AUTOMORPHISMS OF THE SEMIGROUP OF ENDOMORPHISMS OF FREE ASSOCIATIVE ALGEBRAS

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Abstract. Let $A = A(x_1, ..., x_n)$ be a free associative algebra in the variety of associative algebras $\mathcal{A}$ freely generated over K by a set $X = \{x_1, ..., x_n\}$, $End\, A$ be the semigroup of endomorphisms of A, and $Aut\, End\, A$ be the group of automorphisms of the semigroup $End\, A$. We investigate the structure of the groups $Aut\, End\, A$ and $Aut\, \mathcal{A}^\circ$, where $\mathcal{A}^\circ$ is the category of finitely generated free algebras from $\mathcal{A}$. We prove that the group $Aut\, End\, A$ is generated by semi-inner and mirror automorphisms of $End\, F$ and the group $Aut\, \mathcal{A}^\circ$ is generated by semi-inner and mirror automorphisms of the category $\mathcal{A}^\circ$. This result solves an open Problem formulated in [14].

1. Introduction

Let Θ be a variety of linear algebras over a commutative-associative ring K and $F = F(X)$ be a free algebra from Θ generated by a finite set X. Here X is supposed to be a subset of some infinite universum X^0. Let G be an algebra from Θ and $K_\Theta(G)$ be the category of algebraic sets over G. Here and bellow we refer to [15, 16] for Universal Algebraic Geometry (UAG) definitions used in our work.

The category $K_\Theta(G)$ can be considered from the point of view of the possibility to solve systems of equations in the algebra G. Algebras G_1 and G_2 from Θ are categorically equivalent if the categories $K_\Theta(G_1)$ and $K_\Theta(G_2)$ are correctly isomorphic. Algebras G_1 and G_2 are geometrically equivalent if

$$T''_{G_1} = T''_{G_2}$$

holds for all finite sets X and for all binary relations T on F and $'$ is Galois correspondence between sets in $Hom(F, G)$ and the binary relations on F.

It has been shown in [16] that categorical and geometrical equivalences of algebras are related and their relation is determined by the structure of the group $Aut\, \Theta^0$, where Θ^0 is the category of free finitely generated algebras of Θ. There is a natural connection between a structure of the groups $Aut\, End\, F$, $F \in \Theta$, and $Aut\, \Theta^0$.

Let $\mathcal{A}$ be the variety of associative algebras with (or without) 1, $A = A(x_1, ..., x_n)$ be a free associative algebra in $\mathcal{A}$ freely generated over K by a set $X = \{x_1, ..., x_n\}$. One of our aim here is to describe the group $Aut\, End\, A$ and, as a consequence, to obtain a description of the group $Aut\, \mathcal{A}^\circ$ for the variety of associative algebras over a field K.

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We prove that the group $Aut\, End\, A$ is generated by semi-inner and mirror automorphisms of $End\, A$ and the group $Aut\, \mathcal{A}^\circ$ is generated by semi-inner and mirror automorphisms of the category $\mathcal{A}^\circ$.

Earlier, the description of $Aut\, \mathcal{A}^\circ$ for the variety $\mathcal{A}$ of associative algebras over algebraically closed fields has been given in [11] and, over infinite fields, in [3]. Also in the same works, the description of $Aut\, End\, F(x_1, x_2)$ has been obtained.

Note that a description of the groups $Aut\, End\, F$, $F \in \Theta$, and $Aut\, \Theta^\circ$ for some other varieties Θ has been given in [2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 18].

2. Automorphisms of the semigroup $End\, F$ and of the category Θ^0

We recall the basic definitions we use in the case of the variety $\mathcal{A}$ of associative algebras over a field K.

Let $F = F(x_1, ..., x_n)$ be a finitely generated free algebra of a variety Θ of linear algebras over K generated by a set $X = \{x_1, ..., x_n\}$.

Definition 2.1. [2] An automorphism Φ of the semigroup $End\ F$ of endomorphisms of F is called quasi-inner if there exists a bijection $s : F \to F$ such that $\Phi(\nu) = s\nu s^{-1}$, for any $\nu \in End\ F$; s is called adjoint to Φ.

Definition 2.2. [15] A quasi-inner automorphism Φ of $End\, F$ is called semi-inner if its adjoint bijection $s : F \to F$ satisfies the following conditions:

1. $s(a + b) = s(a) + s(b)$,
2. $s(a \cdot b) = s(a) \cdot s(b)$,
3. $s(\alpha a) = \varphi(\alpha)s(a)$,

for all $\alpha \in K$ and $a, b \in F$ and an automorphism $\varphi : K \to K$. If φ is the identity automorphism of K, we say that Φ is an inner.

Let $A = A(x_1, ..., x_n)$ be a finitely generated free associative algebra over a field K of the variety $\mathcal{A}$. Further, without loss of generality, we assume that associative algebras of $\mathcal{A}$ contain 1.

Definition 2.3. [11] A quasi-inner automorphism Φ of $End\, A$ is called mirror if its adjoint bijection $s : A \to A$ is anti-automorphism of A.

Recall the notions of category isomorphism and equivalence [10]. An isomorphism $\varphi : \mathcal{C} \to \mathcal{D}$ of categories is a functor φ from $\mathcal{C}$ to $\mathcal{D}$ which is a bijection both on objects and morphisms. In other words, there exists a functor $\psi : \mathcal{D} \to \mathcal{C}$ such that $\psi\varphi = 1_{\mathcal{C}}$ and $\varphi\psi = 1_{\mathcal{D}}$.

Let φ_1 and φ_2 be two functors from $\mathcal{C}_1$ to $\mathcal{C}_2$. A functor isomorphism $s : \varphi_1 \longrightarrow \varphi_2$ is a collection of isomorphisms $s_D : \varphi_1(D) \longrightarrow \varphi_2(D)$ defined for all $D \in Ob\, \mathcal{C}_1$ such that for every $\nu : D \longrightarrow B,\ \nu \in Mor\, \mathcal{C}_1$, $B \in Ob\, \mathcal{C}_1$, holds

$$s_B \cdot \varphi_1(\nu) = \varphi_2(\nu) \cdot s_D,$$

i.e., the following diagram is commutative

$$\begin{tikzcd} \varphi_1(D) \arrow[r, "s_D"] \arrow[d, "\varphi_1(\nu)"'] & \varphi_2(D) \arrow[d, "\varphi_2(\nu)"] \\ \varphi_1(B) \arrow[r, "s_B"] & \varphi_2(B) \end{tikzcd}$$

The isomorphism of functors φ_1 and φ_2 is denoted by $\varphi_1 \cong \varphi_2$.

An equivalence between categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $\varphi : \mathcal{C} \to \mathcal{D}$ and $\psi : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $\psi\varphi \cong 1_{\mathcal{C}}$ and $\varphi\psi \cong 1_{\mathcal{D}}$. If $\mathcal{C} = \mathcal{D}$, then we get the notions of automiorphism and autoequivalence of the category $\mathcal{C}$.

For every small category $\mathcal{C}$ denote the group of all its automorphisms by $Aut\,\mathcal{C}$.

We will distinguish the following classes of automorphisms of $\mathcal{C}$.

Definition 2.4. [7, 13] An automorphism $\varphi : \mathcal{C} \to \mathcal{C}$ is equinumerous if $\varphi(D) \cong D$ for any object $D \in Ob\,\mathcal{C}$; φ is stable if $\varphi(D) = D$ for any object $D \in Ob\,\mathcal{C}$; and φ is inner if φ and $1_{\mathcal{C}}$ are naturally isomorphic, i.e., $\varphi \cong 1_{\mathcal{C}}$.

In other words, an automorphism φ is inner if for all $D \in Ob\,\mathcal{C}$ there exists an isomorphism $s_D : D \to \varphi(D)$ such that

$$\varphi(\nu) = s_B \nu s_D^{-1} : \varphi(D) \to \varphi(B)$$

for any morphism $\nu : D \to B$, $B \in Ob\,\mathcal{C}$.

Let Θ be a variety of linear algebras over K. Denote by Θ^0 the full subcategory of finitely generated free algebras $F(X), |X| < \infty$, of the variety Θ.

Definition 2.5. [13] Let A_1 and A_2 be algebras from Θ, δ be an automorphism of K and $\varphi : A_1 \to A_2$ be a ring homomorphism of these algebras. A pair (δ, φ) is called semimomorphism from A_1 to A_2 if

$$\varphi(\alpha \cdot u) = \alpha^{\delta} \cdot \varphi\,(u), \quad \forall \alpha \in K,\ \forall u \in A_1.$$

Define the notion of a semi-inner automorphism of the category Θ^0.

Definition 2.6. [13] An automorphism $\varphi \in Aut\,\Theta^0$ is called semi-inner if there exists a family of semi-isomorphisms $\{s_{F(X)} = (\delta, \tilde{\varphi}) : F(X) \to \tilde{\varphi}(F(X)),\ F(X) \in Ob\ \Theta^0\}$, where $\delta \in Aut\,K$ and $\tilde{\varphi}$ is a ring isomorphism from $F(X)$ to $\tilde{\varphi}(F(X))$ such that for any homomorphism $\nu : F(X) \longrightarrow F(Y)$ the following diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{s_{F(X)}} & \tilde{\varphi}(F(X)) \\ \nu \downarrow & & \downarrow \varphi(\nu) \\ F(Y) & \xrightarrow[s_{F(Y)}]{} & \tilde{\varphi}(F(Y)) \end{array}$$

is commutative.

Now we define the notion of a mirror automorphism of the category $\mathcal{A}^{\circ}$.

Definition 2.7. [16] An automorphism $\varphi \in Aut\,\mathcal{A}^{\circ}$ is called mirror if it does not change objects of $\mathcal{A}^{\circ}$ and for every $\nu : A(X) \to A(Y)$, where $A(X), A(Y) \in Ob\ \mathcal{A}^{\circ}$, it holds

$$\varphi(\nu) : A(X) \to A(Y) \ \text{ such that } \ \varphi(\nu)(x) = \delta(\nu(x)),\ \forall x \in X,$$

where $\delta : A(Y) \to A(Y)$ is the mirror automorphism of $A(Y)$.

Further, we will need the following

Proposition 2.8. [7, 13] For any equinumerous automorphism $\varphi \in Aut\,\mathcal{C}$ there exists a stable automorphism φ_S and an inner automorphism φ_I of the category $\mathcal{C}$ such that $\varphi = \varphi_S\varphi_I$.

3. Quasi-inner automorphisms of the semigroup $End\,F$ for associative and Lie varieties

We will need the standard endomorphisms of free algebra $F = F(x_1, ..., x_n)$ of the variety Θ.

Definition 3.1. [9] Standard endomorphisms of F in the base $X = \{x_1, ..., x_n\}$ are the endomorphisms e_{ij} of F which are determined on the free generators $x_k \in X$ by the rule: $e_{ij}(x_k) = \delta_{jk} x_i,\ x_i \in X,\ i, j, k \in [1n],\ \delta_{jk}$ is the Kronecker delta.

Denote by S_0 a subsemigroup of $End\ F$ generated by $e_{ij},\ i, j \in [1n]$. Further, we will use the following statements

Proposition 3.2. [9] Let $\Phi \in Aut\,End\,F(X)$. Elements of the semigroup $\Phi(S_0)$ are standard endomorphisms in some base $U = \{u_1, ..., u_n\}$ of F if and only if Φ is a quasi-inner automorphism of $End\ F$.

The description of quasi-inner automorphisms of $End\,A(X)$, where $A(X)$ is a free associative algebra with or without 1 over a field K, is following

Proposition 3.3. [4, 9] Let $\Phi \in Aut\,End\,A(X)$ be a quasi-inner automorphism of $End\,A(X)$. Then Φ is either a semi-inner or a mirror automorphism, or a composition of them.

Let us investigate the images of standard endomorphisms under automorphisms of $End\,A$. To this end we introduce endomorphisms of rank 1.

Definition 3.4. We say that an endomorphism φ of A has rank 1, and write this as $rk\,(\varphi) = 1$, if its image $Im\,\varphi$ is a commutative subalgebra of F.

Note that according to Bergman's theorem [1], the centralizer of any non-scalar element of A is a polynomial ring in one variable over K. Thus, an endomorphism φ of A is of rank 1 if and only if $\varphi(A) = K[z]$ for some element $z \in A$.

Proposition 3.5. An endomorphism φ of A is of rank 1 if and only if there exists a non-zero endomorphism $\psi \in End\,A$ such that for any $h \in End\,A$

$$\varphi \circ h \circ \psi = 0 \tag{3.1}$$

Proof. Let $\varphi \in End\,A$ be an endomorphism of rank 1. Let us take the endomorphism $\psi \in End\,A$ such that

$$\psi(x_1) = [x_1, x_2] \text{ and } \psi(x_i) = 0 \text{ for all } i \neq 1.$$

Since $\varphi(A)$ is a commutative subalgebra of F, the condition (3.1) is fulfilled for any $h \in End\,A$.

Conversely, let the condition (3.1) is fulfilled for the endomorphism φ. Assume, on the contrary, that $Im\,\varphi$ is not a commutative algebra. Without loss of generality, it can be supposed that $[\varphi(x_1), \varphi(x_2)] \neq 0$, $x_1, x_2 \in X$. Denote by $R = K[\varphi(x_1), \varphi(x_2)$ a subalgebra of A generated by $\varphi(x_1)$ and $\varphi(x_2)$. It is well known (see [5]) that R is a free non-commutative subalgebra of A.

Since ψ is a non-zero endomorphism of A, there exists $x_i \in X$ such that $\psi(x_i) \neq 0$. Set $P = P(x_1, ..., x_n) = \psi(x_i)$. We wish to show that P is an identity of the algebra R. Assume, on the contrary, that there exist elements $z_1, ..., z_n \in R$ such that $P(z_1, ..., z_n) \neq 0$. Consider sets $\varphi^{-1}(z_i)$, $i \in [1n]$, and choose elements

$y_i \in \varphi^{-1}(z_i), i \in [1n]$, from them. We may construct an endomorphism h of A such that $h(x_i) = y_i$, $i \in [1n]$. Then we have

$$0 = \varphi \circ h \circ \psi(x_i) = P(\varphi \circ h(x_1), ..., \varphi \circ h(x_n)) = P(\varphi(y_1), ..., \varphi(y_n)) = P(z_1, ..., z_n).$$

We arrived at a contradiction. Therefore, P is an identity of R. Since R is a free non-commutative subalgebra of A, it has no non-trivial identities. Thus, $P = 0$. We get a contradiction again. Therefore, $Im\,\varphi$ is a commutative algebra and Proposition is proved. □

It follows directly from this Proposition

Corollary 3.6. Let $\Phi \in Aut\,End\,A$ and $rk\,(\varphi) = 1$. Then $rk\,(\Phi(\varphi)) = 1$.

Definition 3.7. A set of endomorphisms $\mathcal{B}_e = \{e'_{ij} \mid e'_{ij} \in End\,A,\ i, j \in [1n]\}$ of A is called a subbase of $End\,A$ if

1. $e'_{ij}e'_{km} = \delta_{jk}e'_{im}$, $\forall i, j, k, m \in [1n]$,
2. $rk\,(e'_{ij}) = 1$, $\forall i, j \in [1n]$, i.e., there exist elements $z_{ij} \in A$, $i, j \in [1n]$, such that $e'_{ij}(A(X)) = K[z_{ij}]$ for all $i, j \in [1n]$.

Further, for simplicity, we write $z_{ii} = z_i$, $i \in [1n]$.

Definition 3.8. We say that a subbase $\mathcal{B}_e$ is a base collection of endomorphisms of A (or a base of $End\,A$, for short) if $Z = \langle z_i \mid z_i \in A, i \in [1n]\rangle$ is a base of A.

Proposition 3.9. A subbase of endomorphisms $\mathcal{B}_e$ is a base if and only if for any collection of endomorphisms $\alpha_i : A \to A$, $\forall i \in [1n]$, and any subbase $\mathcal{B}_f = \{f'_{ij} \mid i, j \in [1n]\}$ of $End\,A$ there exist endomorphisms φ, $\psi \in End\,A$ such that

$$\alpha_i \circ f'_{ii} = \psi \circ e'_{ii} \circ \varphi, \text{ for all } i \in [1n]. \tag{3.2}$$

Proof. Let a subbase of endomorphisms $\mathcal{B}_e$ be base. Since $rk\,(f'_{ij}) = 1$, $\forall i, j \in [1n]$, there exist elements $y_{ij} \in A$, $i, j \in [1n]$, such that $f'_{ij}(A(X)) = K[y_{ij}]$ for all $i, j \in [1n]$. We define an endomorphisms ψ and φ of A in the following way:

$$\varphi(x_i) = z_i \text{ and } \psi(z_i) = \alpha_i(y_i), \text{ for all } i \in [1n]$$

where $y_i = y_{ii}, \forall i \in [1n]$. Since $Z = \langle z_i \mid z_i \in A, i \in [1n]\rangle$ is a base of A, the definition of the endomorphism ψ is correct. Now, it is easy to check that the condition (3.2) with the given ϕ and ψ is fulfilled.

Conversely, assume that the condition (3.2) is fulfilled for the subbase $\mathcal{B}_e$. Let us prove that $Z = \langle z_i \mid z_i \in A, i \in [1n]\rangle$ is a base of A. Choosing in (3.2) $\alpha_i = e_{ii}$ and $f'_{ij} = e_{ij}$ for all $i, j \in [1n]$, we obtain

$$e_{ii} = \psi \circ e'_{ii} \circ \varphi,$$

i.e., $\psi(e'_{ii}\varphi(x_i)) = x_i$ for all $i \in [1n]$. Denote $t_i = e'_{ii}\varphi(x_i)$. We have $\psi(t_i) = x_i$. Since A is Hopfian, the elements t_i, $i \in [1n]$, form a base of A. Taking into account the equality $e'_{ii}(A(X)) = K[z_i]$, we obtain $t_i = \chi_i(z_i) \in K[z_i]$. Since $t_iz_i = z_it_i, i \in [1n]$, by Bergman's theorem we have $z_i = g_i(t_i)$. Thus, $z_i = g_i(\chi_i(z_i))$. Similarly, $t_i = \chi_i(g_i(t_i))$. Therefore, there exists non-zero elements a_i and b_i in K such that $z_i = a_it_i + b_i$, $i \in [1n]$. Thus, $Z = \langle z_i \mid z_i \in F, \forall i \in [1n]\rangle$ is also a base of A as claimed. □

Now we deduce

Corollary 3.10. Let $\Phi \in Aut\,End\,A$. Then $\mathcal{C} = \{\Phi(e_{ij}) \mid i, j \in [1n]\}$ forms a base collection of endomorphisms of A.

Proof. Since $\Phi(e_{ij})\Phi(e_{km}) = \delta_{jk}\Phi(e_{im})$ and by Corollary 3.6, $rk\,(\Phi(e_{ij})) = 1$, the set $\mathcal{C}$ is a subbase of $End\,A$. It is evident that the condition (3.2) is fulfilled for the subbase $\mathcal{C}$. By Proposition 3.9, $\mathcal{C}$ is a base of $End\,A$. □

Lemma 3.11. *Let $\mathcal{B}_e = \{e'_{ij} \mid e'_{ij} \in End\,A,\, i,j \in [1n]\}$ be a base collection of endomorphisms of $End\,A$. Then there exists a base $S = \langle s_k \,|\, s_k \in A, k \in [1n]\rangle$ of A such that the endomorphisms from $\mathcal{B}_e$ are standard endomorphisms in S.*

Proof. Since $(e'_{ii})^2 = e'_{ii}$, we have $e'_{ii}(z_i) = z_i$, $i \in [1n]$. The equality $e'_{ii}e'_{ij}z_j = e'_{ij}z_j$ implies the existence of a polynomial $f_j(z_i) \in K[z_i]$ such that $e'_{ij}z_j = f_j(z_i)$. Similarly, there exists a polynomial $g_i(z_j) \in K[z_j]$ such that $e'_{ji}z_i = g_i(z_j)$. We have

$$z_j = e'_{jj}z_j = e'_{ji}e'_{ij}z_j = e'_{ji}(f_j(z_i)) = f_j(g_i(z_j)) \ \text{ for all } \ i,j \in [1n].$$

and, in similar way, $z_i = g_i(f_j(z_i))$ for all $i,j \in [1n]$. Thus f_j and g_i are linear polynomials over K in variables z_i and z_j, respectively. Therefore,

$$e'_{ij}z_j = a_jz_i + b_j,\ a_i, b_i \in K \ \text{ and } \ a_i \neq 0. \tag{3.3}$$

Note that $e'_{ij}z_k = e'_{ij}e'_{kk}z_k = 0$ if $k \neq j$. Now we have for $i \neq j$

$$0 = {e'_{ij}}^2 z_j = e'_{ij}(a_jz_i + b_j) = e'_{ij}((b_j) = b_j,$$

i.e., $e'_{ij}z_j = a_jz_i$, $a_j \neq 0$. Let $V = Span(z_1, ..., z_n)$. Then V is the vector space over K with a basis $Z = \langle z_k \,|\, z_k \in A, k \in [1n]\rangle$ and e'_{ij}, $i,j \in [1n]$, are linear operators on V. Set

$$S = \langle s_i = e'_{i1}z_1 \mid z_1 \in Z,\, i \geq 1\rangle.$$

Since $s_i = a_1z_i$, $a_1 \neq 0, i \in [1n]$, we have that S is a base of A. In this base we obtain $e'_{ij}s_k = \delta_{jk}s_i$, $i,j,k \in [1n]$. The proof is complete. □

4. Structure of automorphisms of the semigroup $End\,F$ for associative and Lie varieties

Now we give the description of the groups $Aut\,End\,A$ and $Aut\,\mathcal{A}^\circ$.

Theorem 4.1. *The group $Aut\,End\,A$ is generated by semi-inner and mirror automorphisms of $End\,A$.*

Proof. By Corollary 3.10, the set of endomorphisms $\mathcal{C} = \{\Phi(e_{ij}) \mid \forall i \in [1n]\}$ is a base collection of endomorphisms of A. By Lemma 3.11, there exists a base $S = \langle s_k \,|\, s_k \in A, k \in [1n]\rangle$ such that the endomorphisms $\Phi(e_{ij})$ are standard endomorphisms in S. According to Proposition 3.2, we obtain that Φ is quasi-inner. By virtue of Proposition 3.3, the group $Aut\,End\,A$ is generated by semi-inner and mirror automorphisms of $End\,A$ as claimed. □

Using Theorem 4.1 we prove

Theorem 4.2. *The group $Aut\,\mathcal{A}^\circ$ of automorphisms of the category $\mathcal{A}^\circ$ is generated by semi-inner and mirror automorphisms of the category $\mathcal{A}^\circ$.*

Proof. Let $\varphi \in Aut\,\mathcal{A}^\circ$. It is clear that φ is an equinumerous automorphism. By Proposition 2.8, φ can be represented as the composition of a stable automorphism φ_S and an inner automorphism φ_I. Since a stable automorphism does not change free algebras from $\mathcal{A}^\circ$, we obtain that $\varphi_S \in Aut\,End\,A(x_1, ..., x_n)$. By Theorem 4.1, φ_S is generated by semi-inner and mirror automorphisms of $End\,A$. Using this fact and Reduction Theorem [7, 13], we obtain that the group $Aut\,\mathcal{A}^\circ$ generated by semi-inner and mirror automorphisms of the category $\mathcal{A}^\circ$. This ends the proof. $\square$

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