Irreducible Coxeter groups

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Abstract

We prove that a non-spherical irreducible Coxeter group is (directly) indecomposable and that a non-spherical and non-affine Coxeter group is strongly indecomposable in the sense that all its finite index subgroups are (directly) indecomposable.

Let W be a Coxeter group. Write $W = W_{X_1} \times \cdots \times W_{X_b} \times W_{Z_3}$, where $W_{X_1}, \dots, W_{X_b}$ are non-spherical irreducible Coxeter groups and W_{Z_3} is a finite one. By a classical result, known as the Krull-Remak-Schmidt theorem, the group W_{Z_3} has a decomposition $W_{Z_3} = H_1 \times \cdots \times H_q$ as a direct product of indecomposable groups, which is unique up to a central automorphism and a permutation of the factors. Now, $W = W_{X_1} \times \cdots \times W_{X_b} \times H_1 \times \cdots \times H_q$ is a decomposition of W as a direct product of indecomposable subgroups. We prove that such a decomposition is unique up to a central automorphism and a permutation of the factors.

Write $W = W_{X_1} \times \cdots \times W_{X_a} \times W_{Z_2} \times W_{Z_3}$, where $W_{X_1}, \dots, W_{X_a}$ are non-spherical and non-affine Coxeter groups, W_{Z_2} is an affine Coxeter group whose irreducible components are all infinite, and W_{Z_3} is a finite Coxeter group. The group W_{Z_2} contains a finite index subgroup R isomorphic to $\mathbb{Z}^d$, where $d = |Z_2| - b + a$ and $b - a$ is the number of irreducible components of W_{Z_2}. Choose d copies $R_1, \dots, R_d$ of $\mathbb{Z}$ such that $R = R_1 \times \cdots \times R_d$. Then $G = W_{X_1} \times \cdots \times W_{X_a} \times R_1 \times \cdots \times R_d$ is a virtual decomposition of W as a direct product of strongly indecomposable subgroups. We prove that such a virtual decomposition is unique up to commensurability and a permutation of the factors.

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1 Introduction

Let S be a finite set. A *Coxeter matrix* over S is a square matrix $M = (m_{s\,t})_{s,t \in S}$ indexed by the elements of S and such that $m_{s\,s} = 1$ for all $s \in S$ and $m_{s\,t} = m_{t\,s} \in \{2, 3, 4, \dots, +\infty\}$ for all $s, t \in S$, $s \neq t$. A Coxeter matrix $M = (m_{s\,t})_{s,t \in S}$ is usually represented by its *Coxeter graph*, Γ, which is defined by the following data. The set of vertices of Γ is S, two vertices $s, t \in S$ are joined by an edge if $m_{s\,t} \geq 3$, and this edge is labeled by $m_{s\,t}$ if $m_{s\,t} \geq 4$.

Let Γ be a Coxeter graph with set of vertices S. Define the *Coxeter system of type* Γ to be the pair (W, S), where $W = W_\Gamma$ is the group generated by S and subject to the relations

$$\begin{gathered} s^2 = 1 \quad \text{for } s \in S\,, \\ (st)^{m_{s\,t}} = 1 \quad \text{for } s,t \in S,\ s \neq t,\ \text{and } m_{s\,t} < +\infty\,, \end{gathered}$$

where $M = (m_{s\,t})_{s,t \in S}$ is the Coxeter matrix of Γ. The group $W = W_\Gamma$ is simply called the *Coxeter group of type* Γ.

A Coxeter system (W, S) is called *irreducible* if its defining Coxeter graph is connected. The number n of elements of S is called the *rank* of the Coxeter system.

The term "Coxeter" in the above definition certainly comes from the famous Coxeter's theorem [8], [9] which states that the finite (real) reflection groups are precisely the finite Coxeter groups. However, the Coxeter groups were first introduced by Tits in an unpublished manuscript [24] whose results appeared in the seminal Bourbaki's book [2]. The Coxeter groups have been widely studied, they have many attractive properties, and they form an important source of examples for group theorists. Basic references for them are [2] and [16].

The notion of irreducibility is of importance in the theory as all the Coxeter groups can be naturally decomposed as direct products of irreducible ones. Nevertheless, I do not know any work which studies the irreducibility itself. This is the object of the present paper.

Our analysis needs to differentiate three classes of irreducible systems: the spherical ones which correspond to the finite Coxeter groups, the affine ones, and the remainder that we shall simply call non-spherical and non-affine Coxeter systems. These classes are defined as follows.

Let Γ be a Coxeter graph and let (W, S) be the Coxeter system of type Γ. Take an abstract set $\Pi = \{\alpha_s ; s \in S\}$ in one-to-one correspondence with S and call the elements of Π *simple roots*. Let $V = \oplus_{s \in S} \mathbb{R} \alpha_s$ be the real vector space having Π as a basis. Define the *canonical form* to be the symmetric bilinear form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ determined by

$$\langle \alpha_s, \alpha_t \rangle = \begin{cases} -\cos(\pi / m_{s\,t}) & \text{if } m_{s\,t} < +\infty\,, \\ -1 & \text{if } m_{s\,t} = +\infty\,, \end{cases}$$

for all $s, t \in S$. There is a faithful linear representation $W \to GL(V)$, called the *canonical representation*, which leaves invariant the canonical form, and which is defined by $s(x) = x - 2\langle x, \alpha_s \rangle \alpha_s$, for $x \in V$ and $s \in S$.

A Coxeter graph Γ is of *spherical type* if the canonical form of Γ is positive definite. As pointed out before, it has been prove by Coxeter [8], [9] that Γ is of spherical type if and only if the associated Coxeter group W_Γ is finite. The connected spherical type Coxeter graphs are precisely the Coxeter graphs A_n $(n \geq 1)$, B_n $(n \geq 2)$, D_n $(n \geq 4)$, E_6, E_7, E_8, F_4, G_2, H_3, H_4, $I_2(p)$ $(p \geq 5$ and $p \neq 6)$ pictured in [2], p. 193.

A Coxeter graph Γ is of *affine type* if the canonical form of Γ is positive semidefinite but not positive definite. Any affine Coxeter group W_Γ can be presented as a semidirect product $R \rtimes W_0$, where R is a finitely generated free abelian group of rank ≥ 1 and W_0 is a finite Coxeter group. If, moreover, Γ is connected, then the rank of R is precisely $n-1$, where $n = |S|$ is the rank of (W, S). Conversely, if a Coxeter group W_Γ has a finite index subgroup isomorphic to $\mathbb{Z}^d$ $(d \geq 1)$, then Γ is of affine type (see [12]). The connected affine type Coxeter graphs are precisely the Coxeter graphs $\tilde{A}_n$ $(n \geq 1)$, $\tilde{B}_n$ $(n \geq 2)$, $\tilde{C}_n$ $(n \geq 3)$, $\tilde{D}_n$ $(n \geq 4)$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{F}_4$, $\tilde{G}_2$ pictured in [2], p. 199.

Let G be a group. A subgroup H of G is called a *direct factor* of G if there exists a subgroup K of G such that $G = H \times K$. If there are no proper non-trivial direct factors of G, then G is said to be *indecomposable*. If G is infinite and all the finite index subgroups of G are indecomposable, then G is said to be *strongly indecomposable*.

Clearly, every simple group is indecomposable. Other examples include the infinite cyclic group, the cyclic group of order p^m, where p is a prime number, the free groups, and the Artin groups of spherical type [21]. Now, the infinite simple groups are also strongly indecomposable as well as $\mathbb{Z}$ and the torsion free hyperbolic groups. In this paper we prove that an irreducible non-spherical Coxeter group is indecomposable and that an irreducible non-spherical and non-affine Coxeter group is strongly indecomposable (see Theorem 4.1). (After the first version of the present paper, the first part of Theorem 4.1 has been extended to the infinitely generated Coxeter groups by Nuida [20], and a different proof of Theorem 4.1 has been proposed by de Cornulier and de la Harpe [7].)

Not all the irreducible spherical type Coxeter groups are indecomposable. The reader certainly knows the example of the dihedral group of order $8q+4$ which is isomorphic to the direct product of $C_2 = \{\pm 1\}$ with the dihedral group of order $4q + 2$. However, there are few exceptions. In particular, one of the factors must be the center of the group. The decompositions as direct products of the finite irreducible Coxeter groups are known to experts, but I do not know any reference where they are listed. So, we include the list in the last section of the paper.

An irreducible affine type Coxeter group W_Γ cannot be strongly indecomposable since it contains a finite index subgroup isomorphic to $\mathbb{Z}^{n-1}$, where n is the rank of W_Γ, except if $n = 2$. If $n = 2$, then W_Γ is the Coxeter group of type $\tilde{A}_1$ which is isomorphic to $C_2 * C_2$, where $C_2 = \{\pm 1\}$.

A *Remak decomposition* of a group G is a decomposition of G as a direct product of finitely many non-trivial indecomposable subgroups. An automorphism $\varphi : G \to G$ is said to be *central* if it is the identity modulo the center of G, that is, if $g^{-1}\varphi(g) \in Z(G)$ for all $g \in G$. We say that a group G satisfies the *maximal* (resp. *minimal*) *condition on normal subgroups* if each non-empty family of normal subgroups contains at least one maximal (resp. minimal) element for the inclusion. Obviously, finite groups satisfy both conditions. On the other hand, the Coxeter groups do not satisfy these conditions in general. For example, the Coxeter group $C_2 * C_2 * C_2$ does not satisfy neither the maximal condition, nor the minimal condition on normal subgroups. The proof of this last fact is left to the reader (but can be also found in [7]).

A classical result, known as the Krull-Remak-Schmidt theorem, states that, if a group G satisfies either the maximal or the minimal condition on normal subgroups, then it has a Remak decomposition. Furthermore, if it satisfies both, the maximal and the minimal conditions on normal subgroups, then this Remak decomposition is unique up to a central automorphism and a permutation of the factors. Of course, there are groups which have no Remak decompositions, and there are groups which have two different Remak decompositions whose factors are not isomorphic up to a permutation. We refer to [7] for a detailed account on this question. Our aim here is to show that a Coxeter group has a Remak decomposition which is unique up to a central automorphism (see Theorem 5.2). This Remak decomposition is quite natural and can be described as follows.

Let Γ be a Coxeter graph and let (W, S) be the Coxeter system of type Γ. For $X \subset S$, we denote by Γ_X the full subgraph of Γ generated by X and by W_X the subgroup of W generated by X. By [2], Chap. 4, § 8, (W_X, X) is the Coxeter system of type Γ_X. A subgroup of the form W_X is called a *standard parabolic subgroup*, and a subgroup conjugated to a standard parabolic subgroup is simply called a *parabolic subgroup*.

Let $\Gamma_1, \dots, \Gamma_l$ be the connected components of Γ and, for $1 \le i \le l$, let X_i be the set of vertices of Γ_i. Then $W = W_{X_1} \times \cdots \times W_{X_l}$ and each (W_{X_i}, X_i) is an irreducible Coxeter system. The above equality is called the *standard decomposition of W into irreducible components*, and each W_{X_i} is called an *irreducible component* of W. Up to a permutation of the Γ_i's, we can assume that, for some $0 \le a \le b \le l$, the Coxeter graphs $\Gamma_1, \dots, \Gamma_a$ are non-spherical and non-affine, the Coxeter graphs $\Gamma_{a+1}, \dots, \Gamma_b$ are of affine type, and the Coxeter graphs $\Gamma_{b+1}, \dots, \Gamma_l$ are of spherical type. Set $Z_1 = X_1 \cup \cdots \cup X_a$, $Z_2 = X_{a+1} \cup \cdots \cup X_b$, and $Z_3 = X_{b+1} \cup \cdots \cup X_l$. Then $W = W_{Z_1} \times W_{Z_2} \times W_{Z_3}$, W_{Z_1} is a Coxeter group whose irreducible components are all non-spherical and non-affine, W_{Z_2} is an affine Coxeter group, and W_{Z_3} is a finite Coxeter group. The subgroup W_{Z_1} is called the *non-spherical and non-affine part* of W, W_{Z_2} is called the *affine part*, and W_{Z_3} is called the *finite part* of W.

By the above mentioned Krull-Remak-Schmidt theorem, the finite part has a Remak decomposition $W_{Z_3} = H_1 \times \cdots \times H_q$ which is unique up to a central automorphism. Then

$$W = W_{X_1} \times \cdots \times W_{X_b} \times H_1 \times \cdots \times H_q$$

is a Remak decomposition of W.

Let G be a group. Define a *virtual strong Remak decomposition* of G to be a finite index subgroup H provided with a decomposition $H = H_1 \times \cdots \times H_m$ as a direct product of finitely many (infinite) strongly indecomposable subgroups. Recall that two groups G_1 and G_2 are *commensurable* if there is a finite index subgroup of G_1 isomorphic to a finite index subgroup of G_2. Two virtual strong Remak decompositions $H = H_1 \times \cdots \times H_p$ and $H' = H'_1 \times \cdots \times H'_q$ of G are called *equivalent* if $p = q$ and, up to a permutation of the H_i's, H_i and H'_i are commensurable for all $1 \le i \le p$.

Obviously, a strong indecomposable group as well as a finitely generated abelian group has a unique virtual strong Remak decomposition up to equivalence. I do not know much more about virtual strong Remak decompositions, but I believe that this question should be of interest in the study of groups up to commensurability. We prove in this paper (Theorem 6.1) that a Coxeter group has a unique virtual strong Remak decomposition up to equivalence. Here again, the decomposition is quite natural and can be described as follows.

Let Γ be a Coxeter graph and let (W, S) be the Coxeter system of type Γ. Write $W = W_{X_1} \times \cdots \times W_{X_a} \times W_{Z_2} \times W_{Z_3}$, where $W_{X_1}, \dots, W_{X_a}$ are the non-spherical and non-affine irreducible components of W, W_{Z_2} is the affine part, and W_{Z_3} is the finite part. Let $d = |Z_2| - b + a$, where $b - a$ is the number of affine irreducible components of (W, S). By [2], Chap. 6, § 2, W_{Z_2} contains a finite index subgroup R isomorphic to $\mathbb{Z}^d$. Let $R_1, \dots, R_d$ be d copies of $\mathbb{Z}$ such that $R = R_1 \times \cdots \times R_d$. Then

$$G = W_{X_1} \times \cdots \times W_{X_a} \times R_1 \times \cdots \times R_d$$

is a virtual strong Remak decomposition of W. The number d will be called the *affine dimension* of W.

An element $w \in W$ is called *essential* if it does not lie in any proper parabolic subgroup. Part of the proofs of the paper are based on Krammer's study on essential elements (see [17]). I am sure that other proofs with other (more or less simple) techniques can be easily found by experts. However, another goal of the present paper is to present this piece of Krammer's Ph. D. Thesis which deserves to be known. In particular, it is shown in [17] that, in a non-spherical and non-affine Coxeter group W,

- every essential element is of infinite order,
- an element w is essential if and only if any non-trivial power w^m is essential,
- the centralizer in W of an essential element w contains $\langle w \rangle = \{w^m; m \in \mathbb{Z}\}$ as a finite index subgroup.

These Krammer's results are explained in Section 2.

Choose a linear ordering $S = \{s_1, s_2, \dots, s_n\}$ and define the *Coxeter element* of W (with respect to this linear ordering) to be $c = s_n \dots s_2 s_1$. In order to apply the above mentioned Krammer's results, we need at some point to choose an essential element. In particular, we first have to show that such an element exists. A natural approach to this question is to prove that the Coxeter elements are essential. Curiously, this result is unknown (for instance, it is a question in [18]). We give a simple proof of this fact in Section 3.

2 Essential elements

Let Γ be a Coxeter graph and let (W, S) be the Coxeter system of type Γ. Recall from the introduction the set $\Pi = \{\alpha_s; s \in S\}$ of simple roots, the real vector space $V = \oplus_{s \in S} \mathbb{R}\alpha_s$, the canonical form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$, and the canonical representation $W \to GL(V)$. The set $\Phi = \{w\alpha_s; w \in W \text{ and } s \in S\}$ is called the *root system* of (W, S). The elements of $\Phi^+ = \{\beta = \sum_{s \in S} \lambda_s \alpha_s \in \Phi; \lambda_s \geq 0 \text{ for all } s \in S\}$ are the *positive roots*, and the elements of $\Phi^- = -\Phi^+$ are the *negative roots*. For $w \in W$, we set $\Phi_w = \{\beta \in \Phi^+; w\beta \in \Phi^-\}$. The following proposition collects some classical results (see for example Chapter 5 of [16]).

Proposition 2.1.

1. *We have the disjoint union* $\Phi = \Phi^+ \sqcup \Phi^-$.
2. *Let* $\mathrm{lg} : W \to \mathbb{N}$ *denote the word length with respect to* S. *Let* $w \in W$. *Then* $\mathrm{lg}(w) = |\Phi_w|$. *In particular,* Φ_w *is finite.*
3. *Let* $w \in W$ *and* $s \in S$. *Set* $\beta = w\alpha_s$ *and* $r_\beta = wsw^{-1}$. *Then* r_β *acts on* V *by* $r_\beta(x) = x - 2\langle x, \beta \rangle \beta$, *for all* $x \in V$.

Let $u, v \in W$ and $\alpha \in \Phi$. We say that α *separates* u and v if there is some $\varepsilon \in \{\pm 1\}$ such that $u\alpha \in \Phi^\varepsilon$ and $v\alpha \in \Phi^{-\varepsilon}$. Let $w \in W$ and $\alpha \in \Phi$. We say that α is *w-periodic* if there is some $m \geq 1$ such that $w^m \alpha = \alpha$.

Proposition 2.2. *Let* $w \in W$ *and* $\alpha \in \Phi$. *Then exactly one of the following holds.*

1. *α is w-periodic.*

2. *α is not w-periodic, and the set $\{m \in \mathbb{Z}; \alpha$ separates w^m and $w^{m+1}\}$ is finite and even.*

3. *α is not w-periodic, and the set $\{m \in \mathbb{Z}; \alpha$ separates w^m and $w^{m+1}\}$ is finite and odd.*

We say that α is *w-even* is Case 2, and *w-odd* in Case 3.

Proof. Assume that α is not w-periodic. Set $N(\alpha) = \{m \in \mathbb{Z}; \alpha \text{ separates } w^m \text{ and } w^{m+1}\}$. Clearly, we only need to show that $N(\alpha)$ is finite.

Observe that, if $m \in N(\alpha)$, then $w^m \in \Phi_w \cup -\Phi_w$. On the other hand, if $w^{m_1}\alpha = w^{m_2}\alpha$, then $w^{m_1-m_2}\alpha = \alpha$, thus $m_1 = m_2$, since α is not w-periodic. The set $\Phi_w \cup -\Phi_w$ is finite (see Proposition 2.1), thus $N(\alpha)$ is finite, too. $\square$

Proposition 2.3. *Let $\alpha \in \Phi$, $w \in W$, and $p \in \mathbb{N}$, $p \geq 1$.*

- *α is w-periodic if and only if α is w^p-periodic.*
- *α is w-even if and only if α is w^p-even.*
- *α is w-odd if and only if α is w^p-odd.*

Proof. The equivalence

$$\alpha \text{ is } w\text{-periodic} \quad \Leftrightarrow \quad \alpha \text{ is } w^p\text{-periodic}$$

is obvious. So, we can assume that α is not w-periodic and we show the other equivalences.

Observe that, if α is w-even, then there are $m_0 > 0$ and $\varepsilon \in \{\pm 1\}$ such that $w^m\alpha, w^{-m}\alpha \in \Phi^\varepsilon$ for all $m \geq m_0$. On the other hand, if α is w-odd, then there are $m_0 > 0$ and $\varepsilon \in \{\pm 1\}$ such that $w^m\alpha \in \Phi^\varepsilon$ and $w^{-m}\alpha \in \Phi^{-\varepsilon}$ for all $m \geq m_0$. It is easily checked that these criteria for a non-periodic root to be even or odd imply the last two equivalences. $\square$

Recall that an *essential element* is an element $w \in W$ which does not lie in any proper parabolic subgroup of W. Now, we state the first Krammer's result in which we are interested (see [17], Corollary 5.8.7).

Theorem 2.4 (Krammer [17]). *Assume that Γ is connected and non-spherical. Let $w \in W$. Then w is essential if and only if W is generated by the set $\{r_\beta; \beta \in \Phi^+$ and β w-odd$\}$.*

Remark. It will be shown in Section 3 that the Coxeter elements are essential, but the proof will not use Theorem 2.4. Actually, I do not know how to use Theorem 2.4 in a simple way for this purpose.

A direct consequence of Proposition 2.3 and Theorem 2.4 is the following.

Corollary 2.5. *Assume that Γ is connected and non-spherical. Let $w \in W$ and $p \in \mathbb{N}$, $p \geq 1$. Then w is essential if and only if w^p is essential.*

Now, the following result is one of the main tools in the present paper. It can be found in [17], Corollary 6.3.10.

Theorem 2.6 (Krammer [17]). *Assume that Γ is connected, non-spherical, and non-affine. Let $w \in W$ be an essential element. Then $\langle w \rangle = \{w^m; m \in \mathbb{Z}\}$ is a finite index subgroup of the centralizer of w in W.*

3 Coxeter elements

Theorem 3.1. *Let Γ be a Coxeter graph and let (W, S) be the Coxeter system of type Γ. Then any Coxeter element $c = s_n \dots s_2 s_1$ of W is essential.*

The following lemma is a preliminary result to the proof of Theorem 3.1.

Lemma 3.2. *Let Γ be a Coxeter graph with set of vertices S. Then there exist a Coxeter graph $\tilde{\Gamma}$ with set of vertices $\tilde{S}$ and an embedding $S \hookrightarrow \tilde{S}$ such that $\Gamma = \tilde{\Gamma}_S$ and the canonical form $\langle \cdot, \cdot \rangle_{\tilde{\Gamma}}$ of $\tilde{\Gamma}$ is non-degenerate.*

Proof. We denote by $B_\Gamma = (b_{s\,t})_{s,t \in S}$ the Gram matrix of the canonical form $\langle \cdot, \cdot \rangle_\Gamma$ in the basis $\Pi = \{\alpha_s; s \in S\}$. So, $b_{s\,t} = -\cos(\pi/m_{s\,t})$ if $m_{s\,t} < +\infty$, and $b_{s\,t} = -1$ if $m_{s\,t} = +\infty$. For $X \subset S$, we shall simply write B_X for B_{Γ_X}.

We can and do choose a non-empty subset $X_0 \subset S$ such that $\det B_{X_0} \neq 0$, and such that $\det B_X = 0$ whenever $|X| > |X_0|$ for $X \subset S$. Set $X_1 = S \setminus X_0$, take an abstract set $\tilde{X}_1 = \{\tilde{s}; s \in X_1\}$ in one-to-one correspondence with X_1, and set $\tilde{S} = X_0 \sqcup X_1 \sqcup \tilde{X}_1$. Define $\tilde{\Gamma}$ adding to Γ an edge labeled by $+\infty$ between s and $\tilde{s}$ for all $s \in X_1$. We emphasize that there is no other edge in $\tilde{\Gamma}$ apart the ones of Γ, and the edges labeled by $+\infty$ between the elements of X_1 and they corresponding elements of $\tilde{X}_1$. Now, the proof of the following equality is left to the reader.

$$\det B_{\tilde{\Gamma}} = (-1)^{|X_1|} \det B_{X_0} \neq 0\,.$$

$\square$

Proof of Theorem 3.1. We suppose that $c = s_n \dots s_2 s_1$ is not essential. So, there exist $X \subset S$, $X \neq S$, and $u \in W$, such that $c \in u W_X u^{-1}$.

Our proof uses the observation that, if $w \in W$ is not essential, then there exists a non-zero vector $x \in V \setminus \{0\}$ such that $w(x) = x$. The converse is false, especially when the canonical form $\langle \cdot, \cdot \rangle_\Gamma$ is degenerate. Indeed, all the vectors of the radical of $\langle \cdot, \cdot \rangle_\Gamma$ are fixed by all the elements of W. In order to palliate this difficulty, we use the simple trick to embed W into a larger Coxeter group with a non-degenerate canonical form.

So, by Lemma 3.2, there exist a Coxeter graph $\tilde{\Gamma}$ with set of vertices $\tilde{S}$ and an embedding $S \hookrightarrow \tilde{S}$ such that $\tilde{\Gamma}_S = \Gamma$ and the canonical form $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\tilde{\Gamma}}$ of $\tilde{\Gamma}$ is non-degenerate. Let $\{s_{n+1}, \dots, s_m\} = \tilde{S} \setminus S$, let $\tilde{W}$ be the Coxeter group of type $\tilde{\Gamma}$, and let $\tilde{V} = \oplus_{i=1}^m \mathbb{R}\alpha_{s_i}$.

We write $\beta_i = u^{-1}(\alpha_{s_i})$ and $r_i = u^{-1} s_i u = r_{\beta_i}$ for $1 \leq i \leq n$, and we write $\beta_i = \alpha_{s_i}$ for $n+1 \leq i \leq m$. Note that $\{\beta_1, \dots, \beta_n, \beta_{n+1}, \dots, \beta_m\}$ is a basis of $\tilde{V}$. Let $d = s_m \dots s_{n+1} r_n \dots r_2 r_1$.

Observe that $d = s_m \dots s_{n+1}(u^{-1}cu)$, thus $d \in \tilde{W}_{X \sqcup (\tilde{S} \setminus S)}$. Moreover, $\tilde{W}_{X \sqcup (\tilde{S} \setminus S)}$ is a proper (standard) parabolic subgroup, hence there exists $x \in \tilde{V} \setminus \{0\}$ such that $d(x) = x$. It is easily checked that there exist numbers $\lambda_{i\,j} = \lambda_{i\,j}(x) \in \mathbb{R}$, $1 \le i < j \le m$, such that

$$d(x) = x - \sum_{j=1}^{m} \left(2\langle x, \beta_j \rangle + \sum_{i=1}^{j-1} \lambda_{i\,j} \langle x, \beta_i \rangle \right) \beta_j \,.$$

Now, the equality $d(x) = x$ implies

$$\sum_{j=1}^{m} \left(2\langle x, \beta_j \rangle + \sum_{i=1}^{j-1} \lambda_{i\,j} \langle x, \beta_i \rangle \right) \beta_j = 0 \,,$$

thus, since $\{\beta_1, \dots, \beta_m\}$ is a basis of $\tilde{V}$,

$$\langle x, \beta_1 \rangle = \cdots = \langle x, \beta_m \rangle = 0 \,.$$

This contradicts the fact that $x \neq 0$, $\{\beta_1, \dots, \beta_m\}$ is a basis, and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\tilde{\Gamma}}$ is non-degenerate. $\square$

4 Indecomposability

Let Γ be a Coxeter graph and let (W, S) be the Coxeter system of type Γ. The goal of the present section is to prove the following.

Theorem 4.1.

1. *Assume that Γ is connected and non-spherical. Then W is indecomposable.*
2. *Assume that Γ is connected, non-spherical, and non-affine. Then W is strongly indecomposable.*

We start collecting some well-known results on Coxeter groups. For $X \subset S$, we set $\Pi_X = \{\alpha_s; s \in X\}$ and $G_X = \{w \in W; w\Pi_X = \Pi_X\}$. The set $T = \{wsw^{-1}; s \in S \text{ and } w \in W\}$ is called the set of *reflections* of W. Any subgroup of W generated by reflections is called a *reflection subgroup* of W.

For a group G and a subgroup H of G we denote by $N_G(H)$ the normalizer of H in G and by $Z_G(H)$ the centralizer of H in G. The center of G is denoted by $Z(G)$.

Proposition 4.2.

1. (Bourbaki [2], Chap. 5, § 4, Exercice 2). *Any finite subgroup of W is contained in a finite parabolic subgroup.*
2. (Solomon [23]). *The intersection of a family of parabolic subgroups is a parabolic subgroup.*
3. (Deodhar [10]). *If $X \subset S$, then $N_W(W_X) = W_X \rtimes G_X$.*

4. (Deodhar [10]). *Let* $T = \{wsw^{-1}; s \in S$ *and* $w \in W\}$ *be the set of reflections. Then* W *is finite if and only if* T *is finite.*

5. (Dyer [11]). *If* H *is a reflection subgroup of* W*, then there exists a subset* $Y \subset H \cap T$ *such that* (H, Y) *is a Coxeter system.*

Let (W, S) be a non-spherical and non-affine Coxeter system, let G be a finite index subgroup of W, and let A, B be two subgroups of G such that $G = A \times B$. The proof (of the second part) of Theorem 4.1 is divided into two steps. In the first one we show that either A or B is finite. The main tool in this step is Krammer's result stated in Theorem 2.6. The second step consists on showing that A is trivial if it is finite. This second step is based on the following result.

Proposition 4.3. *Assume that* Γ *is connected and non-spherical. Let* H *be a non-trivial finite subgroup of* W*. Then* $N_W(H)$ *has infinite index in* W*.*

Proof.

Assertion 1. *Let* $X \subset S$*,* $\emptyset \neq X \neq S$*. Then* W_X *is not normal in* W*.*

Let V_X denote the linear subspace of V spanned by Π_X. By Proposition 4.2, we have $w(V_X) = V_X$ for all $w \in N_W(W_X)$. Choose $s \in X$ and $t \in S \setminus X$ such that $m_{s\,t} \geq 3$. Then $\alpha_s \in V_X$ and $t(\alpha_s) = \alpha_s - 2\langle \alpha_s, \alpha_t \rangle \alpha_t \notin V_X$, thus $t \notin N_W(W_X)$.

Assertion 2. *Let* H *be a non-trivial finite subgroup of* W*. Then* H *is not normal.*

Let $\mathrm{Pc}(H)$ denote the intersection of all the parabolic subgroups that contain H. By Proposition 4.2, $\mathrm{Pc}(H)$ is a finite parabolic subgroup. Moreover, $\{1\} \neq \mathrm{Pc}(H)$ since it contains H, and $\mathrm{Pc}(H) \neq W$ since $\mathrm{Pc}(H)$ is finite while W is infinite. By Assertion 1, if follows that $N_W(\mathrm{Pc}(H)) \neq W$. Now, observe that $N_W(H) \subset N_W(\mathrm{Pc}(H))$, thus $N_W(H) \neq W$.

Assertion 3. *Recall that* $T = \{wsw^{-1}; s \in S$ *and* $w \in W\}$*. Let* $X \subset S$*, let* $T_X = \{wxw^{-1}; x \in X$ *and* $w \in W\}$*, and let* H *be the subgroup generated by* T_X*. Then* $T \cap H = T_X$*.*

Let Ω be the (regular) graph defined as follows. The set of vertices of Ω is S. Two vertices $s, t \in S$ are joined by an edge in Ω if $m_{s\,t} < +\infty$ and $m_{s\,t}$ is odd. Let $\Omega_1, \dots, \Omega_m$ be the connected components of Ω and, for $1 \leq i \leq m$, let Y_i be the set of vertices of Ω_i. We suppose that $X \cap Y_i \neq \emptyset$ for $1 \leq i \leq p$ and $X \cap Y_i = \emptyset$ for $p+1 \leq i \leq m$. Let $C_2 = \{\pm 1\}$ be the cyclic group of order 2. There is an epimorphism $\kappa : W \to C_2$ which sends s to 1 for all $s \in Y_1 \cup \dots \cup Y_p$ and s to -1 for all $s \in Y_{p+1} \cup \dots \cup Y_m$. We have $\kappa(t) = 1$ for all $t \in T_X$, thus $\kappa(H) = \{1\}$, and $\kappa(t) = -1$ for all $t \in T \setminus T_X$. This shows that $T \cap H = T_X$.

Assertion 4. *Let* $X \subset S$*,* $X \neq \emptyset$*, such that* W_X *is finite. Then* $N_W(W_X)$ *has infinite index in* W*.*

Suppose that $N_W(W_X)$ has finite index in W. Since W_X is finite, this implies that $Z_W(W_X)$ has finite index in W, too. Set $T_X = \{wxw^{-1}; x \in X$ and $w \in W\}$. Then the fact that $Z_W(W_X)$ has finite index in W implies that T_X is finite. Let H be the subgroup of W generated by T_X.

By definition, H is normal. The subgroup H is also a reflection group thus, by Proposition 4.2, there exists a subset $Y \subset H \cap T$ such that (H, Y) is a Coxeter system. We have $H \cap T = T_X$ (Assertion 3) and $T_H = \{hyh^{-1}; y \in Y \text{ and } h \in H\} \subset T_X$, thus T_H is finite. By Proposition 4.2, it follows that H is finite. As pointed out before, H is also normal, and this contradicts Assertion 2.

Assertion 5. *Let H be a non-trivial finite subgroup of W. Then $N_W(H)$ has infinite index in W.*

Let $\mathrm{Pc}(H)$ denote the intersection of all the parabolic subgroups that contain H. As before, $\mathrm{Pc}(H)$ is a non-trivial finite parabolic subgroup. By Assertion 4, it follows that $N_W(\mathrm{Pc}(H))$ has infinite index in W. Since $N_W(H) \subset N_W(\mathrm{Pc}(H))$, we conclude that $N_W(H)$ has infinite index in W. □

Proof of Theorem 4.1. We start with the second part of the theorem. We assume that Γ is connected, non-spherical, and non-affine. Let G be a finite index subgroup of W and let A, B be two subgroups of G such that $G = A \times B$. Take an essential element $w \in W$ (a Coxeter element, for example). Since G has finite index in W, there is some non-trivial power w^m of w which belongs to G. By Corollary 2.5, the element w^m is also essential, thus we can assume that $w \in G$. We write $w = w_A w_B$ where $w_A \in A$ and $w_B \in B$. The element w has infinite order thus either w_A or w_B has infinite order (say w_A has infinite order). The subgroup $\langle w \rangle$ has finite index in $Z_W(w)$ (by Theorem 2.6), $w_A \in Z_W(w)$, and w_A has infinite order, thus there exist $p, q \in \mathbb{Z} \setminus \{0\}$ such that $w^p = w_A^q$. By Corollary 2.5, it follows that w_A is essential. The group $\langle w_A \rangle$ has finite index in $Z_W(w_A)$ (by Theorem 2.6) and $\langle w_A \rangle \times B \subset Z_W(w_A)$, thus B is finite. Finally, from the inclusion $G \subset N_W(B)$ and the fact that G has finite index in W, follows that $N_W(B)$ has finite index in W, hence, by Proposition 4.3, $B = \{1\}$.

We turn now to the first part of Theorem 4.1. By the above, it suffices to consider the case where Γ is connected and of affine type. Let A, B be two subgroups of W such that $W = A \times B$. By [2], Chap. 6, § 2, there exist a finite Coxeter group W_0 and an irreducible representation $\rho : W_0 \to GL(\mathbb{Z}^{n-1})$ such that $W = \mathbb{Z}^{n-1} \rtimes W_0$. (Irreducible means in this context that the corresponding linear representation $W_0 \to GL(\mathbb{R}^{n-1})$ is irreducible. Note also that n is the rank of (W, S), but this fact is not needed for our purpose.) One of the components, A or B, must be infinite (say A). The subgroup $\mathbb{Z}^{n-1}$ has finite index in W, thus $A \cap \mathbb{Z}^{n-1}$ has finite index in A, therefore $A \cap \mathbb{Z}^{n-1} \neq \{1\}$ since A is infinite. The subgroup $A \cap \mathbb{Z}^{n-1}$ is normal since both, A and $\mathbb{Z}^{n-1}$ are normal, thus $(A \cap \mathbb{Z}^{n-1}) \otimes \mathbb{R}$ is a non-trivial linear subspace of $\mathbb{R}^{n-1}$ invariant by the action of W_0, therefore $(A \cap \mathbb{Z}^{n-1}) \otimes \mathbb{R} = \mathbb{R}^{n-1}$ since ρ is irreducible. This implies that $A \cap \mathbb{Z}^{n-1}$ has finite index in $\mathbb{Z}^{n-1}$ and, consequently, in W. Thus A has finite index in W. We conclude that B is finite and hence, by Proposition 4.3, $B = \{1\}$. □

5 Remak decompositions

We start this section with the statement of the Krull-Remak-Schmidt theorem, and refer to [22], Section 3.3, for the proof.

Theorem 5.1 (Krull-Remak-Schmidt).

1. *Let G be a group which satisfies either the maximal or the minimal condition on normal subgroups. Then G has a Remak decomposition.*

2. *Let G be a group which satisfies both, the maximal and the minimal conditions on normal subgroups. If $G = G_1 \times \cdots \times G_p$ and $G = H_1 \times \cdots \times H_q$ are two Remak decompositions, then $p = q$ and there exist a permutation $\sigma \in \mathrm{Sym}_p$ and a central automorphism $\varphi : G \to G$ such that $\varphi(G_i) = H_{\sigma(i)}$ for all $1 \le i \le p$.*

We turn now to the main result of the section. Let Γ be a Coxeter graph and let (W, S) be the Coxeter system of type Γ. Write $W = W_{X_1} \times \cdots \times W_{X_b} \times W_{Z_3}$, where $W_{X_1}, \dots, W_{X_b}$ are the non-spherical irreducible components of W and W_{Z_3} is the finite part. By Theorem 5.1, we can choose a Remak decomposition $W_{Z_3} = H_1 \times \cdots \times H_q$ of W_{Z_3} which is unique up to a central automorphism. Then, by Theorem 4.1, $W = W_{X_1} \times \cdots \times W_{X_b} \times H_1 \times \cdots \times H_q$ is a Remak decomposition.

Theorem 5.2. *Let $W = G_1 \times \cdots \times G_m$ be a Remak decomposition of W. Then $m = b + q$ and there exist a permutation $\sigma \in \mathrm{Sym}_m$ and a central automorphism $\varphi : W \to W$ such that $\varphi(W_{X_i}) = G_{\sigma(i)}$ for all $1 \le i \le b$ and $\varphi(H_j) = G_{\sigma(b+j)}$ for $1 \le j \le q$.*

Corollary 5.3. *The group $\mathrm{Aut}(W_{X_1}) \times \cdots \times \mathrm{Aut}(W_{X_b}) \times \mathrm{Aut}(W_{Z_3})$ has finite index in $\mathrm{Aut}(W)$.*

In many cases, for example when Γ has no infinite labels (see [14]), the group $\mathrm{Out}(W)$ is finite. However, there are Coxeter groups having infinite outer automorphism groups, for example the free product of n copies ($n \ge 3$) of $C_2 = \{\pm 1\}$.

Corollary 5.4. *Let $(\tilde{W}, \tilde{S})$ be a Coxeter system. Write $\tilde{W} = \tilde{W}_{\tilde{X}_1} \times \cdots \times \tilde{W}_{\tilde{X}_d} \times \tilde{W}_{\tilde{Z}_3}$, where $\tilde{W}_{\tilde{X}_1}, \dots, \tilde{W}_{\tilde{X}_d}$ are the non-spherical irreducible components of $\tilde{W}$ and $\tilde{W}_{\tilde{Z}_3}$ is the finite part. Then W and $\tilde{W}$ are isomorphic if and only if W_{Z_3} and $\tilde{W}_{\tilde{Z}_3}$ are isomorphic, $b = d$, and there exists a permutation $\sigma \in \mathrm{Sym}_b$ such that W_{X_i} is isomorphic to $\tilde{W}_{\tilde{X}_{\sigma(i)}}$ for all $1 \le i \le b$.*

Call a Coxeter group *rigid* if it cannot be defined by two non-isomorphic Coxeter graphs.

Corollary 5.5. *The Coxeter group W is rigid if and only if $W_{X_1}, \dots, W_{X_b}, W_{Z_3}$ are all rigid.*

Rigid Coxeter groups include, notably, those Coxeter groups that act effectively, properly, and cocompactly on contractible manifolds (see [6]). By a recent result of Caprace and Mühlherr [5], the infinite Coxeter groups defined by connected Coxeter graphs that have no infinite labels are also rigid. Note that, by Corollary 5.5, this last result extends to the Coxeter groups defined by disjoint unions of non-spherical connected Coxeter graphs having no infinite labels. On the other hand, an interesting construction of non-rigid (and irreducible) Coxeter groups is given in [3]. We refer to [19] for a recent exposition on rigidity in Coxeter groups and on the so-called isomorphism problem.

We start now the proof of Theorem 5.2 with the following lemma which will be also used in the next two sections.

For a group G and a subset $E \subset G$, we denote by $Z_G(E) = \{w \in G; wx = xw \text{ for all } x \in E\}$ the centralizer of E in G.

Lemma 5.6. *Let G be a group and let A, B be two subgroups of G such that $G = A \times B$. Let E be a subset of G. Then $Z_G(E) = (Z_G(E) \cap A) \times (Z_G(E) \cap B)$.*

Proof. Take $w = (w_A, w_B) \in Z_G(E)$. Let $x = (x_A, x_B) \in E$. We have $wx = (w_A x_A, w_B x_B) = xw = (x_A w_A, x_B w_B)$, thus w_A and x_A commute, therefore $w_A x = (w_A x_A, x_B) = (x_A w_A, x_B) = x w_A$. This shows that $w_A \in Z_G(E)$ (and also that $w_B \in Z_G(E)$). $\square$

Proof of Theorem 5.2. Recall that $W_{Z_3} = H_1 \times \cdots \times H_q$ and $W = W_{X_1} \times \cdots \times W_{X_b} \times H_1 \times \cdots \times H_q$ are Remak decompositions. Recall also that the center of W is included in W_{Z_3}. In particular, $Z(W) = Z(H_1) \times \cdots \times Z(H_q)$. Now, we assume given a Remak decomposition $W = G_1 \times \cdots \times G_m$.

Take $i \in \{1, \dots, b\}$ and set $\tilde{W}_i = W_{X_i} \times Z(W) = W_{X_i} \times Z(H_1) \times \cdots \times Z(H_q)$. The group $\tilde{W}_i$ is the centralizer in W of $W_{X_1} \times \cdots \times W_{X_{i-1}} \times W_{X_{i+1}} \times \cdots \times W_{X_b} \times W_{Z_3}$, thus, by Lemma 5.6,

$$\tilde{W}_i = (\tilde{W}_i \cap G_1) \times \cdots \times (\tilde{W}_i \cap G_m)\,. \tag{1}$$

The inclusion $Z(W) \subset \tilde{W}_i$ implies that $Z(G_j) \subset (\tilde{W}_i \cap G_j)$ for all $1 \le j \le m$. Hence, the quotient of the equality (1) by $Z(W)$ gives the isomorphism

$$W_{X_i} \simeq ((\tilde{W}_i \cap G_1)/Z(G_1)) \times \cdots \times ((\tilde{W}_i \cap G_m)/Z(G_m))\,. \tag{2}$$

Since W_{X_i} is indecomposable, it follows that there exists $\chi(i) \in \{1, \dots, m\}$ such that

$$\begin{aligned} (\tilde{W}_i \cap G_{\chi(i)})/Z(G_{\chi(i)}) &\simeq W_{X_i}\,, \\ (\tilde{W}_i \cap G_j)/Z(G_j) &= \{1\} \quad \text{for } j \neq \chi(i)\,. \end{aligned} \tag{3}$$

For $1 \le j \le m$, we denote by $\kappa_j : W \to G_j$ the projection on the j-th component. By (3), the restriction $\kappa_{\chi(i)} : W_{X_i} \to G_{\chi(i)}$ of $\kappa_{\chi(i)}$ to W_{X_i} is injective, and $\kappa_j(W_{X_i}) \subset Z(G_j)$ for $j \neq \chi(i)$. Set $\hat{W}_{X_i} = W_{X_1} \times \cdots \times W_{X_{i-1}} \times W_{X_{i+1}} \times \cdots \times W_{X_b} \times W_{Z_3}$ and define $\varphi_i : W = W_{X_i} \times \hat{W}_{X_i} \to W$ by

$$\varphi_i(w) = \begin{cases} w & \text{if } w \in \hat{W}_{X_i}\,, \\ w \cdot \displaystyle\prod_{j \neq \chi(i)} \kappa_j(w)^{-1} & \text{if } w \in W_{X_i}\,. \end{cases}$$

Then φ_i is a well-defined central automorphism, it sends W_{X_i} into $G_{\chi(i)}$, and it restricts to the identity on $\hat{W}_{X_i}$. Moreover, φ_i and φ_k commute if $i \neq k$. Set

$$\varphi = \prod_{i=1}^{b} \varphi_i\,.$$

Then $\varphi : W \to W$ is a central automorphism and $\varphi(W_{X_i}) \subset G_{\chi(i)}$ for all $1 \le i \le b$. So, upon replacing W_{X_i} by $\varphi(W_{X_i})$ for all $1 \le i \le b$ if necessary, we can assume that $W_{X_i} \subset G_{\chi(i)}$ for all $1 \le i \le b$.

For a group G and two subgroups A, B of G, we denote by $[A, B]$ the subgroup of G generated by $\{\alpha^{-1}\beta^{-1}\alpha\beta; \alpha \in A \text{ and } \beta \in B\}$. In particular, the equality $[A, B] = \{1\}$ means that every element of A commutes with every element of B.

From the equality $W = W_{X_i} \times \hat{W}_{X_i}$ we deduce that $G_{\chi(i)} = \kappa_{\chi(i)}(W) = W_{X_i} \cdot \kappa_{\chi(i)}(\hat{W}_{X_i})$ and $[W_{X_i}, \kappa_{\chi(i)}(\hat{W}_{X_i})] = \{1\}$. Moreover, $W_{X_i} \cap \kappa_{\chi(i)}(\hat{W}_{X_i})$ lies in the center of W_{X_i} which is trivial, thus $W_{X_i} \cap \kappa_{\chi(i)}(\hat{W}_{X_i}) = \{1\}$, therefore $G_{\chi(i)} = W_{X_i} \times \kappa_{\chi(i)}(\hat{W}_{X_i})$. Since $G_{\chi(i)}$ is indecomposable, it follows that $\kappa_{\chi(i)}(\hat{W}_{X_i}) = \{1\}$ and $G_{\chi(i)} = W_{X_i}$. It also follows that $\chi(i) \neq \chi(k)$ if $i \neq k$. So, up to a permutation of the G_j's, we can assume that $W_{X_i} = G_i$ for all $1 \leq i \leq b$.

At this point, we have the Remak decompositions $W = W_{X_1} \times \cdots \times W_{X_b} \times H_1 \times \cdots \times H_q$ and $W = W_{X_1} \times \cdots \times W_{X_b} \times G_{b+1} \times \cdots \times G_m$. We also have $W_{Z_3} = H_1 \times \cdots \times H_q$. Let $\pi : W \to W_{Z_3} = H_1 \times \cdots \times H_q$ be the projection on W_{Z_3}. We have $\pi(G_{b+j}) \simeq G_{b+j}$ for all $1 \leq j \leq m-b$, and $W_{Z_3} = \pi(G_{b+1}) \times \cdots \times \pi(G_m)$ is a Remak decomposition. By Theorem 5.1, it follows that $m = b+q$ and, up to a central automorphism and a permutation of the G_{b+j}'s, $\pi(G_{b+j}) = H_j$ for all $1 \leq j \leq q$. The equality $\pi(G_{b+j}) = H_j$ implies that $G_{b+j} \subset W_{X_1} \times \cdots \times W_{X_b} \times H_j$, and the decomposition $W = W_{X_1} \times \cdots \times W_{X_b} \times G_{b+1} \times \cdots \times G_m$ implies that $G_{b+j} \subset Z_W(W_{X_1} \times \cdots \times W_{X_b}) = H_1 \times \cdots \times H_q$, thus $G_{b+j} \subset (W_{X_1} \times \cdots \times W_{X_p} \times H_j) \cap (H_1 \times \cdots \times H_q) = H_j$, that is, $G_{b+j} = H_j$ (since $\pi(G_{p+j}) = H_j$). □

6 Virtual strong Remak decompositions

Let Γ be a Coxeter graph and let (W, S) be the Coxeter system of type Γ. Write $W = W_{X_1} \times \cdots \times W_{X_a} \times W_{Z_2} \times W_{Z_3}$, where $W_{X_1}, \dots, W_{X_a}$ are the non-spherical and non-affine irreducible components of W, W_{Z_2} is the affine part, and W_{Z_3} is the finite part. We choose a finite index subgroup R of W_{Z_2} isomorphic to $\mathbb{Z}^d$, where d is the affine dimension of W, and a decomposition $R = R_1 \times \cdots \times R_d$ of R as a direct product of d copies of $\mathbb{Z}$. Then, by Theorem 4.1, $G = W_{X_1} \times \cdots \times W_{X_a} \times R_1 \times \cdots \times R_d$ is a virtual strong Remak decomposition of W.

Theorem 6.1. *Let $H = H_1 \times \cdots \times H_m$ be a virtual strong Remak decomposition of W. Then $m = a + d$ and there exist*

- *a virtual strong Remak decomposition $K = A_1 \times \cdots \times A_a \times B_1 \times \cdots \times B_d$ of W,*
- *a central automorphism $\varphi : K \to K$,*
- *a permutation $\sigma \in \mathrm{Sym}_m$,*

such that

- *A_i is a finite index subgroup of W_{X_i} and $\varphi(A_i)$ is a finite index subgroup of $H_{\sigma(i)}$ for all $1 \leq i \leq a$,*
- *B_j is isomorphic to $\mathbb{Z}$, $\varphi(B_j)$ is a finite index subgroup of $H_{\sigma(a+j)}$ for all $1 \leq j \leq d$, and $B = B_1 \times \cdots \times B_d$ is a finite index subgroup of R.*

Corollary 6.2. *Every Coxeter group has a unique virtual strong Remak decomposition up to equivalence.*

Corollary 6.3. *Let $(\tilde{W}, \tilde{S})$ be a Coxeter system. Write $\tilde{W} = \tilde{W}_{\tilde{X}_1} \times \cdots \times W_{\tilde{X}_c} \times \tilde{W}_{\tilde{Z}_2} \times \tilde{W}_{\tilde{Z}_3}$, where $\tilde{W}_{\tilde{X}_1}, \dots, \tilde{W}_{\tilde{X}_c}$ are the non-spherical and non-affine irreducible components of $\tilde{W}$, $\tilde{W}_{\tilde{Z}_2}$ is the affine part, and $\tilde{W}_{\tilde{Z}_3}$ is the finite part. Then W and $\tilde{W}$ are commensurable if and only if $a = c$, the affine dimension of W is equal to the affine dimension of $\tilde{W}$, and there exists a permutation $\sigma \in \mathrm{Sym}_a$ such that W_{X_i} and $\tilde{W}_{\tilde{X}_{\sigma(i)}}$ are commensurable for all $1 \le i \le a$.*

The proof of Theorem 5.2 uses the fact that the center of any infinite irreducible Coxeter group is trivial, but the proof of Theorem 6.1 needs the following stronger result.

Proposition 6.4. *Assume Γ to be connected, non-spherical, and non-affine. Let G be a finite index subgroup of W. Then $Z_W(G) = \{1\}$.*

Proof. Take $w_0 \in Z_W(G)$. Suppose first that w_0 has finite order. Let $H = \langle w_0 \rangle$ be the subgroup of W generated by w_0. The group G is included in $N_W(H)$ and G has finite index in W, thus $N_W(H)$ has finite index in W, therefore, by Proposition 4.3, $H = \{1\}$ and $w_0 = 1$.

Now, suppose that w_0 has infinite order. Take some essential element $w \in W$. Since G has finite index in W, there is some non-trivial power w^m of w which belongs to G. So, upon replacing w by w^m, we may assume that $w \in G$. The subgroup $\langle w \rangle$ has finite index in $Z_W(w)$ (by Theorem 2.6), $w_0 \in Z_W(w)$, and w_0 has infinite order, thus there exist $p, q \in \mathbb{Z} \setminus \{0\}$ such that $w_0^p = w^q$. By Corollary 2.5, this implies that w_0 is essential. Finally, $G \subset Z_W(w_0)$, G has finite index in W, and $\langle w_0 \rangle$ has finite index in $Z_W(w_0)$, thus $\langle w_0 \rangle$ is a finite index subgroup of W. As pointed out in the introduction, such a Coxeter group, being virtually $\mathbb{Z}$, must be the Coxeter group of type $\tilde{A}_1$ which is of affine type: a contradiction. □

Proof of Theorem 6.1. First, observe the following fact: if G is a group, H is a finite index subgroup of G, and K is any subgroup, then $H \cap K$ is a finite index subgroup of K.

Now, let $G = W_{X_1} \times \cdots \times W_{X_a} \times R$, where $W_{X_1}, \dots, W_{X_a}$ are the non-spherical and non-affine irreducible components of W, and R is a finite index subgroup of the affine part isomorphic to $\mathbb{Z}^d$. Let $H = H_1 \times \cdots \times H_m$ be a virtual strong Remak decomposition.

Set $H_i' = H_i \cap G$ for all $1 \le i \le m$ and $H' = H_1' \times \cdots \times H_m'$. Since G has finite index in W, each H_i' has finite index in H_i, thus H' has finite index in H (and in W). So, upon replacing H by H' and H_i by H_i' for all $1 \le i \le m$, we can assume that $H \subset G$.

Set $A_i = W_{X_i} \cap H$ for all $1 \le i \le a$, $B = R \cap H$, and $K = A_1 \times \cdots \times A_a \times B$. The group A_i is a finite index subgroup of W_{X_i} for all $1 \le i \le a$ and B is a finite index subgroup of $R \simeq \mathbb{Z}^d$ (in particular, $B \simeq \mathbb{Z}^d$), thus K is a finite index subgroup of both, G and H.

Now, we show that $Z(H) = B$. Let $w \in Z(H)$. Since $H \subset G$, the element w can be written in the form $w_1 \cdots w_a u$ where $w_i \in W_{X_i}$ for all $1 \le i \le a$ and $u \in R$. We have $w_i \in Z_{W_{X_i}}(A_i)$ and $Z_{W_{X_i}}(A_i)$ is trivial by Proposition 6.2, thus $w_i = 1$ for all $1 \le i \le a$ and $w = u \in R \cap H = B$. This shows that $Z(H) \subset B$. The reverse inclusion $B \subset Z(H)$ is obvious.

Take $i \in \{1, \dots, a\}$ and set $\tilde{A}_i = A_i \times B$ and $\hat{A}_i = A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_a \times B$. Let $G' = W_{X_1} \times \cdots \times W_{X_a} \times B$. It is easily shown using the equality $Z(H) = B = R \cap H$ that

$H \subset G'$. By Proposition 6.2, we have $Z_{G'}(\hat{A}_i) = W_{X_i} \times B$, thus $Z_H(\hat{A}_i) = (W_{X_i} \times B) \cap H$. Furthermore, since $B \subset H$, we have $(W_{X_i} \times B) \cap H = (W_{X_i} \cap H) \times B = A_i \times B = \tilde{A}_i$, hence $Z_H(\hat{A}_i) = \tilde{A}_i$. By Lemma 5.6, it follows that

$$\tilde{A}_i = (\tilde{A}_i \cap H_1) \times \cdots \times (\tilde{A}_i \cap H_m) . \tag{4}$$

The inclusion $B = Z(H) \subset \tilde{A}_i$ implies that $Z(H_j) \subset (\tilde{A}_i \cap H_j)$ for all $1 \le j \le m$. Hence, the quotient of the equality (4) by B gives the isomorphism

$$A_i \simeq ((\tilde{A}_i \cap H_1)/Z(H_1)) \times \cdots \times ((\tilde{A}_i \cap H_m)/Z(H_m)) . \tag{5}$$

Since W_{X_i} is strongly indecomposable and A_i is a finite index subgroup of W_{X_i}, it follows that there exists $\chi(i) \in \{1, \dots, m\}$ such that

$$\begin{aligned} (\tilde{A}_i \cap H_{\chi(i)})/Z(H_{\chi(i)}) &\simeq A_i , \\ (\tilde{A}_i \cap H_j)/Z(H_j) &= \{1\} \quad \text{for } j \neq \chi(i) . \end{aligned} \tag{6}$$

For $1 \le j \le m$, we denote by $\kappa_j : H \to H_j$ the projection on the j-th component. By (6), the restriction $\kappa_{\chi(i)} : A_i \to H_{\chi(i)}$ of $\kappa_{\chi(i)}$ to A_i is injective, and $\kappa_j(A_i) \subset Z(H_j)$ for $j \neq \chi(i)$. Define $\varphi_i : K = A_i \times \hat{A}_i \to K$ by

$$\varphi_i(w) = \begin{cases} w & \text{if } w \in \hat{A}_i , \\ w \cdot \prod\limits_{j \neq \chi(i)} \kappa_j(w)^{-1} & \text{if } w \in A_i . \end{cases}$$

Then φ_i is well-defined since, for $j \neq \chi(i)$, $\kappa_j(w) \in Z(H_j) \subset Z(H) = B \subset K$, for all $w \in A_i$. It is a central automorphism, it sends A_i into $H_{\chi(i)}$, and it restricts to the identity on $\hat{A}_i$. Moreover, φ_i and φ_k commute if $i \neq k$. Set

$$\varphi = \prod_{i=1}^{a} \varphi_i .$$

Then $\varphi : K \to K$ is a central automorphism and $\varphi(A_i) \subset H_{\chi(i)}$ for all $1 \le i \le a$. So, upon replacing A_i by $\varphi(A_i)$ for all $1 \le i \le a$, we can assume that $A_i \subset H_{\chi(i)}$ for all $1 \le i \le a$.

From the equality $K = A_i \times \hat{A}_i$ follows that $\kappa_{\chi(i)}(K) = A_i \cdot \kappa_{\chi(i)}(\hat{A}_i)$ and $[A_i, \kappa_{\chi(i)}(\hat{A}_i)] = \{1\}$. Moreover, $A_i \cap \kappa_{\chi(i)}(\hat{A}_i)$ lies in the center of A_i which is trivial by Proposition 6.2, thus $A_i \cap \kappa_{\chi(i)}(\hat{A}_i) = \{1\}$, therefore $\kappa_{\chi(i)}(K) = A_i \times \kappa_{\chi(i)}(\hat{A}_i)$. The group K is a finite index subgroup of H, thus $\kappa_{\chi(i)}(K)$ is a finite index subgroup of $H_{\chi(i)}$. Now, $H_{\chi(i)}$ is strongly indecomposable, thus $\kappa_{\chi(i)}(\hat{A}_i) = \{1\}$, $A_i = \kappa_{\chi(i)}(K)$, and A_i is a finite index subgroup of $H_{\chi(i)}$. It also follows that $\chi(i) \neq \chi(k)$ if $i \neq k$. So, up to a permutation of the H_j's, we can assume that A_i is a finite index subgroup of H_i for all $1 \le i \le a$.

Recall that $B = Z(H) = Z(H_1) \times \cdots \times Z(H_m)$. Furthermore, for $1 \le i \le a$, we have $Z(H_i) = \kappa_i(B) \subset \kappa_i(\hat{A}_i) = \{1\}$. So, $B = Z(H) = Z(H_{a+1}) \times \cdots \times Z(H_m)$. In particular, B is a subgroup of $H_{a+1} \times \cdots \times H_m$. Consider the inclusions

$$K = A_1 \times \cdots \times A_a \times B \subset A_1 \times \cdots \times A_a \times H_{a+1} \times \cdots \times H_m \subset H_1 \times \cdots \times H_a \times H_{a+1} \times \cdots \times H_m = H .$$

Since K has finite index in H, $K = A_1 \times \cdots \times A_a \times B$ has finite index in $A_1 \times \cdots \times A_a \times H_{a+1} \times \cdots \times H_m$, thus B has finite index in $H_{a+1} \times \cdots \times H_m$. For $1 \le j \le m-a$, we set $B_j = B \cap H_{a+j} = Z(H_{a+j})$. The group B_j has finite index in H_{a+j} and H_{a+j} is strongly indecomposable, thus B_j is non-trivial and indecomposable. On the other hand, B_j is a subgroup of B which is a finitely generated free abelian group, therefore B_j is isomorphic to $\mathbb{Z}$. Finally, $B = B_1 \times \cdots \times B_{m-a} \simeq \mathbb{Z}^d$, thus $m - a = d$. □

7 Finite irreducible Coxeter groups

Recall that the connected spherical type Coxeter graphs are precisely the graphs A_n ($n \ge 1$), B_n ($n \ge 2$), D_n ($n \ge 4$), E_6, E_7, E_8, F_4, H_3, H_4, $I_2(p)$ ($p \ge 5$). Here we use the notation $I_2(6)$ for the Coxeter graph G_2. We may also use the notation $I_2(3)$ for A_2, and $I_2(4)$ for B_2. We number the vertices of each of these graphs as it is usually done. The numbering coincides with the standard numbering of the simple roots if W is a Weil group (see [15], p. 58), and the vertices of H_3 and H_4 are numbered so that s_1 and s_2 are joined by an edge labeled by 5.

Let Γ be a spherical type connected Coxeter graph and let (W, S) be the Coxeter system of type Γ. For $w \in W$, we denote by $\mathrm{lg}(w)$ the word length of w with respect to S. The group W has a unique element of maximal length, w_0, which satisfies $w_0^2 = 1$ and $w_0 S w_0 = S$. The following proposition is a list of well-known properties on W which can be found for instance in [2].

Proposition 7.1.

1. *There exists a permutation $\theta : S \to S$ such that $w_0 s w_0 = \theta(s)$ for all $s \in S$, and $\theta^2 = \mathrm{Id}$.*
2. *The group $N_W(S) = \{w \in W; wSw^{-1} = S\}$ is cyclic of order 2 generated by w_0.*
3. *If $\theta = \mathrm{Id}$, then $Z(W) = N_W(S)$. If $\theta \neq \mathrm{Id}$, then $Z(W) = \{1\}$.*
4. *We have $\theta \neq \mathrm{Id}$ if and only if $\Gamma \in \{A_n; n \ge 2\} \cup \{D_n; n \ge 5 \text{ and } n \text{ odd}\} \cup \{I_2(p); p \ge 5 \text{ and } p \text{ odd}\} \cup \{E_6\}$.*

Theorem 7.2. *Let Γ be a connected spherical type Coxeter graph and let (W, S) be the Coxeter system of type Γ. Then W is decomposable if and only if $\Gamma \in \{I_2(p); p \ge 6 \text{ and } p \equiv 2(\text{mod } 4)\} \cup \{B_n; n \ge 3; \text{ and } n \text{ odd}\} \cup \{H_3, E_7\}$. In that case, a Remak decomposition of W is isomorphic to $Z(W) \times W/Z(W)$.*

Remark. If either $\Gamma = I_2(p)$ ($p \ge 6$ and $p \equiv 2(\text{mod } 4)$) or $\Gamma = B_n$ ($n \ge 3$ and n odd), then the factor $W/Z(W)$ is a Coxeter group. It is the Coxeter group of type $I_2(\frac{p}{2})$ if $\Gamma = I_2(p)$ ($p \ge 6$ and $p \equiv 2(\text{mod } 4)$) and it is the Coxeter group of type D_n if $\Gamma = B_n$ ($n \ge 3$ and n odd). If $\Gamma \in \{H_3, E_7\}$, then $W/Z(W)$ is not a Coxeter group. It is the alternating group on 5 letters if $\Gamma = H_3$ and it is $SO_7(2)$ if $\Gamma = E_7$.

Proof of Theorem 7.2. We shall often use centralizers of parabolic subgroups (of finite Coxeter groups) in our proof. These can be easily computed with the techniques of [13], [10], [4], or [1], together with a suitable computer program. For instance, we used the package "Chevie" of GAP.

Let Γ be a connected finite type Coxeter graph and let (W, S) be the Coxeter system of type Γ. We assume given two subgroups $A, B \subset W$ such that $W = A \times B$, and we argue case by case.

Case $\Gamma = I_2(p)$, $p \geq 3$. Left to the reader.

Case $\Gamma = A_n$, $n \geq 3$. The case $n = 3$ is left to the reader. So, we assume $n \geq 4$. Let $X = \{s_3, s_4, \dots, s_n\}$. We have $Z_W(X) = \{1, s_1\}$. By Lemma 5.6, if follows that either $s_1 \in A$ or $s_1 \in B$ (say $s_1 \in A$). We conclude that $A = W$ since A is normal and W is normally generated by s_1.

Case $\Gamma = B_n$, $n \geq 3$. Let $C_2 = \{\pm 1\}$ be the cyclic group of order 2. Then $W = (C_2)^n \rtimes \mathrm{Sym}_n$, where Sym_n denotes the n-th symmetric group. Let w_0 be the longest element of W. Then $w_0 = [-1, -1, \dots, -1] \in (C_2)^n$, and $Z(W) = \{1, w_0\}$.

Consider the natural projection $\eta : W \to \mathrm{Sym}_n$. We have $\mathrm{Sym}_n = \eta(A) \cdot \eta(B)$, and $[\eta(A), \eta(B)] = \{1\}$. Notice that $\eta(A) \cap \eta(B) \subset Z(\mathrm{Sym}_n) = \{1\}$, thus we actually have $\mathrm{Sym}_n = \eta(A) \times \eta(B)$. By the previous case, it follows that either $\eta(A) = \{1\}$ or $\eta(B) = \{1\}$ (say $\eta(B) = \{1\}$).

Assume $B \neq \{1\}$. Let $\gamma : (C_2)^n \to C_2$ be the epimorphism defined by $\gamma[\varepsilon_1, \dots, \varepsilon_n] = \prod_{i=1}^n \varepsilon_i$, and let $K = \mathrm{Ker}\gamma$. There are precisely four normal subgroups of W contained in $(C_2)^n$: $\{1\}$, $Z(W) = \{1, w_0\}$, K, and $(C_2)^n$. Thus either $B = Z(W)$, or $B = K$, or $B = (C_2)^n$. Observe that $\eta(A) = \mathrm{Sym}_n$, thus there exists $u \in (C_2)^n$ such that $u \cdot (1, 2) \in A$. We cannot have either $B = K$ or $B = (C_2)^n$, because $v = [1, -1, -1, 1, \dots, 1] \in K \subset (C_2)^n$, and v does not commute with $u \cdot (1, 2)$. So, $B = Z(W) = \{1, w_0\}$.

It remains to show that n is odd. The quotient of W by A determines an epimorphism $\mu : W \to C_2$ which satisfies $\mu(w_0) = -1$, and such an epimorphism exists only if n is odd.

Case $\Gamma = D_n$, $n \geq 4$. Let K be the subgroup of $(C_2)^n$ defined above. We have $W = K \rtimes \mathrm{Sym}_n$. Then one can easily prove, using the same arguments as in the case $\Gamma = B_n$, that either $A = \{1\}$ or $B = \{1\}$. (Note that $[-1, -1, \dots, -1] \notin K$ if n is odd.)

Case $\Gamma = H_3$. Let w_0 be the longest element of W. Then $Z(W) = \{1, w_0\}$. Let $\gamma : W \to \{\pm 1\}$ be the epimorphism defined by $\gamma(s_1) = \gamma(s_2) = \gamma(s_3) = -1$, and let $K = \mathrm{Ker}\gamma$. Then $\gamma(w_0) = -1$ and $W = K \times Z(W)$. Note also that W is normally generated by s_1, and K is normally generated by $s_1 w_0$.

The group $Z_W(s_1)$ is generated by $\{s_1, s_3, w_0\}$, and isomorphic to $(C_2)^3$. By Lemma 5.6, we have $Z_W(s_1) = (Z_W(s_1) \cap A) \times (Z_W(s_1) \cap B)$. Observe that $|Z_W(s_1)| = 8$, thus $|Z_W(s_1) \cap (A \cup B)| \geq 5$, therefore $\{s_1, s_1 w_0, s_3, s_3 w_0\} \cap (A \cup B) \neq \emptyset$. Furthermore, s_1 and s_3 are conjugate in W, thus there exists $a \in \{0, 1\}$ such that either $s_1 w_0^a \in A$ or $s_1 w_0^a \in B$ (say $s_1 w_0^a \in A$). We conclude that $K \subset A$, thus either $A = K$ or $A = W$.

Case $\Gamma = H_4$. Let $X = \{s_3, s_4\}$. Set

$$v = s_2 s_1 s_3 s_2 s_1 s_2 s_1 s_4 s_3 s_2 s_1 s_2 s_1 s_3 s_2 s_1 s_2 s_3 s_4 s_3 s_2 \, .$$

Then $Z_W(X)$ is generated by $\{s_1, v\}$ and has the presentation $\langle s_1, v | s_1^2 = v^2 = (s_1 v)^6 = 1 \rangle \simeq W_{I_2(6)}$. Let w_0 be the longest element of W. Then $Z(W) = \{1, w_0\}$ and $(s_1 v)^3 = (v s_1)^3 = w_0$.

By Lemma 5.6, we have $Z_W(X) = (Z_W(X) \cap A) \times (Z_W(X) \cap B)$. By the case $\Gamma = I_2(6)$ treated before, it follows that there exists $a \in \{0, 1\}$ such that either $s_1 w_0^a \in A$ or $s_1 w_0^a \in B$ (say $s_1 w_0^a \in A$). We conclude that $A = W$ since W is normally generated by $s_1 w_0^a$ (whatever is $a \in \{0, 1\}$).

Case $\Gamma = F_4$. Let $X = \{s_1, s_2\}$, let $Y = \{s_1, s_2, s_3\}$, and let w_Y be the longest element of W_Y. The group $Z_W(X)$ is generated by $\{s_4, w_Y\}$, and has the presentation $\langle s_4, w_Y | s_4^2 = w_Y^2 = (s_4 w_Y)^6 = 1 \rangle \simeq W_{I_2(6)}$. Let w_0 be the longest element of W. Then $Z(W) = \{1, w_0\}$ and $(s_4 w_Y)^3 = (w_Y s_4)^3 = w_0$. By Lemma 5.6, we have $Z_W(X) = (Z_W(X) \cap A) \times (Z_W(X) \cap B)$. By the case $\Gamma = I_2(6)$ treated before, it follows that there exists $a \in \{0, 1\}$ such that either $s_4 w_0^a \in A$ or $s_4 w_0^a \in B$ (say $s_4 w_0^a \in A$). The generator s_3 is conjugate to s_4, thus we also have $s_3 w_0^a \in A$. Similarly, there exists $b \in \{0, 1\}$ such that either $s_1 w_0^b, s_2 w_0^b \in A$, or $s_1 w_0^b, s_2 w_0^b \in B$. We cannot have $s_1 w_0^b, s_2 w_0^b \in B$, since $s_2 w_0^b$ and $s_3 w_0^a$ do not commute, thus $s_1 w_0^b, s_2 w_0^b \in A$. We conclude that $w_0 = (s_1 s_2 s_3 s_4)^6 \in A$, thus $s_1, s_2, s_3, s_4 \in A$, therefore $A = W$.

Case $\Gamma = E_6$. The centralizer of $X = \{s_1, s_2, s_3, s_4\}$ in W is $\{1, s_6\}$, thus, by Lemma 5.6, either $s_6 \in A$, or $s_6 \in B$ (say $s_6 \in A$). We conclude that $A = W$ since W is normally generated by s_6.

Case $\Gamma = E_7$. Let w_0 be the longest element of W. Then $Z(W) = \{1, w_0\}$. Let $\gamma : W \to C_2 = \{\pm 1\}$ be the epimorphism defined by $\gamma(s_i) = -1$ for all $1 \le i \le 7$, and let $K = \mathrm{Ker}\gamma$. Then $\gamma(w_0) = -1$ and $W = K \times Z(W)$. Note also that W is normally generated by s_1, and K is normally generated by $s_1 w_0$.

Using the same argument as in the case $\Gamma = H_4$, but with the centralizer of $X = \{s_2, s_3, s_5, s_6, s_7\}$, we show that there exists $a \in \{0, 1\}$ such that either $s_1 w_0^a \in A$ or $s_1 w_0^a \in B$ (say $s_1 w_0^a \in A$). This implies that $K \subset A$, thus either $A = K$ or $A = W$.

Case $\Gamma = E_8$. This case can be handle in the same way as the case $\Gamma = H_4$, but with the centralizer of $X = \{s_2, s_4, s_5, s_6, s_7, s_8\}$. □

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