

Bianchi groups are conjugacy separable

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Abstract

We prove that non-uniform arithmetic lattices of $SL_2(\mathbb{C})$ and in particular the Bianchi groups are conjugacy separable. The proof based on recent deep results of Agol, Long, Reid and Minasyan.

1 Introduction

The Bianchi groups are defined as $PSL_2(O_d)$, where O_d denotes the ring of integers of the field $\mathbb{Q}(\sqrt{-d})$ for each square-free positive integer d . These groups are classical objects investigated first time in 1872 by Luigi Bianchi. The Bianchi groups have long been of interest, not only because of their intrinsic interest as abstract groups, but also because they arise naturally in number theory and geometry. They are discrete subgroups of $PSL_2(\mathbb{C}) \cong Isom^+(\mathbb{H}_3)$, and the quotient \mathbb{H}_3 modulo $PSL_2(O_d)$ is a finite volume hyperbolic 3-orbifold. We refer to [EGM-98] and [F-89] for further information about Bianchi groups.

The recent big advance in the study of Bianchi groups was the proof that they are subgroup separable (see Theorem 3.4 in [LR-2008]). Another important residual property is conjugacy separability on which we concentrate in this paper.

A group G is conjugacy separable if whenever x and y are non-conjugate elements of G , there exists some finite quotient of G in which the images of x and y are non-conjugate. The notion of the conjugacy separability owes its importance to the fact, first pointed out by Mal'cev [M-58], that the

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conjugacy problem has a positive solution in finitely presented conjugacy separable groups.

It follows from a recent work of Minasyan [M] combined with the work of Agol, Long and Reid [ALR-01], [LR-2008] that Bianchi groups are virtually conjugacy separable, i.e., contain a finite index subgroup that are conjugacy separable*. This does not prove however the conjugacy separability of Bianchi groups, since (in contrast with subgroup separability) conjugacy separability is not preserved by commensurability (see [G-86, CZ1-09, MM]). We say that two groups of $PSL_2(\mathbb{C})$ are commensurable if their intersection has finite index in both of them. More generally, two groups are (abstractly) commensurable if they contain isomorphic subgroups of finite index. Recall that non-uniform arithmetic lattices in $SL_2(\mathbb{C})$ are precisely the subgroups commensurable with Bianchi groups.

In this paper we find here group theoretic conditions that imply commensurability invariance of conjugacy separability within torsion free groups (see Theorem 2.3). Using it and different methods for torsion elements we prove our main

Theorem 1.1. *Non-uniform arithmetic lattices of $SL_2(\mathbb{C})$ and in particular the Bianchi groups are conjugacy separable.*

The conjugacy separability of Bianchi groups was conjectured in [WZ-98] where the conjugacy separability of Euclidean Bianchi groups was proved (more precisely, the cases $d = 1, 2, 7, 11$ were established there and $d = 3$ was completed in [LZ-09]).

The same methods also allow us to prove conjugacy separability of torsion free virtually Limit groups.

Theorem 1.2. *Torsion free virtually Limit groups are conjugacy separable.*

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2 Results

We say that $g \in G$ is conjugacy distinguished if its conjugacy class is closed in the profinite topology of G .

Proposition 2.1. *Let G be a group containing a conjugacy separable subgroup H of finite index. Let $a \in G$ be an element such that there exists a natural m with $a^m \in H$ and the following conditions hold:*

- $C_G(a^m)$ is conjugacy separable;
- $\widehat{C_H(a^m)} = \overline{C_H(a^m)} = C_{\widehat{H}}(a^m)$.

Then a is conjugacy distinguished.

Proof. Suppose $b = \gamma^{-1}a\gamma$, for some $\gamma \in \widehat{G}$. Observe that $\widehat{G} = G\widehat{H}$, so that we can write $\gamma = \delta\gamma_0$, where $\gamma_0 \in \widehat{H}$ and $\delta \in G$. Therefore $b^m = (a^\gamma)^m = (a^m)^\gamma = (a^m)^{\delta\gamma_0}$. Now substituting a by a^δ , we can suppose that $\gamma \in \widehat{H}$. Thus a^m and b^m are conjugate in \widehat{H} , and since H is conjugacy separable there exists $h \in H$ such that $a^m = (b^h)^m$. Hence $a^m = (b^m)^h$, so a^m and b^m are conjugate in H . Thus we can suppose that $a^m = b^m$ and so $\gamma \in C_{\widehat{H}}(a^m)$.

Let $C = C_G(a^m)$ be the centralizer of a^m in G , so $b \in C$. By the second hypothesis $\widehat{C_H(a^m)} = C_{\widehat{H}}(a^m) = \widehat{C_H(a^m)}$, so $\gamma \in \widehat{C_G(a^m)}$. Now observe $a, b \in C$ and $\gamma \in \widehat{C_G(a^m)}$, so by the first hypothesis there exists $g \in G$ such that $a^g = b$. \square

Remark 2.2. *The second equality of the second condition of Proposition 2.1 is equivalent to hereditary conjugacy separability of H (see Proposition 2.3 in [M]).*

The proposition implies the following

Theorem 2.3. *Let G be a torsion free group containing a conjugacy separable subgroup H of finite index. Suppose that for every $1 \neq h \in H$,*

- $C_G(h)$ is conjugacy separable;
- $\overline{C_H(h)} = \widehat{C_H(h)} = C_{\widehat{H}}(h)$;

Then G is conjugacy separable.

Remind that a fully residually free group is a group that satisfies the following condition: for each finite subset $K \subset G$ of non trivial elements, there exist a homomorphism $\varphi : G \rightarrow F$ to some free group F such that the restriction $\varphi|_K$ is injective. A finitely generated residually free group is called limit group.

We apply Theorem 2.3 to prove

Theorem 2.4. *A torsion free group commensurable with a limit group is conjugacy separable.*

Proof. Let G be a torsion free virtually limit group and L a limit subgroup of finite index in G . By commutative transitivity property $C_L(h)$ is finitely generated abelian for every $h \in L$. Therefore $C_G(h)$ is finitely generated virtually abelian and hence is conjugacy separable, i.e. the first condition of Theorem 2.3 holds. By Lemma 3.5 in [CZ-07] combined with subgroup separability of limit groups [W-2008] the second hypothesis of Theorem 2.3 is also satisfied. Thus the result follows from Theorem 2.3. \square

Theorem 2.5. *Let G be a virtually free-by-cyclic hereditarily conjugacy separable group. Then every element of prime order p is conjugacy distinguished.*

Proof. Pick $a, b \in G$ such that $a = b^\gamma$, where $\gamma \in \widehat{G}$ and a, b have order p . We need to prove that there exist an element $g \in G$ such that $a = b^g$. Let $H = F \rtimes C$ be a free-by-infinite cyclic hereditarily conjugacy separable subgroup of finite index in G . Without loss of generality we can suppose that H is normal in G , and since $\widehat{G} = G\widehat{H}$ we can write $\gamma = g\delta$, where $\delta \in \widehat{H}$ and $g \in G$ and so $a = b^\gamma = b^{g\delta}$. Therefore, $b \in \langle \widehat{H}, a \rangle \cap G = \langle H, a \rangle$, so changing b to b^g we can suppose that $G = \langle H, a \rangle$.

Put $G_0 = G/F_G$, where F_G is the core of F in G . We shall use subindex zero for the images in G_0 .

Claim. The centralizer $C_{H_0}(a_0)$ is cyclic.

Since $F_G = \cap_{i=1}^p F^{a^i}$, H_0 is a subgroup of free abelian group of rank p on which a_0 acts non-trivially unless $H_0 \cong \mathbb{Z}$. Therefore, the centralizer of a_0 in H_0 is cyclic. Indeed, by §34B in [CR-90] non-trivial torsion-free $C_p\mathbb{Z}$ modules of \mathbb{Z} -rank $\leq p$ has an irreducible submodule of \mathbb{Z} -rank $p - 1$. Therefore the submodule of fixed points has \mathbb{Z} -rank at most 1. This finished the proof of the claim.

Let U be the preimage of the centralizer $C_{G_0}(a_0) = \langle C_{H_0}(a_0), a_0 \rangle = C_{H_0}(a_0) \times \langle a_0 \rangle$ and let V be the preimage of $\langle a_0 \rangle$ in G . Then $U = V \rtimes \mathbb{Z}$ (since $U/V = C_{H_0}(a_0)$ unless $U = V$) and by Theorem of Dyer and Scott [DS] $V = *_{i \in I} (F_i \times T_i) * L$, where $T_i = \langle t_i \rangle$ and t_i has order p . Since torsion elements are conjugate to elements of a free factor, we may assume that $a = t_k$ and $b = t_j$ for some $k, j \in I$.

Consider the conjugacy class $Cl_U(a)$ of a in U .

Case 1. $Cl_U(a) = \cup_{i=1}^n Cl_V(t_i)$ is a union of finitely many conjugacy classes $Cl_V(t_i)$ of t_i in V . Then it suffices to prove that $Cl_V(a)$ is closed in V . Since V is free by finite, and so is conjugacy separable $Cl_V(a)$ is indeed closed.

Case 2. $Cl_U(a) = \cup_{i=1}^\infty Cl_V(t_i)$ is a union of infinitely many conjugacy classes $Cl_V(t_i)$ of t_i in V . First, we prove that \mathbb{Z} acts freely on the conjugacy class of the elements in $Cl_U(a)$. To see this, observe that the stabilizer of $Cl_V(a)$ is either trivial or has finite index in V and the latter is discarded by the hypothesis of this case.

Consider the action of $\widehat{\mathbb{Z}}$ on the conjugacy class $Cl_{\overline{U}}(a)$ in the closure of U in \widehat{G} . Then $Stab_{\widehat{\mathbb{Z}}}(a) = Stab_{\widehat{\mathbb{Z}}}(b)$, since a and b are in the same orbit. Hence, $Stab_{\widehat{\mathbb{Z}}}(b) = \{1\}$ and so \mathbb{Z} acts freely on $Cl_U(b)$.

Now consider the abelianization V/V' of V and observe that the torsion elements are conjugated in V if and only if they coincide in V/V' . Therefore, this elements are conjugate in U if, and only if, they are conjugated in U/V' . Thus it is enough to prove that a and b are conjugated in U/V' .

Suppose not. Observe that $U/V' = V/V' \rtimes \mathbb{Z}$ and $\overline{U}/\overline{V'} = \overline{V}/\overline{V'} \rtimes \widehat{\mathbb{Z}}$ and $V/V' = \oplus_{i \in I} (F_i/F'_i \times T_i) \oplus L/L'$. Abusing notation we denote images of a and b in U/V' by the same letters. Since $a = b^\gamma$ there exists $\kappa \in \widehat{\mathbb{Z}}$ with $a = b^\kappa$. Consider the subgroup $S = \langle a, b, \mathbb{Z} \rangle$ in U/V' . Then $\kappa \in \widehat{S}$. Since a and b are not conjugated in S (remember they are not conjugated in U/V') the group $S = (\langle a \rangle \times \langle b \rangle) \wr \mathbb{Z}$ is a restricted wreath product because \mathbb{Z} acts freely on the conjugacy classes of a and b in U/V' . It follows that $S/\mathbb{Z}^S = \langle a \rangle \times \langle b \rangle$. Since $S/\mathbb{Z}^S = \widehat{S}/\widehat{\mathbb{Z}}^{\widehat{S}}$ we get a contradiction with the equality $a = b^\kappa$. \square

Corollary 2.6. *Let $G = H \rtimes C_p$ be a semidirect product of a hereditarily conjugacy separable free-by-cyclic group $H = F \rtimes C$ and a group C_p of order p . Suppose that for every $h \in H$ the centralizer $C_G(h)$ is conjugacy separable and $\overline{C_G(h)} = \widehat{C_G(h)}$. Then G is hereditarily conjugacy separable.*

Proof. Follows from Theorem 2.5 and Proposition 2.1 combined with Remark 2.2. \square

Theorem 2.7. *Let G be a group and H a virtually free-by-cyclic hereditarily conjugacy separable group of G . If $C_G(h)$ is conjugacy separable and $\overline{C_H(h)} = \widehat{C_H(h)}$ for every $h \in H$, then G is conjugacy separable.*

Proof. Replacing G by its core we may assume that H is normal in G . We need to prove that any element $a \in G$ is conjugacy distinguished. If a has infinite order, then $a^m \in H$ for some $m \in \mathbb{N}$, so the result follows from Proposition 2.1 combined with Remark 2.2. Suppose a has finite order. Let t be a power of a such that $c = a^t$ has prime order p . By Corollary 2.6 $\langle c, H \rangle$ is hereditarily conjugacy separable and so by Remark 2.2 $C_H(c)$ is dense in $C_{\widehat{H}}(c)$. Then by Proposition 2.1 a is conjugacy distinguished. \square

The proof of the next proposition was basically communicated to us by Henry Wilton.

Proposition 2.8. *A non-uniform arithmetic lattice in $SL_2(\mathbb{C})$ is virtually free-by-cyclic hereditarily conjugacy separable.*

Proof. Recall that non-uniform arithmetic lattices in $SL_2(\mathbb{C})$ are precisely the subgroups commensurable with Bianchi groups. Therefore we just need to show the existence of a virtually free-by-cyclic hereditarily conjugacy separable subgroup of finite index in a Bianchi group G . By [ALR-01] Bianchi groups are (virtually) geometrically finite subgroups of right angled Coxeter groups and then by Theorem 1.6 in [LR-2008] Bianchi groups have a subgroup H of finite index that is virtual retract of a right angled Coxeter group. By Corollary 2.3 in [M] a right angled Coxeter group is virtually hereditarily conjugacy separable. Since a virtual retract of a hereditarily conjugacy separable group is hereditarily conjugacy separable (see Theorem 3.4 in [CZ1-09]) it follows that H is hereditarily conjugacy separable. On the other hand by [A-08] there exist a finite index surface-by-infinite cyclic subgroup $U = S \rtimes \mathbb{Z}$ in the Bianchi group G . It is well known that Bianchi groups have virtual cohomological dimension 2, so from [B-78, Corollary 1] S is free. Thus $U \cap H$ is the desired subgroup. \square

Theorem 2.9. *Non-uniform arithmetic lattices of $SL_2(\mathbb{C})$ are conjugacy separable.*

Proof. Let G be a non-uniform arithmetic lattices of $SL_2(\mathbb{C})$. Then $C_G(g)$ is finitely generated virtually abelian by Lemma 2.2 in [CZ2-09]. The equality $\overline{C_G(g)} = \widehat{C_G(g)}$ holds since G is subgroup separable. By Proposition 2.8 there exists a finite index hereditarily conjugacy separable subgroup H of G such that $H = F \rtimes \mathbb{Z}$. Therefore the result follows from Theorem 2.7. \square

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