# A PROFINITE APPROACH TO STABLE PAIRS 

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#### Abstract

We give a short proof, using profinite techniques, that idempotent pointlikes, stable pairs and triples are decidable for the pseudovariety of aperiodic monoids. Stable pairs are also described for the pseudovariety of all finite monoids.


## 1. Introduction

In this paper we introduce a new combinatorial technique for working with elements of free pro- $\mathbf{V}$ monoids where $\mathbf{V}$ is a pseudovariety of monoids closed under Malcev product on the left by the pseudovariety of aperiodic monoids. The approach uses the Henckell-Schützenberger expansion, and essentially allows one to transfer arguments from Combinatorics on Words to the profinite context. Let us describe some of the applications. Detailed definitions are given below.

If $\mathbf{V}$ is a pseudovariety of monoids, then a finite monoid $M$ belongs to the Malcev product $\mathbf{V}(\mathbb{m} \mathbf{A}$ if and only if the maximal $\mathbf{A}$-idempotent pointlikes of $M$ belong to $\mathbf{V}[11,14,18]$. If $\mathbf{V}$ is a local pseudovariety of monoids [19], then $M$ belongs to the semidirect product $\mathbf{V} * \mathbf{A}$ if and only if, for each maximal A-stable pair $(Y, N)$ of $M$, the quotient of $N$ by the kernel of the action of $N$ on $Y$ belongs to $\mathbf{V}[8]$. Henckell proved that A-idempotent pointlikes and $\mathbf{A}$-stable pairs are computable $[7,8]$. We give a much easier proof of his results using profinite techniques. Also we characterize the stable pairs for the pseudovariety $\mathbf{M}$ of all finite monoids, giving a partial answer to a question raised in [2]. We also prove that A-triples (introduced below) are decidable.

The paper is organized as follows. First we introduce stable pairs and pointlike sets, and prove a standard compactness result. Next, we recall the definition of the Henckell-Schützenberger expansion. We then describe stabilizers in certain free pro- $\mathbf{V}$ semigroups and introduce a discontinuous homomorphism. This leads to a proof of Henckell's theorem on idempotent pointlikes. As a warm-up, we handle M-stable pairs before turning to Astable pairs. The final section concerns aperiodic triples, which we believe

[^0]will play a role in the solution to deciding membership in the complexity one pseudovariety.

## 2. Stable pairs and pointlikes

If $X$ is a set, we use $X^{*}$ for the free monoid, $X^{+}$for the free semigroup and $\widehat{X^{*}}$ for the free profinite monoid generated by $X$ [1]. If $\mathbf{V}$ is a pseudovariety of monoids we use $\widehat{F_{\mathbf{V}}}(X)$ to denote the free pro-V monoid generated by $X$. If $M$ is a monoid generated by a set $X$, then the image of an element $w$ of $X^{*}$ or $\widehat{X^{*}}\left(\right.$ or $\widehat{F_{\mathbf{V}}}(X)$ if applicable) in $M$ is denoted $[w]_{M}$. As a shorthand, if $\gamma \in \widehat{X^{*}}$, then the image of $\gamma$ in $\widehat{F_{\mathbf{V}}}(X)$ is denoted $[\gamma]_{\mathbf{V}}$.

If $M$ and $N$ are $X$-generated monoids, we define the canonical relational morphism $\varphi: M \rightarrow N$ by $n \in m \varphi$ if and only if there exists $w \in X^{*}$ such that $[w]_{M}=m$ and $[w]_{N}=n$; this is equivalent to there existing $\alpha \in \widehat{X^{*}}$ with $[\alpha]_{M}=m$ and $[\alpha]_{N}=n$. If $M$ is an $X$-generated monoid and $\mathbf{V}$ is a pseudovariety, then the canonical relational morphism $\rho_{\mathbf{V}}: M \rightarrow \widehat{F_{\mathbf{V}}}(X)$ is the relational morphism given by $\alpha \in m \rho_{\mathbf{V}}$ if and only if there exists $\alpha^{\prime} \in \widehat{X^{*}}$ with $\left[\alpha^{\prime}\right]_{M}=m$ and $\left[\alpha^{\prime}\right]_{\mathbf{V}}=\alpha$. Alternatively, $\alpha \in m \rho_{\mathbf{V}}$ if and only if there is a sequence $w_{n} \in X^{*}$ such that $w_{n} \rightarrow \alpha$ and $\left[w_{n}\right]=m$ for all $n$. We remark that $m \rho_{\mathbf{V}}$ is a closed subset of $\widehat{F_{\mathbf{V}}}(X)$ (see [12,14, 17, 18]).

Definition 2.1 (V-pointlikes). If $M$ is a finite monoid and $Z \subseteq M$ is a subset, then $Z$ is said to be $\mathbf{V}$-pointlike, if for all relational morphisms $\varphi: M \rightarrow N$ with $N \in \mathbf{V}$, there exists $n \in N$ such that $Z \subseteq n \varphi^{-1}$.

The collection $\mathrm{PL}_{\mathbf{V}}(M)$ of $\mathbf{V}$-pointlikes of $M$ is a submonoid of the power set $P(M)$, which is downwards closed in the order $\subseteq$. The following fact about pointlike sets is well known. Proofs can be found in $[6,17,18]$, for instance.

Proposition 2.2. Let $\mathbf{V}$ be a pseudovariety of monoids. The map $M \mapsto$ $\mathrm{P}_{\mathbf{V}}(M)$ is a functor preserving onto maps. More precisely, if $\varphi: M \rightarrow N$ is a homomorphism and $Z \in \operatorname{PL}(M)$, then $Z \varphi \in \operatorname{PL}_{\mathrm{v}}(N)$. If, in addition, $\varphi$ is onto, then given $Z \in \operatorname{PL}_{\mathbf{v}}(N)$, there exists $Z^{\prime} \in \operatorname{PL}_{\mathbf{v}}(M)$ with $Z^{\prime} \varphi=Z$.

So given a homomorphism $\varphi: M \rightarrow N$, there is an induced homomorphism $\varphi_{*}: \operatorname{PL}_{\mathbf{v}}(M) \rightarrow \mathrm{PL}_{\mathbf{v}}(N)$ given by $Z \varphi_{*}=Z \varphi$ (the direct image).

An element $Z \in \mathrm{PL}_{\mathrm{V}}(M)$ is called $\mathbf{V}$-idempotent pointlike if, for all relational morphisms $\varphi: M \rightarrow N$ with $N \in \mathbf{V}$, there exists an idempotent $e \in N$ with $Z \subseteq e \varphi^{-1}$. Notice that if $Z \in \operatorname{PLv}^{(M)}$ and $Z=Z^{2}$, then $Z$ is trivially idempotent pointlike since if $Z \subseteq n \varphi^{-1}$, then $Z \subseteq n^{\omega} \varphi^{-1}$. Also the set of $\mathbf{V}$-idempotent pointlikes of $M$ form a downwards closed subset of $P(M)$.

Next we consider the notion of a $\mathbf{V}$-stable pair. If $M$ is a monoid and $s \in M$, then the stabilizer of $m$ is the submonoid

$$
\operatorname{Stab}(m)=\left\{m^{\prime} \in M \mid m m^{\prime}=m\right\} .
$$

Definition 2.3 (V-stable pairs). Let $M$ be a monoid. A pair $(Y, U)$ with $Y \subseteq M$ and $U \leq M$ (a submonoid) is called $a \mathbf{V}$-stable pair if, for all relational morphisms $\varphi: M \rightarrow N$ with $N \in \mathbf{V}$, there exists $n \in N$ such that $Y \subseteq n \varphi^{-1}$ and $N \leq \operatorname{Stab}(n) \varphi^{-1}$.

If we use the product ordering on pairs $(Y, N)$ with $Y$ a subset and $N$ a submonoid of $M$, then the set of $\mathbf{V}$-stable pairs is downwards closed. Notice that to decide which pairs are $\mathbf{V}$-stable, we just need to be able to compute all the maximal ones. Therefore, we focus our attention on these. Observe that if $(Y, U)$ is a stable pair, then so is $(Y U, U)$. Thus the maximal stable pairs are transformation monoids. It is straightforward to verify that if $\mathbf{V}$ is a local pseudovariety of monoids in the sense of Tilson [19], then $M \in \mathbf{V} * \mathbf{W}$ if and only if, for each maximal $\mathbf{W}$-stable pair $(Y, U)$ of $M$, the quotient of $U$ by the kernel of the action on $Y$ belongs to $\mathbf{V}$, c.f. $[8,9]$.

Let us consider a more general notion. A directed graph $\Gamma$ consists of a vertex set $V(\Gamma)$, an edge set $E(\Gamma)$ and functions $\iota, \tau: E(\Gamma) \rightarrow V(\Gamma)$ selecting the initial and terminal vertices $e$ of an edge, respectively. We consider only finite graphs. A labelling of a graph $\Gamma$ over a monoid $M$ is a function $\ell: V(\Gamma) \cup E(\Gamma) \rightarrow P(M)$. If the image of $\ell$ is contained in $M$, we call $\ell$ a singleton labelling. A singleton labelling $\ell$ is said to commute if, for each edge $e$, eıle $\ell=e \tau \ell$. If $\varphi: M \rightarrow N$ is a relational morphism, $\ell$ is a labelling of $\Gamma$ over $M$ and $\ell^{\prime}$ is a singleton labelling of $\Gamma$ over $N$, then $\ell$ is said to be $\varphi$-related to $\ell^{\prime}$ if $x \ell \subseteq x \ell^{\prime} \varphi^{-1}$ for all vertices and edges $x$ of $\Gamma$. The following notion generalizes a notion of Almeida [2], which in turn generalizes a notion of Ash [3].
Definition 2.4 (V-inevitable graph). Let $M$ be a finite monoid and $\mathbf{V}$ a pseudovariety. A labelling $\ell$ of a graph $\Gamma$ over $M$ is $\mathbf{V}$-inevitable if, for all relational morphisms $\varphi: M \rightarrow N$ with $N \in \mathbf{V}$, there is a singleton labelling $\ell^{\prime}$ of $\Gamma$ over $N$ which commutes and which is $\varphi$-related to $\ell$.

For instance, $Z \subseteq M$ is $\mathbf{V}$-pointlike if and only if the graph with a single vertex labelled by $Z$ is $\mathbf{V}$-inevitable. Let $Y \subseteq M$ and $N \leq M$. Let $\Gamma$ be a graph with one vertex and $|N|$ loops. Then $(Y, N)$ is a $\mathbf{V}$-stable pair if and only if the labelling of $\Gamma$ that assigns $Y$ to the vertex and labels the edges by the elements of $N$ is $\mathbf{V}$-inevitable. Conversely, a labelling of a graph with one vertex with label $Y$ and that assigns singletons to the loops at the vertex is $\mathbf{V}$-inevitable if and only if $(Y,\langle Z\rangle)$ is a $\mathbf{V}$-stable pair where $Z$ is the set of labels of the edges. A singleton labelling of a graph $\Gamma$ by $M$ is $\mathbf{V}$-inevitable if and only if it is $\mathbf{V}$-inevitable in the sense of Almeida [2]. Conversely, one can go from inevitable labellings in our sense to that of Almeida by changing the graph. For instance, $Z \subseteq M$ is $\mathbf{V}$-pointlike if and only if the singleton labelling of a graph with two vertices and $|Z|$ directed edges, where the initial vertex is labelled 1 , the edges are labelled by the elements of $Z$ and the terminal vertex is labelled by some element of $Z$, is $\mathbf{V}$-inevitable [2]. Similarly, $(Y, N)$ is a $\mathbf{V}$-stable pair, if and only if the singleton labelling of the graph $\Gamma$ with two vertices $v_{1}, v_{2},|Y|$ edges from $v_{1}$
to $v_{2}$ and $|N|$ loops from $v_{2}$ to $v_{2}$ where $v_{1}$ is labelled by $1, v_{2}$ is labelled by some element of $Y$, the $|Y|$ edges are labelled by the elements of $Y$ and the $|N|$ loops are labelled by the elements of $N$ is $\mathbf{V}$-inevitable. We leave the general construction to the reader.

Our notion has the advantage that it is closed downwards in the partial order. That is, a labelling $\ell: V(\Gamma) \cup E(\Gamma) \rightarrow P(M)$ can be viewed as an element of $P(M)^{V(\Gamma) \cup E(\Gamma)}$. If we order this set by the product ordering, then the $\mathbf{V}$-inevitable elements form a down-set. Decidability is then reduced to calculating the maximal elements.

The next two results give the relationship between the notions we have been discussing and profinite techniques. We include them for completeness, and readers already conversant with this subject should feel free to skip them.

Lemma 2.5. Let $M$ be a finite $X$-generated monoid and $\mathbf{V}$ a pseudovariety of monoids. Let $\rho_{\mathbf{V}}: M \rightarrow \widehat{F_{\mathbf{V}}}(X)$ be the canonical relational morphism. Write $\widehat{F_{\mathbf{V}}}(X)=\underset{\leftarrow}{\lim } M_{\alpha}$ where the $M_{\alpha}$ are $X$-generated monoids in $\mathbf{V}$. Let $\rho_{\alpha}: M \rightarrow M_{\alpha}$ be the canonical relational morphism and $\pi_{\alpha}: \widehat{F_{\mathbf{V}}}(X) \rightarrow M_{\alpha}$ the canonical projection for each $\alpha$. Then:
(1) If $C=\underset{\lim _{2}}{ } C_{\alpha} \subseteq \widehat{F_{\mathbf{V}}}(X)$ is an inverse limit of subsets $C_{\alpha} \subseteq M_{\alpha}$ (with the induced inverse system), then $C \rho_{\mathbf{V}}^{-1}=\bigcap C_{\alpha} \rho_{\alpha}^{-1}$;
(2) $\rho_{\mathbf{V}}^{-1}=\bigcap \pi_{\alpha} \rho_{\alpha}^{-1}$;
(3) If $\gamma \in \widehat{F_{\mathbf{V}}}(X)$, then $\operatorname{Stab}(\gamma)=\lim _{\rightleftarrows} \operatorname{Stab}\left(\gamma \pi_{\alpha}\right)$.

Proof. Since $\rho_{\alpha}=\rho_{\mathbf{V}} \pi_{\alpha}$ and $C \pi_{\alpha} \subseteq C_{\alpha}$, we have, for all $\alpha$,

$$
C \rho_{\mathbf{V}}^{-1} \subseteq C \pi_{\alpha} \pi_{\alpha}^{-1} \rho_{\mathbf{V}}^{-1}=C \pi_{\alpha} \rho_{\alpha}^{-1} \subseteq C_{\alpha} \rho_{\alpha}^{-1}
$$

For the converse, suppose $m \in \bigcap C_{\alpha} \rho_{\alpha}^{-1}$. Let $Y_{\alpha}=\left\{y \in C_{\alpha} \mid m \in y \rho_{\alpha}^{-1}\right\}$. Then the $Y_{\alpha}$ are easily verified to form an inverse system. By assumption on $m$, the $Y_{\alpha}$ are non-empty finite sets. Hence $\emptyset \neq \lim Y_{\alpha} \subseteq \lim _{\leftarrow} C_{\alpha}=C$. Now $\left(\lim _{\longleftarrow} Y_{\alpha}\right) \pi_{\beta} \subseteq Y_{\beta} \subseteq m \rho_{\beta}=m \rho_{\mathbf{V}} \pi_{\beta}$, for all $\beta$, and hence $\lim _{\leftrightarrows} \subseteq m \rho_{\mathbf{V}}$, since $m \rho_{\mathbf{V}}$ is closed $[12,14,17,18]$. This shows that $m \in C \rho_{\mathbf{V}}^{-1}$ and completes the proof of (1).

One deduces (2) from (1) by observing that if $\gamma \in \widehat{F_{\mathbf{V}}}(X)$, then $\{\gamma\}=$ $\lim _{\leftrightarrows}\left\{\gamma \pi_{\alpha}\right\}$. Item (3) is clear from the description of $\lim M_{\alpha}$ as a subsemigroup of $\prod M_{\alpha}$ (see also [12, Proposition 9.6]).

The following compactness result encompasses several well-known such results $[2,11,14,17,18]$.

Theorem 2.6. Let $M$ be a finite $X$-generated monoid and $\mathbf{V}$ a pseudovariety of monoids. Let $\rho_{\mathbf{V}}: M \rightarrow \widehat{F_{\mathbf{V}}}(X)$ be the canonical relational morphism. Then:
(1) A subset $Z \subseteq M$ is $\mathbf{V}$-pointlike if and only if there exists $\alpha \in \widehat{F_{\mathbf{V}}}(X)$ with $Z \subseteq \alpha \rho_{\mathbf{V}}^{-1}$;
(2) A subset $Z \subseteq M$ is $\mathbf{V}$-idempotent pointlike if and only if there exists an idempotent $\alpha \in \widehat{F_{\mathbf{V}}}(X)$ with $Z \subseteq \alpha \rho_{\mathbf{V}}^{-1}$;
(3) $(Y, N)$ is a $\mathbf{V}$-stable pair for $M$ if and only if there exists $\alpha \in \widehat{F_{\mathbf{V}}}(X)$ with $Y \subseteq \alpha \rho_{\mathbf{v}}^{-1}$ and $N \leq \operatorname{Stab}(\alpha) \rho_{\mathbf{v}}^{-1}$;
(4) A labelling $\ell$ of a graph $\Gamma$ is $\mathbf{V}$-inevitable if and only if there is a singleton labelling of $\Gamma$ over $\widehat{F_{\mathbf{V}}}(X)$ that is $\rho_{\mathbf{V}}$-related to $\ell$ and which commutes.

Proof. We prove (3) and (4). A proof of (1) and (2) can be found in [18] (alternatively, (2) follows from (4)).

For (3), let $Y \subseteq M$ and $N \leq M$. If $\varphi: M \rightarrow T$ is a relational morphism, there is always an $X$-generated submonoid $T^{\prime}$ of $T$ and a canonical relational morphism $\psi: M \rightarrow T^{\prime}$ such that $\psi \subseteq \varphi$ (as relations) [14]. So we may take all the relational morphisms in the definition of a $\mathbf{V}$-stable pair to be canonical relational morphisms of $X$-generated monoids. Suppose that $\widehat{F_{\mathbf{V}}}(X)=\lim _{\leftarrow} M_{\alpha}$ where the $M_{\alpha}$ run over all $X$-generated monoids in $\mathbf{V}$. Let $\rho_{\alpha}: M \rightarrow M_{\alpha}$ and $\rho_{\mathbf{V}}: M \rightarrow \widehat{F_{\mathbf{V}}}(X)$ be the canonical relational morphisms and denote by $\pi_{\alpha}: \widehat{F_{\mathbf{V}}}(X) \rightarrow M_{\alpha}$ the canonical projection. Set

$$
\begin{aligned}
C_{\alpha} & =\left\{m \in M_{\alpha} \mid Y \subseteq m \rho_{\alpha}^{-1}, N \leq \operatorname{Stab}(m) \rho_{\alpha}^{-1}\right\} \\
C & =\left\{\gamma \in \widehat{F_{\mathbf{V}}}(X) \mid Y \subseteq \gamma \rho_{\mathbf{v}}^{-1}, N \leq \operatorname{Stab}(\gamma) \rho_{\mathbf{v}}^{-1}\right\} .
\end{aligned}
$$

Then the $C_{\alpha}$ are easily verified to form an inverse system. We claim that $C=\lim _{幺} C_{\alpha}$. Since an inverse limit of finite sets is non-empty if and only if each of the finite sets is non-empty, this will yield (3). Indeed, applying Lemma 2.5 we see that, for $\gamma \in \widehat{F_{\mathbf{V}}}(X)$, the equalities

$$
\begin{aligned}
\gamma \rho_{\mathbf{V}}^{-1} & =\bigcap \gamma \pi_{\alpha} \rho_{\alpha}^{-1} \\
\operatorname{Stab}(\gamma) \rho_{\mathbf{V}}^{-1} & =\bigcap \operatorname{Stab}\left(\gamma \pi_{\alpha}\right) \rho_{\alpha}^{-1}
\end{aligned}
$$

hold. Thus $\gamma \in C$ if and only if $Y \subseteq \gamma \rho_{\mathbf{V}}^{-1}, N \leq \operatorname{Stab}(\gamma) \rho_{\mathbf{V}}^{-1}$, if and only if $Y \subseteq \gamma \pi_{\alpha} \rho_{\alpha}^{-1}, N \leq \operatorname{Stab}\left(\gamma \pi_{\alpha}\right) \rho_{\alpha}^{-1}$ all $\alpha$, if and only if $\gamma \in \lim _{\leftarrow} C_{\alpha}$, as required.

For (4), let $\Gamma$ be a graph. If $N$ is a monoid, we use $N^{\Gamma}$ as a shorthand for $N^{V(\Gamma) \cup E(\Gamma)}$. As before, we need only consider canonical relational morphisms of $X$-generated monoids when considering $\mathbf{V}$-inevitability.

Consider a labelling $\ell \in P(M)^{\Gamma}$. Write $\widehat{F_{\mathbf{V}}}(X)=\lim M_{\alpha}$ where the $M_{\alpha}$ run over all $X$-generated monoids in $\mathbf{V}$. Let $\rho_{\alpha} \overleftarrow{:} M \rightarrow M_{\alpha}$ and $\rho_{\mathbf{V}}: M \rightarrow \widehat{F_{\mathbf{V}}}(X)$ be the canonical relational morphisms. Let $C_{\alpha}(\Gamma) \subseteq M_{\alpha}^{\Gamma}$ be the set of all commuting singleton labellings of $\Gamma$ that are $\rho_{\alpha}$-related to $\ell$. Then the $C_{\alpha}(\Gamma)$ form an inverse system. Indeed, if $\pi_{\alpha, \beta}: M_{\alpha} \rightarrow M_{\beta}$ is the canonical projection, then the image under $\pi_{\alpha, \beta}$ of a commuting singleton labelling of $M_{\alpha}$ clearly commutes and also $\rho_{\alpha}^{-1} \subseteq \pi_{\alpha, \beta} \rho_{\beta}^{-1}$ so $\rho_{\alpha}$-related labellings to $\ell$ are sent to $\rho_{\beta}$-related labellings.

Let $C(\Gamma) \subseteq \widehat{F_{\mathbf{V}}}(X)^{\Gamma}$ be the set of all commuting singleton labellings of $\Gamma$ that are $\rho_{\mathrm{V}}$-related to $\ell$. Then $C(\Gamma)$ is a closed subset of the profinite monoid $\widehat{F_{\mathbf{V}}}(X)^{\Gamma}$ and, in fact, $C(\Gamma)=\lim _{\leftarrow} C_{\alpha}(\Gamma)$. Indeed, writing $\pi_{\alpha}: \widehat{F_{\mathbf{V}}}(X) \rightarrow M_{\alpha}$ for the canonical projection, we have that $\rho_{\mathbf{V}}^{-1}=\bigcap \pi_{\alpha} \rho_{\alpha}^{-1}$ by Lemma 2.5 and a labelling $\ell^{\prime} \in \widehat{F_{\mathbf{V}}}(X)^{\Gamma}$ commutes if and only if all its images in the $M_{\alpha}$ commute (viewing $\widehat{F_{\mathbf{V}}}(X)$ as a submonoid of $\Pi M_{\alpha}$ ). Since the inverse limit of an inverse system of finite sets is non-empty if and only if each of the sets is non-empty, we conclude that $\ell$ is $\mathbf{V}$-inevitable if and only if $C(\Gamma) \neq \emptyset$. This completes the proof.

## 3. The Henckell-Schützeneberger Expansion

Our key tool for understanding stable pairs and related notions is the Henckell-Schützenberger expansion. Further applications of this expansion can be found in [13]. Recall that if $M$ and $N$ are monoids, then their Schützenberger product $[4,5]$ is the monoid

$$
M \diamond N=\left[\begin{array}{cc}
M & P(M \times N) \\
0 & N
\end{array}\right]
$$

with multiplication given by

$$
\left[\begin{array}{cc}
m & U \\
0 & n
\end{array}\right]\left[\begin{array}{cc}
m^{\prime} & U^{\prime} \\
0 & n^{\prime}
\end{array}\right]=\left[\begin{array}{cc}
m m^{\prime} & m U^{\prime}+U n^{\prime} \\
0 & n n^{\prime}
\end{array}\right]
$$

where addition is union and where $P(M \times N)$ is viewed as an $M$ - $N$-bimodule in the obvious way.

If $M$ is an X-generated monoid, then the Henckell-Schützenberger expansion $\widetilde{M}$ is the submonoid of $M \diamond M$ generated by matrices of the form

$$
\left[\begin{array}{cc}
x & (1, x)+(x, 1) \\
0 & x
\end{array}\right]
$$

with $x \in X$. So $\widetilde{M}$ is an $X$-generated monoid mapping naturally onto $M$ via the projection $\eta: \widetilde{M} \rightarrow M$ to the diagonal. Thus $\widetilde{M}$ is an expansion cut-to-generators in the sense of [4]. Since $M \diamond M$ is really a double semidirect product of $M$ with $P(M \times M)$, it follows $\eta$ is an LSl-morphism [14], meaning that the inverse image of each idempotent is locally a semilattice. In particular $\eta$ is an aperiodic morphism (see also $[4,5]$ ).

Let $w \in X^{+}$. By a cut of $w$ we mean a pair $(u, v) \in X^{*} \times X^{*}$ such that $w=u v$. The set of cuts of $w$ will be denoted $\vec{c}(w)$; we set $\vec{c}(\varepsilon)=\emptyset$. The next proposition is well known [4,5] and can be proved by a simple induction on length.

Proposition 3.1. Let $w \in X^{*}$ and $M$ an $X$-generated monoid. Then

$$
[w]_{\widetilde{M}}=\left[\begin{array}{cc}
{[w]_{M}} & \sum_{(u, v) \in \vec{c}(w)}\left([u]_{M},[v]_{M}\right) \\
0 & {[w]_{M}}
\end{array}\right] .
$$

In particular, for $w, w^{\prime} \in X^{+}$, the equality $[w]_{\widetilde{M}}=\left[w^{\prime}\right]_{\widehat{M}}$ holds if and only if, for each factorization $w=u v$, there is a factorization $w^{\prime}=u^{\prime} v^{\prime}$ such that $[u]_{M}=\left[u^{\prime}\right]_{M}$ and $[v]_{M}=\left[v^{\prime}\right]_{M}$, and vice versa.

Henckell [8] observed that stabilizers in $\widetilde{M}$ enjoy a certain nice property.
Lemma 3.2 (Henckell). Let $M$ be an $X$-generated monoid and let $w \in X^{*}$. Then $\operatorname{Stab}\left([w]_{\widetilde{M}}\right) \eta$ is an $E$-chain in the monoid $\operatorname{Stab}\left([w]_{M}\right)$.
Proof. If $w=\varepsilon$, there is nothing to prove, so assume $w \in X^{+}$. Suppose that $u, v \in X^{*}$ with $[w u]_{\widetilde{M}}=[w]_{\widetilde{M}}=[w v]_{\widetilde{M}}$. Then $w u=w_{1} w_{2}$ where $\left[w_{1}\right]_{M}=[w]_{M}$ and $\left[w_{2}\right]_{M}=[v]_{M}$. There are two cases. Suppose first that $|w| \leq\left|w_{1}\right|$. Then $w_{1}=w x$ and $w x w_{2}=w u$, so $u=x w_{2}$. Therefore, $[w]_{M}=\left[w_{1}\right]_{M}=[w]_{M}[x]_{M}$, establishing that $[x]_{M} \in \operatorname{Stab}\left([w]_{M}\right)$. In addition, $[u]_{M}=[x]_{M}\left[w_{2}\right]_{M}=[x]_{M}[v]_{M}$ and so $[u]_{M} \leq_{\mathrm{E}}[v]_{M}$ in $\operatorname{Stab}\left([w]_{M}\right)$. If $|w|>\left|w_{1}\right|$, then $w=w_{1} y$ and $w_{1} w_{2}=w_{1} y u$, so $w_{2}=y u$. A similar argument to the above one then shows that $[y]_{M} \in \operatorname{Stab}\left([w]_{M}\right)$ and $[v]_{M} \leq_{\mathrm{E}}[u]_{M}$ in $\operatorname{Stab}\left([w]_{M}\right)$. This completes the proof.

## 4. The structure of stabilizers and idempotent pointlikes

We begin with some applications of the Henckell-Schützenberger expansion to stabilizers and idempotent pointlikes.
4.1. The structure of stabilizers. Our first goal is to characterize stabilizers for free pro- $\mathbf{V}$ semigroups when $\mathbf{V}=\mathbf{A} m \mathbf{V}$, that is, $\mathbf{V}$ is closed under the Henckell-Schützenberger expansion. The approach is similar to the one taken in [12] for related results. A monoid $M$ will be called an internal E-chain if the L -classes of $M$ form a chain for the L -ordering. The reason the word internal is used is because if $M \leq N$, then $M$ can be an L-chain in $N$ without being an internal L -chain.

Theorem 4.1. Let $\mathbf{V}$ be a pseudovariety of monoids such that $\mathbf{V}=\mathbf{A}(\mathrm{m}) \mathbf{V}$ and let $X$ be a finite set. Then, for each $\gamma \in \widehat{F_{\mathbf{V}}}(X)$, the submonoid $\operatorname{Stab}(\gamma)$ is an internal $E$-chain.
Proof. Since $X$ is finite, we may write $\widehat{F_{\mathbf{V}}}(X)=\lim _{\hbar}{ }_{n \in \mathbb{N}} M_{n}$ where the $M_{n}$ are finite $X$-generated monoids in $\mathbf{V}$. Let $\pi_{n}: \widehat{F_{\mathbf{V}}}(X) \rightarrow M_{n}$ be the canonical projection. Then, for $\gamma \in \widehat{F_{\mathbf{V}}}(X)$, we have $\operatorname{Stab}(\gamma)=\lim _{\overbrace{n \in \mathbb{N}}} \operatorname{Stab}\left(\gamma \pi_{n}\right)$, by Lemma 2.5. Let $\delta, \sigma \in \operatorname{Stab}(\gamma)$ and consider $M_{n}$. Since $\mathbf{V}=\mathbf{A}(\mathrm{m}) \mathbf{V}$, we have that $\widetilde{M_{n}} \in \mathbf{V}$. Then $[\delta]_{\widetilde{M_{n}}},[\sigma]_{\widetilde{M_{n}}} \in \operatorname{Stab}\left([\gamma]_{\widetilde{M_{n}}}\right)$ and so Lemma 3.2 implies that $[\delta]_{M_{n}},[\sigma]_{M_{n}}$ are comparable in the $\leq_{\mathrm{E}}$-order on $\operatorname{Stab}\left([\gamma]_{M_{n}}\right)$. By going to a subsequence we may assume without loss of generality that, say, $[\delta]_{M_{n}} \leq_{\mathrm{E}}[\sigma]_{M_{n}}$ in $\operatorname{Stab}\left([\gamma]_{M_{n}}\right)$ for all $n$. It then follows that $\delta \leq_{\mathrm{E}} \sigma$ in $\operatorname{Stab}(\gamma)$ (c.f. [1, Theorem 5.6.1] or [12, Proposition 9.1]). Hence $\operatorname{Stab}(\gamma)$ is an internal E -chain.

In [12, Corollary 14.5] it was shown that, for $\alpha \in \widehat{X^{*}}, \operatorname{Stab}(\alpha)$ is an $\mathscr{R}$-trivial band. We can now refine this result.

Corollary 4.2. Let $X$ be a finite set and $\alpha \in \widehat{X^{*}}$. Then $\operatorname{Stab}(\alpha)$ is an $E$-chain of idempotents. In particular, it is an $\mathscr{R}$-trivial band.
4.2. A discontinuous homomorphism. The next lemma is the principal advantage obtained by our profinite approach over Henckell's approach [8].

Lemma 4.3. Let $\mathbf{V}$ be a pseudovariety of monoids such that $\mathbf{V}=\mathbf{A}(\mathbb{m} \mathbf{V}$. Let $X$ be a finite set, $M$ an $X$-generated finite monoid and $\rho_{\mathbf{V}}: M \rightarrow \widehat{F_{\mathbf{V}}}(X)$ the canonical relational morphism. Then

$$
\gamma \rho_{\mathbf{V}}^{-1} \sigma \rho_{\mathbf{V}}^{-1}=(\gamma \sigma) \rho_{\mathbf{v}}^{-1}
$$

for all $\gamma, \sigma \in \widehat{F_{\mathbf{V}}}(X)$.
Proof. The inclusion $\gamma \rho_{\mathbf{v}}^{-1} \sigma \rho_{\mathbf{V}}^{-1} \subseteq(\gamma \sigma) \rho_{\mathbf{V}}^{-1}$ is true for any relational morphism. Since $1 \in \varepsilon \rho_{\mathbf{V}}^{-1}$, the reverse inclusion is trivial if either $\sigma$ or $\gamma$ is $\varepsilon$, so assume $\sigma \neq \varepsilon \neq \gamma$. Let $m \in(\gamma \sigma) \rho_{\mathbf{V}}$. Then there exists a sequence of words $w_{n} \in X^{+}$such that $w_{n} \rightarrow \gamma \sigma$ and $\left[w_{n}\right]_{M}=m$. Since $X$ is a finite set, we can write $\widehat{F_{\mathbf{V}}}(X)=\lim _{\substack{ \\n_{\mathbb{N}}}} M_{n}$ where the $M_{n}$ are $X$-generated monoids from $\mathbf{V}$. Again, $\mathbf{V}$ is closed under the expansion $N \mapsto \widetilde{N}$. By going to a subsequence, we may assume that $\left[w_{n}\right]_{\widetilde{M_{n}}}=[\gamma \sigma]_{\widetilde{M_{n}}}$, all $n$. Similarly, we can find sequences $u_{n}, v_{n} \in X^{+}$such that $u_{n} \rightarrow \gamma, v_{n} \rightarrow \sigma$ and $\left[u_{n}\right]_{\widetilde{M_{n}}}=[\gamma]_{\widetilde{M_{n}}},\left[v_{n}\right]_{\widetilde{M_{n}}}=[\sigma]_{\widetilde{M_{n}}}$. Hence $\left[u_{n} v_{n}\right]_{\widehat{M_{n}}}=[\gamma \sigma]_{\widehat{M_{n}}}=\left[w_{n}\right]_{\widehat{M_{n}}}$ and so, by Proposition [3.1, $w_{n}=c_{n} s_{n}$ with $\left[c_{n}\right]_{M_{n}}=\left[u_{n}\right]_{M_{n}}=[\gamma]_{M_{n}}$ and $\left[s_{n}\right]_{M_{n}}=\left[v_{n}\right]_{M_{n}}=[\sigma]_{M_{n}}$. Thus $c_{n} \rightarrow \gamma$ and $s_{n} \rightarrow \sigma$. Since $M$ is finite, by going to a subsequence, we may assume that $\left[c_{n}\right]_{M}$ and $\left[s_{n}\right]_{M}$ are constant, say $\left[c_{n}\right]_{M}=m_{1}$ and $\left[s_{n}\right]_{M}=m_{2}$. Then $m_{1} \in \gamma \rho_{\mathbf{V}}^{-1}, m_{2} \in \sigma \rho_{\mathbf{V}}^{-1}$ and $m_{1} m_{2}=\left[c_{n} s_{n}\right]_{M}=\left[w_{n}\right]_{M}=m$. So $m \in \gamma \rho_{\mathbf{V}}^{-1} \sigma \rho_{\mathbf{V}}^{-1}$. This establishes $(\gamma \sigma) \rho_{\mathbf{V}}^{-1} \subseteq \gamma \rho_{\mathbf{V}}^{-1} \sigma \rho_{\mathbf{V}}^{-1}$ and completes the proof of the lemma.

Let us reformulate the above result into our critical lemma.
Lemma 4.4. Let $M$ be an $X$-generated finite monoid, let $\mathbf{V}$ be a pseudovariety such that $\mathbf{V}=\mathbf{A} \equiv \mathbf{V}$ and let $\rho_{\mathbf{V}}: M \rightarrow \widehat{F_{\mathbf{V}}}(X)$ be the canonical relational morphism. Then the map $f_{\mathbf{V}}: \widehat{F_{\mathbf{V}}}(X) \rightarrow \mathrm{PL}_{\mathbf{V}}(M)$ defined by $\gamma f_{\mathbf{V}}=\gamma \rho_{\mathbf{V}}^{-1}$ is a monoid homomorphism.

Proof. Theorem 2.6 shows that $f_{\mathbf{V}}$ is well defined. Lemma 4.3 shows that $f_{\mathbf{V}}$ is a semigroup homomorphism. Since $\varepsilon$ is an isolated point of $\widehat{F_{\mathbf{V}}}(X)$ (as the congruence class of $\varepsilon$ is trivial in $\widetilde{M}$ for any $M \in \mathbf{V}$ ), we conclude $\varepsilon f_{\mathbf{V}}=\varepsilon \rho_{\mathbf{V}}^{-1}=\{1\}$ and thus $f_{\mathbf{V}}$ is a monoid homomorphism.

We remark that $f_{\mathbf{V}}$ is not necessarily continuous. For instance, if $\gamma f_{\mathbf{A}}=$ $Z$, then $\gamma^{\omega} f_{\mathbf{A}} \supseteq \bigcup_{n \in \mathbb{N}} Z^{\omega} Z^{n}$, which can be strictly bigger than $Z^{\omega}$. This discontinuity is what underlies the analysis of aperiodic pointlike sets $[6,10]$.
4.3. Idempotent pointlikes. As a warm-up we prove the result of Henckell [7] relating $\mathbf{V}$-idempotent pointlikes with idempotent $\mathbf{V}$-pointlikes.

Theorem 4.5 (Henckell). Let $\mathbf{V}$ be a pseudovariety of monoids such that $\mathbf{A}(\mathrm{m}) \mathbf{V}=\mathbf{V}$ and let $M$ be a finite monoid. Then the maximal $\mathbf{V}$-idempotent pointlikes of $M$ are precisely the maximal idempotents of $\mathrm{P}_{\mathbf{V}}(M)$.

Proof. We already observed that idempotents of $\mathrm{PL}_{\mathbf{V}}(M)$ are $\mathbf{V}$-idempotent pointlike. Conversely, suppose that $Z$ is a maximal $\mathbf{V}$-idempotent pointlike subset of $M$. Let $X$ be a finite generating set for $M$ and let $\rho_{\mathbf{V}}: M \rightarrow \widehat{F_{\mathbf{V}}}(X)$ and $f_{\mathbf{V}}: \widehat{F_{\mathbf{V}}}(X) \rightarrow \mathrm{PL}_{\mathbf{V}}(M)$ be as per Lemma 4.4. By maximality and Theorem [2.6, we must have that $Z=e \rho_{\mathbf{V}}^{-1}=e f_{\mathbf{V}}$ for some idempotent $e \in \widehat{F_{\mathbf{V}}}(X)$. Since $f_{\mathbf{V}}$ is a homomorphism, $Z \in \mathrm{PL}_{\mathbf{V}}(A)$ is idempotent.

Since Henckell proved $[6,10]$ that A-pointlikes are decidable, we have the following corollaries.

Corollary 4.6. If $\mathbf{V}=\mathbf{A}(\mathrm{m} \mathbf{V}$ and $\mathbf{V}$-pointlikes are decidable, then $\mathbf{V}$ idempotent pointlikes are decidable. In particular, A-idempotent pointlikes are decidable.

Corollary 4.7. If $\mathbf{V}$ has decidable membership, then do does $\mathbf{V}(\mathrm{m} \mathbf{A}$.

## 5. M-Stable Pairs

Let $\mathbf{M}$ be the pseudovariety of all finite monoids. As a second warmup exercise we characterize the M-stable pairs. This partially answers a question raised in [2]. By considering the identity map, we see that an Mstable pair of a monoid $M$ must be of the form $(\{x\}, N)$ where $N \leq \operatorname{Stab}(x)$. Corollary 4.2 suggests that L -chains of idempotents should play a role. The next lemma describes what kind of monoid you can obtain by such a chain.

Lemma 5.1. Suppose that $e_{1} \geq_{E} e_{2} \geq_{E} \cdots \geq_{E} e_{n}$ is an E-chain of idempotents in a monoid $M$. Then $\left\langle e_{1}, \ldots, e_{n}\right\rangle$ is an $\mathscr{R}$-trivial band.

Proof. Set $N=\left\langle e_{1}, \ldots, e_{n}\right\rangle$. First we observe that $e_{i} e_{j}=e_{i}$ if $i \geq j$. Thus, each element of $T$ can be written in the form $e_{i_{1}} e_{i_{2}} \cdots e_{i_{m}}$ where the indices are increasing: $i_{1}<i_{2}<\cdots<i_{m}$. Clearly then one has

$$
\left(e_{i_{1}} e_{i_{2}} \cdots e_{i_{m}}\right)^{2}=e_{i_{1}} e_{i_{2}} \ldots e_{i_{m}}
$$

since $i_{m} \geq i_{j}$ for all $1 \leq j \leq m$. Let $s=e_{i_{1}} e_{i_{2}} \cdots e_{i_{m}}$ and $t=e_{j_{1}} e_{j_{2}} \cdots e_{j_{\ell}}$. Then $s t=s$ if $i_{m} \geq j_{\ell}$, or else $s t=s e_{j_{k}} \cdots e_{j_{\ell}}$, where $k$ is the smallest index such that $i_{m}<j_{k}$. In the first case, clearly sts $=s^{2}=s=s t$, while in the latter case we have $j_{\ell} \geq i_{r}$ for all $r$ and so $e_{j_{\ell}} s=e_{j_{\ell}}$, from which we conclude that sts $=s t$. This proves that $N$ is an $\mathscr{R}$-trivial band.

Theorem 5.2. Let $M$ be a finite monoid. Then $(\{y\}, N)$ is an $\mathbf{M}$-stable pair of $M$ if and only if there there is an $E$-chain $Y$ of idempotents in $\operatorname{Stab}(y)$ such that $N \leq\langle Y\rangle$.

Proof. Suppose that $Y \subseteq \operatorname{Stab}(y)$ is an E-chain of idempotents. Without loss of generality, we may assume that $N=\langle Y\rangle$ (since stable pairs are downwards closed). Since every monoid belongs to M, it clearly suffices to show that if $\varphi: S \rightarrow M$ is an onto homomorphism, then there exists $y^{\prime} \in S$ such that $y^{\prime} \varphi=y$ and $N \leq \operatorname{Stab}\left(y^{\prime}\right) \varphi$.

Choose $\widetilde{y} \in S$ with $\widetilde{y} \varphi=y$. Next, suppose $Y=e_{1} \geq_{\mathrm{£}} e_{2} \geq_{\mathrm{£}} \cdots \geq_{\mathrm{£}} e_{n}$ and choose an idempotent $f_{1} \in S$ with $f_{1} \varphi=e_{1}$. Assume inductively that, for $1 \leq i<n$ we have found $f_{1} \geq_{\mathrm{E}} \cdots \geq_{\mathrm{E}} f_{i}$ in $S$ with $f_{j} \varphi=e_{j}$, for $1 \leq j \leq i$. Then $e_{i+1} \in M e_{i} \subseteq\left(S f_{i}\right) \varphi$. So there exists an idempotent $f_{i+1}$ of $S f_{i}$ with $f_{i+1} \varphi=e_{i+1}$. This completes the induction. Set $Y^{\prime}=$ $\left\{f_{1}, \ldots, f_{n}\right\}$. Clearly $N^{\prime}=\left\langle Y^{\prime}\right\rangle$ maps onto $N$ via $\varphi$. Also $N^{\prime}$ is an $\mathscr{R}$-trivial band by Lemma 5.1. In particular, if $s$ belongs to the minimal ideal of $N^{\prime}$ and $t \in N^{\prime}$, then st $\mathscr{R} s$ and hence, since $N^{\prime}$ is $\mathscr{R}$-trivial, st $=s$. Thus $N^{\prime} \leq \operatorname{Stab}(s) \leq \operatorname{Stab}(\widetilde{y} s)$. But $(\widetilde{y} s) \varphi=\widetilde{y} \varphi s \varphi=y s \varphi \in y N=\{y\}$. This shows that $(\{y\}, N)$ is an M-stable pair.

For the converse, choose a generating set $X$ for $M$. Let $\pi: \widehat{X^{*}} \rightarrow M$ be the canonical projection. Then $\pi^{-1}$ is the canonical relational morphism $\rho_{\mathrm{M}}: M \rightarrow \widehat{X^{*}}$. So Theorem 2.6 shows there exists $\alpha \in \widehat{X^{*}}$ with $\alpha \pi=y$ and $N \leq \operatorname{Stab}(\alpha) \pi$. Corollary 4.2 yields $\operatorname{Stab}(\alpha)$ is an E-chain of idempotents, from which the result easily follows.

## 6. A-stable pairs

The situation for $\mathbf{A}$-stable pairs is more complicated since we no longer have that the stabilizers in $\widehat{F_{\mathbf{A}}}(X)$ must be bands. Let us recall some terminology from $[15,16]$ (see also [14]). Let ER be the pseudovariety of monoids whose idempotent-generated submonoids are $\mathscr{R}$-trivial. It is well-known that $M \in \mathbf{E R}$ if and only if each regular $\mathscr{R}$-class of $M$ contains a unique idempotent [14].

Proposition 6.1. Let $M \in \mathbf{E R} \cap \mathbf{A}$. Then, for any $x$ in the minimal ideal of $M$, one has $\operatorname{Stab}(x)=M$.

Proof. Since $M \in \mathbf{E R}$, the minimal ideal $I$ of $M$ contains a unique E -class. If $x \in I$ and $m \in M$, then $x m \mathscr{R} x$ by stability of finite semigroups and $x m \mathrm{~L} x$ since $I$ has a unique E -class. Since $M$ is aperiodic, $x m=x$.

A monoid $M$ is said to be absolute Type $I[9,14-16]$ if it can be generated by a chain of its L -classes. In particular, an internal L -chain is absolute Type I. The facts contained in our next proposition are from [15]; see [14] for proofs.

## Proposition 6.2.

(1) An aperiodic absolute Type I-monoid belongs to ER.
(2) If $\varphi: M \rightarrow N$ is an onto homomorphism and $M$ is absolute Type $I$, then $N$ is absolute Type $I$.
(3) If $\varphi: M \rightarrow N$ is an onto homomorphism and $N$ is absolute Type $I$, then there is an absolute Type I-submonoid $M^{\prime} \leq M$ with $M^{\prime} \varphi=N$.

We now present a sufficient condition for $(Y, N)$ to be an $\mathbf{A}$-stable pair for a monoid $M$.

Proposition 6.3. Let $M$ be a finite monoid. Suppose that $Y \in \operatorname{PL}_{\mathbf{A}}(M)$ and $W \leq \mathrm{PL}_{\mathbf{A}}(M)$ is a submonoid which is an internal $E$-chain such that:
(1) $\cup W=N$;
(2) $W \leq \operatorname{Stab}(Y)$.

Then $(Y, N)$ is an A-stable pair.
Proof. Let $\varphi: M \rightarrow A$ with $A \in \mathbf{A}$ be a relational morphism. Factor $\varphi=\alpha^{-1} \beta$ where $\alpha: R \rightarrow M$ is an onto homomorphism, $\beta: R \rightarrow A$ is a homomorphism and $R$ is finite. Let $\alpha_{*}: \mathrm{PL}_{\mathbf{A}}(R) \rightarrow \mathrm{PL}_{\mathbf{A}}(M)$ and $\beta_{*}: \mathrm{PL}_{\mathbf{A}}(R) \rightarrow \mathrm{PL}_{\mathbf{A}}(A)=A$ be the induced maps from Proposition 2.2. We shall use several times that if $X \beta_{*}=x$, then $X \subseteq x \beta^{-1}$. Since $W$ is absolute Type I, we can find, by Proposition 6.2, an absolute Type I submonoid $W^{\prime} \leq \mathrm{PL}_{\mathbf{A}}(R)$ with $W^{\prime} \alpha_{*}=W$. Then $W^{\prime \prime}=W^{\prime} \beta_{*}$ is absolute Type I and hence belongs to $\mathbf{E R} \cap \mathbf{A}$ (again by Proposition 6.2). Choose $a \in A$ with $Y \subseteq a \varphi^{-1}$ and choose $z^{\prime \prime}$ from the minimal ideal of $W^{\prime \prime}$. By definition of $W^{\prime \prime}$, there exists $Z^{\prime} \in W^{\prime}$ with $Z^{\prime} \beta_{*}=z^{\prime \prime}$. Setting $Z=Z^{\prime} \alpha_{*} \in W$, we have $Z=Z^{\prime} \alpha \subseteq z^{\prime \prime} \beta^{-1} \alpha=z^{\prime \prime} \varphi^{-1}$ and so, as $W \leq \operatorname{Stab}(Y)$,

$$
\begin{equation*}
Y=Y Z \subseteq a \varphi^{-1} z^{\prime \prime} \varphi^{-1} \subseteq\left(a z^{\prime \prime}\right) \varphi^{-1} \tag{6.1}
\end{equation*}
$$

Now Proposition 6.1 shows that $W^{\prime \prime} \subseteq \operatorname{Stab}\left(z^{\prime \prime}\right) \subseteq \operatorname{Stab}\left(a z^{\prime \prime}\right)$. So we are left with showing that $N \leq W^{\prime \prime} \varphi^{-1}$. Let $U \in W$. Then we can find $U^{\prime} \in W^{\prime}$ such that $U^{\prime} \alpha_{*}=U$. Set $U^{\prime} \beta_{*}=u^{\prime \prime} \in W^{\prime \prime}$. Then $U=U^{\prime} \alpha \subseteq u^{\prime \prime} \beta^{-1} \alpha=$ $u^{\prime \prime} \varphi^{-1}$. Thus $U \subseteq W^{\prime \prime} \varphi^{-1}$ and so we may conclude

$$
\begin{equation*}
N=\bigcup W \subseteq W^{\prime \prime} \varphi^{-1} \subseteq \operatorname{Stab}\left(a z^{\prime \prime}\right) \varphi^{-1} \tag{6.2}
\end{equation*}
$$

Combining (6.1) and (6.2) yields that $(Y, N)$ is an $\mathbf{A}$-stable pair.
We now prove the converse for maximal A-stable pairs; Henckell proves an apparently stronger formulation in [8].

Theorem 6.4. Suppose that $M$ is a finite monoid. Then the maximal Astable pairs of $M$ are the maximal pairs $(Y, N)$ such that $Y \in \mathrm{PL}_{\mathbf{A}}(M)$ and there exists a submonoid $W \leq \mathrm{PL}_{\mathbf{A}}(M)$ with $W$ an internal $E$-chain and:
(1) $\bigcup W=N$;
(2) $W \leq \operatorname{Stab}(Y)$.

Proof. By Proposition 6.3 any such pair $(Y, N)$ is A-stable. Conversely, suppose that $(Y, N)$ is a maximal $\mathbf{A}$-stable pair for $M$. Choose a finite generating set $X$ for $M$ and let $\rho_{\mathbf{A}}: M \rightarrow \widehat{F_{\mathbf{A}}}(X)$ be the canonical relational morphism. Let $f_{\mathbf{A}}: \widehat{F_{\mathbf{A}}}(X) \rightarrow \mathrm{PL}_{\mathbf{A}}(M)$ be the homomorphism from Lemma 4.4: so $f_{\mathbf{A}}=\rho_{\mathbf{A}}^{-1}$. Maximality and Theorem 2.6 implies there
exists $\gamma \in \widehat{F_{\mathbf{A}}}(X)$ such that $Y=\gamma \rho_{\mathbf{A}}^{-1}=\gamma f_{\mathbf{A}}$ and $N=\operatorname{Stab}(\gamma) \rho_{\mathbf{A}}^{-1}$. By Theorem 4.1, $\operatorname{Stab}(\gamma)$ is an internal L -chain. Then we see that

$$
W=\operatorname{Stab}(\gamma) f_{\mathbf{A}} \leq \operatorname{Stab}\left(\gamma f_{\mathbf{A}}\right)=\operatorname{Stab}(Y)
$$

is a submonoid of $\mathrm{PL}_{\mathbf{A}}(M)$ and an internal L -chain. Moreover, we have

$$
\bigcup W=\bigcup_{\beta \in \operatorname{Stab}(\gamma)} \beta f_{\mathbf{A}}=\bigcup_{\beta \in \operatorname{Stab}(\gamma)} \beta \rho_{\mathbf{A}}^{-1}=\operatorname{Stab}(\gamma) \rho_{\mathbf{A}}^{-1}=N .
$$

This completes the proof of the theorem.
Since $\mathrm{PL}_{\mathbf{A}}(M)$ is computable $[6,10]$, Theorem 6.4 admits as corollaries:
Corollary 6.5. Stable pairs are decidable for A. Equivalently, A-inevitability is decidable for labellings of graphs with a single vertex, with singletons on the edges.

Corollary 6.6. If $\mathbf{V}$ is a local pseudovariety with decidable membership, then $\mathbf{V} * \mathbf{A}$ is decidable.

## 7. A-triples

To compute the Krohn-Rhodes complexity of a monoid, we shall need some other notions, related to those we have been considering.

Definition 7.1 (V-triple). Let us call a triple $(A, B, C)$ of subsets of a finite monoid $M a \mathbf{V}$-triple if, for all relational morphisms $\varphi: M \rightarrow N$ with $N \in \mathbf{V}$, there exist $a, b, c \in N$ such that $A \subseteq a \varphi^{-1}, B \subseteq c \varphi^{-1}, C \subseteq c \varphi^{-1}$ and $a b c=a b$.

Equivalently, $(A, B, C)$ is a $\mathbf{V}$-triple if and only if the graph with two vertices $v_{1}, v_{2}$, an edge $e_{1}$ from $v_{1}$ to $v_{2}$ and a loop $e_{2}$ from $v_{2}$ to $v_{2}$ with $v_{1}$ labelled by $A, e_{1}$ by $B, v_{2}$ by $A B$ and $e_{2}$ by $C$ is $\mathbf{V}$-inevitable. Thus an analogue of Theorem 2.6 holds for $\mathbf{V}$-triples.

We are particularly interested in A-triples and so we begin by investigating solutions to equations of the from $x y z=x y$ in $\widehat{F_{\mathbf{A}}}(X)$. It turns out that the Henckell-Schützenberger expansion allows one to treat equations over $\widehat{F_{\mathbf{A}}}(X)$ in a similar way to equations over free monoids.
Proposition 7.2. Let $X$ be a finite set and let $\alpha, \beta, \gamma \in \widehat{F_{\mathbf{A}}}(X)$. Then $\alpha \beta \gamma=\alpha \beta$ if and only if one of the following three situations occur:
(1) $\beta \gamma=\beta$;
(2) there exists $\tau \in \widehat{F_{\mathbf{A}}}(X)$ such that $\alpha \beta \tau=\alpha$ and $\gamma=\tau \beta$;
(3) there exist $\sigma, \tau \in \widehat{F_{\mathbf{A}}}(X)$ and $i \geq 1$ such that $\alpha=\alpha \tau \sigma, \beta=(\tau \sigma)^{i} \tau$ and $\gamma=\sigma \tau$.

Proof. Clearly any of (1), (2) or (3) implies $\alpha \beta \gamma=\alpha \beta$. For the converse, if $\alpha$ or $\gamma$ are $\varepsilon$, we are in case (1). If $\beta=\varepsilon$, then we are in case (2) with $\tau=\gamma$. So we may assume that none of $\alpha, \beta$ and $\gamma$ are $\varepsilon$. Since $X$ is finite, we may write $\widehat{F_{\mathbf{A}}}(X)=\lim _{幺}{ }_{n \in \mathbb{N}} M_{n}$ with the $M_{n}$ finite $X$-generated
aperiodic monoids. Moreover, $\widetilde{M_{n}} \in \mathbf{A}$ for all $n$. Choose sequences of words $a_{n}, b_{n}, c_{n}$ from $X^{+}$such that $a_{n} \rightarrow \alpha, b_{n} \rightarrow \beta$ and $c_{n} \rightarrow \gamma$. By passing to subsequences, we may assume that $\left[a_{n}\right]_{\widetilde{M_{n}}}=[\alpha]_{\widetilde{M_{n}}},\left[b_{n}\right]_{\widetilde{M_{n}}}=[\beta]_{\widetilde{M_{n}}}$ and $\left[c_{n}\right]_{\widetilde{M_{n}}}=[\gamma]_{\widetilde{M_{n}}}$, for all $n$.

Then, for each $n$, we have the equality $\left[a_{n} b_{n} c_{n}\right]_{\widetilde{M_{n}}}=\left[a_{n} b_{n}\right]_{\widetilde{M_{n}}}$. Proposition 3.1 says that $a_{n} b_{n} c_{n}=a_{n}^{\prime} b_{n}^{\prime}$ with $\left[a_{n}^{\prime}\right]_{M_{n}}=\left[a_{n}\right]_{M_{n}}=[\alpha]_{M_{n}}$ and $\left[b_{n}^{\prime}\right]_{M_{n}}=\left[b_{n}\right]_{M_{n}}=[\beta]_{M_{n}}$. In particular, $a_{n}^{\prime} \rightarrow \alpha$ and $b_{n}^{\prime} \rightarrow \beta$.

For each $n$, there are three cases: $\left|a_{n}^{\prime}\right| \leq\left|a_{n}\right|,\left|a_{n}^{\prime}\right| \geq\left|a_{n} b_{n}\right|$ and finally $\left|a_{n}\right|<\left|a_{n}^{\prime}\right|<\left|a_{n} b_{n}\right|$. By passing to a subsequence, we may assume that the same case occurs for all $n$.

Suppose that $\left|a_{n}^{\prime}\right| \leq\left|a_{n}\right|$ for all $n$. Then, for each $n$, there exists $t_{n} \in X^{*}$ so that $a_{n}=a_{n}^{\prime} t_{n}, b_{n}^{\prime}=t_{n} b_{n} c_{n}$. By passing to a subsequence, we may assume that $t_{n} \rightarrow \tau \in \widehat{F_{\mathbf{A}}}(X)$. Then $\beta=\tau \beta \gamma$ and so we have $\beta=\tau^{\omega} \beta \gamma^{\omega}$. Thus $\beta \gamma=\tau^{\omega} \beta \gamma^{\omega} \gamma=\tau^{\omega} \beta \gamma^{\omega}=\beta$ and we are in case (1).

Next suppose that $\left|a_{n}^{\prime}\right| \geq\left|a_{n} b_{n}\right|$ for all $n$. Then, for each $n$, we can find $t_{n} \in X^{*}$ such that $a_{n}^{\prime}=a_{n} b_{n} t_{n}$ and $c_{n}=t_{n} b_{n}^{\prime}$. By passing to a subsequence, we may assume $t_{n} \rightarrow \tau \in \widehat{F_{\mathbf{A}}}(X)$. Then $\alpha=\alpha \beta \tau$ and $\gamma=\tau \beta$, and so we are in case (2).

Finally, suppose $\left|a_{n}\right|<\left|a_{n}^{\prime}\right|<\left|a_{n} b_{n}\right|$ for all $n$. Then, for each $n$, we can find $p_{n}, t_{n} \in X^{*}$ such that $a_{n}^{\prime}=a_{n} p_{n}, b_{n}^{\prime}=t_{n} c_{n}$ and $b_{n}=p_{n} t_{n}$. By extracting a subsequence, we may assume that $p_{n} \rightarrow \pi$ and $t_{n} \rightarrow \tau_{1}$ in $\widehat{F_{\mathbf{A}}}(X)$. Then we have in $\widehat{F_{\mathbf{A}}}(X)$ the equalities $\tau_{1} \gamma=\beta=\pi \tau_{1}$ and

$$
\begin{equation*}
\alpha=\alpha \pi . \tag{7.1}
\end{equation*}
$$

Define $\tau_{0}=\beta$ and suppose inductively that we have found $\tau_{i} \in \widehat{F_{\mathbf{A}}}(X)$, for $i \geq 1$, such that $\tau_{i} \gamma=\tau_{i-1}=\pi \tau_{i}$. Notice that a simple induction yields

$$
\begin{equation*}
\tau_{i} \gamma^{i}=\beta=\pi^{i} \tau_{i} \tag{7.2}
\end{equation*}
$$

Then we can choose sequences of words $P_{n}, T_{n}$ such that $P_{n} \rightarrow \pi, T_{n} \rightarrow \tau_{i}$ and $\left[P_{n}\right]_{\widetilde{M_{n}}}=[\pi]_{\widetilde{M_{n}}},\left[T_{n}\right]_{\widetilde{M_{n}}}=\left[\tau_{i}\right]_{\widehat{M_{n}}}$, all $n$. Then, for each $n$, we have

$$
\left[T_{n} c_{n}\right]_{\widehat{M_{n}}}=\left[\tau_{i} \gamma\right]_{\widehat{M_{n}}}=\left[\pi \tau_{i}\right]_{\widehat{M_{n}}}=\left[P_{n} T_{n}\right]_{\widetilde{M_{n}}} .
$$

Proposition 3.1 then shows $T_{n} c_{n}=P_{n}^{\prime} T_{n}^{\prime}$ where

$$
\left[P_{n}^{\prime}\right]_{M_{n}}=\left[P_{n}\right]_{M_{n}}=[\pi]_{M_{n}}, \quad\left[T_{n}^{\prime}\right]_{M_{n}}=\left[T_{n}\right]_{M_{n}}=\left[\tau_{i}\right]_{M_{n}} .
$$

In particular, we have $P_{n}^{\prime} \rightarrow \pi$ and $T_{n}^{\prime} \rightarrow \tau_{i}$.
For any $n$, there are two cases: $\left|P_{n}^{\prime}\right|<\left|T_{n}\right|$ and $\left|P_{n}^{\prime}\right| \geq\left|T_{n}\right|$. By passing to a subsequence, we may assume that the same case applies for all $n$. Suppose first that $\left|P_{n}^{\prime}\right|<\left|T_{n}\right|$ for all $n$. Then we can find $R_{n} \in X^{*}$ so that $T_{n}=P_{n}^{\prime} R_{n}$ and $T_{n}^{\prime}=R_{n} c_{n}$. Extracting a subsequence, we may assume that $R_{n}$ converges to some $\tau_{i+1}$ in $\widehat{F_{\mathbf{A}}}(X)$. Then $\pi \tau_{i+1}=\tau_{i}=\tau_{i+1} \gamma$, allowing us to continue the induction.

Next assume that $\left|P_{n}^{\prime}\right| \geq\left|T_{n}\right|$ for all $n$. Then $P_{n}^{\prime}=T_{n} S_{n}$ and $c_{n}=S_{n} T_{n}^{\prime}$ for some $S_{n} \in X^{*}$. By passing to a subsequence, we may assume that
$S_{n} \rightarrow \sigma$ in $\widehat{F_{\mathbf{A}}}(X)$. Then $\pi=\tau_{i} \sigma$ and $\gamma=\sigma \tau_{i}$. Therefore, by (7.1) and (7.2), we have the equalities

$$
\alpha=\alpha \tau_{i} \sigma, \quad \beta=\pi^{i} \tau_{i}=\left(\tau_{i} \sigma\right)^{i} \tau_{i}, \quad \gamma=\sigma \tau_{i},
$$

and so we are in case (3) and may stop.
Hence, either one of cases (1), (2) or (3) arises, or we can find an infinite sequence $\left\{\tau_{i}\right\}$ of elements of $\widehat{F_{\mathbf{A}}}(X)$ with $\beta=\tau_{i} \gamma^{i}$. By passing to a subsequence, we may assume that $\tau_{i} \rightarrow \tau \in \widehat{F_{\mathbf{A}}}(X)$. Since $\lim _{i \rightarrow \infty} \gamma^{i}=\gamma^{\omega}$, we obtain $\beta=\tau \gamma^{\omega}$ and hence $\beta \gamma=\tau \gamma^{\omega} \gamma=\tau \gamma^{\omega}=\beta$, so we are again in case (1). This completes the proof.

Corollary 7.3. Let $M$ be a finite monoid. Then the maximal A-triples are the maximal triples $(A, B, C) \in \mathrm{PL}_{\mathbf{A}}(M)^{3}$ such that one of the following occurs:
(1) $B C=B$;
(2) there exists $T \in \mathrm{PL}_{\mathbf{A}}(M)$ such that $A B T=A$ and $C=T B$;
(3) there exist $S, T \in \mathrm{PL}_{\mathbf{A}}(M)$ and $i \geq 1$ such that $A=A T S, B=$ $(T S)^{i} T$ and $C=S T$.

In particular, A-triples are decidable.
Proof. First we show that if $(A, B, C) \in \mathrm{PL}_{\mathbf{A}}(M)^{3}$ satisfies any of (1)-(3), then it is an A-triple. Let $\varphi: M \rightarrow N$ with $N \in \mathbf{A}$ be a relational morphism.

Suppose that (1) holds. Choose $a, b, c \in N$ with $A \subseteq a \varphi^{-1}, B \subseteq b \varphi^{-1}$ and $C \subseteq c \varphi^{-1}$. Then $B \subseteq b c^{\omega} \varphi^{-1}$ and $a\left(b c^{\omega}\right) c=a\left(b c^{\omega}\right)$. Thus $(A, B, C)$ is an A-triple.

Next assume that (2) holds. Choose $a, b, t \in N$ with $A \subseteq a \varphi^{-1}, B \subseteq b \varphi^{-1}$ and $T \subseteq t \varphi^{-1}$. Then we have $A \subseteq a(b t)^{\omega} \varphi^{-1}, C \subseteq t b \varphi^{-1}$ and the equality $\left[a(b t)^{\omega}\right] b(t b)=\left[a(b t)^{\omega}\right] b$, and so $(A, B, C)$ is an A-triple.

Finally, assume that (3) holds. Choose $a, s, t \in N$ with $A \subseteq a \varphi^{-1}$, $S \subseteq s \varphi^{-1}$ and $T \subseteq t \varphi^{-1}$. Then we have $A \subseteq a(t s)^{\omega} \varphi^{-1}, B \subseteq(t s)^{i} t \varphi^{-1}$ and $C \subseteq s t \varphi^{-1}$. Moreover, $\left[a(t s)^{\omega}\right]\left[(t s)^{i} t\right](s t)=\left[a(t s)^{\omega}\right]\left[(t s)^{i} t\right]$. So we see that in all cases $(A, B, C)$ is an $\mathbf{A}$-triple.

Next suppose that $(A, B, C)$ is a maximal $\mathbf{A}$-triple. Choose a generating set $X$ for $M$ and let $\rho_{\mathbf{A}}: M \rightarrow \widehat{F_{\mathbf{A}}}(X)$ be the canonical relational morphism. Let $f_{\mathbf{A}}: \widehat{F_{\mathbf{A}}}(X) \rightarrow \mathrm{PL}_{\mathbf{A}}(M)$ be the homomorphism from Lemma 4.4. By Theorem 2.6 and maximality, we can find $\alpha, \beta, \gamma \in \widehat{F_{\mathbf{A}}}(X)$ such that

$$
A=\alpha \rho_{\mathbf{A}}^{-1}=\alpha f_{\mathbf{A}}, \quad B=\beta \rho_{\mathbf{A}}^{-1}=\beta f_{\mathbf{A}}, \quad C=\gamma \rho_{\mathbf{A}}^{-1}=\gamma f_{\mathbf{A}} .
$$

and $\alpha \beta \gamma=\alpha \beta$. We analyze the situation according to the three cases of Proposition 7.2. If $\beta \gamma=\beta$, then

$$
B C=\beta f_{\mathbf{A}} \gamma f_{\mathbf{A}}=(\beta \gamma) f_{\mathbf{A}}=\beta f_{\mathbf{A}}=B
$$

and we are in case (1). If there exists $\tau \in \widehat{F_{\mathbf{A}}}(X)$ such that $\alpha \beta \tau=\alpha$ and $\gamma=\tau \beta$, then setting $T=\tau f_{\mathbf{A}}$, we have

$$
\begin{aligned}
& A=\alpha f_{\mathbf{A}}=(\alpha \beta \tau) f_{\mathbf{A}}=\alpha f_{\mathbf{A}} \beta f_{\mathbf{A}} \tau f_{\mathbf{A}}=A B T \\
& C=\gamma f_{\mathbf{A}}=(\tau \beta) f_{\mathbf{A}}=\tau f_{\mathbf{A}} \beta f_{\mathbf{A}}=T B
\end{aligned}
$$

and so we are in case (2).
Finally, if there exist $\sigma, \tau \in \widehat{F_{\mathbf{A}}}(X)$ and $i \geq 1$ such that $\alpha=\alpha \tau \sigma, \beta=$ $(\tau \sigma)^{i} \tau$ and $\gamma=\sigma \tau$, then setting $S=\sigma f_{\mathbf{A}}$ and $T=\tau f_{\mathbf{A}}$, we have

$$
\begin{aligned}
& A=\alpha f_{\mathbf{A}}=(\alpha \tau \sigma) f_{\mathbf{A}}=\alpha f_{\mathbf{A}} \tau f_{\mathbf{A}} \sigma f_{\mathbf{A}}=A T S \\
& B=\beta f_{\mathbf{A}}=\left((\tau \sigma)^{i} \tau\right) f_{\mathbf{A}}=\left(\tau f_{\mathbf{A}} \sigma f_{\mathbf{A}}\right)^{i} \tau f_{\mathbf{A}}=(T S)^{i} T \\
& C=\gamma f_{\mathbf{A}}=(\sigma \tau) f_{\mathbf{A}}=\sigma f_{\mathbf{A}} \tau f_{\mathbf{A}}=S T
\end{aligned}
$$

and hence we are in case (3). This completes the proof.

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