

FINITELY GENERATED INFINITE SIMPLE GROUPS OF INFINITE COMMUTATOR WIDTH

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ABSTRACT. It is shown that there exists a finitely generated infinite simple group of infinite commutator width, and that the commutator width of a finitely generated infinite *boundedly* simple group can be arbitrarily large. Besides, such groups can be constructed with decidable word and conjugacy problems.

CONTENTS

1. Introduction	1
2. Construction of the groups	4
2.1. Boundedly simple group of large commutator width	4
2.2. Simple group of infinite commutator width	5
3. Combinatorial complexes, maps, and van Kampen diagrams	6
3.1. Combinatorial complexes	6
3.2. Maps	13
3.3. Transformations of maps	15
3.4. Diagrams	19
4. Estimating Lemmas	22
5. S -maps	30
6. Asphericity and torsion	34
7. Proof of the theorems	35
Acknowledgements	48
References	49

1. INTRODUCTION

In 1951, Oystein Ore (see [Ore51]) conjectured that all elements in every non-abelian finite simple group are commutators. In terms of *commutator width*, the question is whether the commutator width of every non-abelian finite simple group is 1. This question still remains open. However, using the Classification of Finite Simple Groups, it was shown by John Wilson that there exists a (not found explicitly) common upper bound on the commutator widths of all finite simple groups (see [Wil96]).

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In 1977, Martin Isaacs (see [Isa77]) noted that no simple group, finite or infinite, was known to have commutator width greater than 1. In 1999, Valerij Bardakov posed the following question (see Problem 14.13 in [MK99]):

Does there exist a (finitely presented) simple group of infinite commutator width?

Simple groups of infinite commutator width, realised as groups of certain surface diffeomorphisms, appeared in [BG92, GG04] in 1992. The infinity of the commutator width is established there by constructing *nontrivial homogeneous quasi-morphisms*. However, the question of existence of *finitely generated* simple groups of commutator width greater than 1 seems to have been open until now.

In this paper it is shown that presentations (by generators and defining relations) of finitely generated infinite simple groups of infinite commutator width, as well as of large finite commutator width, can be constructed using methods of small-cancellation theory. This approach is rather flexible and can yield groups with various additional properties.¹

Definition 1. The *commutator* of two group elements x and y , denoted $[x, y]$, is $xyx^{-1}y^{-1}$. The *commutator length* of an element g of the derived subgroup of a group G , denoted $\text{cl}_G(g)$, is the minimal n such that there exist elements $x_1, \dots, x_n, y_1, \dots, y_n$ in G such that $g = [x_1, y_1] \dots [x_n, y_n]$. The commutator length of the identity element is 0. The *commutator width* of a group G , denoted $\text{cw}(G)$, is the maximum of the commutator lengths of the elements of its derived subgroup $[G, G]$.

Definition 2. The *conjugate* of a group element g by a group element h , denoted g^h , is hgh^{-1} . A nontrivial group G is called *n -boundedly simple* if for every two nontrivial elements $g, h \in G$, the element h is the product of n or fewer conjugates of $g^{\pm 1}$, i.e.,

$$(\exists m \leq n) (\exists \sigma_1, \dots, \sigma_m \in \{\pm 1\}) (\exists x_1, \dots, x_m \in G) (g = (h^{\sigma_1})^{x_1} \dots (h^{\sigma_m})^{x_m}).$$

A group G is called *boundedly simple* if it is n -boundedly simple for some natural n .

Every boundedly simple group is simple, but the converse is not generally true (e.g., for an infinite alternating group).

Remark 3. A group is boundedly simple if and only if each of its ultrapowers is simple. If a group is n -boundedly simple, then all its ultrapowers are n -boundedly simple.

Theorem 4. *For every natural n , there exists a torsion-free 2-generated simple group G with a rank-2 free subgroup H such that:*

- (1) *for every $g \in G$ and every $x \in G \setminus \{1\}$, there exist y_1, \dots, y_{2n+2} in G such that $g = x^{y_1} \dots x^{y_{2n+2}}$; and*
- (2) *for every $h \in H \setminus \{1\}$ and for every $m \geq 2n$, $\text{cl}_G(h^m) > n$*

¹In particular, groups can be constructed so that to admit no nontrivial homogeneous quasi-morphisms (it suffices to make the *stable commutator length* of each element of a group equal 0). This is not actually done in the paper.

(in particular, G is $(2n+2)$ -boundedly simple, and $n+1 \leq \text{cw}(G) \leq 2n+2$). Moreover, there exists such a group G with decidable word and conjugacy problems.

Note that Theorem 4 improves the result of Theorem 2 in [Mur05].

Theorem 5. *There exists a torsion-free 2-generated simple group G with a rank-2 free subgroup H such that for every $h \in H \setminus \{1\}$,*

$$\lim_{n \rightarrow +\infty} \text{cl}_G(h^n) = +\infty$$

(in particular, G has infinite commutator width). Moreover, there exists such a group G with decidable word and conjugacy problems.

The theorems are proved by providing examples of groups which satisfy the required properties. These groups are presented by generators and defining relations in Section 2. (The constructed presentations are recursive, as follows from the proof of Proposition 80.)

The properties of simplicity or bounded simplicity for the constructed groups follow directly from the imposed relations, but the existence of free non-cyclic subgroups and estimates on the commutator lengths of their elements are obtained through a nontrivial analysis of van Kampen diagrams on spheres with handles.

To show that the commutator length of a given element g of a constructed group is greater than n , it is proved that if Δ is a van Kampen diagram on a sphere with handles and a hole such that some group word representing the element g “reads” on the boundary of Δ , then the number of handles is greater than n . This is done by assuming that the number of handles is not greater than n , which gives a lower bound on the Euler characteristic, and coming to a contradiction.

The contradiction is obtained as follows. The hole in the diagram is covered with an extra face so as to make the diagram closed. Some arcs of the diagram are selected and distributed among the faces. This is done in such a manner that the sum of the lengths of the arcs associated to each face is small, significantly less than half of the perimeter of that face, but almost all edges of the diagram lie on selected arcs. This eventually leads to a contradiction with the fact that in a closed diagram the number of edges is half the sum of the perimeters of faces.

Certainly, the least obvious part of the proof is distributing “almost all” edges among the faces, while associating “few” edges to each face. For this purpose the group presentations are constructed with small-cancellation-type conditions. These conditions allow one to choose a system of selected arcs in a specific way. The selected arcs together cover almost all edges of the diagram, and are short relative to the perimeters of incident faces. Using the bound on the Euler characteristic of the diagram (determined by the number of handles), and one combinatorial lemma by Philip Hall, the selected arcs can be distributed among the faces so as to have “very few” relatively “short” and “a few” relatively “very short” arcs associated to each face.

The approach used in this paper is similar to that of [Mur05]. There are improvements and generalizations which allow one to obtain better estimates and to deal with diagrams on arbitrary surfaces.

2. CONSTRUCTION OF THE GROUPS

Let $\mathfrak{A} = \{a, b\}$ be a two-letter alphabet. Choose recursive sequences of positive numbers $\{\lambda_n\}_{n \in \mathbb{N}}$ and $\{\mu_n\}_{n \in \mathbb{N}}$ so that for every $n \in \mathbb{N}$,

$$(1) \quad 2\lambda_n + (14n + 8)\mu_n + \frac{2n + 1}{4n + 4} < \frac{1}{2}.$$

(This inequality is to be used in the proofs of some of the properties.) Let for definiteness

$$(2) \quad \lambda_n = \frac{1}{20n + 20} \quad \text{and} \quad \mu_n = \frac{1}{(14n + 8)(8n + 8)}.$$

Remark 6. The values of λ_n and μ_n satisfying the inequality (1) can be chosen by the *lowest parameter principle* of Alexander Ol'shanskii, e.g., $1/n \succ \mu_n \succ \lambda_n$, which means that for every n , if μ_n is sufficiently small, then the desired inequality holds for all sufficiently small λ_n (see [Ol'89, Ol'91]).

2.1. Boundedly simple group of large commutator width. Take an arbitrary natural number n .

Let $(v_1, w_1), (v_2, w_2), (v_3, w_3), \dots$ be a list of all ordered pairs of reduced group words over \mathfrak{A} . Moreover, let the function $i \mapsto (v_i, w_i)$ be recursive.

Let $\{u_{ij}\}_{i \in \mathbb{N}; j=1, \dots, 2n+2}$ be a recursive *indexed family* of reduced group words, and z_1, z_2 be two cyclically reduced group words over \mathfrak{A} such that:

- (1) for every $i \in \mathbb{N}$,
 - (a) $|u_{i,1}| = |u_{i,2}| = \dots = |u_{i,2n+2}| \geq i$, and
 - (b) $\lambda_n(4n + 4)|u_{i,1}| \geq |v_i| + (2n + 2)|w_i|$;
- (2) the family $\{u_{ij}\}_{i \in \mathbb{N}; j=1, \dots, 2n+2}$ satisfies the following small-cancellation condition: if $u_{i_1 j_1}^{\sigma_1} = p_1 s q_1$ and $u_{i_2 j_2}^{\sigma_2} = p_2 s q_2$ (here $\sigma_1, \sigma_2 \in \{\pm 1\}$), then either

$$(i_1, j_1, \sigma_1, p_1, q_1) = (i_2, j_2, \sigma_2, p_2, q_2),$$

or

$$\mu_n(4n + 4)|u_{i_1 j_1}| \geq |s| \leq \mu_n(4n + 4)|u_{i_2 j_2}|;$$

- (3) z_1 starts and ends with a^{+1} , and z_2 starts and ends with b^{+1} (hence, if $t(x, y)$ is an arbitrary reduced group word over $\{x, y\}$, then substituting z_1 for x and z_2 for y yields a reduced groups word $t(z_1, z_2)$ over \mathfrak{A});
- (4) if s is a common subword of u_{ij} and of the concatenation of several copies of $z_1^{\pm 1}$ and $z_2^{\pm 1}$, then

$$|s| \leq \mu_n(4n + 4)|u_{ij}|.$$

For example, z_1, z_2 , and $\{u_{ij}\}_{i \in \mathbb{N}; j=1, \dots, 2n+2}$ may be defined as follows:

$$z_1 = a^2, \quad z_2 = b^2,$$

$$(3) \quad u_{ij} = \prod_{k=4(14n+8)(j-1)+1}^{4(14n+8)j} a^k b^{2(|u_{i-1,1}|+|v_i|+|w_i|)+4(14n+8)(2n+2)+1-k},$$

where in the case $i = 0$, the summand $|u_{i-1,1}|$ shall be replaced with 0. (Here multiplication in Π -notation is understood in the usual left-to-right sense, e.g., $\prod_{i=1}^3 A_i = A_1 A_2 A_3$, and not $A_3 A_2 A_1$.)

For every $i \in \mathbb{N}$, let

$$r_i = w_i^{u_{i,1}} \dots w_i^{u_{i,2n+2}} v_i^{-1}$$

where $w_i^{u_{i,1}} \dots w_i^{u_{i,2n+2}} v_i^{-1}$ denotes the concatenation of the group words $u_{i,1}$, w_i , $u_{i,1}^{-1}$, $u_{i,2}$, w_i , $u_{i,2}^{-1}$, \dots , $u_{i,2n+2}$, w_i , $u_{i,2n+2}^{-1}$, and v_i^{-1} in this order.

Now inductively construct a group presentation $\langle \mathfrak{A} \parallel \mathcal{R}_n \rangle$ as follows. Start with $\mathcal{R}_n^{(0)} = \emptyset$. On step number i ($i \in \mathbb{N}$), if the relation ' $w_i = 1$ ' is a consequence of the relations ' $r = 1$ ', $r \in \mathcal{R}_n^{(i-1)}$, then define $\mathcal{R}_n^{(i)} = \mathcal{R}_n^{(i-1)}$; otherwise, define $\mathcal{R}_n^{(i)} = \mathcal{R}_n^{(i-1)} \cup \{r_i\}$. Finally, let

$$\mathcal{R}_n = \bigcup_{i \in \mathbb{N}} \mathcal{R}_n^{(i)}.$$

Let G_n be the group presented by $\langle \mathfrak{A} \parallel \mathcal{R}_n \rangle$.

2.2. Simple group of infinite commutator width. Let w_1, w_2, w_3, \dots be a recursive list of all reduced group words over \mathfrak{A} .

Let $\{u_{ij}\}_{i \in \mathbb{N}; j=1, \dots, 4i+4}$ be a recursive indexed family of reduced group words, and z_1, z_2 be two cyclically reduced group words over \mathfrak{A} such that:

- (1) for every $i \in \mathbb{N}$,
 - (a) $|u_{i,1}| = |u_{i,2}| = \dots = |u_{i,4i+4}|$,
 - (b) $\lambda_i(4i+4)|u_{i,1}| \geq 1 + (2i+2)|w_i|$,
 - (c) $\lambda_i(4i+4)|u_{i,1}| \leq \lambda_{i+1}(4(i+1)+4)|u_{i+1,1}|$,
 - (d) $\mu_i(4i+4)|u_{i,1}| \leq \mu_{i+1}(4(i+1)+4)|u_{i+1,1}|$,
 - (e) $|u_{i,1}| \leq |u_{i+1,1}|$, and
 - (f) $\mu_i(4i+4)|u_{i,1}| \geq i$;
- (2) the family $\{u_{ij}\}_{i \in \mathbb{N}; j=1, \dots, 4i+4}$ satisfies the following condition: if $u_{i_1 j_1}^{\sigma_1} = p_1 s q_1$ and $u_{i_2 j_2}^{\sigma_2} = p_2 s q_2$ ($\sigma_1, \sigma_2 \in \{\pm 1\}$), then either

$$(i_1, j_1, \sigma_1, p_1, q_1) = (i_2, j_2, \sigma_2, p_2, q_2),$$

or

$$\mu_{i_1}(4i_1+4)|u_{i_1 j_1}| \geq |s| \leq \mu_{i_2}(4i_2+4)|u_{i_2 j_2}|;$$

- (3) z_1 starts and ends with a^{+1} , and z_2 starts and ends with b^{+1} ;
- (4) if s is a common subword of u_{ij} and of the concatenation of several copies of $z_1^{\pm 1}$ and $z_2^{\pm 1}$, then

$$|s| \leq \mu_i(4i+4)|u_{ij}|.$$

For example, z_1, z_2 , and $\{u_{ij}\}_{i \in \mathbb{N}; j=1, \dots, 4i+4}$ may be defined as follows:

$$z_1 = a^2, \quad z_2 = b^2,$$

$$(4) \quad u_{ij} = \prod_{k=4(14i+8)(j-1)+1}^{4(14i+8)j} a^k b^{2(|u_{i-1,1}|+|w_i|)+4(14i+8)(4i+4)+1-k},$$

where in the case $i = 0$, the summand $|u_{i-1,1}|$ shall be replaced with 0.

For every $i \in \mathbb{N}$, let

$$r_{i,1} = w_i^{u_{i,1}} \dots w_i^{u_{i,2i+2}} a^{-1}, \quad r_{i,2} = w_i^{u_{i,2i+3}} \dots w_i^{u_{i,4i+4}} b^{-1}.$$

Now inductively construct a group presentation $\langle \mathfrak{A} \parallel \mathcal{R}_\infty \rangle$ as follows. Start with $\mathcal{R}_\infty^{(0)} = \emptyset$. On step number i , if the relation ' $w_i = 1$ ' is a consequence of the relations ' $r = 1$ ', $r \in \mathcal{R}_\infty^{(i-1)}$, then define $\mathcal{R}_\infty^{(i)} = \mathcal{R}_\infty^{(i-1)}$; otherwise, define $\mathcal{R}_\infty^{(i)} = \mathcal{R}_\infty^{(i-1)} \cup \{r_{i,1}, r_{i,2}\}$. Finally, let

$$\mathcal{R}_\infty = \bigcup_{i \in \mathbb{N}} \mathcal{R}_\infty^{(i)}.$$

Let G_∞ be the group presented by $\langle \mathfrak{A} \parallel \mathcal{R}_\infty \rangle$.

Note that:

Proposition 7. *For every natural n , if the group G_n is nontrivial, then it is $(2n + 2)$ -boundedly simple. Moreover, for every $g \in G_n$ and every $x \in G_n \setminus \{1\}$, there exist $y_1, \dots, y_{2n+2} \in G_n$ such that $g = x^{y_1} \dots x^{y_{2n+2}}$.*

Proposition 8. *If the group G_∞ is nontrivial, then it is simple.*

Other properties of these groups shall be established in Section 7.

3. COMBINATORIAL COMPLEXES, MAPS, AND VAN KAMPEN DIAGRAMS

3.1. Combinatorial complexes. The purpose of introducing *combinatorial cell complexes* is to model CW-complexes by combinatorial objects which preserve most of the combinatorial theory without need for deep topological proofs.

Combinatorial cell complexes described here are equivalent to a particular type of *cone categories* defined in [McC00]. They should not be confused with *cell categories* defined therein because, for example, in a cell category a 2-cell cannot have less than 3 corners, but in a combinatorial cell complex 2-cells with just 1 or 2 corners are allowed.

Cone categories provide a perfect algebraic alternative to CW-complexes. Unfortunately, despite the beauty of their concise and purely algebraic definition, the language of cone categories seems less appealing to geometric intuition than the language of CW-complexes. Some of the natural geometric operations, such as “cutting and pasting,” are harder to visualise when thinking about cone complexes instead of their geometric realisations. The author chooses to define and use *combinatorial cell complexes* whose analogy to CW-complexes is more evident, but he believes that it is possible to rewrite all the statements, proofs, and definitions in this paper in terms of cone categories.

The following terms shall all be used as synonyms: an *i-dimensional combinatorial cell complex* is the same as a *combinatorial i-complex*, or just an *i-complex*. Combinatorial 0-complexes and 1-complexes shall be viewed as particular cases of 2-complexes. Only 0-, 1-, and 2-complexes will be used, and only their definitions are discussed. Thus all combinatorial complexes may be assumed 2-dimensional.

Some terms whose meaning in relation to combinatorial complexes can be unambiguously inferred from their meaning in relation to CW-complexes or simplicial complexes may be used without definition (e.g., connectedness, a link of a vertex, etc.).

The definition of combinatorial complexes here is very similar to that in [Mur05].

A (combinatorial) 0-complex A is a 3-tuple $(A(0), \emptyset, \emptyset)$ where $A(0)$ is an arbitrary set. Elements of $A(0)$ are called 0-cells, or *vertices*, of A .

A 0-complex with exactly 2 vertices shall be called a *combinatorial 0-sphere*.

A *morphism* ϕ from a 0-complex A to a 0-complex B is a 3-tuple $(\phi(0), \emptyset, \emptyset)$ where $\phi(0)$ is an arbitrary function $A(0) \rightarrow B(0)$. If A , B , and C are 0-complexes, and $\phi: A \rightarrow B$ and $\psi: B \rightarrow C$ are morphisms, then the product $\psi\phi: A \rightarrow C$ is defined naturally: $(\psi\phi)(0) = \psi(0) \circ \phi(0)$. A morphism $\phi: A \rightarrow B$ is called an *isomorphism* of A with B if there exists a morphism $\psi: B \rightarrow A$ such that $\psi\phi$ is the *identity morphism* of the complex A and $\phi\psi$ is the identity morphism of the complex B .

A (combinatorial) 1-complex A is a 3-tuple $(A(0), A(1), \emptyset)$ such that:

- (1) $A(0)$ is an arbitrary set;
- (2) $A(1)$ is a set of ordered triples of the form (i, E, α) where E is a combinatorial 0-sphere and α is a morphism of E to the 0-complex $(A(0), \emptyset, \emptyset)$, such that the function $(i, E, \alpha) \mapsto i$ is injective on $A(1)$.

Elements of $A(0)$ are called 0-cells, or *vertices*, of A , elements of $A(1)$ are called 1-cells, or *edges*. The 0-complex $(A(0), \emptyset, \emptyset)$ is called the 0-skeleton of A and is denoted A^0 .

If $e = (i, E, \alpha)$ is an edge of a 1-complex A , then i is called the *index* of e , E is called the *characteristic boundary* of e and shall be denoted by \dot{e} , and α is called the *attaching morphism* of e . (The purpose of “indexing” 1- and 2-cell is to allow distinct cells with identical characteristic boundaries and attaching morphisms.)

Combinatorial 1-complexes may sometimes be called *graphs*.

A 1-complex which “looks like” a circle shall be called a *combinatorial 1-sphere*, or a *combinatorial circle*. More precisely, a combinatorial circle is a finite *connected* non-empty graph in which the *degree* of every vertex is 2.

If A and B are 1-complexes, then a *morphism* $\phi: A \rightarrow B$ is a 3-tuple $(\phi(0), \phi(1), \emptyset)$ such that:

- (1) $\phi^0 = (\phi(0), \emptyset, \emptyset)$ is a morphism of A^0 to B^0 ;
- (2) $\phi(1)$ is a function on $A(1)$ which maps each $e = (i, E, \alpha) \in A(1)$ to an ordered pair (e', ξ) such that $e' = (i', E', \alpha') \in B(1)$, ξ is an isomorphism of E with E' , and $\phi^0\alpha = \alpha'\xi$.

Multiplication of morphisms of 1-complexes is defined naturally. For example, if A , B , C are 1-complexes, ϕ and ψ are morphisms, $\phi: A \rightarrow B$, $\psi: B \rightarrow C$, $e = (i, E, \alpha)$ is an edge of A , $e' = (i', E', \alpha')$ is an edge of B , $e'' = (i'', E'', \alpha'')$ is an edge of C , $\phi(1)(e) = (e', \xi)$, $\psi(1)(e') = (e'', \zeta)$, then $(\psi\phi)(1)(e) = (e'', \zeta\xi)$ (note that $\zeta\xi$ is an isomorphism of the characteristic boundary of e with that of e''). *Isomorphisms* of 1-complexes are defined in the natural way.

A (combinatorial) 2-complex A is a 3-tuple $(A(0), A(1), A(2))$ such that:

- (1) $(A(0), A(1), \emptyset)$ is a 1-complex, called the 1-skeleton of A and denoted A^1 ;

- (2) $A(2)$ is a set of ordered triples of the form (i, F, α) where F is a combinatorial 1-sphere and α is a morphism of F to A^1 , such that the function $(i, F, \alpha) \mapsto i$ is injective on $A(2)$.

Elements of $A(0)$ are called *0-cells*, or *vertices*, of the complex A , elements of $A(1)$ are called *1-cells*, or *edges*, elements of $A(2)$ are called *2-cells*, or *faces*.

If $f = (i, F, \alpha)$ is a face of a 2-complex A , then i is called the *index* of f , F is called the *characteristic boundary* of f and shall be denoted by \dot{f} , and α is called the *attaching morphism* of f .

If A and B are 2-complexes, then a *morphism* $\phi: A \rightarrow B$ is a 3-tuple $(\phi(0), \phi(1), \phi(2))$ such that:

- (1) $\phi^1 = (\phi(0), \phi(1), \emptyset)$ is a morphism of A^1 to B^1 ;
- (2) $\phi(2)$ is a function on $A(2)$ which maps each $f = (i, F, \beta) \in A(2)$ to an ordered pair (f', ξ) such that $f' = (i', F', \beta') \in B(2)$, ξ is an isomorphism of F with F' , and $\phi^1 \beta = \beta' \xi$.

Products of morphisms of 2-complexes are defined analogously to the case of 1-complexes. The notion of *isomorphism* for 2-complexes is the natural one.

The *empty* combinatorial complex, a *finite* complex, a *subcomplex*, etc., are defined naturally.

Any combinatorial complex C gives rise to a CW-complex, called its *geometric realisation*, which is unique up to isomorphism (homeomorphism preserving the cellular structure). A geometric realisation is constructed as follows:

Let C be a combinatorial 2-complex. Construct a geometric realisation X^0 of C^0 by imposing a structure of a 0-dimensional CW-complex on an arbitrary set which is in bijective correspondence with $C(0)$. Next, construct a geometric realisation X^1 of C^1 by attaching 1-cells to X^0 as follows. For every $e \in C(1)$, construct a geometric realisation S of \dot{e} (a 2-point 0-dimensional CW-complex); take B to be a cone over S viewed as a topological space (homeomorphic to a closed interval); attach B to X^0 via the continuous function $S \rightarrow X^0$ induced by the attaching morphism of e ; this procedure yields a 1-cell in X^1 for each 1-cell of C . Finally, construct a geometric realisation $X = X^2$ of C by attaching 2-cells to X^1 in correspondence with 2-cells of C in the same manner as how 1-cells were attached to X^0 .

The class of CW-complexes that are geometric realisations of combinatorial complexes is quite narrow, and does not even include all *transverse* 2-dimensional CW-complexes. For example, a geometric realisation of a 2-complex cannot have a 2-cell attached to a single 0-cell.

Every morphism ϕ from a combinatorial 2-complex A to a combinatorial 2-complex B naturally determines functions $A(i) \rightarrow B(i)$, $i = 0, 1, 2$. By abuse of terminology, if e is an i -cell of A and e' is its image under the function $A(i) \rightarrow B(i)$ induced by a morphism $\phi: A \rightarrow B$, then e' shall be called the *image* of e under ϕ , or the ϕ -*image* of e .

If $e = (i, E, \alpha)$ is an edge of a 2-complex C , then the vertices of the characteristic boundary $\dot{e} = E$ of e are called the *ends* of e . The images of the ends of e under the attaching morphism α of e are called the *end-vertices*

of e . An edge e is *incident* to a vertex v if v is an end-vertex of e . An edge with only 1 end-vertex is called a *loop*.

If f is a face of a 2-complex Φ , then the vertices and edges of the characteristic boundary of f are called the *corners* and *sides* of f , respectively. The images of the corners and sides of a face f under the attaching morphism of f are called *corner-vertices* and *side-edges* of f , respectively, and are said to be *incident* to f .

The *size* of a face f is the number of its sides (which is equal to the number of its corners).

If v is a vertex of a complex C , then the number of edges incident to v “counted with multiplicity” is called the *degree* of v . More precisely, the degree of a vertex v in a complex C is the total number of ordered pairs (e, x) where e is an edge of C , x is an end of e , and v is the image of x under the attaching morphism of e .

The *link* of a vertex v in a combinatorial complex C , denoted $\text{Link}_C v$, or $\text{Link } v$, is a combinatorial complex which can be viewed as “the boundary of a nice small neighborhood of v .” A precise definition is not given here since the meaning of the term in the present context can be easily inferred from its meaning in context of simplicial complexes or CW-complexes.² Note that the link of a vertex v in a 2-complex is a 1-complex whose vertices are in bijective correspondence with all the ordered pairs (e, x) where e is an edge incident to v and x is an end of e which is mapped to v by the attaching morphism of e , and whose edges are in bijective correspondence with all the ordered pairs (f, x) where f is a face incident to v and x is a corner of f which is mapped to v by the attaching morphism of f .

An *orientation* of an edge e is a function from the set of ends of e to \mathbb{Z} which maps one of the ends to $+1$ and the other to -1 . An *oriented edge* is an edge together with an orientation. The end-vertex of an oriented edge e that is the image of the “negative” end is called the *tail-vertex*, or the *initial vertex*, of e . The end-vertex that is the image of the “positive” end is called the *head-vertex*, or the *terminal vertex*, of the oriented edge; on figures it is indicated with an arrowhead. An oriented edge *exits* its tail-vertex and *enters* its head-vertex.

Consider a combinatorial circle C and a function f which chooses an orientation of each edge of C . The choice of orientations f is called *coherent* if every vertex of C is the tail-vertex of exactly one (and the head-vertex of exactly one) of the oriented edges obtained from edges of C by assigning orientations according to f . A coherent choice of orientations of all edges of C is called an *orientation* of C , and C together with one of its orientations is called an *oriented combinatorial circle*.

An *orientation* of a face f is an orientation of the characteristic boundary of f . An *oriented face* is a face together with an orientation.

Every edge and every face has exactly two *opposite* orientations. Two oriented edges (or faces) with the same underlying non-oriented edge (or face) but with opposite orientations are called mutually *inverse*.

²In the language of cone categories, the link of a vertex is the full subcategory of the co-slice category of that vertex obtained by removing the initial object.

A *path* is a non-empty finite sequence of alternating vertices and oriented edges in which every oriented edge is immediately preceded by its tail-vertex and immediately succeeded by its head-vertex.

The *length* of a path $p = (v_0 e_1, v_1, \dots, e_n, v_n)$ is n ; it is denoted by $|p|$. The vertices v_1, \dots, v_{n-1} of this path are called *intermediate*. A *trivial path* is a path of length zero. By abuse of notation, a path of the form (v_1, e, v_2) shall be denoted by e , and a trivial path (v) shall be denoted by v .

The *inverse path* to a path p is defined naturally and is denoted by p^{-1} . If the *terminal vertex* of a path p_1 coincides with the *initial vertex* of a path p_2 , then the *product* $p_1 p_2$ is defined (naturally). A path s is an *initial subpath* of a path p if $p = sq$ for some path q . A path s is a *terminal subpath* of p if $p = qs$ for some q .

A *cyclic path* is a path whose terminal and initial vertices coincide. A *cycle* is the set of all *cyclic shifts* of a cyclic path. The cycle represented by a cyclic path p shall be denoted by $\langle p \rangle$. The *length* of a cycle c , denoted by $|c|$, is the length of an arbitrary representative of c . A *trivial cycle* is a cycle of length zero. A path p is a *subpath* of a cycle c if for some representative r of c and for some $n \in \mathbb{N}$, p is a subpath of r^n (i.e., of the product of n copies of r).

A path is *reduced* if it has no subpath of the form ee^{-1} where e is an oriented edge. A cyclic path is *cyclically reduced* if it is reduced and its first and last oriented edges are not mutually inverse. (Trivial paths are cyclically reduced.) A cycle is *reduced* if it consists of cyclically reduced cyclic paths. A path is *simple* if it is nontrivial, reduced, and every its intermediate vertex appears in it only once. A cycle is *simple* if it consists of simple cyclic paths.

An *oriented arc* of a complex C is a simple path whose all intermediate vertices have degree 2 in C . An *oriented pseudo-arc* of a complex C is a nontrivial reduced path all intermediate vertices of which have degree 2 in C . A (non-oriented) *arc* is a pair of mutually inverse oriented arcs. A (non-oriented) *pseudo-arc* is a pair of mutually inverse oriented pseudo-arcs. Sometimes edges may be viewed as arcs, and oriented edges as oriented arcs.

Note that if C is a connected 1-complex and is not a combinatorial circle, then every pseudo-arc of C is an arc. All pseudo-arcs under consideration will be pseudo-arcs in characteristic boundaries of faces.

A (pseudo-)arc or an edge u *lies on* a path p if at least one of the oriented (pseudo-)arcs or oriented edges, respectively, associated with u is a subpath of p . An arc shall be called *free* if none of its edges is incident to any face.

Any morphism of combinatorial complexes induces maps of paths and cycles. An arc u is *incident* to a face f if the associated oriented arcs are the images of some paths in \dot{f} under the map induced by the attaching morphism of f .

Definition 9. A *c-path*, *c-cycle*, *c-arc*, or *c-pseudo-arc* of a face f is a path, a cycle, an arc, or a pseudo-arc in the characteristic boundary of f , respectively. A *c-edge* of a face is the same as its side.

It shall be assumed that distinct faces always have disjoint sets of corners and disjoint sets of sides, as well as that distinct edges have disjoint sets of ends. This will assure that every c-path is a c-path of exactly one face.

Moreover, it is convenient to assume that no other set-theoretic complications happen; in particular, no vertex can be simultaneously an edge.

To use combinatorial complexes effectively, a few operations on them need to be defined, and some properties of these operations need to be established.

The following operations are easy to define:

Removing a face: this operation is self-explanatory.

Removing an arc: If u is a free arc (or free edge) in a complex C , then to *remove* u from C means to remove all edges and all *intermediate* vertices of u from C .

Attaching a face (along a cyclic path): This is the operation inverse to removing a face. If p is a cyclic path in a complex C , then to *attach* a face f along p means to attach a new face f in such a way that the image of some simple cyclic c -paths of f be mapped to p by the attaching morphism of f .

Attaching an arc (at a pair of vertices): this is the operation inverse to removing an arc.

Next, there are operations whose geometric meaning is clear, but whose precise combinatorial definition may be complicated. Instead of giving precise definitions, these operations are informally described here:

Dividing an edge by a vertex: To *divide* an edge e by a vertex v geometrically means to put a new vertex v inside e .

Dividing a face by an edge: To *divide* a face f by an edge e through its (not necessarily distinct) corners x and y geometrically means to connect the corners x and y within f by a new edge e .

Pulling an edge into a face: To *pull* an edge e into a face f through its corner x geometrically means to put a new vertex v inside f and connect it within f to the corners x by a new edge e .

Merging two edges across a vertex: this is the operation inverse to dividing an edge by a vertex.

Merging two faces across an edge: this is the operation inverse to dividing a face by an edge.

Pushing an edge out of a face: this is the operation inverse to pulling an edge into a face.

Definition 10. The complex obtained from a given 2-complex C by an arbitrary (finite) sequence of operations of dividing edges by vertices, dividing faces by edges, and pulling edges into faces is called a *subdivision* of C . Two combinatorial 2-complexes are called *geometrically equivalent* if they have isomorphic subdivisions.

As shown in Section 9 of [Ol'89, Ol'91], if the topological spaces of the geometric realisations of two combinatorial complexes A and B are homeomorphic 2-dimensional surfaces with or without boundaries, then A and B are geometrically equivalent in the defined above sense. Vice versa, it is obvious that geometrically equivalent combinatorial complexes have homeomorphic geometric realisations.

Definition 11. A *combinatorial surface* is a non-empty combinatorial complex in which every vertex link is either a combinatorial circle or a *combinatorial segment* (a finite connected 1-complex in which 2 vertices have degree

1 and all the others have degree 2). A combinatorial surface is *closed* if the link of every vertex is a combinatorial circle.

Only combinatorial surfaces will be discussed, rather than topological ones. The adjective “combinatorial” shall be omitted for brevity.

Alternatively, a combinatorial surface may be defined as a combinatorial complex whose geometric realisation is a 2-dimensional surface, with or without boundary. Only finite combinatorial surfaces are discussed in this paper.

Consider a combinatorial surface S and a function f which chooses an orientation of each face of S . Then the function f induces an orientation on every side of every face of S . The attaching morphisms of faces carry the orientations of sides of these faces over to the edges that are the images of these sides (under the attaching morphisms). The choice of orientations f is called *coherent* if for every edge e of S that is the image of two distinct face sides, the orientations of these two sides determined by f induce (via the attaching morphisms) opposite orientations of e .

Definition 12. A coherent choice of orientations of all faces of a combinatorial surface S is called an *orientation* of S , and S together with an orientation is called an *oriented combinatorial surface*. A combinatorial surface which can be oriented is called *orientable*.

It can be shown that a combinatorial surface is orientable if and only if its geometric realisation is.

An orientable connected combinatorial surface has exactly two orientations.

Let a *sample combinatorial disc* be a 2-complex C consisting of 1 vertex, 1 edge, and 1 face whose attaching morphism is an isomorphism of its characteristic boundary with the 1-skeleton of C .

Definition 13. A *combinatorial disc* is an arbitrary 2-complex which is geometrically equivalent to a sample combinatorial disc.

Let a *sample combinatorial sphere* be a 2-complex C consisting of 1 vertex, 1 edge, and 2 faces whose attaching morphisms are isomorphisms of their characteristic boundaries with the 1-skeleton of C .

Definition 14. A *combinatorial sphere* is an arbitrary 2-complex which is geometrically equivalent to a sample combinatorial sphere.

(Combinatorial discs and spheres are exactly those 2-complexes whose geometric realisations are 2-discs and 2-spheres, respectively.)

In a similar fashion, other combinatorial surfaces, e.g., combinatorial tori, may be defined. Such terms shall be used without further definitions.

Definition 15. A *nontrivial singular combinatorial disc* is an arbitrary 2-complex that can be obtained from a combinatorial sphere by removing 1 face (or which can be turned into a combinatorial sphere by attaching 1 face). A *trivial singular combinatorial disc* is a combinatorial complex that consists of a single vertex.

Lemma 16. *If C is a proper connected subcomplex of a combinatorial surface, and C has at least 1 edge, then C can be turned into a connected*

combinatorial surface by operations of attaching faces. If C is a proper subcomplex of a connected combinatorial surface, then C is not a closed combinatorial surface.

This lemma is not proved here because it is intuitively obvious, while its proof would probably be rather technical but hardly interesting.

Definition 17. The *Euler characteristic* of a 2-complex C is denoted by χ_C and is defined by $\chi_C = \|C(0)\| - \|C(1)\| + \|C(2)\|$ where $\|C(i)\|$ is the number of i -cells of C , $i = 0, 1, 2$.

Lemma 18. *The maximal possible Euler characteristic of a closed connected combinatorial surface is 2, and among all closed connected surfaces, only spheres have Euler characteristic 2, and only projective planes have Euler characteristic 1. The maximal possible Euler characteristic of a proper connected subcomplex of a combinatorial surface is 1, and every such complex is a singular combinatorial disc.*

Proof. The first part of this lemma follows from the classification of compact (or finite combinatorial) surfaces. The second part follows from the first part together with Lemma 16. \square

Thus a combinatorial disc could be defined as a non-closed connected finite combinatorial surface of Euler characteristic 1, and a combinatorial sphere could be defined as a connected finite combinatorial surface of Euler characteristic 2.

3.2. Maps.

Definition 19. A *nontrivial connected map* Δ consists of:

- (1) a finite connected combinatorial complex C ;
- (2) a function that for every face Π of C chooses one of its simple cyclic c -paths, called the *contour c -path*, or *c -contour*, of Π ;
- (3) a (possibly empty) indexed family of cyclic paths in C , called the *contour paths*, or *contours*, of Δ .

The only requirement to this structure is that a complex obtained from C by attaching one new face along each of the *indexed contours* of Δ must be a closed combinatorial surface.³

An *indexed contour* of a map is an element of the indexed family of contours of the map together with its index. One contour of a map may correspond to two distinct indexed contours.

The c -contour of a face Π shall be denoted by $\partial^*\Pi$. The image of the c -contour of a face Π in C^1 is called the *contour path*, or *contour*, of Π and shall be denoted by $\partial\Pi$. The cycle represented by the contour of a face Π is called the *contour cycle* of Π . Similarly, the cycle represented by a contour of a map Δ is called a *contour cycle* of Δ . The contours of Δ shall be denoted by $\partial_1\Delta$, $\partial_2\Delta$, $\partial_3\Delta$, et cetera. If Δ has only one contour, then it can be alternatively denoted by $\partial\Delta$.

³It seems that a natural way to extend the notion of a map would be to weaken this condition and require instead that in the complex obtained from C by attaching faces along the contours of Δ , the links of all vertices are connected.

Definition 20. A *trivial map* is a combinatorial complex consisting of a single vertex together with the trivial cyclic path in it called its *contour*.

A *map* in general consists of a finite non-zero number of *connected components*, each of which is either a nontrivial connected map, or a trivial map.

Definition 21. A map without contours is called *closed*. If Δ is a map without trivial connected components, then a closed map obtained from Δ by attaching new faces along its contours, and choosing the c-contours of the new faces so that the contours of Δ become the contours of the new faces, is called a *closure* of Δ .

Remark 22. It is not possible to define a closure of a trivial map in a similar way because no face in a combinatorial complex can have boundary consisting of a single vertex.

A map is closed if and only if its underlying complex is a closed surface (see Lemma 16).

A closure of a map Δ is unique up to isomorphism.

The important notion of a *submap* of a map shall be defined in the subsequent subsection.

Definition 23. Two maps are called *essentially isomorphic* if there exists an isomorphism between their underlying complexes which preserves the contours of the maps up to re-indexing and/or replacing with cyclic shifts or cyclic shifts of the inverses.

Two maps without trivial connected components are essentially isomorphic if and only if there exists an isomorphism between their underlying combinatorial complexes that extends to an isomorphism between the underlying complexes of their closures.

Definition 24. A map is *simple* if all its contours are simple (cyclic) paths, and distinct contours have no common vertices. A map is *semi-simple* if in it every edge is incident to a face. A map without faces is called *degenerate*.

Definition 25. A *disc map* is either a trivial map, or any map with exactly one contour which has a spherical closure. An *annular map* is an arbitrary map with exactly two contours which has a spherical closure. An *elementary map* is a spherical map with exactly 2 faces, whose 1-skeleton is a combinatorial circle.

The underlying 2-complex of a disc map is a singular combinatorial disc, and the underlying 2-complex of simple disc map is a combinatorial disc.

Lemma 26. A connected map of Euler characteristic 2 is closed spherical. A connected map of Euler characteristic 1 is either disc, or (closed) projective-planar. The maximal possible Euler characteristic of a connected map with n contours is $2 - n$.

This lemma follows from Lemma 18.

Every non-free arc of a map is either *internal* (is the image of two distinct c-arcs, and does not lie on any contour cycle of the map), or *external* (is the image of only one c-arc, and lies on some contour cycle of the map).

Definition 27. A map is called *contour-oriented* if every oriented edge occurs in the contour of some face or in some contour of the map.

If Δ is a contour-oriented map and $\bar{\Delta}$ is its closure, then the c-contours of the faces of Δ induce an orientation of the underlying complex of $\bar{\Delta}$.

It is convenient to have terms to express the idea that two paths in the characteristic boundary of a face “go in the same direction,” and also to have a “preferred direction” in the boundary. For that purpose let all nontrivial reduced c-path of each face of a map be divided into *positive* and *negative*:

Definition 28. A c-path p of a face Π of a map is called *positive* if it is a nontrivial subpath of $\langle \partial^* \Pi \rangle$. A c-path p is *negative* if p^{-1} is positive.

3.3. Transformations of maps.

3.3.1. Removing a face. If Π is a face of a map Δ , then the *submap* of Δ obtained by *removing* Π is the map Ψ such that:

- (1) the underlying complex of Ψ is obtained from the underlying complex of Δ by removing the face Π ,
- (2) the c-contours of faces of Ψ are those inherited from Δ , and
- (3) the indexed family of contours of Ψ is obtained from the indexed family of contours of Δ by adding the contour of Π as a new indexed member, and possibly re-indexing the family.

(*Re-indexing* an indexed family means replacing the index set with a new set of the same cardinality, and pre-composing the indexing function with a bijection from the new index set onto the original index set of the family.)

3.3.2. Removing an arc. If u is a free arc of a map Δ , then a *submap* of Δ obtained by *removing* u is a map Ψ such that:

- (1) the underlying complex of Ψ is obtained from the underlying complex of Δ by removing the arc u ;
- (2) the c-contours of faces of Ψ are those inherited from Δ ;
- (3) the family of contours of Ψ consists of, up to re-indexing, all the indexed contours of Δ that do not have common edges with u , together with additional 1 or 2 chosen as follows:
 - (a) if removing u from the underlying complex increases the number of connected components, then, first, take paths p_1 , p_2 , and v such that:
 - (i) v and v^{-1} are the oriented arcs associated with u , and
 - (ii) $\langle vp_1v^{-1}p_2 \rangle$ is a contour cycle of Δ ,
 and second, assign new indices to some cyclic shift of $p_1^{\pm 1}$ and some cyclic shift of $p_2^{\pm 1}$, and take them as 2 additional indexed contours of Ψ ;
 - (b) if removing u from the underlying complex does not increase the number of connected components, and u lies on only one contour cycle of Δ (which implies that a closure of Δ is non-orientable), then, first, take paths p_1 , p_2 , and v such that:
 - (i) v and v^{-1} are the oriented arcs associated with u , and
 - (ii) $\langle vp_1vp_2 \rangle$ is a contour cycle of Δ ,
 and second, assign a new index to a cyclic shift of $(p_1p_2^{-1})^{\pm 1}$, and take it as an additional indexed contour of Ψ ;

- (c) if u lies on two distinct contour cycles of Δ , then, first, take paths p_1 , p_2 , and v , and indices i and j such that:
 - (i) v and v^{-1} are the oriented arcs associated with u ,
 - (ii) $\langle vp_1 \rangle = \langle \partial_i \Delta \rangle$,
 - (iii) either $\langle vp_2 \rangle = \langle \partial_j \Delta \rangle$, or $\langle p_2^{-1}v^{-1} \rangle = \langle \partial_j \Delta \rangle$, and
 - (iv) $i \neq j$,
 and second, assign a new index to a cyclic shift of $(p_1 p_2^{-1})^{\pm 1}$, and take it as an additional indexed contour of Ψ .

Definition 29. A map Ψ is a *submap* of a map Δ if it can be obtained from Δ by operations of removing faces, removing free arcs, and removing connected components (the last operation is self-explanatory).

Lemma 30. *If Γ is a submap of a map Δ , and $\bar{\Delta}$ is a closure of Δ , then:*

- (1) *the underlying complex of Γ is a subcomplex of the underlying complex of Δ ;*
- (2) *the contours and c-contours of faces of Γ are those inherited from Δ ;*
- (3) *for every contour q of Γ that is neither a contour of Δ , nor the contour of a face of Δ , there is $n \in \mathbb{N}$ and there are subpaths $p_0, p_1, \dots, p_n = p_0$ of q , oriented arcs $u_0, u_1, \dots, u_n = u_0$ of Δ , and c-paths $p'_0, p'_1, \dots, p'_n = p'_0$, $u'_0, u'_1, \dots, u'_n = u'_0$, $v'_0, v'_1, \dots, v'_n = v'_0$ of faces of $\bar{\Delta}$ that are not in Γ , such that:*
 - (a) $\langle q \rangle = \langle p_0 p_1 \dots p_{n-1} \rangle$,
 - (b) u_0, u_1, \dots, u_n are pairwise non-overlapping and maximal among oriented arcs of Δ that do not have common edges with Γ ,
 - (c) the terminal vertices of u_0, u_1, \dots, u_n are vertices of q ,
 - (d) for every $i = 0, \dots, n$, the images of p'_i , u'_i , and v'_i are p_i , u_i , and u_i^{-1} , respectively, but $v'_i \neq u_i'^{-1}$,
 - (e) for every $i = 0, \dots, n-1$, the product $u'_i p'_i v'_{i+1}$ is a reduced c-path of a face of $\bar{\Delta}$.

This lemma can be proved by induction on the number of operations of removing (free) arcs used in obtaining Γ from Δ .

Lemma 31. *Every subcomplex of the underlying complex of every map Δ has a structure of a submap of Δ , which is unique up to essential isomorphism.*

The non-obvious part of this lemma is the “uniqueness.” It can be proved by showing that the set of essential isomorphism classes of submaps of a given map containing a given subcomplex, together with operations of removing faces, arcs, and connected components, form a *confluent* and *terminating* rewriting system.

For brevity, subcomplexes of the underlying complexes of maps shall be called subcomplexes of the maps.

3.3.3. Diamond move.

Definition 32. Two c-edges, or two oriented c-edges, of faces of a complex C are called *contiguous* if their images (in C^1) under the respective attaching morphisms coincide.

Contiguity of oriented c-edges is an equivalence relation. Observe that any closed map is determined up to isomorphism by c-contours of its faces and the contiguity relation on oriented c-edges. In other words, the contiguity relation tells how to glue faces together, which together with a choice of c-contours of faces “completely” determines a closed map. This observation allows one to define *diamond moves* on maps in terms of changing the contiguity relation on oriented c-edges.

Consider an arbitrary *closed* map Δ . Let e_1 and e_2 be two distinct oriented edges in Δ with a common terminal vertex v .

Choose a “local orientation around v ,” i.e., choose an orientation of the link of v in Δ . Consider an arbitrary oriented c-edge x of a face of Δ such that v is the head-vertex of the image of x (for example, the image of x may be e_1 or e_2). The “positive” end of x naturally corresponds to an end of some edge of $\text{Link } v$, which in turn is either “positive” or “negative” with respect to the chosen orientation of $\text{Link } v$. Call x “positive” or “negative” accordingly. (This terminology shall only be used in the subsequent definition.) Thus every oriented edge entering v has one “positive” and one “negative” pre-image under attaching morphisms. Let a , b , c , and d be, respectively, the “positive” and the “negative” pre-images of e_1 , and the “positive” and the “negative” pre-images of e_2 .

Definition 33. A map obtained from Δ by the *diamond move* along e_1 and e_2 is a (unique up to isomorphism) closed map Γ which has the same characteristic boundaries and c-contours of faces, but in which a is contiguous to d , b is contiguous to c , and the contiguity relation on the oriented c-edges distinct from $a^{\pm 1}$, $b^{\pm 1}$, $c^{\pm 1}$, $d^{\pm 1}$ is the same as in Δ (see Fig. 1).

Let e_3 and e_4 be the images in Γ^1 of a and d , and b and c , respectively. There are natural bijections:

- (1) between the vertices of Δ not incident to e_1 and e_2 , and the vertices of Γ not incident to e_3 and e_4 ;
- (2) between the edges of Δ distinct from e_1 and e_2 , and the edges of Γ distinct from e_3 and e_4 ;
- (3) between all faces of Δ and all faces of Γ .

Informally and imprecisely speaking, the diamond move consists in cutting the map Δ along the path $e_1 e_2^{-1}$ and then gluing the sides of the obtained diamond-shaped hole in a different way than how it was before.

Consider now an arbitrary map Δ and two distinct oriented edges e_1 and e_2 in Δ with a common terminal vertex, none of which is a loop. The *diamond move* along e_1 and e_2 consists of, first, closing the connected component of Δ that contains e_1 and e_2 ; second, applying the diamond move along e_1 and e_2 to the closure; and finally, removing the faces that were added when closing the component.

Suppose that neither e_1 nor e_2 is a loop. If the initial vertices of e_1 and e_2 are distinct, then the diamond move is called *proper*, otherwise it is called *improper*. If the initial vertices of e_1 and e_2 coincide, and the cyclic path $e_1 e_2^{-1}$ does not switch orientation in Δ (informally speaking, this means that some neighbourhood of $e_1 e_2^{-1}$ is orientable), then the (improper) diamond

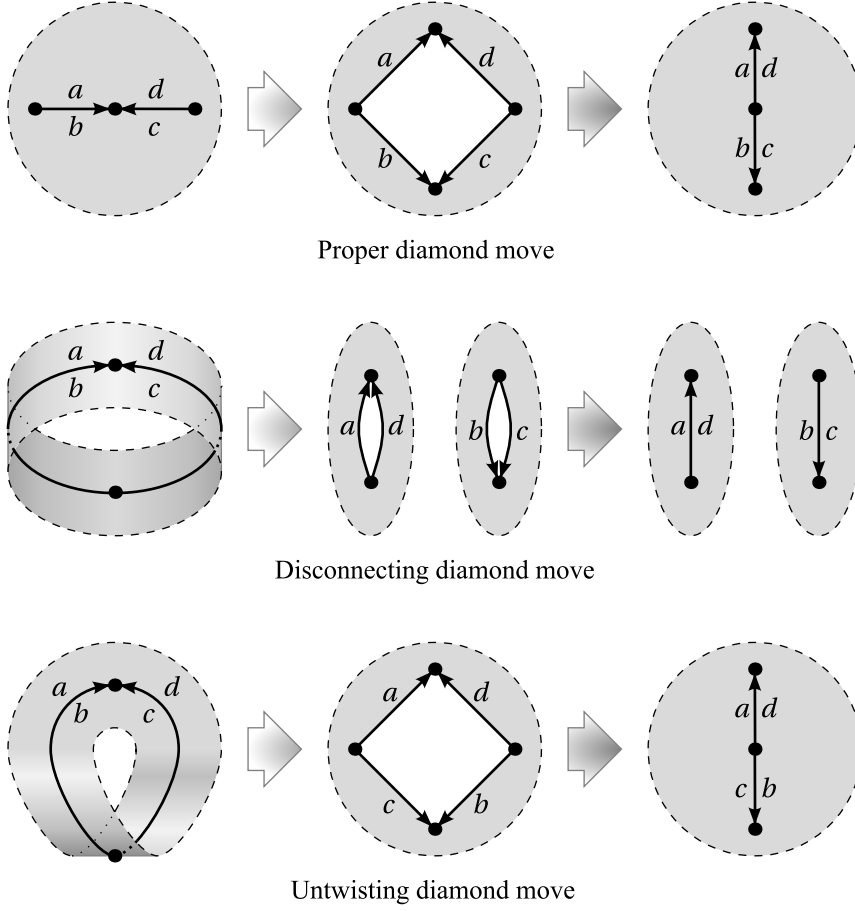


FIGURE 1. Diamond Move.

move is called *disconnecting*; otherwise, if the cyclic path $e_1 e_2^{-1}$ does switch orientation, the (improper) diamond move is called *untwisting*.

Suppose that e_1 is a loop, while e_2 is not. In this case the diamond move is called *proper* for the reason that will be clear from Lemma 34.

If both e_1 and e_2 are loops, then the task to suitably classify such a diamond move as either *proper*, or *improper disconnecting*, or *improper untwisting* is left to the reader. (This case is admittedly more difficult, but analogous to the previous two.)

Lemma 34. *Consider an arbitrary map Δ and a map Γ obtained from Δ by a diamond move. Then*

- (1) $\|\Gamma(1)\| = \|\Delta(1)\|$ and $\|\Gamma(2)\| = \|\Delta(2)\|$;
- (2) *if the diamond move is proper, then $\|\Gamma(0)\| = \|\Delta(0)\|$, $\chi_\Gamma = \chi_\Delta$, and $\bar{\Gamma}$ is geometrically equivalent to $\bar{\Delta}$ if $\bar{\Gamma}$ and $\bar{\Delta}$ are closures of Γ and Δ ;*
- (3) *if the diamond move is untwisting, then $\|\Gamma(0)\| = \|\Delta(0)\| + 1$, and Γ has the same number of connected components as Δ ;*
- (4) *if the diamond move is disconnecting, then $\|\Gamma(0)\| = \|\Delta(0)\| + 2$, and Γ has either the same number of connected components as Δ , or 1 more;*

$$(5) \chi_\Delta \leq \chi_\Gamma \leq \chi_\Delta + 2.$$

Proof of this lemma is left to the reader.

Properties of diamond moves can be described even in greater detail, but this lemma is sufficient for many applications.

3.4. Diagrams.

Definition 35. If $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ is a group presentation, then a *van Kampen diagram* (or simply a *diagram*) over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ is a map together with a labelling of its oriented edges such that every two mutually inverse oriented edges are labelled with mutually inverse group letters from $\mathfrak{A}^{\pm 1}$, and each group word that “reads” on the contour of some face belongs to $\mathcal{R}^{\pm 1}$.

In every van Kampen diagram, let the *label* of an oriented edge e be denoted by $\ell(e)$, and the label of a path p be denoted by $\ell(p)$.

Definition 36. Two faces of a diagram or of two distinct diagrams are said to be *congruent* if their contour labels are either cyclic shifts of each other, or cyclic shifts of the inverses of each other.

Definition 37. Two diagrams are called *essentially isomorphic* if there exists a label-preserving isomorphism between their underlying complexes which preserves the contours of the diagrams up to re-indexing and/or cyclically shifting and/or inverting.

Definition 38. A pair of distinct faces $\{\Pi_1, \Pi_2\}$ in a diagram Δ is called *immediately cancellable* if there are paths p_1 and p_2 in Δ such that

- (1) $\langle \partial \Pi_1 \rangle \in \{\langle p_1 \rangle, \langle p_1^{-1} \rangle\}$ and $\langle \partial \Pi_2 \rangle \in \{\langle p_2 \rangle, \langle p_2^{-1} \rangle\}$,
- (2) p_1 and p_2 have a common nontrivial initial subpath, and
- (3) $\ell(p_1) = \ell(p_2)$.

A diagram Δ is called *weakly reduced* if it does not have immediately cancellable pairs of faces.

Weakly reduced diagrams are exactly the diagrams *reduced* in the sense of [LS01].

Definition 39. A *diamond move* in a diagram Δ (a *diagrammatic diamond move*) is a diamond move in the underlying map of Δ along two oriented edges with identical labels, followed by the natural labelling of the obtained map so as to obtain a diagram.⁴

Definition 40. A pair of distinct faces $\{\Pi_1, \Pi_2\}$ in a diagram Δ is called *cancellable* if there exists a sequence of diamond moves that separates these two faces into an elementary spherical subdiagram (i.e., leads to a diagram in which the faces corresponding to Π_1 and Π_2 form an elementary spherical connected component). A diagram Δ is called *reduced* if it does not have cancellable pairs of faces.

Lemma 41. *Immediately cancellable pairs are cancellable. Reduced diagrams are weakly reduced.*

⁴Diamond moves in diagrams correspond to *bridge moves* in *pictures*, see [Rou79, Hue81].

Proof of this lemma is left to the reader.

It should be noted that there is a substantial distinction between the diagrams (and their transformations) defined in this paper and such classical objects as *pictures* (called *standard diagrams* in [Rou79]) and *0-refined diagrams* (in [Ol'89, Ol'91]). If D^2 is a 2-dimensional disc, $K(\mathfrak{A}; \mathcal{R})$ is the geometric realisation of a group presentation $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$, and $K^1(\mathfrak{A}; \mathcal{R}) = K(\mathfrak{A}; \emptyset)$ is its 1-skeleton, which is a wedge of circles, then both pictures and 0-refined diagrams over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ can be used to represent arbitrary *transverse* (in the sense of [BRS76]) continuous maps $(D^2, \partial D^2) \rightarrow (K(\mathfrak{A}; \mathcal{R}), K^1(\mathfrak{A}; \mathcal{R}))$ up to isotopy of the domain D^2 . Transverse maps in turn represent arbitrary continuous maps up to homotopy. Certain combinatorially defined transformations of pictures, as well as of 0-refined diagrams, represent homotopies between transverse continuous maps. The diagrams defined in this paper without introducing 0-cells are not suitable for representing arbitrary homotopy classes of maps $(D^2, \partial D^2) \rightarrow (K(\mathfrak{A}; \mathcal{R}), K^1(\mathfrak{A}; \mathcal{R}))$. Nevertheless, they are an appropriate tool for studying relations in groups and formulating useful results (see Lemma 42).

If $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ is a group presentation, G is the group presented by $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$, and w is a group word over \mathfrak{A} , then let $[w]_G$, or $[w]_{\mathcal{R}}$, or simply $[w]$, denote the element of G represented by w .

The results of the following lemma are assumed to be well-known.

Lemma 42. *Let G be the group presented by $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$. Let w, w_1 , and w_2 be arbitrary group words over \mathfrak{A} , and n be a natural number. Then*

- (1) *if there exists a disc diagram Δ over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ such that $\ell(\partial\Delta) = w$, then $[w] = 1$;*
- (2) *if $[w] = 1$, then there exists a reduced disc diagram Δ over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ such that $\ell(\partial\Delta) = w$;*
- (3) *if there exists a contour-oriented annular diagram Δ over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ such that $\ell(\partial_1\Delta) = w_1$ and $\ell(\partial_2\Delta)^{-1} = w_2$, then $[w_1]$ and $[w_2]$ are conjugate in G ;*
- (4) *if $[w_1]$ and $[w_2]$ are conjugate in G , then either $[w_1] = [w_2] = 1$, or there exists a contour-oriented reduced annular diagram Δ over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ such that $\ell(\partial_1\Delta) = w_1$ and $\ell(\partial_2\Delta)^{-1} = w_2$;*
- (5) *if there exists a one-contour diagram Δ over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ such that $\ell(\partial\Delta) = w$ and the underlying complex of a closure of Δ is a combinatorial sphere with n handles, then $[w] \in [G, G]$ and $\text{cl}_G([w]) \leq n$;*
- (6) *if $\text{cl}_G([w]) = n$, then there exists a one-contour reduced diagram Δ over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ such that the underlying complex of a closure of Δ is a combinatorial sphere with n handles, and $\ell(\partial\Delta) = w$.*

Outline of a proof. Parts (1), (2), (3), and (4) follow, for example, from Theorem V.1.1 and Lemmas V.1.2, V.5.1, and V.5.2 of [LS01]. See also Lemmas 11.1 (van Kampen Lemma) and 11.2 in [Ol'89, Ol'91] (all results there are formulated in terms of 0-refined diagrams).

Here is an outline of a proof of parts (5) and (6).

Suppose that Δ is a one-contour diagram over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ such that the underlying complex of a closure of Δ is a combinatorial sphere with n handles, and $\ell(\partial\Delta) = w$. Let o be the initial vertex of $\partial\Delta$. Consider the (combinatorial) fundamental group $\pi_1(\Delta, o)$ of Δ with base-point o . It can be

shown from the definition of a combinatorial handled sphere (which is easy to formulate) that in $\pi_1(\Delta, o)$, the homotopy class of $\partial\Delta$ is the product of n commutators. Therefore $\text{cl}_G([w]) \leq n$, since there is a homomorphism $\pi_1(\Delta, o) \rightarrow G$ which maps the homotopy class of $\partial\Delta$ to $[w]$.

Now suppose $\text{cl}_G([w]) = n$. Let $x_1, \dots, x_n, y_1, \dots, y_n$ be group words over \mathfrak{A} such that $[w] = [[x_1], [y_1]] \dots [[x_n], [y_n]]$ in G . Let Ψ be a disc diagram over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ such that $\ell(\partial\Delta) = [x_1, y_1] \dots [x_n, y_n] w^{-1}$ (here part (2) of this lemma is used). At this point 0-refinement of Ψ is needed.

Definition and explanation of 0-refinement are given in [Ol'89, Ol'91]. In a 0-refined diagram, faces and edges are usually divided into 2 classes: 0-edges and 0-faces, and all the other, “regular,” edges and faces. Here the terminology shall be slightly different. The class of 0-faces shall be subdivided into a class of 0-faces and a class of 1-faces; “regular” edges shall be called 1-edges, and “regular” faces shall be called 2-faces. Thus, all the edges of a 0-refined diagram are divided into 0-edges and 1-edges, and all the faces are divided into 0-faces, 1-faces, and 2-faces. The requirements on the labelling of a 0-refined diagram over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ are the following:

- (1) the label of every oriented 0-edge is 1 (the symbol ‘1’ here is regarded as a new group letter such that $1^{-1} = 1^{+1} = 1$);
- (2) the label of every oriented 1-edge is an element of $\mathfrak{A}^{\pm 1}$, and, as usual, mutually inverse oriented edges are labelled with mutually inverse group letters;
- (3) the label of the contour of every 0-face is of the form 1^k ;
- (4) the label of the contour of every 1-face is of the form $1^k x 1^l x^{-1} 1^m$ where $x \in \mathfrak{A}^{\pm 1}$; and
- (5) the label of the contour of every 2-face is an element of $\mathcal{R}^{\pm 1}$.

Let $\tilde{\Psi}$ be a 0-refinement of Ψ such that $\partial\tilde{\Psi}$ is a simple cyclic path, and $\ell(\partial\tilde{\Psi}) = \ell(\partial\Psi)$. Let $p_1, \dots, p_{2n}, q_1, \dots, q_{2n}$, and t be the paths such that

- (1) $\partial\tilde{\Delta} = p_1 p_2 q_1^{-1} q_2^{-1} \dots p_{2n-1} p_{2n} q_{2n-1}^{-1} q_{2n}^{-1} t^{-1}$,
- (2) $\ell(p_{2i-1}) = \ell(q_{2i-1}) = x_i$ and $\ell(p_{2i}) = \ell(q_{2i}) = y_i$ for $i = 1, \dots, n$, and
- (3) $\ell(t) = w$.

Let Δ_0 be the (0-refined) diagram obtained from $\tilde{\Psi}$ by “gluing” together each pair of paths p_i and q_i , $i = 1, \dots, n$, and choosing t (or rather the copy of t in Δ_0) as the contour of Δ_0 . Let $\bar{\Delta}_0$ be a closure of Δ_0 . Then the underlying complex of $\bar{\Delta}_0$ is a combinatorial sphere with n handles. Let Θ be the “improper” face of $\bar{\Delta}_0$ (which is not a face of Δ_0); this face is to be regarded as a 2-face in the sense of 0-refinement.

Eliminate all 0-edges and 0-faces of $\bar{\Delta}_0$ by *collapsing* 0-edges. If e is a 0-edge which is not a loop and not the only edge of some connected component of the diagram, then the meaning of collapsing e is clear. If e is the only edge of some connected component of the diagram, then collapsing e means removing this component all together. Consider a 0-edge e which is a loop. If e is the only edge incident to some 0-face Π_1 , and e is incident to another face Π_2 which is incident to some edge distinct from e , then collapsing e results in removing e and Π_1 , and shortening the contour of Π_2 by 1. If e is incident to two distinct faces Π_1 and Π_2 , both of which are also incident to some other

edges, then collapsing e results in removing e , possibly doubling the end-vertex of e (unless e switches orientation in the diagram), and shortening the contours of Π_1 and Π_2 by 1. Let $\bar{\Delta}_1$ be the closed map obtained from $\bar{\Delta}_0$ by collapsing one-by-one all 0-edges. Then $\bar{\Delta}_1$ does not have any 0-edges or 0-faces. Clearly, $\bar{\Delta}_1$ is orientable, since so is $\bar{\Delta}_0$.

Observe that if the operation of collapsing an edge increases the number of connected components, then it increases it only by 1, and simultaneously increases the Euler characteristic by 2, and if it decreases the Euler characteristic, then it decreases it at most by 2 and simultaneously decreases the number of connected components (recall Lemma 18). Therefore, if k is the number of connected components of $\bar{\Delta}_1$, then $\chi_{\bar{\Delta}_1} \geq \chi_{\bar{\Delta}_0} + 2(k - 1)$. Let $\bar{\Delta}_2$ be the connected component of $\bar{\Delta}_1$ that contains the face Θ . By Lemma 18, $\chi_{\bar{\Delta}_2} \geq \chi_{\bar{\Delta}_0}$.

The number of connected components and the Euler characteristic of any map that can be obtained from a given closed map by diamond moves are both bounded from above. Indeed, the number of connected components is bounded by the number of faces, and hence, by Lemma 18, the Euler characteristic is bounded by 2 times the number of faces. Let $\bar{\Delta}_3$ be a map of maximal Euler characteristic that can be obtained from $\bar{\Delta}_2$ by diamond moves. Since diamond moves do not decrease the Euler characteristic, and improper diamond moves increase it (see Lemma 34), no improper diamond move is applicable to $\bar{\Delta}_3$, nor to any diagram obtained from $\bar{\Delta}_3$ by any sequence of diamond moves.

Let $\bar{\Delta}$ be the connected component of $\bar{\Delta}_3$ that contains the face Θ . Then $\bar{\Delta}$ does not contain any 1-faces (otherwise an improper diamond move would be applicable to $\bar{\Delta}$). Every diamond move that increases the number of connected components, increases it by 1 and simultaneously increases the Euler characteristic by 2. Therefore $\chi_{\bar{\Delta}} \geq \chi_{\bar{\Delta}_2}$. Since $\bar{\Delta}$ is oriented, its underlying complex is a combinatorial sphere with at most n handles, but the number of handles cannot be less than n , as follows from part (5).

Let Δ be the subdiagram of $\bar{\Delta}$ obtained by removing Θ . The diagram Δ is reduced, because otherwise some improper diamond move would be applicable to some diagram obtained from $\bar{\Delta}$ by proper diamond moves. The diagram Δ is such as desired. \square

4. ESTIMATING LEMMAS

Lemmas of this and the subsequent sections are rather technical. It is advisable that the reader first takes a look at the proofs of Propositions 77, 78, and 80 in Section 7.

If X is a set, then $\|X\|$ shall denote the cardinality of X . Assume the usual definitions and notation concerning binary relations (subsets of Cartesian products). In particular, if R is a relation and X is a set, then

$$R(X) = \{ y \mid (\exists x \in X)(x R y) \}.$$

Lemma 43 (Philip Hall, 1935). *Let A and B be two finite sets, and R be a relation from A to B (i.e., $R \subset A \times B$). Then the following are equivalent:*

- (I) *There exists an injection $h : A \rightarrow B$ such that for each $x \in A$, $x R h(x)$.*

(II) For each subset X of A , $\|R(X)\| \geq \|X\|$.

Corollary 44. Let A and B be two finite sets, and R be a relation from A to B . Let w be a function from B to $\mathbb{N} \cup \{0\}$. Then the following are equivalent:

- (I) There exists a function $h: A \rightarrow B$ such that:
 - (1) for each $x \in A$, $x R h(x)$, and
 - (2) for each $y \in B$, the full pre-image of y under h consists of at most $w(y)$ elements.
- (II) For each subset X of A , $\sum_{y \in R(X)} w(y) \geq \|X\|$.
- (III) For each subset Y of B , $\sum_{y \in Y} w(y) \geq \|\{x \mid R(\{x\}) \subset Y\}\|$.

Proofs of the lemma and the corollary may be found, for example, in [Hal35, Mur05]. (The equivalence of items (II) and (III) of the corollary is not proved in those papers, but is easy to verify.)

Definition 45. A c-path is called *regular* if its image in the 1-skeleton of the complex is reduced and nontrivial. A c-pseudo-arc is *regular* if the associated oriented c-pseudo-arcs are such.

Definition 46. An S_1 -map is a map together with a system of *selected* c-paths of its faces satisfying the following conditions:

- (1) all selected c-paths are regular (in particular, they are oriented c-pseudo-arcs),
- (2) the inverse path of every selected c-path is selected, and
- (3) every nontrivial subpath of every selected c-path is selected.

A c-pseudo-arc of a face in an S_1 -map is *selected* if the associated oriented c-pseudo-arcs are selected. An arc in an S_1 -map is *selected* if this arc is internal and both c-arcs that map to it (by attaching morphisms) are selected.

This definition of S_1 -maps is similar to the definition of S-maps in [Mur05], but it is adapted to the more general definition of maps (one of the generalisations is that maps now are allowed to be non-orientable). In [Mur05], “S” stood for “selection,” and here it stands for “structure,” because an S_1 -map is a map with additional structure. So are S_2 -maps and S -maps, which shall be defined and used below.

Definition 47. A set X of c-pseudo-arcs *encloses* a simple disc submap Φ if

- (1) elements of X are c-pseudo-arcs of faces which do not belong to Φ (are “outside” of Φ),
- (2) for every c-pseudo-arc from X , one of the associated oriented c-pseudo-arcs maps to a subpath of $\langle \partial\Phi \rangle$, and
- (3) $\partial\Phi$ can be decomposed into a product of paths each of which is the image of a subpath of an oriented c-pseudo-arc associated with an element of X .

Definition 48. Let Δ be an S_1 -map, Φ its simple disc submap, and $n \in \mathbb{N}$. The S_1 -map Δ is said to satisfy the *condition $Z(n)$ relative to Φ* if every set of selected c-pseudo-arcs enclosing Φ in Δ has at least $n + 1$ element.

Lemma 49. *Let Ψ be a non-degenerate disc map. Suppose $\partial\Psi = p_1 \dots p_n$, $n \in \mathbb{N}$, where p_1, \dots, p_n are reduced paths. Then there exist a maximal simple disc submap Φ of Ψ and simple paths q_1, \dots, q_m , $1 \leq m \leq n$, such that $\partial\Phi = q_1 \dots q_m$ and there are i_1, \dots, i_m such that $1 \leq i_1 < \dots < i_m \leq n$ and for every $j = 1, \dots, m$, q_j is a subpath of p_{i_j} .*

(This Lemma is similar to Proposition 3.1 in [Mur05].)

Proof. Without loss of generality, assume that all the paths p_1, \dots, p_n are nontrivial, and that the contour of Ψ is cyclically reduced. For if it is not, then the terminal vertex of one of the paths p_1, \dots, p_n has degree 1 in Ψ . Remove this vertex together with the incident edge, and “shorten” or remove each of the paths from among p_1, \dots, p_n that start or end at this vertex (the contour of Ψ is also “shortened”). If the lemma holds for the new disc map and the new set of paths, it is clear that it holds for the initial ones. Thus it can be assumed that $\partial\Psi$ is cyclically reduced.

The conclusion is obvious if Ψ is simple. Assume Ψ is not simple. Then it has two maximal simple disc submaps whose contours are subpaths of $\langle\partial\Psi\rangle$, and which are either disjoint, or have only one vertex in common. Let Φ_1 and Φ_2 be such maximal simple disc submaps.

If $\partial\Phi_2$ is a subpath of one of the paths p_1, \dots, p_n , then take $\Phi = \Phi_2$, $m = 1$, $q_1 = \partial\Phi$, and see that the conclusion holds.

Suppose $\partial\Phi_2$ is not a subpath of any one of the paths p_1, \dots, p_n . If the initial vertex of $\partial\Psi$ is not in Φ_1 , then let $\Phi = \Phi_1$. If the initial vertex of $\partial\Psi$ is in Φ_1 , then let Φ be the map obtained from Φ_1 by cyclically shifting its contour so that $\partial\Phi$ starts at the same vertex as $\partial\Psi$. The initial vertices of the $n+1$ paths $p_1, \dots, p_n, \partial\Phi_1$ divide the simple path $\partial\Phi$ into at most n simple subpaths. Denote these subpaths by q_1, \dots, q_m so that $\partial\Phi = q_1 \dots q_m$. The submap Φ and the path q_1, \dots, q_m are the desired ones. \square

The following notation is used in Estimating Lemma 50 and throughout the rest of this paper: if Π is a face of an S_1 -map Δ , then let $\kappa_\Delta(\Pi)$, or $\kappa(\Pi)$, denote the number of maximal selected c-pseudo-arcs of Π , and $\kappa'_\Delta(\Pi)$, or $\kappa'(\Pi)$, denote the number of maximal piece-wise selected regular c-pseudo-arcs of Π . Note that $\kappa'_\Delta(\Pi) \leq \kappa_\Delta(\Pi)$. Note also that if all c-pseudo-arcs of Π are selected, as well as if no c-pseudo-arc of Π is selected, then $\kappa_\Delta(\Pi) = \kappa'_\Delta(\Pi) = 0$.

Recall that an *elementary map* is a spherical map with exactly 2 faces, whose 1-skeleton is a combinatorial circle. Elementary maps are “bad” in the sense that the conclusion of Estimating Lemma 50 may fail for them (but only if all c-pseudo-arcs are selected, and hence $\kappa = \kappa' = 0$). They are also “inconvenient” in the sense that their distinct maximal selected arcs can overlap.

Estimating Lemma 50 (First Estimating Lemma). *Let Δ be a non-elementary connected S_1 -map, or an elementary S_1 -map which has a maximal selected c-pseudo-arc. Let A be a set of selected internal arcs of Δ such that no two distinct elements of A are subarcs of the same selected arc. Let C and D be sets of faces of Δ such that:*

- (1) C contains all faces incident to arcs from A , and

- (2) Δ satisfies the condition $Z(2)$ relative to every simple disc submap that does not contain any faces from D and does not contain at least one arc from A .

Let c_Δ be the number of contours of Δ . Then either A is empty, or

$$\|A\| \leq \sum_{y \in C} (3 + \kappa_\Delta(y) + \kappa'_\Delta(y)) + 2\|D \setminus C\| - c_\Delta - 3\chi_\Delta.$$

Furthermore, if B is a subset of C , there exist a subset $E \subset A$ and a function $f: A \setminus E \rightarrow B$ such that:

- (1) either E is empty, or

$$\begin{aligned} \|E\| \leq & \sum_{y \in (C \setminus B) \setminus D} (3 + \kappa_\Delta(y) + \kappa'_\Delta(y)) \\ & + \sum_{y \in (C \setminus B) \cap D} (1 + \kappa_\Delta(y) + \kappa'_\Delta(y)) \\ & + 2\|D\| - c_\Delta - 3\chi_\Delta; \end{aligned}$$

- (2) for every $x \in A \setminus E$, $f(x)$ is incident to x ;
 (3) for every $y \in B$, the full pre-image of y under f consists of at most $3 + \kappa_\Delta(y) + \kappa'_\Delta(y)$ elements;
 (4) for every $y \in D$, the full pre-image of y under f consists of at most $1 + \kappa_\Delta(y) + \kappa'_\Delta(y)$ elements.

Proof. If A is empty, then there is nothing to prove (meaning the proof is easy). Assume it is non-empty.

It suffices to prove this lemma in the case Δ is closed. (To prove the statement in the case Δ is not closed, apply this lemma to a closure of Δ , the same sets A , B , C , and the set D extended by including the attached “improper” faces). Hence assume without loss of generality that Δ is closed.

Let K be the set of all connected components of a submap obtained from Δ by removing all the faces that are in C and all the arcs that are in A .

For every element Ψ of K , let $d(\Psi)$ denote the number of arcs in A that have exactly one end-vertex in Ψ , plus twice the number of arcs in A that have both end-vertices in Ψ . (Thus d is analogous to vertex degree.)

Clearly,

$$\sum_{\Psi \in K} \chi_\Psi - \|A\| + \|C\| = \chi_\Delta,$$

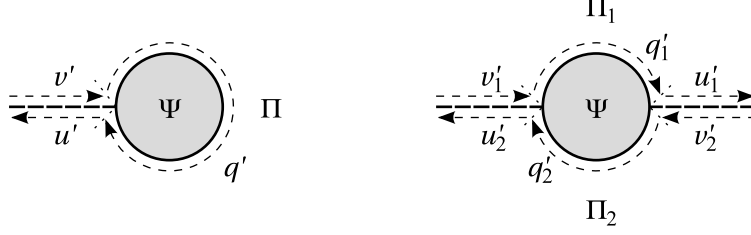
and

$$\sum_{\Psi \in K} d(\Psi) = 2\|A\|.$$

Using these two equations, one has

$$\begin{aligned} \|A\| &= 3\|C\| + 3 \sum_{\Psi \in K} \chi_\Psi - 2\|A\| - 3\chi_\Delta \\ &= 3\|C\| + \sum_{\Psi \in K} (3\chi_\Psi - d(\Psi)) - 3\chi_\Delta. \end{aligned}$$

By Lemmas 18 and 26, the Euler characteristic of each element of K is at most 1, and if the Euler characteristic of $\Psi \in K$ is 1, then Ψ is a disc map.

FIGURE 2. Illustrations of $\Psi \in K_1''$ (left) and $\Psi \in K_2''$ (right).

Let

$$K'_i = \{ \Psi \in K \mid d(\Psi) = i \text{ and } \chi_\Psi = 1 \} \quad \text{for } i = 0, 1, 2, \dots$$

Each element of each K'_i is a disc map. Observe that $K'_0 = \emptyset$. Therefore,

$$\|A\| \leq 3\|C\| + 2\|K'_1\| + \|K'_2\| - 3\chi_\Delta.$$

To complete the proof, essentially it is only left to demonstrate that

$$2\|K'_1\| + \|K'_2\| \leq \sum_{\Pi \in C} (\kappa_\Delta(\Pi) + \kappa'_\Delta(\Pi)) + 2\|D \setminus C\|,$$

and then to apply the corollary of Hall's Lemma.

For $i = 1, 2$, let K''_i be the set of those elements of K'_i whose face sets are disjoint with D (i.e., such $\Psi \in K'_i$ that $\Psi(2) \cap D = \emptyset$). Clearly,

$$\|K'_1 \setminus K''_1\| + \|K'_2 \setminus K''_2\| \leq \|D \setminus C\|.$$

Now it is to be shown that

$$2\|K''_1\| + \|K''_2\| \leq \sum_{\Pi \in C} (\kappa_\Delta(\Pi) + \kappa'_\Delta(\Pi)).$$

Let L be the set of all positive (for definiteness) maximal selected c-paths of all face from C . Let L' be the set of all elements of L that are terminal subpaths of maximal piece-wise selected regular c-path. Clearly,

$$\|L\| = \sum_{\Pi \in C} \kappa_\Delta(\Pi) \quad \text{and} \quad \|L'\| = \sum_{\Pi \in C} \kappa'_\Delta(\Pi).$$

Note that the image in Δ^1 of every element of L has a common vertex with at least one element of K (because of the maximality of elements of L).

Let a function $h: L \rightarrow K$ be defined as follows: $h(x)$ is the element of K such that the image of some terminal subpath of x has a common vertex with $h(x)$ and no common vertices with any other element of K .

Assign weights to all elements of L so that the weight of every element of L' is 2, and the weight of every element of $L \setminus L'$ is 1. Let the weight of every element of K be the sum of the weights of all elements of its full pre-image under h .

Consider an arbitrary $\Psi \in K''_1$. Let v be the oriented arc that represents an element of A and whose terminal vertex is in Ψ . Let Π be the face incident to v . Let v' , q' , and u' be c-paths of Π such that $v'q'u'$ is a c-path of Π as well, the images of v' and u' are v and v^{-1} , respectively, and the image of q' represents $\langle \partial \Psi \rangle$ (see Fig. 2). Both v' and u' are selected c-paths. Let q be the image of q' .

Suppose Ψ is degenerate. Then vqv^{-1} is not reduced, which means that $v'q'u'$ is not regular. Consider the maximal positive piece-wise selected regular c-path x containing either v' or u'^{-1} as a subpath. Clearly, the image of the terminal vertex of x is in Ψ , and therefore x has a (terminal) subpath which is an element of L' . The image of this element of L' under h is Ψ . Therefore, the weight of Ψ is at least 2.

Suppose Ψ is non-degenerate. By Lemma 49, there is a simple disc submap Φ of Ψ whose contour is a subpath of q . Since Δ satisfies Z(2) relative to Φ , the (nontrivial) c-path q' cannot be selected and cannot be the product of two selected c-paths. Therefore, the full pre-image of Ψ under h must contain either an element of L' , or at least 2 distinct elements of L . (If the pre-image does not contain any element of L' , then $v'q'u'$ is piece-wise selected; if additionally the pre-image consisted of a single element, then q' would be selected or would be the product of two selected c-paths, which is impossible.) In either case the weight of Ψ is at least 2.

Consider an arbitrary $\Psi \in K_2''$. Let v_1 and v_2 be the two oriented arcs that represent elements of A and whose terminal vertices are in Ψ . Let $v'_1, q'_1, u'_1, v'_2, q'_2, u'_2$ be c-paths, and q_1 and q_2 be paths such that:

- (1) $v'_1q'_1u'_1$ and $v'_2q'_2u'_2$ are c-paths (i.e., the products are defined),
- (2) the images of $v'_1, q'_1, u'_1, v'_2, q'_2, u'_2$ are $v_1, q_1, v_2^{-1}, v_2, q_2, u_1^{-1}$, respectively, and
- (3) $\langle q_1q_2 \rangle = \langle \partial\Psi \rangle$ (see Fig. 2).

The c-paths v'_1, u'_1, v'_2 , and u'_2 are selected. Let Π_1 and Π_2 be the faces to which the c-paths $v'_1q'_1u'_1$ and $v'_2q'_2u'_2$ respectively belong.

Suppose Ψ is degenerate. If $v_1q_1v_2^{-1}$ or $v_2q_2v_1^{-1}$ is not reduced, which means that $v'_1q'_1u'_1$ or $v'_2q'_2u'_2$ is not regular, then the pre-image of Ψ under h contains at least one element of L' . Suppose now that both $v_1q_1v_2^{-1}$ and $v_2q_2v_1^{-1}$ are reduced. Then they are inverse to each other. Therefore, they are oriented arcs in Δ , unless $v_1 = v_2^{-1}$. If $v_1 = v_2^{-1}$, then Δ is elementary, $K = \{\Psi\}$, $L \neq \emptyset$, and hence the full pre-image of Ψ under h is non-empty. Hence, suppose that $v_1q_1v_2^{-1}$ and $v_2q_2v_1^{-1}$ are mutually inverse oriented arcs of Δ . The associated non-oriented arc cannot be selected because otherwise it would be a selected arc containing two distinct elements of A as subarcs. Therefore, at least one of the c-paths $v'_1q'_1u'_1$ or $v'_2q'_2u'_2$ is not selected, which implies that the maximal selected c-path containing one of the c-paths v'_1, u_1^{-1}, v'_2 , or u_2^{-1} as a subpath is mapped by h to Ψ . Thus, the weight of every degenerate element of K_2'' is at least 1.

Suppose Ψ is non-degenerate. By Lemma 49, there is a simple disc submap Φ of Ψ whose contour either is a subpath of one of the paths q_1 or q_2 , or is the product of a subpath of q_1 and a subpath of q_2 . Since Δ satisfies Z(2) relative to Φ , at least one of the c-paths q'_1 or q'_2 is nontrivial but not selected. Therefore, the full pre-image of Ψ under h is not empty, and the weight of Ψ is at least 1.

On one hand, the sum of the weights of all elements of $K_1'' \sqcup K_2''$ is at least $2\|K_1''\| + \|K_2''\|$. On the other hand, it equals the sum of the weights of all

elements of L , which is $\|L\| + \|L'\|$. Therefore,

$$\begin{aligned} 2\|K'_1\| + \|K'_2\| &= 2\|K''_1\| + \|K''_2\| + 2\|K'_1 \setminus K''_1\| + \|K'_2 \setminus K''_2\| \\ &\leq \|L\| + \|L'\| + 2\|D \setminus C\| \\ &\leq \sum_{\Pi \in C} (\kappa_\Delta(\Pi) + \kappa'_\Delta(\Pi)) + 2\|D \setminus C\|. \end{aligned}$$

This gives

$$\|A\| \leq \sum_{\Pi \in C} (3 + \kappa_\Delta(\Pi) + \kappa'_\Delta(\Pi)) + 2\|D \setminus C\| - 3\chi_\Delta.$$

Now the first part of the statement of the lemma is proved. Apply it to all subset of A . More precisely, take an arbitrary subset X of A , take the subset Y of C consisting of all the faces incident to elements of X , and apply the proved part of the lemma to conclude that

$$\|X\| \leq \sum_{y \in Y} (3 + \kappa_\Delta(y) + \kappa'_\Delta(y)) + 2\|D \setminus Y\| - 3\chi_\Delta.$$

Let w be the function on C defined as follows:

$$w(\Pi) = \begin{cases} 3 + \kappa_\Delta(\Pi) + \kappa'_\Delta(\Pi) & \text{for } \Pi \in C \setminus D, \\ 1 + \kappa_\Delta(\Pi) + \kappa'_\Delta(\Pi) & \text{for } \Pi \in C \cap D. \end{cases}$$

In terms of w , have

$$\begin{aligned} \|X\| &\leq \sum_{y \in Y} w(y) + 2\|D\| - 3\chi_\Delta \\ &\leq \sum_{y \in (Y \cap B)} w(y) + \sum_{y \in (C \setminus B)} w(y) + 2\|D\| - 3\chi_\Delta. \end{aligned}$$

Now apply the corollary of Hall's Lemma to verify the remaining part of the lemma. Let ω be an arbitrary element not in B . Define a binary relation $R \subset A \times (B \cup \{\omega\})$ as follows: $x R y$ if and only if $x \in A$ and either $y = \omega$, or $y \in B$ and y is incident to x . Use the corollary of Hall's Lemma and the last inequality to conclude that there is a function $h: A \rightarrow B \cup \{\omega\}$ such that:

- (1) $\|h^{-1}(\omega)\| \leq \max\{0, \sum_{y \in (C \setminus B)} w(y) + 2\|D\| - 3\chi_\Delta\}$;
- (2) for every $x \in A$, either $h(x)$ is incident to x , or $h(x) = \omega$;
- (3) for every $y \in B$, $\|h^{-1}(y)\| \leq w(y)$.

To complete the proof of the second part, take $E = h^{-1}(\omega)$ and $f = h|_{A \setminus E}$. \square

Definition 51. A *graded map* is a map Δ together with a function $\text{rk}_\Delta: \Delta(2) \rightarrow J$ where J is an arbitrary set or algebraic structure. The *rank* of a face Π of Δ is $\text{rk}(\Pi)$. Two faces are called *rank-equivalent* if their ranks are equal.

Definition 52. An S_2 -*map* is a graded map together with a system of *exceptional arcs* such that:

- (1) distinct exceptional arcs do not overlap,
- (2) every exceptional arc is incident to a face, and
- (3) faces incident to the same exceptional arc are of the same rank.

Assign a *rank* to every exceptional arc of an S_2 -map according to the rank of the incident faces. Exceptional arcs of the same rank shall be called *rank-equivalent*.

Consider an arbitrary *connected* S_2 -map Δ . For every j , let Γ_j denote the subcomplex of Δ obtained by removing all the faces of rank j and all the internal exceptional arcs of rank j .

Definition 53. The S_2 -map Δ is said to satisfy the *condition Y* if for every j such that Δ has an internal exceptional arc of rank j , the number of connected component of Γ_j that either have Euler characteristic 1 or contain a rank- j (external) exceptional arc of Δ is less than or equal to the number of faces of Δ of rank j .

Note that every connected component of Γ_j which contains an (external) exceptional arc of Δ of rank j is the underlying subcomplex of a map with at least 2 contours, and hence has non-positive Euler characteristic.

Estimating Lemma 54 (Second Estimating Lemma). *Let Δ be a connected S_2 -map satisfying the condition Y. For every j , let A_j denote the set of all the internal exceptional arcs of Δ of rank j , and B_j denote the set of all the faces of Δ of rank j . For every j , let $\varepsilon(j) = 1$ if Δ has an external exceptional arc of rank j , and let $\varepsilon(j) = 0$ otherwise. Then for every j , either A_j is empty, or*

$$\|A_j\| \leq 2\|B_j\| - \varepsilon(j) - \chi_\Delta.$$

Furthermore, there exists a set E such that:

- (1) either E is empty, or $\|E\| \leq -\chi_\Delta$, and
- (2) for every j , $\|A_j \setminus E\| \leq 2\|B_j\| - \varepsilon(j)$.

Proof. For every set J , let $A_J = \bigcup_{j \in J} A_j$, $B_J = \bigcup_{j \in J} B_j$, and let Γ_J be the subcomplex obtained from the underlying complex of Δ by removing all the faces that are in B_J and all the arcs that are in A_J .

It follows from the condition Y that for every j such that $A_j \neq \emptyset$, the number of connected components of $\Gamma_{\{j\}}$ of Euler characteristic 1 is at most $\|B_j\| - \varepsilon(j)$. Observe also that for every j , $\|B_j\| - \varepsilon(j) \geq 0$.

Let J be an arbitrary set such that A_J is non-empty. It is to be shown that

$$\chi_{\Gamma_J} \leq \|B_J\| - \sum_{j \in J} \varepsilon(j).$$

Let K be the set of all connected components of Γ_J . By Lemmas 18 and 26, the Euler characteristic of each elements of K is at most 1, and every element of K of Euler characteristic 1 is the underlying complex of a disc submap of Δ . Let K' be the set of all the elements of K of Euler characteristic 1.

Define a function $f: K' \rightarrow J$ as follows. Consider an arbitrary $\Psi \in K'$. Let $A_J^{(\Psi)}$ be the set of all the elements of A_J that have an end-vertex in Ψ . Since Δ is connected and $A_J \neq \emptyset$ (and hence at least one arc has been removed in the process of obtaining Γ_J), the set $A_J^{(\Psi)}$ is non-empty. Since Ψ is the underlying complex of a disc submap of Δ (and a disc map has only

1 contour), all elements of $A_J^{(\Psi)}$ are of the same rank (see Lemma 30). Let $f(\Psi)$ be the rank of the elements of $A_J^{(\Psi)}$. Then $A_{f(\Psi)} \supset A_J^{(\Psi)} \neq \emptyset$.

Because Δ satisfies the condition Y, and every $\Psi \in K'$ is a connected component of $\Gamma_{\{f(\Psi)\}}$, it follows that

$$\|\{\Psi \in K' \mid f(\Psi) = j\}\| \leq \|B_j\| - \varepsilon(j) \quad \text{for every } j.$$

Therefore,

$$\chi_{\Gamma_J} = \sum_{\Psi \in K} \chi_{\Psi} \leq \|K'\| \leq \sum_{j \in J} (\|B_j\| - \varepsilon(j)).$$

Let J be an arbitrary set. Then $\chi_{\Delta} = \chi_{\Gamma_J} - \|A_J\| + \|B_J\|$, and therefore

$$\|A_J\| = \|B_J\| + \chi_{\Gamma_J} - \chi_{\Delta}.$$

In the case A_J is non-empty, obtain:

$$\|A_J\| \leq \sum_{j \in J} (2\|B_j\| - \varepsilon(j)) - \chi_{\Delta} \leq \sum_j (2\|B_j\| - \varepsilon(j)) - \chi_{\Delta}.$$

In particular this proves the first part of the statement (take J to be the one-element set $\{j\}$). To prove the second part, take J to be the set of all j such that $\|A_j\| > 2\|B_j\| - \varepsilon(j)$, and observe from the last inequality that a desired set $E \subset A_J$ exists. \square

5. S -MAPS

Definition 55. An S -map is a map together with structures of an S_1 -map and an S_2 -map such that every internal exceptional arc is selected, and every external exceptional arc lies on the image of a selected c-path.

Every submap of an S -map has a natural structure of an S -map. If Γ is an S -submap of an S -map Δ , then an arc of Γ is exceptional in Γ if and only if it is exceptional in Δ and is incident to a face of Γ .

Definition 56. An S -map Δ is said to satisfy the *condition* $D(\lambda, \mu, \nu)$ relative to a submap Γ if λ , μ , and ν are functions defined on $\Gamma(2)$ (and possibly elsewhere) with values in $[0, 1]$ such that the following three conditions hold:

$D_1(\lambda)$: if Π is a face of Γ , and L is the number of non-selected c-edges of Π , then

$$L \leq \lambda(\Pi)|\partial\Pi|;$$

$D_2(\mu)$: if Π is a face of Γ , u is a selected internal arc of Δ incident to Π , and M is the number of the edges of u that do not lie on any exceptional arc, then

$$M \leq \mu(\Pi)|\partial\Pi|;$$

$D_3(\nu)$: if Π is a face of Δ , p is a simple path in Δ which is the image of a selected c-path of Π , and N is the sum of the lengths of all the exceptional arcs of Γ that lie on p , then

$$N \leq \nu(\Theta)|\partial\Theta|$$

for every face Θ of Γ such that $\text{rk}(\Theta) = \text{rk}(\Pi)$.

The S -map Δ is said to satisfy the *condition* $D(\lambda, \mu, \nu)$ (absolutely) if it satisfies it relative to itself.

Let $D'_2(\mu)$ denote the condition obtained from $D_2(\mu)$ by replacing “... $M \leq \mu(\Pi)|\partial\Pi|$ ” with “... $M \leq \mu(\Theta)|\partial\Theta|$ for every face Θ of Γ such that $\text{rk}(\Theta) \geq \text{rk}(\Pi)$.”

Let $D'_3(\nu)$ denote the condition obtained from $D_3(\nu)$ by replacing “... such that $\text{rk}(\Theta) = \text{rk}(\Pi)$ ” with “... such that $\text{rk}(\Theta) \geq \text{rk}(\Pi)$.”

Definition 57. The *condition* $D'(\lambda, \mu, \nu)$ is the conjunction of the conditions $D_1(\lambda)$, $D'_2(\mu)$, and $D'_3(\nu)$. An S -map Δ is said to satisfy the *condition* $D'(\lambda, \mu, \nu)$ absolutely if it satisfies it relative to itself.

Note that if an S -map Δ satisfies $D(\lambda, \mu, \nu)$ or $D'(\lambda, \mu, \nu)$ relative to a submap Γ , then Δ satisfies the same condition relative to every submap of Γ as well.

The condition D will be used in the proof of Theorem 4, and the somewhat stronger condition D' will be used in the proof of Theorem 5.

Inductive Lemma 58 (Inductive Lemma). *Let Δ be an S -map, and Φ be a simple disc S -submap of Δ . Assume Δ satisfies the condition $Z(2)$ relative to every proper simple disc submap of Φ , Φ satisfies the condition Y , and Δ satisfies $D(\lambda, \mu, \nu)$ relative to Φ . Suppose*

$$\lambda + (3 + \kappa + \kappa')\mu + 2\nu \leq \frac{1}{2}$$

point-wise on Φ (i.e., for every face of Φ). Then Δ satisfies $Z(2)$ relative to Φ .

Proof. Suppose Δ does not satisfy $Z(2)$ relative to Φ .

Let $\bar{\Phi}$ be a (spherical) closure of Φ . Note that the 1-skeleton of $\bar{\Phi}$ is a subcomplex of the 1-skeleton of Δ . Let Θ be the face of $\bar{\Phi}$ that is not in Φ (the *improper* face). Endow $\bar{\Phi}$ with a structure of an S_1 -map by selecting all the c -paths of faces of Ψ that are selected in Ψ , and selecting those c -paths of Θ whose images in Φ coincide with images of selected c -paths of faces that are in $\Delta(2) \setminus \Phi(2)$.

Since Δ satisfies $Z(2)$ relative to every proper simple disc submap of Φ , so does $\bar{\Phi}$.

Since Δ does not satisfy $Z(2)$ relative to Φ , it follows that $\kappa'_{\bar{\Phi}}(\Theta) = 0$ and $\kappa_{\bar{\Phi}}(\Theta) \leq 2$.

Let A' be the set of all the exceptional arcs of Δ that are internal in Φ , and A'' be the set of all the exceptional arcs of Δ that are external in Φ .

Let A be a set of pair-wise non-overlapping selected (internal) arcs of $\bar{\Phi}$ such that every selected edge of $\bar{\Phi}$ lies on an element of A , every element of $A' \sqcup A''$ lies on an element of A , and the cardinality of A is the minimal possible. Then it is easy to see that no two distinct elements of A are subarcs of the same selected arc.

Consider a special case: suppose that $\bar{\Phi}$ is an elementary map in which all c -paths are selected. This implies that Δ itself is an elementary map in which all c -paths are selected. Then Φ has a single face Π , the set A consists of a single element u , and one of the oriented arcs representing u is a cyclic shift of $\partial\Pi$. Hence, as follows from $D_2(\mu)$ and $D_3(\nu)$,

$$|\partial\Pi| = |u| \leq (\mu(\Pi) + \nu(\Pi))|\partial\Pi| < \frac{1}{2}|\partial\Pi|.$$

This gives a contradiction, and hence either $\bar{\Phi}$ is non-elementary, or at least it has a maximal selected c-path. Therefore, Estimating Lemma 50 can be applied.

Apply Estimating Lemma 50 to $\bar{\Phi}$, A , $\Phi(2)$ (in the role of the set B), $\bar{\Phi}(2)$ (in the role of the set C), and $\{\Theta\}$ (in the role of the set D). Let f be a function $A \rightarrow \Phi(2)$ such that:

- (1) for every $x \in A$, $f(x)$ is incident to x , and
- (2) for every $y \in \Phi(2)$, the full pre-image of y under f consists of at most $3 + \kappa_{\bar{\Phi}}(y) + \kappa'_{\bar{\Phi}}(y)$ elements.

(Since $1 + \kappa_{\bar{\Phi}}(\Theta) + \kappa'_{\bar{\Phi}}(\Theta) + 2 - 3\chi_{\bar{\Phi}} \leq -1 \leq 0$, the “set E ” is empty.)

For every j , let B_j be the set of all rank- j faces of Φ , and A'_j be the set of all rank- j elements of A' . As in Estimating Lemma 54, for every j , let $\varepsilon(j) = 1$ if A'' has an element of rank j , and $\varepsilon(j) = 0$ otherwise.

By Estimating Lemma 54 applied to Φ ,

$$\|A'_j\| \leq \max\{0, 2\|B_j\| - \varepsilon(j) - 1\} \quad \text{for every } j.$$

Let p_1 and p_2 be paths such that $\langle p_1 p_2 \rangle = \langle \partial\Phi \rangle$, p_1 is the image of a selected c-path of some face $\Pi_1 \in \Delta(2) \setminus \Phi(2)$, and p_2 either is trivial, or is the image of a selected c-path of some $\Pi_2 \in \Delta(2) \setminus \Phi(2)$ (Π_1 and Π_2 are not assumed to be distinct). Moreover, choose such paths p_1 and p_2 so that every element of A'' lie on one of them. Such paths p_1 and p_2 exist because Δ does not satisfy $Z(2)$ relative to Φ .

For $i = 1, 2$, let $A''^{(i)}$ be the set of those elements of A'' that lie on p_i . Clearly, for each i , all elements of $A''^{(i)}$ have the same rank. If $A''^{(i)} \neq \emptyset$ and j is the rank of elements of $A''^{(i)}$, then $\varepsilon(j) = 1$.

Let J be the set of ranks of all elements of $A' \sqcup A''$. For every $j \in J$, let

$$n(j) = \min_{\Pi \in B_j} \nu(\Pi) |\partial\Pi|.$$

Then, as follows from $D_3(\nu)$,

$$\sum_{x \in A''^{(i)}} |x| \leq \sum_{j \in J} \varepsilon(j) n(j) \leq \sum_{j \in J} n(j) \quad \text{for } i = 1, 2.$$

Estimate the total number of edges of all the elements of $A' \sqcup A''$. Denote this number by N . By $D_3(\nu)$, obtain:

$$\begin{aligned} N &= \sum_{j \in J} \sum_{x \in A'_j} |x| + \sum_{x \in A''^{(1)}} |x| + \sum_{x \in A''^{(2)}} |x| \\ &\leq \sum_{j \in J} (2\|B_j\| - \varepsilon(j) - 1) n(j) + \sum_{j \in J} \varepsilon(j) n(j) + \sum_{j \in J} n(j) \\ &= \sum_{j \in J} 2\|B_j\| n(j) \leq \sum_{\Pi \in \Phi(2)} 2\nu(\Pi) |\partial\Pi|. \end{aligned}$$

Estimate the total number of the edges of elements of A that are not edges of elements of $A' \sqcup A''$. Denote this number by M . By $D_2(\mu)$, obtain:

$$M = \sum_{\Pi \in \Phi(2)} \sum_{x: f(x)=\Pi} |x| \leq \sum_{\Pi \in \Phi(2)} (3 + \kappa(\Pi) + \kappa'(\Pi)) \mu(\Pi) |\partial\Pi|.$$

Estimate the total number of the edges of Φ that are not edges of elements of A . Denote this number by L . By $D_1(\lambda)$, obtain:

$$L \leq \sum_{\Pi \in \Phi(2)} \lambda(\Pi) |\partial \Pi|.$$

Thus, on one hand,

$$\begin{aligned} \|\Phi(1)\| &= L + M + N \\ &\leq \sum_{\Pi \in \Phi(2)} (\lambda(\Pi) + (3 + \kappa(\Pi) + \kappa'(\Pi))\mu(\Pi) + 2\nu(\Pi)) |\partial \Pi| \\ &\leq \sum_{\Pi \in \Phi(2)} \frac{1}{2} |\partial \Pi|; \end{aligned}$$

on the other hand,

$$\|\Phi(1)\| = \frac{1}{2} \sum_{\Pi \in \Phi(2)} |\partial \Pi| + \frac{1}{2} |\partial \Phi|.$$

This gives a contradiction. \square

Lemma 59. *Let Δ be an S -map, c_Δ be the number of contours of Δ . Suppose $c_\Delta + 3\chi_\Delta \geq 0$. Assume Δ satisfies the conditions Y and $D(\lambda, \mu, \nu)$ (absolutely). Let $\gamma = \lambda + (3 + \kappa_\Delta + \kappa'_\Delta)\mu + 2\nu$. Suppose $\gamma(\Pi) \leq 1/2$ for every face Π of Δ . Let T be the set of all the edges of Δ that are incident to faces. Let S be the set of all those elements of T that are external edges of Δ and are the images of selected c -edges, Then*

$$\sum_i |\partial_i \Delta| \geq \|S\| \geq \|T\| - \sum_{\Pi \in \Delta(2)} \gamma(\Pi) |\partial \Pi| \geq \sum_{\Pi \in \Delta(2)} (1 - 2\gamma(\Pi)) |\partial \Pi|.$$

Proof. It suffices to prove that

$$\|S\| \geq \|T\| - \sum_{\Pi \in \Delta(2)} \gamma(\Pi) |\partial \Pi|.$$

One of the other two inequalities is obvious, and the other follows from a simple computation similar to that in Remark 6.1 of [Mur05] or in Proposition 4.1 of [Mur07].

Using induction and Inductive Lemma, obtain that Δ satisfies the condition $Z(2)$ relative to every simple disc submap.

Let N be the sum of the lengths of all the internal exceptional arcs of Δ , M be the number of the selected internal edges of Δ that do not belong to any exceptional arc, and L be the number of (non-selected) edges of Δ that are the images of non-selected c -edges. Then

$$\|S\| = \|T\| - L - M - N.$$

Using Estimating Lemma 54 and the condition $D_3(\nu)$, obtain:

$$N \leq \sum_{\Pi \in \Delta(2)} 2\nu(\Pi) |\partial \Pi|.$$

Similarly to the proof of Inductive Lemma (but with no need for using a closure of Δ) obtain that

$$M \leq \sum_{\Pi \in \Delta(2)} (3 + \kappa_{\Delta}(\Pi) + \kappa'_{\Delta}(\Pi)) \mu(\Pi) |\partial \Pi|$$

(using Estimating Lemma 50 and the condition $D_2(\mu)$), and

$$L \leq \sum_{\Pi \in \Delta(2)} \lambda(\Pi) |\partial \Pi|$$

(using the condition $D_1(\lambda)$). Therefore,

$$L + M + N \leq \sum_{\Pi \in \Delta(2)} \gamma(\Pi) |\partial \Pi|,$$

which completes the proof. \square

6. ASPHERICITY AND TORSION

Definitions of *aspherical* (A), *combinatorially aspherical* (CA), *diagrammatically aspherical* (DA), *singularly aspherical* (SA), and *Cohen-Lyndon aspherical* (CLA) presentations may be found in [CCH81]. It should be noted that none of these definitions requires the set of relators to consist of only reduced elements. Moreover, *group presentations* are regarded in a way that a priori allows for repetition of relators (instead of sets of relators, presentations have *indexed families* of relators). Only diagrammatic and singular asphericities shall be used in this paper.

The version of asphericity defined in [Ol'89, Ol'91] is equivalent to diagrammatic asphericity by Theorem 32.2 therein.

The following is another equivalent definition of diagrammatic asphericity:

Definition 60. A group presentation is *diagrammatically aspherical* if every spherical diagram over this presentation can be transformed by a sequence of diamond moves into a diagram whose all connected components are elementary spherical diagrams.

Proof of equivalence is left to the reader.

Definition 61. A group presentation $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ is *singularly aspherical* if it is diagrammatically aspherical, no element of \mathcal{R} represents a proper power in the free group $\langle \mathfrak{A} \parallel \emptyset \rangle$, and no two distinct elements of \mathcal{R} are conjugate or conjugate to each other's inverses in $\langle \mathfrak{A} \parallel \emptyset \rangle$.

Definition 62. Call a group (A), (CA), (DA), (SA), or (CLA), accordingly, if it has a presentation which is such.

Interesting results on relations between different concepts of asphericity (of which, by the way, combinatorial asphericity is the weakest, and singular asphericity is in a sense one of the strongest), and classification of torsion elements in combinatorially aspherical groups may be found in [Hue79, CCH81]. Combinatorial asphericity is also discussed in [Hue80] in great detail.

Remark 63. It follows directly from the definition of singular asphericity that a group is singularly aspherical if and only if it has a diagrammatically aspherical presentation without proper powers among relators, relators being viewed as elements of the free group on the set of generators.

Lemma 64. *Singularly aspherical groups are torsion-free.*

Proof. Let G be an arbitrary singularly aspherical group. Let $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ be a singularly aspherical presentation of G . Then the relation module M of $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ is a free G -module by Corollary 32.1 in [Ol'89, Ol'91]. Therefore, there exists a finite-length free resolution of \mathbb{Z} over $\mathbb{Z}G$:

$$0 \rightarrow M \rightarrow \bigoplus_{x \in \mathfrak{A}} \mathbb{Z}G \rightarrow \mathbb{Z}G \rightarrow \mathbb{Z} \rightarrow 0,$$

where $\mathbb{Z}G$ and $\bigoplus_{x \in \mathfrak{A}} \mathbb{Z}G$ are identified with the (free) G -modules of, respectively, 0- and 1-dimensional cellular chains of the Cayley complex of $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$.

Suppose now that G has torsion. Let C be a nontrivial finite cyclic subgroup of G . Every free G -module may be naturally regarded as a free C -module. Hence the above resolution may be viewed a free resolution of \mathbb{Z} over $\mathbb{Z}C$. This contradicts the fact that all odd-dimensional homology groups of any nontrivial finite cyclic group are nontrivial (see [Bro94]). \square

7. PROOF OF THE THEOREMS

Theorems 4 and 5 are proved in this section by parallel series of similar arguments. It is convenient in both cases to use the notation of Section 2.

There is a conflict in notation between Subsections 2.1 and 2.2, but it shall not cause confusion if the notation of Subsection 2.1 is used only in the context of proving Theorem 4, while considering $\langle \mathfrak{A} \parallel \mathcal{R}_n \rangle$, $n \in \mathbb{N}$, and the notation of Subsection 2.2 is used only in the context of proving Theorem 5, while considering $\langle \mathfrak{A} \parallel \mathcal{R}_\infty \rangle$.

It is convenient to assume in this section that to every diagram under consideration there is assigned a *sort* which is either (I. n), $n \in \mathbb{N}$, or (II). More precisely, every diagram or presentation considered in this section is always equipped with a *sort attribute*. Diagrams of sorts (I. n), $n \in \mathbb{N}$, will be used for proving Theorem 4, and most diagrams of sort (I. n) under consideration will be diagrams over $\langle \mathfrak{A} \parallel \mathcal{R}_n \rangle$. Similarly, diagrams of sort (II) will be used for proving Theorem 5, and most diagrams of this sort under consideration will be diagrams over $\langle \mathfrak{A} \parallel \mathcal{R}_\infty \rangle$. Whenever the sort is not assigned explicitly, it shall be assumed in the most natural way, but a priori the sort is not determined by the diagram itself. The purpose of this convention is to unambiguously use the same term in relation to a diagram in different senses depending on the context (on the sort of the diagram).

Define sets of indices I_n , $n \in \mathbb{N}$, and I_∞ as follows:

$$I_n = \{ j \mid \mathcal{R}_n^{(j)} \neq \mathcal{R}_n^{(j-1)} \}, \quad I_\infty = \{ j \mid \mathcal{R}_\infty^{(j)} \neq \mathcal{R}_\infty^{(j-1)} \}.$$

Then

$$\mathcal{R}_n = \{ r_j \mid j \in I_n \}, \quad \mathcal{R}_\infty = \{ r_{j,1}, r_{j,2} \mid j \in I_\infty \}.$$

Let $W_n = \{w_j \mid j \in I_n\}$ where w_j are the group words defined in Subsection 2.1. Let $W_\infty = \{w_j \mid j \in I_\infty\}$ where w_j are the group words defined in Subsection 2.2.

Definition 65. A graded S_1 -diagram Δ of sort (I, n) is called *correct* if

- (1) the rank of every face of Δ is in $I_n \sqcup \{0\}$;
- (2) for every face Π of rank j , $\ell(\partial\Pi) \in \{r_j^{\pm 1}\}$ if $j \in I_n$, and $\ell(\partial\Pi)$ is the concatenation of several copies of $z_1^{\pm 1}$ and $z_2^{\pm 1}$ if $j = 0$;
- (3) for every face Π of rank $j \neq 0$, Π has c-paths $s_1, \dots, s_{2n+2}, s'_1, \dots, s'_{2n+2}, t_1, \dots, t_{2n+2}$, and t_0 such that:
 - (a) $s_1 t_1 s'_1 s_2 t_2 s'_2 \dots s_{2n+2} t_{2n+2} s'_{2n+2} t_0 \in \{\partial^* \Pi, (\partial^* \Pi)^{-1}\}$,
 - (b) (i) $\ell(s_i) = u_{ji}$ and $\ell(s'_i) = u_{ji}^{-1}$ for every $i = 1, \dots, 2n+2$,
 - (ii) $\ell(t_1) = \dots = \ell(t_{2n+2}) = w_j$,
 - (iii) $\ell(t_0) = v_j^{-1}$,
 - (c) a c-path of Π is selected if and only if it is a nontrivial subpath of one of the following $8n+8$ paths: $s_1^{\pm 1}, \dots, s_{2n+2}^{\pm 1}, s'_1{}^{\pm 1}, \dots, s'_{2n+2}{}^{\pm 1}$
 (in particular, $\kappa_\Delta(\Pi) = 4n+4$);
- (4) for every face Π of rank 0, all c-pseudo-arcs of Π are selected (in particular, $\kappa_\Delta(\Pi) = 0$);
- (5) if two faces are congruent (i.e., their contour labels are cyclic shifts of each other or cyclic shifts of the inverses of each other), then either these faces have the same rank, or the rank of at least one of these faces is 0.

Definition 66. A graded S_1 -diagram Δ of sort (I, ∞) is called *correct* if

- (1) the rank of every face of Δ is in $I_\infty \sqcup \{0\}$;
- (2) for every face Π of rank j , $\ell(\partial\Pi) \in \{r_{j,1}^{\pm 1}, r_{j,2}^{\pm 1}\}$ if $j \in I_\infty$, and $\ell(\partial\Pi)$ is the concatenation of several copies of $z_1^{\pm 1}$ and $z_2^{\pm 1}$ if $j = 0$;
- (3) for every face Π of rank $j \neq 0$, Π has c-paths $q, s_1, \dots, s_{2j+2}, s'_1, \dots, s'_{2j+2}, t_1, \dots, t_{2j+2}$, and t_0 such that:
 - (a) $s_1 t_1 s'_1 s_2 t_2 s'_2 \dots s_{2j+2} t_{2j+2} s'_{2j+2} t_0 = q \in \{\partial^* \Pi, (\partial^* \Pi)^{-1}\}$,
 - (b) either
 - (i) $\ell(q) = r_{j,1}$,
 - (ii) $\ell(s_i) = u_{ji}$ and $\ell(s'_i) = u_{ji}^{-1}$ for every $i = 1, \dots, 2j+2$,
 - (iii) $\ell(t_1) = \dots = \ell(t_{2j+2}) = w_j$,
 - (iv) $\ell(t_0) = a^{-1}$,
 or
 - (i) $\ell(q) = r_{j,2}$,
 - (ii) $\ell(s_i) = u_{j,2j+2+i}$ and $\ell(s'_i) = u_{j,2j+2+i}^{-1}$ for every $i = 1, \dots, 2j+2$,
 - (iii) $\ell(t_1) = \dots = \ell(t_{2j+2}) = w_j$,
 - (iv) $\ell(t_0) = b^{-1}$,
 - (c) a c-path of Π is selected if and only if it is a nontrivial subpath of one of the following $8j+8$ paths: $s_1^{\pm 1}, \dots, s_{2j+2}^{\pm 1}, s'_1{}^{\pm 1}, \dots, s'_{2j+2}{}^{\pm 1}$
 (hence $\kappa_\Delta(\Pi) = 4\text{rk}(\Pi) + 4$);

- (4) for every face Π of rank 0, all c-pseudo-arcs of Π are selected (in particular, $\kappa_\Delta(\Pi) = 0$);
- (5) if two faces are congruent, then either they have the same rank, or at least one of them has rank 0.

(The last conditions in these two definitions may be redundant, but are easy to satisfy, and they facilitate the proof of Lemma 74.)

Definition 67. Faces of rank 0 in a correct graded S_1 -diagram of any sort are called *alien*, all the other faces are called *native*. Correct graded S_1 -diagrams without alien faces are called *restricted*.

For every diagram over $\langle \mathfrak{A} \parallel \mathcal{R}_n \rangle$, and for every diagram over $\langle \mathfrak{A} \parallel \mathcal{R}_\infty \rangle$, there is an essentially isomorphic diagram that has a selection and a grading which turn it into a restricted correct graded S_1 -diagram of sort (I.n) or (II), respectively.

Definition 68. An S -diagram Δ of any sort is called *correct* if

- (1) Δ is correct as a graded S_1 -diagram;
- (2) an internal arc u of Δ is exceptional if and only if there exist c-paths $s_1, s_{1-}, s_{10}, s_{1+}, s_2, s_{2-}, s_{20}, s_{2+}$ such that:
 - (a) $s_1 = s_{1-}s_{10}s_{1+}$ and $s_2 = s_{2-}s_{20}s_{2+}$,
 - (b) s_1 and s_2 are maximal selected c-paths,
 - (c) s_{10} and s_{20} are distinct c-paths with a common image (in the 1-skeleton of Δ) which coincides with one of the oriented arcs associated with u , and
 - (d) $\ell(s_1) = \ell(s_2)$, $|s_{1-}| = |s_{2-}|$, $|s_{1+}| = |s_{2+}|$;
- (3) exceptional arcs are not incident to faces of rank 0

(in particular, every internal exceptional arc of Δ is a maximal selected arc, and there are no exceptional arcs of rank 0).

Every correct graded S_1 -diagram of any sort has a structure of a correct S -diagram that extends the given structure of a graded S_1 -diagram and is unique up to choice of external exceptional arcs. Every maximal selected internal arc of a correct S -diagram either is exceptional, or does not overlap with any exceptional arc.

Definition 69. An internal exceptional arc u of a correct S -diagram is called *non-extendible* if it is the image of two maximal selected c-arcs; otherwise u is called *extendible*.

Every extendible exceptional arc of a correct S -diagram of any sort can be “extended” to a longer exceptional arc by a diamond move. (Diamond moves are viewed here as operations on correct S -diagrams of a given sort.)

Definition 70. An S -diagram Δ of any sort is called *special* if it is correct, weakly reduced, and has no extendible internal exceptional arcs.

Clearly, every S -subdiagram of every special S -diagram is special.

Lemma 71. Every correct S -diagram of any sort can be transformed by a series of diamond moves into an S -diagram each connected component of which is either special and reduced, or elementary spherical. In particular, every reduced correct S -diagram can be transformed by diamond moves into a special S -diagram.

Note that even when the diagrams under consideration are graded, the property of being reduced is the same as for non-graded ones, unlike [Ol'89, Ol'91].

Proof of the lemma. The number of connected components and the Euler characteristic of any map that can be obtained from a given map Γ by diamond moves are bounded from above. Indeed, the number of connected components is bounded by $\|\Gamma(2)\| + c_\Gamma$, and the Euler characteristic is bounded by $2\|\Gamma(2)\| + c_\Gamma$, as follows from Lemma 26. Here c_Γ denotes the number of contours of Γ .

Since diamond moves do not decrease the Euler characteristic, and improper diamond moves increase it (see Lemma 34), it follows that in any sequence of diamond moves applied to a given diagram, there is only bounded number of improper ones.

Consider an arbitrary correct S -diagram Δ . Assume without loss of generality that no improper diamond move is applicable to Δ , nor to any (correct) S -diagram obtained from Δ by proper diamond moves. In particular, neither the number of connected components of Δ , nor the Euler characteristic of Δ can be increased by any sequence of diamond moves. Then every connected component of Δ either is reduced or otherwise can be turned into an elementary spherical diagram by a sequence of proper diamond moves. Thus it is left to show that every reduced connected component of Δ can be made special by (proper) diamond moves.

Let Ψ be a reduced connected component of Δ . Diamond moves allow one to “extend” all extendible internal exceptional arcs one-by-one. Any S -diagram obtained from Ψ in this manner will be special. \square

For every n , denote $1/(4n + 4)$ by ν_n .

If Δ is a restricted special S -diagram of sort (I. n), then let λ_Δ , μ_Δ , and ν_Δ be the constant functions on $\Delta(2)$ defined as follows:

$$\lambda_\Delta = \lambda_n, \quad \mu_\Delta = \mu_n, \quad \nu_\Delta = \nu_n = \frac{1}{4n + 4}.$$

Then, as follows from (1),

$$(5) \quad 2\lambda_\Delta + (2\kappa_\Delta + 6n)\mu_\Delta + (2n + 1)\nu_\Delta < \frac{1}{2}.$$

If Δ is a restricted special S -diagram of sort (II), then let λ_Δ , μ_Δ , and ν_Δ be the functions on $\Delta(2)$ defined as follows:

$$\lambda_\Delta(\Pi) = \lambda_{\text{rk}(\Pi)}, \quad \mu_\Delta(\Pi) = \mu_{\text{rk}(\Pi)}, \quad \nu_\Delta(\Pi) = \nu_{\text{rk}(\Pi)}.$$

Then, as follows from (1),

$$(6) \quad 2\lambda_\Delta + (2\kappa_\Delta + 6\text{rk})\mu_\Delta + (2\text{rk} + 1)\nu_\Delta < \frac{1}{2}.$$

Lemma 72. *Let Δ be an arbitrary correct and weakly reduced (or special) S -diagram, and Γ be a restricted S -subdiagram. Then Δ satisfies the condition $D(\lambda_\Gamma, \mu_\Gamma, \nu_\Gamma)$ relative to Γ . In the case Δ is of sort (II), it satisfies $D'(\lambda_\Gamma, \mu_\Gamma, \nu_\Gamma)$ relative to Γ .*

This lemma follows directly from the construction of the presentations $\langle \mathfrak{A} \parallel \mathcal{R}_n \rangle$, $n \in \mathbb{N}$, and $\langle \mathfrak{A} \parallel \mathcal{R}_\infty \rangle$ in Section 2.

Lemma 73. *Let j be a natural number and Δ be a restricted special disc S -diagram satisfying the condition Υ and containing a face of rank at least j . Let $w = w_j$ where w_j is defined in Subsection 2.1 or 2.2 depending on the sort of Δ . Then $|\partial\Delta| > |w|$.*

Proof. By Lemma 72, the S -map Δ satisfies the condition $D(\lambda_\Delta, \mu_\Delta, \nu_\Delta)$ absolutely.

Let

$$\gamma = \lambda_\Delta + (3 + \kappa_\Delta + \kappa'_\Delta)\mu_\Delta + 2\nu_\Delta.$$

It follows from inequality (5) or (6), depending to the sort of Δ , that $\gamma(\Pi) < 1/2 - \lambda_\Delta(\Pi)$ for every face Π of Δ .

By Lemma 59,

$$|\partial\Delta| \geq \sum_{\Pi \in \Delta(2)} (1 - 2\gamma(\Pi))|\partial\Pi| > \sum_{\Pi \in \Delta(2)} 2\lambda_\Delta(\Pi)|\partial\Pi|.$$

Let Π be a face of Δ of rank $\geq j$. It follows from one of the conditions imposed on the group presentations in Section 2 that $|w| \leq \lambda_\Delta(\Pi)|\partial\Pi|$. Hence $|\partial\Delta| > |w|$. \square

Lemma 74. *Let Δ be a connected restricted special S -diagram (of any sort) over a group presentation $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$. Suppose that every proper (finite) subpresentation of $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$ defines a torsion-free group. Then Δ satisfies the condition Υ .*

Proof. Without loss of generality, assume that every connected special S -diagram over $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$ whose set of face ranks is a proper subset of the set of face ranks of Δ , does satisfy Υ . (Alternatively, one can induct on the number of different face ranks of a diagram.)

Let j be the rank of an arbitrary internal exceptional arc of Δ . Let A be the set of all the internal exceptional arcs of Δ of rank j , and B be the set of all the faces of Δ of rank j . Let Γ be the S -subdiagram obtained from Δ by removing all the faces and internal exceptional arcs of rank j .

Let \mathcal{T} be the minimal subset of \mathcal{S} such that Γ is a diagram over $\langle \mathfrak{A} \parallel \mathcal{T} \rangle$. The set of face ranks of Γ is a proper subset of that of Δ ; therefore, \mathcal{T} is a proper subset of \mathcal{S} . Hence the groups presented by $\langle \mathfrak{A} \parallel \mathcal{T} \rangle$ is torsion-free.

Let $k = \kappa_\Delta(\Pi)$ for an arbitrary face Π of rank j ($k = 4n + 4$ if the sort is (I.n), and $k = 4j + 4$ if the sort is (II)). Let C be the set of corners of elements of B chosen as follows: If Π is an arbitrary element of B , let $s_1, \dots, s_{k/2}, s'_1, \dots, s'_{k/2}, t_1, \dots, t_{k/2}$, and t_0 be the c-paths of Π such as in the definition of correct graded S_1 -diagrams ($s_1, \dots, s_{k/2}, s'_1, \dots, s'_{k/2}$ are maximal selected); let C contain the initial corners of the c-paths $s_1 t_1 s'_1, \dots, s_{k/2} t_{k/2} s'_{k/2}$, and no other corners of Π . Thus C contains exactly $k/2$ corners of each element of B .

The image (under the attaching morphism) of each element of C is a vertex of Γ . As follows from Lemma 26, every connected component of Γ of Euler characteristic 1 is a disc S -diagram. It suffices to prove now that for every connected component Ψ of Γ , if either Ψ is a disc submap, or it contains the image of a selected c-edge of a face of Δ of rank j , then the number of elements of C whose images are in Ψ is at least $k/2$.

Since Δ is weakly reduced, it follows from the definition of exceptional arcs in a correct S -diagram that for every connected component Ψ of Γ , the number of elements of C whose images are in Ψ is divisible by $k/2$ (recall Lemma 30).

Consider an arbitrary connected component Ψ of Γ which contains the image of a selected c-edge of an element of B . Let Π be an element of B and x be a selected c-edge of Π such that the image of x is in Ψ . Let s be the maximal selected c-path of Π such that x lies on s , and the label of s is of the form u_{ji} ($i \in \{1, \dots, 2n+2\}$ if the diagrams are of sort (I.n), and $i \in \{1, \dots, 4j+4\}$ if the diagrams are of sort (II)). Let c be the element of C which is the corner of Π “closest” to the initial vertex of s . More precisely, c is the only element of C which is a corner of Π and either coincides with the initial vertex of s , or can be connected to the initial vertex of s by a path (c-path of Π) without selected oriented c-edges. Then the image of c is in Ψ , and hence the number of elements of C mapped to Φ is at least $k/2$ (since it is not 0 and is divisible by $k/2$).

Now consider an arbitrary connected component Ψ of Γ endowed with the inherited structure of a special disc S -diagram. Suppose Ψ does not contain the image of any element of C . It follows from Lemma 30 and from Δ ’s being weakly reduced that some cyclic shift of $(\partial\Psi)^{\pm 1}$ can be decomposed into the product of paths each of which is labelled by w_j . Therefore, w_j represents a finite-order element in the group presented by $\langle \mathfrak{A} \parallel \mathcal{T} \rangle$. Since that group is torsion-free, w_j represents the identity element in it.

Let Φ be a restricted special disc S -diagram over $\langle \mathfrak{A} \parallel \mathcal{T} \rangle$ whose contour label is w_j , and whose set of face ranks is a subset of that of Γ (here Lemma 71 is used to find such a *special* Φ). By the inductive assumption at the beginning of this proof, Φ satisfies Y.

By the construction of group presentations in Section 2, w_j is not trivial modulo $\langle \mathfrak{A} \parallel \mathcal{R}_n^{(j-1)} \rangle$ or $\langle \mathfrak{A} \parallel \mathcal{R}_\infty^{(j-1)} \rangle$, whichever corresponds to the sort of the diagrams under consideration. Therefore, Φ contains at least one face whose rank is j or greater. This contradicts Lemma 73.

Thus Ψ does contain the image of some element of C , and hence Ψ contains the images of at least $k/2$ element of C .

On one hand, $\|C\| = (k/2)\|B\|$ (recall that B is the set of all rank- j faces). On the other hand, all elements of C can be distributed among connected components of Γ so that there are at least $k/2$ elements assigned to each component that either is disc, or contains an external exceptional arc of Δ of rank j . Therefore, the number of such components does not exceed the number of faces of rank j . The same is true for every j such that Δ has an internal exceptional arc of rank j . Hence the condition Y. \square

Lemma 75. *No element of $\bigcup_{n \in \mathbb{N}} \mathcal{R}_n \cup \mathcal{R}_\infty$ represents a proper power in the free group on \mathfrak{A} . Distinct element of \mathcal{R}_n , $n \in \mathbb{N}$, or of \mathcal{R}_∞ do not represent conjugate elements of the free group on \mathfrak{A} , nor elements conjugate to each other’s inverses.*

Proof. Most likely there is a straightforward way to prove these facts using only the small-cancellation conditions imposed on the (subwords of) defining

relations in Section 2, or they can be obtained for free by imposing additional restrictions on the defining relators of the constructed presentations. Following is a proof which is more in the spirit of this paper.

Suppose $r \in \mathcal{R}_\infty$ or $r \in \mathcal{R}_n$ for some n , and r represents a proper power in the free group on \mathfrak{A} . Then r is freely conjugate to s^m where s is cyclically reduced and $m > 1$. Let Φ be a special simple single-face disc S -diagram over $\langle \mathfrak{A} \parallel \{r\} \rangle$ such that $\ell(\partial\Phi) = s^m$.

Let Δ be a special spherical S -diagram obtained from two copies of Φ by attaching them to each other along their contour cycles with a shift by $|s|$ edges. More precisely, let Φ_1 and Φ_2 be two copies of the S -diagram Φ . Let $q_1 = \partial\Phi_1$, and let q_2 be a cyclic shift of $\partial\Phi_2$ by $|s|$ edges (in either direction). Observe that $\ell(q_1) = s^m = \ell(q_2)$. Let Δ be the correct spherical S -diagram obtained by “gluing” Φ_1 and Φ_2 together along the pair of paths q_1 and q_2 .

Because of the shift in “gluing” the copies of Φ , the S -diagram Δ does not have any exceptional arcs, and hence satisfies the condition Υ . For the same reason, Δ satisfies the condition $D(\lambda_\Delta, \mu_\Delta, 0)$ absolutely.

Let Π_1 and Π_2 be the two faces of Δ . Let $\hat{\kappa} = \kappa_\Delta(\Pi_1) = \kappa_\Delta(\Pi_2)$, $\hat{\lambda} = \lambda_\Delta(\Pi_1) = \lambda_\Delta(\Pi_2)$, and $\hat{\mu} = \mu_\Delta(\Pi_1) = \mu_\Delta(\Pi_2)$.

By Lemma 59 and inequalities (5) and (6),

$$0 \geq (1 - 2(\hat{\lambda} + (3 + 2\hat{\kappa})\hat{\mu}))(|\partial\Pi_1| + |\partial\Pi_2|) > 0.$$

This gives a contradiction.

Suppose two distinct defining relators of one of the constructed presentations represent conjugate elements of the free group $\langle \mathfrak{A} \parallel \emptyset \rangle$. This situation also gives rise to a special spherical S -diagram Δ without exceptional arcs and satisfying $D(\lambda_\Delta, \mu_\Delta, 0)$. (Such Δ is also obtained by “gluing” together two single-face simple disc diagrams.) This again leads to a contradiction with Lemma 59.

The case of two relators conjugate to the inverses of each other is dealt with similarly. \square

Lemma 76. *Let $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$ be a finite subpresentation of $\langle \mathfrak{A} \parallel \mathcal{R}_\infty \rangle$ or of $\langle \mathfrak{A} \parallel \mathcal{R}_n \rangle$ for some $n \in \mathbb{N}$. Then $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$ is singularly aspherical, and every connected restricted special S -diagram (of appropriate sort) over $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$ satisfies the condition Υ .*

Proof. Induction on \mathcal{S} : if $\mathcal{S} = \emptyset$, then the conclusion is obvious; assume $\mathcal{S} \neq \emptyset$, and the statement is true for all proper subpresentations of $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$.

By the inductive assumption and Lemma 64, every proper subpresentation of $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$ defines a torsion-free group. By Lemma 74, every connected restricted special S -diagram over $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$ satisfies the condition Υ .

It is left to show that $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$ is singularly aspherical. Suppose it is not. Due to Lemma 75, this means that $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$ is not diagrammatically aspherical.

Let Δ_0 be a restricted correct reduced spherical S -diagram over $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$ (it exists since the presentation is not diagrammatically aspherical). Let Δ_1 be a special S -diagram obtained from Δ_0 by diamond moves (see Lemma 71). Then, as follows from Lemmas 34 and 26, every connected component of Δ_1 is a reduced spherical diagram. Let Δ be an arbitrary connected component of Δ_1 . It is already shown that such Δ must satisfy the condition Υ . By

Lemma 72, Δ satisfies the condition $D(\lambda_\Delta, \mu_\Delta, \nu_\Delta)$. By Lemma 59 and inequalities (5) and (6),

$$0 \geq \sum_{\Pi \in \Delta(2)} (1 - 2(\lambda_\Delta(\Pi) + (3 + 2\kappa_\Delta(\Pi))\mu_\Delta(\Pi) + 2\nu_\Delta(\Pi)))|\partial\Pi| > 0,$$

which gives a contradiction. Thus $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$ is singularly aspherical. \square

Proposition 77. *For every $n \in \mathbb{N}$, the group G_n constructed in Subsection 2.1 is singularly aspherical, torsion-free, and the elements $[z_1]_{G_n}$ and $[z_2]_{G_n}$ freely generate a free subgroup H such that*

$$(\forall h \in H \setminus \{1\}) (\forall m \geq 2n) (\text{cl}_{G_n}(h^m) > n).$$

Proof. By Lemma 76, every finite subpresentation of $\langle \mathfrak{A} \parallel \mathcal{R}_n \rangle$ is singularly aspherical. Therefore, $\langle \mathfrak{A} \parallel \mathcal{R}_n \rangle$ itself is singularly aspherical. Therefore, by Lemma 64, the group G_n is torsion-free.

Let w be an arbitrary nontrivial reduced product of several copies of $z_1^{\pm 1}$ and $z_2^{\pm 1}$. Let m be an arbitrary integer such that $m \geq 2n$. Since, by Proposition 7, G_n is simple or trivial, the commutator length of $[w^m]$ in G_n is defined. To complete the proof, it is only left to show that $\text{cl}_{G_n}([w^m]) > n$.

Suppose that on the contrary $\text{cl}_{G_n}([w^m]) \leq n$. By Lemma 42, there exists a one-contour reduced diagram over $\langle \mathfrak{A} \parallel \mathcal{R}_n \rangle$, the underlying complex of whose closure is a combinatorial handled sphere with n or fewer handles, and whose contour label is w^m . Denote such a diagram by Δ_0 . Then $\chi_{\Delta_0} \geq 1 - 2n$.

After cyclically shifting, if necessary, the c-contours of some of the faces of Δ_0 , endow Δ_0 with the structure of a restricted correct S -diagram (of sort (I. n)) without external exceptional arcs. Transform Δ_0 into a special S -diagram Δ_1 by diamond moves. This is possible by Lemma 71 and because Δ_0 is reduced. Let Δ be the connected component of Δ_1 containing $\partial\Delta_1$. Then a closure of Δ is a handled sphere. Since diamond moves do not decrease the Euler characteristic, the maximal possible Euler characteristic of a connected component is 2, and every diamond move that increases the number of connected components increases it by 1 and increases the Euler characteristic by 2, it follows that $\chi_\Delta \geq \chi_{\Delta_0}$. Note also that $\ell(\partial\Delta) = w^m$.

Case 1: Δ has no faces. Let $F_{\mathfrak{A}}$ be the free group presented by $\langle \mathfrak{A} \parallel \emptyset \rangle$. Then, by Lemma 42, $[w^m]_{F_{\mathfrak{A}}} \in [F_{\mathfrak{A}}, F_{\mathfrak{A}}]$ (hence $[w]_{F_{\mathfrak{A}}} \in [F_{\mathfrak{A}}, F_{\mathfrak{A}}]$) and $\text{cl}_{F_{\mathfrak{A}}}([w^m]_{F_{\mathfrak{A}}}) \leq n$. This contradicts with Corollary 5.2 in [DH91]. (That corollary implies, in particular, that for every nontrivial element x of the derived subgroup of an arbitrary free group F , and for every $m \in \mathbb{N}$, $\text{cl}_F(x^m) \geq (m+1)/2$.)

Case 2: Δ has at least one face. Let $\bar{\Delta}$ be a closure of Δ . Clearly, $\bar{\Delta}$ cannot be elementary spherical (otherwise some cyclic shift of $w^{\pm m}$ would be a relator, which is clearly not possible under the conditions imposed in Subsection 2.1). The underlying complex of $\bar{\Delta}$ is a handled sphere with at most n handles since $\chi_{\bar{\Delta}} \geq 2 - 2n$. Let Θ be the face of $\bar{\Delta}$ that is not in Δ (the “improper” face). Extend the existing structure of a (restricted special) S -diagram on Δ to a structure of an (unrestricted special) S -diagram on $\bar{\Delta}$, assigning to Θ rank 0 and choosing all c-paths of Θ as selected. Then $\kappa_{\bar{\Delta}}(\Theta) = \kappa'_{\bar{\Delta}}(\Theta) = 0$.

By Lemma 72, $\bar{\Delta}$ satisfies $D(\lambda_n, \mu_n, \nu_n)$ relative to Δ . By Lemma 76, Δ satisfies Y . By induction and Inductive Lemma, using inequality (5), obtain that $\bar{\Delta}$ satisfies $Z(2)$ relative to every simple disc subdiagram of Δ .

Let N be the sum of the lengths of all exceptional arcs of $\bar{\Delta}$ (of Δ), M be the sum of the lengths of all the non-exceptional maximal selected arcs of $\bar{\Delta}$ that are incident to faces of Δ (recall that every non-exceptional maximal selected internal arc of a correct S -diagram does not overlap with any exceptional arc), and L be the number of non-selected edges of $\bar{\Delta}$. Note that every non-selected edge of $\bar{\Delta}$, as well as every exceptional arc, is incident to a face of Δ .

Let T be the set of all the edges of $\bar{\Delta}$ that are incident to faces of Δ . Then $L + M + N = \|T\|$. To come to a contradiction, it is left to show that $L + M + N \leq (1/2) \sum_{\Pi \in \Delta(2)} |\partial\Pi|$, because $\sum_{\Pi \in \Delta(2)} |\partial\Pi| < 2\|T\|$ (here it is used that Δ is non-degenerate).

The following upper estimate on L follows from the condition $D_1(\lambda_\Delta)$:

$$L \leq \sum_{\Pi \in \Delta(2)} \lambda_n |\partial\Pi|.$$

To estimate M , Estimating Lemma 50 and the condition $D_2(\mu_n)$ shall be applied. Let A be the set of all the non-exceptional maximal selected arcs that are incident to faces of Δ . Clearly, distinct elements of A do not overlap and are not subarcs of the same selected arc. Let $B = \Delta(2)$, $C = \bar{\Delta}(2)$, $D = \{\Theta\}$. By Estimating Lemma 50 applied to $\bar{\Delta}$, A , B , C , D , there exist a subset E of A and a function $h: A \setminus E \rightarrow \Delta(2)$ such that:

(1) either E is empty, or

$$\|E\| \leq 1 + \kappa_{\bar{\Delta}}(\Theta) + \kappa'_{\bar{\Delta}}(\Theta) + 2 - 3\chi_{\bar{\Delta}} \leq 3 - 3(2 - 2n) = 6n - 3;$$

(2) for every $x \in A \setminus E$, the face $h(x)$ is incident to x ;

(3) for every face Π , the number of arcs mapped to Π by h is at most $3 + \kappa_{\Delta}(\Pi) + \kappa'_{\Delta}(\Pi) \leq 3 + 2(4n + 4) = 8n + 11$.

Let $f: A \rightarrow \Delta(2)$ be an arbitrary extension of h such that for every $x \in A$, the face $f(x)$ is incident to x . Then for every face Π of Δ , the number of arcs mapped to Π by f is at most $14n + 8$. As follows from $D_2(\mu_n)$, $|x| \leq \mu_n |\partial(f(x))|$ for every $x \in A$. Therefore,

$$M \leq \sum_{\Pi \in \Delta(2)} (14n + 8) \mu_n |\partial\Pi|.$$

It easily follows from Estimating Lemma 54 applied to Δ , that for every j , the number of exceptional arcs of Δ of rank j is at most $2n + 1$ times the number of faces of Δ of rank j . Then it follows from $D_3(\nu_n)$ that

$$N \leq \sum_{\Pi \in \Delta(2)} (2n + 1) \nu_n |\partial\Pi|.$$

Thus, by inequality (1) and because Δ is non-degenerate,

$$L + M + N < \sum_{\Pi \in \Delta(2)} \frac{1}{2} |\partial\Pi|.$$

This leads to a contradiction in Case 2. □

Proposition 78. *The group G_∞ constructed in Subsection 2.2 is singularly aspherical, torsion-free, and the elements $[z_1]_{G_\infty}$ and $[z_2]_{G_\infty}$ freely generate a free subgroup H such that*

$$(\forall h \in H \setminus \{1\}) \left(\lim_{n \rightarrow +\infty} \text{cl}_{G_\infty}(h^n) = +\infty \right).$$

Proof. The same way as in Proposition 77, obtain that $\langle \mathfrak{A} \parallel \mathcal{R}_\infty \rangle$ is singularly aspherical and G_∞ is torsion-free.

Let w be an arbitrary nontrivial reduced product of several copies of $z_1^{\pm 1}$ and $z_2^{\pm 1}$, and n be an arbitrary positive integer. To complete this proof it is only left to show that for every large enough m , $\text{cl}_{G_\infty}([w^m]) > n$. Without loss of generality, assume that w is cyclically reduced, and that $|w| \leq \mu_j |r_{j,1}| = \mu_j |r_{j,2}|$ for every $j \geq n$.

Let m be an arbitrary integer such that $|w^m| \geq |r_{n,1}|$ and $1/m \leq \mu_n$. Suppose that $\text{cl}_{G_\infty}([w^m]) \leq n$. By the same argument as in the proof of Proposition 77, there exists a one-contour reduced restricted special S -diagram over $\langle \mathfrak{A} \parallel \mathcal{R}_\infty \rangle$ (of sort (II)), the underlying complex of whose closure is a combinatorial handled sphere with at most n handles, and whose contour label is w^m . Let Δ be such an S -diagram.

Let $\bar{\Delta}$ be a closure of Δ . The diagram $\bar{\Delta}$ cannot be elementary spherical because of the conditions imposed in Subsection 2.2. Let Θ be the face of $\bar{\Delta}$ that is not in Δ . Extend the existing structure of a (restricted special) S -diagram on Δ to a structure of an (unrestricted special) S -diagram on $\bar{\Delta}$, assigning to Θ rank 0 and choosing all c-paths of Θ as selected. Then $\kappa_{\bar{\Delta}}(\Theta) = \kappa'_{\bar{\Delta}}(\Theta) = 0$.

By Lemma 72, $\bar{\Delta}$ satisfies $D'(\lambda_\Delta, \mu_\Delta, \nu_\Delta)$ relative to Δ . By Lemma 76, $\bar{\Delta}$ satisfies Υ . By induction and Inductive Lemma, using inequality (6), obtain that $\bar{\Delta}$ satisfies $Z(2)$ relative to every simple disc subdiagram of Δ .

Let N be the sum of the lengths of all exceptional arcs of $\bar{\Delta}$, M be the sum of the lengths of all non-exceptional maximal selected arcs of $\bar{\Delta}$, and L be the number of non-selected edges of $\bar{\Delta}$. Then $L + M + N = \|\bar{\Delta}(1)\|$. To obtain a contradiction, it suffices to show that $L + M + N < (1/2) \sum_{\Pi \in \bar{\Delta}(2)} |\partial \Pi|$.

The following upper estimate on L follows from the condition $D_1(\lambda_\Delta)$ relative to Δ , because every c-edge of Θ is selected:

$$L \leq \sum_{\Pi \in \Delta(2)} \lambda_\Delta(\Pi) |\partial \Pi|.$$

Let A be the set of all maximal selected arcs of $\bar{\Delta}$. Then $M + N = \sum_{x \in A} |x|$.

Apply Estimating Lemma 50 to the S -diagram $\bar{\Delta}$ and the sets A , $B = \Delta(2)$, $C = \bar{\Delta}(2)$, $D = \{\Theta\}$. Let E be a subset of A and h be a function $A \setminus E \rightarrow \Delta(2)$ such that:

(1) either E is empty, or

$$\|E\| \leq 3 - 3\chi_{\bar{\Delta}} \leq 6n - 3;$$

(2) for every $x \in A \setminus E$, the face $h(x)$ is incident to x ;

(3) for every face Π , the number of arcs mapped to Π by h is at most $3 + \kappa_\Delta(\Pi) + \kappa'_\Delta(\Pi) \leq 8 \text{rk}(\Pi) + 11$.

Let M_1 be the sum of the lengths of all non-exceptional elements of $A \setminus E$, and M_2 be the sum of the lengths of all non-exceptional elements of E . Then $M_1 + M_2 = M$, and, by the condition $D_2(\mu_\Delta)$,

$$M_1 \leq \sum_{\Pi \in \Delta(2)} (8 \operatorname{rk}(\Pi) + 11) \mu_\Delta(\Pi) |\partial \Pi|,$$

while M_2 is less than or equal to $6n - 3$ times the maximal length of a non-exceptional maximal selected arc.

Apply Estimating Lemma 54 to Δ . Let F be a set of exceptional arcs of Δ such that:

- (1) either F is empty, or $\|F\| \leq -\chi_\Delta \leq 2n - 1$, and
- (2) for every j , the number of exceptional arcs of rank j that are not in F is at most twice the number of faces of Δ of rank j .

Let N_1 be the sum of the lengths of all the exceptional arcs that are not elements of F , and N_2 be the sum of the lengths of all the elements of F . Then $N_1 + N_2 = N$, and, by the condition $D_3(\nu_\Delta)$,

$$N_1 \leq \sum_{\Pi \in \Delta(2)} 2\nu_\Delta(\Pi) |\partial \Pi|,$$

while N_2 is less than or equal to $2n - 1$ times the maximal length of an exceptional arc.

It is left to find suitable “global” estimates on the lengths of non-exceptional maximal selected arc, and on the lengths of exceptional ones.

Observe that the length of every arc of $\bar{\Delta}$ that is incident to Θ and not incident to any other faces is at most $|w| - 1 < (1/m)|\partial \Theta| \leq \mu_n |\partial \Theta|$. Indeed, the label of each of the oriented arcs associated with such an arc is a common subword of a power of w and of a power of w^{-1} (because $\bar{\Delta}$ is orientable). Any such word of length $|w|$ would be a cyclic shift of w and of w^{-1} in the same time, but in a free group a nontrivial element is not conjugate to its own inverse (if it was, it would commute with the square of the conjugating element, and hence would commute with the conjugating element itself).

Case 1: the rank of every face of Δ is less than n . It follows from conditions of Subsection 2.2 and from the inequality $|\partial \Theta| \geq |r_{n,1}|$, that the length of any non-exceptional maximal selected arc of $\bar{\Delta}$ cannot be greater than $\mu_n |\partial \Theta|$, the length of any exceptional arc of $\bar{\Delta}$ cannot be greater than $\nu_n |\partial \Theta|$. Therefore,

$$M_2 \leq (6n - 3) \mu_n |\partial \Theta| \quad \text{and} \quad N_2 \leq (2n - 1) \nu_n |\partial \Theta|.$$

Thus, by inequalities (1) and (6), obtain a contradiction:

$$\begin{aligned}
L + M + N &\leq \sum_{\Pi \in \Delta(2)} \lambda_{\Delta}(\Pi) |\partial \Pi| \\
&\quad + \sum_{\Pi \in \Delta(2)} (8 \operatorname{rk}(\Pi) + 11) \mu_{\Delta}(\Pi) |\partial \Pi| + (6n - 3) \mu_n |\partial \Theta| \\
&\quad + \sum_{\Pi \in \Delta(2)} 2\nu_{\Delta}(\Pi) |\partial \Pi| + (2n - 1) \nu_n |\partial \Theta| \\
&< \sum_{\Pi \in \Delta(2)} \frac{1}{2} |\partial \Pi|.
\end{aligned}$$

Case 2: Δ has a face of rank at least n . Let $\hat{\Pi}$ be a face of Δ of maximal rank, $\operatorname{rk}(\hat{\Pi}) \geq n$. By the condition $D'_2(\mu_{\Delta})$ and by inequality $|w| \leq \mu_{\Delta}(\hat{\Pi}) |\partial \hat{\Pi}|$, the length of every non-exceptional maximal selected arc of $\bar{\Delta}$ is at most $\mu_{\Delta}(\hat{\Pi}) |\partial \hat{\Pi}|$. By the condition $D'_3(\nu_{\Delta})$, the length of every exceptional arc of $\bar{\Delta}$ is at most $\nu_{\Delta}(\hat{\Pi}) |\partial \hat{\Pi}|$. By inequality (6), obtain a contradiction:

$$\begin{aligned}
L + M + N &\leq \sum_{\Pi \in \Delta(2)} \lambda_{\Delta}(\Pi) |\partial \Pi| \\
&\quad + \sum_{\Pi \in \Delta(2)} (8 \operatorname{rk}(\Pi) + 11) \mu_{\Delta}(\Pi) |\partial \Pi| + (6n - 3) \mu_{\Delta}(\hat{\Pi}) |\partial \hat{\Pi}| \\
&\quad + \sum_{\Pi \in \Delta(2)} 2\nu_{\Delta}(\Pi) |\partial \Pi| + (2n - 1) \nu_{\Delta}(\hat{\Pi}) |\partial \hat{\Pi}| \\
&\leq \sum_{\Pi \in \Delta(2)} (\lambda_{\Delta}(\Pi) + (14 \operatorname{rk}(\Pi) + 8) \mu_{\Delta}(\Pi) \\
&\quad + (2 \operatorname{rk}(\Pi) + 1) \nu_{\Delta}(\Pi)) |\partial \Pi| \\
&< \sum_{\Pi \in \Delta(2)} \frac{1}{2} |\partial \Pi|.
\end{aligned}$$

□

It remains to show that the word and conjugacy problems in the constructed groups are decidable. Proving this fact could be facilitated by imposing additional restrictions on the constructed presentations, but this is not necessary.

Observe that inequality (1) implies that for every $n \in \mathbb{N}$,

$$\lambda_n + (8n + 11) \mu_n + 2\nu_n < \frac{19}{44} < 0.45.$$

The following lemma is helpful for solving the word and conjugacy problems in the constructed groups.

Lemma 79. *Let $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$ be a subpresentation of one of the presentations $\langle \mathfrak{A} \parallel \mathcal{R}_n \rangle$, $n \in \mathbb{N}$, or $\langle \mathfrak{A} \parallel \mathcal{R}_{\infty} \rangle$. Let K be the group presented by $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$. Then*

- (1) if w is a nontrivial group word over \mathfrak{A} , and Δ is a minimal by the number of faces disc diagram over $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$ such that $\ell(\partial\Delta) = w$, then

$$|w| > \frac{1}{10} \sum_{\Pi \in \Delta(2)} |\partial\Pi|$$

(in particular, K is hyperbolic if \mathcal{S} is finite);

- (2) if w_1 and w_2 are group words over \mathfrak{A} such that $[w_1]_K \neq 1_K$, and Δ is a minimal by the number of faces contour-oriented annular diagram over $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$ such that $\ell(\partial_1\Delta) = w_1$ and $\ell(\partial_2\Delta)^{-1} = w_2$, then

$$|w_1| + |w_2| > \frac{1}{10} \sum_{\Pi \in \Delta(2)} |\partial\Pi|.$$

Proof. First, let w be a group word over \mathfrak{A} , and Δ be a minimal by the number of faces disc diagram over $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$ such that $\ell(\partial\Delta) = w$. Because of minimality, Δ is reduced. After cyclically shifting, if necessary, the c-contours of some of the faces of Δ , endow Δ with a structure of a restricted correct S -diagram of appropriate sort. Transform Δ into a special S -diagram Γ by diamond moves. The connected component of Γ containing $\partial\Gamma$ is a disc diagram. By the minimality of Δ , this implies that Γ is connected. The S -diagram Γ satisfies the conditions Υ and $D(\lambda_\Gamma, \mu_\Gamma, \nu_\Gamma)$. Let

$$\gamma = \lambda_\Gamma + (3 + \kappa_\Gamma + \kappa'_\Gamma)\mu_\Gamma + 2\nu_\Gamma.$$

By inequalities (5) and (6), $\gamma(\Pi) < 0.45$ for every $\Pi \in \Gamma$. By Lemma 59,

$$|w| = |\partial\Gamma| \geq \sum_{\Pi \in \Gamma(2)} (1 - 2\gamma(\Pi))|\partial\Pi| \geq \frac{1}{10} \sum_{\Pi \in \Gamma(2)} |\partial\Pi| = \frac{1}{10} \sum_{\Pi \in \Delta(2)} |\partial\Pi|,$$

and the equality in the both inequalities simultaneously is not possible because $|w| > 0$.

Second, let w_1 and w_2 be group words over \mathfrak{A} such that $[w_1]_G \neq 1_G$, and Δ be a minimal by the number of faces contour-oriented restricted correct annular S -diagram of appropriate sort over $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$ such that $\ell(\partial_1\Delta) = w_1$ and $\ell(\partial_2\Delta)^{-1} = w_2$. Because of minimality, Δ is reduced. Transform Δ into a special S -diagram Γ by diamond moves. The connected component of Γ that contains $\partial_1\Gamma$ is either disc or annular, and is contour-oriented. By the minimality of Δ , and because $[w_1]_G \neq 1_G$, this implies that Γ is connected. The S -diagram Γ satisfies the conditions Υ and $D(\lambda_\Gamma, \mu_\Gamma, \nu_\Gamma)$. Let γ be as above. Then $\gamma(\Pi) < 0.45$ for every $\Pi \in \Gamma$. By Lemma 59,

$$|w_1| + |w_2| = |\partial_1\Gamma| + |\partial_2\Gamma| \geq \sum_{\Pi \in \Gamma(2)} (1 - 2\gamma(\Pi))|\partial\Pi| \geq \frac{1}{10} \sum_{\Pi \in \Delta(2)} |\partial\Pi|,$$

and the equality in the both inequalities is not possible simultaneously. \square

Proposition 80. *The groups G_n , $n \in \mathbb{N}$, and G_∞ constructed in Section 2 have decidable word and conjugacy problems.*

Proof. Here follows a proof of decidability of the word and conjugacy problems for the group G_∞ . For the groups G_n , $n \in \mathbb{N}$, there is a completely analogous proof which therefore shall not be given here.

For every $i \in \mathbb{N}$, let w_i , $r_{i,1}$ and $r_{i,2}$ be the same w_i , $r_{i,1}$ and $r_{i,2}$ as in Subsection 2.2.

It is clear that the sequence $r_{1,1}, r_{1,2}, r_{2,1}, r_{2,2}, r_{3,1}, \dots$ is recursive, and the sequence $|r_{1,1}|, |r_{1,2}|, |r_{2,1}|, |r_{2,2}|, |r_{3,1}|, \dots$ is bounded from below by an increasing recursive sequence (of rational numbers) tending to $+\infty$.

Consider an arbitrary subset \mathcal{S} of \mathcal{R}_∞ . Let K be the group presented by the (sub)presentation $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$. It follows from Lemmas 42 and 79 that:

- (1) for every group word x over \mathfrak{A} , $[x]_K = 1_K$ (if and) only if there exists a disc diagram Δ over $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$ such that $\ell(\partial\Delta) = x$ and $\|\Delta(1)\| \leq 6|x|$ (because $(10+1)/2 < 6$);
- (2) for every group words x and y over \mathfrak{A} such that $[x]_K \neq 1_K$, $[x]_K$ and $[y]_K$ are conjugate in K (if and) only if there exists a contour-oriented annular diagram Δ over $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$ such that $\ell(\partial_1\Delta) = x$, $\ell(\partial_2\Delta)^{-1} = y$, and $\|\Delta(1)\| < 6(|x| + |y|)$.

The following algorithm decides the conjugacy problem for $\langle \mathfrak{A} \parallel \mathcal{R}_\infty \rangle$: Let x and y be arbitrary group words over \mathfrak{A} given as an input. Using the (effective) lower bound on $|r_{i,j}|$, find a k such that for every $r \in \mathcal{R}_\infty$, if $(1/10)|r| \leq |x| + |y|$, then $r \in \mathcal{R}_\infty^{(k)}$. (Then, by Lemmas 42 and 79, x and y represent conjugate elements of G_∞ if and only if they represent conjugate elements of $\langle \mathfrak{A} \parallel \mathcal{R}_\infty^{(k)} \rangle$.) Determine the (finite) set $\mathcal{R}_\infty^{(k)}$ by finding all the sets $\mathcal{R}_\infty^{(1)}, \mathcal{R}_\infty^{(2)}, \dots, \mathcal{R}_\infty^{(k)}$ one-by-one in this order. Do so in k steps. On the step number i , the set $\mathcal{R}_\infty^{(i-1)}$ is already determined. To determine $\mathcal{R}_\infty^{(i)}$, decide first whether $[w_i]_{\mathcal{R}_\infty^{(i-1)}} = 1$ by checking if there exists a disc diagrams Δ over $\langle \mathfrak{A} \parallel \mathcal{R}_\infty^{(i-1)} \rangle$ with at most $6|w_i|$ edges and with the contour label w_i . If $[w_i]_{\mathcal{R}_\infty^{(i-1)}} = 1$, then $\mathcal{R}_\infty^{(i)} = \mathcal{R}_\infty^{(i-1)}$, otherwise $\mathcal{R}_\infty^{(i)} = \mathcal{R}_\infty^{(i-1)} \cup \{r_{i,1}, r_{i,2}\}$. After the set $\mathcal{R}_\infty^{(k)}$ is found, decide whether $[x]_{G_\infty} = 1_{G_\infty}$. Do so by checking if there exists a disc diagram Δ over $\langle \mathfrak{A} \parallel \mathcal{R}_\infty^{(k)} \rangle$ with at most $6|x|$ edges and with the contour label x . If found that $[x]_{G_\infty} = 1_{G_\infty}$, then similarly decide whether $[y]_{G_\infty} = 1_{G_\infty}$. In the case $[x]_{G_\infty} = 1_{G_\infty} = [y]_{G_\infty}$, the elements $[x]_{G_\infty}$ and $[y]_{G_\infty}$ are conjugate in G_∞ , and in the case $[x]_{G_\infty} = 1_{G_\infty} \neq [y]_{G_\infty}$, they are not. If found that $[x]_{G_\infty} \neq 1_{G_\infty}$, decide whether $[x]_{G_\infty}$ and $[y]_{G_\infty}$ are conjugate in G_∞ by checking whether there exists a contour-oriented annular diagram Δ over $\langle \mathfrak{A} \parallel \mathcal{R}_\infty^{(k)} \rangle$ with less than $6(|x| + |y|)$ edges and with the contour labels x and y^{-1} .

Thus the group G_∞ has decidable word and conjugacy problems. \square

Theorems 4 and 5 are direct corollaries of Propositions 7, 8, 77, 78, 80.

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