezium Δ does not exceed $Ct(|W_0|_a + |W_t|_a)$ for a constant C independent of the computation.

Proof. If the computation is short, then the statement follows from Lemma 4.4 applied to the restrictions of the computation to subbases $k_0q_1k_1$ and $k_1q_2k_2$. Therefore as in the proof of Lemma 4.5, one may suppose that the brief history of the computation is $(1)(12)\ldots(6)$.

We denote by T_i the *i*-th θ -band of Δ . Observe that at most four (θ, a) -cells of T_i can be attached to its (θ, q) -cells along *a*-edges. Hence the number of cells in T_i does not exceed $4 + 6 + |W_i| = 10 + |W_i|$ because T has at most six k- and q-cells.

Let the application of rules (12) and (56) be the *l*-th and the *m*-th, respectively, in the history $\rho_1 \ldots \rho_t$. Then Δ is the union of of 3 subtrapezia Δ_1 , Δ_2 and Δ_3 which correspond to subwords $\rho_1 \ldots \rho_l$, $\rho_l \ldots \rho_m$, and $\rho_m \ldots \rho_t$, respectively. By Lemma 4.5, there is an integer *r* such that m - l = 5 + 2g(r) + 2g(g(r)), and r < n since $m - l \le t < g(g(n))$. Also, by Lemma 4.5, $|W_i|_a \le r + g(r)$ for $l \le i \le m$. Using the observation of the previous paragraph, we see that the area of Δ_2 does not exceed (10 + r + g(r))(5 + 2g(g(r))).

Consider the restriction of the subcomputation with subhistory $\rho_1 \dots \rho_l$ to the subbase $k_1q_2k_2$. By Lemma 4.1, there is an integer $d, 0 \leq d \leq l$, such that the applications of ρ_1, \dots, ρ_d decrease the *a*-widths of the suffices $V_{i-1} = k_1 \dots k_2$ of subwords W_{i-1} , and the applications of $\rho_{d+1}, \dots, \rho_l$ increase them. But $|V_l|_a = g(r)$ by Lemma 4.5, and therefore $l - d \leq g(r)$.

Let Δ_{11} and Δ_{12} be the subtrapesia of Δ_1 of heights d and l - d, respectively. It follows from Lemma 4.5 that the *a*-width of Δ_{12} does not exceed r + g(r), and therefore its area does not exceed (10 + r + g(r))g(r) since its height is not greater than g(r).

The *a*-width of Δ_{11} is not greater than 3l + r + g(r) because a single application of a rule changes the *a*-width of a word with base $k_0q_1k_1q_2k_2$ at most by 3. Thus the area of Δ_{11} does not exceed l(10+3l+r+g(r)). Therefore the area of Δ_1 is not greater than (10+r+g(r))g(r) +l(10+3l+r+g(r)). Similarly, the area of Δ_3 is bounded from above by (10+r+g(r))g(r) +(t-m)(10+3(t-m)+r+g(r)). Thus the area of Δ is at most $100g(g(r))g(r)+100tg(r)+3t^2$.

It follows from the definition of Δ_{11} that $|W|_a \ge d$. Also recall that $d \ge l - g(r)$. Hence $|W_0|_a + |W_t|_a$ is at least

$$l-g(r) + (t-m) - g(r) = t - (m-l) - 2g(r) = t - (5 + 2g(r) + 2g(g(r))) - 2g(r) > \max(t/3, g(g(r))) - \max(t$$

by the choice of t and by inequality $r \leq n-1$. Therefore the area of Δ is not greater than $Ct(|W_0|_a + |W_t|_a)$ for a constant C independent of the computation.

Lemma 4.7. Let $W_0 \to W_1 \to \ldots \to W_t$ be a long computation with base $k_0q_1k_1q_2k_2$ or $k_1q_2k_2k_0q_1k_1$, or $k_2k_0q_1k_1q_2k_2$. Then $|W_0|_a + |W_t|_a + 1 \ge c_0 \log' \log' t$ for a positive constant c_0 .

Proof. We will use the notation of Lemma 4.6. To proof the statement, it suffices to assume that d = 0 because, by Lemma 2.7 (2), the applications of ρ_1, \ldots, ρ_d cannot increase the lengths of subwords $k_0 \ldots k_1$ of the words W_i since the rules of age (1) follows by a connecting rule in the whole computation; and they decrease the lengths of their subwords of the form $k_1 \ldots k_2$. Similar assumption is applicable to $W_m \to W_{m+1} \to \ldots \to W_t$. Then $l = l - d \leq g(r)$ and $t - m \leq g(r)$ as in the proof of Lemma 4.6. Hence $t \leq 5 + 2g(r) + 2g(g(r)) + 2g(r)$.

On the other hand, by Lemma 4.3 (2) restricted to the base $k_0q_1k_1$, we have $|W_0|_a$, $|W_t|_a \ge r$. To finalize the proof, it suffices to note that, by Lemma 2.7, there exists a positive c_0 such that $c_0 \log' \log' (5 + 4g(r) + 2g(g(r))) \le 2r + 1$ for every $r \ge 0$.

Lemma 4.8. Let $W_0 \to W_1 \to \ldots \to W_t$ be a long computation with base $k \ldots k$, where the first and the last k-letters coincide. Then, for some constant c and for every $i \in \{1, \ldots, t\}$, we have $|W_i|_a \leq c(|W_1|_a + |W_t|_a + b\log_2 t)$, where b is the length of the base.

Proof. Neither the base nor its inverse word has subwords of the form kqq^{-1} or $k^{-1}k$, or $qq^{-1}q$, or $q^{-1}qk$, or kk^{-1} by Lemma 4.4. Recall also that every letter k_2 can be followed in the base by letter k_0 only. Then it follows from the lemma assumption that every word W_i (or the inverse word) can be covered by its subwords with base of the form $k_0q_1k_1q_2k_2$ or $k_1q_2k_2k_0q_1k_1$, or

 $k_2k_0q_1k_1q_2k_2$, in such way that every basic letter is covered at most two times and every *a*-letter is covered once. Now the assertion is a consequence of Lemma 4.5.

Lemma 4.9. Let Δ be a trapezium of height $h \geq 1$ with either (a) base $k \dots k$, where the first and the last k-letters coincide, and which contains neither subwords $(qq^{-1}q)^{\pm 1}$ nor shorter subwords of the form $k \dots k$, or (b) base $qq^{-1}q$. Then the area of Δ does not exceed $ch(|W|_a + |W'|_a + \log' h)$ for a constant c, where W, W' are the labels of its top and bottom, respectively. The third summand can be replaced by 1 if the base is $qq^{-1}q$ or Δ corresponds to a short computation.

Proof. (a) It follows from the lemma assumption that the length b of the base is bounded from above. Since the area of the *i*-th band of Δ can exceed the length of W_i at most by 2b, the lemma statement follows from lemmas 4.8 and 4.4.

(b) Similarly, the statement follows from Lemma 4.4 in this case.

5 Areas of diagrams over the group G

As in [9], we use constants L, K, δ . I suffices to set $L \geq 6$ since the are no defining relations of length > 6 now. Then $K = 2K_0$, where K_0 bounds from above the length of bases having neither subwords $(qq^{-1}q)^{\pm 1}$ nor subwords xux whith a k-letter x. As in [9], a sufficiently small positive δ is selected so that $\delta(4L + LK + 1) < 2$. As in Section 4.1 [9], the *lengths* of words, paths and perimeters of diagrams are modified now. (The number of edges in a path is called now a *combinatorial length*.) The reader of this section should have the paper [9] at hand. In particular, the concept of diagram dispersion $\mathcal{E}(\Delta)$ is crucial for the proofs of Lemma 6.2 [9] and the lemmas of this section. However it is not defined here since we do not use the definition and use the same property of dispersion as in [9] (e.g., the quadratic upper bound in term of the perimeter $|\partial \Delta|$). As in [9] we take a big enough constant M. Here "big enough" means that M satisfies the inequalities used in the proof of lemmas 5.1 and 5.2. Each of them has the form M > C for some constant C that does not depend on M (but depends on the constants introduced earlier). Since the number of inequalities is finite, one can choose such a number M.

Lemma 5.1. The area of a reduced diagram Δ does not exceed $M\Psi(n) + M\psi(n)\mathcal{E}(\Delta)$, where $n = |\partial \Delta|$ and $\psi(n) = \log' n / \log' \log' n$.

Proof. We follow the proof of Lemma 6.2 [9]. Steps 1 and 2 are analogous to those in [9]: The only difference is that one must multiply the entropy $\mathcal{E}(\Delta)$ by $\psi(\Delta)$ and replace the factor $\log'(\ldots)$ by $\psi(\ldots)$. Then we use all the notations of Step 3 [9] for the supposed minimal counter example: $\Delta, \mathcal{T}, \mathcal{T}', \mathcal{Q}, \mathcal{Q}', \mathcal{Q}_2 - \mathcal{Q}_4, l, l', (l' > l/2), l_3, l_4, \Gamma, \Gamma', \Gamma_1 - \Gamma_4, \Delta_0, n, n_0, \alpha_i, p_i, p^i, u_i d_i, d'_i$ for i = 3, 4 and $A_0 - A_4$. Then reader can just compare our arguments here and there. In particular, as in (6.23) [9],

$$n - n_0 \ge 2 + \delta(\max(0, d'_3 - 2L, \alpha_3 - (d_3 - d'_3) - 2Ll_3) + \max(0, d'_4 - L, \alpha_4 - (d_4 - d'_4) - 2Ll_4))$$
(5.2)

By Lemma 4.9 we have now

$$A_2 \le C_2 l' (d_3 + d_4 + \log' l') \tag{5.3}$$

for some constant C_2 , and

$$A_2 \le C_2 l'(d_3 + d_4 + 1) \tag{5.4}$$

if the computation defined by the trapezium Γ_2 is short.

The inequalities (5.3), (5.4) provide us with the following modification of our task (in comparison with (6.28) in [9]): To obtain the desired contradiction, we must now prove that

$$(Mn(n-n_0) + \frac{M}{K^2}l'(l-l'))\psi(n) \ge C_3l'(d_3 + d_4 + \log' l') + C_3(l_3^2 + l_4^2) + 2\alpha_3l_3 + 2\alpha_4l_4 \quad (5.5)$$

where (as in inequality (6.28), [9]) $C_3 \ge C_2$ is a constant that does not depend on M, and if the trapezium Γ_2 corresponds to a short computation, we must prove (5.5) with the logarithmic summand replaced by 1.

First, as in [9], we can choose M big enough so that

$$\frac{M}{3K^2}l'(l-l') \ge C_3(l_3^2 + l_4^2) \tag{5.6}$$

Then, as in [9], we assume without loss of generality that $\alpha_3 \ge \alpha_4$, and consider two cases. (a) Suppose we have $\alpha_3 \le 2C_3(l-l')$.

Since $d_i \leq \alpha_i + d'_i$ for i = 3, 4, we also, by inequality (5.2), have $d_3 + d_4 + 1 \leq \alpha_3 + \alpha_4 + d'_3 + d'_4 + 1 < 4C_3(l - l') + \delta^{-1}(n - n_0) + 2L - 2\delta^{-1} + 1 < 4C_3(l - l') + \delta^{-1}(n - n_0)$ because $\delta^{-1} > L + 1/2$ by the choice of δ . Therefore

$$\left(\frac{M}{5K^2}l'(l-l') + \frac{M}{2}n(n-n_0)\right) \ge C_3l'(d_3 + d_4 + 1)$$
(5.7)

because $n \ge l', n - n_0 \ge 2$ by (5.2), $M \ge C_3 \delta^{-1}$ and $M \ge 20C_3^2 K^2$.

Since $l_3 + l_4 = l - l' < l'$, we have also

$$\frac{M}{5K^2}l'(l-l') \ge 2(l-l')2C_3(l-l') \ge 2\alpha_3 l_3 + 2\alpha_4 l_4$$
(5.8)

because $M \ge 20K^2C_3$.

If the trapezium Γ_2 corresponds to a short computation, then the inequality (5.5) (with the logarithmic summand replaced by 1) follows from (5.6), (5.7) and (5.8). Then we assume that the computation is long. Since the base of Γ_2 satisfies the Lemma 4.9 condition (as in [9]), it follows from Lemma 4.4 that the base or its inverse has one of the forms $k_0q_1k_1q_2k_2k_0$, $k_1q_2k_2k_0q_1k_1$, $k_2k_0q_1k_1q_2k_2$.

Now we consider two possibilities.

(a1) Let $l - l' \ge \frac{1}{20C_3} \log' \log' l'$. Then

$$\frac{M}{5K^2}l'(l-l')\psi(n) \ge C_3l'\log' l'$$
(5.9)

by the definition of the function $\psi(n)$, because $M \ge 100K^2C_3^2$. By adding inequalities (5.6), (5.7) - (5.9), we obtain a stronger inequality than the desired inequality (5.5).

(a2) Let $l - l' \leq \frac{1}{20C_3} \log' \log' l'$. Let us estimate the number of \mathcal{A} -edges lying on the path u_3 . Recall that it is equal d'_3 plus $|p^3|_a$ (see [9]). It follows from Lemma 4.10 [9] that $|p^3|_a \geq (d_3 - d'_3) - C_0 l_3$ for a constant C_0 (One may assume that $C_3 > C_0 c_0^{-1}/10$ where c_o is given by Lemma 4.7.) Thus $|u_3|_a \geq d_3 - C_0 l_3$. By using a similar lower bound for $|u_4|_a$, we have $|u_3|_a + |u_4|_a \geq (d_3 + d_4) - C_0(l - l')$. When applying the assumption (a2) and Lemma 4.7 to the right-hand side of this equality, we have $|u_3|_a + |u_4|_a + 1 \geq (c_0 - \frac{C_0}{20C_3}) \log' \log' l'$. By Lemma 4.6

[9], we obtain $|u_i| \ge l_i + \delta(|u_i|_a - Ll_i)$ for i = 3, 4. Since we may chose C_3 such that $C_3 \ge c_0^{-1}L$, and $l_3 + l_4 = l - l' \le \frac{1}{20C_3} \log' \log' l'$, we have now

$$|u_3| + |u_4| + 1 \ge l_3 + l_4 + \delta(c_0 - \frac{C_0}{20C_3} - \frac{L}{20C_3})\log'\log'l' \ge (l - l') + \frac{\delta c_0}{2}\log'\log'l' \le l_3 + l_4 + \delta(c_0 - \frac{C_0}{20C_3} - \frac{L}{20C_3})\log'\log'l' \le (l - l') + \frac{\delta c_0}{2}\log'\log'l' \le l_3 + l_4 + \delta(c_0 - \frac{C_0}{20C_3} - \frac{L}{20C_3})\log'\log'l' \le (l - l') + \frac{\delta c_0}{2}\log'l' \le l_3 + l_4 + \delta(c_0 - \frac{C_0}{20C_3} - \frac{L}{20C_3})\log'\log'l' \le (l - l') + \frac{\delta c_0}{2}\log'l' \le l_3 + l_4 + \delta(c_0 - \frac{C_0}{20C_3} - \frac{L}{20C_3})\log'\log'l' \le (l - l') + \frac{\delta c_0}{2}\log'l' \le l_3 + l_4 + \delta(c_0 - \frac{C_0}{20C_3} - \frac{L}{20C_3})\log'l' \le l_3 + l_4 + \delta(c_0 - \frac{C_0}{20C_3} - \frac{L}{20C_3})\log'l' \le l_3 + l_4 + \delta(c_0 - \frac{C_0}{20C_3} - \frac{L}{20C_3})\log'l' \le l_3 + l_4 + \delta(c_0 - \frac{C_0}{20C_3} - \frac{L}{20C_3})\log'l' \le l_3 + l_4 + \delta(c_0 - \frac{L}{20C_3})\log'l' \le l_4 + \delta(c_0 - \frac{L}{20C_3})\log'l'$$

It follows from this inequality and the comparison of perimeters $|\Delta|$ and $|\Delta_0|$, that $n - n_0 \ge 2 + |u_3| + |u_4| - (l - l') \ge \frac{\delta c_0}{2} \log' \log' l'$, and since $l' \le n$ and $M \ge 2\delta^{-1}c_0^{-1}C_3$, we obtain

$$\frac{M}{2}n(n-n_0)\psi(n) \ge C_3 l' log' l'$$
(5.10)

The sum of inequalities (5.6), (5.7), (5.8), and (5.10) gives us a stronger inequality than (5.5).

(b) Assume now that $\alpha_3 > 2C_3(l-l')$. Then, as in [9], we have

$$d_3 + d_4 + 1 \le \frac{5}{3}\alpha_3 + \delta^{-1}(n - n_0)$$
(5.11)

Here we add 1 to the left-hand side (comparatively to [9]). This is possible since, as in [9], $n - n_0 \ge 2$ and δ can be selected small enough. Then, as in [9], one ontains

$$n - n_0 \ge \frac{1}{7} \delta \alpha_3. \tag{5.12}$$

From inequalities (5.11),(5.12), $M \ge 10C_3\delta^{-1}$ and $l' \le n/2$, we have

$$\frac{M}{3}n(n-n_0) \ge \frac{10C_3}{3}\delta^{-1}n(n-n_0) \ge C_3l'(d_3+d_4+1)$$
(5.13)

Inequalities (5.12), $M \ge 21\delta^{-1}$, $\alpha_3 \le \alpha_4$, and $l_3 + l_4 = l - l' \le l/2 \le \frac{1}{4}n$ give us

$$\frac{M}{6}n(n-n_0) \ge \frac{7}{2}\delta^{-1}(n-n_0)n \ge 2\alpha_3(l_3+l_4) \ge 2\alpha_3l_3 + 2\alpha_4l_4$$
(5.14)

If the Γ_2 -computation is short then the corresponding version of (5.5) (where the logarithmic summand is replaced by 1) follows from inequalities (5.6), (5.13) and (5.14). Thus, as in case (a), by Lemma 4.4, the base or its inverse can be supposed having one of the forms $k_0q_1k_1q_2k_2k_0$, $k_1q_2k_2k_0q_1k_1$, $k_2k_0q_1k_1q_2k_2$.

Then, from (5.11), (5.12) and Lemma 4.7, we have

$$n - n_0 \ge \frac{\delta}{13}(d_3 + d_4 + 1) \ge \frac{\delta c_0}{13} \log' \log' l'$$

Since $M \ge 13\delta^{-1}c_0^{-1}C_3$ and $l' \le n/2$, it follows from the definition of $\psi(n)$ that

$$\frac{M}{2}n(n-n_0)\psi(n) \ge C_3 l' log' l'$$
(5.15)

The inequality (5.5) follows now from inequalities (5.6), (5.13), (5.14), and (5.15). The lemma is proved by contradiction.

Lemma 5.2. Let the perimeter n of a reduced diagram Δ satisfy inequality $n \leq g(g(r))$ for some positive integer r. Then the area of diagram Δ does not exceed $M(n^2 + m^2 \log' m) + M\mathcal{E}(\Delta)$, where $m = 10g(g(r-1))\log' n$.

Proof. By Lemma 5.1, the statement is true for $n \leq 10g(g(r-1))\log'n$ since $m \geq n$ in this case. Then arguing by contradiction, we consider a counter-example Δ with minimal perimeter $n > 10g(g(r-1))\log'n$.

As in the proof of Lemma 5.1, we follow the proof of Lemma 6.2 [9]. Steps 1 and 2 are analogous to those in [9] since the extra term $m^2 \log' m$ does not affect. Then again we use the notations of Step 3 [9]. In particular, inequality (5.2), (5.6) hold again, and by lemmas 4.4 and 4.6, there is a constant C (independent of M) such that

$$A_2 \le Cl'(d_3 + d_4 + 1) \tag{5.16}$$

if $l' \geq m/\log' n$.

For arbitrary l', by Lemma 4.9,

$$A_2 \le cl'(d_3 + d_4 + \log' l'),\tag{5.17}$$

and what's more, the logarithmic summand can be replaced by 1 if the trapezium Γ_2 defines a short computation.

To obtain the desired contradiction, we first consider

Case 1: Either $l' < n/\log' n$ or the computation corresponding to Γ_2 is short.

In view of inequalities (5.17), the modified (in comparison with [9]) task is to show that

$$(Mn(n-n_0) + \frac{M}{K^2}l'(l-l')) \ge C_3l'(d_3 + d_4 + \log' l') + C_3(l_3^2 + l_4^2) + 2\alpha_3l_3 + 2\alpha_4l_4$$
(5.18)

where C_3 is a constant that does not depend on M, and the logarithm is replaced by 1 when the trapezium Γ_2 corresponds to a short computation.

We notice that

$$\frac{M}{2}n(n-n_0) \ge C_3 l' log' l'$$
(5.19)

if Γ_2 corresponds to a long computation, because we have $n - n_0 \ge 2$, $n \ge l' \log l'$ and $M \ge C_3$. Then, as in [9], we assume without loss of generality that $\alpha_3 \ge \alpha_4$, and consider two cases.

(a) Suppose we have $\alpha_3 \leq 2C_3(l-l')$. Then inequalities (5.7) and (5.8) hold as in Lemma 5.1.

The sum of inequalities (5.19) (if l' < n/log'n; otherwise we do not need it), (5.6), (5.7), and (5.8) gives us both versions of the desired inequality (5.18).

(b) Assume now that $\alpha_3 > 2C_3(l-l')$. Then to come to a contradiction, we argue as in case (b) of the proof of Lemma 5.1, but inequality $\frac{M}{2}n(n-n_0) \ge C_3l'log'l'$ (the analog of (5.15)) follows now just from the assumption that $l' \le n/log'n$ since $M \ge 2C_3$.

Case 2: $l' \ge n/\log' n$ and Γ_2 corresponds to a long computation.

Since n has been supposed to be greater than $10g(g(r-1))\log' n$, we have l' > 10g(g(r-1)). Then, by Lemma 4.6, we may use inequality (5.16) instead of (5.17), which has no term $l'\log l'$. So this term is absent in (5.18), and we do not need (5.19), and in subcase (b), we do not need any analog of inequality (5.15).

The lemma is proved by contradiction.

6 Proof of Theorem 1.1

Lemma 6.1. Let n be the combinatorial perimeter of a reduced diagram Δ , and $|\partial \Delta|$ the modified perimeter. Then $n = O(|\partial \Delta|)^{-1}$

¹We use the Computer Science "big-O" notation assuming that f(n) = O(g(n)) if $\frac{1}{C}g(n) < f(n) < Cg(n)$ for some positive constant C.

Proof. As in [9], it follows from the definition that $\delta n \leq |\partial \Delta| \leq n$.

Proof of the theorem. (1) It is proved in [9] (Lemma 5.3) that

$$\mathcal{E}(\Delta) \le (n/2)^2. \tag{6.20}$$

By lemmas 6.1, 5.1 and inequality (6.20), we have the desired upper bound in the property (1) of Theorem 1.1. The lower bound follows from the consideration of the diagrams corresponding to the consequences of the commutativity relations (2.1).

(2) We set $n_i = 5 + 4g(i) + 2g(g(i))$. Then there is a trapezium Δ of height n_i whose area is $O(n_ig(i)) = O(n_i \log n_i)$ and the combinatorial lengths of top and bottom are equal to $5 + i = O(\log \log n_i)$. (See Lemma 4.5 and the comments to the definition of machine \mathcal{M} in Section 4.) Lemma 6.1 and the trick from [9] with $O(n_i/\log \log n_i)$ copies of Δ gluing along sides of these trapezia, give us a diagram with area $O(\Psi(n_i))$.

(3) Since $g(g(r-1))^2 \leq g(g(r))$ we conclude from $\mathcal{E}(\Delta) \leq O(|\partial \Delta|^2)$ and from Lemma 5.2 that $f(n'_i)$ is at most $O((n'_i)^2)$ for $n'_i = g(g(i))$.

(4) Moreover, the same argument shows that f(x) does not exceed a quadratic function on the set $\bigcup_{i=1}^{\infty} \left[\frac{d_i}{\lambda_i}, \lambda_i d_i\right]$, where $d_i = (n'_i)^{\frac{3}{4}}$ and $\lambda_i = (n'_i)^{\varepsilon}$ with $\varepsilon < 1/4$.

(5) It follows from the definitions of n_i and n'_i that $n_i/3 < n'_i$ for big enough *i*-s. Hence the property (5) of Theorem 1.1 holds with $c_5 = 1/3$.

Theorem 1.1 is proved.

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