

# Property (FA) and lattices in $SU(2, 1)$

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## Abstract

In this article we consider Property (FA) for lattices in  $SU(2, 1)$ . First, we prove that  $SU(2, 1; \mathcal{O}_3)$  has Property (FA). Also, we prove that cocompact lattices in  $SU(2, 1)$  corresponding to complex hyperbolic surfaces with Picard number one and  $b_1 = 0$  cannot split as a nontrivial free product with amalgamation. In particular, this applies to the arithmetic lattices in  $SU(2, 1)$  of the second type arising from congruence subgroups studied by Rapoport–Zink and Rogawski; one such example is Mumford’s fake projective plane.

## 1 Introduction

Two important questions in the study of lattices in semisimple Lie groups, and more generally the fundamental groups of any class of manifolds, are whether a lattice or fundamental group splits as a nontrivial free product with amalgamation or admits a homomorphism onto  $\mathbf{Z}$ . Property (FA), originally due to Bass and Serre, encodes precisely when a finitely generated group has neither of these properties - see Theorem 2.3. More generally, one can also ask for these properties in a finite sheeted cover; the virtual- $b_1$  conjecture asks, most famously in the setting of closed hyperbolic 3-manifolds, whether a finite sheeted cover of a closed hyperbolic 3-manifold has fundamental group admitting a homomorphism onto  $\mathbf{Z}$ . Using Kazhdan’s Property (T), one can prove that irreducible lattices in the so-called superrigid Lie groups,  $Sp(n, 1)$  for  $n \geq 2$ ,  $F_{4(-20)}$ , and semisimple Lie groups with  $\mathbf{R}$ -rank at least 2, never have Property (FA) - see [10]. Therefore, all the interesting questions relative to Property (FA) and irreducible lattices in semisimple Lie groups occur for the fundamental groups of real and complex hyperbolic manifolds - lattices in  $SO(n, 1)$  and  $SU(n, 1)$ .

Splittings as a free product with amalgamation for cocompact Fuchsian groups, lattices in  $PSL(2; \mathbf{R})$ , are well understood. For example, by considering a separating curve on a compact Riemann surface it follows that many finite covolume Fuchsian groups split as nontrivial amalgamated products. Cocompact Fuchsian triangle groups are well known to have Property (FA), but  $PSL(2; \mathbf{Z})$ , the  $(2, 3, \infty)$  triangle group,

splits as a free product. Furthermore, all Fuchsian groups are known to be virtually surjective. See [13] for more on splittings of Fuchsian groups.

However, the situation becomes much more complicated for lattices in  $\mathrm{PSL}(2; \mathbb{C})$ . If  $d$  is a square free natural number, Frohman and Fine [7] prove that the Bianchi group  $\mathrm{PSL}(2; \mathcal{O}_d)$  splits as a nontrivial free product with amalgamation for  $d \neq 3$ , where  $\mathcal{O}_d$  denotes the ring of integers in  $\mathbb{Q}(\sqrt{-d})$ , and Serre proves in [20] that  $\mathrm{PSL}(2; \mathcal{O}_3)$  has Property (FA). Using similar techniques to Serre, we prove the following theorem.

**Theorem 1.1.**  *$\mathrm{SU}(2, 1; \mathcal{O}_3)$  and  $\mathrm{PU}(2, 1; \mathcal{O}_3)$  have Property (FA).*

The relative similarity of the proofs for  $\mathrm{PSL}(2; \mathcal{O}_3)$  and  $\mathrm{SU}(2, 1; \mathcal{O}_d)$  begs the question as to how much further this analogy between  $\mathrm{PSL}(2; \mathcal{O}_d)$  to  $\mathrm{SU}(2, 1; \mathcal{O}_d)$  carries. A theorem of Kazhdan [11] implies that all  $\mathrm{SU}(2, 1; \mathcal{O}_d)$  are virtually surjective, but no explicit homomorphisms are known for  $d \neq 3$  - neither is there a known presentation for these groups when  $d \neq 3$  - so we pose:

**Question.** Does  $\mathrm{SU}(2, 1; \mathcal{O}_d)$  (see Example 2.1) or  $\mathrm{PU}(2, 1; \mathcal{O}_d)$  have Property (FA) for  $d \neq 3$ ?

Also of particular interest is recent work of F. Calageri and N. Dunfield in [3]. They construct, assuming certain conjectures in number theory, an infinite tower of hyperbolic rational homology 3-spheres  $M_n$  so that the injectivity radius grows arbitrarily large as  $n \rightarrow \infty$ . However, for all  $n$ ,  $\pi_1(M_n)$  splits as a nontrivial free product with amalgamation (so  $M_n$  is Haken - see §2.14 of [3]). In particular, the question of whether there are non-Haken hyperbolic 3-manifolds with arbitrarily large injectivity radius remains open. In contrast, as a consequence of this article, all fake projective planes (complex hyperbolic cousins to rational homology 3-spheres) cannot admit such a decomposition.

The arithmetic lattices of second type, which are constructed using cyclic division algebras of degree three equipped with an involution of the second kind, also admit some higher rank features. The combination of work of Corlette [4] and Gromov–Schoen [9] implies that irreducible quaternionic hyperbolic lattices satisfy the arithmeticity and superrigidity of higher rank lattices, as proven by Margulis (see [14] chapter 0). This leads to the natural question as to what extent these types of rigidity results can hold for real and complex hyperbolic lattices - i.e. lattices in  $\mathrm{SO}(n, 1)$  and  $\mathrm{SU}(n, 1)$ . For example, it is known that non-arithmetic lattices exist in  $\mathrm{SO}(n, 1)$  for all  $n$  [8] and in  $\mathrm{SU}(n, 1)$  for  $n = 2, 3$  [5].

When arising from congruence subgroups, arithmetic lattices of the second type also have several properties that are remarkably similar to the superrigid lattices, including non-archimedean and archimedean superrigidity-like properties and vanishing first cohomology - see §4. As Rogawski proves in [19], these lattices have  $b_1 = 0$ , and Blasius and Rogawski prove in [1] that these lattices have Picard number one. In fact, it is a question attributed to Rogawski as to whether all lattices in  $\mathrm{SU}(2, 1)$  satisfying these criteria are necessarily arithmetic and of the second type. We strengthen the superrigid-like analogy for these lattices with the following theorem.

**Theorem 1.2.** *Let  $\Gamma < \mathrm{SU}(2, 1)$  be a torsion-free cocompact lattice such that  $\mathbf{H}_{\mathbb{C}}^2/\Gamma$  has Picard number one and  $b_1 = 0$ . Then  $\Gamma$  does not split as a nontrivial free product with amalgamation.*

With the assumption that  $b_1 = 0$ , this immediately implies that  $\Gamma$  has Property (FA). The manifold assumption of Theorem 4.1 restricts us to the torsion free setting, however with Selberg's lemma we also prove the following corollary.

**Corollary 1.3.** *Every arithmetic lattice  $\Gamma < \mathrm{SU}(2, 1)$  of second type arising from a congruence subgroup has Property (FA).*

*Proof.* If  $\Gamma$  is torsion free, this is a direct application of Theorem 1.2. If  $\Gamma$  has torsion, it suffices to show that  $\Gamma$  has a finite index normal subgroup with Property (FA). Selberg's lemma implies that  $\Gamma$  has a finite index torsion-free subgroup  $\Gamma'$  that by construction arises from a congruence subgroup. Theorem 1.2 implies that  $\Gamma'$  has Property (FA), so  $\Gamma$  must have Property (FA).  $\square$

As mentioned above, one important class of complex hyperbolic manifolds that satisfy the conditions of Theorem 1.2 are fake projective planes, compact algebraic surfaces with the same betti numbers as  $\mathbb{CP}^2$ . It follows from Yau's solution to the Calabai conjecture (see [12]) that all fake projective places are complex hyperbolic surfaces and that there are only finitely many up to homeomorphism. In fact, it is proven independently in [12] and [21] that all fake projective planes are arithmetic of the second type. The first such example was constructed by Mumford in [16], and his construction implies that the corresponding lattice is  $\mathrm{SU}(2, 1)$  is of the second kind arising from a congruence subgroup, and more recently, Prasad and Yeung [17] classified all fake projective planes using arithmetic techniques. As any fake projective plane satisfies the conditions of Theorem 1.2 by assumption, we have:

**Corollary 1.4.** *The fundamental group of any fake projective plane does not split as a nontrivial free product with amalgamation.*

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## 2 Preliminaries

Here, we collect the definitions and facts necessary for later sections.

### 2.1 Complex Hyperbolic Surfaces

Here, we briefly recall the construction of the complex hyperbolic plane,  $\mathbf{H}_{\mathbb{C}}^2$ . Consider the Hermitian form on  $\mathbb{C}^3$  of signature  $(2, 1)$  given by

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

which in coordinates is

$$\langle z, w \rangle = z_1 \bar{w}_3 + z_2 \bar{w}_2 + z_3 \bar{w}_1,$$

and let  $N_-$  denote the collection of  $z \in \mathbf{C}^3$  such that  $\langle z, z \rangle < 0$ . Then,  $\mathbf{H}_{\mathbf{C}}^2$  is the projective image of  $N_-$ , which can be canonically identified with the open unit ball in  $\mathbf{C}^2$  with the Bergman metric. It is clear from this construction that we obtain biholomorphic isometries of  $\mathbf{H}_{\mathbf{C}}^2$  from the group

$$\mathrm{SU}(J) = \{A \in \mathrm{SL}(3; \mathbf{C}) : A^* J A = J\},$$

where  $*$  denotes conjugate transpose. We shall denote  $\mathrm{SU}(J)$  by  $\mathrm{SU}(2, 1)$ . We should remark that this is somewhat nonstandard notation, but will be convenient for consistency with the notation of Falbel and Parker [6] that we will need later.

Then, since we project to obtain  $\mathbf{H}_{\mathbf{C}}^2$ , the group of biholomorphic isometries of  $\mathbf{H}_{\mathbf{C}}^2$  are isomorphic to  $\mathrm{PU}(2, 1)$ , where  $\mathrm{SU}(2, 1)$  is a 3-fold cover of  $\mathrm{PU}(2, 1)$  by the subgroup generated by  $\zeta_3 I$ , for  $I$  the identity matrix and  $\zeta_3$  a primitive cube root of unity. This allows us to identify the fundamental groups of complex hyperbolic surfaces and 2-orbifolds, finite volume quotients of  $\mathbf{H}_{\mathbf{C}}^2$  by discrete groups of isometries, with lattices in  $\mathrm{SU}(2, 1)$ .

## 2.2 Arithmetic lattices in $\mathrm{SU}(2, 1)$

See [15] for a complete treatment of arithmetic lattices in  $\mathrm{SU}(n, 1)$ ; our treatment is heavily influenced by these notes. For  $n = 2$  there are two distinct constructions of arithmetic lattices, which we will call arithmetic lattices of the first and second type.

To construct arithmetic lattices of the first type, we start with a totally real number field  $F$  and an imaginary quadratic extension  $E/F$  with Galois embeddings  $\sigma_1, \dots, \sigma_n : E \rightarrow \mathbf{C}$  and ring of integers  $\mathcal{O}_E$ . Then, choose an  $E$ -defined Hermitian matrix  $H \in \mathrm{GL}(3; \mathbf{C})$  such that  $H$  has signature  $(2, 1)$  at  $\sigma_1$  and signature  $(3, 0)$  at  $\sigma_i$  for all  $i$  arising from a different Galois embedding of  $F$ . Finally, for an  $\mathcal{O}_E$ -order  $\mathcal{O}$  we define

$$\mathrm{SU}(H; \mathcal{O}) = \{A \in \mathrm{SL}(3; \mathcal{O}) : A^* H A = H\}$$

where  $*$  denotes the conjugate transpose. Under equivalence of Hermitian forms, we can associate  $\mathrm{SU}(H, \mathcal{O})$  with a lattice in  $\mathrm{SU}(2, 1)$  under the isomorphism  $\mathrm{SU}(H) \cong \mathrm{SU}(2, 1)$ . Then, we call any lattice in  $\mathrm{SU}(2, 1)$  commensurable with  $\mathrm{SU}(H, \mathcal{O})$  an *arithmetic lattice of the first type*.

**Example 2.1.** Let  $F = \mathbf{Q}$ ,  $E = \mathbf{Q}(\sqrt{-d})$  for  $d$  a square free natural number, and let  $\mathcal{O}_d$  denote the ring of integers in  $E$ . We take  $J$  as in §2.1, and then  $\mathrm{SU}(J; \mathcal{O}_d) = \mathrm{SU}(2, 1; \mathcal{O}_d)$  is a non-cocompact arithmetic lattice of the first type. As these lattices visibly contain unipotent elements, Godement's compactness criterion implies that  $\mathrm{SU}(2, 1; \mathcal{O}_d)$  is a non-cocompact lattice.

Arithmetic lattices of the second type are constructed as follows. Again, choose a totally real number field  $F$ , an imaginary quadratic extension  $E/F$  with ring of

integers  $\mathcal{O}_E$ . Also, choose a degree three Galois extension  $L/E$  with  $\text{Gal}(L/E) = \langle \theta \rangle$  and let  $K/F$  be the degree three totally real subfield of  $L$ . For an element  $\alpha \in E$  such that

$$N_{E/F}(\alpha) \in N_{K/F}(K^\times), \quad \alpha \notin N_{L/E}(L^\times),$$

where  $N_{k/k'}$  denotes the field norm, we define the degree three cyclic algebra

$$A = (L/E, \theta, \alpha) = \left\{ \sum_{i=0}^2 \beta_i X^i : X^3 = \alpha, X\beta = \theta(\beta)X \text{ for } \beta, \beta_i \in L \right\}.$$

A theorem of Wedderburn implies that this is a division algebra by our specific choice of  $\alpha$ . Also, our selection of  $\alpha$  also ensures, by a theorem of Albert, that  $A$  admits an involution  $\tau$  such that the restriction  $\tau|_E$  from the natural inclusion  $E \rightarrow A$  is complex conjugation. We call such an involution an *involution of the second kind*, and we define a *Hermitian element*  $h$  of an algebra equipped with such an involution  $\tau$  to be an element such that  $h = \tau(h)$ ; notice that this is precisely the usual notion of Hermitian when  $h$  is a matrix and  $\tau$  is conjugate transposition.

Then, for a Hermitian element  $h \in A$  and an  $\mathcal{O}_E$ -order  $\mathcal{O}$  of  $A$ , we define

$$\text{SU}(h, \mathcal{O}) = \{x \in \mathcal{O} : \tau(x)hx = h\}.$$

Since  $A \otimes_E \mathbf{C} \cong \text{M}(3, \mathbf{C})$  gives us an isomorphism  $\text{SU}(h) \cong \text{SU}(2, 1)$ , we can identify  $\text{SU}(h, \mathcal{O})$  with a lattice in  $\text{SU}(2, 1)$ , and we call any lattice commensurable with  $\text{SU}(h, \mathcal{O})$  an *arithmetic lattice of the second type*. Since  $A$  is a division algebra, Godement's compactness criterion implies that all such lattices are cocompact.

**Example 2.2 (Mumford's Fake  $\mathbf{CP}^2$  [16]).** We will not construct Mumford's example [16]; we only give the arithmetic construction commensurable with it. However, Mumford's construction also implies that the corresponding lattice in  $\text{SU}(2, 1)$  commensurable with ours actually arises from a congruence subgroup.

For  $\zeta_7$  a primitive 7<sup>th</sup> root of unity,  $F = \mathbf{Q}$ ,  $E = \mathbf{Q}(\sqrt{-7})$ , and  $L = \mathbf{Q}(\zeta_7)$ , let  $\lambda = (-1 + \sqrt{-7})/2$ ,  $\alpha = \lambda/\bar{\lambda}$ , and  $\theta$ , the generator of  $\text{Gal}(L/E)$ , which is given by  $\zeta_7 \mapsto \zeta_7^2$ . Then,  $A = (L/E, \theta, \alpha)$  has the involution of the second kind  $\tau(X) = \bar{\alpha}X^2$ ,  $\tau(\beta) = \bar{\beta}$  for  $\beta \in E$ . Finally, define the Hermitian form  $h$  and  $\mathcal{O}_E$ -order  $\mathcal{O}$  in  $A$  given respectively by

$$\begin{aligned} h &= \bar{\lambda}X^2 - \bar{\lambda}X + (\lambda - \bar{\lambda}) \\ \mathcal{O} &= \mathcal{O}_L \oplus \bar{\lambda}X\mathcal{O}_L \oplus \bar{\lambda}X^2\mathcal{O}_L. \end{aligned}$$

The involution  $\tau$  of second kind is explicitly given by

$$\tau(\beta) = \bar{\beta}, \quad \tau(X) = \bar{\alpha}X^2.$$

Then,  $\Gamma = \text{SU}(h, \mathcal{O})$  is an arithmetic lattice of the second type commensurable with Mumford's fake  $\mathbf{CP}^2$ .

### 2.3 Property (FA)

If  $\mathcal{T}$  is a tree with an action by a group  $G$  (without inversion), we denote by  $\mathcal{T}^G$  the subtree of fixed points of the  $G$ -action. We say  $G$  has Property (FA) if  $\mathcal{T}^G \neq \emptyset$  for every tree  $\mathcal{T}$ . We now quote three fundamental results in the study of groups with Property (FA).

**Theorem 2.3 (Theorem 15 on p. 58 of [20]).** *A group  $G$  has Property (FA) if and only if*

1.  $G$  is finitely generated.
2.  $G$  does not split as a nontrivial free product with amalgamation.
3.  $G$  does not admit a homomorphism onto  $\mathbb{Z}$ .

The following two propositions will be crucial in the proof of Theorem 1.1.

**Proposition 2.4 (Proposition 26 on p. 64 of [20]).** *Suppose  $G$  is a group with subgroups  $A = \langle a_i \rangle$  and  $B = \langle b_j \rangle$  with  $G = \langle A, B \rangle$  and that  $G$  acts on a tree  $\mathcal{T}$ . If  $\mathcal{T}^A, \mathcal{T}^B \neq \emptyset$  and every  $a_i b_j$  has a fixed point on  $\mathcal{T}$ , then  $\mathcal{T}^G \neq \emptyset$ .*

**Proposition 2.5 (see ex. 4 on p. 66 of [20]).** *Suppose  $G$  is a finitely presented group and  $N \trianglelefteq G$  a nilpotent subgroup such that every subgroup  $H \trianglelefteq N$  with  $N/H \cong \mathbb{Z}$  is not normal in  $G$ . If  $G/N$  has Property (FA), then  $G$  also has Property (FA).*

## 3 The proof of Theorem 1.1

Let  $\Gamma_3$  denote the group  $\mathrm{SU}(2, 1; \mathcal{O}_3)$ , where we consider  $\mathrm{SU}(2, 1; \mathcal{O}_d)$  to be the subgroup of  $\mathrm{SL}(3; \mathcal{O}_d)$  preserving the Hermitian form

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

of signature (2,1). Also, let  $\mathcal{D}(\mathcal{O}_3)$  denote the diagonal subgroup of  $\Gamma_3$  and  $\mathcal{N}(\mathcal{O}_3)$  the subgroup of strictly upper triangular matrices, which is a lattice in the 3-dimensional Heisenberg group  $\mathcal{N}$ ; the Borel subgroup of upper triangular matrices is then

$$\mathcal{B}(\mathcal{O}_3) = \mathcal{N}(\mathcal{O}_3) \rtimes \mathcal{D}(\mathcal{O}_3).$$

Similar to Serre's proof for  $\mathrm{PSL}(2; \mathcal{O}_3)$ , we will make use of a particular presentation of  $\mathrm{PU}(2, 1; \mathcal{O}_3)$ .

**Theorem 3.1 (Falbel–Parker [6]).**

$$\mathrm{PU}(2, 1, \mathcal{O}_3) = \langle W, P, QP^{-1} : W^2, (QP^{-1})^6, (WP)^3, [W, QP^{-1}], P^3Q^{-2} \rangle,$$

where  $\langle P, QP^{-1} \rangle$  generates the Borel subgroup.

First, we claim that the Borel subgroup,  $\mathcal{B}(\mathcal{O}_3)$ , has Property (FA). It follows immediately from the presentation that the Borel subgroup has finite abelianization, so it cannot map onto  $\mathbf{Z}$ . Indeed, the abelianization has  $P^3 = Q^2$  and  $P^6 = Q^6$ , implying that  $Q^4 = Q^6$ , so  $Q^2 = 1$ , which implies that  $P^3 = 1$ .

To show that it cannot split as a nontrivial free product with amalgamation, Falbel and Parler also prove that the Borel subgroup fits into a short exact sequence

$$1 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{B}(\mathcal{O}_3) \longrightarrow \Delta(2, 3, 6) \longrightarrow 1.$$

It follows from Proposition 2.4 that  $\Delta(2, 3, 6)$  has Property (FA), it cannot split as a free product with amalgamation. Since the  $\mathbf{Z}$  factor is central in  $\mathcal{B}(\mathcal{O}_3)$ , if  $\mathcal{B}(\mathcal{O}_3)$  splits as a free product with amalgamation then the  $\mathbf{Z}$  subgroup must be contained in the amalgamating subgroup. However, this implies that the short exact sequence induces a nontrivial free product with amalgamation for  $\Delta(2, 3, 6)$ , which is a contradiction.

Now, we apply Proposition 2.4 to  $\mathrm{PU}(2, 1; \mathcal{O}_3) = \langle W, \mathcal{B}(\mathcal{O}_3) \rangle = \langle A, B \rangle$ , where  $\langle W \rangle \cong \mathbf{Z}/2\mathbf{Z}$  has Property (FA) since it is a finite group. In other words, given an action of  $\mathrm{PU}(2, 1; \mathcal{O}_3)$  on a tree  $\mathcal{T}$ , we know that  $\mathcal{T}^A, \mathcal{T}^B \neq \emptyset$ , so we need only prove that the products  $WP$  and  $W(QP^{-1})$  have fixed points on  $\mathcal{T}$ . This follows from our presentation, as finite order elements necessarily have fixed points on  $\mathcal{T}$  and

$$(WP)^3 = 1, \quad (WQP^{-1})^6 = W^6(QP^{-1})^6 = 1.$$

Therefore,  $\mathrm{PU}(2, 1; \mathcal{O}_3)$  has Property (FA).

Finally, to show that  $\mathrm{SU}(2, 1; \mathcal{O}_3)$  has Property (FA), we apply Proposition 2.5 to the short exact sequence

$$1 \longrightarrow \mathbf{Z}/3\mathbf{Z} \longrightarrow \mathrm{SU}(2, 1; \mathcal{O}_3) \longrightarrow \mathrm{PU}(2, 1; \mathcal{O}_3) \longrightarrow 1$$

and the proof is complete.  $\square$

## 4 The proof of Theorem 1.2

In order to prove Theorem 1.2, we will need some additional results about the Kähler structure of compact complex hyperbolic surfaces. Recall that a Riemannian manifold  $(X, g)$  is a *Kähler manifold* if it admits an integrable almost complex structure  $J \in \mathrm{End}(TX)$  with  $J^2 = -\mathrm{Id}$  such that the form  $\omega(X, Y) = g(JX, Y)$  is closed. We call a group  $\Gamma$  a *Kähler group* if it is the fundamental group of a compact Kähler manifold. In particular, all cocompact lattices in  $\mathrm{SU}(n, 1)$  are Kähler groups, as they are complex projective varieties. The following striking theorem connects the geometry of a Kähler manifold with the structure of its fundamental group.

**Theorem 4.1 (Gromov–Schoen [9]).** *Let  $X$  be a compact Kähler manifold with fundamental group  $\Gamma = \Gamma_1 *_{\Delta} \Gamma_2$  where  $\Delta$  is index at least 2 in  $\Gamma_1$  and index at least 3 in  $\Gamma_2$ , where either index is allowed to be infinite. Then  $X$  maps holomorphically onto a compact Riemann surface.*

If  $H^{1,1}(X)$  denotes the collection of 2-forms on a complex manifold  $X$  that split into holomorphic and antiholomorphic part, define the Picard number of  $X$  to be the rank of  $H^{1,1}(X) \cap H^2(X, \mathbf{Q})$ . We will say that a torsion-free lattice  $\Gamma < \mathrm{SU}(2, 1)$  has Picard number one if the corresponding quotient manifold  $\mathbf{H}_{\mathbf{C}}^2/\Gamma$  has Picard number one. Finally, we make use of the following lemma, due to Yeung [21], whose proof we include for completeness.

**Lemma 4.2 (Yeung [21]).** *If  $X$  is an algebraic surface with Picard number one then  $X$  admits no nontrivial holomorphic map onto a compact Riemann surface.*

*Proof.* Let  $f : X \rightarrow \Sigma$  be a nontrivial holomorphic map from  $X$  to a compact Riemann surface  $\Sigma$ . Then, the fundamental class  $[\Sigma]$  pulls back to a non-torsion element  $\sigma \in H^{1,1} \cap H^2(X; \mathbf{Z})$ . Since  $X$  is of Picard number one, this is some nonzero multiple of the generator  $\theta$  of  $H^{1,1} \cap H^2(X; \mathbf{Z})$ , which implies that the push-forward of  $\theta$  is a nontrivial cycle. Then, generic fibers of  $f$  are one-dimensional over  $\mathbf{C}$ , and if  $\alpha$  is the cohomology class representing a generic fiber it is also a nonzero multiple of  $\theta$ . Then, since  $\theta$  has a nontrivial push-forward,  $\alpha$  must also have a nonzero push-forward. However, generic fibers necessarily have trivial push-forward, which is a contradiction.  $\square$

Finally, we cite three theorems that will allow us to apply the above results to the complex hyperbolic surfaces under consideration.

**Theorem 4.3 (Rogawski [19]).** *Let  $\Gamma$  be an arithmetic lattice in  $\mathrm{SU}(2, 1)$  of the second type arising from a congruence subgroup and  $X = \mathbf{H}_{\mathbf{C}}^2/\Gamma$ . Then*

$$H^1(X, \mathbf{Q}) = H^1(\Gamma, \mathbf{Q}) = 0.$$

*In particular,  $\mathrm{rank}(H_1(X, \mathbf{Q})) = b_1(X) = 0$ .*

The following theorem is often credited to [19], but the book contains no mention of such a result. In fact, for our lattices [19] tells us about the cohomology in every dimension except 2. Rogawski kindly provided a copy of the correct reference - see Theorem 3 of [1].

**Theorem 4.4 (Blasius–Rogawski [1]).** *If  $\Gamma$  is an arithmetic lattice in  $\mathrm{SU}(2, 1)$  of the second kind arising from a congruence subgroup, then  $\Gamma$  has Picard number one.*

As mentioned briefly in §1, it is a question attributed to Rogawski as to whether arithmetic lattices of the second kind are the only lattices with Picard number one and  $b_1 = 0$ . Remarkable as such a question may sound, Klingler [12] and Yeung [21] give serious credibility to this question. The first indication that such a result may hold was given by the following archimedean superrigidity-like theorem.

**Theorem 4.5 (Reznikov [18]).** *Let  $X = \mathbf{H}_{\mathbf{C}}^2/\Gamma$  be a complex hyperbolic manifold with Picard number one and  $b_1 = 0$ . Then, any representation of  $\Gamma$  to  $\mathrm{SL}(3; \mathbf{C})$  has compact Zariski closure or can be extended to a totally geodesic homomorphism of  $\mathrm{SU}(2, 1)$  into  $\mathrm{SL}(3; \mathbf{C})$ .*

Now, we are ready to begin the proof of Theorem 1.2.



*Proof of Theorem 1.2.* Let  $\Gamma < \mathrm{SU}(2, 1)$  be a torsion-free cocompact lattice with Picard number one and  $b_1 = 0$ , and suppose that  $\Gamma$  splits as a nontrivial free product with amalgamation  $\Gamma_1 *_\Delta \Gamma_2$ . Theorem 4.1 combined with Lemma 4.2 allows us to assume that we have  $[\Gamma_i : \Delta] = 2$  for  $i = 1, 2$ . To see this, notice that if  $\Delta$  has index at least 3 (possibly infinite) in either of the  $\Gamma_i$ , Theorem 4.1 gives a holomorphic map onto a compact Riemann surface, which is in contradiction with Lemma 4.2. Also, notice that  $\Gamma = \Gamma_1 *_\Delta \Gamma_2$  with  $[\Gamma_i : \Delta] = 2$  for  $i = 1, 2$  if and only if  $\Gamma$  surjects the infinite dihedral group  $D_\infty$ .

**Remark.** It is tempting at this point to say that we are done, since  $\Gamma$  must then have an index two subgroup  $\Gamma'$  admitting a homomorphism onto  $\mathbf{Z}$  arising from the index two subgroup of  $D_\infty$ . However, we do not know whether this subgroup is congruence, so we do not know that  $b_1(\Gamma') = 0$ . In fact, it is a theorem of Buser and Sarnak [2] that all but finitely many arithmetic Fuchsian groups admit an index (at most) two non-congruence subgroup. As the congruence subgroup problem is unknown for the lattices under consideration here, we cannot rule out this phenomenon.

We now construct a family of faithful representations of  $D_\infty$  into  $\mathrm{SL}(3; \mathbf{C})$  that factor through the inclusion  $\mathrm{GL}(2; \mathbf{C}) \rightarrow \mathrm{SL}(3; \mathbf{C})$  given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & \frac{1}{\det A} \end{pmatrix},$$

and that have eigenvalues off the unit circle  $S^1$ . To complete the proof with such a representation  $\rho$ , let  $\bar{\rho} : \Gamma \rightarrow \mathrm{SL}(3; \mathbf{C})$  be the representation obtained by composing the natural surjection  $\Gamma \rightarrow D_\infty$  with  $\rho$ . It follows that  $\bar{\rho}(\Gamma)$  cannot have compact Zariski closure, since it has arbitrarily large eigenvalues. It also follows that it does not arise from a totally geodesic embedding  $\mathrm{SU}(2, 1) \rightarrow \mathrm{SL}(3; \mathbf{C})$ , since this would produce a totally geodesic embedding of  $\mathrm{SU}(2, 1)$  in  $\mathrm{GL}(2; \mathbf{C})$ , which is impossible. This contradicts Theorem 4.5 and completes our proof.

To construct the representation of  $D_\infty$ , present  $D_\infty$  as  $\langle r, s : s^2, sr sr \rangle$  and consider the matrices in  $\mathrm{GL}(2; \mathbf{C})$  given by

$$R = \begin{pmatrix} \alpha & \beta \\ 2\mathrm{Im}(\alpha)i & \bar{\alpha} \end{pmatrix}, \quad S = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}.$$

A pair of calculations shows that  $S^2 = I$  and

$$SRS^{-1} = \det(R)R^{-1},$$

so if  $\det(R) = \alpha\bar{\alpha} - 2\mathrm{Im}(\alpha)\beta i = 1$ , we obtain a representation  $\rho$  of  $D_\infty$  into  $\mathrm{SL}(2; \mathbf{C})$ . Furthermore, if the eigenvalues of  $R$  lie off the unit circle, it follows that this representation will be faithful and that the image cannot lie in a conjugate of the unitary group, and thus does not have compact Zariski closure. Since the equation for the eigenvalues of  $R$  is

$$1 - 2\mathrm{Re}(\alpha)\lambda + \lambda^2 = 0,$$

we can obtain any nonzero eigenvalue  $\lambda_0$  we like by selecting

$$\operatorname{Re}(\alpha) = \frac{1 + \lambda_0^2}{2\lambda_0},$$

as long as this number lies in  $\mathbf{R}$ . As  $\beta$  and  $\operatorname{Im}(\alpha)$  do not factor into this equation, we still have the necessary freedom to assure that  $\det(R) = 1$ . This provides us with the representation  $\rho$  required above.  $\square$

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