# ON ISOMORPHISMS OF $\mathcal{R}$ - AND $\mathcal{L}$-CROSS-SECTIONS OF WREATH PRODUCTS OF FINITE INVERSE SYMMETRIC SEMIGROUPS. 

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#### Abstract

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We classify $\mathcal{R}$ - and $\mathcal{L}$-cross-sections of wreath products of finite inverse symmetric semigroups $\mathcal{I} \mathcal{S}_{m} \imath_{p} \mathcal{I} \mathcal{S}_{n}$ up to isomorphism. We show that every isomorphism of $\mathcal{R}(\mathcal{L}$ - $)$ cross-sections of $\mathcal{I} \mathcal{S}_{m} \imath_{p} \mathcal{I} \mathcal{S}_{n}$ is a conjugacy. As an auxiliary result, we get that every isomorphism of $\mathcal{R}-(\mathcal{L}-)$ cross-sections of $\mathcal{I} \mathcal{S}_{n}$ is also a conjugacy. We also compute the number of non-isomorphic $\mathcal{R}$ $(\mathcal{L}-)$ cross-sections of $\mathcal{I} \mathcal{S}_{m} \mathfrak{l}_{p} \mathcal{I} \mathcal{S}_{n}$.


Regular rooted tree; partial automorphism; finite inverse symmetric semigroup; partial wreath product; Green's relations; cross-sections.

## 1. Introduction

Transformation semigroups play an important role in semigroup theory. One of the reasons is that transformation semigroups appeared in recent studies in general symmetry theory as (full or partial) endomorphisms semigroup of different combinatorial objects.

The study of cross-sections of semigroups was started by Renner [10. Later on different authors have studied cross-sections of particular semigroups. $\mathcal{H}$-crosssections of inverse symmetric semigroups were deeply studied by Cowan and Reilly [1], $\mathcal{R}$ - and $\mathcal{L}$ - cross-sections of $\mathcal{I} \mathcal{S}_{n}$ were classified by Ganyushkin and Mazorchuk 3]. $\mathcal{R}$ - and $\mathcal{H}$-cross-sections for the full finite transformation semigroup $\mathcal{T}_{n}$ and for the infinite full transformation semigroup $\mathcal{T}_{X}$ were classified by Pyekhtyeryev [8, 9].

In the present paper we continue the study of $\mathcal{R}$ - and $\mathcal{L}$ - cross-sections of partial wreath products of finite inverse symmetric semigroups initiated in [6]. We classify $\mathcal{R}$ - and $\mathcal{L}$ - cross-sections of partial wreath products of finite inverse semigroups up to isomorphism. The paper is organized as follows. All necessary definitions are collected in Section 2, Section 3 contains known results on $\mathcal{R}$ - and $\mathcal{L}$-cross-sections of $\mathcal{I} \mathcal{S}_{n}$ and $\mathcal{I} \mathcal{S}_{m}{\imath_{p}}^{\mathcal{I}} \mathcal{S}_{n}$. Classification of $\mathcal{R}$ - and $\mathcal{L}$ - cross-sections of $\mathcal{I} \mathcal{S}_{n}$ and $\mathcal{I} \mathcal{S}_{m} \imath_{p} \mathcal{I S}_{n}$ up to isomorphism is given in Section 4 and Section 5 respectively.

## 2. Basic definitions

For a set $X$, let $\mathcal{I S}(X)$ denote the set of all partial bijections on $X$ with the natural composition law: $f \circ g: \operatorname{dom}(f) \cap f^{-1}(\operatorname{dom}(g)) \ni x \mapsto x f g$ for $f, g \in \mathcal{I S}(X)$. The set $(\mathcal{I S}(X), \circ)$ is clearly an inverse semigroup. This semigroup is called the full inverse symmetric semigroup on $X$. If $X=\mathcal{N}_{n}$, where $\mathcal{N}_{n}=\{1, \ldots, n\}$, then semigroup $\mathcal{I S}\left(\mathcal{N}_{n}\right)$ is called the full inverse symmetric semigroup of rank $n$ and is denoted $\mathcal{I} \mathcal{S}_{n}$. We distinguish the element whose domain is $\varnothing$, it will be
denoted by 0 . It is the zero of semigroup $\mathcal{I} \mathcal{S}_{n}$. Also we distinguish an identity map $1: \mathcal{I S}_{n} \rightarrow \mathcal{I} \mathcal{S}_{n}$ defined by $x 1=x$ for all $x \in \mathcal{I} \mathcal{S}_{n}$. Clearly, this is the unity of $\mathcal{I} \mathcal{S}_{n}$.

It is possible to introduce for elements of $\mathcal{I} \mathcal{S}_{n}$ an analogue of the cyclic decomposition for elements of the symmetric group $\mathcal{S}_{n}$. We start with introducing two classes of elements. Let $A=\left\{x_{1}, x_{2}, \ldots, x_{k}\right\} \subset \mathcal{N}_{n}$ be a subset. Denote by $\left(x_{1}, x_{2}, \ldots, x_{k}\right)$ the unique element $f \in \mathcal{I} \mathcal{S}_{n}$ such that $x_{i} f=x_{i+1}, i=1,2, \ldots, k-1$, $x_{k} f=x_{1}$ and $x f=x, x \notin A$. Assume that $A \neq \varnothing$ and denote by $\left[x_{1}, x_{2}, \ldots, x_{k}\right]$ the unique element $f \in \mathcal{I} \mathcal{S}_{n}$ such that $\operatorname{dom}(f)=\mathcal{N}_{n} \backslash\left\{x_{k}\right\}$ and $x_{i} f=x_{i+1}$, $i=1,2, \ldots, k-1$, and $x f=x, x \notin A$. The element $\left(x_{1}, x_{2}, \ldots, x_{k}\right)$ is called a cycle and the element $\left[x_{1}, x_{2}, \ldots, x_{k}\right]$ is called a chain, and the set $A$ is called the support of $\left(x_{1}, x_{2}, \ldots, x_{k}\right)$ or $\left[x_{1}, x_{2}, \ldots, x_{k}\right]$. Any element of $\mathcal{I} \mathcal{S}_{n}$ decomposes uniquely into a product of cycles and chains with disjoint supports. This decomposition is called a chain decomposition [2]. Denote by $\left\langle x_{1}, x_{2}, \ldots, x_{k}\right\rangle$ the element $f \in \mathcal{I} \mathcal{S}_{n}$ such that $\operatorname{dom}(f)=A \backslash\left\{x_{k}\right\}$ and $x_{i} f=x_{i+1}, i=1,2, \ldots, k-1$.

Recall the definition of a partial wreath product of semigroups. Let $S$ be a semigroup, $(X, P)$ be the semigroup of partial transformations of a set $X$. Define the set $S^{P X}$ as the set of partial functions from $X$ to $S$ :

$$
S^{P X}=\{f: A \rightarrow S \mid \operatorname{dom}(f)=A, A \subset X\}
$$

Given $f, g \in S^{P X}$, the product $f g$ is defined as:

$$
\operatorname{dom}(f g)=\operatorname{dom}(f) \cap \operatorname{dom}(g),(f g)(x)=f(x) g(x) \text { for all } x \in \operatorname{dom}(f g)
$$

For $a \in P, f \in S^{P X}$, define $f^{a}$ as:

$$
\left(f^{a}\right)(x)=f(x a), \operatorname{dom}\left(f^{a}\right)=\{x \in \operatorname{dom}(a) ; x a \in \operatorname{dom}(f)\}
$$

Definition 1. The partial wreath product of the semigroup $S$ with the semigroup $(X, P)$ of partial transformations of the set $X$ is the set

$$
\left\{(f, a) \in S^{P X} \times(X, P) \mid \operatorname{dom}(f)=\operatorname{dom}(a)\right\}
$$

with product defined by $(f, a) \cdot(g, b)=\left(f g^{a}, a b\right)$. We will denote the partial wreath product of semigroups $S$ and $(X, P)$ by $S \imath_{p} P$.

Remark 1. Some authors use the term wreath product. We follow terminology from the book of J.D.P Meldrum [7], where this construction is called partial wreath product.

It is known [7] that a partial wreath product of semigroups is a semigroup. Moreover, a partial wreath product of inverse semigroups is an inverse semigroup. An important example of an inverse semigroup is the semigroup PAut $T_{n}^{k}$ of partial automorphisms of a $k$-level $n$-regular rooted tree $T_{n}^{k}$. Here by a partial automorphism we mean a root-preserving tree homomorphism defined on a rooted subtree of $T_{n}^{k}$. It is shown in [5] that

$$
\text { PAut } T_{n}^{k} \simeq \underbrace{\mathcal{I} \mathcal{S}_{n} \imath_{p} \mathcal{I} \mathcal{S}_{n} \imath_{p} \cdots \imath_{p} \mathcal{I} \mathcal{S}_{n}}_{k}
$$

This is an analogue of the well-known fact that Aut $T_{n}^{k} \simeq \mathcal{S}_{n} \imath \cdots$ S $\mathcal{S}_{n}$.
Remark 2. Let $\mathcal{N}_{n}^{\prime}=\mathcal{N}_{n} \cup\{\emptyset\}$, and we can consider semigroup $\mathcal{I} \mathcal{S}_{n}$ as a subsemigroup $K_{n+1}$ of full transformation semigroup $\mathcal{T}_{n+1}$ acting on the set $\mathcal{N}_{n}^{\prime}$ : for every $a \in \mathcal{I} \mathcal{S}_{n}$ we define $a^{\prime} \in K_{n+1}$ as $x a^{\prime}=x a$ for $x \in \operatorname{dom}(a)$ and $x a^{\prime}=\emptyset$ for $x \notin \operatorname{dom}(a)$. Let $S=K_{m+1}$ 〕 $K_{n+1}$, where $\imath$ is a wreath product of semigroups (see,
e.g., 4, Chapter 1], [7, Chapter 10]). We define equivalence $\sim$ on the semigroup $S:(f, a) \sim(g, a) \Leftrightarrow f(x)=g(x)$ for $x \in \mathcal{N}_{n}^{\prime}$ such that $x a \neq \emptyset$. Then we identify every element $(f, a) \in \mathcal{I} \mathcal{S}_{m} \imath_{p} \mathcal{I} \mathcal{S}_{n}$ with an element $\left(f^{\prime}, a^{\prime}\right) \in S / \sim$ in a following way. We set $x a^{\prime}=x a$ for $x \in \operatorname{dom}(a)$ and $x a=\emptyset$ otherwise, and $f^{\prime}(x)=f(x)$ for $x \in \operatorname{dom}(a), f^{\prime}(x)$ for $x \notin \operatorname{dom}(a)$ can be chosen arbitrarily. So we can consider partial wreath product $\mathcal{I} \mathcal{S}_{m} \imath_{p} \mathcal{I} \mathcal{S}_{n}$ as a quotient $S / \sim$.

## 3. $\mathcal{R}$ - AND $\mathcal{L}$-CROSS-SECTIONS OF $\mathcal{I} \mathcal{S}_{m} \imath_{p} \mathcal{I} \mathcal{S}_{n}$

In this section we give a brief description of known results on $\mathcal{R}$ - and $\mathcal{L}$-crosssections of the semigroup $\mathcal{I} \mathcal{S}_{m} \imath_{p} \mathcal{I} \mathcal{S}_{n}$.

Let $S$ be an inverse semigroup with identity. Recall that Green's $\mathcal{R}$-relation on inverse semigroup $S$ is defined by $a \mathcal{R} b \Leftrightarrow a S=b S$, similarly, Green's $\mathcal{L}$-relation is defined by $a \mathcal{L} b \Leftrightarrow S a=S b$. It is well-known (e.g. [2]) that Green's relations on $\mathcal{I} \mathcal{S}_{n}$ can be described as follows: $a \mathcal{R} b \Leftrightarrow \operatorname{dom}(a)=\operatorname{dom}(b) ; a \mathcal{L} b \Leftrightarrow \operatorname{im}(a)=\operatorname{im}(b)$.
$\mathcal{R}$ - and $\mathcal{L}$-relations on $\mathcal{I} \mathcal{S}_{m} \imath_{p} \mathcal{I} \mathcal{S}_{n}$ are described in
Proposition 1. [5] Let $(f, a),(g, b) \in \mathcal{I} \mathcal{S}_{m} \imath_{p} \mathcal{I S}_{n}$. Then
(1) $(f, a) \mathcal{R}(g, b)$ if and only if $\operatorname{dom}(a)=\operatorname{dom}(b)$ and for any $z \in \operatorname{dom}(a)$ $f(z) \mathcal{R} g(z)$;
(2) $(f, a) \mathcal{L}(g, b)$ if and only if $\operatorname{im}(a)=\operatorname{im}(b)$ and for any $z \in \operatorname{im}(a) g^{b^{-1}}(z) \mathcal{L}$ $f^{a^{-1}}(z)$, where $a^{-1}$ is the inverse for $a$.

Now let $\rho$ be an equivalence relation on $S$. A subsemigroup $T \subset S$ is called cross-section with respect to $\rho$ (or simply $\rho$-cross-section) provided that $T$ contains exactly one element from every equivalence class. Correspondingly, cross-sections with respect to $\mathcal{R}-(\mathcal{L}-)$ Green's relations are called $\mathcal{R}$ - $(\mathcal{L}-)$ cross-sections. Note that every $\mathcal{R}-(\mathcal{L}-)$ equivalence class of inverse semigroup contains exactly one idempotent. Then the number of elements in every $\mathcal{R}$ - $(\mathcal{L}-)$ cross-section of inverse semigroup $S$ is $|E(S)|$, where $E(S)$ is the subsemigroup of all idempotents of semigroup $S$.

Observe that a subsemigroup $H$ of semigroup $\mathcal{I S}_{n}$ is an $\mathcal{R}$-cross-section if and only if for every subset $A \subseteq \mathcal{N}_{n}$ it contains exactly one element $a$ such that $\operatorname{dom}(a)=A$.

Before describing $\mathcal{R}$ - and $\mathcal{L}$-cross-sections in semigroup $\mathcal{I} \mathcal{S}_{m} \chi_{p} \mathcal{I} \mathcal{S}_{n}$, we give first the description of $\mathcal{R}$ - and $\mathcal{L}$-cross-sections in semigroup $\mathcal{I} \mathcal{S}_{n}$ presented in [3]. Let $\mathcal{N}_{n}=M_{1} \sqcup M_{2} \sqcup \ldots \sqcup M_{s}$ be an arbitrary decomposition of $\mathcal{N}_{n}=$ $\{1,2, \ldots, n\}$ into disjoint union of non-empty blocks, where the order of blocks is irrelevant. Assume that a linear order is fixed on the elements of every block: $M_{i}=\left\{m_{1}^{i}<m_{2}^{i}<\cdots<m_{\left|M_{i}\right|}^{i}\right\}$.

For each pair $i, j, 1 \leq i \leq s, 1 \leq j \leq\left|M_{i}\right|$ define $a_{i, j}=\left[m_{1}^{i}, m_{2}^{i}, \ldots, m_{j}^{i}\right]$ and denote by $\left.R\left(\overrightarrow{M_{1}}, \overrightarrow{M_{2}}, \ldots, \overrightarrow{M_{s}}\right)=\left\langle a_{i, j}\right| 1 \leq i \leq s, 1 \leq j \leq\left|M_{i}\right|\right\rangle \sqcup\{e\}$.

Theorem 1. [3] For an arbitrary decomposition $\mathcal{N}_{n}=M_{1} \sqcup M_{2} \sqcup \ldots \sqcup M_{s}$ and arbitrary linear orders on the elements of every block of this decomposition the semigroup $R\left(\overrightarrow{M_{1}}, \overrightarrow{M_{2}}, \ldots, \overrightarrow{M_{s}}\right)$ is an $\mathcal{R}$-cross-section of $\mathcal{I} \mathcal{S}_{n}$. Moreover, every $\mathcal{R}$ -cross-section is of the form $R\left(\overrightarrow{M_{1}}, \overrightarrow{M_{2}}, \ldots, \overrightarrow{M_{s}}\right)$ for some decomposition $\mathcal{N}_{n}=M_{1} \sqcup$ $M_{2} \sqcup \ldots \sqcup M_{s}$ and some linear orders on the elements of every block.

Since the map $a \mapsto a^{-1}$ is an anti-isomorphism of the semigroup $\mathcal{I} \mathcal{S}_{n}$, which sends $\mathcal{R}$-classes to $\mathcal{L}$-classes, then $\mathcal{L}$-cross-sections are described similarly.

Now we turn to the description of $\mathcal{R}$ - and $\mathcal{L}$-cross-sections of semigroup $\mathcal{I} \mathcal{S}_{m} \imath_{p} \mathcal{I} \mathcal{S}_{n}$. It follows from Proposition 1 that a subsemigroup $H \subset \mathcal{I} \mathcal{S}_{m} \imath_{p} \mathcal{I} \mathcal{S}_{n}$ is an $\mathcal{R}$-crosssection if and only if for any $A \subset \mathcal{N}_{n}$ and any collection of sets $B_{x_{1}}, \ldots, B_{x_{|A|}} \subset \mathcal{N}_{m}$ there exists exactly one element $(f, a) \in H$ satisfying $\operatorname{dom}(a)=A$ and $\operatorname{dom}\left(x_{i} f\right)=$ $B_{x_{i}}$ for all $x_{i} \in A$. Later on we will use this fact frequently.

Define the map $\varphi_{\mu}: \prod_{i=1}^{k}\left(S{\left.l_{p} \mathcal{I} \mathcal{S}\left(M_{i}\right)\right) \rightarrow S{\imath_{p}}^{\mathcal{I} \mathcal{S}_{n}} \text { in the following manner: } \varphi_{\mu}, ~}_{\text {m }}\right.$ maps the product $\prod_{i=1}^{k}\left(f_{i}, a_{i}\right)$ to the element $(f, a)$ such that $\left.a\right|_{M_{i}}=a_{i},\left.f\right|_{M_{i}}=f_{i}$.

Theorem 2. 6] Let $R\left(\overrightarrow{M_{1}}, \overrightarrow{M_{2}}, \ldots, \overrightarrow{M_{k}}\right)$ be an $\mathcal{R}$-cross-section of semigroup $\mathcal{I} \mathcal{S}_{n}$, $R_{1}, \ldots, R_{k}$ be $\mathcal{R}$-cross-sections of semigroup $\mathcal{I} \mathcal{S}_{m}$. Then

$$
R=\varphi_{\mu}\left(\left(R_{1} \imath_{p} R\left(\overrightarrow{M_{1}}\right)\right) \times\left(R_{2} \imath_{p} R\left(\overrightarrow{M_{2}}\right)\right) \times \ldots \times\left(R_{k} \imath_{p} R\left(\overrightarrow{M_{k}}\right)\right)\right)
$$

is an $\mathcal{R}$-cross-section of semigroup $\mathcal{I S}_{m} \lambda_{p} \mathcal{I} \mathcal{S}_{n}$.
Moreover, every $\mathcal{R}$-cross-section of semigroup $\mathcal{I S}_{m} \imath_{p} \mathcal{I} \mathcal{S}_{n}$ is isomorphic to

$$
\left(R_{1} \imath_{p} R\left(\overrightarrow{M_{1}}\right)\right) \times\left(R_{2} \imath_{p} R\left(\overrightarrow{M_{2}}\right)\right) \times \ldots \times\left(R_{k} \imath_{p} R\left(\overrightarrow{M_{k}}\right) .\right.
$$

A map $(f, a) \mapsto(f, a)^{-1}$ is an anti-isomorphism of semigroup $\mathcal{I} \mathcal{S}_{m} \imath_{p} \mathcal{I} \mathcal{S}_{n}$, that sends $\mathcal{R}$-classes to $\mathcal{L}$-classes. It is also clear that it maps $\mathcal{R}$-cross-sections to $\mathcal{L}$ -cross-sections and vice-versa. Hence dualizing Theorem 2, one gets a description of $\mathcal{L}$-cross-sections.

## 4. Isomorphisms of $\mathcal{R}$ - and $\mathcal{L}$-cross-sections of $\mathcal{I} \mathcal{S}_{n}$

Clearly, it is enough to study problem of isomorphism only for $\mathcal{R}$-cross-sections. The result for $\mathcal{L}$-cross-sections is analogous.

For an arbitrary $\mathcal{R}$-cross-section $R=R\left(\overrightarrow{M_{1}}, \overrightarrow{M_{2}}, \ldots, \overrightarrow{M_{s}}\right)$ we call an idempotent $e \in E(R)$ block idempotent if $\operatorname{dom}(e) \subset M_{i}$ for some $i$. The idempotents of $R$ are described below. Let

$$
R\left(\vec{M}_{i}\right)=\left\{a_{i, j}\left|1 \leq j \leq\left|M_{i}\right|\right\} \cup e\right.
$$

Then

$$
R\left(\overrightarrow{M_{1}}, \overrightarrow{M_{2}}, \ldots, \overrightarrow{M_{s}}\right) \simeq R\left(\overrightarrow{M_{1}}\right) \times \ldots \times R\left(\overrightarrow{M_{s}}\right)
$$

Isomorphism is established by $\varphi(a)=\left(\left.a\right|_{M_{1}}, \ldots,\left.a\right|_{M_{s}}\right)$.
For an $\mathcal{R}$-cross-section $R=R\left(\overrightarrow{M_{1}}, \overrightarrow{M_{2}}, \ldots, \overrightarrow{M_{s}}\right)$ element $e \in R$ is an idempotent iff $\left.e\right|_{M_{i}} \in E\left(R\left(\overrightarrow{M_{i}}\right)\right)$ for all $i=1, \ldots, s$. An element $a \in R\left(\vec{M}_{i}\right)$ with domain $\operatorname{dom}(a)=\left\{m_{j_{1}}^{i}, \ldots, m_{j_{k}}^{i} \mid j_{1}<\ldots<j_{k}\right\}$ acts in a following way: $\left(m_{j_{l}}^{i}\right) a=$ $m_{\left|M_{i}\right|-k+l}^{i}$. Taking it into account we get that idempotents of $R\left(\overrightarrow{M_{i}}\right)$ are described as $e=\left(m_{j}^{i}\right)\left(m_{j+1}^{i}\right) \ldots\left(m_{\left|M_{i}\right|}^{i}\right)$.

Recall that on the set of idempotents the partial order $\preceq$ is defined as $e \preceq f \Leftrightarrow$ $e f=f e=e$.

Proposition 2. An idempotent $e \in E(R)$ is a block idempotent if and only if there exist no idempotents $e_{1} \neq 0, e_{2} \neq 0$ such that $e_{1} \preceq e, e_{2} \preceq e, e_{1} e_{2}=0$.

Proof. Necessity. Let $e$ be block idempotent. Let $e_{1} \preceq e, e_{2} \preceq e$ be idempotents of $R$ such that $e_{1} \neq 0, e_{2} \neq 0$. If $M_{i}=\left\{x_{1}, \ldots, x_{k} \mid x_{1}<x_{2}<\ldots<x_{k}\right\}$, then
for an arbitrary block idempotent $e$ it holds $\operatorname{dom}(e)=\left\{x_{j}, \ldots, x_{k}\right\}, j \leq k$. Since $e, e_{1}, e_{2} \in E(R)$, then $e_{1} e_{2} \neq 0$.

Sufficiency. Assume the contrary. It means that there is no index $i$ such that $\operatorname{dom}(e) \subset M_{i}$. Consider a block $M_{i}$ for which $\operatorname{dom}(e) \cap M_{i} \neq \varnothing$. Then $e_{1}=\left.e\right|_{M_{i}}$ and $e_{2}=\left.e\right|_{\overline{M_{i}}}$ are idempotents. Evidently, $e_{1} e_{2}=0$. From condition $\operatorname{dom}(e) \cap M_{i} \neq \varnothing$ it follows $e_{1} \neq 0$, and from the fact that $e$ is not block idempotent, it follows $e_{2} \neq 0$.

Lemma 1. Let $R_{1}, R_{2}$ be $\mathcal{R}$-cross-sections of the semigroup $\mathcal{I} \mathcal{S}_{n}, \varphi: R_{1} \rightarrow R_{2}$ be an isomorphism. Then there exists a permutation $\Theta \in \mathcal{S}_{n}$ such that for any block idempotent $e \in R_{1}$ and every $x \in\{1, \ldots, n\}$ it holds $x e \Theta=x \Theta \varphi(e)$.
Proof. Let $R_{1}=R\left(\overrightarrow{M_{1}}, \overrightarrow{M_{2}}, \ldots, \overrightarrow{M_{s}}\right)$. The domains of block idempotents defined on different blocks are disjoint. Thus if $i \neq j$ and $\operatorname{dom}\left(e_{i}\right) \subset M_{i}, \operatorname{dom}\left(e_{j}\right) \subset M_{j}$, then $e_{i} e_{j}=0$. From the definition of the partial order, we have for block idempotents defined on the same block that $e_{i} \preceq e_{j}$ if and only if $\operatorname{dom}\left(e_{i}\right) \subset \operatorname{dom}\left(e_{j}\right)$. Then the set $E_{b}(R)$ of block idempotents of $\mathcal{R}$-cross-section $R$ as a poset can be drawn as:


It follows from Proposition 2 that the property to be a block idempotent is preserved under an isomorphism. On the other hand, the isomorphism $\varphi$ preserves the order $\preceq$. Thus $\varphi$ defines a poset isomorphism between $E_{b}\left(R_{1}\right)$ and $E_{b}\left(R_{2}\right)$. If $M_{i}=\left\{x_{1}<\ldots<x_{k}\right\}$, then for a block idempotent $e \in R_{1}$ with $\operatorname{dom}(e)=$ $\left\{x_{j}, \ldots, x_{k}\right\}$ we put $\nu_{1}(e)=x_{j}$. We define $\nu_{2}(e)$ for a block idempotent $e \in R_{2}$ similarly. It is easily checked that $\Theta$ defined by $i \Theta=\varphi\left(i \nu_{1}^{-1}\right) \nu_{2}, i \in\{1, \ldots, n\}$, is as required.

Theorem 3. Let $R_{1}, R_{2}$ be $\mathcal{R}$-cross-sections of the semigroup $\mathcal{I S} S_{n}, \varphi: R_{1} \rightarrow R_{2}$ be an isomorphism. Then there exists an element $\Theta \in \mathcal{S}_{n}$ such that for any $\alpha \in R_{1}$ and $x \in\{1, \ldots, n\}$ the following equality holds $x \alpha \Theta=x \Theta \varphi(\alpha)$.
Proof. Let $\Theta$ be the permutation provided by Lemma 1 .
Let $\xi \in R_{1}=R\left(\overrightarrow{M_{1}}, \overrightarrow{M_{2}}, \ldots, \overrightarrow{M_{s}}\right)$ be such that $\operatorname{dom}(\xi) \subset M_{i}$ and $\operatorname{im}(\xi) \subset M_{i}$ for some $i$. It means $\operatorname{id}_{M_{i}} \xi=\xi$, i.e. $\xi$ acts inside the block $M_{i}$. Thus,

$$
\varphi(\xi)=\varphi\left(\mathrm{id}_{M_{i}} \xi\right)=\varphi\left(\mathrm{id}_{M_{i}}\right) \varphi(\xi)=\operatorname{id}_{\Theta\left(M_{i}\right)} \varphi(\xi)
$$

Therefore, $\zeta=\varphi(\xi)$ acts inside the block $\Theta\left(M_{i}\right)$.

Let

$$
\begin{gathered}
M_{i}=\left\{x_{1}, x_{2}, \ldots, x_{k_{i}} \mid x_{1}<x_{2}<\ldots<x_{k_{i}}\right\} \\
\Theta\left(M_{i}\right)=\left\{y_{1}, y_{2}, \ldots, y_{k_{i}} \mid y_{1}<y_{2}<\ldots<y_{k_{i}}\right\}
\end{gathered}
$$

where $y_{i}=x_{i} \Theta, i=1, \ldots, k_{i}$. Let $\operatorname{dom}(\xi)=\left\{x_{i_{1}}, \ldots, x_{i_{l}}\right\}, i_{1}<\ldots<i_{l}, \operatorname{dom}(\zeta)=$ $\left\{y_{j_{1}}, \ldots, y_{j_{p}}\right\}, j_{1}<\ldots<j_{p}$.

Then $x_{i_{m}} \xi=x_{i_{\left|M_{i}\right|}-l+m}, m=1, \ldots, l, y_{j_{m}} \zeta=y_{j_{\left|M_{i}\right|}-p+m}, m=1, \ldots, p$. Denote $e_{m}=\operatorname{id}_{\left\{x_{m}, \ldots, x_{\left|M_{i}\right|}\right\}}, f_{m}=\operatorname{id}_{\left\{y_{m}, \ldots, y_{\left|M_{i}\right|}\right\}}$. We have then

$$
\begin{equation*}
e_{\left|M_{i}\right|} \xi=0, \ldots, e_{i_{l}+1} \xi=0, e_{i_{l}} \xi \neq 0 \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
f_{\left|M_{i}\right|} \zeta=0, \ldots, f_{j_{p}+1} \zeta=0, f_{j_{p}} \zeta \neq 0 \tag{2}
\end{equation*}
$$

Applying to (1) the isomorphism $\varphi$ and using Lemma 1 we get

$$
\begin{equation*}
f_{\left|M_{i}\right|} \zeta=0, \ldots, f_{i_{l}+1} \zeta=0, f_{i_{l}} \zeta \neq 0 \tag{3}
\end{equation*}
$$

It follows from equalities (2) and (3) that $l=p$ and $i_{l}=j_{p}$.
Further, we have

$$
\begin{aligned}
& e_{i_{l}} \xi=e_{i_{l}-1} \xi \\
& f_{i_{l}} \zeta=f_{i_{l}-1} \zeta=\ldots=e_{i_{l-1}-1} \xi \neq e_{i_{l-1}} \xi \\
& j_{l-1}-1 \\
&=f_{j_{l-1}} \zeta
\end{aligned}
$$

Similarly, we get $j_{l-1}=i_{l-1}$. By induction we obtain $i_{m}=j_{m}$ and $l=p$. Then for any $x \in \operatorname{dom}(\xi)$ it holds: $x \xi \Theta=x \Theta \varphi(\xi)$, because $y_{i_{m}} \zeta=x_{i_{m}} \xi \Theta$.

We will show now that equality $x \alpha \Theta=x \Theta \varphi(\alpha)$ is true for any $\alpha \in R_{1}$.
Let $\alpha \in R_{1}, x \in M_{i}$. Then

$$
\begin{gathered}
x \alpha \Theta=\left.x \alpha\right|_{M_{i}} \Theta=x \operatorname{id}_{M_{i}} \alpha \Theta=x \Theta \varphi\left(\operatorname{id}_{M_{i}} \alpha\right)= \\
=x \Theta\left(\varphi\left(\operatorname{id}_{M_{i}}\right) \varphi(\alpha)\right)=x \Theta\left(\operatorname{id}_{\Theta\left(M_{i}\right)} \varphi(\alpha)\right)=x \Theta \varphi(\alpha)
\end{gathered}
$$

Thus, for every $\alpha \in R_{1}, x \in\{1, \ldots, n\}$ it is true that $x \alpha \Theta=x \Theta \varphi(\alpha)$
Remark 3. It is proved in [3] that two $\mathcal{R}$ - $(\mathcal{L}-)$ cross-sections in $\mathcal{I} \mathcal{S}_{n}$ are isomorphic if and only if they are conjugate. Theorem 3 strengthens this result: every isomorphism of $\mathcal{R}$ - ( $\mathcal{L}$-) cross-sections is a conjugacy.

## 5. IsOMORPHISMS OF $\mathcal{R}$ - AND $\mathcal{L}$-CROSS-SECTIONS OF $\mathcal{I} \mathcal{S}_{m} \imath_{p} \mathcal{I} \mathcal{S}_{n}$

In this section we generalize the result of the previous section that every isomorphism of $\mathcal{R}$ - $(\mathcal{L}-)$ cross-sections of inverse symmetric semigroup $\mathcal{I} \mathcal{S}_{n}$ is a conjugacy, to partial wreath products of finite inverse symmetric semigroups.

Theorem 4. Let $R^{\prime}, R^{\prime \prime}$ be $\mathcal{R}$-cross-sections of the semigroup $\mathcal{I S}_{m} \imath_{p} \mathcal{I} \mathcal{S}_{n}, \varphi: R^{\prime} \rightarrow$ $R^{\prime \prime}$ be an isomorphism. Then there exists such an element $\Theta=(\vartheta, \theta) \in \mathcal{S}_{m} \prec \mathcal{S}_{n}$ that

$$
\varphi((f, a))=\Theta^{-1}(f, a) \Theta
$$

In other words, if $(f, a) \in R^{\prime}$ and $(g, b)=\varphi((f, a))$, then $\operatorname{dom} b=\theta(\operatorname{dom}(a))$ and for any $x \in \operatorname{dom}(a)$

$$
x a \theta=x \theta b, \quad g(\theta(x))=\vartheta^{-1}(x) f(x) \vartheta(x a)
$$

Proof. Step 1. For an idempotent $e=\left(f_{e}, a_{e}\right) \in \mathcal{I} \mathcal{S}_{m} \imath_{p} \mathcal{I} \mathcal{S}_{n}$ denote

$$
N_{e}=\left\{\zeta=(f, a) \in \mathcal{I} \mathcal{S}_{m}{\imath_{p}}^{\left.\mathcal{I} \mathcal{S}_{n} \mid e \zeta=\zeta\right\} . . . . ~}\right.
$$

If $\zeta=(f, a) \in N_{e}$, then $\operatorname{dom}(a) \subset \operatorname{dom}\left(a_{e}\right)$ and $\operatorname{dom}(x f) \subset \operatorname{dom}\left(x f_{e}\right)$ for any $x \in \operatorname{dom}(a)$. But every element of $\mathcal{R}$-cross-section is completely defined by the sets $\operatorname{dom}(a)$ and $\operatorname{dom}(x f), x \in \operatorname{dom}(a)$. Thus the number of elements of $\mathcal{R}$-cross-sections $R$, which are in $N_{e}$, is equal to

$$
\left|N_{e} \cap R\right|=\sum_{A \subset \operatorname{dom}\left(a_{e}\right)} \prod_{x \in A} 2^{\left|\operatorname{dom}\left(x f_{e}\right)\right|}=\prod_{x \in \operatorname{dom}\left(a_{e}\right)}\left(1+2^{\left|\operatorname{dom}\left(x f_{e}\right)\right|}\right)
$$

Take an element $e^{\prime}=\left(0, \operatorname{id}_{\mathcal{N}_{n}}\right)$. We have $e^{\prime} \in R^{\prime}$.
Set $R^{\prime} \cap N_{e^{\prime}}$ contains $2^{n}$ elements (they are elements $(0, a)$ ). Since $\varphi$ is an isomorphism, then $e^{\prime \prime}=\left(f^{\prime \prime}, a^{\prime \prime}\right)=\varphi\left(e^{\prime}\right)$ is an idempotent, and $\left|R^{\prime \prime} \cap N_{e^{\prime \prime}}\right|=2^{n}$. Therefore, we have

$$
\prod_{x \in \operatorname{dom}\left(a^{\prime \prime}\right)}\left(1+2^{\left|\operatorname{dom}\left(f^{\prime \prime}(x)\right)\right|}\right)=2^{n}
$$

This clearly implies $\left|\operatorname{dom}\left(f^{\prime \prime}(x)\right)\right|=0$ for all $x \in \operatorname{dom}\left(a^{\prime \prime}\right)$ and $\left|\operatorname{dom}\left(a^{\prime \prime}\right)\right|=n$, which means $e^{\prime \prime}=e^{\prime}$.

Step 2. Define

$$
R_{1}^{\prime}=\left\{a \in \mathcal{I} \mathcal{S}_{n} \mid(f, a) \in R^{\prime}\right\}
$$

and similarly $R_{1}^{\prime \prime}$. Both $R_{1}^{\prime}$ and $R_{1}^{\prime \prime}$ are $\mathcal{R}$-cross-sections of $\mathcal{I} \mathcal{S}_{n}$. For every element $(f, a) \in R$ the product

$$
(f, a)(0, e)=(0, e)(f, a)=(0, a) \in R
$$

We have then $\left(0, R_{1}^{\prime}\right)=e^{\prime} R^{\prime}$ and $\left(0, R_{1}^{\prime \prime}\right)=e^{\prime} R^{\prime \prime}$. Since $\varphi$ is an isomorphism, then $\varphi\left(\left(0, R_{1}^{\prime}\right)\right)=\left(0, R_{1}^{\prime \prime}\right)$ and a map $\varphi_{1}: R_{1}^{\prime} \rightarrow R_{1}^{\prime \prime}$, which is defined as $\left(0, \varphi_{1}(x)\right)=$ $\varphi((0, x))$, is an isomorphism. From Theorem3it follows that there exists an element $\theta \in \mathcal{S}_{n}$ such that $\varphi_{1}(a)=\theta^{-1} a \theta$.

Take an arbitrary element $(f, a) \in R^{\prime}$ and put $(g, b)=\varphi((f, a))$. We get

$$
\left(0, \varphi_{1}(a)\right)=\varphi((0, a))=\varphi\left(e^{\prime}(f, a)\right)=e^{\prime} \varphi((f, a))=e^{\prime}(g, b)=(0, b)
$$

which implies $b=\varphi_{1}(a)=\theta^{-1} a \theta$. Renumbering elements of the set $\mathcal{N}_{n}$ in a proper way, we could obtain $R_{1}^{\prime}=R_{1}^{\prime \prime}=R\left(M_{1}\right) \times \cdots \times R\left(M_{s}\right)$ and $\varphi_{1}=$ id. Then, evidently, $b=a$.

Step 3. Define the "maximal" block idempotents: $e_{i}=\left(1_{M_{i}}, \mathrm{id}_{M_{i}}\right)$, where $1_{M_{i}}(x)=\operatorname{id}_{\mathcal{N}_{m}}, x \in M_{i}$. According to Lemma 1 maximal idempotents are preserved under isomorphism. As $\varphi$ acts identically on the second component, we have $\varphi\left(e_{i}\right)=e_{i}$. Thus

$$
\begin{equation*}
\varphi\left(\left(\left.f\right|_{M_{i}},\left.a\right|_{M_{i}}\right)\right)=\varphi\left(e_{i}(f, a)\right)=e_{i} \varphi((f, a))=e_{i}(g, a)=\left(\left.g\right|_{M_{i}},\left.a\right|_{M_{i}}\right) \tag{4}
\end{equation*}
$$

Since $M_{i}^{R}=M_{i}$ and $\mathcal{N}_{n}=M_{1} \sqcup M_{2} \sqcup \ldots \sqcup M_{s}$, then there exists a monomorphism from $R$ to $\prod_{i=1}^{s}\left(\mathcal{I} \mathcal{S}_{m} \imath_{p} \mathcal{I S}\left(M_{i}\right)\right)$ defined in the following way:

$$
(f, a) \mapsto\left(\left(\left.f\right|_{M_{1}},\left.a\right|_{M_{1}}\right), \ldots,\left(\left.f\right|_{M_{s}},\left.a\right|_{M_{s}}\right)\right)
$$

Using the monomorphism (defined as above) from $\mathcal{R}$-cross-section to Cartesian product of $\mathcal{R}$-cross-sections of $\mathcal{I} \mathcal{S}_{m} \imath_{p} \mathcal{I S}\left(M_{i}\right)$ and equality (4), we have that it is enough to prove the proposition for "restriction $\varphi$ to cofactors". That is we may suppose that $R_{1}^{\prime}=R_{1}^{\prime \prime}=R(M)$, and that the isomorphism $\varphi$ is identical on the
second component (because evidently, this property is preserved under restriction). Without loss of generality assume that $M=\{1, \ldots, l\}$ with natural order.

Let we have some $\mathcal{R}$-cross-section $R$. Let $\left(f_{i}, a_{i}\right), i=1, \ldots, l$, be elements of $\mathcal{R}$-cross-section $R$ such that

$$
a_{i}=\langle 1, i, i+1, \ldots, l-1, l\rangle, \operatorname{dom}(f(1))=\mathcal{N}_{m} .
$$

Put $\varphi_{i}=f_{i}(1)$ for an element $\left(f_{i}, a_{i}\right) \in R$. Consider now the map $\Theta: \mathcal{I} \mathcal{S}_{m} \imath_{p} \mathcal{I S}(M) \rightarrow$ $\mathcal{I} \mathcal{S}_{m} 2_{p} \mathcal{I S}(M)$, which acts as follows: $(f, a) \mapsto(g, a)$, where for $x \in \operatorname{dom}(a)$ $g(x)=\varphi_{x} f(x) \varphi_{x^{a}}^{-1}$. It is easy to check that this map is an isomorphism and an isomorphic image of $\mathcal{R}$-cross-section is an $\mathcal{R}$-cross-section. In such a way we define the maps $\Theta^{\prime}$ for $R^{\prime}$ and $\Theta^{\prime \prime}$ for $R^{\prime \prime}$.

Therefore, applying to both cross-sections $R^{\prime}$ and $R^{\prime \prime}$ recently defined maps $\Theta^{\prime}$ and $\Theta^{\prime \prime}$, we obtain $\mathcal{R}$-cross-sections $\Theta^{\prime}\left(R^{\prime}\right)=R_{2}^{\prime} z_{p} R(M)$ and $\Theta^{\prime \prime}\left(R^{\prime \prime}\right)=R_{2}^{\prime \prime} z_{p} R(M)$ (see Lemma 3.6 in [6]). Since both $\Theta^{\prime}$ and $\Theta^{\prime \prime}$ are clearly conjugacies and act identically on the second component, then we may assume $R^{\prime}=R_{2}^{\prime} \imath_{p} R(M)$, $R^{\prime \prime}=R_{2}^{\prime \prime} \imath_{p} R(M)$, and isomorphism $\varphi$ acts identically on the second component.

Step 4. Consider the set $R_{2}=\left\{\left(f, \mathrm{id}_{\{l\}}\right) \in R^{\prime}\right\}$. Evidently, $R_{2} \simeq R_{2}^{\prime}$ (the isomorphism is defined by $\left.\psi\left(\left(f, \operatorname{id}_{\{l\}}\right)\right)=f(l)\right)$. Since $\varphi$ is identical on the second component, then $\varphi\left(R_{2}\right)=\left\{\left(f, \operatorname{id}_{\{l\}}\right) \in R^{\prime \prime}\right\} \simeq R_{2}^{\prime \prime}$, hence $R_{2}^{\prime \prime} \simeq R_{2}^{\prime}$. Moreover, as $\sigma=\psi^{-1} \varphi \psi$ is an isomorphism of $\mathcal{R}$-cross-sections of $\mathcal{I} \mathcal{S}_{m}$, then there exists $\theta_{0} \in \mathcal{S}_{m}$ such that $\sigma(a)=\theta_{0}^{-1} a \theta_{0}$. Put $x \vartheta=\theta_{0}, x \in M$. We will show that $\Theta=\left(\vartheta, \mathrm{id}_{M}\right) \in \mathcal{S}_{m} \imath \mathcal{S}_{l}$ is as required.

For $j \in M$ denote

$$
\tau_{j}= \begin{cases}(f,\langle j, l\rangle) \in R^{\prime}, \operatorname{dom}(f(j))=\mathcal{N}_{m}, & \text { if } j<l \\ \left(f, \operatorname{id}_{\{l\}}\right) \in R^{\prime}, \operatorname{dom}(f(l))=\mathcal{N}_{m}, & \text { if } j=l\end{cases}
$$

As $f(j) \in R_{2}^{\prime}$, then $f(j)=\operatorname{id}_{\mathcal{N}_{m}}$. We claim that $\varphi\left(\tau_{j}\right)=\tau_{j}$. Indeed, $\tau_{j}$ is a unique element of $R^{\prime}$ of the form $(f,\langle j, l\rangle)$, which cannot be represented as a product of an element of $R^{\prime}$ of such a form and a non-idempotent element from $R_{2}$. Since $\varphi$ is identical on the second component, then the set of the elements of the form $(f,\langle j, l\rangle)$ is preserved under the action of $\varphi$, and the same is true for $R_{2}$. Moreover, $\varphi$ preserves the operation and idempotents, hence $\varphi$ should preserve an element $\tau_{j}$ also.

Further, for arbitrary $i, j \in M, i<j$, denote by $\Lambda(i, j):=\{i, j+1, j+2, \ldots, l\}$ and consider an element $\lambda_{i j}=(f, a) \in R^{\prime}$ such that $\operatorname{dom}(a)=\Lambda(i, j)$ (i.e. $i a=j$, $k a=k$ for $k>j$ ) and $\operatorname{dom}(x f)=\mathcal{N}_{m}$ (i.e. $\left.f(x)=\operatorname{id}_{\mathcal{N}_{m}}\right)$ for $x \in \Lambda(i, j)$. We claim that $\varphi\left(\lambda_{i j}\right)=\lambda_{i j}$. Indeed, if we denote $\Lambda(j):=\{j, j+1, \ldots, l\}$, then element $\lambda_{i j}$ can be characterized by the property: it is the only element of $R^{\prime}$ of the form $\left(f, \operatorname{id}_{\Lambda(i, j)}\right)$, which cannot be represented as the product of the element of such a form and non-idempotent element of $R^{\prime}$ of the form $\left(f, \mathrm{id}_{\Lambda(j)}\right)$. As this property is preserved under $\varphi$, we have $\varphi\left(\lambda_{i j}\right)=\lambda_{i j}$.

Consider now an arbitrary element $(f, a) \in R^{\prime}$. Let $(g, a)=\varphi((f, a))$. Take any $i \in \operatorname{dom}(a)$ and let $i a=j$. Take $\left(h, \operatorname{id}_{\{l\}}\right)$ such that $\operatorname{dom}(h(l))=\operatorname{dom}(f(i))$ (i.e. $h(l)=f(i))$. The elements $z_{1}=(f, a) \tau_{j}$ and $z_{2}=\tau_{i}\left(h, \mathrm{id}_{l}\right)$ are in $R^{\prime}$, since $z_{1} \mathcal{R} z_{2}$, then $z_{1}=z_{2}$. Applying to this equality $\varphi$, we get $(g, a) \tau_{j}=\tau_{i} \varphi\left(\left(h, \operatorname{id}_{\{l\}}\right)\right)$, which implies $g(i)=\theta_{0}^{-1} h(l) \theta_{0}=\theta_{0}^{-1} f(i) \theta_{0}$.

Remark 4. In the terms of semigroup of transformations of rooted trees this theorem states that every isomorphism of $\mathcal{R}$ - $(\mathcal{L}-)$ cross-sections of semigroup of partial "rooted" automorphisms of a rooted regular two-level tree $T$ is a conjugacy.

Remark 5. Because of recursive definition of partial wreath product, one can generalize this Theorem for $\mathcal{R}-(\mathcal{L}-)$ cross-sections of semigroup $\mathcal{I} \mathcal{S}_{n_{k}} \imath_{p} \mathcal{I} \mathcal{S}_{n_{k-1}} \imath_{p} \ldots \imath_{p} \mathcal{I} \mathcal{S}_{n_{1}}$.

Let $R=R\left(\overrightarrow{M_{1}}, \overrightarrow{M_{2}}, \ldots, \overrightarrow{M_{s}}\right)$ be an $\mathcal{R}$-cross-section of $\mathcal{I} \mathcal{S}_{n}$. The vector $\left(u_{1}, \ldots, u_{n}\right)$, where $u_{k}=\left|\left\{i| | M_{i} \mid=k\right\}\right|, 1 \leq k \leq n$, is called the type of $\mathcal{R}$-cross-section $R$. The type of $\mathcal{L}$-cross-section is defined in a similar way. It is proven in [3] that two $\mathcal{R}$ - $(\mathcal{L}$ - $)$ cross-sections is isomorphic if and only if they have the same type. It is also shown that the number of non-isomorphic $\mathcal{R}$ - $(\mathcal{L}$ - $)$ cross-sections is equal to $p_{n}$, where $p_{n}$ is the number of decompositions of $n$ into the sum of positive integers, where the order of summands is not important.

Corollary 1. The number of non-isomorphic $\mathcal{R}$ - ( $\mathcal{L}$-) cross-sections of $\mathcal{I} \mathcal{S}_{m} \lambda_{p} \mathcal{I} \mathcal{S}_{n}$ is

$$
\sum_{\substack{j_{1}, j_{2}, \ldots, j_{n} \geq 0 \\ j_{1}+2 j_{2}+\cdots+n j_{n}=n}} \prod_{i=1}^{m}\binom{p_{m}+j_{i}-1}{j_{i}},
$$

where $p_{n}$ denotes the number of decompositions of $n$ into the sum of positive integers, where the order of summands is not important.

Proof. Clearly, it is enough to compute the number of non-isomorphic $\mathcal{R}$-crosssections of $\mathcal{I} \mathcal{S}_{m} \imath_{p} \mathcal{I} \mathcal{S}_{n}$. The number of non-isomorphic $\mathcal{L}$-cross-sections is the same.

Partition the set of all $\mathcal{R}$-cross-sections of semigroup $\mathcal{I} \mathcal{S}_{m}$ into $p_{m}$ classes of isomorphic $\mathcal{R}$-cross-sections and enumerate them with the integers from 1 to $p_{m}$.

For a fixed partition $\left\{M_{1}, M_{2}, \ldots, M_{s}\right\}$ consider all $\mathcal{R}$-cross-sections having the form

$$
\begin{equation*}
\left(R_{1} \imath_{p} R\left(M_{1}\right)\right) \times \ldots \times\left(R_{k} \imath_{p} R\left(M_{s}\right)\right) \tag{5}
\end{equation*}
$$

of semigroup $\mathcal{I} \mathcal{S}_{m} \imath_{p} \mathcal{I} \mathcal{S}_{n}$ such that $\mathcal{R}$-cross-section $R_{1}=R\left(\overrightarrow{M_{1}}, \overrightarrow{M_{2}}, \ldots, \overrightarrow{M_{s}}\right)$. Define as above $j_{k}=\left|\left\{i| | M_{i} \mid=k\right\}\right|$.

To each $\mathcal{R}$-cross-section assign a sequence of number pairs

$$
\left(\left(\left|M_{1}\right|, i_{1}\right), \ldots,\left(\left|M_{s}\right|, i_{s}\right)\right), \quad i_{l} \in\left\{1,2, \ldots, p_{m}\right\}
$$

where $i_{l}$ is the number of the equivalence class of $R_{i} \subset \mathcal{I} \mathcal{S}_{m}$. Two $\mathcal{R}$-cross-sections of the form (5) are isomorphic iff their corresponding sequences are equal up to the permutation of elements. Indeed, we know from Theorem 4 that any isomorphism of $\mathcal{R}$-cross-sections is generated by a "tree isomorphism" $\Theta$. Moreover, it follows from the proof of this theorem that each cofactor $R_{k} \imath_{p} R\left(M_{s}\right)$ is mapped to a similar cofactor $R_{k}^{\prime} \imath_{p} R\left(M_{k}^{\prime}\right)$ by $\Theta$, and $R_{k}^{\prime} \simeq R_{k},\left|M_{k}^{\prime}\right|=\left|M_{s}\right|$. On the other hand, if $M_{j}=M_{\sigma(j)}^{\prime}$ and for each $j=1, \ldots, k \quad \psi_{j}: R_{j}^{\prime} \rightarrow R_{\sigma(j)}$ is an isomorphism, then isomorphism between $\mathcal{R}$-cross-sections $\left(R_{1} \imath_{p} R\left(M_{1}\right)\right) \times \ldots \times\left(R_{s} \imath_{p} R\left(M_{s}\right)\right)$ and $\left(R_{1}^{\prime} \imath_{p} R\left(M_{1}^{\prime}\right)\right) \times \ldots \times\left(R_{s}^{\prime} \imath_{p} R\left(M_{s}^{\prime}\right)\right)$ is established by the map

$$
\left(f_{1}, a_{1}\right) \times \ldots \times\left(f_{s}, a_{s}\right) \mapsto\left(\psi_{1}\left(f_{\sigma(1)}\right), a_{\sigma(1)}\right) \times \ldots \times\left(\psi_{1}\left(f_{\sigma(s)}\right), a_{\sigma(s)}\right)
$$

So, among permutationally-equivalent sequences we can choose a "canonical" representation, for instance, we can arrange as

$$
\left(\left(1, i_{11}\right), \ldots,\left(1, i_{1 j_{1}}\right),\left(2, i_{21}\right), \ldots,\left(2, i_{2 j_{2}}\right) \ldots,\right)
$$

$$
1 \leq i_{11} \leq \ldots \leq i_{1 j_{1}} \leq p_{m}, 1 \leq i_{21} \leq \ldots, i_{2 j_{2}} \leq p_{m}, \ldots
$$

Thus, we have to compute the number of representatives in order to find the number of non-isomorphic $\mathcal{R}$-cross-sections. To get their number we have to find the number of non-decreasing functions from $\left\{1,2, \ldots, j_{l}\right\}$ to $\left\{1,2, \ldots, p_{m}\right\}$.

The number of non-decreasing functions from $\left\{1,2, \ldots, j_{l}\right\}$ to $\left\{1,2, \ldots, p_{m}\right\}$ is equal to the number of solutions of the equation

$$
x_{1}+\ldots+x_{p_{m}}=j_{l},
$$

that is $\binom{p_{m}+j_{l}-1}{j_{l}}$. So the number of non-isomorphic $\mathcal{R}$-cross-sections of form (5) is

$$
\prod_{i=1}^{m}\binom{p_{m}+j_{i}-1}{j_{i}}
$$

Summing this over forms (5), i.e. over all partitions of the integer $n$, we get the the number of all non-isomorphic $\mathcal{R}$-cross-section of the semigroup $\mathcal{I} \mathcal{S}_{m} \imath_{p} \mathcal{I} \mathcal{S}_{n}$ :

$$
\sum_{\substack{j_{1}, j_{2}, \ldots, j_{n} \geq 0 \\ j_{1}+2 j_{2}+\cdots+n j_{n}=n}} \prod_{i=1}^{m}\binom{p_{m}+j_{i}-1}{j_{i}} .
$$

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