# INTERSECTION OF SUBGROUPS IN FREE GROUPS AND HOMOTOPY GROUPS 

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#### Abstract

We show that the intersection of three subgroups in a free group is related to the computation of the third homotopy group $\pi_{3}$. This generalizes a result of Gutierrez-Ratcliffe who relate the intersection of two subgroups with the computation of $\pi_{2}$. Let $K$ be a twodimensional CW-complex with subcomplexes $K_{1}, K_{2}, K_{3}$ such that $K=K_{1} \cup K_{2} \cup K_{3}$ and $K_{1} \cap K_{2} \cap K_{3}$ is the 1 -skeleton $K^{1}$ of $K$. We construct a natural homomorphism of $\pi_{1}(K)$ modules $$
\pi_{3}(K) \rightarrow \frac{R_{1} \cap R_{2} \cap R_{3}}{\left[R_{1}, R_{2} \cap R_{3}\right]\left[R_{2}, R_{3} \cap R_{1}\right]\left[R_{3}, R_{1} \cap R_{2}\right]},
$$ where $R_{i}=\operatorname{ker}\left\{\pi_{1}\left(K^{1}\right) \rightarrow \pi_{1}\left(K_{i}\right)\right\}, i=1,2,3$ and the action of $\pi_{1}(K)=F / R_{1} R_{2} R_{3}$ on the right hand abelian group is defined via conjugation in $F$. In certain cases, the defined map is an isomorphism. Finally, we discuss certain applications of the above map to group homology.


## 1. Introduction

Simplicial homotopy theory makes it possible to translate certain homotopy questions to the group-theoretical language. As a rule, the group-theoretical problems appearing in this direction have a difficult nature. Still combinatorial group theory is a crucial tool in homotopy theory (see, for example, [3, 5). On the other hand, certain group-theoretical results may be obtained by use of homotopy methods, like methods of simplicial homotopy theory and the theory of derived functors (see, for example, [8]).

It is the purpose of this paper to combine the results of Gutierrez-Ratcliffe and Wu which present certain links between homotopical and group-theoretical structures. Let us recall them first.

1. (Exact sequence due to Guttierez-Ratcliffe, [10]) Let $K$ be a connected 2-dimensional CWcomplex, and $K_{1}$ and $K_{2}$ subcomplexes such that $K=K_{1} \cup K_{2}$ and $K_{1} \cap K_{2}$ is the 1-skeleton $K^{1}$ of $K$, then there is an exact sequence of $\pi_{1}(K)$-modules

$$
\begin{equation*}
0 \rightarrow i_{1} \pi_{2}\left(K_{1}\right) \oplus i_{2} \pi_{2}\left(K_{2}\right) \xrightarrow{\alpha} \pi_{2}(K) \rightarrow \frac{R \cap S}{[R, S]} \rightarrow 0 \tag{1}
\end{equation*}
$$

where $\alpha$ is induced by inclusion, $R$ is the kernel of $\pi_{1}\left(K^{1}\right) \rightarrow \pi_{1}\left(K_{1}\right), S$ is the kernel of $\pi_{1}\left(K^{1}\right) \rightarrow \pi_{1}\left(K_{1}\right)$ and the action of $\pi_{1}(K) \simeq \pi_{1}\left(K^{1}\right) / R S$ on $\frac{R \cap S}{[R, S]}$ is induced by conjugation. The paper [10] contains another exact sequence of the same nature. Let $G=\langle X \mid \mathcal{N}\rangle$ be a group presentation with relation module $N / \gamma_{2}(N)$, where $N$ is the normal closure of the set $\mathcal{N}$ in the free group $F(X)$. Let $K_{r}$ be a standard 2-complex for a presentation $\langle X \mid r\rangle, r \in \mathcal{N}$, let $s_{r}$ be the root of $r$ in the free group $F(X)$. There is an exact sequence of $G$-modules:

$$
\begin{equation*}
0 \rightarrow \oplus_{r \in \mathcal{N}} i_{*} \pi_{2}\left(K_{r}\right) \xrightarrow{\alpha} \pi_{2}(K) \rightarrow \oplus_{r \in \mathcal{N}} \mathbb{Z}[G] /\left(s_{r}-1\right) \mathbb{Z}[G] \xrightarrow{\gamma} N / \gamma_{2}(N) \rightarrow 0, \tag{2}
\end{equation*}
$$

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where $\alpha$ is induced by inclusion and $\gamma$ maps $1+\left(s_{r}-1\right) \mathbb{Z}[G]$ onto $r \gamma_{2}(N)$ for each $r \in \mathcal{N}$. For a group-theoretical application of the above exact sequences consider a two-relator presentation $\mathcal{P}=\left\langle X \mid r_{1}, r_{2}\right\rangle$ of a group $F(X) /\left\langle r_{1}, r_{2}\right\rangle^{F(X)}$. As a consequence of the exact sequences (11) and (21), we have that the quotient $\frac{\langle r\rangle^{F(X)} \cap\left\langle r_{2}\right\rangle^{F(X)}}{\left[\left[r_{1}\right\rangle^{F(X)},\left\langle r_{2}\right\rangle^{F(X)]}\right.}$ is a subgroup of the quotient $\mathbb{Z}[G] /\left(s_{r_{1}}-1\right) \mathbb{Z}[G] \oplus \mathbb{Z}[G] /\left(s_{r_{2}}-1\right) \mathbb{Z}[G]$, i.e. it is a free Abelian. Hence, we proved the following generalization of the theorem due to Hartley-Kuzmin [12]: let $F$ be a free group, $r_{1}$ and $r_{2}$ words in $F, R_{i}=\left\langle r_{i}\right\rangle^{F}, i=1,2$, then the group $\frac{R_{1} \cap R_{2}}{\left[R_{1}, R_{2}\right]}$ is a free Abelian group.
2. (The presentation of homotopy groups of the 2 -sphere due to Wu ) It is one of the deep problems of algebraic topology to compute homotopy groups $\pi_{n}\left(S^{2}\right)$. Note that the 2 -sphere presents the most 'unstable' case from the point of view of homotopy theory. The developed methods of Adams-type spectral sequences usually do not work in this case. In low degrees one has (see [15]):

$$
\begin{array}{c|cccccccc}
n & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline \pi_{n}\left(S^{2}\right) & \mathbb{Z} & \mathbb{Z} & \mathbb{Z}_{2} & \mathbb{Z}_{2} & \mathbb{Z}_{4} \oplus \mathbb{Z}_{3} & \mathbb{Z}_{2} & \mathbb{Z}_{2} & \mathbb{Z}_{3}
\end{array}
$$

The structure of $\pi_{n}\left(S^{2}\right)$ is known up to some stage ( $n \approx 30$ ), mostly due to Toda and his students. The general structure of $\pi_{n}\left(S^{2}\right)$ is unclear and mysterious.

Recall the description of homotopy groups of the 2-sphere due to Wu [16]. Let $F\left[S^{1}\right]$ be Milnor's $F[K]$-construction applied to the simplicial circle $S^{1}$. This is the free simplicial group with $F\left[S^{1}\right]_{n}$ a free group of rank $n \geq 1$ with generators $x_{0}, \ldots, x_{n-1}$. Changing the basis of $F\left[S^{1}\right]_{n}$ in the following way: $y_{i}=x_{i} x_{i+1}^{-1}, y_{n-1}=x_{n-1}$, we get another basis $\left\{y_{0}, \ldots, y_{n-1}\right\}$ in which the simplicial maps can be written easier. A combinatorial group-theoretical argument then gives the following description of the $n$-th homotopy group of the loop space $\Omega \Sigma S^{1}$, which is isomorphic to the homotopy group of $\pi_{n+1}\left(S^{2}\right)$ (see [16] for explicit computations):

$$
\begin{equation*}
\pi_{n+1}\left(S^{2}\right) \cong \frac{\left\langle y_{-1}\right\rangle^{F} \cap\left\langle y_{0}\right\rangle^{F} \cap \cdots \cap\left\langle y_{n-1}\right\rangle^{F}}{\left[\left[y_{-1}, y_{0}, \ldots, y_{n-1}\right]\right]}, n \geq 1 \tag{3}
\end{equation*}
$$

where $F$ is a free group with generators $y_{0}, \ldots, y_{n-1}, y_{-1}=\left(y_{0} \ldots y_{n-1}\right)^{-1}$, the group $\left[\left[y_{-1}, y_{0}, \ldots, y_{n-1}\right]\right]$ is the normal closure in $F$ of the set of left-ordered commutators

$$
\begin{equation*}
\left[z_{1}^{\varepsilon_{1}}, \ldots, z_{t}^{\varepsilon_{t}}\right] \tag{4}
\end{equation*}
$$

with the properties that $\varepsilon_{i}= \pm 1, z_{i} \in\left\{y_{-1}, \ldots, y_{n-1}\right\}$ and all elements in $\left\{y_{-1}, \ldots, y_{n-1}\right\}$ appear at least once in the sequence of elements $z_{i}$ in (4).

The main idea of our approach can be formulated as the following conjectural observation: the nature of the presentation (3) comes from the fact that the 2-sphere is homotopically equivalent to the standard 2-complex, constructed from the presentation $\left\langle y_{0}, \ldots, y_{n-1} \mid y_{0}, \ldots, y_{n-1}, y_{n-1}^{-1} \ldots y_{0}^{-1}\right\rangle$. Given a free group $F$ and normal subgroups ( $n \geq 2$ )

$$
R_{1}, \ldots, R_{n} \unlhd F
$$

denote the quotient group

$$
I_{n}\left(F, R_{1}, \ldots, R_{n}\right):=\frac{R_{1} \cap \cdots \cap R_{n}}{\left.\prod_{I \cup J=\{1, \ldots, n\}, I \cap J=\emptyset} \bigcap_{i \in I} R_{i}, \bigcap_{j \in J} R_{j}\right]}
$$

Here $\bigcap$ denotes the intersection of subgroups in the free group $F$ and $\Pi$ is the product of commutator subgroups as indicated. In fact, the abelian group $I_{n}$ has the natural structure of an $F / R_{1} \ldots R_{n}$-module, with the group action defined via conjugation in $F$.

The computation of the abelian group $I_{n}$ is highly non-trivial. In fact, in the special case $F=\left\langle x_{1}, \ldots, x_{n-1}\right\rangle, R_{i}=\left\langle x_{i}\right\rangle^{F}, i=1, \ldots, n-1, R_{n}=\left\langle x_{1} \ldots x_{n-1}\right\rangle^{F}$ a standard commutator calculus argument, given essentially in Corollary 3.5 of [16] shows that

$$
\left.\left[\left[y_{-1}, y_{0}, \ldots, y_{n-1}\right]\right]=\prod_{I \cup J=\{1, \ldots, n+1\}, I \cap J=\emptyset} \bigcap_{i \in I} R_{i}, \bigcap_{j \in J} R_{j}\right],
$$

and hence we have the following isomorphism

$$
I_{n}\left(F, R_{1}, \ldots, R_{n}\right)=\pi_{n}\left(S^{2}\right)
$$

On the other hand, for $n=2$, one has a general description of the $F / R_{1} R_{2}$-module $I_{2}\left(F, R_{1}, R_{2}\right)=$ $\frac{R_{1} \cap R_{2}}{\left[R_{1}, R_{2}\right]}$ in terms of homotopy groups of certain spaces given by the sequence (1) due to GutierrezRatcliffe. For the generalization of the Gutierrez-Ratcliffe's approach to the higher dimensional homotopy groups consider a connected 2 -dimensional CW-complex $K$ with subcomplexes

$$
K_{1}, \ldots, K_{n} \subset K
$$

for which $K_{1} \cup \cdots \cup K_{n}=K$ and $K_{1} \cap \cdots \cap K_{n}$ is the 1 -skeleton $K^{1}$ of $K$, with $F=\pi_{1}\left(K^{1}\right)$ and

$$
R_{i}=\operatorname{ker}\left\{\pi_{1}\left(K^{1}\right) \rightarrow \pi_{1}\left(K_{i}\right)\right\}, \quad i=1, \ldots, n .
$$

We conjecture that each element $\alpha \in \pi_{n}\left(S^{2}\right)$ determines a natural function ( $n \geq 2$ )

$$
\alpha_{*}: \pi_{2}(K) /\left(i_{1} \pi_{2}\left(K_{1}\right)+\cdots+i_{n} \pi_{2}\left(K_{n}\right)\right) \rightarrow I_{n}\left(F, R_{1}, \ldots, R_{n}\right) .
$$

In general, $\alpha_{*}$ is not a homomorphism of abelian groups.
Proposition. Let $n=2$. If $\alpha$ is a generator of $\pi_{2}\left(S^{2}\right)=\mathbb{Z}$, then $\alpha_{*}$ exists and is given by the map $\pi_{2}(K) \rightarrow I_{2}\left(F, R_{1}, R_{2}\right)$ of Gutierrez-Ratcliffe [10].

Moreover, as a main result of this paper we prove the following
Theorem. Let $n=3$. If $\alpha \in \pi_{3}\left(S^{2}\right)$ is a generator, then there is a well-defined function $\alpha_{*}$ which is a quadratic map inducing a natural homomorphism of $\pi_{1}(K)$-modules

$$
\alpha_{\#}: \pi_{3}(K) \rightarrow I_{3}\left(F, R_{1}, R_{2}, R_{3}\right) .
$$

For the example of $W u$, one has $K=S^{2}$ and in this case $\alpha_{\#}$ is an isomorphism.
In the construction of the above homomorphism of $\pi_{1}(K)$-modules we essentially use the fact that $\pi_{3}(K)=\Gamma \pi_{2}(K)$, where $\Gamma$ is Whitehead's universal quadratic functor.

One can interpret certain elements of $\pi_{*}\left(S^{2}\right)$ as the elements of certain free groups in Milnor's $F\left[S^{1}\right]$-construction. For example, the element in $F\left(y_{0}, y_{1}, y_{2}\right)$ corresponding to the generator of $\pi_{4}\left(S^{2}\right)$ in (3) is

$$
\left[\left[y_{0}, y_{1}\right],\left[y_{0}, y_{1} y_{2}\right]\right] .
$$

The element in $F\left(y_{0}, y_{1}, y_{2}, y_{3}\right)$, corresponding to the generator of $\pi_{5}\left(S^{2}\right)$ is

$$
\left[\left[\left[y_{0}, y_{1}\right],\left[y_{0}, y_{1} y_{2}\right]\right],\left[\left[y_{0}, y_{1}\right],\left[y_{0}, y_{1} y_{2} y_{3}\right]\right]\right] .
$$

This follows from the result of Wu [16]. In Section 3 we shall formulate a conjecture, which, when applied to these elements, leads to non-trivial commutator problems in free groups that we are not ready to solve. This difficulty is the reason why we consider in this paper only the case of three subcomplexes.

Finally, we use the main construction of the paper for an arbitrary free group $F$ and its normal subgroups $R_{1}, R_{2}, R_{3}$, to define the following natural map of abelian groups:

$$
H_{4}(G) \rightarrow \frac{R_{1} \cap R_{2} \cap R_{3}}{\left[R_{1}, R_{2} \cap R_{3}\right]\left[R_{2}, R_{3} \cap R_{1}\right]\left[R_{3}, R_{1} \cap R_{2}\right]\left[F, R_{1} \cap R_{2} \cap R_{3}\right]}
$$

This map is related to the Brown-Ellis construction (see [4), however, the methods used in the current paper differ from those used in [4]. The relation with Brown-Ellis construction is given in [11.

## 2. The category $\mathcal{K}_{n}$

For $n \geq 2$, denote by $\mathcal{K}_{n}$ the category with objects $\bar{K}=\left(K, K_{1}, \ldots, K_{n}\right)$. Here $K$ is a twodimensional CW-complex, $K_{i}$ is a subcomplex of $K, i=1, \ldots, n$, such that $K=K_{1} \cup \cdots \cup K_{n}$, and $K^{1}=K_{1} \cap \cdots \cap K_{n}$. A morphism in $\operatorname{Hom}_{\mathcal{K}_{n}}(\bar{K}, \bar{L})$ for $\bar{K}, \bar{L} \in \mathcal{K}_{n}$ is a map

$$
f: K^{1} \rightarrow L^{1}
$$

between 1 -skeletons of $K$ and $L$, such that $f$ can be extended to a map $\bar{f}: K \rightarrow L$, with the property $\bar{f}\left(K_{i}\right) \subseteq L_{i}, i=1, \ldots, n$.

Denote by $\mathcal{R}_{n}(n \geq 2)$ the category with objects $\left(F, R_{1}, \ldots, R_{n}\right)$, where $F$ is a free group and $R_{i}$ is a normal subgroup in $F$. A morphism in $\mathcal{R}_{n}$ between two objects ( $F, R_{1}, \ldots, R_{n}$ ) and $\left(F^{\prime}, R_{1}^{\prime}, \ldots, R_{n}^{\prime}\right)$ is a group homomorphism $g: F \rightarrow F^{\prime}$ such that $g\left(R_{i}\right) \subseteq R_{i}^{\prime}, i=1, \ldots, n$. This category was also considered in [6].

There is a natural functor between these two categories,

$$
\mathcal{F}_{n}: \mathcal{K}_{n} \rightarrow \mathcal{R}_{n},
$$

defined by setting

$$
\mathcal{F}_{n}:\left(K, K_{1}, \ldots, K_{n}\right) \mapsto\left(\pi_{1}\left(K^{1}\right), R_{1}, \ldots, R_{n}\right),
$$

where $R_{i}=\operatorname{ker}\left\{\pi_{1}\left(K^{1}\right) \rightarrow \pi_{1}\left(K_{i}\right)\right\}$.
For $n \geq 2$, define the functor

$$
I_{n}: \mathcal{R}_{n} \rightarrow \mathcal{A} b
$$

where $\mathcal{A} b$ is the category of abelian groups, by setting

$$
I_{n}: \bar{R}=\left(F, R_{1}, \ldots, R_{n}\right) \mapsto I_{n}(\bar{R}):=\frac{R_{1} \cap \cdots \cap R_{n}}{\left.\prod_{I \cup J=\{1, \ldots, n\}, I \cap J=\emptyset} \bigcap_{i \in I} R_{i}, \bigcap_{j \in J} R_{j}\right]}
$$

Clearly, for any $\bar{R} \in \mathcal{R}_{n}$, the abelian group $I_{n}(\bar{R})$ has a natural structure of $F / R_{1} \ldots R_{n}$-module, where the group action viewed via conjugation in $F$.

## 3. The surjection $q$ and the conjecture on $\alpha_{*}$

3.1. Consider the two-dimensional sphere $S^{2}$ as the standard two-complex constructed from the following presentation of the trivial group:

$$
\begin{equation*}
\left\langle x_{1}, \ldots, x_{n-1} \mid x_{1}, \ldots, x_{n-1}, x_{n-1}^{-1} \cdots x_{1}^{-1}\right\rangle \tag{5}
\end{equation*}
$$

This presentation defines an element $\bar{S}_{n}$ from $\mathcal{K}_{n}$ :

$$
\begin{equation*}
\bar{S}_{n}=\left(S^{2}, L_{1}, \ldots, L_{n}\right), \tag{6}
\end{equation*}
$$

with $L_{i}=\bigvee_{i=1}^{n-1} S^{1} \cup e_{i}$, where $e_{i}$ is the 2-cell corresponding to the relation word $x_{i}, i=$ $1, \ldots, n-1, e_{n}$ is the 2 -cell corresponding to the relation word $x_{n-1}^{-1} \cdots x_{1}^{-1}$.

In this section we show the following result.
Proposition 1. For an object $\bar{S}_{n}$ in $\mathcal{K}_{n}$ associated to Wu's example in $\mathcal{R}_{n}$ there is a surjection $q: \operatorname{Hom}_{\mathcal{K}_{n}}\left(\bar{S}_{n}, \bar{K}\right) \rightarrow \pi_{2}(K) /\left(i_{1}\left(K_{1}\right)+\ldots i_{n}\left(K_{n}\right)\right)$,
which is natural in $\bar{K} \in \mathcal{K}_{n}$.
For $\alpha \in \pi_{n}\left(S^{2}\right)=I_{n} \mathcal{F}_{n}\left(\bar{S}_{n}\right)$ we thus obtain the following diagram

where $\alpha^{*}(f)=f_{*}(\alpha)$.
Conjecture 1. For each $\alpha \in \pi_{n}\left(S^{2}\right)$ there exists a function $\alpha_{*}$ for which the diagram commutes. Hence $\alpha_{*}$ is well defined and natural provided $q(f)=q(g)$ implies $\alpha^{*}(f)=\alpha^{*}(g)$.
3.2. Recall that for a given two-dimensional complex $K$, the free crossed module

$$
\partial: \pi_{2}\left(K, K^{1}\right) \rightarrow \pi_{1}\left(K^{1}\right)
$$

can be defined as follows. The group $\pi_{2}\left(K, K^{1}\right)$ is generated by the set

$$
\left\{e_{\alpha}^{w} \mid \alpha \text { is a } 2 \text {-cell in } K, w \in \pi_{1}\left(K^{1}\right)\right\}
$$

with the set of relations

$$
\begin{equation*}
\left\{e_{\alpha}^{v} e_{\beta}^{w} e_{\alpha}^{-v} e_{\beta}^{-u}, u=v r_{\alpha} v^{-1} w\right\} \tag{7}
\end{equation*}
$$

where $r_{\alpha} \in \pi_{1}\left(K^{1}\right)$ is the attaching element representing $e_{\alpha}$ (see, for example, [9]). The homomorphism $\partial$ is defined by setting $\partial: e_{\alpha}^{w} \mapsto r_{\alpha}^{w}$. Hence every element from $\operatorname{ker}(\partial)=\pi_{2}(K)$ can be represented by an element $e_{\alpha_{1}}^{ \pm w_{1}} \ldots e_{\alpha_{m}}^{ \pm w_{m}}$, such that $r_{\alpha_{1}}^{ \pm w_{1}} \ldots r_{\alpha_{m}}^{ \pm w_{m}}$ is trivial in $\pi_{1}\left(K^{1}\right)$.

Let $f \in \operatorname{Hom}_{\mathcal{K}_{n}}\left(\bar{S}_{n}, \bar{K}\right)$. It means that there exists a homomorphism between two free groups $f: F_{n-1}:=F\left(x_{1}, \ldots, x_{n}\right) \rightarrow \pi_{1}\left(K^{1}\right)$ such that

$$
\begin{equation*}
f\left(x_{i}\right) \in \operatorname{ker}\left\{\pi_{1}\left(K^{1}\right) \rightarrow \pi_{i}\left(K_{i}\right)\right\}, \quad i=1, \ldots, n-1 \tag{8}
\end{equation*}
$$

and $f$ can be extended to a homomorphism between two crossed modules:


For a given group homomorphism $f: F_{n-1} \rightarrow \pi_{1}\left(K^{1}\right)$ with the property (8), the necessary and sufficient condition for the existence of the extension (9) is the condition

$$
f\left(x_{1} \cdots x_{n}\right) \subseteq R_{n}:=\operatorname{ker}\left\{\pi_{1}\left(K^{1}\right) \rightarrow \pi_{1}(K)\right\}
$$

For $\bar{K}=\left(K, K_{1}, \ldots, K_{n}\right) \in \mathcal{K}_{n}$, we now define the canonical (forgetful) map

$$
q: \operatorname{Hom}_{\mathcal{K}_{n}}\left(\bar{S}_{n}, \bar{K}\right) \rightarrow \pi_{2}(K) /\left(i_{1} \pi_{2}\left(K_{1}\right)+\ldots i_{n} \pi_{2}\left(K_{n}\right)\right)
$$

which carries a morphism $\bar{S}^{2} \rightarrow \bar{K}$ to the underlying map $S^{2} \rightarrow K$. Here the natural maps $i_{j}: \pi_{2}\left(K_{j}\right) \rightarrow K$ are induced by inclusions $K_{j} \rightarrow K$. Using the language of crossed modules, we can describe the map $q$ as follows. Denote by $\left\{s_{1}, \ldots, s_{n}\right\}$ the set of 2 -cells in $S^{2}$ viewed as the standard two-complex for the group presentation (5). The map $f^{\prime}$ defines elements $f^{\prime}\left(s_{\alpha}\right) \in$ $\pi_{2}\left(K, K^{1}\right)$. Observe that $\partial_{1}\left(s_{1} \ldots s_{n}\right)=1$ and the element $s_{1} \ldots s_{n}$ presents the generator of $\pi_{2}\left(S^{2}\right)$. Since the diagram (9) is commutative, $\partial_{2}\left(f^{\prime}\left(s_{1}\right) \ldots f^{\prime}\left(s_{n}\right)\right)=1$ and the element $f^{\prime}\left(s_{1}\right) \ldots f^{\prime}\left(s_{n}\right)$ represents certain element from $\operatorname{ker}\left(\partial_{2}\right)=\pi_{2}(K)$, which is exactly $q(f)$. Let us show that this map does not depend on an extension (91). Suppose we have another extension of the homomorphism $f$ :

$$
\begin{array}{ccc}
\pi_{2}\left(S^{2}, \vee_{i=1}^{n-1} S^{1}\right) & \xrightarrow{\partial_{1}} & F_{n-1} \\
f^{\prime \prime} \downarrow & & f \downarrow  \tag{10}\\
\pi_{2}\left(K, K^{1}\right) & \xrightarrow{\partial_{2}} & \pi_{1}\left(K^{1}\right)
\end{array}
$$

with $f^{\prime \prime}\left(s_{j}\right) \neq f^{\prime}\left(s_{j}\right)$ at least for one $j(1 \leq j \leq n)$. It follows that $\partial_{2}\left(f^{\prime}\left(s_{j}\right) f^{\prime \prime}\left(s_{j}\right)^{-1}\right)=1$, hence

$$
f^{\prime}\left(s_{j}\right) f^{\prime \prime}\left(s_{j}\right)^{-1} \in i m\left\{i_{j}: \pi_{2}\left(K_{j}\right) \rightarrow \pi_{2}(K)\right\} .
$$

Therefore, the images of elements $f^{\prime}\left(s_{1} \ldots s_{n}\right)$ and $f^{\prime \prime}\left(s_{1} \ldots s_{n}\right)$ are equal in the quotient $\pi_{2}(K) /\left(i_{1} \pi_{2}\left(K_{1}\right)+\right.$ $\left.\ldots i_{n} \pi_{2}\left(K_{n}\right)\right)$ and the map $q$ is well-defined.

Lemma 1. The map $q$ is surjective.
Proof. Consider the diagram (10). Now let $c=e_{\alpha_{1}}^{ \pm w_{1}} \ldots e_{\alpha_{m}}^{ \pm w_{m}}$ be an arbitrary element from $\operatorname{ker}\left(\partial_{2}\right)$. Let us enumerate all cells of $K$ in the following order: $e_{1, \alpha}, \ldots, e_{n, \alpha}$ with $e_{i, \alpha} \in K_{i}, i=$ $1, \ldots, m$. Clearly, the set of relations (7) in $\pi_{2}\left(K, K^{1}\right)$ gives a possibility to present the element $c$ in the form

$$
c=\prod_{*} e_{1, *}^{ \pm w_{1, *}} \cdots \prod_{*} e_{n, *}^{ \pm w_{n, *}}
$$

with some $w_{i, *} \in \pi_{1}\left(K^{1}\right)$. We define the map $f: F_{n-1} \rightarrow \pi_{1}\left(K^{1}\right)$ by setting $f\left(x_{i}\right)=\prod_{*} r_{i, *}^{ \pm w_{n, *}}$. We can extend it to $f^{\prime}: \pi_{2}\left(S^{2}, \vee_{i=1}^{n-1} S^{1}\right) \rightarrow \pi_{2}\left(K, K^{1}\right)$ by $f^{\prime}\left(s_{i}\right)=\prod_{*} r_{i, *}^{ \pm w_{n, *}}$. This is correct, since

$$
\partial_{1}\left(f^{\prime}\left(s_{n}\right)\right)=\partial_{2}\left(f^{\prime}\left(s_{1}\right) \ldots f^{\prime}\left(s_{n-1}\right)\right)^{-1}=f\left(\partial_{1}\left(s_{1} \ldots s_{n-1}\right)^{-1}\right)
$$

The homotopy class corresponding to the element $c \in \pi_{2}\left(K, K^{1}\right)$ coincides with $q(f)$ and the surjectivity of $q$ is proved.

## 4. Proof of the conjecture for $n=2$ and $n=3$

4.1. There are different ways of description of elements from $\pi_{2}$ for a standard complex of a given group presentation, for example, pictures, kernels of Jacobian maps, defined via Fox calculus etc. We describe the map $q$ in the conjecture by use of identity sequences which represent elements in $\pi_{2}(K)$, see [14]. The material about identity sequences we recall here is well-known. Nevertheless, we give it in detail since it will be the basic technical devise in our proofs.

Let $F$ be a free group with basis $X$ and $\mathcal{R}$ a certain set of words in $F$. Consider the group presentation

$$
\begin{equation*}
\mathcal{P}=\langle X \mid \mathcal{R}\rangle \tag{11}
\end{equation*}
$$

$c_{i}, i=1, \ldots, m$ are words in $F$, which are conjugates of elements from $\mathcal{R}$, i.e. $c_{i}=t_{i}^{ \pm w_{i}}, t_{i} \in$ $\mathcal{R}, w_{i} \in F$. The sequence

$$
\begin{equation*}
c=\left(c_{1}, \ldots, c_{m}\right) \tag{12}
\end{equation*}
$$

is called an identity sequence if the product $c_{1} \ldots c_{m}$ is the identity in $F$. For a given identity sequence (12), define its inverse:

$$
c^{-1}=\left(c_{m}^{-1}, \ldots, c_{1}^{-1}\right)
$$

For a given element $w \in F$, the conjugate $c^{w}$ is the sequence:

$$
c^{w}=\left(c_{1}^{w}, \ldots, c_{m}^{w}\right),
$$

which clearly is again an identity sequence. Define the following operation in the class of identity sequences, called Peiffer operations:
(i) replace each $w_{i}$ by any word equal to it in $F$;
(ii) delete two consecutive terms in the sequence if one is equal identically to the inverse of the other;
(iii) add two consecutive terms in the sequence if one is equal identically to the inverse of the other;
(iv) replace two consecutive terms $c_{i}, c_{i+1}$ by terms $c_{i+1}, c_{i+1}^{-1} c_{i} c_{i+1}$;
(v) replace two consecutive terms $c_{i}, c_{i+1}$ by terms $c_{i} c_{i+1} c_{i}^{-1}, c_{i}$.

Two identity sequences are called equivalent if one can be obtained from the other by a finite number of Peiffer operations. This defines an equivalence relation in the class of identity sequences. The set of equivalence classes of identity sequences for a given group presentation (11) denote by $E_{\mathcal{P}}$. The set $E_{\mathcal{P}}$ can be viewed as a group, with a binary operation defined as a class of justaposition of two sequences: for identity sequences $c_{1}, c_{2}$ and their equivalence classes $\left\langle c_{1}\right\rangle,\left\langle c_{2}\right\rangle \in E_{\mathcal{P}},\left\langle c_{1}\right\rangle+\left\langle c_{2}\right\rangle=\left\langle c_{1} c_{2}\right\rangle$. The inverse element of the class $\langle c\rangle$ is $\left\langle c^{-1}\right\rangle$ and the identity in $E_{\mathcal{P}}$ is the empty sequence. It is easy to see that $E_{\mathcal{P}}$ is Abelian. For two identity sequences $c=\left(c_{1}, \ldots, c_{m}\right)$ and $d=\left(d_{1}, \ldots, d_{k}\right)$, we have

$$
\langle c d\rangle=\left\langle\left(c_{1}, \ldots, c_{m}, d_{1}, \ldots, d_{k}\right)\right\rangle=\left\langle\left(d_{1}, \ldots, d_{k}, c_{1}^{d_{1} \ldots d_{m}}, \ldots, c_{m}^{d_{1} \ldots d_{m}}\right)\right\rangle
$$

by the relation (iv). Since $d_{1} \ldots d_{m}=1$ in $F$, we have

$$
\langle c d\rangle=\left\langle\left(d_{1}, \ldots, d_{k}, c_{1}, \ldots, c_{m}\right)\right\rangle=\langle d c\rangle
$$

Furthermore, $E_{\mathcal{P}}$ is a $F$-module, where the action is given by

$$
\langle c\rangle \circ f=\left\langle c^{f}\right\rangle, f \in F .
$$

It is easy to show that

$$
\langle c\rangle \circ r=\langle c\rangle, r \in R,
$$

i.e. the subgroup $R$ acts trivially at $E_{\mathcal{P}}$. To see this, let $r=r_{1}^{ \pm v_{1}} \ldots r_{k}^{ \pm v_{k}}, r_{i} \in \mathcal{R}, v_{i} \in F$. For any identity sequence $c=\left(c_{1}, \ldots, c_{m}\right)$, by (ii), (iii), (iv),

$$
\begin{aligned}
& \left\langle\left(c_{1}, \ldots, c_{m}\right)\right\rangle=\left\langle\left(c_{1}, \ldots, c_{m}, r_{1}^{ \pm v_{1}}, \ldots, r_{k}^{ \pm v_{k}}, r_{k}^{\mp v_{k}}, \ldots, r_{1}^{\mp v_{1}}\right)\right\rangle= \\
& \quad\left\langle\left(r_{1}^{ \pm v_{1}}, \ldots, r_{k}^{ \pm v_{k}}, c_{1}^{r}, \ldots, c_{m}^{r}, r_{k}^{\mp v_{k}}, \ldots, r_{1}^{\mp v_{1}}\right)\right\rangle=\left\langle\left(c_{1}^{r}, \ldots, c_{m}^{r}\right)\right\rangle .
\end{aligned}
$$

Thus $E_{\mathcal{P}}$ can be viewed as a $G$-module. It is not hard to show that for a given presentation $\mathcal{P}$, the second homotopy module $\pi_{2}\left(K_{\mathcal{P}}\right)$ is isomorphic to the identity sequence module $E_{\mathcal{P}}$ (see, for example, [14]).
4.2. For a given $\bar{K}$ choose the elements $e_{i, \alpha} \in \pi_{2}\left(K, K^{1}\right), i=1, \ldots, n, \alpha \in A$ which represent the corresponding two-dimensional cells in $K_{i}, i=1, \ldots, n$ with the natural property

$$
\partial\left(e_{i, \alpha}\right) \in R_{i},
$$

where $R_{i}=\operatorname{ker}\left\{\pi_{1}\left(K^{1}\right) \rightarrow \pi_{1}\left(K_{i}\right)\right\}, i=1, \ldots n$, and the normal closure of the set $\left\{\partial\left(e_{i, \alpha}\right) \mid \alpha \in\right.$ $A\}$ in $\pi_{1}\left(K^{1}\right)$ is equal to $R_{i}$. Clearly, $K$ is homotopically equivalent to a wedge

$$
K \simeq \bigvee_{j \in J} S^{2} \vee K_{\mathcal{P}}
$$

where $K_{\mathcal{P}}$ is the standard two-complex constructed from the group presentation

$$
\left\langle X \mid \partial\left(e_{i, \alpha}\right), i=1, \ldots, n, \alpha \in A\right\rangle,
$$

with $X$ being a basis of $\pi_{1}\left(K^{1}\right)$. We have the following natural isomorphism of $\pi_{1}(K)$-modules:

$$
\pi_{2}(K) /\left(i_{1} \pi_{2}\left(K_{1}\right)+\cdots+i_{n} \pi_{2}\left(K_{n}\right)\right) \simeq \pi_{2}\left(K_{\mathcal{P}}\right) /\left(i_{1} \pi_{2}\left(K_{\mathcal{P}_{1}}\right)+\cdots+i_{n} \pi_{2}\left(K_{\mathcal{P}_{n}}\right)\right)
$$

where $\mathcal{P}_{i}$ is the following presentation of the group $\pi_{1}\left(K_{i}\right)$ :

$$
\left\langle X \mid \partial\left(e_{i, \alpha}\right), \alpha \in A\right\rangle
$$

for $i=1, \ldots, n$.
Let $f, g \in \operatorname{Hom}_{\mathcal{K}_{n}}\left(\bar{S}_{n}, \bar{K}\right)$. We can present

$$
\begin{aligned}
& f\left(x_{i}\right)=r_{1}^{(i)^{ \pm w_{1, i}} \cdots r_{k_{i}}^{(i)^{ \pm w_{k_{i}, i}}}, \quad i=1, \ldots, n-1,} \\
& f\left(x_{1} \cdots x_{n}\right)=r_{1}^{(n)^{ \pm w_{1, n}}} \cdots r_{k_{n}}^{(n)^{ \pm w_{k n, n}}}
\end{aligned}
$$

for some $r_{j}^{(i)} \in\left\{\partial\left(e_{i, \alpha}\right), \alpha \in A\right\}$ and $w_{j, i} \in \pi_{1}\left(K^{1}\right)$. Analogically for $g \in \operatorname{Hom}_{\mathcal{K}_{n}}\left(\bar{S}_{n}, \bar{K}\right)$ :

$$
\begin{aligned}
& g\left(x_{i}\right)=r_{1}^{\prime(i)^{ \pm w_{1, i}^{\prime}} \ldots r_{k_{i}^{\prime}}^{\prime(i) \pm w_{k_{i}^{\prime}, i}^{\prime}}, i=1, \ldots, n-1,} \\
& g\left(x_{1} \cdots x_{n}\right)=r_{1}^{\prime(n)^{ \pm w_{1, n}^{\prime}} \ldots r_{k_{n}^{\prime}}^{\prime\left(n^{\prime}\right) \pm w_{k_{n}^{\prime}, n}^{\prime}}}
\end{aligned}
$$

The following Lemma follows directly from the definition of the map $q$ and the above description of the second homotopy module for the standard complex in terms of identity sequences.
Lemma 2. Using the above notation, $q(f)=q(g)$ if and only if the identity sequence

$$
\left(r_{1}^{(1)^{ \pm w_{1,1}}}, \ldots, r_{k_{n}}^{(n)^{ \pm w_{k_{n}, n}}}, r_{k_{n}^{\prime}}^{\left(n^{\prime}\right) \mp w_{k_{n}^{\prime}, n}^{\prime}}, \ldots, \ldots, r_{1}^{\prime(1) \mp w_{1,1}^{\prime}}\right)
$$

is equivalent to an identity sequence of the form

$$
\left(s_{1}^{(1)}, \ldots, s_{l_{1}}^{(1)}, \ldots, s_{1}^{(n)}, \ldots, s_{l_{n}}^{(n)}\right)
$$

with $s_{j}^{(i)} \in\left\{\partial\left(e_{i, \alpha}\right)^{ \pm w}, w \in \pi_{1}\left(K^{1}\right)\right\}$ such that $s_{1}^{(i)} \ldots s_{l_{i}}^{(i)}$ is trivial in $\pi_{1}\left(K^{1}\right)$ for every $i=$ $1, \ldots, n$.

Let $\left(K, K_{1}, K_{2}\right) \in \mathcal{K}_{2}$. The $\pi_{1}(K)$-module $\pi_{2}(K) /\left(i_{1} \pi_{2}\left(K_{1}\right)+i_{2} \pi_{2}\left(K_{2}\right)\right)$ can be identified to the module of the identity sequences of the type

$$
\begin{equation*}
\left(c_{1}, \ldots, c_{m}\right), c_{j} \in\left\{\partial\left(c_{i, \alpha}\right)^{w}, w \in \pi_{1}\left(K^{1}\right), \alpha \in A, i=1,2\right\} \tag{13}
\end{equation*}
$$

modulo the sequences of the form $\left(c_{1}, \ldots, c_{m_{1}}, c_{m_{1}+1}, \ldots, c_{m}\right)$ with

$$
c_{1}, \ldots, c_{m_{1}} \in\left\{\partial\left(c_{1, \alpha}\right)^{w}, w \in \pi_{1}\left(K^{1}\right)\right\}, c_{m_{1}+1}, \ldots, c_{m} \in\left\{\partial\left(c_{2, \alpha}\right)^{w}, w \in \pi_{1}\left(K^{1}\right)\right\}
$$

and

$$
c_{1} \ldots c_{m_{1}}=c_{m_{1}+1} \ldots c_{m}=1
$$

in $\pi_{1}\left(K^{1}\right)$.

Every identity sequence (13) with the help of Peiffer operations of the type (iv) can be reduced to the sequence of the form $\left(c_{1}, \ldots, c_{m_{1}}, c_{m_{1}+1}, \ldots, c_{m}\right)$ with $c_{1}, \ldots, c_{m_{1}} \in\left\{\partial\left(c_{1, \alpha}\right)^{w}, w \in\right.$ $\left.\pi_{1}\left(K^{1}\right)\right\}, c_{m_{1}+1}, \ldots, c_{m} \in\left\{\partial\left(c_{2, \alpha}\right)^{w}, w \in \pi_{1}\left(K^{1}\right)\right\}$.
4.3. For the most elementary case $n=2$ we view the 2 -sphere $S^{2}$ as a standard complex constructed from the group presentation

$$
\left\langle x \mid x, x^{-1}\right\rangle
$$

Clearly then

$$
I_{2}\left(\bar{S}^{2}\right)=\frac{\langle x\rangle \cap\left\langle x^{-1}\right\rangle}{[\langle x\rangle,\langle x\rangle]} \simeq \mathbb{Z}
$$

with $x$ a generator of this infinite cyclic group. For the generator $x \in \pi_{2}\left(S^{2}\right)$, the map

$$
\Lambda_{x}: \pi_{2}(K) /\left(i_{1} \pi_{2}\left(K_{1}\right)+i_{2} \pi_{2}\left(K_{2}\right)\right) \rightarrow \frac{R_{1} \cap R_{2}}{\left[R_{1}, R_{2}\right]}
$$

is given in the above notation by

$$
\Lambda_{x}:\left(c_{1}, \ldots, c_{m_{1}}, c_{m_{1}+1}, \ldots, c_{m}\right) \mapsto c_{1} \cdots c_{m_{1}} \cdot\left[R_{1}, R_{2}\right] .
$$

First observe that $\Lambda_{x}$ is the homomorphism of $\pi_{1}(K)=\pi_{1}\left(K^{1}\right) / R_{1} R_{2}$-modules. Secondly, $\Lambda_{x}$ clearly is an epimorphism. The fact that $\Lambda_{x}$ is a monomorphism is not difficult (see Theorem 1.3 [14] for the complete proof). Hence we have the following exact sequence of $\pi_{1}(K)$-modules due to Gutierrez and Ratcliffe [10]:

$$
\begin{equation*}
0 \rightarrow i_{1} \pi_{2}\left(K_{1}\right)+i_{2} \pi_{2}\left(K_{2}\right) \xrightarrow{\alpha} \pi_{2}(K) \rightarrow \frac{R_{1} \cap R_{2}}{\left[R_{1}, R_{2}\right]} \rightarrow 0 \tag{14}
\end{equation*}
$$

Theorem 1. Conjecture 1 is true for $n=3$.
Proof. In this case we view $S^{2}$ as the standard complex constructed for the group presentation

$$
\left\langle x_{1}, x_{2} \mid x_{1}, x_{2}, x_{2}^{-1} x_{1}^{-1}\right\rangle
$$

with

$$
I_{3}\left(\mathcal{F}_{3}\left(\bar{S}^{2}\right)\right)=I_{3}\left(F\left(x_{1}, x_{2}\right),\left\langle x_{1}\right\rangle^{F\left(x_{1}, x_{2}\right)},\left\langle x_{2}\right\rangle^{F\left(x_{1}, x_{2}\right)},\left\langle x_{2}^{-1} x_{1}^{-1}\right\rangle^{F\left(x_{1}, x_{2}\right)}\right) \simeq \mathbb{Z}
$$

with a generator given by the commutator $\left[x_{1}, x_{2}\right]$.
4.4. Let $\bar{K}=\left(K, K_{1}, K_{2}, K_{3}\right) \in \mathcal{K}_{3}$. Denote $F=\pi_{1}\left(K^{1}\right)$. Denote the sets of words in $F$ : $\mathcal{R}_{i}=\left\{\partial\left(e_{i, \alpha}, \alpha \in A\right\}, i=1,2,3\right.$. By $\mathcal{R}_{i}^{F}$ we mean the set $\left\{r^{w}, r \in \mathcal{R}_{i}, w \in F\right\}$. The $\pi_{1}(K)$-module $\pi_{2}(K) /\left(i_{1} \pi_{2}\left(K_{1}\right)+i_{2} \pi_{2}\left(K_{2}\right)+i_{3} \pi_{2}\left(K_{3}\right)\right)$ can be identified with the module of the identity sequences

$$
\begin{equation*}
c=\left(c_{1}, \ldots, c_{m}\right), c_{j} \in \mathcal{R}_{1}^{F} \cup \mathcal{R}_{2}^{F} \cup \mathcal{R}_{3}^{F} \tag{15}
\end{equation*}
$$

modulo the sequences of the type

$$
\begin{equation*}
\left(c_{1}, \ldots, c_{m_{1}}, c_{m_{1}+1}, \ldots, c_{m_{2}}, c_{m_{2}+1}, \ldots, c_{m}\right) \tag{16}
\end{equation*}
$$

with $c_{1}, \ldots, c_{m_{1}} \in \mathcal{R}_{1}^{F}, c_{m_{1}+1}, \ldots, c_{m} \in \mathcal{R}_{2}^{F}, c_{m_{2}+1}, \ldots, c_{m} \in \mathcal{R}_{3}^{F}$ and

$$
\begin{equation*}
c_{1} \ldots c_{m_{1}}=c_{m_{1}+1} \ldots c_{m_{2}}=c_{m_{2}+1} \ldots c_{m}=1 \tag{17}
\end{equation*}
$$

in $F$.
Divide the sequence (15) into the three ordered subsequences

$$
\begin{equation*}
\left(c_{r_{1}}, \ldots, c_{r_{l}}\right), \quad\left(c_{s_{1}}, \ldots, c_{s_{k}}\right), \quad\left(c_{t_{1}}, \ldots, c_{t_{n}}\right) \tag{18}
\end{equation*}
$$

where $c_{r_{i}} \in \mathcal{R}_{1}^{F}, i=1, \ldots, l, c_{s_{i}} \in \mathcal{R}_{2}^{F}, i=1, \ldots, k, c_{t_{i}} \in \mathcal{R}_{3}^{F}, i=1, \ldots, h$ and

$$
r_{1}<r_{2}<\cdots<r_{l}, \quad s_{1}<s_{2}<\cdots<s_{k}, \quad t_{1}<t_{2}<\cdots<t_{h}
$$

$$
\left\{r_{1}, \ldots, r_{l}\right\} \cup\left\{s_{1}, \ldots, s_{k}\right\} \cup\left\{t_{1}, \ldots, t_{h}\right\}=\{1, \ldots, m\} .
$$

Denote

$$
\begin{aligned}
& \bar{c}_{i}=c_{r_{i}}, i=1, \ldots, l, \\
& \bar{c}_{l+i}=c_{s_{i}} \prod_{r_{j}>s_{i}} c_{r_{j}} \\
& \bar{c}_{l+k+i}=i=1, \ldots, k, \\
& \left(\Pi_{t_{1}}\right. \\
& \left(\Pi_{r_{z}>t_{i}} c_{r_{z}}\right) \prod_{s_{j}>t_{1}} c_{s_{j}}^{\Pi_{r_{z}>s_{j}} c_{r_{z}}}, i=1, \ldots, h .
\end{aligned}
$$

Clearly,

$$
\bar{c}_{1}, \ldots, \bar{c}_{l} \in R_{1}, \bar{c}_{l+1}, \ldots, \bar{c}_{l+k} \in R_{2}, \bar{c}_{l+k+1}, \ldots, \bar{c}_{l+k+h} \in R_{3}
$$

and the sequence

$$
\begin{equation*}
\left(\bar{c}_{1}, \ldots, \bar{c}_{l+k+h}\right) \tag{19}
\end{equation*}
$$

is made of the sequence (15), applying the Peiffer operations of type (iv). At the first step we replace all terms $c_{r_{i}}$ to the left side of the sequence. At the second step we replace all terms $c_{s_{i}}$ between elements $c_{r_{i}}$-s and $c_{t_{i}}$-s and get the sequence (19). Denote

$$
\begin{aligned}
r_{c} & :=\bar{c}_{1} \ldots \bar{c}_{l} \in R_{1}, \\
s_{c} & :=\bar{c}_{l+1} \ldots \bar{c}_{l+k} \in R_{2}, \\
t_{c} & :=\bar{c}_{l+k+1} \ldots \bar{c}_{l+k+h} \in R_{3} .
\end{aligned}
$$

In these notations, for the generator $x:=\left[x_{1}, x_{2}\right]$ of $I_{3}\left(\mathcal{F}_{3}\left(\bar{S}^{2}\right)\right)$ construct the map

$$
\begin{equation*}
\Lambda_{x}: \pi_{2}(K) /\left(i_{1} \pi_{2}\left(K_{1}\right)+i_{2} \pi_{2}\left(K_{2}\right)+i_{3} \pi_{2}\left(K_{3}\right)\right) \rightarrow \frac{R_{1} \cap R_{2} \cap R_{3}}{\left[R_{1}, R_{2} \cap R_{3}\right]\left[R_{2}, R_{3} \cap R_{1}\right]\left[R_{3}, R_{1} \cap R_{2}\right]}, \tag{20}
\end{equation*}
$$

where $F=\pi_{1}\left(K^{1}\right), R_{i}=\operatorname{ker}\left\{F \rightarrow \pi_{1}\left(K_{i}\right)\right\}, i=1,2,3$, by setting

$$
\Lambda_{x}:\left(c_{1}, \ldots, c_{m}\right) \mapsto\left[r_{c}, s_{c}\right] \cdot\left[R_{1}, R_{2} \cap R_{3}\right]\left[R_{2}, R_{3} \cap R_{1}\right]\left[R_{3}, R_{1} \cap R_{2}\right]
$$

Since $r_{c} s_{c} t_{c}=1$ in $F$, we have $\left[r_{c}, s_{c}\right] \in R_{1} \cap R_{2} \cap R_{3}$.
Let us show that the above map $\Lambda_{x}$ is well-defined. Let $c^{\prime}$ be an identity sequence equivalent to the sequence $c$. Defining elements $r_{c^{\prime}}, s_{c^{\prime}}, t_{c^{\prime}}$ as above, we have to show that

$$
\begin{equation*}
\left[r_{c}, s_{c}\right] \equiv\left[r_{c^{\prime}}, s_{c^{\prime}}\right] \quad \bmod \left[R_{1}, R_{2} \cap R_{3}\right]\left[R_{2}, R_{3} \cap R_{1}\right]\left[R_{3}, R_{1} \cap R_{2}\right] . \tag{21}
\end{equation*}
$$

Since we above defined map $\Lambda_{x}$ is trivial for any sequence of the type (16) with conditions (17), the equivalence (21) is necessary and sufficient for the correctness of the map $\Lambda_{x}$.

First observe that if the sequences $c$ and $c^{\prime}$ differ by the Peiffer operations of the type (ii) or (iii), the equivalence 21 holds. The only non-trivial Peiffer operations needed to check are operations (iv) and (v). Since (v) is converse to (iv), it is enough to prove the equivalence (21) for the case $c^{\prime}$ is obtained from $c$ by the single Peiffer operation of the type (iv):

$$
c_{i}^{\prime}=c_{i+1}, c_{i+1}^{\prime}=c_{i+1}^{-1} c_{i} c_{i+1}, \quad c_{j}^{\prime}=c_{j}, j \neq i, i+1
$$

for some $1 \leq i<m$.
The cases $i, i+1 \in\left\{r_{1}, \ldots, r_{l}\right\}, i, i+1 \in\left\{s_{1}, \ldots, s_{k}\right\}, i, i+1 \in\left\{t_{1}, \ldots, t_{h}\right\}$ are trivial. In these cases $r_{c}=r_{c^{\prime}}, s_{c}=s_{c^{\prime}}$, hence the needed equivalence (21) follows. If $i+1 \in\left\{r_{1}, \ldots, r_{l}\right\}$, there is also nothing to prove, since the definition of $r_{c}, s_{c}$ involves the process of repeating of such operations. If $i \in\left\{t_{1}, \ldots, t_{h}\right\}$ or $i+1 \in\left\{t_{1}, \ldots, t_{h}\right\}$ then we clearly have $\left[r_{c}, s_{c}\right] \equiv\left[r_{c^{\prime}}, s_{c^{\prime}}\right]$ $\bmod \left[R_{1}, R_{2} \cap R_{3}\right]\left[R_{2}, R_{3} \cap R_{1}\right]$.

The only non-trivial case to consider is $i \in\left\{r_{1}, \ldots, r_{l}\right\}, i+1 \in\left\{s_{1}, \ldots, s_{k}\right\}$. Clearly then, $\left[r_{c^{\prime}}, s_{c^{\prime}}\right]=\left[r_{c^{\prime \prime}}, s_{c^{\prime \prime}}\right]$, where the sequence $c^{\prime \prime}$ is obtained by applying again the operation (iv) to the sequence $c^{\prime}$ :

$$
c_{i}^{\prime \prime}=c_{i+1}^{-1} c_{i} c_{i+1}, c_{i+1}^{\prime \prime}=c_{i+1}^{-1} c_{i}^{-1} c_{i+1} c_{i} c_{i+1} .
$$

Let $c_{i}=c_{r_{j}}, c_{i+1}=c_{s_{e}}$. Repeating the operation (iv), we can deform the sequences $c$ and $c^{\prime \prime}$ to the form

$$
r_{2}=r_{1}+1, \ldots, r_{j-1}=r_{j-2}+1, r_{j+2}=r_{j+1}+1, \ldots, r_{l}=r_{l-1}+1
$$

without changing $\left[r_{c}, s_{c}\right]$ and $\left[r_{c^{\prime \prime}}, s_{c^{\prime \prime}}\right]$. Now we can form the triple of words in $F$ :

$$
\mathcal{R}_{1}^{\prime}=\mathcal{R}_{1} \cup\left\{c_{r_{1}} \ldots c_{r_{j-1}}, c_{j+1} \ldots c_{l}\right\}, \mathcal{R}_{2}^{\prime}=\mathcal{R}_{2}, \mathcal{R}_{3}^{\prime}=\mathcal{R}_{3}
$$

Clearly, this triple preserves the triple of normal subgroups $R_{1}, R_{2}, R_{3}$ and we can consider the new identity sequences for the triple of words $\mathcal{R}_{1}^{\prime} \cup \mathcal{R}_{2}^{\prime} \cup \mathcal{R}_{3}^{\prime}$ formed by gluing the elements $c_{r_{1}}, \ldots, c_{r_{j-1}}$, and $c_{r_{j+1}}, \ldots, c_{r_{l}}$ :

$$
c^{\prime \prime \prime}=\left(*, \ldots, *, c_{r_{1}} \ldots c_{r_{j-1}}, *, \ldots, *, c_{r_{j+1}} \ldots c_{r_{l}}, *, \ldots, *\right) .
$$

It is easy to see that

$$
\left[r_{c}, s_{c}\right]=\left[r_{c^{\prime \prime \prime}}, s_{c^{\prime \prime \prime}}\right]
$$

in $F$. Hence, we can always assume that $l=3, c_{r_{2}}=c_{i}$ and reduce arbitrary case to this one using the described procedure. In these notations, we have sequences

$$
\begin{aligned}
& c=\left(*, \ldots, *, c_{r_{1}}, *, \ldots, *, c_{r_{2}}, c_{s_{e}}, *, \ldots, *, c_{r_{3}}, *, \ldots, *\right), \\
& c^{\prime \prime}=\left(*, \ldots, *, c_{r_{1}}, *, \ldots, *, c_{r_{2}}^{s_{e}}, c_{s_{e}}^{c_{r_{2}} c_{s_{e}}}, *, \ldots, *, c_{r_{3}}, *, \ldots, *\right) .
\end{aligned}
$$

We have the following:

$$
\begin{aligned}
& {\left[r_{c}, s_{c}\right]=\left[c_{r_{1}} c_{r_{2}} c_{r_{3}}, S_{1}\right]} \\
& {\left[r_{c^{\prime \prime \prime}}, s_{c^{\prime \prime \prime \prime}}\right]=\left[c_{r_{1}} c_{r_{2}} c_{r_{3}}, S_{2}\right]}
\end{aligned}
$$

where

$$
\begin{aligned}
& S_{1}=\left(\prod_{s_{j}<r_{1}} c_{s_{j}}^{c_{r_{1}} c_{r_{2}} c_{r_{3}}}\right)\left(\prod_{r_{1}<s_{j}<s_{e}} c_{s_{j}}^{c_{r_{2}} c_{r_{3}}}\right) \cdot c_{s_{e}}^{c_{r_{3}}} \cdot\left(\prod_{s_{e}<s_{j}<r_{3}} c_{s_{j}}^{c_{r_{3}}}\right)\left(\prod_{s_{j}<r_{1}} c_{r_{3}<s_{j}} c_{s_{j}} c_{r_{1}<s_{j}<s_{e}}^{c_{s_{e}}} c_{s_{j}}^{c_{s_{e}}} c_{r_{3}}\right) \cdot c_{s_{e}}^{c_{r_{2}} c_{s_{e} e} c_{r_{3}}} \cdot\left(\prod_{s_{e}<s_{j}<r_{3}} c_{s_{j}}^{c_{r_{3}}}\right)\left(\prod_{r_{3}<s_{j}} c_{s_{j}}\right) .
\end{aligned}
$$

We then have

$$
\begin{aligned}
& {\left[c_{r_{1}} c_{r_{2}}^{c_{s e}} c_{r_{3}}, S_{2}\right]=c_{r_{3}}^{-1} c_{r_{2}}^{-1}\left[c_{r_{2}}^{-1}, c_{s_{e}}\right] c_{r_{1}}^{-1} S_{2}^{-1} c_{r_{1}} c_{r_{2}}^{c_{s_{e}}} c_{r_{3}} S_{2}=} \\
& \\
& c_{r_{3}}^{-1} c_{r_{2}}^{-1}\left[c_{r_{2}}^{-1}, c_{s_{e}} c_{r_{1}}^{-1} S_{2}^{-1} c_{r_{3}}^{-1} c_{r_{2}}^{-c_{s e}} c_{r_{2}}^{c_{s_{e} e}} c_{r_{3}} c_{r_{1}} c_{r_{2}}^{c_{s_{e}}} c_{r_{3}} S_{2} \equiv\right. \\
& \quad c_{r_{3}}^{-1} c_{r_{2}}^{-1} c_{r_{1}}^{-1} S_{2}^{-1} c_{r_{3}}^{-1} c_{r_{2}}^{-c_{s_{e}}}\left[c_{r_{2}}^{-1}, c_{s_{e}} c_{r_{2}}^{c_{s}} c_{r_{3}} c_{r_{1}} c_{r_{2}}^{c_{s}} c_{r_{3}} S_{2} \quad \bmod \left[R_{3}, R_{1} \cap R_{2}\right],\right.
\end{aligned}
$$

since $S_{2}^{-1} c_{r_{3}}^{-1} c_{r_{2}}^{-c_{s e}} \in R_{3},\left[c_{r_{2}}^{-1}, c_{s_{e}}\right] \in R_{1} \cap R_{2}$. Therefore,

$$
\left[c_{r_{1}} c_{r_{2}}^{c_{s e}} c_{r_{3}}, S_{2}\right] \equiv c_{r_{3}}^{-1} c_{r_{2}}^{-1} c_{r_{1}}^{-1} S_{2}^{-1} c_{r_{3}}^{-1} c_{r_{2}}^{-c_{s e}} c_{r_{2}} c_{r_{3}} c_{r_{1}} c_{r_{2}}^{c_{s e}} c_{r_{3}} S_{2} \quad \bmod \left[R_{2}\right]
$$

However,

$$
c_{r_{1}} c_{r_{2}}^{c_{s e}} c_{r_{3}} S_{2}=c_{r_{1}} c_{r_{2}} c_{r_{3}} S_{1} \in R_{3},
$$

therefore, $S_{2}=c_{r_{3}}^{-1} c_{r_{2}}^{-c_{s e}} c_{r_{2}} c_{r_{3}} S_{1}$ and we have

$$
\left[c_{r_{1}} c_{r_{2}}^{c_{s e}} c_{r_{3}}, S_{2}\right] \equiv\left[c_{r_{1}} c_{r_{2}} c_{r_{3}}, S_{1}\right] \quad \bmod \left[R_{3}, R_{1} \cap R_{2}\right] .
$$

Hence, we always have the needed equivalence (21) and we proved that the map $\Lambda_{x}$ is welldefined.

For the generator $x \in \pi_{3}\left(S^{2}\right)$ denote by $\Lambda$ the composite map of the natural projection $\pi_{2}(K) \rightarrow \pi_{2}(K) / i_{1} \pi_{2}\left(K_{1}\right)+i_{2} \pi_{2}\left(K_{2}\right)+i_{3} \pi_{2}\left(K_{3}\right)$ and the map $\Lambda_{x}:$

$$
\Lambda: \pi_{2}(K) \rightarrow \frac{R_{1} \cap R_{2} \cap R_{3}}{\left[R_{1}, R_{2} \cap R_{3}\right]\left[R_{2}, R_{3} \cap R_{1}\right]\left[R_{3}, R_{1} \cap R_{2}\right]}
$$

Proposition 2. Let $b \in i_{12} \pi_{2}\left(K_{1} \cup K_{2}\right)+i_{13} \pi_{2}\left(K_{1} \cup K_{3}\right)+i_{23} \pi_{2}\left(K_{2} \cup K_{3}\right) \subseteq \pi_{2}(K)$ where the maps $i_{12}, i_{13}, i_{23}$ are induced by the inclusions

$$
i_{12}: K_{1} \cup K_{2} \rightarrow K, i_{13}: K_{1} \cup K_{3} \rightarrow K, i_{23}: K_{2} \cup K_{3} \rightarrow K
$$

We then have $\Lambda(a+b)=\Lambda(a)$ for every $a \in \pi_{2}(K)$.
Proof. Let $a$ be an element from $\pi_{2}(K)$ presented by identity sequence (18) and the element $b$ be an element from $i_{12}\left(K_{1} \cup K_{2}\right) \subseteq \pi_{2}(K)$ presented by the identity sequence $\left(d_{1}, \ldots, d_{l^{\prime}}, e_{1}, \ldots, e_{k^{\prime}}\right)$ with $d_{i} \in \mathcal{R}_{1}^{F}, e_{i} \in \mathcal{R}_{2}^{F}$. The element $a+b$ can be presented by the following identity sequence

$$
c(a+b)=\left(c_{r_{1}}, \ldots, c_{r_{l}}, d_{1}, \ldots, d_{l^{\prime}}, c_{s_{1}}^{d_{1} \ldots d_{l^{\prime}}}, \ldots, c_{s_{k}}^{d_{1} \ldots d_{l^{\prime}}}, e_{1}, \ldots, e_{k^{\prime}}, f_{1}, \ldots, f_{h^{\prime}}\right)
$$

with $f_{1}, \ldots, f_{h^{\prime}} \in \mathcal{R}_{3}^{F}$. Denote $a_{1}=c_{r_{1}} \ldots c_{r_{l}}, a_{2}=d_{1} \ldots d_{l^{\prime}}, b_{1}=c_{s_{1}} \ldots c_{s_{k}}, b_{2}=e_{1} \ldots e_{k^{\prime}}$. We then have

$$
\begin{aligned}
& {\left[a_{1} a_{2}, b_{1}^{a_{2}} b_{2}\right]=a_{2}^{-1} a_{1}^{-1} b_{2}^{-1} a_{2}^{-1} b_{1}^{-1} a_{2} a_{1} b_{1} a_{2} b_{2}} \\
& \qquad a_{1}^{-1} b_{2}^{-1} a_{2}^{-1} b_{1}^{-1} a_{1} b_{1} a_{2} b_{2} \equiv a_{1}^{-1} b_{1}^{-1} a_{1} b_{1} \quad \bmod \left[R_{3}, R_{1} \cap R_{2}\right],
\end{aligned}
$$

since $a_{2} \in R_{1} \cap R_{2}, a_{1}^{-1} b_{2}^{-1} a_{2}^{-1} b_{1}^{-1} \in R_{3}, a_{2} b_{2}=1$. Hence $\Lambda(a+b)=\Lambda(a)$.
In the case $b \in i_{13} \pi_{2}\left(K_{1} \cup K_{3}\right)+i_{23} \pi_{2}\left(K_{2} \cup K_{3}\right)$, we have obviously, that the elements which represent $\Lambda(a+b)$ and $\Lambda(a)$ are equal modulo $\left[R_{1}, R_{2} \cap R_{3}\right]\left[R_{2}, R_{3} \cap R_{1}\right]$ hence $\Lambda(a+b)=\Lambda(a)$.

The following example shows that the map $\Lambda$ is not always surjective.
Example. Let $F$ be a free group with generators $x_{1}, x_{2}$. Consider the following sets of words:

$$
\mathcal{R}_{1}=\left\{x_{1}\right\}, \mathcal{R}_{2}=\left\{\left[x_{1}, x_{2}\right]\right\}, \mathcal{R}_{3}=\left\{\left[x_{1}, x_{2}, x_{1}\right]\right\} .
$$

Denoting $R_{1}, R_{2}, R_{3}$ the normal closures of the sets $\mathcal{R}_{1}, \mathcal{R}_{2}, \mathcal{R}_{3}$ respectively, we have

$$
\left[R_{1}, R_{2} \cap R_{3}\right],\left[R_{2}, R_{3} \cap R_{1}\right],\left[R_{3}, R_{1} \cap R_{2}\right] \subseteq \gamma_{4}(F)
$$

where $\gamma_{4}(F)$ the 4 -th lower central series term of $F$. However,

$$
\left[x_{1}, x_{2}, x_{1}\right] \in\left(R_{1} \cap R_{2} \cap R_{3}\right) \backslash \gamma_{4}(F),
$$

since $\left[x_{1}, x_{2}, x_{1}\right]$ is a basic commutator of length three in $F$. Suppose we have

$$
\Lambda(x)=\left[x_{1}, x_{2}, x_{3}\right] \cdot\left[R_{1}, R_{2} \cap R_{3}\right]\left[R_{2}, R_{3} \cap R_{1}\right]\left[R_{3}, R_{1} \cap R_{2}\right]
$$

for some element $x$ of the second homotopy module of the standard complex constructed for the group presentation

$$
\left\langle x_{1}, x_{2} \mid x_{1},\left[x_{1}, x_{2}\right],\left[x_{1}, x_{2}, x_{1}\right]\right\rangle .
$$

We then have

$$
\begin{equation*}
\left[x_{1}, x_{2}, x_{1}\right] \equiv[r, s] \quad \bmod \gamma_{4}(F) \tag{22}
\end{equation*}
$$

for some $r \in R_{1}, s \in R_{2}$, such that

$$
\begin{equation*}
r s \in R_{3} . \tag{23}
\end{equation*}
$$

However, the condition (23) implies that $r \in \gamma_{2}(F)$, since $s \in \gamma_{2}(F)$. Therefore $[r, s] \in \gamma_{4}(F)$ and the equivalence (22) is not possible. Hence, the map $\Lambda$ is not surjective.
Theorem 2. The map $\Lambda$ is a homogenous quadratic map, i.e.

$$
\Lambda(a, b)=\Lambda(a+b)-\Lambda(a)-\Lambda(b)
$$

is bilinear and $\Lambda(x)=\Lambda(-x)$ for any $a, b, x \in \pi_{2}(K)$.

Proof. For $x, y \in \pi_{2}\left(K_{\langle X}\left|\mathcal{R}_{1} \cup \mathcal{R}_{2} \cup \mathcal{R}_{3}\right\rangle\right)$, consider the cross-effect

$$
\Lambda(a, b)=\Lambda(a+b)-\Lambda(a)-\Lambda(b) \in \frac{R_{1} \cap R_{2} \cap R_{3}}{\left[R_{1}, R_{2} \cap R_{3}\right]\left[R_{2}, R_{3} \cap R_{1}\right]\left[R_{3}, R_{1} \cap R_{2}\right]} .
$$

Represent elements $a, b$ by identity sequences:

$$
c(a)=\left(c_{1}, \ldots, c_{m}\right), c(b)=\left(c_{1}^{\prime}, \ldots, c_{m^{\prime}}^{\prime}\right)
$$

Consider the corresponding divisions of the sequences $c(a)$ and $c(b)$ :

$$
\begin{aligned}
& \left\{c_{r_{1}}, \ldots, c_{r_{l}}\right\} \cup\left\{c_{s_{1}}, \ldots, c_{s_{k}}\right\} \cup\left\{c_{t_{1}}, \ldots, c_{t_{n}}\right\}=\left\{c_{1}, \ldots, c_{m}\right\} \\
& \left\{c_{\bar{r}_{1}}^{\prime}, \ldots, c_{\bar{r}_{l_{l}}}^{\prime}\right\} \cup\left\{c_{\bar{s}_{1}}^{\prime}, \ldots, c_{\bar{s}_{k^{\prime}}}^{\prime}\right\} \cup\left\{c_{\bar{t}_{1}}^{\prime}, \ldots, c_{\bar{t}_{n^{\prime}}}^{\prime}\right\}=\left\{c_{1}^{\prime}, \ldots, c_{m}^{\prime}\right\}
\end{aligned}
$$

with $c_{r_{i}}, c_{\bar{r}_{i}}^{\prime} \in \mathcal{R}_{1}^{F}, c_{s_{i}}, c_{\bar{s}_{i}}^{\prime} \in \mathcal{R}_{2}^{F}, c_{t_{i}}, c_{\bar{t}_{i}}^{\prime} \in \mathcal{R}_{3}^{F}$. Consider then the induced division of the sequence $c(a+b)=\left(c_{1}, \ldots, c_{m}, c_{1}^{\prime}, \ldots, c_{m^{\prime}}^{\prime}\right)$, which represents the element $a+b \in \pi_{2}\left(K_{\left\langle X \mid \mathcal{R}_{1} \cup \mathcal{R}_{2} \cup \mathcal{R}_{3}\right\rangle}\right):$

$$
\left\{c_{r_{1}}, \ldots, c_{r_{l}}, c_{\bar{r}_{1}}^{\prime}, \ldots, c_{\bar{r}_{l^{\prime}}}^{\prime}\right\} \cup\left\{c_{s_{1}}, \ldots, c_{s_{k}}, c_{\bar{s}_{\bar{x}_{1}}}^{\prime}, \ldots c_{\bar{s}_{k^{\prime}}}^{\prime}\right\} \cup\left\{c_{t_{1}}, \ldots, c_{t_{n}}, c_{\bar{t}_{1}}^{\prime}, \ldots, c_{\bar{t}_{n^{\prime}}}^{\prime}\right\}
$$

For the description of the functor $\Lambda(a, b)$, using the Peiffer operation (iv) to the sequences $c(a)$ and $c(b)$, we can reduce the general case to the case of $l=1, k=1, l^{\prime}=1, k^{\prime}=1$ with $r_{1}<s_{1}$, $\bar{r}_{1}<\bar{s}_{1}$. Denote $x_{1}=c_{r_{1}}, y_{1}=c_{s_{1}}, x_{2}=c_{\bar{r}_{1}}^{\prime}, y_{2}=c_{\bar{s}_{1}}^{\prime}$.

We then have

$$
\begin{aligned}
& \Lambda(a)=\left[x_{1}, y_{1}\right], \Lambda(b)=\left[x_{2}, y_{2}\right], \\
& \Lambda(a+b)=\left[x_{1} x_{2}, y_{1}^{x_{2}} y_{2}\right] .
\end{aligned}
$$

We have

$$
\begin{aligned}
\Lambda(a+b) & =\left[x_{1}, y_{2}\right]^{x_{2}}\left[x_{2}, y_{2}\right]\left[x_{1}, y_{1}^{x_{2}}\right]^{x_{2} y_{2}}\left[x_{2}, y_{1}^{x_{2}}\right]^{y_{2}} \\
& \equiv\left[x_{1}, y_{2}\right]^{x_{2}}\left[x_{2}, y_{2}\right]\left[x_{1}, y_{1}^{x_{2}}\right]\left[x_{2}, y_{1}\right] \quad \bmod \left[R_{3}, R_{1} \cap R_{2}\right] \\
& \equiv\left[x_{1}, y_{2}\right]^{x_{2}}\left[x_{2}, y_{2}\right] x_{1}^{-1} x_{2}^{-1} y_{1}^{-1} x_{2} x_{1} y_{1} \quad \bmod \left[R_{3}, R_{1} \cap R_{2}\right] \\
& \equiv\left[x_{1}, y_{2}\right]^{x_{2}}\left[x_{2}, y_{2}\right]\left[x_{2}, y_{1}\right]^{x_{1}}\left[x_{1}, y_{1}\right] \quad \bmod \left[R_{3}, R_{1} \cap R_{2}\right] .
\end{aligned}
$$

Since $x_{1} y_{1}, x_{2} y_{2} \in R_{3}$,

$$
\begin{aligned}
\Lambda(a, b)=\Lambda(a+b)-\Lambda(a)-\Lambda(b) & \equiv\left[x_{1}, y_{2}\right]^{x_{2}}\left[x_{2}, y_{1}\right]^{x_{1}} \quad \bmod \left[R_{3}, R_{1} \cap R_{2}\right] \\
& \equiv\left[x_{1}, y_{2}\right]^{y_{2}^{-1}}\left[x_{2}, y_{1}\right]^{y_{1}^{-1}} \quad \bmod \left[R_{3}, R_{1} \cap R_{2}\right] \\
& \equiv\left[y_{2}^{-1}, x_{1}\right]\left[y_{1}^{-1}, x_{2}\right] \quad \bmod \left[R_{3}, R_{1} \cap R_{2}\right] .
\end{aligned}
$$

Now let us show the linearity of the functor $\Lambda(*, *)$, i.e. that

$$
\begin{align*}
& \Lambda(a+b, d)=\Lambda(a, c)+\Lambda(b, d),  \tag{24}\\
& \Lambda(a, b+d)=\Lambda(a, b)+\Lambda(a, d) \tag{25}
\end{align*}
$$

for arbitrary elements $a, b, d \in \pi_{2}\left(K_{\langle X|}\left|\mathcal{R}_{1} \cup \mathcal{R}_{2} \cup \mathcal{R}_{3}\right\rangle\right)$. Let $c(a), c(b)$ and $c(d)$ be the identity sequences represented the elements $a, b$ and $d$ respectively. Again, without loss of generality we can assume that these elements are represented by identity sequences with single element from each class $\mathcal{R}_{i}$. Denote the correspondent pairs by $x_{1}, y_{1} \subset c(a)$ (the set-theoretical inclusion means that $x_{1}, y_{1}$ are elements of the sequence $\left.c(a)\right), x_{2}, y_{2} \subset c(b), x_{3}, y_{3} \subset c(d)$. In this
notation, modulo [ $R_{1}, R_{2} \cap R_{3}$ ][ $\left.R_{2}, R_{3} \cap R_{1}\right]\left[R_{3}, R_{1} \cap R_{2}\right.$ ], we have

$$
\begin{aligned}
\Lambda(a+b, d) & \equiv\left[y_{3}^{-1}, x_{1} x_{2}\right]\left[y_{2}^{-1} y_{1}^{-x_{2}}, x_{3}\right] \\
& \equiv\left[y_{3}^{-1}, x_{2}\right]\left[y_{3}^{-1}, x_{1}\right]^{x_{2}}\left[y_{2}^{-1}, x_{3}\right]^{y_{1}^{-x_{2}}}\left[y_{1}^{-x_{2}}, x_{3}\right] \\
& \equiv\left[y_{3}^{-1}, x_{2}\right] x_{2}^{-1} y_{3} x_{1}^{-1} y_{3} x_{1} y_{1} x_{2} y_{2} x_{3}^{-1} y_{2}^{-1} x_{2}^{-1} y_{1}^{-1} x_{2} x_{3} \\
& \equiv\left[y_{3}^{-1}, x_{2}\right] x_{2}^{-1} y_{3} x_{1}^{-1} y_{3} x_{1} y_{1}\left(x_{2} y_{2} x_{3}^{-1} y_{2}^{-1} x_{2}^{-1} x_{3}\right) x_{3}^{-1} y_{1}^{-1} x_{2} x_{3} \\
& \equiv\left[y_{3}^{-1}, x_{2}\right] x_{2}^{-1}\left(x_{2} y_{2} x_{3}^{-1} y_{2}^{-1} x_{2}^{-1} x_{3}\right) y_{3} x_{1}^{-1} y_{3} x_{1} y_{1} x_{3}^{-1} y_{1}^{-1} x_{2} x_{3} \\
& \equiv\left[y_{3}^{-1}, x_{2}\right]\left[y_{2}^{-1}, x_{3}\right] x_{3}^{-1} x_{2}^{-1} x_{3} y_{3} x_{1}^{-1} y_{3} x_{1} y_{1} x_{3}^{-1} y_{1}^{-1} x_{2} x_{3} \\
& \equiv\left[y_{3}^{-1}, x_{2}\right]\left[y_{2}^{-1}, x_{3}\right] x_{3}^{-1} x_{2}^{-1} x_{3}\left[y_{3}^{-1}, x_{1}\right]\left[y_{1}^{-1}, x_{3}\right] x_{3}^{-1} x_{2} x_{3} \\
& \equiv\left[y_{3}^{-1}, x_{2}\right]\left[y_{2}^{-1}, x_{3}\right]\left[y_{3}^{-1}, x_{1}\right]\left[y_{1}^{-1}, x_{3}\right] \\
& \equiv \Lambda(a, d)+\Lambda(b, d),
\end{aligned}
$$

since $\left[y_{3}^{-1}, x_{1}\right]\left[y_{1}^{-1}, x_{3}\right] \in R_{2} \cap R_{3}$ and (24) follows. The equality (25) can be proved analogically.
Now let us prove that $\Lambda(-x)=\Lambda(x)$. Clearly, we can assume that our identity sequence representing the element $x \in \pi_{2}(K)$ has the form

$$
\left(r_{1}, s_{1}, t_{1}\right)
$$

with $r_{1} \in \mathcal{R}_{1}, s_{1} \in \mathcal{R}_{2}, t_{1} \in \mathcal{R}_{3}$. The inverse sequence, which represents the element $-x$ has the form

$$
\left(t_{1}^{-1}, s_{1}^{-1}, r_{1}^{-1}\right)
$$

We have

$$
\Lambda(-x)=\left[r_{1}^{-1}, s_{1}^{-r_{1}^{-1}}\right]=\left[s_{1}^{-1}, r_{1}\right]=\left[r_{1}, s_{1}\right]^{s_{1}^{-1}} \equiv\left[r_{1}, s_{1}\right] \equiv \Lambda(x) \quad \bmod \left[R_{2}, R_{3} \cap R_{1}\right] .
$$

Theorem 3. The function $\Lambda$ induces the homomorphism of $F / R_{1} R_{2} R_{3}$-modules

$$
\bar{\Lambda}: \pi_{3}(K) \rightarrow \frac{R_{1} \cap R_{2} \cap R_{3}}{\left[R_{1}, R_{2} \cap R_{3}\right]\left[R_{2}, R_{3} \cap R_{1}\right]\left[R_{3}, R_{1} \cap R_{2}\right]}
$$

Proof. Let $x \in \pi_{2}\left(K_{\langle X|}\left|\mathcal{R}_{1} \cup \mathcal{R}_{2} \cup \mathcal{R}_{3}\right\rangle\right)$. Present $x$ by the sequence

$$
c(x)=\left(c_{1}, \ldots, c_{m}\right)
$$

For a given element $f \in \pi_{1}(K)$, present this element as a coset $f=w \cdot R_{1} R_{2} R_{3}$ for some element $w \in F$. The element $f \circ x \in \pi_{2}\left(K_{\langle X|}\left|\mathcal{R}_{1} \cup \mathcal{R}_{2} \cup \mathcal{R}_{3}\right\rangle\right)$ can be presented by sequence

$$
c(x)^{w}=\left(c_{1}^{w}, \ldots, c_{m}^{w}\right)
$$

It follows directly from the definition of $\Lambda(x)$, that

$$
\Lambda(f \circ x) \equiv \Lambda(x)^{w} \quad \bmod \left[R_{1}, R_{2} \cap R_{3}\right]\left[R_{2}, R_{3} \cap R_{1}\right]\left[R_{3}, R_{1} \cap R_{2}\right] .
$$

Since $\pi_{3}(K)=\Gamma \pi_{2}(K)$, we have the needed homomorphism of $F / R_{1} R_{2} R_{3}$-modules due to Theorem 2.

Example. For two-dimensional sphere $S^{2}$, clearly, $\Lambda$ defines the isomorphism (3)):

$$
\bar{\Lambda}: \pi_{3}\left(S^{2}\right) \rightarrow I_{3}\left(\mathcal{F}_{3}\left(\bar{S}_{3}\right)\right)
$$

with $\bar{S}_{3} \in \mathcal{K}_{3}$ defined in (6).
Example. Consider a group presentation

$$
\mathcal{P}=\left\langle x_{1}, \ldots, x_{k} \mid r_{1}, \ldots, r_{l}\right\rangle
$$

of a group $G$. Let $\mathcal{P}^{\prime}$ be another presentation of $G$ with $k+2 l$ generators and $3 l$ relators given by

$$
\mathcal{P}^{\prime}=\left\langle x_{1}, \ldots, x_{k}, y_{1}, \ldots, y_{l}, z_{1}, \ldots, z_{l} \mid y_{1}, \ldots, y_{l}, z_{1} y_{1}^{-1}, \ldots, z_{l} y_{l}^{-1}, z_{1}^{-1} r_{1}, \ldots, z_{l}^{-1} r_{l}\right\rangle
$$

The standard complex $K_{\mathcal{P}^{\prime}}$ is the union $K_{1} \cup K_{2} \cup K_{3}$, where $K_{1}, K_{2}, K_{3}$ are standard complexes of the following presentations

$$
\begin{aligned}
& \left\langle x_{1}, \ldots, x_{k}, y_{1}, \ldots, y_{l}, z_{1}, \ldots, z_{l} \mid y_{1}, \ldots, y_{l}\right\rangle \\
& \left\langle x_{1}, \ldots, x_{k}, y_{1}, \ldots, y_{l}, z_{1}, \ldots, z_{l} \mid z_{1} y_{1}^{-1}, \ldots, z_{l} y_{l}^{-1}\right\rangle \\
& \left\langle x_{1}, \ldots, x_{k}, y_{1}, \ldots, y_{l}, z_{1}, \ldots, z_{l} \mid z_{1}^{-1} r_{1}, \ldots, z_{l}^{-1} r_{l}\right\rangle
\end{aligned}
$$

respectively. Denoting $\bar{K}=\left(K_{\mathcal{P}^{\prime}}, K_{1}, K_{2}, K_{3}\right) \in \mathcal{K}_{3}$, we have the following isomorphism of $G$-modules:

$$
\pi_{3}\left(K_{\mathcal{P}}\right) \simeq \pi_{3}\left(K_{\mathcal{P}^{\prime}}\right) \simeq I_{3}\left(\mathcal{F}_{3}(\bar{K})\right)
$$

This isomorphism follows directly from the description of Kan's loop construction $G K_{\mathcal{P}}$ and the fact that for a simplicial group $G_{*}$ with $G_{2}$ generated by degeneracy elements, one has $\pi_{2}\left(G_{*}\right) \simeq I_{3}\left(G_{2}, \operatorname{ker}\left(d_{0}\right), \operatorname{ker}\left(d_{1}\right), \operatorname{ker}\left(d_{2}\right)\right)$ (see, for example, [13]).

## 5. Application to group homology

5.1. For a given element $\left(F ; R_{1}, \ldots, R_{n}\right) \in \mathcal{R}_{n}$ it follows from [4] and [6] that under certain conditions on $\left(F ; R_{1}, \ldots, R_{n}\right)$, the group homologies, or more generally, the derived functors of the lower central quotients, can be obtained with the help of the groups $I_{n}\left(F ; R_{1}, \ldots, R_{n}\right)_{F}$, i.e. the $F$-coinvariant part of $I_{n}\left(F ; R_{1}, \ldots, R_{n}\right)$.

For every $\left(F ; R_{1}, R_{2}\right) \in \mathcal{R}_{2}$, it is well-known that there exists a canonical map

$$
\begin{equation*}
H_{3}(G) \rightarrow I_{2}\left(F ; R_{1}, R_{2}\right)=\frac{R_{1} \cap R_{2}}{\left[R_{1}, R_{2}\right]\left[F, R_{1} \cap R_{2}\right]} \tag{26}
\end{equation*}
$$

which is a part of a long exact sequence of homology groups, see [2] and [7]. This map can be easily obtained from the Gutierrez-Ratcliffe map (11). For that we consider arbitrary $\left(K, K_{1}, K_{2}\right) \in \mathcal{K}_{2}$ with $F=\pi_{1}\left(K^{1}\right), R_{i}=\operatorname{ker}\left\{\pi_{1}\left(K^{1}\right) \rightarrow \pi_{1}\left(K_{i}\right)\right\}, i=1,2$. The chain complex

$$
\begin{equation*}
0 \rightarrow \pi_{2}(K) \rightarrow C_{2}(\tilde{K}) \rightarrow C_{1}(\tilde{K}) \rightarrow \mathbb{Z}\left[F / R_{1} R_{2}\right] \rightarrow \mathbb{Z} \rightarrow 0 \tag{27}
\end{equation*}
$$

of the universal cover $\tilde{K}$ of $K$ can be viewed as a complex of free $F / R_{1} R_{2}$-modules. Applying the group homology functor $H_{*}\left(F / R_{1} R_{2},-\right)$ to (27), we obtain the natural maps

$$
\partial_{n}: H_{n}\left(F / R_{1} R_{2}\right) \rightarrow H_{n-3}\left(F / R_{1} R_{2}, \pi_{2}(K)\right), n \geq 3
$$

which are isomorphisms for $n \geq 4$. The map (26) is the composition of $\partial_{3}$ and the $F$-coinvariant map from (11).
5.2. Recall the definitions of certain quadratic functors in the category of abelian groups. Let $A$ be an abelian group. Define the symmetric tensor square

$$
S P^{2}(A)=A \otimes A /\{a \otimes b-b \otimes a, a, b \in A\}
$$

and the augmentation power functor

$$
P_{2}(A)=\Delta(A) / \Delta^{3}(A), \Delta(A)=\operatorname{ker}\{\mathbb{Z}[A] \rightarrow \mathbb{Z}\}
$$

It is well-known (see, for example, [1]) that for a free abelian group $A$, there are the following short exact sequences of abelian groups:

$$
\begin{align*}
& 0 \rightarrow S P^{2}(A) \rightarrow P_{2}(A) \rightarrow A \rightarrow 0  \tag{28}\\
& 0 \rightarrow S P^{2}(A) \rightarrow \Gamma(A) \rightarrow A \otimes \mathbb{Z}_{2} \rightarrow 0 \tag{29}
\end{align*}
$$

Now let $A$ be a $G$-module. The $G$-action can be naturally extended to the abelian groups $S P^{2}(A), P_{2}(A), \Gamma(A), A \otimes \mathbb{Z}_{2}$, thus allowing to consider the sequences (28) and (29) as sequences of $G$-modules. Applying homology functor $H_{*}(G,-)$ to (28) and (29) we obtain long exact sequences

$$
\begin{aligned}
& \cdots \rightarrow H_{1}\left(G, P_{2}(G)\right) \rightarrow H_{1}(G, A) \rightarrow H_{0}\left(G, S P^{2}(A)\right) \rightarrow H_{0}\left(G, P_{2}(A)\right) \rightarrow \ldots \\
& \cdots \rightarrow H_{1}(G, \Gamma(A)) \rightarrow H_{1}\left(G, A \otimes \mathbb{Z}_{2}\right) \rightarrow H_{0}\left(G, S P^{2}(A)\right) \rightarrow H_{0}(G, \Gamma(A)) \rightarrow \ldots
\end{aligned}
$$

5.3. It is natural to ask about applications of the map $\bar{\Lambda}$ constructed in Theorem 3, To this end, we consider $\left(K ; K_{1}, K_{2}, K_{3}\right) \in \mathcal{K}_{3}$ with

$$
\pi_{1}\left(K^{1}\right)=F, R_{i}=\operatorname{ker}\left\{\pi_{1}\left(K^{1}\right) \rightarrow \pi_{1}\left(K_{i}\right)\right\}, i=1,2,3
$$

Let us denote $G=F / R_{1} R_{2} R_{3}$ and let us define the map

$$
\Psi_{4}: H_{4}(G) \rightarrow I_{3}\left(F ; R_{1}, R_{2}, R_{3}\right)_{F}=\frac{R_{1} \cap R_{2} \cap R_{3}}{\left[R_{1}, R_{2} \cap R_{3}\right]\left[R_{2}, R_{3} \cap R_{1}\right]\left[R_{3}, R_{1} \cap R_{2}\right]\left[F, R_{1} \cap R_{2} \cap R_{3}\right]}
$$

as a composite map in the following diagram with exact rows and columns:


Remark. Proposition 2 implies that the natural composition map

$$
H_{4}\left(F / R_{1} R_{2}\right) \oplus H_{4}\left(F / R_{2} R_{3}\right) \oplus H_{4}\left(F / R_{1} R_{3}\right) \rightarrow H_{4}(G) \xrightarrow{\Psi_{4}} I_{3}\left(F ; R_{1}, R_{2}, R_{3}\right)_{F}
$$

is the zero map.

## References

[1] H.-J. Baues: Homotopy Category of Simply Connected 4-Manifolds, Cambridge University Press, (2003).
[2] W. Bogley and M. Gutierrez: Mayer-Vietoris sequences homotopy of 2 -complexes and in homology of groups, J. Pure Appl. Algebra 77 (1992) 39-65.
[3] A. Bousfield, E. Curtis, D. Kan, D. Quillen, D. Rector and J. Schlesinger: The mod-p lower central series and the Adams spectral sequence, Topology 5 (1966), 331-342.
[4] R. Brown and G. Ellis: Hopf formulae for the higher homology of a group, Bull. London Math. Soc. 20 (1988), 124-128.
[5] E. Curtis: Simplicial homotopy theory, Adv. Math. 6 (1971), 107-209.
[6] G. Donadze, N. Inassaridze, T. Porter: $N$-fold Čech derived functors and generalised Hopf type formulas. K-Theory 35, 341-373 (2006).
[7] A. Duncan, G.J.Ellis, N.D.Gilbert: A Mayer-Vietoris sequence in group homology and the decomposition of relation modules, Glasgow Math. J. 37 (1995) 159-171.
[8] L. Gruenenfelder: Lower central series, augmentation quotients and homology of groups, Comment. Math. Helv., 55 (1980), 159-177.
[9] M. Gutierrez and P. Hirschhorn: Free simplicial groups and the second relative homotopy group of an adjunction space, J. Pure. Appl. Alg. 39, (1986), 119-123.
[10] M. A. Gutierrez and J. Ratcliffe: On the second homotopy group, Quart. J. Math. Oxford, 32 (1981), 45-55.
[11] G. Ellis and R. Mikhailov: A colimit of classifying spaces, preprint.
[12] B. Hartley and Yu. Kuzmin: On the quotient of a free group by the commutator of two normal subgroups, J. Pure Appl. Algebra, 74 (1991), 247-256.
[13] A. Multu and T. Porter: Iterated Peiffer pairings in the Moore complex of a simlicial group, Appl. Cat. Str. 9 (2001), 111-130.
[14] S. J. Pride: Identities among relations, Group theory from a geometrical point of view, 687-717 (1991).
[15] H. Toda: Composition methods in homotopy groups of spheres, Annals of Mathematics Studies, No. 49 Princeton University Press, Princeton, N.J. (1962)
[16] J. Wu: Combinatorial description of homotopy groups of certain spaces, Math. Proc. Camb. Phyl. Soc. 130, (2001), 489-513.

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