# Automata computation of branching laws for endomorphisms of Cuntz algebras 

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#### Abstract

In our previous articles, we have presented a class of endomorphisms of the Cuntz algebras which are defined by polynomials of canonical generators and their conjugates. We showed the classification of some case under unitary equivalence by help of branching laws of permutative representations. In this article, we construct an automaton which is called the Mealy machine associated with the endomorphism in order to compute its branching law. We show that the branching law is obtained as outputs from the machine for the input of information of a given representation.


## 1 Introduction

In [8, 9], we introduced a class of endomorphisms of the Cuntz algebra $\mathcal{O}_{N}$ which are called permutative endomorphisms. They are given by noncommutative polynomials in canonical generators of $\mathcal{O}_{N}$. Such endomorphisms were motivated by an interest of the following endomorphism $\rho_{\nu}$ of $\mathcal{O}_{3}$ discovered by Noboru Nakanishi:

$$
\left\{\begin{array}{l}
\rho_{\nu}\left(s_{1}\right) \equiv s_{2} s_{3} s_{1}^{*}+s_{3} s_{1} s_{2}^{*}+s_{1} s_{2} s_{3}^{*},  \tag{1.1}\\
\rho_{\nu}\left(s_{2}\right) \equiv s_{3} s_{2} s_{1}^{*}+s_{1} s_{3} s_{2}^{*}+s_{2} s_{1} s_{3}^{*}, \\
\rho_{\nu}\left(s_{3}\right) \equiv s_{1} s_{1} s_{1}^{*}+s_{2} s_{2} s_{2}^{*}+s_{3} s_{3} s_{3}^{*}
\end{array}\right.
$$

where $s_{1}, s_{2}, s_{3}$ are canonical generators of $\mathcal{O}_{3}$. Because $\rho_{\nu}\left(s_{1}\right), \rho_{\nu}\left(s_{2}\right), \rho_{\nu}\left(s_{3}\right)$ satisfy the relation of canonical generators of $\mathcal{O}_{3}$, we can verify that $\rho_{\nu}$ is an endomorphism of $\mathcal{O}_{3} . \rho_{\nu}$ is very concrete but its property is not so clear. In Theorem 1.2 of [8], we proved that $\rho_{\nu}$ is irreducible but not an automorphism by using branching laws of $\rho_{\nu}$ with respect to permutative representations. Especially, $\rho_{\nu}$ is not unitarily equivalent to the canonical endomorphism of $\mathcal{O}_{3}$.

[^0]In general, representations of $\mathrm{C}^{*}$-algebras do not have unique decomposition (up to unitary equivalence) into sums or integrals of irreducibles. However, the permutative representations of $\mathcal{O}_{N}$ do [1, 3, 4]. Because a representation arising from the right transformation of a permutative representation by a permutative endomorphism is also a permutative representation, their branching laws make sense. By such branching laws, permutative endomorphisms are characterized and classified effectively.

Definition 1.1. Let $s_{1}, \ldots, s_{N}$ be canonical generators of $\mathcal{O}_{N}$ and $(\mathcal{H}, \pi)$ be a representation of $\mathcal{O}_{N}$.
(i) $(\mathcal{H}, \pi)$ is a permutative representation of $\mathcal{O}_{N}$ if there is a complete orthonormal basis $\left\{e_{n}\right\}_{n \in \Lambda}$ of $\mathcal{H}$ and a family $f=\left\{f_{i}\right\}_{i=1}^{N}$ of maps on $\Lambda$ such that $\pi\left(s_{i}\right) e_{n}=e_{f_{i}(n)}$ for each $n \in \Lambda$ and $i=1, \ldots, N$.
(ii) For $J=\left(j_{i}\right)_{i=1}^{k} \in\{1, \ldots, N\}^{k},(\mathcal{H}, \pi)$ is $P(J)$ if there is a unit cyclic vector $\Omega \in \mathcal{H}$ such that $\pi\left(s_{J}\right) \Omega=\Omega$ and $\left\{\pi\left(s_{j_{i}} \cdots s_{j_{k}}\right) \Omega\right\}_{i=1}^{k}$ is an orthonormal family in $\mathcal{H}$ where $s_{J} \equiv s_{j_{1}} \cdots s_{j_{k}}$.
(iii) $(\mathcal{H}, \pi)$ is a cycle if there is $J \in\{1, \ldots, N\}^{k}$ such that $(\mathcal{H}, \pi)$ is $P(J)$.

For any $J \in\{1, \ldots, N\}^{k}, P(J)$ exists uniquely up to unitary equivalence. In Theorem 1.3 of $[9$, we showed the following:

Theorem 1.2. Let $\mathfrak{S}_{N, l}$ be the set of all permutations on the set $\{1, \ldots, N\}^{l}$. For $\sigma \in \mathfrak{S}_{N, l}$, let $\psi_{\sigma}$ be the endomorphism of $\mathcal{O}_{N}$ defined by

$$
\begin{equation*}
\psi_{\sigma}\left(s_{i}\right) \equiv u_{\sigma} s_{i} \quad(i=1, \ldots, N) \tag{1.2}
\end{equation*}
$$

where $u_{\sigma} \equiv \sum_{J \in\{1, \ldots, N\}^{l}} s_{\sigma(J)}\left(s_{J}\right)^{*}$. If a representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_{N}$ is $P(J)$ for $J \in\{1, \ldots, N\}^{k}$ and $\sigma \in \mathfrak{S}_{N, l}$, then there are $J_{1}, \ldots, J_{M} \in$ $\bigcup_{m \geq 1}\{1, \ldots, N\}^{m}$ and subrepresentations $\pi_{1}, \ldots, \pi_{M}$ of $\pi \circ \psi_{\sigma}$ such that

$$
\begin{equation*}
\pi \circ \psi_{\sigma}=\pi_{1} \oplus \cdots \oplus \pi_{M}, \tag{1.3}
\end{equation*}
$$

$\pi_{i}$ is $P\left(J_{i}\right)$ and $J_{i} \in \coprod_{n=1}^{N^{l-1}}\{1, \ldots, N\}^{n k}$ for $i=1, \ldots, M$. Further $1 \leq M \leq$ $N^{l-1}$.
$\psi_{\sigma}$ in (1.2) is called the permutative endomorphism of $\mathcal{O}_{N}$ by $\sigma$. The canonical endomorphism of $\mathcal{O}_{N}$ and $\rho_{\nu}$ in (1.1) are permutative endomorphisms.

By the uniqueness of decomposition of permutative representation, the rhs in (1.3) is unique up to unitary equivalence. When $(\mathcal{H}, \pi)$ is $P(J)$ and
$\rho \in \operatorname{End} \mathcal{O}_{N}$, we denote $(\mathcal{H}, \pi \circ \rho)$ by $P(J) \circ \rho$ simply. Then (1.3) can be rewritten as follows:

$$
\begin{equation*}
P(J) \circ \psi_{\sigma}=P\left(J_{1}\right) \oplus \cdots \oplus P\left(J_{M}\right) . \tag{1.4}
\end{equation*}
$$

We call (1.4) by the branching law of $\psi_{\sigma}$ with respect to $P(J)$. The branching law of $\psi_{\sigma}$ is unique up to unitary equivalence of $\psi_{\sigma}$. Concrete such branching laws are already given in [8, [] by direct computation. These branching laws are interesting subjects themselves and they are useful to classify endomorphisms effectively. On the other hand, an automaton is a typical object to consider algorithm of computation in the computer science (5), 6, 7, 10. An automaton is a machine which changes the internal state by an input. A Mealy machine is a kind of automaton with output.

In this article, we show a better algorithm to compute branching law, that is, an algorithm to seek $J_{1}, \ldots, J_{M}$ from a given $J$ in (1.4) by reducing the problem to a semi-Mealy machine $\mathrm{M}_{\sigma}$ as an input $(=J)$ and outputs $\left(=J_{1}, \ldots, J_{M}\right)$ :


If $J=J_{0}^{r}$, that is, $J$ is a sequence of $r$-times repetition of a sequence $J_{0} \in\{1, \ldots, N\}^{k^{\prime}}$ and $r \geq 2$, then there are $z_{1}, \ldots, z_{r} \in U(1)$ such that $P(J)=\bigoplus_{j=1}^{r} P\left(J_{0}\right) \circ \gamma_{z_{j}}$ where $\gamma$ is the gauge action on $\mathcal{O}_{N}$ by Theorem 2.4 (iv) in [9. Because $\gamma_{z} \circ \psi_{\sigma}=\psi_{\sigma} \circ \gamma_{z}$ for each $z$, the branching law of $P(J) \circ \psi_{\sigma}$ is reduced to that of $P\left(J_{0}\right) \circ \psi_{\sigma}$. Therefore it is sufficient to show the case that $J$ is nonperiodic, that is, $J$ is impossible to be written as $J_{0}^{r}$ for $r \geq 2$. Hence we assume that $J$ is nonperiodic.

For $\sigma \in \mathfrak{S}_{N, l}$ with $l \geq 2$ and $J \in\{1, \ldots, N\}^{l}$, we define $\sigma_{1}(J), \ldots, \sigma_{l}(J) \in$ $\{1, \ldots, N\}$ by $\sigma(J)=\left(\sigma_{1}(J), \ldots, \sigma_{l}(J)\right)$ and let $\sigma_{n, m}(J) \equiv\left(\sigma_{n}(J), \ldots, \sigma_{m}(J)\right)$ for $1 \leq n<m \leq l$. Define $\{1, \ldots, N\}^{0} \equiv\{0\}$ for convenience.

Definition 1.3. For $\sigma \in \mathfrak{S}_{N, l}$, a data $\mathrm{M}_{\sigma} \equiv(Q, \Sigma, \Delta, \delta, \lambda)$ is called the semi-Mealy machine by $\sigma$ if $Q, \Sigma, \Delta$ are finite sets,

$$
Q \equiv\left\{q_{K}: K \in\{1, \ldots, N\}^{l-1}\right\}, \quad \Sigma \equiv\left\{a_{j}\right\}_{j=1}^{N}, \quad \Delta \equiv\left\{b_{j}\right\}_{j=1}^{N}
$$

and two maps $\delta: Q \times \Sigma^{*} \rightarrow Q, \lambda: Q \times \Sigma^{*} \rightarrow \Delta^{*}$ are defined by
$\delta\left(q_{K}, a_{i}\right) \equiv\left\{\begin{array}{ll}q_{0} & (l=1), \\ q_{\left(\sigma^{-1}\right)_{2, l}(K, i)} & (l \geq 2),\end{array} \quad \lambda\left(q_{K}, a_{i}\right) \equiv \begin{cases}b_{\sigma^{-1}(i)} & (l=1), \\ b_{\left(\sigma^{-1}\right)_{1}(K, i)} & (l \geq 2)\end{cases}\right.$
for $i=1, \ldots, N$ and $K \in\{1, \ldots, N\}^{l-1}$ where $\Sigma^{*}$ and $\Delta^{*}$ are free semigroups generated by $\Sigma$ and $\Delta$, respectively.

We posteriori define $\delta(q, w a) \equiv \delta(\delta(q, w), a)$ and $\lambda(q, w a) \equiv \lambda(q, w) \lambda(\delta(q, w), a)$ for $q \in Q, w \in \Sigma^{*}$ and $a \in \Sigma$. For a given $J=\left(j_{i}\right)_{i=1}^{k} \in\{1, \ldots, N\}^{k}$, define $Q_{J} \equiv\left\{q \in Q\right.$ : there exists $n \in \mathbf{N}$ s.t. $\left.\delta\left(q,\left(a_{J}\right)^{n}\right)=q\right\}$ where $a_{J} \equiv$ $a_{j_{1}} \cdots a_{j_{k}} \in \Sigma^{*}$ and define an equivalence relation $\sim$ in $Q_{J}$ by $q \sim q^{\prime}$ if there is $n \in \mathbf{N}$ such that $\delta\left(q,\left(a_{J}\right)^{n}\right)=q^{\prime}$. Define $[q] \equiv\left\{q^{\prime} \in Q_{J}: q \sim q^{\prime}\right\}$. Then $[q]$ is a cyclic component of $Q_{J}$ with respect to the iteration of the right action of $a_{J}$ by $\delta$. There are $p_{1}, \ldots, p_{M} \in Q_{J}$ such that the set $Q_{J}$ of periodic points is decomposed into orbits as follows:

$$
\begin{equation*}
Q_{J}=\left[p_{1}\right] \sqcup \cdots \sqcup\left[p_{M}\right] . \tag{1.5}
\end{equation*}
$$

Under these preparations, the main theorem is given as follows:
Theorem 1.4. If $J$ is nonperiodic, then $J_{1}, \ldots, J_{M}$ in 1.4) are obtained by

$$
b_{J_{i}}=\lambda\left(p_{i},\left(a_{J}\right)^{r_{i}}\right) \quad(i=1, \ldots, M)
$$

where $p_{1}, \ldots, p_{M} \in Q_{J}$ are taken as (1.5) and $r_{i} \equiv \#\left[p_{i}\right]$ for $i=1, \ldots, M$.
In Theorem [1.4, if $p_{1}^{\prime}, \ldots, p_{M}^{\prime}$ satisfy (1.4) and $\left[p_{i}^{\prime}\right]=\left[p_{i}\right]$ for each $i$, then the associated $J_{1}^{\prime}, \ldots, J_{M}^{\prime}$ satisfy that $P\left(J_{i}^{\prime}\right)=P\left(J_{i}\right)$ for each $i$. We show a more practical algorithm to compute branching laws by using the Mealy diagram as follows:

The transition diagram (Mealy diagram) $\mathcal{D}(\mathrm{M})$ of a semi-Mealy machine $\mathrm{M}=(Q, \Sigma, \Delta, \delta, \lambda)$ is a directed graph with labeled edges, which has a set $Q$ of vertices and a set $E \equiv\{(q, \delta(q, a), a) \in Q \times Q \times \Sigma: q \in Q, a \in \Sigma\}$ of directed edges with labels. The meaning of $(q, \delta(q, a), a)$ is an edge from $q$ to $\delta(q, a)$ with a label " $a / \lambda(q, a)$ " for $a \in \Sigma$ :

$$
\delta(q, a)=p, \quad \lambda(q, a)=b \quad \Longleftrightarrow \quad q \quad a / b \quad \longrightarrow p
$$

For $\rho_{\nu}$ in (1.1), we compute branching laws by the semi-Mealy machine. Define $\sigma_{0} \in \mathfrak{S}_{3,2}$ by | $J$ | 11 | 12 | 13 | 21 | 22 | 23 | 31 | 32 | 33 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\sigma_{0}(J)$ | 23 | 31 | 12 | 32 | 13 | 21 | 11 | 22 | 33 | Then $\rho_{\nu}=\psi_{\sigma_{0}}$ and $\mathrm{M}_{\sigma_{0}}=\left(\left\{q_{1}, q_{2}, q_{3}\right\},\left\{a_{1}, a_{2}, a_{3}\right\},\left\{b_{1}, b_{2}, b_{3}\right\}, \delta, \lambda\right)$ is given as follows:

| $p$ | $\delta\left(p, a_{1}\right)$ | $\delta\left(p, a_{2}\right)$ | $\delta\left(p, a_{3}\right)$ | $\lambda\left(p, a_{1}\right)$ | $\lambda\left(p, a_{2}\right)$ | $\lambda\left(p, a_{3}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $q_{1}$ | $q_{1}$ | $q_{3}$ | $q_{2}$ | $b_{3}$ | $b_{1}$ | $b_{2}$ |
| $q_{2}$ | $q_{3}$ | $q_{2}$ | $q_{1}$ | $b_{2}$ | $b_{3}$ | $b_{1}$ |
| $q_{3}$ | $q_{2}$ | $q_{1}$ | $q_{3}$ | $b_{2}$ | $b_{1}$ | $b_{3}$ |

From this, $\mathcal{D}\left(\mathrm{M}_{\sigma_{0}}\right)$ is as follows:


According to Theorem 1.4, we compute branching laws for $\rho_{\nu}$ by $\mathcal{D}\left(\mathrm{M}_{\sigma_{0}}\right)$. When the input word is $a_{1}, \delta\left(q_{1}, a_{1}\right)=q_{1}, \delta\left(q_{2}, a_{1}\right)=q_{3}, \delta\left(q_{3}, a_{1}\right)=q_{2}$. Therefore $Q_{1}=\left[q_{1}\right] \sqcup\left[q_{2}\right], r_{1}=1, r_{2}=2$ and there are two cycles $q_{1}$ and $q_{2} q_{3}$ in $Q$ with respect to $a_{1}$. From this, we have output words, $\lambda\left(q_{1}, a_{1}\right)=b_{3}$ and $\lambda\left(q_{2},\left(a_{1}\right)^{2}\right)=b_{2} b_{1}$. Hence $P(1) \circ \rho_{\nu}=P(3) \oplus P(21)=P(3) \oplus P(12)$. where we use a fact that $P\left(j_{p(1)}, \ldots, j_{p(k)}\right)=P\left(j_{1}, \ldots, j_{k}\right)$ for each $p \in \mathbf{Z}_{k}$. Further the following holds:

| input | cycles | outputs | branching law |
| :---: | :---: | :---: | :---: |
| $a_{1}$ | $q_{1}, q_{2} q_{3}$ | $b_{3}, b_{2} b_{1}$ | $P(1) \circ \rho_{\nu}=P(3) \oplus P(12)$ |
| $a_{1} a_{2}$ | $q_{1} q_{1} q_{3} q_{2} q_{2} q_{3}$ | $b_{3} b_{1} b_{1} b_{3} b_{2} b_{2}$ | $P(12) \circ \rho_{\nu}=P(113223)$ |
| $a_{1} a_{2} a_{3}$ | $q_{1} q_{1} q_{3} q_{3} q_{2} q_{2}, q_{2} q_{3} q_{1}$ | $b_{3} b_{1} b_{3} b_{1} b_{3} b_{1}, b_{2} b_{2} b_{2}$ | $P(123) \circ \rho_{\nu}=P(131313) \oplus P(222)$ |
| $a_{1} a_{3} a_{2}$ | $q_{1} q_{1} q_{2} q_{2} q_{3} q_{3}, q_{3} q_{2} q_{1}$ | $b_{3} b_{2} b_{3} b_{2} b_{3} b_{2}, b_{1} b_{1} b_{1}$ | $P(132) \circ \rho_{\nu}=P(232323) \oplus P(111)$ |

In $\sqrt{2}$ we rewrite branching laws by branching function systems and their transformations, and we review known facts about endomorphisms. \$3 is devoted to prove Theorem 1.4 by branching function systems. In $\$ 4$ we show examples of Mealy diagram of the semi-Mealy machine $\mathrm{M}_{\sigma}$ and branching laws of $\psi_{\sigma}$ for concrete $\sigma \in \mathfrak{S}_{N, l}$.

## 2 Branching function systems

In order to compute branching laws of endomorphisms, we introduce branching function systems and their transformations by permutations.

Let $\{1, \ldots, N\}_{1}^{*} \equiv \bigcup_{k \geq 1}\{1, \ldots, N\}^{k}$. For $J \in\{1, \ldots, N\}_{1}^{*}$, the length of $J$ is defined by $k$ when $J \in\{1, \ldots, N\}^{k}$. For $J_{1}=\left(j_{1}, \ldots, j_{k}\right), J_{2}=$ $\left(j_{1}^{\prime}, \ldots, j_{l}^{\prime}\right)$, let $J_{1} \cup J_{2} \equiv\left(j_{1}, \ldots, j_{k}, j_{1}^{\prime}, \ldots, j_{l}^{\prime}\right)$. Especially, we define $(i, J) \equiv$ (i) $\cup J$ for convenience. For $J$ and $k \geq 2, J^{k}=J \cup \cdots \cup J$ ( $k$-times). For
$J=\left(j_{1}, \ldots, j_{k}\right)$ and $\tau \in \mathbf{Z}_{k}$, define $\tau(J) \equiv\left(j_{\tau(1)}, \ldots, j_{\tau(k)}\right)$. For $J_{1}, J_{2} \in$ $\{1, \ldots, N\}_{1}^{*}, J_{1} \sim J_{2}$ if there are $k \geq 1$ and $\tau \in \mathbf{Z}_{k}$ such that $J_{1}, J_{2} \in$ $\{1, \ldots, N\}^{k}$ and $\tau\left(J_{1}\right)=J_{2}$. For $J_{1}=\left(j_{1}, \ldots, j_{k}\right), J_{2}=\left(j_{1}^{\prime}, \ldots, j_{k}^{\prime}\right), J_{1} \prec J_{2}$ if $\sum_{l=1}^{k}\left(j_{l}^{\prime}-j_{l}\right) N^{k-l} \geq 0 . J \in\{1, \ldots, N\}_{1}^{*}$ is minimal if $J \prec J^{\prime}$ for each $J^{\prime} \in\{1, \ldots, N\}_{1}^{*}$ such that $J \sim J^{\prime}$. Define $[1, \ldots, N]^{*} \equiv\left\{J \in\{1, \ldots, N\}_{1}^{*}\right.$ : $J$ is minimal and nonperiodic $\}$. $[1, \ldots, N]^{*}$ is in one-to-one correspondence with the set of all equivalence classes of nonperiodic elements in $\{1, \ldots, N\}_{1}^{*}$ with respect to the equivalence relation $\sim$.

Let $\Lambda$ be an infinite set and $N \geq 2 . f=\left\{f_{i}\right\}_{i=1}^{N}$ is a branching function system on $\Lambda$ if $f_{i}$ is an injective transformation on $\Lambda$ for $i=1, \ldots, N$ such that a family of their images coincides a partition of $\Lambda$. Let $\operatorname{BFS}_{N}(\Lambda)$ be the set of all branching function systems on $\Lambda . f=\left\{f_{i}\right\}_{i=1}^{N} \in \operatorname{BFS}_{N}\left(\Lambda_{1}\right)$ and $g=\left\{g_{i}\right\}_{i=1}^{N} \in \operatorname{BFS}_{N}\left(\Lambda_{2}\right)$ are equivalent if there is a bijection $\varphi$ from $\Lambda_{1}$ to $\Lambda_{2}$ such that $\varphi \circ f_{i} \circ \varphi^{-1}=g_{i}$ for $i=1, \ldots, N$. For $f=\left\{f_{i}\right\}_{i=1}^{N}$, we denote $f_{J} \equiv f_{j_{1}} \circ \cdots \circ f_{j_{k}}$ when $J=\left(j_{1}, \ldots, j_{k}\right) \in\{1, \ldots, N\}^{k}$ and define $f_{0} \equiv i d$. For $x, y \in \Lambda, x \sim y$ (with respect to $f$ ) if there are $J_{1}, J_{2} \in\{1, \ldots, N\}^{*}$ and $z \in \Lambda$ such that $f_{J_{1}}(z)=x$ and $f_{J_{2}}(z)=y$. For $x \in \Lambda$, define $A_{f}(x) \equiv$ $\{y \in \Lambda: x \sim y\} . f=\left\{f_{i}\right\}_{i=1}^{N} \in \operatorname{BFS}_{N}(\Lambda)$ is cyclic if there is an element $x \in \Lambda$ such that $\Lambda=A_{f}(x) .\left\{n_{1}, \ldots, n_{k}\right\} \subset \Lambda$ is a cycle of $f$ if there is $J=\left(j_{1}, \ldots, j_{k}\right)$ such that $f_{j_{1}}\left(n_{1}\right)=n_{k}, f_{j_{2}}\left(n_{2}\right)=n_{1}, \ldots, f_{j_{k}}\left(n_{k}\right)=n_{k-1} . f$ has a cycle if there is a cycle of $f$ in $\Lambda$.

Let $\Xi$ be a set and $\Lambda_{\omega}$ be an infinite set for $\omega \in \Xi$. For $f^{[\omega]}=$ $\left\{f_{i}^{[\omega]}\right\}_{i=1}^{N} \in \operatorname{BFS}_{N}\left(\Lambda_{\omega}\right), f$ is the direct sum of $\left\{f^{[\omega]}\right\}_{\omega \in \Xi}$ if $f=\left\{f_{i}\right\}_{i=1}^{N} \in$ $\operatorname{BFS}_{N}(\Lambda)$ for a set $\Lambda \equiv \coprod_{\omega \in \Xi} \Lambda_{\omega}$ which is defined by $f_{i}(n) \equiv f_{i}^{[\omega]}(n)$ when $n \in \Lambda_{\omega}$ for $i=1, \ldots, N$ and $\omega \in \Xi$. For $f \in \operatorname{BFS}_{N}(\Lambda), f=\bigoplus_{\omega \in \Xi} f^{[\omega]}$ is a decomposition of $f$ into a family $\left\{f^{[\omega]}\right\}_{\omega \in \Xi}$ if there is a family $\left\{\Lambda_{\omega}\right\}_{\omega \in \Xi}$ of subsets of $\Lambda$ such that $f$ is the direct sum of $\left\{f^{[\omega]}\right\}_{\omega \in \Xi}$. For each $f=\left\{f_{i}\right\}_{i=1}^{N} \in \operatorname{BFS}_{N}(\Lambda)$, there is a decomposition $\Lambda=\coprod_{\omega \in \Xi} \Lambda_{\omega}$ such that $\# \Lambda_{\omega}=\infty,\left.f\right|_{\Lambda_{\omega}} \equiv\left\{\left.f_{i}\right|_{\Lambda_{\omega}}\right\}_{i=1}^{N} \in \operatorname{BFS}_{N}\left(\Lambda_{\omega}\right)$ and $\left.f\right|_{\Lambda_{\omega}}$ is cyclic for each $\omega \in \Xi$.

Definition 2.1. (i) For $J \in\{1, \ldots, N\}^{k}, f \in \operatorname{BFS}_{N}(\Lambda)$ is $P(J)$ if $f$ is cyclic and has a cycle $\left\{n_{1}, \ldots, n_{k}\right\}$ such that $f_{J}\left(n_{k}\right)=n_{k}$.
(ii) For $f \in \operatorname{BFS}_{N}(\Lambda)$ and $J \in\{1, \ldots, N\}_{1}^{*}, g$ is a $P(J)$-component of $f$ if $g$ is a direct sum component of $f$ and $g$ is $P(J)$.

For $f \in \operatorname{BFS}_{N}(\Lambda)$ and $\Lambda_{1}, \Lambda_{2} \subset \Lambda$, if $\left.f\right|_{\Lambda_{i}}$ is $P\left(J_{i}\right)$ for $i=1,2$, then either $\Lambda_{1} \cap \Lambda_{2}=\emptyset$ or $\Lambda_{1}=\Lambda_{2}$.

Recall $\mathfrak{S}_{N, l}$ in Theorem [1.2. For $\sigma \in \mathfrak{S}_{N, l}$ and $f=\left\{f_{i}\right\}_{i=1}^{N} \in$
$\operatorname{BFS}_{N}(\Lambda)$, define $f^{(\sigma)}=\left\{f_{i}^{(\sigma)}\right\}_{i=1}^{N} \in \operatorname{BFS}_{N}(\Lambda)$ by

$$
\begin{equation*}
f_{i}^{(\sigma)} \equiv f_{\sigma(i)} \quad(l=1), \quad f_{i}^{(\sigma)}\left(f_{J}(n)\right) \equiv f_{\sigma(i, J)}(n) \quad(l \geq 2) \tag{2.1}
\end{equation*}
$$

for $n \in \Lambda, i=1, \ldots, N$ and $J \in\{1, \ldots, N\}^{l-1}$. If $\sigma \in \mathfrak{S}_{N}=\mathfrak{S}_{N, 1}$ and $f \in$ $\operatorname{BFS}_{N}(\Lambda)$ is $P(J)$, then $f^{(\sigma)}$ is $P\left(J_{\sigma^{-1}}\right)$ where $J_{\sigma^{-1}} \equiv\left(\sigma^{-1}\left(j_{1}\right), \ldots, \sigma^{-1}\left(j_{k}\right)\right)$ for $J=\left(j_{1}, \ldots, j_{k}\right)$. For any $J \in\{1, \ldots, N\}_{1}^{*}$, there is $f \in \operatorname{BFS}_{N}(\Lambda)$ for some set $\Lambda$ such that $f$ is $P(J)$. In this case, for $\sigma \in \mathfrak{S}_{N, l}$, there is $1 \leq M \leq N^{l-1}$ such that $f^{(\sigma)}$ is decomposed into a direct sum of $M$ cycles by Lemma 2.2 in [9]. Furthermore, the length of each cycle is a multiple of that of $J$.

For $N \geq 2$, let $\mathcal{O}_{N}$ be the Cuntz algebra [2], that is, the $\mathrm{C}^{*}$-algebra which is universally generated by $s_{1}, \ldots, s_{N}$ satisfying $s_{i}^{*} s_{j}=\delta_{i j} I$ for $i, j=$ $1, \ldots, N$ and $s_{1} s_{1}^{*}+\cdots+s_{N} s_{N}^{*}=I$. In this article, any representation and endomorphism are assumed unital and $*$-preserving.
$\left(l_{2}(\Lambda), \pi_{f}\right)$ is the permutative representation of $\mathcal{O}_{N}$ by $f=\left\{f_{i}\right\}_{i=1}^{N} \in$ $\operatorname{BFS}_{N}(\Lambda)$ if $\pi_{f}\left(s_{i}\right) e_{n} \equiv e_{f_{i}(n)}$ for $n \in \Lambda$ and $i=1, \ldots, N$. For $J \in$ $\{1, \ldots, N\}_{1}^{*}, P(J)$ in Definition 1.1 is irreducible if and only if $J$ is nonperiodic. For $J_{1}, J_{2} \in\{1, \ldots, N\}_{1}^{*}, P\left(J_{1}\right) \sim P\left(J_{2}\right)$ if and only if $J_{1} \sim J_{2}$ where $P\left(J_{1}\right) \sim P\left(J_{2}\right)$ means the unitary equivalence of two representations which satisfy the condition $P\left(J_{1}\right)$ and $P\left(J_{2}\right)$, respectively. $[1, \ldots, N]^{*}$ is in one-to-one correspondence with the set of equivalence classes of irreducible permutative representations of $\mathcal{O}_{N}$ with a cycle. If $f \in \operatorname{BFS}_{N}(\Lambda)$ and $g \in \operatorname{BFS}_{N}\left(\Lambda^{\prime}\right)$ satisfy $f \sim g$, then $\left(l_{2}(\Lambda), \pi_{f}\right) \sim\left(l_{2}\left(\Lambda^{\prime}\right), \pi_{g}\right)$. If $f$ is cyclic, then $\left(l_{2}(\Lambda), \pi_{f}\right)$ is cyclic. If $f$ is $P(J)$, then $\left(l_{2}(\Lambda), \pi_{f}\right)$ is $P(J)$. If $\Lambda=\Lambda_{1} \sqcup \Lambda_{2}$ and $\left.f^{(i)} \equiv f\right|_{\Lambda_{i}} \in \operatorname{BFS}_{N}\left(\Lambda_{i}\right)$ for $i=1,2$, then $\left(l_{2}(\Lambda), \pi_{f}\right) \sim$ $\left(l_{2}\left(\Lambda_{1}\right), \pi_{f(1)}\right) \oplus\left(l_{2}\left(\Lambda_{2}\right), \pi_{f^{(2)}}\right)$.

Let $\operatorname{End} \mathcal{A}$ be the set of all unital $*$-endomorphisms of a unital $*$-algebra $\mathcal{A}$. For $\rho \in \operatorname{End} \mathcal{A}, \rho$ is proper if $\rho(\mathcal{A}) \neq \mathcal{A}$. $\rho$ is irreducible if $\rho(\mathcal{A})^{\prime} \cap \mathcal{A}=\mathbf{C} I$ where $\rho(\mathcal{A})^{\prime} \cap \mathcal{A} \equiv\{x \in \mathcal{A}$ : for all $a \in \mathcal{A}, \rho(a) x=x \rho(a)\}$. $\rho$ and $\rho^{\prime}$ are equivalent if there is a unitary $u \in \mathcal{A}$ such that $\rho^{\prime}=\operatorname{Ad} u \circ \rho$. In this case, we denote $\rho \sim \rho^{\prime}$. Let $\operatorname{Rep} \mathcal{A}($ resp. $\operatorname{IrrRep} \mathcal{A})$ be the set of all unital (resp. irreducible) *-representations of $\mathcal{A}$. We simply denote $\pi$ for $(\mathcal{H}, \pi) \in \operatorname{Rep} \mathcal{A}$. If $\rho, \rho^{\prime} \in \operatorname{End} \mathcal{A}$ and $\pi, \pi^{\prime} \in \operatorname{Rep} \mathcal{A}$ satisfy $\rho \sim \rho^{\prime}$ and $\pi \sim \pi^{\prime}$, then $\pi \circ \rho \sim \pi^{\prime} \circ \rho^{\prime}$. Assume that $\mathcal{A}$ is simple. If there is $\pi \in \operatorname{IrrRep} \mathcal{A}$ such that $\pi \circ \rho \in \operatorname{IrrRep} \mathcal{A}$, then $\rho$ is irreducible. If there is $\pi \in \operatorname{Rep} \mathcal{A}$ such that $\pi \circ \rho \nsim \pi \circ \rho^{\prime}$, then $\rho \nsim \rho^{\prime}$. If there is $\pi \in \operatorname{IrrRep} \mathcal{A}$ such that $\pi \circ \rho \notin \operatorname{IrrRep} \mathcal{A}$, then $\rho$ is proper.

For $\psi_{\sigma}$ in (1.2), define

$$
\begin{equation*}
E_{N, l} \equiv\left\{\psi_{\sigma} \in \operatorname{End} \mathcal{O}_{N}: \sigma \in \mathfrak{S}_{N, l}\right\} \quad(l \geq 1) \tag{2.2}
\end{equation*}
$$

If $\sigma \in \mathfrak{S}_{N}$, then $\psi_{\sigma}$ is an automorphism of $\mathcal{O}_{N}$ which satisfies $\psi_{\sigma}\left(s_{i}\right)=s_{\sigma(i)}$ for $i=1, \ldots, N$. Especially, if $\sigma=i d$, then $\psi_{i d}=i d$. If $\sigma \in \mathfrak{S}_{N, 2}$ is defined by $\sigma(i, j) \equiv(j, i)$ for $i, j=1, \ldots, N$, then $\psi_{\sigma}$ is just the canonical endomorphism of $\mathcal{O}_{N}$. For $\sigma \in \mathfrak{S}_{N, l}$ and $f \in \operatorname{BFS}_{N}(\Lambda), \pi_{f} \circ \psi_{\sigma}=\pi_{f(\sigma)}$ where $f^{(\sigma)}$ is in (2.1). If $\rho$ is a permutative endomorphism and $(\mathcal{H}, \pi)$ is a permutative representation of $\mathcal{O}_{N}$, then $\pi \circ \rho$ is also a permutative representation.

A representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_{N}$ has a $P(J)$-component if $(\mathcal{H}, \pi)$ has a subrepresentation $\left(\mathcal{H}_{0},\left.\pi\right|_{\mathcal{H}_{0}}\right)$ which is $P(J)$. A component of a representation $P(J) \circ \rho$ of $\mathcal{O}_{N}$ means a subrepresentation of $(\mathcal{H}, \pi)$ which is equivalent to $P\left(J^{\prime}\right)$ for some $J^{\prime}$.

For comparison of the method to find $\left(J_{i}\right)_{i=1}^{M}$ in (1.4) for a given $J$, we show the usual method to determine $\left(J_{i}\right)_{i=1}^{M}$ as follows: (a) Prepare a representation $(\mathcal{H}, \pi)$ which is $P(J)$. We often take $\mathcal{H}=l_{2}(\mathbf{N})$ and $\pi=\pi_{f}$ for suitable branching function system $f$ on $\mathbf{N}$. (b) Compute $\pi\left(\psi_{\sigma}\left(s_{i}\right)\right) e_{n}$ for each $n \in \mathbf{N}$ and $i=1, \ldots, N$. By the proof of Lemma 2.2 in @, we see that it is sufficient to check for $1 \leq n \leq N^{l-1} k$ when $|J|=k$. (c) Find all cycles in $\mathcal{H}$ by using results in (b). In this way, the direct computation of branching law is too much of a bother because of a great number of calculated amount when $N, k, l$ are large.

## 3 Proof of Theorem 1.4

In this section, we assume that $\sigma \in \mathfrak{S}_{N, l}, l \geq 2, J=\left(j_{i}\right)_{i=1}^{k} \in\{1, \ldots, N\}^{k}$ and $J$ is nonperiodic. For $r \geq 2$, extend $J=\left(j_{i}\right)_{i=1}^{k}$ as $\left(j_{n}\right)_{n=1}^{r-k}$ by $j_{k(c-1)+i} \equiv$ $j_{i}$ for each $c=1, \ldots, r$ and $i=1, \ldots, k$ for convenience.

Lemma 3.1. Let $f \in \operatorname{BFS}_{N}(\Lambda)$ be $P(J), f^{(\sigma)}$ be in (2.1) and let $M_{\sigma}=$ $(Q, \Sigma, \Delta, \delta, \lambda)$ be in Definition 1.3. For $p \in Q_{J}$, define $r_{J}(p) \in \mathbf{N}$ by $r_{J}(p) \equiv$ \# $[p]$.
(i) For $p \in Q_{J}$ and $\alpha \equiv r_{J}(p) \cdot k$, define $p_{1}, \ldots, p_{\alpha} \in Q$ and $T=\left(t_{i}\right)_{i=1}^{\alpha} \in$ $\{1, \ldots, N\}^{\alpha}$ by $p_{1} \equiv p, b_{t_{1}}=\lambda\left(p_{\alpha}, a_{j_{\alpha}}\right)$ and

$$
p_{i} \equiv \delta\left(p_{i-1}, a_{j_{i-1}}\right), \quad b_{t_{i}}=\lambda\left(p_{i-1}, a_{j_{i-1}}\right) \quad(i=2, \ldots, \alpha),
$$

then there is $\Lambda(p) \subset \Lambda$ such that $\left.f^{(\sigma)}\right|_{\Lambda(p)}$ is $P(T)$.
(ii) In (i), define $T^{\prime} \in\{1, \ldots, N\}^{\alpha}$ by $b_{T^{\prime}}=\lambda\left(p, a_{J}^{r_{J}(p)}\right)$. Then $\left.f^{(\sigma)}\right|_{\Lambda(p)}$ is $P\left(T^{\prime}\right)$.
(iii) If there is $\Lambda_{0}$ such that $\left.f^{(\sigma)}\right|_{\Lambda_{0}}$ is $P(T)$ for $T=\left(t_{i}\right)_{i=1}^{\alpha} \in\{1, \ldots, N\}^{\alpha}$, then there is $p \in Q_{J}$ such that $\Lambda_{0}$ is equal to $\Lambda(p)$ in (i).
(iv) In (i), $p \sim p^{\prime}$ if and only if $\Lambda(p)=\Lambda\left(p^{\prime}\right)$.
(v) Choose $p_{1}, \ldots, p_{M}$ as (1.5). Then the decomposition $f^{(\sigma)}=f^{[1]} \oplus \cdots \oplus$ $f^{[M]}$ holds as a branching function system where $\left.f^{[i]} \equiv f^{(\sigma)}\right|_{\Lambda\left(p_{i}\right)}$ for each $i$.

Proof. Let $n_{0} \in \Lambda$ such that $f_{J}\left(n_{0}\right)=n_{0}$. Because $J$ is nonperiodic, such $n_{0}$ is unique in $\Lambda$.
(i) Let $r \equiv r_{J}(p)$. There is a sequence $\left(I_{1}, \ldots, I_{\alpha}\right)$ in $\{1, \ldots, N\}^{l-1}$ such that $p_{i}=q_{I_{i}}$ for each $i$. By definition of $\delta$ and $\lambda$ and assumption,

$$
\begin{equation*}
\sigma\left(t_{1}, I_{1}\right)=\left(I_{\alpha}, j_{\alpha}\right), \sigma\left(t_{2}, I_{2}\right)=\left(I_{1}, j_{1}\right), \ldots, \sigma\left(t_{\alpha}, I_{\alpha}\right)=\left(I_{\alpha-1}, j_{\alpha-1}\right) \tag{3.1}
\end{equation*}
$$

Define $m(p) \equiv f_{\sigma\left(t_{1}, I\right)}\left(n_{0}\right) \in \Lambda$. Then $m(p)=f_{I_{\alpha}}\left(f_{j_{\alpha}}\left(n_{0}\right)\right)$. By this and definition of $f^{(\sigma)}$, we can verify that $f_{T}^{(\sigma)}(m(p))=m(p)$. Define

$$
m_{\alpha} \equiv m(p), \quad m_{\alpha-1} \equiv f_{t_{\alpha}}^{(\sigma)}(m(p)), \ldots, m_{1} \equiv f_{\left(t_{1}, \ldots, t_{\alpha}\right)}^{(\sigma)}(m(p))
$$

and $\Lambda(p) \equiv\left\{f_{K}^{(\sigma)}(m(p)): K \in\{1, \ldots, N\}_{1}^{*}\right\}$. It is sufficient to show that $m_{i} \neq m_{j}$ when $i \neq j$. By definition,
$m_{i}=f_{t_{i+1}}^{(\sigma)}\left(m_{i+1}\right)=f_{\left(I_{i}, j_{i}\right)}\left(f_{\left(j_{i+1}, \ldots, j_{\alpha}\right)}\left(n_{0}\right)\right) \quad(i=1, \ldots, \alpha-1) \quad m_{\alpha}=f_{t_{1}}^{(\sigma)}\left(m_{1}\right)$.
Assume that $m_{i}=m_{i^{\prime}}$ and $c \equiv i^{\prime}-i \geq 0$. This implies that $m_{\tau(i)}=m_{\tau\left(i^{\prime}\right)}$ for each $\tau \in \mathbf{Z}_{\alpha}$. From this, $\left(I_{\tau(i)}, j_{\tau(i)}\right)=\left(I_{\tau\left(i^{\prime}\right)}, j_{\tau\left(i^{\prime}\right)}\right)$ and $f_{\left(t_{i+1}, \ldots, t_{\alpha}\right)}^{(\sigma)}(m(p))=$ $f_{\left(t_{i^{\prime}+1}, \ldots, t_{\alpha}\right)}^{(\sigma)}(m(p))$. This implies that $f_{\left(I_{\alpha}, j_{\alpha}\right)}\left(n_{0}\right)=f_{\left(I_{c}, j_{c}\right)}\left(f_{\left(j_{c+1}, \ldots, j_{\alpha}\right)}\left(n_{0}\right)\right)$. Therefore $n_{0}=f_{\left(j_{c+1}, \ldots, j_{\alpha}\right)}\left(n_{0}\right)$. By the uniqueness of the cycle in $\Lambda$ with respect to $f, c=k(d-1)$ for $1 \leq d \leq r$. Hence $I_{\tau(i)}=I_{\tau(i+k(d-1))}$ for each $\tau$. Therefore $p_{\tau(i)}=q_{I_{\tau(i)}}=q_{I_{\tau(i+k(d-1))}}=p_{\tau(i+k(d-1))}$ for each $\tau$. By the choice of $r, d=1$ and $i=i^{\prime}$. Hence the statement holds.
(ii) We see that $t_{1}^{\prime}=t_{\alpha}, t_{2}^{\prime}=t_{1}, \ldots, t_{\alpha}^{\prime}=t_{\alpha-1}$. Hence $P(T) \sim P\left(T^{\prime}\right)$ by definition.
(iii) Fix $\tau \in \mathbf{Z}_{\alpha}$. Define $T^{\prime}=\left(t_{i}^{\prime}\right)_{i=1}^{\alpha} \in\{1, \ldots, N\}^{\alpha}$ by

$$
\begin{equation*}
t_{i}^{\prime} \equiv t_{\tau^{-1}(i)} \quad(i=1, \ldots, \alpha) \tag{3.2}
\end{equation*}
$$

Then $\left.f^{(\sigma)}\right|_{\Lambda_{0}}$ is also $P\left(T^{\prime}\right)$ and there is $m_{0} \in \Lambda_{0}$ such that $f_{T^{\prime}}^{(\sigma)}\left(m_{0}\right)=m_{0}$. Define $m_{\alpha} \equiv m_{0}$ and $m_{i} \equiv f^{(\sigma)}\left(t_{i+1}, \ldots, t_{\alpha}\right)\left(m_{0}\right)$ for $i=1, \ldots, \alpha-1$.

Then $m_{i} \neq m_{i^{\prime}}$ when $i \neq i^{\prime}$. By definition of $f$, there are $n^{\prime} \in \Lambda, I_{0} \in$ $\{1, \ldots, N\}^{l-1}$ and $u_{0} \in\{1, \ldots, N\}$ such that $m_{\alpha}=f_{\left(I_{0}, u_{0}\right)}\left(n^{\prime}\right)$. Define a sequence $\left(I_{i}^{\prime}\right)_{i=1}^{\alpha}$ in $\{1, \ldots, N\}^{l-1}$ and $U=\left(u_{i}\right)_{i=1}^{\alpha} \in\{1, \ldots, N\}^{\alpha}$ by

$$
I_{\alpha}^{\prime} \equiv I_{0}, \quad u_{\alpha} \equiv u_{0}, \quad\left(I_{i}^{\prime}, u_{i}\right) \equiv \sigma\left(t_{i+1}^{\prime}, I_{i+1}^{\prime}\right) \quad(i=\alpha-1, \alpha-2, \ldots, 1)
$$

By assumption, we see that $f_{\left(I_{\alpha}^{\prime}, u_{\alpha}\right)}\left(n^{\prime}\right)=f_{\sigma\left(t_{1}^{\prime}, I_{1}^{\prime}\right)}\left(f_{U}\left(n^{\prime}\right)\right)$. By definition of $f,\left(I_{\alpha}^{\prime}, u_{\alpha}\right)=\sigma\left(t_{1}^{\prime}, I_{1}^{\prime}\right)$ and $n^{\prime}=f_{U}\left(n^{\prime}\right)$. By the uniqueness of cycle in $\Lambda$ with respect to $f, U \sim J^{r}$. Hence there is $\tau^{\prime} \in \mathbf{Z}_{\alpha}$ such that $j_{i}=u_{\tau^{\prime}(i)}$ for $i=1, \ldots, \alpha$. Here choose $\tau$ in (3.2) by $\tau \equiv \tau^{\prime}$ and define $I_{i} \equiv I_{\tau(i)}^{\prime}$ for each $i$. Then (3.1) holds. From this, we can verify that $p \equiv q_{I_{1}}$ belongs to $Q_{J}$. Define $m(p) \equiv f_{\left(I_{\alpha}, j_{\alpha}\right)}\left(n_{0}\right)$ as (i). Then $n_{0}=f_{\left(j_{1}, \ldots, j_{\tau}-1(\alpha)\right)}\left(n^{\prime}\right)$ and $m_{\alpha}=f_{\left(t_{\tau-1}(1), \ldots, t_{\alpha}\right)}^{(\sigma)}(m(p))$. Therefore $m_{\alpha} \in \Lambda(p)$. Since $m_{\alpha} \in \Lambda_{0} \cap \Lambda(p)$, $\Lambda_{0}=\Lambda(p)$.
(iv) If $p \sim p^{\prime}$, then there is $c$ such that $p^{\prime}=p_{k c+1}$ in (i) and we can verify that $m\left(p^{\prime}\right)=f_{\left(t_{1+k c}, \ldots, t_{\alpha}\right)}^{(\sigma)}(m(p)) \in \Lambda(p)$. Since $m\left(p^{\prime}\right) \in \Lambda\left(p^{\prime}\right) \cap \Lambda(p)$, $\Lambda\left(p^{\prime}\right)=\Lambda(p)$.

Assume that $\Lambda(p)=\Lambda\left(p^{\prime}\right)$. Let $m(p), m\left(p^{\prime}\right) \in \Lambda$ be in the proof of (i). Then there are $T, T^{\prime} \in\{1, \ldots, N\}_{1}^{*}$ such that $f_{T}^{(\sigma)}(m(p))=m(p)$ and $f_{T^{\prime}}^{(\sigma)}\left(m\left(p^{\prime}\right)\right)=m\left(p^{\prime}\right)$. Then $\left.f^{(\sigma)}\right|_{\Lambda(p)}$ is $P(T)$ and $\left.f^{(\sigma)}\right|_{\Lambda\left(p^{\prime}\right)}$ is $P\left(T^{\prime}\right)$. Since $\left.f^{(\sigma)}\right|_{\Lambda(p)}=\left.f^{(\sigma)}\right|_{\Lambda\left(p^{\prime}\right)}, T^{\prime} \sim T$. Assume that $T=\left(t_{i}\right)_{i=1}^{\alpha}$ and $T^{\prime}=\left(t_{i}^{\prime}\right)_{i=1}^{\alpha}$. Let $\left\{m_{i}\right\}_{i=1}^{\alpha}$ be the cycle in $\Lambda(p)$ of $f^{(\sigma)}$ in (i). By the uniqueness of the cycle in $\Lambda(p)$ with respect to $f^{(\sigma)},\left\{m_{i}\right\}_{i=1}^{\alpha}$ is also the cycle in $\Lambda\left(p^{\prime}\right)$ of $f^{(\sigma)}$. By the proof of (i), $m\left(p^{\prime}\right) \in\left\{m_{i}\right\}_{i=1}^{\alpha}$. Hence there is $\tau \in \mathbf{Z}_{\alpha}$ such that $m\left(p^{\prime}\right)=m_{\tau(\alpha)}$. From this, $t_{i}^{\prime}=t_{\tau(i)}$ for $i=1, \ldots, \alpha$. Because $T \sim T^{\prime}$, $r_{J}\left(p^{\prime}\right)=r_{J}(p)$. Let $r \equiv r_{J}(p)$. Assume that $p=q_{I_{1}}$ and $p^{\prime}=q_{I_{1}^{\prime}}$. By definition of $m(p)$ and $m\left(p^{\prime}\right)$ and their relation, we see that $I_{1}^{\prime}=I_{\tau(1)}$. Therefore $p^{\prime}=q_{I_{\tau(1)}}$. By choice of $p$ and $p^{\prime}, \delta\left(p, a_{J}^{r}\right)=p$ and $\delta\left(p^{\prime}, a_{J}^{r}\right)=p^{\prime}$. Because $J$ is nonperiodic, $\tau(i)=i+k c$ for a certain $c$ modulo $\alpha$. Therefore $p^{\prime}=q_{I_{\tau(1)}}=q_{I_{1+k c}}=\delta\left(p, a_{J}^{c}\right)$. Therefore $p^{\prime} \sim p$.
(v) If $i \neq j$, then $\Lambda\left(p_{i}\right) \neq \Lambda\left(p_{j}\right)$ by (iv). Hence $\Lambda\left(p_{i}\right) \cap \Lambda\left(p_{j}\right)=\emptyset$. Therefore $\Lambda\left(p_{1}\right) \sqcup \cdots \sqcup \Lambda\left(p_{M}\right) \subset \Lambda$. By (iii) and the decomposability of the branching function $f^{(\sigma)}, \Lambda\left(p_{1}\right) \sqcup \cdots \sqcup \Lambda\left(p_{M}\right)=\Lambda$. This implies the statement.

Proof of Theorem 1.4. Assume that $J=\left(j_{i}\right)_{i=1}^{k} \in\{1, \ldots, N\}^{k}$. When $l=1$, $Q_{J}=\left\{q_{0}\right\}$. Let $J_{\sigma^{-1}} \equiv\left(\sigma^{-1}\left(j_{1}\right), \ldots, \sigma^{-1}\left(j_{k}\right)\right)$. Then we can check that $\lambda\left(q_{0}, a_{J}\right)=b_{J_{\sigma^{-1}}}$ and $P(J) \circ \psi_{\sigma}=P\left(J_{\sigma^{-1}}\right)$ independently. Hence the asser-
tion is verified. Assume that $l \geq 2$. By applying the correspondence between branching function systems and permutative representations, we see that the decomposition in Lemma 3.1 (v) implies that in (1.4). By definition of $J_{i}$ and applying Lemma 3.1 (i), (ii) to each component in the decomposition, the statement holds.

By Theorem 1.4 it is not necessary for computation of branching law (1.4) to prepare any representation space. Further Theorem 1.4 implies the following:

Proposition 3.2. If the Mealy diagram of $\mathrm{M}_{\sigma}$ has $M$ connected components, then $P(J) \circ \psi_{\sigma}$ has $M$ components of direct sum at least for each $J$.

## 4 Examples

We show examples of permutative endomorphism of $\mathcal{O}_{N}$ and compute their branching laws by using the Mealy diagram according to Theorem 1.4. Recall $E_{N, l}$ in (2.2). Here we often denote $\left(j_{1}, \ldots, j_{k}\right)$ by $j_{1} \cdots j_{k}$ simply.

## $4.1 \quad E_{2,2}$

In [8], we show that there are 16 equivalence classes in $E_{2,2}$ and there are 5 irreducible and proper classes $\mathcal{E}$ in them. We treat 3 elements in $\mathcal{E}$ here. For each $\sigma \in \mathfrak{S}_{2,2}, \mathrm{M}_{\sigma}=(Q, \Sigma, \Delta, \delta, \lambda)$ consists of $Q=\left\{q_{1}, q_{2}\right\}, \Sigma=\left\{a_{1}, a_{2}\right\}$ and $\Delta=\left\{b_{1}, b_{2}\right\}$.

Define a transposition $\sigma \in \mathfrak{S}_{2,2}$ by $\sigma(1,1) \equiv(1,2)$. Then $\psi_{\sigma}$ and the Mealy diagram $\mathcal{D}\left(\mathrm{M}_{\sigma}\right)$ of $\mathrm{M}_{\sigma}$ are as follows:
$\left\{\begin{array}{l}\psi_{\sigma}\left(s_{1}\right) \equiv s_{1} s_{2} s_{1}^{*}+s_{1} s_{1} s_{2}^{*}, \\ \psi_{\sigma}\left(s_{2}\right) \equiv s_{2},\end{array} a_{2} / b_{1}\left(q_{1} \frac{a_{1} / b_{1}}{a_{1} / b_{2}} q_{2} / b_{2}\right.\right.$
$\psi_{\sigma}$ is irreducible and proper (Table II in [8]). We denote $\psi_{\sigma}$ by $\psi_{12}$ in convenience. We show several branching laws by $\psi_{12}$ :

| input | cycles | outputs | branching law |
| :---: | :---: | :---: | :---: |
| $a_{1}$ | $q_{1} q_{2}$ | $b_{1} b_{2}$ | $P(1) \circ \psi_{12}=P(12)$ |
| $a_{2}$ | $q_{1}, q_{2}$ | $b_{1}, b_{2}$ | $P(2) \circ \psi_{12}=P(1) \oplus P(2)$ |
| $a_{1} a_{2}$ | $q_{1} q_{2} q_{2} q_{1}$ | $b_{1} b_{2} b_{2} b_{1}$ | $P(12) \circ \psi_{12}=P(1122)$ |
| $a_{1} a_{1} a_{2} a_{2}$ | $q_{1} q_{2} q_{1} q_{1}, q_{2} q_{1} q_{2} q_{2}$ | $b_{1} b_{2} b_{1} b_{1}, b_{2} b_{1} b_{2} b_{2}$ | $P(1122) \circ \psi_{12}=P(1112) \oplus P(1222)$ |

Focusing attention on closed paths in $\mathcal{D}\left(\mathrm{M}_{\sigma}\right)$, we can verify the following:

Proposition 4.1. For each $J \in\{1,2\}_{1}^{*}$, there are $J_{1}, J_{2}$ or $J_{3}$ such that

$$
P(J) \circ \psi_{12}= \begin{cases}P\left(J_{1}\right) \oplus P\left(J_{2}\right) & \left(n_{1}(J)=\text { even }\right) \\ P\left(J_{3}\right) & \left(n_{1}(J)=\text { odd }\right)\end{cases}
$$

where $n_{1}(J) \equiv \sum_{l=1}^{k}\left(2-j_{l}\right)$ for $J=\left(j_{1}, \ldots, j_{k}\right) \in\{1,2\}^{k}$.
Let $\sigma \in \mathfrak{S}_{2,2}$ be a transposition defined by $\sigma(1,1) \equiv(2,1)$. Then $\psi_{\sigma}$, $\mathcal{D}\left(\mathrm{M}_{\sigma}\right)$ and branching laws of $\psi_{\sigma}$ are given as follows:


| input | cycles | outputs | branching law |
| :---: | :---: | :---: | :---: |
| $a_{1}$ | $q_{1}$ | $b_{2}$ | $P(1) \circ \psi_{\sigma}=P(2)$ |
| $a_{2}$ | $q_{2}$ | $b_{2}$ | $P(2) \circ \psi_{\sigma}=P(2)$ |
| $a_{1} a_{2}$ | $q_{2} q_{1}$ | $b_{1} b_{2}$ | $P(12) \circ \psi_{\sigma}=P(11)$ |
| $a_{1} a_{1} a_{2}$ | $q_{2} q_{1} q_{1}$ | $b_{1} b_{2} b_{1}$ | $P(112) \circ \psi_{\sigma}=P(112)$ |
| $a_{1} a_{2} a_{2}$ | $q_{2} q_{1} q_{2}$ | $b_{1} b_{1} b_{2}$ | $P(122) \circ \psi_{\sigma}=P(112)$ |

Let $\sigma \in \mathfrak{S}_{2,2}$ be defined by $\sigma(1,1) \equiv(2,2), \sigma(1,2) \equiv(1,1), \sigma(2,1) \equiv$ $(2,1), \sigma(2,2) \equiv(1,2)$. Then $\psi_{\sigma}, \mathcal{D}\left(\mathrm{M}_{\sigma}\right)$ and branching laws are as follows:

$$
\left\{\begin{array} { l } 
{ \psi _ { \sigma } ( s _ { 1 } ) \equiv s _ { 2 } s _ { 2 } s _ { 1 } ^ { * } + s _ { 1 } s _ { 1 } s _ { 2 } ^ { * } , } \\
{ \psi _ { \sigma } ( s _ { 2 } ) \equiv s _ { 2 } s _ { 1 } s _ { 1 } ^ { * } + s _ { 1 } s _ { 2 } s _ { 2 } ^ { * } , }
\end{array} \left\{\begin{array}{c|c|c|c}
q_{1} \\
\text { input } & \text { cycles } & \text { outputs } & \text { branching law } \\
\hline a_{1} & q_{1} q_{2} & b_{1} b_{2} & P(1) \circ \psi_{\sigma}=P(12) \\
\hline a_{2} & q_{1} q_{2} & b_{2} b_{1} & P(2) \circ \psi_{\sigma}=P(12) \\
\hline a_{1} a_{2} & q_{1} q_{2}, q_{2} q_{1} & b_{1} b_{1}, b_{2} b_{2} & P(12) \circ \psi_{\sigma}=P(11) \oplus P(22)
\end{array}\right.\right.
$$

## $4.2 \quad E_{3,2}$

Note that $\# E_{2,2}=2^{2}!=24$ and $\# E_{3,2}=3^{2}!\sim 3.6 \times 10^{5}$. Hence it is difficult to classify every element in $E_{3,2}$ by computing its branching laws in comparison with the case $E_{2,2}$. We see that $\mathrm{M}_{\sigma}=\left(\left\{q_{1}, q_{2}, q_{3}\right\},\left\{a_{1}, a_{2}, a_{3}\right\}\right.$, $\left.\left\{b_{1}, b_{2}, b_{3}\right\}, \delta, \lambda\right)$ for each $\sigma \in \mathfrak{S}_{3,2} . \rho_{\nu}$ in (1.1) belongs to $E_{3,2}$.

Let $\sigma \in \mathfrak{S}_{3,2}$ be a transposition by $\sigma(1,1) \equiv(1,2)$. Then $\psi_{\sigma}, \mathcal{D}\left(\mathrm{M}_{\sigma}\right)$ and branching laws are as follows:

$$
\left\{\begin{array}{rl}
\psi_{\sigma}\left(s_{1}\right) & \equiv s_{12,1}+s_{11,2}+s_{13,3} \\
\psi_{\sigma}\left(s_{2}\right) & \equiv s_{2}, \\
\psi_{\sigma}\left(s_{3}\right) & \equiv s_{3}, \\
& \begin{array}{c|c|c|c|c|}
a_{2}
\end{array} \\
& \begin{array}{c}
\text { input }
\end{array} \\
\hline a_{2} & q_{1} q_{2} \\
a_{3} & q_{1} q_{2} \\
q_{3} & b_{1}, b_{2} \\
b_{3} & P(2) \circ \psi_{\sigma}=P(1) \oplus P(2) \\
\hline
\end{array}\right.
$$

where $s_{i j, k} \equiv s_{i} s_{j} s_{k}^{*}$. From this, we see that $\psi_{\sigma}^{n}$ is proper and irreducible for each $n \geq 1$, and $\psi_{\sigma}$ and $\rho_{\nu}$ are not equivalent.

## $4.3 \quad E_{4,2}$

Define $\sigma \in \mathfrak{S}_{4,2}$ by

| $J$ | 11 | 12 | 13 | 14 | 21 | 22 | 23 | 24 | 31 | 32 | 33 | 34 | 41 | 42 | 43 | 44 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\sigma(J)$ | 11 | 21 | 31 | 41 | 12 | 22 | 43 | 42 | 32 | 23 | 13 | 33 | 44 | 24 | 14 | 34 |

Then $\psi_{\sigma}$ and $\mathcal{D}\left(\mathrm{M}_{\sigma}\right)$ are as follows:

$$
\begin{aligned}
& \psi_{\sigma}\left(s_{1}\right) \equiv s_{11,1}+s_{21,2}+s_{31,3}+s_{41,4}, \quad \psi_{\sigma}\left(s_{2}\right) \equiv s_{12,1}+s_{22,2}+s_{43,3}+s_{42,4} \\
& \psi_{\sigma}\left(s_{3}\right) \equiv s_{32,1}+s_{23,2}+s_{13,3}+s_{33,4}, \quad \psi_{\sigma}\left(s_{4}\right) \equiv s_{44,1}+s_{24,2}+s_{14,3}+s_{34,4}
\end{aligned}
$$



When $J=(1), \delta\left(q_{i}, a_{1}\right)=q_{i}$ and $\lambda\left(q_{i}, a_{1}\right)=b_{1}$ for each $i=1,2,3,4$. Therefore $P(1) \circ \psi_{\sigma}=P(1) \oplus P(1) \oplus P(1) \oplus P(1)$. In the same way, we have

$$
P(2) \circ \psi_{\sigma}=P(2) \oplus P(2) \oplus P(2), \quad P(4) \circ \psi_{\sigma}=P(4) \oplus P(444)
$$

This is an example of Proposition 3.2.

### 4.4 Canonical endomorphism

The Mealy diagram associated with the canonical endomorphism $\rho$ of $\mathcal{O}_{N}$ (see 92 ) is given as follows:

$$
\rho(x) \equiv s_{1} x s_{1}^{*}+\cdots+s_{N} x s_{N}^{*}
$$



In this case, there is no transition among different states. We see that $P(J) \circ \rho=P(J)^{\oplus N}$ for each $J \in\{1, \ldots, N\}_{1}^{*}$ where $P(J)^{\oplus N}$ is the direct sum of $N$ copies of $P(J)$. In general, $\pi \circ \rho=\pi^{\oplus N}$ for any representation $\pi$ of $\mathcal{O}_{N}$.

## $4.5 \quad E_{2,3}$

Let $\sigma \in \mathfrak{S}_{2,3}$ be a transposition by $\sigma(1,1,1) \equiv(1,2,1)$. Then $\psi_{\sigma} \in E_{2,3}$, $\mathcal{D}\left(\mathrm{M}_{\sigma}\right)$ and branching laws are as follows:

$$
\begin{aligned}
& \left\{\begin{array}{l}
\psi_{\sigma}\left(s_{1}\right) \equiv s_{121} s_{11}^{*}+s_{112} s_{12}^{*}+s_{111} s_{21}^{*}+s_{122} s_{22}^{*}, \\
\psi_{\sigma}\left(s_{2}\right) \equiv s_{2},
\end{array}\right. \\
& \text { input } \\
& \hline a_{1} \\
& \hline a_{2} / q_{1} \\
& \hline a_{1} a_{2} \\
& \hline a_{1} q_{21} \\
& \hline
\end{aligned}
$$

We see that $\psi_{\sigma}^{n}$ is irreducible and proper for each $n \geq 1$.
Let $\sigma \in \mathfrak{S}_{2,3}$ be defined by the product $\sigma=\sigma^{\prime} \circ \sigma^{\prime \prime}$ of two transpositions $\sigma^{\prime}$ and $\sigma^{\prime \prime \prime}$ defined by $\sigma^{\prime}(1,1,1) \equiv(1,2,1)$ and $\sigma^{\prime \prime}(1,1,2) \equiv(1,2,2)$, respectively. In this case $\psi_{\sigma}=\psi_{12} \in E_{2,2}$ in 4.1. $\mathcal{D}\left(\mathrm{M}_{\sigma}\right)$ is as follows:


We can verify that branching laws of $\psi_{\sigma}$ coincide with those of $\psi_{12}$.

## $4.6 \quad E_{2,4}$

Define a transposition $\sigma \in \mathfrak{S}_{2,4}$ by $\sigma(1,1,1,1) \equiv(1,2,1,1)$. Then $\psi_{\sigma} \in E_{2,4}$, $\mathcal{D}\left(\mathrm{M}_{\sigma}\right)$ and branching laws are given as follows:
$\psi_{\sigma}\left(s_{1}\right) \equiv s_{1211} s_{111}^{*}+s_{1112} s_{112}^{*}+s_{112} s_{12}^{*}+s_{1111} s_{211}^{*}+s_{1212} s_{212}^{*}+s_{122} s_{22}^{*}, \quad \psi_{\sigma}\left(s_{2}\right) \equiv s_{2}$,


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