## Gröbner-Shirshov bases for dialgebras

#### L. A. Bokut\*

School of Mathematical Sciences, South China Normal University
Guangzhou 510631, P. R. China
Sobolev Institute of Mathematics, Russian Academy of Sciences
Siberian Branch, Novosibirsk 630090, Russia
Email: bokut@math.nsc.ru

## Yuqun Chen<sup>†‡</sup> and Cihua Liu

School of Mathematical Sciences, South China Normal University
Guangzhou 510631, P. R. China
Email: yqchen@scnu.edu.cn
langhua01duo@yahoo.com.cn

**Abstract:** In this paper, we define the Gröbner-Shirshov bases for a dialgebra. The composition-diamond lemma for dialgebras is given then. As a result, we obtain a Gröbner-Shirshov basis for the universal enveloping algebra of a Leibniz algebra.

**Key words:** dialgebra, Gröbner-Shirshov bases, composition-diamond lemma, Leibniz algebra

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### 1 Introduction

Recently, J.-L. Loday (1995, [10]) gave the definition of a new class of algebras, dialgebras, which is closely connected to his notion of Leibniz algebras (1993, [9]) and in the same way as associative algebras are connected to Lie algebras. In the manuscript [11], J.-L. Loday found a normal form of elements of a free dialgebra. Here we continue to study free dialgebras and prove the composition-diamond lemma for them. As it is well known, this kind of lemma is the cornerstone of the theory of Gröbner and Gröbner-Shirshov bases (see, for example, [5] and cited literature). In commutative-associative case, this lemma is equivalent to the Main Buchberger's Theorem ([6], [7]). For Lie and associative algebras, this is the Shirshov's lemma [12] (see also L.A. Bokut [3], [4] and G. Bergman [2]). As an application, we get another proof of the Poincare-Birkhoff-Witt theorem for Leibniz algebras, see M. Aymon, P.-P. Grivel [1] and P. Kolesnikov [8].

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<sup>&</sup>lt;sup>†</sup>Corresponding author.

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### 2 Preliminaries

**Definition 2.1** Let k be a field. A k-linear space D equipped with two bilinear multiplications  $\vdash$  and  $\dashv$  is called a dialgebra, if both  $\vdash$  and  $\dashv$  are associative and

$$a \dashv (b \vdash c) = a \dashv b \dashv c$$
  

$$(a \dashv b) \vdash c = a \vdash b \vdash c$$
  

$$a \vdash (b \dashv c) = (a \vdash b) \dashv c$$

for any  $a, b, c \in D$ .

**Definition 2.2** Let D be a dialgebra,  $B \subset D$ . Let us define diwords (dimonomials) of D in the set B by induction:

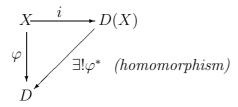
- (i)  $b = (b), b \in B$  is a diword in B of length |b| = 1.
- (ii) (u) is called a diword in B of length n, if  $(u) = ((v) \dashv (w))$  or  $(u) = ((v) \vdash (w))$ , where (v), (w) are diwords in B of length k, l respectively and k + l = n.

**Proposition 2.3** ([11]) Let D be a dialgebra and  $B \subset D$ . Any diword of D in the set B is equal to a diword in B of the form

$$(u) = b_{-m} \vdash \dots \vdash b_{-1} \vdash b_0 \dashv b_1 \dashv \dots \dashv b_n \tag{1}$$

where  $b_i \in B$ ,  $-m \le i \le n$ ,  $m \ge 0$ ,  $n \ge 0$ . Any bracketing of the right side of (1) gives the same result.  $\square$ 

**Definition 2.4** Let X be a set. A free dialgebra D(X) generated by X over k is defined in a usual way by the following commutative diagram:



where D is any dialgebra.

In [11], a construction of a free dialgebra is given.

**Proposition 2.5** ([11]) Let D(X) be free dialgebra generated by X over k. Any diword in X is equal to the unique diword in X of the form

$$[u] = x_{-m} \vdash \dots \vdash x_{-1} \vdash x_0 \dashv x_1 \dashv \dots \dashv x_n = x_{-m} \cdots x_{-1} \dot{x_0} x_1 \cdots x_n \tag{2}$$

where  $x_i \in X$ ,  $m \geq 0$ ,  $n \geq 0$ . We call [u] a normal diword (in X) with the associative word  $u, u \in X^*$ . Clearly, if [u] = [v], then u = v. In (2),  $x_0$  is called the center of the normal diword [u]. Let [u], [v] be two normal diwords, then  $[u] \vdash [v]$  is the normal diword [uv] with the center at the center of [v]. Accordingly,  $[u] \dashv [v]$  is the normal diword [uv] with the center at the center of [u].  $\square$ 

#### Example 2.6

$$(x_{-1} \vdash x_0 \dashv x_1) \vdash (y_{-1} \vdash y_0 \dashv y_1) = x_{-1} \vdash x_0 \vdash x_1 \vdash y_{-1} \vdash y_0 \dashv y_1,$$
$$(x_{-1} \vdash x_0 \dashv x_1) \dashv (y_{-1} \vdash y_0 \dashv y_1) = x_{-1} \vdash x_0 \dashv x_1 \dashv y_{-1} \dashv y_0 \dashv y_1. \quad \Box$$

**Definition 2.7** A k-linear space L equipped with bilinear multiplication [,] is called a Leibniz algebra if for any  $a, b, c \in L$ ,

$$[[a, b], c] = [[a, c], b] + [a, [b, c]]$$

i.e., the Jacobi identity is valid in L.

It is clear that if  $(D, \dashv, \vdash)$  is a dialgebra then  $D^{(-)} = (D, [,])$  is a Leibniz algebra, where  $[a, b] = a \dashv b - b \vdash a$  for any  $a, b \in D$ .

## 3 Composition-Diamond lemma for dialgebras

Let X be a well ordered set, D(X) the free dialgebra over k,  $X^*$  the free monoid generated by X and  $[X^*]$  the set of normal diwords in X. Let us define deg-lex order on  $[X^*]$  in the following way: for any  $[u], [v] \in [X^*]$ ,

$$[u] < [v] \iff wt([u]) < wt([v])$$
 lexicographicaly,

where

$$wt([u]) = (n + m + 1, m, x_{-m}, \dots, x_0, \dots, x_n)$$

if  $[u] = x_{-m} \cdots x_{-1} \dot{x_0} x_1 \cdots x_n$ . It is easy to see that the order < is monomial in the sense:

$$[u] < [v] \Longrightarrow x \vdash [u] < x \vdash [v], \ [u] \dashv x < [v] \dashv x, \ \text{for any} \ \ x \in X.$$

Any polynomial  $f \in D(X)$  has the form

$$f = \sum_{[u] \in [X^*]} f([u])[u] = \alpha[\overline{f}] + \sum_{i} \alpha_i[u_i],$$

where  $[\overline{f}]$ ,  $[u_i]$  are normal diwords in X,  $[\overline{f}] > [u_i]$ ,  $\alpha$ ,  $\alpha_i$ ,  $f([u]) \in k$ . We call  $[\overline{f}]$  the leading term of f. Denote by supp f the set  $\{[u]|f([u]) \neq 0\}$  and deg(f) by  $|[\overline{f}]|$ . f is called monic if  $\alpha = 1$ . f is called left (right) normed if  $f = \sum \alpha_i u_i \dot{x}_i$  ( $f = \sum \alpha_i \dot{x}_i u_i$ ), where each  $\alpha_i \in k$ ,  $x_i \in X$  and  $u_i \in X^*$ . The same terminology will be used for normal diwords.

If [u], [v] are both left normed or both right normed, then it is clear that for any  $w \in [X^*]$ ,

$$[u] < [v] \Longrightarrow [u] \vdash w < [v] \vdash w, \ w \vdash [u] < w \vdash [v], [u] \dashv w < [v] \dashv w, \ w \dashv [u] < w \dashv [v].$$

Let  $S \subset D(X)$ . By an S-diword g we will mean g is a diword in  $\{X \cup S\}$  with only one occurrence of  $s \in S$ . If this is the case and g = (asb) for some  $a, b \in X^*$  and  $s \in S$ , we also call g an s-diword.

From Proposition 2.3 it follows easily that any S-diword is equal to

$$[asb] = x_{-m} \vdash \dots \vdash x_{-1} \vdash x_0 \dashv x_1 \dashv \dots \dashv x_n|_{x_k \mapsto s} \tag{3}$$

where  $-m \le k \le n$ ,  $x_k \in X$ ,  $s \in S$ . To be more precise,  $[asb] = [a\dot{s}b]$  if k = 0;  $[asb] = [asb_1\dot{x_0}b_2]$  if k < 0 and  $[asb] = [a_1\dot{x_0}a_2sb]$  if k > 0. Note that any bracketing of [asb] gives the same result, for example,  $[asb] = [(a_1a_2)sb] = [a_1(a_2s)b]$  if  $a = a_1a_2$ . If the center of the s-diword [asb] is in a, then we denote by  $[\dot{a}sb] = [a_1\dot{x_0}a_2sb]$ . Similarly,  $[as\dot{b}] = [asb_1\dot{x_0}b_2]$  (of course, some  $a_i, b_i$  may be empty).

**Definition 3.1** The S-diword (3) is called a normal S-diword if one of the following conditions holds:

- (i) k = 0.
- (ii) k < 0 and s is left normed.
- (iii) k > 0 and s is right normed.

We call a normal s-diword [asb] a left (right) normed s-diword, if both s and [asb] are left (right) normed. In particulary, s is a left (right) normed s-diword, if s is left (right) normed polynomial.

The following lemma follows from the above properties of the order of normal diwords.

**Lemma 3.2** For a normal S-diword [asb], the leading term of [asb] is equal to  $[a[\overline{s}]b]$ , that is,  $\overline{[asb]} = [a[\overline{s}]b]$ . More specifically, if

$$[asb] = x_{-m} \vdash \cdots \vdash x_{-1} \vdash x_0 \dashv x_1 \dashv \cdots \dashv x_n|_{x_k \mapsto s},$$

then

$$\overline{x_{-m} \vdash \cdots \vdash x_{-1} \vdash s \dashv x_1 \dashv \cdots \dashv x_n} = x_{-m} \vdash \cdots \vdash x_{-1} \vdash [\overline{s}] \dashv x_1 \dashv \cdots \dashv x_n$$

$$\overline{x_{-m} \vdash \cdots \vdash s \vdash \cdots \vdash x_0 \dashv \cdots \dashv x_n} = x_{-m} \vdash \cdots \vdash [\overline{s}] \vdash \cdots \vdash x_0 \dashv \cdots \dashv x_n$$

$$\overline{x_{-m} \vdash \cdots \vdash x_0 \dashv \cdots \dashv s \dashv \cdots \dashv x_n} = x_{-m} \vdash \cdots \vdash x_0 \dashv \cdots \dashv [\overline{s}] \dashv \cdots \dashv x_n \quad \Box$$

For convenience, we denote  $[a[\overline{s}]b]$  by  $[a\overline{s}b]$  for a normal S-diword [asb]. Now, we define compositions of dipolynomials in D(X).

**Definition 3.3** Let the order < be as before and  $f, g \in D(X)$  with f, g monic.

1) Composition of left (right) multiplication.

Let f be a not right normed polynomial and  $x \in X$ . Then  $x \dashv f$  is called the composition of left multiplication. Clearly,  $x \dashv f$  is a right normed polynomial (or 0).

Let f be a not left normed polynomial and  $x \in X$ . Then  $f \vdash x$  is called the composition of right multiplication. Clearly,  $f \vdash x$  is a left normed polynomial (or  $\theta$ ).

2) Composition of including.

Let

$$[w] = [\overline{f}] = [a\overline{g}b],$$

where [agb] is a normal g-diword. Then

$$(f,g)_{[w]} = f - [agb]$$

is called the composition of including. The transformation  $f \mapsto f - [agb]$  is called the elimination of leading diword (ELW) of g in f.

3) Composition of intersection.

Let

$$[w] = [\overline{f}b] = [a\overline{g}], |\overline{f}| + |\overline{g}| > |w|,$$

where [fb] is a normal f-diword and [ag] a normal g-diword. Then

$$(f,g)_{[w]} = [fb] - [ag]$$

is called the composition of intersection.

**Remark** In the Definition 3.3, for the case of 2) or 3), we have  $\overline{(f,g)_{[w]}} < [w]$ . For the case of 1),  $deg(x \dashv f) \leq deg(f) + 1$  and  $deg(f \vdash x) \leq deg(f) + 1$ .

**Definition 3.4** Let the order < be as before,  $S \subset D(X)$  a monic set and  $f, g \in S$ .

1) Let  $x \dashv f$  be a composition of left multiplication. Then  $x \dashv f$  is called trivial modulo S, denoted by  $x \dashv f \equiv 0 \mod(S)$ , if

$$x \dashv f = \sum \alpha_i [a_i s_i b_i],$$

where each  $\alpha_i \in k$ ,  $a_i, b_i \in [X^*]$ ,  $s_i \in S$ ,  $[a_i s_i b_i]$  right normed  $s_i$ -diword and  $|[a_i \overline{s_i} b_i]| \leq deg(x \dashv f)$ .

Let  $f \vdash x$  be a composition of right multiplication. Then  $f \vdash x$  is called trivial modulo S, denoted by  $f \vdash x \equiv 0 \mod(S)$ , if

$$f \vdash x = \sum \alpha_i [a_i s_i b_i],$$

where each  $\alpha_i \in k$ ,  $a_i, b_i \in [X^*]$ ,  $s_i \in S$ ,  $[a_i s_i b_i]$  left normed  $s_i$ -diword and  $|[a_i \overline{s_i} b_i]| \le deg(f \vdash x)$ .

2) Composition  $(f,g)_{[w]}$  of including (intersection) is called trivial modulo (S,[w]), denoted by  $(f,g)_{[w]} \equiv 0 \mod(S,[w])$ , if

$$(f,g)_{[w]} = \sum \alpha_i [a_i s_i b_i],$$

where each  $\alpha_i \in k$ ,  $a_i, b_i \in [X^*]$ ,  $s_i \in S$ ,  $[a_i s_i b_i]$  normal  $s_i$ -diword,  $[a_i \overline{s_i} b_i] < [w]$  and each  $[a_i s_i b_i]$  is right (left) normed  $s_i$ -diword whenever both f and [agb] ([fb] and [ag]) are right (left) normed S-diwords.

The following proposition is useful when one checks the compositions of left and right multiplications.

**Proposition 3.5** Let the order < be as before,  $S \subset D(X)$  a monic set and  $f \in S$ . Let  $x \dashv f$  be a composition of left multiplication. Then  $x \dashv f \equiv 0 \mod(S)$  if and only if

$$x \dashv f = \sum \alpha_i [a_i s_i b_i],$$

where each  $\alpha_i \in k$ ,  $a_i, b_i \in X^*$ ,  $s_i \in S$  is right normed,  $[a_i s_i b_i] = [\dot{a}_i s_i b_i]$  and  $|[a_i \overline{s_i} b_i]| \leq deg(x \dashv f)$ .

Accordingly, for the composition of right multiplication, we have a similar conclusion.

**Proof** Assume that  $x \dashv f = \sum \alpha_i [a_i s_i b_i]$ , where each  $\alpha_i \in k$ ,  $a_i, b_i \in [X^*]$ ,  $[a_i s_i b_i] = [a_i s_i b_i]$ ,  $s_i \in S$  right normed and  $|[a_i \overline{s_i} b_i]| \leq deg(x \dashv f)$ . Then, we have the expression

$$x \dashv f = [\dot{x}f] = \sum_{I_1} \alpha_p [\dot{x_p} a_p s_p b_p] + \sum_{I_2} \beta_q [a_q \dot{x_q} a_q' s_q b_q],$$

where each  $\alpha_p, \beta_q \in k$ ,  $x_p, x_q \in X$ ,  $a_p, a_q, a_q', b_p, b_q \in X^*$ ,  $a_q \neq 1$ ,  $s_p, s_q \in S$  are right normed. From this it follows that  $\sum_{I_2} \beta_q [a_q \dot{x}_q a_q' s_q b_p] = 0$ . Now, the results follow.  $\square$ 

**Definition 3.6** Let  $S \subset D(X)$  be a monic set and the order < as before. We call the set S a Gröbner-Shirshov set (basis) in D(X) if any composition of polynomials in S is trivial modulo S (and [w]).

The following two lemmas play key role in the proof of Theorem 3.9.

**Lemma 3.7** Let  $S \subset D(X)$  and [asb] an S-diword. Assume that each composition of right or left multiplication is trivial modulo S. Then, [asb] has a presentation:

$$[asb] = \sum \alpha_i [a_i s_i b_i],$$

where each  $\alpha_i \in K$ ,  $s_i \in S$ ,  $a_i, b_i \in [X^*]$  and each  $[a_i s_i b_i]$  is normal  $s_i$ -diword.

**Proof** Following Proposition 2.3, we assume that

$$[asb] = x_{-m} \vdash \cdots \vdash x_{-1} \vdash x_0 \dashv x_1 \dashv \cdots \dashv x_n|_{x_{\nu} \mapsto s}.$$

There are three cases to consider.

Case 1. k = 0. Then [asb] is a normal S-diword.

Case 2. k < 0. Then  $[asb] = a \vdash (s \vdash x_{k+1}) \vdash b, k < -1$  or  $[asb] = a \vdash (s \vdash x_0) \dashv b$ . If s is left normed then [asb] is a normal S-diword. If s is not left normed then for the composition  $s \vdash x_{k+1}$  (k < 0) of right multiplication, we have

$$s \vdash x_{k+1} = \sum \alpha_i [a_i s_i b_i],$$

where each  $\alpha_i \in k$ ,  $a_i, b_i \in [X^*]$ ,  $s_i \in S$  and  $[a_i s_i b_i]$  is left normed  $s_i$ -diword. Then

$$[asb] = \sum \alpha_i (a \vdash [a_i s_i b_i] \vdash b)$$

or

$$[asb] = \sum \alpha_i (a \vdash [a_i s_i b_i] \dashv b)$$

is a linear combination of normal S-diwords.

Case 3. k > 0 is similar to the Case 2.  $\square$ 

**Lemma 3.8** Let  $S \subset D(X)$  and each composition  $(f,g)_{[w]}$  in S of including (intersection) trivial modulo (S,[w]). Let  $[a_1s_1b_1]$  and  $[a_2s_2b_2]$  be normal S-diwords such that  $[w] = [a_1\bar{s}_1b_1] = [a_2\bar{s}_2b_2]$ . Then,

$$[a_1s_1b_1] \equiv [a_2s_2b_2] \mod(S, [w]).$$

**Proof** Because  $a_1\bar{s_1}b_1=a_2\bar{s_2}b_2$  as words, there are three cases to consider.

Case 1. Subwords  $\overline{s_1}$ ,  $\overline{s_2}$  have empty intersection. Assume, for example, that  $b_1 = b\overline{s_2}b_2$  and  $a_2 = a_1\overline{s_1}b$ . Because any normal S-diword may be bracketing in any way, we have

$$[a_2s_2b_2] - [a_1s_1b_1] = (a_1s_1(b(s_2 - [\overline{s_2}])b_2)) - ((a_1(s_1 - [\overline{s_1}])b)s_2b_2).$$

For any  $t \in supp(s_2 - \overline{s_2})$   $(t \in supp(s_1 - \overline{s_1}))$ , we prove that  $(a_1s_1btb_2)$   $((a_1tbs_2b_2))$  is a normal  $s_1$ -diword  $(s_2$ -diword ). There are five cases to consider.

- 1.1  $[w] = [\dot{a_1}\overline{s_1}b\overline{s_2}b_2];$
- 1.2  $[w] = [a_1 \overline{s_1} b \overline{s_2} b_2];$
- $1.3 [w] = [a_1 \overline{s_1} \dot{b} \overline{s_2} b_2];$
- 1.4  $[w] = [a_1 \overline{s_1} b \cdot \overline{s_2} b_2];$
- $1.5 [w] = [a_1 \overline{s_1} b \overline{s_2} b_2].$

For 1.1, since  $[a_1s_1b_1]$  and  $[a_2s_2b_2]$  are normal S-diwords, both  $s_1$  and  $s_2$  are right normed by the definition, in particular, t is right normed. It follows that  $(a_1s_1btb_2) = [\dot{a_1}s_1btb_2]$  is a normal  $s_1$ -diword.

For 1.2, it is clear that  $(a_1s_1btb_2)$  is a normal  $s_1$ -diword and t is right normed.

For 1.3, 1.4 and 1.5, since  $[a_1s_1b_1]$  is normal  $s_1$ -diword,  $s_1$  is left normed by the definition, which implies that  $(a_1s_1btb_2)$  is a normal  $s_1$ -diword. Moreover, t is right normed, if 1.3, and left normed, if 1.5.

Thus, for all cases, we have  $\overline{[a_1s_1btb_2]} = [a_1\overline{s_1}btb_2] < [a_1\overline{s_1}b\overline{s_2}b_2] = [w].$ 

Similarly, for any  $t \in supp(s_1 - \overline{s_1})$ ,  $(a_1tbs_2b_2)$  is a normal  $s_2$ -diword and  $[a_1tb\overline{s_2}b_2] < [w]$ .

Case 2. Subwords  $\overline{s_1}$  and  $\overline{s_2}$  have non-empty intersection c. Assume, for example, that  $b_1 = bb_2$ ,  $a_2 = a_1a$ ,  $w_1 = \overline{s_1}b = a\overline{s_2} = acb$ .

There are following five cases to consider:

- $2.1 \ [w] = [\dot{a_1} \overline{s_1} bb_2];$
- $2.2 \ [w] = [a_1 \overline{s_1} b \dot{b_2}];$
- $2.3 [w] = [a_1 \dot{a}cbb_2];$
- $2.4 [w] = [a_1 a\dot{c}bb_2];$

 $2.5 [w] = [a_1 ac\dot{b}b_2].$ 

Then

$$[a_2s_2b_2] - [a_1s_1b_1] = (a_1([as_2] - [s_1b])b_2) = (a_1(s_1, s_2)_{[w_1]}b_2),$$

where  $[w_1] = [acb]$  is as follows:

- $2.1 [w_1]$  is right normed;
- $2.2 [w_1]$  is left normed;
- $2.3 [w_1] = [\dot{a}cb];$
- $2.4 [w_1] = [a\dot{c}b];$
- $2.5 [w_1] = [ac\dot{b}].$

Since S is a Gröbner-Shirshov basis, there exist  $\beta_j \in k$ ,  $u_j, v_j \in [X^*]$ ,  $s_j \in S$  such that  $[s_1b] - [as_2] = \sum_j \beta_j [u_j s_j v_j]$ , where each  $[u_j s_j v_j]$  is normal S-diword and  $[u_j \overline{s_j} v_j] < [w_1] = [acb]$ . Therefore,

$$[a_2s_2b_2] - [a_1s_1b_1] = \sum_j \beta_j(a_1[u_js_jv_j]b_2).$$

Now, we prove that each  $(a_1[u_js_jv_j]b_2)$  is normal  $s_j$ -diword and  $\overline{(a_1[u_js_jv_j]b_2)} < [w] = [a_1\overline{s_1}bb_2].$ 

For 2.1, since  $[\dot{a_1}s_1bb_2]$  and  $[\dot{a_1}as_2b_2]$  are normal S-diwords, both  $[s_1b]$  and  $[as_2]$  are right normed S-diwords. Then, by the definition, each  $[u_js_jv_j]$  is right normed S-diword, and so each  $(a_1[u_js_jv_j]b_2) = [\dot{a_1}u_js_jv_jb_2]$  is a normal S-diword.

For 2.2, both  $[s_1b]$  and  $[as_2]$  must be left normed S-diwords. Then, by the definition, each  $[u_js_jv_j]$  is left normed S-diword, and so each  $(a_1[u_js_jv_j]b_2) = [a_1u_js_jv_j\dot{b}_2]$  is a normal S-diword.

For 2.3, 2.4 or 2.5, by noting that  $(a_1[u_js_jv_j]b_2) = ((a_1) \vdash [u_js_jv_j] \dashv (b_2))$  and  $[u_js_jv_j]$  is normal S-diword,  $(a_1[u_js_jv_j]b_2)$  is also normal S-diword.

Now, for all cases, we have  $\overline{[a_1u_js_jv_jb_2]} = [a_1u_j\overline{s_j}v_jb_2] < [w] = [a_1acbb_2].$ 

Case 3. One of the subwords  $\overline{s_1}$  and  $\overline{s_2}$  contains another as a subword. Assume, for example, that  $b_2 = bb_1$ ,  $a_2 = a_1a$ ,  $w_1 = \overline{s_1} = a\overline{s_2}b$ .

Again there are following five cases to consider:

- $2.1 [w] = [\dot{a_1}a\overline{s_2}bb_1];$
- $2.2 [w] = [a_1 a \overline{s_2} b \dot{b_1}];$
- $2.3 [w] = [a_1 \dot{a} \overline{s_2} b b_1];$
- $2.4 \ [w] = [a_1 a \frac{\dot{s}_2}{s_2} b b_1];$
- $2.5 [w] = [a_1 a \overline{s_2} \dot{b} b_1].$

Then

$$[a_1s_1b_1] - [a_2s_2b_2] = (a_1(s_1 - as_2b)b_1) = (a_1(s_1, s_2)_{[w_1]}b_1)$$

It is similar to the proof of the Case 2, that we have  $[a_1s_1b_1] \equiv [a_2s_2b_2] \mod(S, [w])$ .

The following theorem is the main result.

**Theorem 3.9** (Composition-Diamond Lemma) Let  $S \subset D(X)$  be a monic set and the order < as before. Then  $(i) \Rightarrow (ii) \Leftrightarrow (ii)' \Leftrightarrow (iii) \Rightarrow (iv)$ , where

- (i) S is a Gröbner-Shirshov basis.
- (ii) For any  $f \in D(X)$ ,  $0 \neq f \in Id(S) \Rightarrow \overline{f} = [a\overline{s}b]$  for some  $s \in S$ ,  $a, b \in [X^*]$  and [asb] a normal S-diword.
- (ii)' For any  $f \in D(X)$ , if  $0 \neq f \in Id(S)$ , then  $f = \alpha_1[a_1s_1b_1] + \alpha_2[a_2s_2b_2] + \cdots + \alpha_n[a_ns_nb_n] \quad with \quad [a_1\overline{s_1}b_1] > [a_2\overline{s_2}b_2] > \cdots > [a_n\overline{s_n}b_n],$ where  $[a_is_ib_i]$  is normal S-diword,  $i = 1, 2, \dots, n$ .
- (iii) The set  $Irr(S) = \{u \in [X^*] | u \neq [a\overline{s}b], s \in S, a, b \in [X^*], [asb] \text{ is normal S-diword} \}$  is a linear basis of the dialgebra D(X|S).
- (iv) For each composition  $(f,g)_{[w]}$  of including (intersection), we have

$$(f,g)_{[w]} = \sum \alpha_i [a_i s_i b_i],$$

where each  $\alpha_i \in k$ ,  $a_i, b_i \in [X^*]$ ,  $s_i \in S$ ,  $[a_i s_i b_i]$  normal S-diword and  $[a_i \overline{s_i} b_i] < [w]$ .

**Proof**  $(i) \Rightarrow (ii)$ . Let S be a Gröbner-Shirshov basis and  $0 \neq f \in Id(S)$ . We can assume, by Lemma 3.7, that

$$f = \sum_{i=1}^{n} \alpha_i [a_i s_i b_i],$$

where each  $\alpha_i \in k$ ,  $a_i, b_i \in [X^*]$ ,  $s_i \in S$  and  $[a_i s_i b_i]$  normal S-diword. Let

$$[w_i] = [a_i \overline{s_i} b_i], [w_1] = [w_2] = \dots = [w_l] > [w_{l+1}] \ge \dots$$

We will use the induction on l and  $[w_1]$  to prove that  $\overline{f} = [a\overline{s}b]$ , for some  $s \in S$  and  $a, b \in [X^*]$ . If l = 1, then  $\overline{f} = \overline{[a_1s_1b_1]} = \overline{[a_1\overline{s_1}b_1]}$  and hence the result holds. Assume that  $l \geq 2$ . Then, by Lemma 3.8, we have  $\overline{[a_1s_1b_1]} \equiv \overline{[a_2s_2b_2]} \mod(S, \overline{[w_1]})$ .

Thus, if  $\alpha_1 + \alpha_2 \neq 0$  or l > 2, then the result holds. For the case  $\alpha_1 + \alpha_2 = 0$  and l = 2, we use the induction on  $[w_1]$ . Now, the result follows.

 $(ii) \Rightarrow (ii)'$ . Assume (ii) and  $f \in Id(S)$ . Let  $f = \alpha_1 \overline{f} + \sum_{[u_i] < \overline{f}} \alpha_i [u_i]$ . Then, by (ii),  $\overline{f} = [a_1 \overline{s_1} b_1]$ , where  $[a_1 s_1 b_1]$  is a normal S-diword. Therefore,

$$f_1 = f - \alpha_1[a_1s_1b_1], \overline{f_1} < \overline{f}, f_1 \in Id(S).$$

Now, by using induction on  $\overline{f}$ , we have (ii)'.

- $(ii)' \Rightarrow (ii)$ . This part is clear.
- $(ii)' \Rightarrow (iii)$ . Assume (ii)'. We firstly prove that, for any  $h \in D(X)$ , we have

$$h = \sum_{I_1} \alpha_i [u_i] + \sum_{I_2} \beta_j [a_j s_j b_j]$$
 (4)

where  $[u_i] \in Irr(S)$ ,  $i \in I_1, [a_j s_j b_j]$  normal S-diwords,  $j \in I_2$ .

Let  $h = \alpha_1 \overline{h} + \cdots$ . We use the induction on  $\overline{h}$ .

If  $\overline{h} \in Irr(S)$ , then take  $[u_1] = \overline{h}$  and  $h_1 = h - \alpha_1[u_1]$ . Clearly,  $\overline{h_1} < \overline{h}$ .

If  $\overline{h} \not\in Irr(S)$ , then  $\overline{h} = [a_1\overline{s_1}b_1]$  with  $[a_1s_1b_1]$  a normal S-diword. Let  $h_1 = h - \beta_1[a_1s_1b_1]$ . Then  $\overline{h_1} < \overline{h}$ .

Suppose that  $0 \neq \sum \alpha_i[u_i] = \sum \beta_j[a_js_jb_j]$ , where  $[u_1] > [u_2] > \cdots$ ,  $[u_i] \in Irr(S)$  and  $[a_1\overline{s_1}b_1] > [a_2\overline{s_2}b_2] > \cdots$ . Then,  $[u_1] = [a_1\overline{s_1}b_1]$ , a contradiction.

Now, (iii) follows.

 $(iii) \Rightarrow (ii) \ and \ (iv)$ . Assume (iii). For any  $0 \neq f \in Id(S)$ ,  $\bar{f} \notin Irr(S)$  implies that  $\bar{f} = [a\bar{s}b]$ , where [asb] is a normal S-diword. This shows (ii).

By noting that  $(f,g)_{[w]} \in Id(S)$  and by using (4) and ELW, we have

$$(f,g)_{[w]} = \sum \alpha_i [a_i s_i b_i]$$

where each  $\alpha_i \in k$ ,  $a_i, b_i \in [X^*]$ ,  $s_i \in S$ ,  $[a_i s_i b_i]$  normal S-diword and  $[a_i \overline{s_i} b_i] < [w]$ .

## 4 Applications

Now, by using Theorem 3.9, we obtain a Gröbner-Shirshov basis for the universal enveloping algebra of a Leibniz algebra.

**Theorem 4.1** Let  $\mathcal{L}$  be a Leibniz algebra over a field k with the product  $\{,\}$ . Let  $\mathcal{L}_0$  be the subspace of  $\mathcal{L}$  generated by the set  $\{\{a,a\},\{a,b\}+\{b,a\}\mid a,b\in\mathcal{L}\}$ . Let  $\{x_i|i\in I_0\}$  be a basis of  $\mathcal{L}_0$  and  $X=\{x_i|i\in I\}$  a linearly ordered basis of  $\mathcal{L}$  such that  $I_0\subseteq I$ . Let  $D(X|x_i\dashv x_j-x_j\vdash x_i-\{x_i,x_j\})$  be the dialgebra and the order  $\{x_i\mid a,b\}$  as before. Then

(i)  $D\langle X|x_i \dashv x_j - x_j \vdash x_i - \{x_i, x_j\}\rangle = D(X|S)$ , where S consists of the following polynomials:

1. 
$$f_{ii} = x_i \vdash x_i - x_i \dashv x_j + \{x_i, x_j\}$$
  $(i, j \in I)$ 

2. 
$$f_{ji\vdash t} = x_j \vdash x_i \vdash x_t - x_i \vdash x_j \vdash x_t + \{x_i, x_j\} \vdash x_t$$
  $(i, j, t \in I, j > i)$ 

3. 
$$h_{i_0 \vdash t} = x_{i_0} \vdash x_t$$
  $(i_0 \in I_0, \ t \in I)$ 

4. 
$$f_{t \dashv ji} = x_t \dashv x_j \dashv x_i - x_t \dashv x_i \dashv x_j + x_t \dashv \{x_i, x_j\}$$
  $(i, j, t \in I, j > i)$ 

5. 
$$h_{t \dashv i_0} = x_t \dashv x_{i_0}$$
  $(i_0 \in I_0, \ t \in I)$ 

- (ii) S is a Gröbner-Shirshov basis.
- (iii) The set

$$\{x_j \dashv x_{i_1} \dashv \dots \dashv x_{i_k} \mid j \in I, i_p \in I - I_0, \ 1 \le p \le k, \ i_1 \le \dots \le i_k, \ k \ge 0\}$$

is a linear basis of the universal enveloping algebra  $U(\mathcal{L}) = D(X|S)$ . In particular,  $\mathcal{L}$  can be embedded into  $U(\mathcal{L})$ .

#### **Proof** (i) By using the following

$$f_{ji\vdash t} = f_{ji} \vdash x_t \text{ and } f_{ji} \vdash x_t + f_{ij} \vdash x_t = (\{x_i, x_j\} + \{x_j, x_i\}) \vdash x_t,$$

we have 2 and 3 are in  $Id(f_{ii})$ . By symmetry, 4 and 5 are in  $Id(f_{ii})$ . This shows (i).

(ii) We will prove that all compositions in S are trivial modulo S. We denote by  $(i \wedge j)$  the composition of the polynomials of type i and type j. For convenience, we extend linearly the functions  $f_{ji}$ ,  $f_{ji\vdash t}$ ,  $f_{t\dashv ji}$ ,  $h_{i_0\vdash t}$  and  $h_{t\dashv i_0}$  to  $f_{j\{p,q\}}$   $(f_{\{p,q\}i})$ ,  $f_{ji\vdash \{p,q\}}$  and  $h_{\{p,q\}\dashv i_0}$ , etc respectively, where, for example, if  $\{x_p, x_q\} = \sum \alpha_{pq}^s x_s$ , then

$$\begin{split} f_{j\{p,q\}} &= x_j \vdash \{x_p, x_q\} - \{x_p, x_q\} \dashv x_j + \{\{x_p, x_q\}, x_j\} = \sum \alpha_{pq}^s f_{js}, \\ f_{ji \vdash \{p,q\}} &= \sum \alpha_{pq}^s (x_j \vdash x_i \vdash x_s - x_i \vdash x_j \vdash x_s + \{x_i, x_j\} \vdash x_s) = f_{ji} \vdash \{x_p, x_q\}, \\ h_{\{p,q\} \dashv i_0} &= \sum \alpha_{pq}^s h_{s \dashv i_0}. \end{split}$$

By using the Jacobi identity in  $\mathcal{L}$ , for any  $a, b, c \in \mathcal{L}$ ,

$$\{\{a,b\},c\} = \{a,\{b,c\}\} + \{\{a,c\},b\} \tag{5}$$

we have

$${a, {b,b}} = 0$$
 and  ${a, {b,c} + {c,b}} = 0$ 

and in particular, for any  $i_0 \in I_0$ ,  $j \in I$ ,

$$\{x_i, x_{i_0}\} = 0 \tag{6}$$

and

$$\{x_{i_0}, x_j\} \in \mathcal{L}_0 \tag{7}$$

which implies that  $\mathcal{L}_0$  is an ideal of  $\mathcal{L}$ . Clearly,  $\mathcal{L}/\mathcal{L}_0$  is a Lie algebra.

Since  $\{x_{i_0}, x_j\} = \{x_{i_0}, x_j\} + \{x_j, x_{i_0}\} \in \mathcal{L}_0$ , the (7) follows.

The formulas (5), (6) and (7) are useful in the sequel.

In S, all the compositions are as follows.

1) Compositions of left or right multiplication.

All possible compositions in S of left multiplication are ones related to 1, 2 and 3.

By noting that for any  $s, i, j, t \in I$ , we have

$$\begin{array}{rcl} x_s\dashv f_{ji} &=& f_{s\dashv ji} & (j>i),\\ x_s\dashv f_{ji} &=& -f_{s\dashv ij} + x_s\dashv (\{x_i,x_j\} + \{x_j,x_i\}) & (j< i),\\ x_s\dashv f_{ii} &=& x_s\dashv \{x_i,x_i\},\\ x_s\dashv f_{ji\vdash t} &=& f_{s\dashv ji}\dashv x_t & (j>i) & \text{and}\\ x_s\dashv h_{i_0\vdash t} &=& h_{s\dashv i_0}\dashv x_t, \end{array}$$

it is clear that all cases are trivial modulo S.

By symmetry, all compositions in S of right multiplication are trivial modulo S.

2) Compositions of including or intersection.

All possible compositions of including or intersection are as follows.

$$(1 \wedge 3)$$
  $w = x_{i_0} \vdash x_i \ (i_0 \in I_0)$ . We have, by (6),

$$(f_{i_0i}, h_{i_0 \vdash i})_w = -x_i \dashv x_{i_0} + \{x_i, x_{i_0}\} = -h_{i \dashv i_0}.$$

$$(1 \wedge 4)$$
  $w = x_j \vdash x_i \dashv x_q \dashv x_p \ (q > p)$ . We have

$$(f_{ji}, f_{i \dashv qp})_{w}$$

$$= -x_{i} \dashv x_{j} \dashv x_{q} \dashv x_{p} + \{x_{i}, x_{j}\} \dashv x_{q} \dashv x_{p} + x_{j} \vdash x_{i} \dashv x_{p} \dashv x_{p} - x_{j} \vdash x_{i} \dashv \{x_{p}, x_{q}\}$$

$$= -x_{i} \dashv f_{j \dashv qp} + f_{\{i,j\} \dashv qp} + f_{ji} \dashv x_{p} \dashv x_{q} - f_{ji} \dashv \{x_{p}, x_{q}\}.$$

$$(1 \land 5)$$
  $w = x_i \vdash x_i \dashv x_{i_0} \ (i_0 \in I_0)$ . We have

$$(f_{ji}, h_{i \dashv i_0})_w = -x_i \dashv x_j \dashv x_{i_0} + \{x_i, x_j\} \dashv x_{i_0} = -x_i \dashv h_{j \dashv i_0} + h_{\{i, j\} \dashv i_0}.$$

(2 \lambda 1) There are two cases to consider:  $w = x_j \vdash x_i \vdash x_t$  and  $w = x_j \vdash x_i \vdash x_t \vdash x_p$ . For  $w = x_j \vdash x_i \vdash x_t \ (j > i)$ , by (5), we have

$$(f_{ji\vdash t}, f_{it})_w = -x_i \vdash x_j \vdash x_t + \{x_i, x_j\} \vdash x_t + x_j \vdash x_t \dashv x_i - x_j \vdash \{x_t, x_i\}$$
  
=  $-x_i \vdash f_{it} + f_{\{i,i\}t} + f_{it} \dashv x_i - f_{i\{t,i\}} + f_{i\{t,j\}} - f_{it} \dashv x_j + f_{t\dashv ii}.$ 

For  $w = x_i \vdash x_i \vdash x_t \vdash x_p \ (j > i)$ , we have

$$(f_{ji\vdash t}, f_{tp})_w$$

$$= -x_i \vdash x_j \vdash x_t \vdash x_p + \{x_i, x_j\} \vdash x_t \vdash x_p + x_j \vdash x_i \vdash x_p \dashv x_t - x_j \vdash x_i \vdash \{x_p, x_t\}$$

$$= -x_i \vdash x_j \vdash f_{tp} + \{x_i, x_j\} \vdash f_{tp} + f_{ji\vdash p} \dashv x_t - f_{ji\vdash \{p,t\}}.$$

(2 \lambda 2) There are two cases to consider:  $w = x_j \vdash x_i \vdash x_i \vdash x_s \vdash x_p$  and  $w = x_j \vdash x_i \vdash x_t \vdash x_p$ .

For 
$$w = x_j \vdash x_i \vdash x_t \vdash x_s \vdash x_p \ (j > i, t > s)$$
, we have

$$(f_{ji\vdash t}, f_{ts\vdash p})_w$$

$$= -x_i \vdash x_j \vdash x_t \vdash x_s \vdash x_p + \{x_i, x_j\} \vdash x_t \vdash x_s \vdash x_p + x_j \vdash x_i \vdash x_s \vdash x_t \vdash x_p$$

$$-x_j \vdash x_i \vdash \{x_s, x_t\} \vdash x_p$$

$$= -x_i \vdash x_j \vdash f_{ts\vdash p} + \{x_i, x_j\} \vdash f_{ts\vdash p} + f_{ji\vdash s} \vdash x_t \vdash x_p - f_{ji\vdash \{s,t\}} \vdash x_p.$$

For  $w = x_j \vdash x_i \vdash x_t \vdash x_p \ (j > i > t)$ , suppose that

$$\{x_i, x_j\} = \sum_{m \in I_1} \alpha_{ij}^m x_m + \alpha_{ij}^t x_t + \sum_{n \in I_2} \alpha_{ij}^n x_n \ (m < t < n).$$

Denote by

$$B_{t \vdash \{i,j\} \vdash p} = x_t \vdash \{x_i, x_j\} \vdash x_p - \{x_i, x_j\} \vdash x_t \vdash x_p - \{x_t, \{x_i, x_j\}\} \vdash x_p.$$

Then

$$B_{t\vdash\{i,j\}\vdash p} = \sum_{m\in I_1} \alpha_{ij}^m f_{tm\vdash p} - \sum_{n\in I_2} \alpha_{ij}^n f_{nt\vdash p} - \sum_{q\in I_0} \beta_q h_{q\vdash p}$$

is a linear combination of normal s-diwords of length 2 or 3, where

$$\sum_{q \in I_0} \beta_q x_q = \sum_{m \in I_1} \alpha_{ij}^m (\{x_t, x_m\} + \{x_m, x_t\}) + \alpha_{ij}^t \{x_t, x_t\}.$$

Now, by (5), we have

$$\begin{split} &(f_{ji\vdash t},f_{it\vdash p})_w\\ &= -x_i \vdash x_j \vdash x_t \vdash x_p + \{x_i,x_j\} \vdash x_t \vdash x_p + x_j \vdash x_t \vdash x_i \vdash x_p - x_j \vdash \{x_t,x_i\} \vdash x_p\\ &= -x_i \vdash f_{jt\vdash p} - B_{t\vdash \{i,j\}\vdash p} + f_{jt\vdash i} \vdash x_p - B_{j\vdash \{t,i\}\vdash p} + \sum_{l\in I_0} \gamma_l h_{l\vdash p}\\ &+ B_{i\vdash \{t,j\}\vdash p} - f_{it\vdash j} \vdash x_p + x_t \vdash f_{ji\vdash p}, \end{split}$$

where 
$$\sum_{l \in I_0} \gamma_l x_l = -(\{x_j, \{x_t, x_i\}\} + \{\{x_t, x_i\}, x_j\}) + (\{x_i, \{x_t, x_j\}\} + \{\{x_t, x_j\}, x_i\}).$$

(2 \lambda 3) There are three cases to consider:  $w = x_j \vdash x_{i_0} \vdash x_t \ (i_0 \in I_0), \ w = x_{j_0} \vdash x_i \vdash x_t \ (j_0 \in I_0)$  and  $w = x_j \vdash x_i \vdash x_{t_0} \vdash x_n \ (t_0 \in I_0).$ 

Case 1.  $w = x_j \vdash x_{i_0} \vdash x_t \ (j > i_0, i_0 \in I_0)$ . By (7), we can assume that  $\{x_{i_0}, x_j\} = \sum_{l \in I_0} \gamma_l x_l$ . Then, we have

$$(f_{ji_0\vdash t},h_{i_0\vdash t})_w = -x_{i_0}\vdash x_j\vdash x_t + \{x_{i_0},x_j\}\vdash x_t = -h_{i_0\vdash j}\vdash x_t + \sum_{l\in I_0}\gamma_l h_{l\vdash t}.$$

Case 2. 
$$w = x_{j_0} \vdash x_i \vdash x_t \ (j_0 > i, j_0 \in I_0)$$
. By (6), we have 
$$(f_{j_0 i \vdash t}, h_{j_0 \vdash i})_w = -x_i \vdash x_{j_0} \vdash x_t + \{x_i, x_{j_0}\} \vdash x_t = -x_i \vdash h_{j_0 \vdash t}.$$

Case 3. 
$$w = x_j \vdash x_i \vdash x_{t_0} \vdash x_n \ (j > i, t_0 \in I_0)$$
. We have 
$$(f_{ji \vdash t_0}, h_{t_0 \vdash n})_w = -x_i \vdash x_j \vdash x_{t_0} \vdash x_n + \{x_i, x_j\} \vdash x_{t_0} \vdash x_n = (-x_i \vdash x_j + \{x_i, x_j\}) \vdash h_{t_0 \vdash n}.$$

$$(2 \wedge 4) \quad w = x_j \vdash x_i \vdash x_t \dashv x_q \dashv x_p \quad (j > i, q > p). \text{ We have}$$

$$(f_{ji\vdash t}, f_{t\dashv qp})_w$$

$$= -x_i \vdash x_j \vdash x_t \dashv x_q \dashv x_p + \{x_i, x_j\} \vdash x_t \dashv x_q \dashv x_p$$

$$+x_j \vdash x_i \vdash x_t \dashv x_p \dashv x_q - x_j \vdash x_i \vdash x_t \dashv \{x_p, x_q\}$$

$$= -x_i \vdash x_j \vdash f_{t\dashv qp} + \{x_i, x_j\} \vdash f_{t\dashv qp} + f_{ji\vdash t} \dashv x_p \dashv x_q - f_{ji\vdash t} \dashv \{x_p, x_q\}.$$

$$(2 \wedge 5) \quad w = x_j \vdash x_i \vdash x_t \dashv x_{n_0} \quad (j > i, n_0 \in I_0). \text{ We have}$$

$$(f_{ji\vdash t}, h_{t\dashv n_0})_w = -x_i \vdash x_j \vdash x_t \dashv x_{n_0} + \{x_i, x_j\} \vdash x_t \dashv x_{n_0}$$

$$= (-x_i \vdash x_i + \{x_i, x_i\}) \vdash h_{t\dashv n_0}.$$

(3 \lambda 1) There are two cases to consider:  $w = x_{n_0} \vdash x_t \ (n_0 \in I_0)$  and  $w = x_{n_0} \vdash x_t \vdash x_s \ (n_0 \in I_0)$ .

For  $w = x_{n_0} \vdash x_t \ (n_0 \in I_0)$ , we have

$$(h_{n_0\vdash t},f_{n_0t})_w=x_t\dashv x_{n_0}-\{x_t,x_{n_0}\}=h_{t\dashv n_0}.$$

For  $w = x_{n_0} \vdash x_t \vdash x_s \ (n_0 \in I_0)$ , we have

$$(h_{n_0\vdash t},f_{ts})_w = x_{n_0}\vdash x_s\dashv x_t - x_{n_0}\vdash \{x_s,x_t\} = h_{n_0\vdash s}\dashv x_t - h_{n_0\vdash \{s,t\}}.$$

$$(3 \wedge 2)$$
  $w = x_{n_0} \vdash x_t \vdash x_s \vdash x_p \ (t > s, n_0 \in I_0)$ . We have

$$(h_{n_0 \vdash t}, f_{ts \vdash p})_w = x_{n_0} \vdash x_s \vdash x_t \vdash x_p - x_{n_0} \vdash \{x_s, x_t\} \vdash x_p$$
  
=  $h_{n_0 \vdash s} \vdash x_t \vdash x_p - h_{n_0 \vdash \{s, t\}} \vdash x_p.$ 

$$(3 \wedge 3)$$
  $w = x_{n_0} \vdash x_{t_0} \vdash x_r \ (n_0, t_0 \in I_0)$ . We have

$$(h_{n_0 \vdash t_0}, h_{t_0 \vdash r})_w = 0.$$

$$(3 \wedge 4)$$
  $w = x_{n_0} \vdash x_t \dashv x_q \dashv x_p \ (q > p, n_0 \in I_0)$ . We have

$$(h_{n_0 \vdash t}, f_{t \dashv qp})_w = x_{n_0} \vdash x_t \dashv x_p \dashv x_q - x_{n_0} \vdash x_t \dashv \{x_p, x_q\}$$
  
=  $h_{n_0 \vdash t} \dashv (x_p \dashv x_q - \{x_p, x_q\}).$ 

$$(3 \land 5)$$
  $w = x_{n_0} \vdash x_t \dashv x_{s_0} \ (n_0, s_0 \in I_0)$ . We have

$$(h_{n_0\vdash t}, h_{t\dashv s_0})_w = 0.$$

Since  $(4 \land 4)$ ,  $(4 \land 5)$ ,  $(5 \land 4)$ ,  $(5 \land 5)$  are symmetric with  $(2 \land 2)$ ,  $(2 \land 3)$ ,  $(3 \land 2)$ ,  $(3 \land 3)$  respectively, they have the similar representations. We omit the details.

From the above representations, we know that all compositions in S are trivial modulo S. So, S is a Gröbner-Shirshov basis.

(iii) Clearly, the mentioned set is just the set Irr(S). Now, the results follow from Theorem 3.9.  $\square$ 

By using the Theorem 4.1, we have the following corollary.

**Corollary 4.2** ([1],[8]) Let the notations be as in Theorem 4.1. Then  $U(\mathcal{L})$  is isomorphic to  $\mathcal{L} \otimes U(\mathcal{L}/\mathcal{L}_0)$ , where  $U(\mathcal{L}/\mathcal{L}_0)$  is the universal enveloping of the Lie algebra  $\mathcal{L}/\mathcal{L}_0$ .  $\square$ 

### References

- [1] M. Aymon and P.-P. Grivel, Un theoreme de Poincare-Birkhoff-Witt pour les algebres de Leibniz, *Comm. Algebra*, 31(2003), N2, 527-544.
- [2] G. M. Bergman, The diamond lemma for ring theory, Adv. in Math., 29, 178-218(1978).
- [3] L. A. Bokut, Unsolvability of the word problem, and subalgebras of finitely presented Lie algebras, *Izv. Akad. Nauk. SSSR Ser. Mat.*, 36, 1173-1219(1972).
- [4] L. A. Bokut, Imbeddings into simple associative algebras, *Algebra i Logika*, 15, 117-142(1976).

- [5] L. A. Bokut and K. P. Shum, Gröbner and Gröbner-Shirshov bases in algebra: an elementary approach, *SEA Bull. Math.*, 29, 227-252(2005).
- [6] B. Buchberger, An algorithm for finding a basis for the residue class ring of a zero-dimensional polynomial ideal [in German], Ph.D. thesis, University of Innsbruck, Austria, (1965).
- [7] B. Buchberger, An algorithmical criteria for the solvability of algebraic systems of equations [in German], Aequationes Math., 4, 374-383(1970).
- [8] P. Kolesnikov, Conformal representations of Leibniz algebras, arXiv:math/0611501.
- [9] J.-L. Loday, Une version non commutative des algebres de Lie: les algebres de Leibniz, Ens. Math. 39, 269-293(1993).
- [10] J.-L. Loday, Algebres ayant deux operations associatives (digebres), C. R. Acad. Sci. Paris 321, 141-146(1995).
- [11] J.-L. Loday, Dialgebras, in: Dialgebras and related operads, Lecture Notes in Mathematics, Vol. 1763. Berlin: Springer Verl., 2001, 7-66.
- [12] A. I. Shirshov, Some algorithmic problem for Lie algebras, *Sibirsk. Mat. Z.*, 3(1962), 292-296(in Russian); English translation in SIGSAM Bull., 33(2), 3-6(1999).

# Gröbner-Shirshov bases for dialgebras\*

### L. A. Bokut<sup>†</sup>

School of Mathematical Sciences, South China Normal University
Guangzhou 510631, P. R. China
Sobolev Institute of Mathematics, Russian Academy of Sciences
Siberian Branch, Novosibirsk 630090, Russia
Email: bokut@math.nsc.ru

## Yuqun Chen<sup>‡</sup> and Cihua Liu

School of Mathematical Sciences, South China Normal University
Guangzhou 510631, P. R. China
Email: yqchen@scnu.edu.cn
langhua01duo@yahoo.com.cn

**Abstract:** In this paper, we define the Gröbner-Shirshov basis for a dialgebra. The Composition-Diamond lemma for dialgebras is given then. As results, we give Gröbner-Shirshov bases for the universal enveloping algebra of a Leibniz algebra, the bar extension of a dialgebra, the free product of two dialgebras, and Clifford dialgebra. We obtain some normal forms for algebras mentioned the above.

Key words: dialgebra; Gröbner-Shirshov basis; Leibniz algebra; Clifford dialgebra.

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### 1 Introduction

J.-L. Loday (1995, [11]) gave the definition of a new class of algebras, dialgebras, which is closely connected to his notion of Leibniz algebras (1993, [10]) in the same way as associative algebras connected to Lie algebras. In the manuscript [12], J.-L. Loday found a normal form of elements of a free dialgebra. Here we continue to study free dialgebras and prove the Composition-Diamond lemma for dialgebras. As it is well known, this kind of lemma is the cornerstone of the theory of Gröbner and Gröbner-Shirshov bases (see, for example, [6] and cited literature). In commutative-associative case, this lemma

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<sup>&</sup>lt;sup>‡</sup>Corresponding author.

is equivalent to the Main Buchberger's Theorem ([7, 8]). For Lie and associative algebras, this is the Shirshov's lemma [14] (see also L.A. Bokut [3, 4], G. Bergman [2], L.A. Bokut and Y. Chen [5]). As results, we obtain Gröbner-Shirshov bases for the universal enveloping algebra of a Leibniz algebra, the bar extension of a dialgebra, the free product of two dialgebras, and Clifford dialgebra. By using our Composition-Diamond lemma for dialgebras (Theorem 3.9), we obtain some normal forms for algebras mentioned the above. Moreover, we get another proof of the M. Aymon, P.-P. Grivel's result ([1]) on the Poincare-Birkhoff-Witt theorem for Leibniz algebras (see P. Kolesnikov [9] for other proof).

### 2 Preliminaries

**Definition 2.1** Let k be a field. A k-linear space D equipped with two bilinear multiplications  $\vdash$  and  $\dashv$  is called a dialgebra, if both  $\vdash$  and  $\dashv$  are associative and

$$a \dashv (b \vdash c) = a \dashv b \dashv c$$
  

$$(a \dashv b) \vdash c = a \vdash b \vdash c$$
  

$$a \vdash (b \dashv c) = (a \vdash b) \dashv c$$

for any  $a, b, c \in D$ .

**Definition 2.2** Let D be a dialgebra,  $B \subset D$ . Let us define diwords of D in the set B by induction:

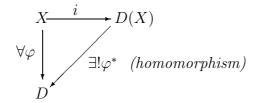
- (i) b = (b),  $b \in B$  is a diword in B of length |b| = 1.
- (ii) (u) is called a diword in B of length |(u)| = n, if  $(u) = ((v) \dashv (w))$  or  $(u) = ((v) \vdash (w))$ , where (v), (w) are diwords in B of length k, l respectively and k + l = n.

**Proposition 2.3** ([12]) Let D be a dialgebra and  $B \subset D$ . Any diword of D in the set B is equal to a diword in B of the form

$$(u) = b_{-m} \vdash \dots \vdash b_{-1} \vdash b_0 \dashv b_1 \dashv \dots \dashv b_n \tag{1}$$

where  $b_i \in B$ ,  $-m \le i \le n$ ,  $m \ge 0$ ,  $n \ge 0$ . Any bracketing of the right side of (1) gives the same result.  $\square$ 

**Definition 2.4** Let X be a set. A free dialgebra D(X) generated by X over k is defined in a usual way by the following commutative diagram:



where D is any dialgebra.

In [12], a construction of a free dialgebra is given.

**Proposition 2.5** ([12]) Let D(X) be a free dialgebra over k generated by X. Any diword in D(X) is equal to the unique diword of the form

$$[u] = x_{-m} \vdash \dots \vdash x_{-1} \vdash x_0 \dashv x_1 \dashv \dots \dashv x_n \triangleq x_{-m} \cdots x_{-1} \dot{x_0} x_1 \cdots x_n \tag{2}$$

where  $x_i \in X$ ,  $m \geq 0$ ,  $n \geq 0$ , and  $x_0$  is called the center of the normal diword [u]. We call [u] a normal diword (in X) with the associative word  $u, u \in X^*$ . Clearly, if [u] = [v], then u = v. In (2). Let [u], [v] be two normal diwords. Then  $[u] \vdash [v]$  is the normal diword [uv] with the center at the center of [v]. Accordingly,  $[u] \dashv [v]$  is the normal diword [uv] with the center at the center of [u].  $\square$ 

#### Example 2.6

$$(x_{-1} \vdash x_0 \dashv x_1) \vdash (y_{-1} \vdash y_0 \dashv y_1) = x_{-1} \vdash x_0 \vdash x_1 \vdash y_{-1} \vdash y_0 \dashv y_1,$$
$$(x_{-1} \vdash x_0 \dashv x_1) \dashv (y_{-1} \vdash y_0 \dashv y_1) = x_{-1} \vdash x_0 \dashv x_1 \dashv y_{-1} \dashv y_0 \dashv y_1. \quad \Box$$

## 3 Composition-Diamond lemma for dialgebras

Let X be a well ordered set, D(X) the free dialgebra over k,  $X^*$  the free monoid generated by X and  $[X^*]$  the set of normal diwords in X. Let us define the deg-lex ordering on  $[X^*]$  in the following way: for any  $[u], [v] \in [X^*]$ ,

$$[u] < [v] \iff wt([u]) < wt([v])$$
 lexicographicaly,

where

$$wt([u]) = (n + m + 1, m, x_{-m}, \dots, x_0, \dots, x_n)$$

if 
$$[u] = x_{-m} \cdots x_{-1} \dot{x_0} x_1 \cdots x_n$$
.

Throughout the paper, we will use this ordering.

It is easy to see that the ordering < is satisfied the following properties:

$$[u] < [v] \Longrightarrow x \vdash [u] < x \vdash [v], \ [u] \dashv x < [v] \dashv x, \text{ for any } x \in X.$$

Any polynomial  $f \in D(X)$  has the form

$$f = \sum_{[u] \in [X^*]} f([u])[u] = \alpha[\overline{f}] + \sum_{i} \alpha_i[u_i],$$

where  $[\overline{f}]$ ,  $[u_i]$  are normal diwords in X,  $[\overline{f}] > [u_i]$ ,  $\alpha$ ,  $\alpha_i$ ,  $f([u]) \in k$ ,  $\alpha \neq 0$ . We call  $[\overline{f}]$  the leading term of f. Denote suppf by the set  $\{[u]|f([u]) \neq 0\}$  and deg(f) by  $|[\overline{f}]|$ . f is called monic if  $\alpha = 1$ . f is called left (right) normed if  $f = \sum \alpha_i u_i \dot{x}_i$  ( $f = \sum \alpha_i \dot{x}_i u_i$ ), where each  $\alpha_i \in k$ ,  $x_i \in X$  and  $u_i \in X^*$ .

If [u], [v] are both left normed or both right normed, then it is clear that for any  $[w] \in [X^*]$ ,

$$[u] < [v] \Longrightarrow [u] \vdash [w] < [v] \vdash [w], [w] \vdash [u] < [w] \vdash [v],$$
$$[u] \dashv [w] < [v] \dashv [w], [w] \dashv [u] < [w] \dashv [v].$$

Let  $S \subset D(X)$ . By an S-diword g we will mean a diword in  $\{X \cup S\}$  with only one occurrence of  $s \in S$ . If this is the case and g = (asb) for some  $a, b \in X^*$ ,  $s \in S$ , we also call g an s-diword.

From Proposition 2.3 it follows that any s-diword is equal to

$$[asb] = x_{-m} \vdash \dots \vdash x_{-1} \vdash x_0 \dashv x_1 \dashv \dots \dashv x_n|_{x_k \mapsto s}$$
(3)

where  $-m \le k \le n$ ,  $s \in S$ ,  $x_i \in X$ ,  $-m \le i \le n$ . To be more precise,  $[asb] = [a\dot{s}b]$  if k = 0;  $[asb] = [asb_1\dot{x_0}b_2]$  if k < 0 and  $[asb] = [a_1\dot{x_0}a_2sb]$  if k > 0. If the center of the s-diword [asb] is in a, then we denote it by  $[\dot{a}sb] = [a_1\dot{x_0}a_2sb]$ . Similarly,  $[a\dot{s}\dot{b}] = [asb_1\dot{x_0}b_2]$  (of course, either  $a_i$  or  $b_i$  may be empty).

**Definition 3.1** The s-diword (3) is called a normal s-diword if one of the following conditions holds:

- (i) k = 0,
- (ii) k < 0 and s is left normed,
- (iii) k > 0 and s is right normed.

We call a normal s-diword [asb] a left (right) normed s-diword if both s and [asb] are left (right) normed. In particulary, s is a left (right) normed s-diword if s is left (right) normed polynomial.

The following lemma follows from the above properties of the ordering <.

**Lemma 3.2** For a normal s-diword [asb], the leading term of [asb] is equal to  $[a[\overline{s}]b]$ , that is,  $[asb] = [a[\overline{s}]b]$ . More specifically, if

$$[asb] = x_{-m} \vdash \cdots \vdash x_{-1} \vdash x_0 \dashv x_1 \dashv \cdots \dashv x_n|_{x_k \mapsto s},$$

then corresponding to k = 0, k < 0, k > 0, respectively, we have

$$\overline{x_{-m} \vdash \cdots \vdash x_{-1} \vdash s \dashv x_1 \dashv \cdots \dashv x_n} = x_{-m} \vdash \cdots \vdash x_{-1} \vdash [\overline{s}] \dashv x_1 \dashv \cdots \dashv x_n,$$

$$\overline{x_{-m} \vdash \cdots \vdash s \vdash \cdots \vdash x_0 \dashv \cdots \dashv x_n} = x_{-m} \vdash \cdots \vdash [\overline{s}] \vdash \cdots \vdash x_0 \dashv \cdots \dashv x_n,$$

$$\overline{x_{-m} \vdash \cdots \vdash x_0 \dashv \cdots \dashv s \dashv \cdots \dashv x_n} = x_{-m} \vdash \cdots \vdash x_0 \dashv \cdots \dashv [\overline{s}] \dashv \cdots \dashv x_n. \quad \Box$$

Now, we define compositions of polynomials in D(X).

**Definition 3.3** Let the ordering < be as before and  $f, g \in D(X)$  with f, g monic.

1) Composition of left (right) multiplication.

Let f be not a right normed polynomial and  $x \in X$ . Then  $x \dashv f$  is called the composition of left multiplication. Clearly,  $x \dashv f$  is a right normed polynomial (or 0).

Let f be not a left normed polynomial and  $x \in X$ . Then  $f \vdash x$  is called the composition of right multiplication. Clearly,  $f \vdash x$  is a left normed polynomial (or  $\theta$ ).

2) Composition of inclusion.

Let

$$[w] = [\overline{f}] = [a[\overline{g}]b],$$

where [agb] is a normal g-diword. Then

$$(f,g)_{[w]} = f - [agb]$$

is called the composition of inclusion. The transformation  $f \mapsto f - [agb]$  is called the elimination of leading diword (ELW) of g in f, and [w] is called the ambiguity of f and g.

3) Composition of intersection.

Let

$$[w] = [[\overline{f}]b] = [a[\overline{g}]], |\overline{f}| + |\overline{g}| > |w|,$$

where [fb] is a normal f-diword and [ag] a normal g-diword. Then

$$(f,g)_{[w]} = [fb] - [ag]$$

is called the composition of intersection, and [w] is called the ambiguity of f and g.

**Remark** In the Definition 3.3, for the case of 2) or 3), we have  $\overline{(f,g)_{[w]}} < [w]$ . For the case of 1),  $deg(x \dashv f) \leq deg(f) + 1$  and  $deg(f \vdash x) \leq deg(f) + 1$ .

**Definition 3.4** Let the ordering < be as before,  $S \subset D(X)$  a monic set and  $f, g \in S$ .

1) Let  $x \dashv f$  be a composition of left multiplication. Then  $x \dashv f$  is called trivial modulo S, denoted by  $x \dashv f \equiv 0 \mod(S)$ , if

$$x \dashv f = \sum \alpha_i [a_i s_i b_i],$$

where each  $\alpha_i \in k$ ,  $a_i, b_i \in X^*$ ,  $s_i \in S$ ,  $[a_i s_i b_i]$  right normed  $s_i$ -diword and  $|[a_i[\overline{s_i}]b_i]| \leq deg(x \dashv f)$ .

Let  $f \vdash x$  be a composition of right multiplication. Then  $f \vdash x$  is called trivial modulo S, denoted by  $f \vdash x \equiv 0 \mod(S)$ , if

$$f \vdash x = \sum \alpha_i [a_i s_i b_i],$$

where each  $\alpha_i \in k$ ,  $a_i, b_i \in X^*$ ,  $s_i \in S$ ,  $[a_i s_i b_i]$  left normed  $s_i$ -diword and  $|[a_i[\overline{s_i}]b_i]| \leq deg(f \vdash x)$ .

2) Composition  $(f,g)_{[w]}$  of inclusion (intersection) is called trivial modulo (S,[w]), denoted by  $(f,g)_{[w]} \equiv 0 \mod(S,[w])$ , if

$$(f,g)_{[w]} = \sum \alpha_i [a_i s_i b_i],$$

where each  $\alpha_i \in k$ ,  $a_i, b_i \in X^*$ ,  $s_i \in S$ ,  $[a_i s_i b_i]$  normal  $s_i$ -diword,  $[a_i [\overline{s_i}] b_i] < [w]$  and each  $[a_i s_i b_i]$  is right (left) normed  $s_i$ -diword whenever either both f and [agb] or both [fb] and [ag] are right (left) normed S-diwords.

We call the set S a Gröbner-Shirshov basis in D(X) if any composition of polynomials in S is trivial modulo S (and [w]).

The following lemmas play key role in the proof of Theorem 3.9.

**Lemma 3.5** Let  $S \subset D(X)$  and [asb] an s-diword,  $s \in S$ . Assume that each composition of right and left multiplication is trivial modulo S. Then, [asb] has a presentation:

$$[asb] = \sum \alpha_i [a_i s_i b_i],$$

where each  $\alpha_i \in K$ ,  $s_i \in S$ ,  $a_i, b_i \in X^*$  and each  $[a_i s_i b_i]$  is normal  $s_i$ -diword.

**Proof.** Following Proposition 2.3, we assume that

$$[asb] = x_{-m} \vdash \cdots \vdash x_{-1} \vdash x_0 \dashv x_1 \dashv \cdots \dashv x_n|_{x_k \mapsto s}.$$

There are three cases to consider.

Case 1. k = 0. Then [asb] is a normal s-diword.

Case 2. k < 0. Then  $[asb] = a \vdash (s \vdash x_{k+1}) \vdash b, k < -1$  or  $[asb] = a \vdash (s \vdash x_0) \dashv b$ . If s is left normed then [asb] is a normal s-diword. If s is not left normed then for the composition  $s \vdash x_{k+1}$  (k < 0) of right multiplication, we have

$$s \vdash x_{k+1} = \sum \alpha_i [a_i s_i b_i],$$

where each  $\alpha_i \in k$ ,  $a_i, b_i \in X^*$ ,  $s_i \in S$  and  $[a_i s_i b_i]$  is left normed  $s_i$ -diword. Then

$$[asb] = \sum \alpha_i (a \vdash [a_i s_i b_i] \vdash b)$$

or

$$[asb] = \sum \alpha_i (a \vdash [a_i s_i b_i] \dashv b)$$

is a linear combination of normal  $s_i$ -diwords.

Case 3. k > 0 is similar to the Case 2.  $\square$ 

**Lemma 3.6** Let  $S \subset D(X)$  and each composition  $(f,g)_{[w]}$  in S of inclusion (intersection) trivial modulo (S,[w]). Let  $[a_1s_1b_1]$  and  $[a_2s_2b_2]$  be normal S-diwords such that  $[w] = [a_1[\bar{s_1}]b_1] = [a_2[\bar{s_2}]b_2]$ , where  $s_1, s_2 \in S$ ,  $a_1, a_2, b_1, b_2 \in X^*$ . Then,

$$[a_1s_1b_1] \equiv [a_2s_2b_2] \mod(S, [w]),$$

i.e.,  $[a_1s_1b_1] - [a_2s_2b_2] = \sum \alpha_i[a_is_ib_i]$ , where each  $\alpha_i \in k$ ,  $a_i, b_i \in X^*$ ,  $s_i \in S$ ,  $[a_is_ib_i]$  normal  $s_i$ -diword and  $[a_i[\overline{s_i}]b_i] < [w]$ .

**Proof.** In the following, all letters a, b, c with indexis are words and  $s_1, s_2, s_j \in S$ . Because  $a_1\bar{s}_1b_1 = a_2\bar{s}_2b_2$  as ordinary words, there are three cases to consider.

Case 1. Subwords  $\overline{s_1}$ ,  $\overline{s_2}$  have empty intersection. Assume, for example, that  $b_1 = b\overline{s_2}b_2$  and  $a_2 = a_1\overline{s_1}b$ . Because any normal S-diword may be bracketing in any way, we have

$$[a_2s_2b_2] - [a_1s_1b_1] = (a_1s_1(b(s_2 - [\overline{s_2}])b_2)) - ((a_1(s_1 - [\overline{s_1}])b)s_2b_2).$$

For any  $[t] \in supp(s_2 - [\overline{s_2}])$ , we prove that  $(a_1s_1b[t]b_2)$  is a normal  $s_1$ -diword. There are five cases to consider.

- 1.1  $[w] = [\dot{a_1}[\overline{s_1}]b[\overline{s_2}]b_2];$
- 1.2  $[w] = [a_1[\overline{s_1}]b[\overline{s_2}]b_2];$
- 1.3  $[w] = [a_1[\overline{s_1}]\dot{b}[\overline{s_2}]b_2];$
- $1.4 [w] = [a_1[\overline{s_1}]b[\dot{\overline{s_2}}]b_2];$
- 1.5  $[w] = [a_1[\overline{s_1}]b[\overline{s_2}]\dot{b_2}].$

For 1.1, since  $[a_1s_1b_1]$  and  $[a_2s_2b_2]$  are normal S-diwords, both  $s_1$  and  $s_2$  are right normed by the definition, in particular, [t] is right normed. It follows that  $(a_1s_1b[t]b_2) = [a_1s_1b[t]b_2]$  is a normal  $s_1$ -diword.

For 1.2, it is clear that  $(a_1s_1b[t]b_2)$  is a normal  $s_1$ -diword and [t] is right normed.

For 1.3, 1.4 and 1.5, since  $[a_1s_1b_1]$  is normal  $s_1$ -diword,  $s_1$  is left normed by the definition, which implies that  $(a_1s_1b[t]b_2)$  is a normal  $s_1$ -diword. Moreover, [t] is right normed, if 1.3, and left normed, if 1.5.

Clearly, for all cases, we have  $\overline{[a_1s_1b[t]b_2]} = [a_1[\overline{s_1}]b[t]b_2] < [a_1[\overline{s_1}]b[\overline{s_2}]b_2] = [w].$ 

Similarly, for any  $[t] \in supp(s_1-[\overline{s_1}])$ ,  $(a_1[t]bs_2b_2)$  is a normal  $s_2$ -diword and  $[a_1[t]b[\overline{s_2}]b_2] < [w]$ .

Case 2. Subwords  $\overline{s_1}$  and  $\overline{s_2}$  have non-empty intersection c. Assume, for example, that  $b_1 = bb_2$ ,  $a_2 = a_1a$ ,  $w_1 = \overline{s_1}b = a\overline{s_2} = acb$ .

There are following five cases to consider:

- $2.1 \ [w] = [\dot{a_1}[\overline{s_1}]bb_2];$
- $2.2 [w] = [a_1[\overline{s_1}]b\dot{b_2}];$
- $2.3 [w] = [a_1 \dot{a}cbb_2];$
- $2.4 \ [w] = [a_1 a \dot{c} b b_2];$
- $2.5 \ [w] = [a_1 ac\dot{b}b_2].$

Then

$$[a_2s_2b_2] - [a_1s_1b_1] = (a_1([as_2] - [s_1b])b_2) = (a_1(s_1, s_2)_{[w_1]}b_2),$$

where  $[w_1] = [acb] = [[\overline{s_1}]b] = [a[\overline{s_2}]]$  is as follows:

- $2.1 [w_1]$  is right normed;
- $2.2 [w_1]$  is left normed;
- $2.3 [w_1] = [\dot{a}cb];$
- $2.4 [w_1] = [a\dot{c}b];$
- $2.5 [w_1] = [acb].$

Since each composition  $(f, g)_{[w]}$  in S is trivial modulo (S, [w]), there exist  $\beta_j \in k$ ,  $u_j, v_j \in X^*$ ,  $s_j \in S$  such that  $[s_1b] - [as_2] = \sum_j \beta_j [u_j s_j v_j]$ , where each  $[u_j s_j v_j]$  is normal S-diword and  $[u_j [\overline{s_j}] v_j] < [w_1] = [acb]$ . Therefore,

$$[a_2s_2b_2] - [a_1s_1b_1] = \sum_j \beta_j(a_1[u_js_jv_j]b_2).$$

Now, we prove that each  $(a_1[u_js_jv_j]b_2)$  is normal  $s_j$ -diword and  $\overline{(a_1[u_js_jv_j]b_2)} < [w] = [a_1[\overline{s_1}]b]b_2].$ 

For 2.1, since  $[a_1s_1bb_2]$  and  $[a_1as_2b_2]$  are normal S-diwords, both  $[s_1b]$  and  $[as_2]$  are right normed S-diwords. Then, by definition, each  $[u_js_jv_j]$  is right normed S-diword, and so each  $(a_1[u_js_jv_j]b_2) = [a_1u_js_jv_jb_2]$  is normal S-diword.

For 2.2, both  $[s_1b]$  and  $[as_2]$  must be left normed S-diwords. Then, by definition, each  $[u_js_jv_j]$  is left normed S-diword, and so each  $(a_1[u_js_jv_j]b_2) = [a_1u_js_jv_j\dot{b}_2]$  is normal S-diword.

For 2.3, 2.4 or 2.5, by noting that  $(a_1[u_js_jv_j]b_2) = ((a_1) \vdash [u_js_jv_j] \dashv (b_2))$  and  $[u_js_jv_j]$  is normal S-diword,  $(a_1[u_js_jv_j]b_2)$  is also normal S-diword.

Now, for all cases, we have  $\overline{[a_1u_js_jv_jb_2]} = [a_1u_j\overline{[s_j]}v_jb_2] < [w] = [a_1[acb]b_2].$ 

Case 3. One of the subwords  $\overline{s_1}$  and  $\overline{s_2}$  contains another as a subword. Assume, for example, that  $b_2 = bb_1$ ,  $a_2 = a_1a$ ,  $w_1 = \overline{s_1} = a\overline{s_2}b$ .

Again there are following five cases to consider:

- $2.1 [w] = [\dot{a_1}a[\overline{s_2}]bb_1];$
- $2.2 [w] = [a_1 a[\overline{s_2}] b\dot{b_1}];$
- $2.3 [w] = [a_1 \dot{a}[\overline{s_2}]bb_1];$
- $2.4 \ [w] = [a_1 a[\overline{s_2}]bb_1];$
- $2.5 [w] = [a_1 a \overline{s_2}] \dot{b} b_1.$

Then

$$[a_1s_1b_1] - [a_2s_2b_2] = (a_1(s_1 - as_2b)b_1) = (a_1(s_1, s_2)_{[w_1]}b_1).$$

It is similar to the proof of the Case 2 that we have  $[a_1s_1b_1] \equiv [a_2s_2b_2] \mod(S, [w])$ .  $\square$ 

**Definition 3.7** Let  $S \subset D(X)$ . Then

 $Irr(S) \triangleq \{u \in [X^*] | u \neq [a[\overline{s}]b], s \in S, a, b \in X^*, [asb] \text{ is normal s-diword}\}.$ 

**Lemma 3.8** Let  $S \subset D(X)$  and  $h \in D(X)$ . Then h has a representation

$$h = \sum_{I_1} \alpha_i[u_i] + \sum_{I_2} \beta_j[a_j s_j b_j]$$

where  $[u_i] \in Irr(S)$ ,  $i \in I_1$ ,  $[a_js_jb_j]$  normal  $s_j$ -diwords,  $s_j \in S$ ,  $j \in I_2$  with  $[a_1[\overline{s_1}]b_1] > [a_2[\overline{s_2}]b_2] > \cdots > [a_n[\overline{s_n}]b_n]$ .

**Proof.** Let  $h = \alpha_1[\overline{h}] + \cdots$ . We prove the result by induction on  $[\overline{h}]$ .

If  $[\overline{h}] \in Irr(S)$ , then take  $[u_1] = [\overline{h}]$  and  $h_1 = h - \alpha_1[u_1]$ . Clearly,  $[\overline{h_1}] < [\overline{h}]$  or  $h_1 = 0$ . If  $[\overline{h}] \not\in Irr(S)$ , then  $[\overline{h}] = [a_1[\overline{s_1}]b_1]$  with  $[a_1s_1b_1]$  a normal  $s_1$ -diword. Let  $h_1 = h - \beta_1[a_1s_1b_1]$ . Then  $[\overline{h_1}] < [\overline{h}]$  or  $h_1 = 0$ .  $\square$ 

The following theorem is the main result.

**Theorem 3.9** (Composition-Diamond lemma) Let  $S \subset D(X)$  be a monic set and the ordering < as before, Id(S) is the ideal generated by S. Then  $(i) \Rightarrow (ii) \Leftrightarrow (ii)' \Leftrightarrow (iii)$ , where

- (i) S is a Gröbner-Shirshov basis in D(X).
- (ii)  $f \in Id(S) \Rightarrow [\overline{f}] = [a[\overline{s}]b]$  for some  $s \in S$ ,  $a, b \in X^*$  and [asb] a normal S-diword.
- (ii)'  $f \in Id(S) \Rightarrow f = \alpha_1[a_1s_1b_1] + \alpha_2[a_2s_2b_2] + \cdots + \alpha_n[a_ns_nb_n]$  with  $[a_1[\overline{s_1}]b_1] > [a_2[\overline{s_2}]b_2] > \cdots > [a_n[\overline{s_n}]b_n]$ , where  $[a_is_ib_i]$  is normal  $s_i$ -diword,  $i = 1, 2, \cdots, n$ .
- (iii) The set Irr(S) is a linear basis of the dialgebra D(X|S) = D(X)/Id(S) generated by X with defining relations S.

**Proof.**  $(i) \Rightarrow (ii)$ . Let S be a Gröbner-Shirshov basis and  $0 \neq f \in Id(S)$ . We may assume, by Lemma 3.5, that

$$f = \sum_{i=1}^{n} \alpha_i [a_i s_i b_i],$$

where each  $\alpha_i \in k$ ,  $a_i, b_i \in X^*$ ,  $s_i \in S$  and  $[a_i s_i b_i]$  normal S-diword. Let

$$[w_i] = [a_i | \overline{s_i}] b_i, [w_1] = [w_2] = \dots = [w_l] > [w_{l+1}] \ge \dots, l \ge 1.$$

We will use induction on l and  $[w_1]$  to prove that  $[\overline{f}] = [a[\overline{s}]b]$  for some  $s \in S$  and  $a, b \in X^*$ . If l = 1, then  $[\overline{f}] = [a_1s_1b_1] = [a_1[\overline{s_1}]b_1]$  and hence the result holds. Assume that  $l \geq 2$ . Then, by Lemma 3.6, we have  $[a_1s_1b_1] \equiv [a_2s_2b_2] \mod(S, [w_1])$ .

Thus, if  $\alpha_1 + \alpha_2 \neq 0$  or l > 2, then the result follows from induction on l. For the case  $\alpha_1 + \alpha_2 = 0$  and l = 2, we use induction on  $[w_1]$ . Now, the result follows.

 $(ii) \Rightarrow (ii)'$ . Assume (ii) and  $0 \neq f \in Id(S)$ . Let  $f = \alpha_1[\overline{f}] + \sum_{[u_i] < [\overline{f}]} \alpha_i[u_i]$ . Then, by (ii),  $[\overline{f}] = [a_1[\overline{s_1}]b_1]$ , where  $[a_1s_1b_1]$  is a normal S-diword. Therefore,

$$f_1 = f - \alpha_1[a_1s_1b_1], \ [\overline{f_1}] < [\overline{f}] \text{ or } f_1 = 0, \ f_1 \in Id(S).$$

Now, by using induction on  $[\overline{f}]$ , we have (ii)'.

- $(ii)' \Rightarrow (ii)$ . This part is clear.
- $(ii) \Rightarrow (iii)$ . Assume (ii). Then by Lemma 3.8, Irr(S) spans D(X|S) as k-space.

Suppose that  $0 \neq \sum \alpha_i[u_i] \in Id(S)$  where  $[u_1] > [u_2] > \cdots$ ,  $[u_i] \in Irr(S)$ . Then by (ii),  $[u_1] = [a_1[\overline{s_1}]b_1]$  where  $[a_1s_1b_1]$  is a normal S-diword, a contradiction.

This shows (iii).

 $(iii) \Rightarrow (ii)$ . Assume (iii). Let  $0 \neq f \in Id(S)$ . Since the elements in Irr(S) are linearly independent in D(X|S), by Lemma 3.8,  $[\bar{f}] = [a[\bar{s}]b]$ , where [asb] is a normal S-diword. Thus, (ii) follows.  $\square$ 

**Remark:** In general,  $(iii) \not\Rightarrow (i)$ . For example, it is noted that

$$Irr(S) = \{x_j \dashv x_{i_1} \dashv \cdots \dashv x_{i_k} \mid j \in I, i_p \in I - I_0, \ 1 \le p \le k, \ i_1 \le \cdots \le i_k, \ k \ge 0\}$$

is a linear basis of D(X|S) in Theorem 4.3. Let

$$S_1 = \{x_i \vdash x_i - x_i \dashv x_j + \{x_i, x_j\}, \ x_t \dashv x_{i_0}, i, j, t \in I, i_0 \in I_0\}.$$

Then  $Irr(S_1) = Irr(S)$  is a linear basis of D(X|S). But in the proof of Theorem 4.3, we know that  $S_1$  is not a Gröbner-Shirshov basis of D(X|S).

#### Applications 4

In this section, we give Gröbner-Shirshov bases for the universal enveloping dialgebra of a Leibniz algebra, the bar extension of a dialgebra, the free product of two dialgebras, and the Clifford dialgebra. By using our Theorem 3.9, we obtain some normal forms for dialgebras mentioned the above.

**Definition 4.1** ([10]) A k-linear space L equipped with bilinear multiplication [,] is called a Leibniz algebra if for any  $a, b, c \in L$ ,

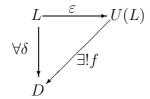
$$[[a,b],c] = [[a,c],b] + [a,[b,c]]$$

i.e., the Leibniz identity is valid in L.

It is clear that if  $(D, \dashv, \vdash)$  is a dialgebra then  $D^{(-)} = (D, [,])$  is a Leibniz algebra, where  $[a,b] = a \dashv b - b \vdash a \text{ for any } a,b \in D.$ 

If f is a Leibniz polynomial in variables X, then by  $f^{(-)}$  we mean a dialgebra polynomial in X obtained from f by transformation  $[a, b] \mapsto a \dashv b - b \vdash a$ .

**Definition 4.2** Let L be a Leibniz algebra. A dialgebra U(L) together with a Leibniz homomorphism  $\varepsilon: L \to U(L)$  is called the universal enveloping dialgebra for L, if the following diagram commute:



where D is a dialgebra,  $\delta$  is a Leibniz homomorphism and  $f:U(L)\to D$  is a dialgebra homomorphism such that  $f\varepsilon = \delta$  (i.e.,  $\varepsilon: L \to U(L)$  is a universal arrow in the sense of S. MacLane [13], p55).

An equivalent definition is as follows: Let L = Lei(X|S) is a Leibniz algebra presented by generators X and definition relations S. Then  $U(L) = D(X|S^{(-)})$  is the dialgebra with generators X and definition relations  $S^{(-)} = \{s^{(-)} | s \in S\}.$ 

**Theorem 4.3** Let  $\mathcal{L}$  be a Leibniz algebra over a field k with the product  $\{,\}$ . Let  $\mathcal{L}_0$  be the subspace of  $\mathcal{L}$  generated by the set  $\{\{a,a\},\{a,b\}+\{b,a\}\mid a,b\in\mathcal{L}\}$ . Let  $\{x_i|i\in I_0\}$ be a basis of  $\mathcal{L}_0$  and  $X = \{x_i | i \in I\}$  a well ordered basis of  $\mathcal{L}$  such that  $I_0 \subseteq I$ . Let  $U(L) = D(X|x_i \dashv x_j - x_j \vdash x_i - \{x_i, x_j\})$  be the universal enveloping dialgebra for L and the ordering < on  $[X^*]$  as before. Then

(i)  $D(X|x_i \dashv x_j - x_j \vdash x_i - \{x_i, x_j\}) = D(X|S)$ , where S consists of the following polynomials:

$$\begin{array}{lll} (a) & f_{ji} = x_{j} \vdash x_{i} - x_{i} \dashv x_{j} + \{x_{i}, x_{j}\} \\ (b) & f_{ji \vdash t} = x_{j} \vdash x_{i} \vdash x_{t} - x_{i} \vdash x_{j} \vdash x_{t} + \{x_{i}, x_{j}\} \vdash x_{t} \\ (c) & h_{i_{0} \vdash t} = x_{i_{0}} \vdash x_{t} \\ (d) & f_{t \dashv ji} = x_{t} \dashv x_{j} \dashv x_{i} - x_{t} \dashv x_{j} + x_{t} \dashv \{x_{i}, x_{j}\} \\ (e) & h_{t \dashv i_{0}} = x_{t} \dashv x_{i_{0}} \\ \end{array}$$
 
$$\begin{array}{ll} (i, j \in I) \\ (i, j, t \in I, j > i) \\ (i, j, t \in$$

(b) 
$$f_{ji\vdash t} = x_j \vdash x_i \vdash x_t - x_i \vdash x_j \vdash x_t + \{x_i, x_j\} \vdash x_t$$
  $(i, j, t \in I, j > i)$ 

(c) 
$$h_{i_0 \vdash t} = x_{i_0} \vdash x_t$$
  $(i_0 \in I_0, \ t \in I)$ 

$$(d) f_{t \dashv i} = x_t \dashv x_i \dashv x_i \dashv x_i \dashv x_i \dashv x_i + x_t \dashv \{x_i, x_i\} (i, j, t \in I, j > i)$$

(e) 
$$h_{t \dashv i_0} = x_t \dashv x_{i_0}$$
  $(i_0 \in I_0, \ t \in I)$ 

- (ii) S is a Gröbner-Shirshov basis in D(X).
- (iii) The set

$$\{x_j \dashv x_{i_1} \dashv \dots \dashv x_{i_k} \mid j \in I, i_p \in I - I_0, \ 1 \le p \le k, \ i_1 \le \dots \le i_k, \ k \ge 0\}$$

is a linear basis of the universal enveloping algebra  $U(\mathcal{L})$ . In particular,  $\mathcal{L}$  is a Leibniz subalgebra of  $U(\mathcal{L})$ .

#### **Proof.** (i) By using the following

$$f_{ji\vdash t} = f_{ji} \vdash x_t \text{ and } f_{ji} \vdash x_t + f_{ij} \vdash x_t = (\{x_i, x_j\} + \{x_j, x_i\}) \vdash x_t,$$

we have (b) and (c) are in  $Id(f_{ji})$ . By symmetry, (d) and (e) are in  $Id(f_{ji})$ . This shows (i).

(ii) We will prove that all compositions in S are trivial modulo S (and [w]). For convenience, we extend linearly the functions  $f_{ji}$ ,  $f_{ji\vdash t}$ ,  $f_{t\dashv ji}$ ,  $h_{i_0\vdash t}$  and  $h_{t\dashv i_0}$  to  $f_{j\{p,q\}}$  ( $f_{\{p,q\}i}$ ),  $f_{ji\vdash \{p,q\}}$  and  $h_{\{p,q\}\dashv i_0}$ , etc respectively. For example, if  $\{x_p, x_q\} = \sum \alpha_{pq}^s x_s$ , then

$$\begin{split} f_{j\{p,q\}} &= x_j \vdash \{x_p, x_q\} - \{x_p, x_q\} \dashv x_j + \{\{x_p, x_q\}, x_j\} = \sum \alpha_{pq}^s f_{js}, \\ f_{ji \vdash \{p,q\}} &= \sum \alpha_{pq}^s (x_j \vdash x_i \vdash x_s - x_i \vdash x_j \vdash x_s + \{x_i, x_j\} \vdash x_s) = f_{ji} \vdash \{x_p, x_q\}, \\ h_{\{p,q\} \dashv i_0} &= \sum \alpha_{pq}^s h_{s \dashv i_0}. \end{split}$$

By using the Leibniz identity,

$$\{\{a,b\},c\} = \{a,\{b,c\}\} + \{\{a,c\},b\},\tag{4}$$

we have

$${a, {b,b}} = 0$$
 and  ${a, {b,c} + {c,b}} = 0$ 

for any  $a, b, c \in \mathcal{L}$ . It means that for any  $i_0 \in I_0, j \in I$ ,

$$\{x_j, x_{i_0}\} = 0 (5)$$

and by noting that  $\{x_{i_0}, x_j\} = \{x_j, x_{i_0}\} + \{x_{i_0}, x_j\}$ , we have

$$\{x_{i_0}, x_j\} \in \mathcal{L}_0. \tag{6}$$

This implies that  $\mathcal{L}_0$  is an ideal of  $\mathcal{L}$ . Clearly,  $\mathcal{L}/\mathcal{L}_0$  is a Lie algebra.

The formulas (4), (5) and (6) are useful in the sequel.

In S, all the compositions are as follows.

1) Compositions of left or right multiplication.

All possible compositions in S of left multiplication are ones related to (a), (b) and (c). By noting that for any  $s, i, j, t \in I$ , we have

$$\begin{array}{rcl} x_{s}\dashv f_{ji} & = & f_{s\dashv ji} & (j>i), \\ x_{s}\dashv f_{ji} & = & -f_{s\dashv ij}+x_{s}\dashv (\{x_{i},x_{j}\}+\{x_{j},x_{i}\}) & (ji) & \text{and} \\ x_{s}\dashv h_{i_{0}\vdash t} & = & h_{s\dashv i_{0}}\dashv x_{t}, \end{array}$$

it is clear that all cases are trivial modulo S.

By symmetry, all compositions in S of right multiplication are trivial modulo S.

2) Compositions of inclusion and intersection.

We denote, for example,  $(a \wedge b)$  the composition of the polynomials of type (a) and type (b). It is noted that since (b) and (c) are both left normed, we have to prove that the corresponding compositions of the cases of  $(b \wedge b)$ ,  $(b \wedge c)$ ,  $(c \wedge c)$  and  $(c \wedge b)$  must be a linear combination of left normed S-diwords in which the leading term of each S-diword is less than w. Symmetrically, we consider the cases for the right normed (d) and (e).

All possible compositions of inclusion and intersection are as follows.

$$(a \wedge c)$$
  $[w] = x_{i_0} \vdash x_i \ (i_0 \in I_0)$ . We have, by (5),  

$$(f_{i_0i}, h_{i_0 \vdash i})_{[w]} = -x_i \dashv x_{i_0} + \{x_i, x_{i_0}\} = -h_{i \dashv i_0} \equiv 0 \mod(S, [w]).$$

$$(a \wedge d) \quad [w] = x_j \vdash x_i \dashv x_q \dashv x_p \quad (q > p). \text{ We have}$$
 
$$(f_{ji}, f_{i \dashv qp})_{[w]}$$
 
$$= -x_i \dashv x_j \dashv x_q \dashv x_p + \{x_i, x_j\} \dashv x_q \dashv x_p + x_j \vdash x_i \dashv x_p \dashv x_p - x_j \vdash x_i \dashv \{x_p, x_q\}$$
 
$$= -x_i \dashv f_{j \dashv qp} + f_{\{i,j\} \dashv qp} + f_{ji} \dashv x_p \dashv x_q - f_{ji} \dashv \{x_p, x_q\}$$
 
$$\equiv 0 \mod(S, [w]).$$

$$(a \wedge e)$$
  $[w] = x_j \vdash x_i \dashv x_{i_0} \ (i_0 \in I_0)$ . We have 
$$(f_{ji}, h_{i\dashv i_0})_{[w]} = -x_i \dashv x_j \dashv x_{i_0} + \{x_i, x_j\} \dashv x_{i_0} = -x_i \dashv h_{j\dashv i_0} + h_{\{i,j\}\dashv i_0} \equiv 0 \mod(S, [w]).$$

 $(b \wedge a)$  There are two cases to consider:  $[w] = x_j \vdash x_i \vdash x_t$  and  $[w] = x_j \vdash x_i \vdash x_t \vdash x_p$ . For  $[w] = x_j \vdash x_i \vdash x_t \ (j > i)$ , by (4), we have

$$(f_{ji\vdash t}, f_{it})_{[w]} = -x_i \vdash x_j \vdash x_t + \{x_i, x_j\} \vdash x_t + x_j \vdash x_t \dashv x_i - x_j \vdash \{x_t, x_i\}$$

$$= -x_i \vdash f_{jt} + f_{\{i,j\}t} + f_{jt} \dashv x_i - f_{j\{t,i\}} + f_{i\{t,j\}} - f_{it} \dashv x_j + f_{t\dashv ji}$$

$$\equiv 0 \mod(S, [w]).$$

For 
$$[w] = x_j \vdash x_i \vdash x_t \vdash x_p \ (j > i)$$
, we have

$$(f_{ji\vdash t}, f_{tp})_{[w]}$$

$$= -x_i \vdash x_j \vdash x_t \vdash x_p + \{x_i, x_j\} \vdash x_t \vdash x_p + x_j \vdash x_i \vdash x_p \dashv x_t - x_j \vdash x_i \vdash \{x_p, x_t\}$$

$$= -x_i \vdash x_j \vdash f_{tp} + \{x_i, x_j\} \vdash f_{tp} + f_{ji\vdash p} \dashv x_t - f_{ji\vdash \{p,t\}}$$

$$\equiv 0 \mod(S, [w]).$$

( $b \wedge b$ ) There are two cases to consider:  $[w] = x_j \vdash x_i \vdash x_t \vdash x_s \vdash x_p$  and  $[w] = x_j \vdash x_i \vdash x_t \vdash x_p$ .

For 
$$[w] = x_j \vdash x_i \vdash x_t \vdash x_s \vdash x_p \quad (j > i, t > s)$$
, we have 
$$(f_{ji\vdash t}, f_{ts\vdash p})_{[w]}$$

$$= -x_i \vdash x_j \vdash x_t \vdash x_s \vdash x_p + \{x_i, x_j\} \vdash x_t \vdash x_s \vdash x_p + x_j \vdash x_i \vdash x_s \vdash x_t \vdash x_p$$

$$-x_j \vdash x_i \vdash \{x_s, x_t\} \vdash x_p$$

$$= -x_i \vdash x_j \vdash f_{ts\vdash p} + \{x_i, x_j\} \vdash f_{ts\vdash p} + f_{ji\vdash s} \vdash x_t \vdash x_p - f_{ji\vdash \{s,t\}} \vdash x_p$$

$$\equiv 0 \mod(S, [w])$$

since it is a combination of left normed S-diwords in which the leading term of each S-diword is less than w.

For  $[w] = x_j \vdash x_i \vdash x_t \vdash x_p \ (j > i > t)$ , suppose that

$$\{x_i, x_j\} = \sum_{m \in I_1} \alpha_{ij}^m x_m + \alpha_{ij}^t x_t + \sum_{n \in I_2} \alpha_{ij}^n x_n \ (m < t < n).$$

Denote

$$B_{t \vdash \{i,j\} \vdash p} = x_t \vdash \{x_i, x_j\} \vdash x_p - \{x_i, x_j\} \vdash x_t \vdash x_p - \{x_t, \{x_i, x_j\}\} \vdash x_p.$$

Then

$$B_{t\vdash\{i,j\}\vdash p} = \sum_{m\in I_1} \alpha_{ij}^m f_{tm\vdash p} - \sum_{n\in I_2} \alpha_{ij}^n f_{nt\vdash p} - \sum_{q\in I_0} \beta_q h_{q\vdash p}$$

is a linear combination of left normed S-diwords of length 2 or 3, where

$$\sum_{q \in I_0} \beta_q x_q = \sum_{m \in I_1} \alpha_{ij}^m(\{x_t, x_m\} + \{x_m, x_t\}) + \alpha_{ij}^t \{x_t, x_t\}.$$

Denote

$$\sum_{l \in I_0} \gamma_l x_l = -(\{x_j, \{x_t, x_i\}\} + \{\{x_t, x_i\}, x_j\}) + (\{x_i, \{x_t, x_j\}\} + \{\{x_t, x_j\}, x_i\}).$$

Now, by (4), we have

$$(f_{ji\vdash t}, f_{it\vdash p})_{[w]}$$

$$= -x_i \vdash x_j \vdash x_t \vdash x_p + \{x_i, x_j\} \vdash x_t \vdash x_p + x_j \vdash x_t \vdash x_i \vdash x_p - x_j \vdash \{x_t, x_i\} \vdash x_p$$

$$= -x_i \vdash f_{jt\vdash p} - B_{t\vdash \{i,j\}\vdash p} + f_{jt\vdash i} \vdash x_p - B_{j\vdash \{t,i\}\vdash p} + \sum_{l\in I_0} \gamma_l h_{l\vdash p}$$

$$+B_{i\vdash \{t,j\}\vdash p} - f_{it\vdash j} \vdash x_p + x_t \vdash f_{ji\vdash p}$$

$$\equiv 0 \mod(S, [w])$$

since it is a combination of left normed S-diwords in which the leading term of each S-diword is less than w.

( $b \wedge c$ ) There are three cases to consider:  $[w] = x_j \vdash x_{i_0} \vdash x_t \ (i_0 \in I_0), \ [w] = x_{j_0} \vdash x_i \vdash x_t \ (j_0 \in I_0)$  and  $[w] = x_j \vdash x_i \vdash x_{t_0} \vdash x_n \ (t_0 \in I_0)$ .

Case 1.  $[w] = x_j \vdash x_{i_0} \vdash x_t \ (j > i_0, i_0 \in I_0)$ . By (6), we can assume that  $\{x_{i_0}, x_j\} = \sum_{l \in I_0} \gamma_l x_l$ . Then, we have

$$(f_{ji_0\vdash t},h_{i_0\vdash t})_{[w]} = -x_{i_0}\vdash x_j\vdash x_t + \{x_{i_0},x_j\}\vdash x_t = -h_{i_0\vdash j}\vdash x_t + \sum_{l\in I_0}\gamma_l h_{l\vdash t} \equiv 0 \ mod(S,[w]).$$

Case 2. 
$$[w] = x_{j_0} \vdash x_i \vdash x_t \ (j_0 > i, j_0 \in I_0)$$
. By (5), we have

$$(f_{j_0i\vdash t},h_{j_0\vdash i})_{[w]} = -x_i\vdash x_{j_0}\vdash x_t + \{x_i,x_{j_0}\}\vdash x_t = -x_i\vdash h_{j_0\vdash t}\equiv 0 \mod(S,[w]).$$

Case 3. 
$$[w] = x_j \vdash x_i \vdash x_{t_0} \vdash x_n \ (j > i, t_0 \in I_0)$$
. We have

$$(f_{ji\vdash t_0}, h_{t_0\vdash n})_{[w]} = -x_i \vdash x_j \vdash x_{t_0} \vdash x_n + \{x_i, x_j\} \vdash x_{t_0} \vdash x_n$$
  
=  $(-x_i \vdash x_j + \{x_i, x_j\}) \vdash h_{t_0\vdash n}$   
=  $0 \mod(S, [w]).$ 

$$(b \wedge d) \quad [w] = x_j \vdash x_i \vdash x_t \dashv x_q \dashv x_p \quad (j > i, q > p). \text{ We have}$$

$$(f_{ji\vdash t}, f_{t\dashv qp})_{[w]}$$

$$= -x_i \vdash x_j \vdash x_t \dashv x_q \dashv x_p + \{x_i, x_j\} \vdash x_t \dashv x_q \dashv x_p$$

$$+x_j \vdash x_i \vdash x_t \dashv x_p \dashv x_q - x_j \vdash x_i \vdash x_t \dashv \{x_p, x_q\}$$

$$= -x_i \vdash x_j \vdash f_{t\dashv qp} + \{x_i, x_j\} \vdash f_{t\dashv qp} + f_{ji\vdash t} \dashv x_p \dashv x_q - f_{ji\vdash t} \dashv \{x_p, x_q\}$$

$$\equiv 0 \mod(S, [w]).$$

$$(b \wedge e) \quad [w] = x_j \vdash x_i \vdash x_t \dashv x_{n_0} \quad (j > i, n_0 \in I_0). \text{ We have}$$
 
$$(f_{ji \vdash t}, h_{t \dashv n_0})_{[w]} = -x_i \vdash x_j \vdash x_t \dashv x_{n_0} + \{x_i, x_j\} \vdash x_t \dashv x_{n_0}$$
 
$$= (-x_i \vdash x_j + \{x_i, x_j\}) \vdash h_{t \dashv n_0}$$
 
$$\equiv 0 \quad mod(S, [w]).$$

 $(c \wedge a)$  There are two cases to consider:  $[w] = x_{n_0} \vdash x_t \ (n_0 \in I_0)$  and  $[w] = x_{n_0} \vdash x_t \vdash x_s \ (n_0 \in I_0)$ .

For  $[w] = x_{n_0} \vdash x_t \ (n_0 \in I_0)$ , we have

$$(h_{n_0\vdash t}, f_{n_0t})_{[w]} = x_t \dashv x_{n_0} - \{x_t, x_{n_0}\} = h_{t\dashv n_0} \equiv 0 \mod(S, [w]).$$

For  $[w] = x_{n_0} \vdash x_t \vdash x_s \ (n_0 \in I_0)$ , we have

$$(h_{n_0\vdash t},f_{ts})_{[w]} = x_{n_0}\vdash x_s\dashv x_t - x_{n_0}\vdash \{x_s,x_t\} = h_{n_0\vdash s}\dashv x_t - h_{n_0\vdash \{s,t\}} \equiv 0 \mod(S,[w]).$$

$$(c \wedge b)$$
  $[w] = x_{n_0} \vdash x_t \vdash x_s \vdash x_p \ (t > s, n_0 \in I_0).$  We have

$$(h_{n_0 \vdash t}, f_{ts \vdash p})_{[w]} = x_{n_0} \vdash x_s \vdash x_t \vdash x_p - x_{n_0} \vdash \{x_s, x_t\} \vdash x_p$$

$$= h_{n_0 \vdash s} \vdash x_t \vdash x_p - h_{n_0 \vdash \{s, t\}} \vdash x_p$$

$$\equiv 0 \mod(S, [w]).$$

$$(c \wedge c)$$
  $[w] = x_{n_0} \vdash x_{t_0} \vdash x_r \ (n_0, t_0 \in I_0).$  We have

$$(h_{n_0 \vdash t_0}, h_{t_0 \vdash r})_{[w]} = 0.$$

$$(c \wedge d)$$
  $[w] = x_{n_0} \vdash x_t \dashv x_q \dashv x_p \ (q > p, n_0 \in I_0)$ . We have

$$(h_{n_0 \vdash t}, f_{t \dashv qp})_{[w]} = x_{n_0} \vdash x_t \dashv x_p \dashv x_q - x_{n_0} \vdash x_t \dashv \{x_p, x_q\}$$

$$= h_{n_0 \vdash t} \dashv (x_p \dashv x_q - \{x_p, x_q\})$$

$$\equiv 0 \mod(S, [w]).$$

$$(c \wedge e)$$
  $[w] = x_{n_0} \vdash x_t \dashv x_{s_0} \ (n_0, s_0 \in I_0).$  We have

$$(h_{n_0\vdash t}, h_{t\dashv s_0})_{[w]} = 0.$$

Since  $(d \wedge d)$ ,  $(d \wedge e)$ ,  $(e \wedge d)$ ,  $(e \wedge e)$  are symmetric with  $(b \wedge b)$ ,  $(b \wedge c)$ ,  $(c \wedge b)$ ,  $(c \wedge c)$  respectively, they have the similar representations. We omit the details.

So, we show that S is a Gröbner-Shirshov basis.

(iii) Clearly, the mentioned set is just the set Irr(S). Now, the results follow from Theorem 3.9.  $\square$ 

A Gröbner-Shirshov basis S is called reduced if S is a monic set and no monomial in any element of the basis contains the leading words of the other elements of the basis as subwords.

**Remark:** Let the notation be in Theorem 4.3. Let  $S^{red}$  consist of the following polynomials:

(a) 
$$f_{ii} = x_i \vdash x_i - x_i \dashv x_i + \{x_i, x_i\}$$
  $(i \in I, j \in I - I_0)$ 

(b) 
$$f_{ji\vdash t} = x_j \vdash x_i \vdash x_t - x_i \vdash x_j \vdash x_t + \{x_i, x_j\} \vdash x_t$$
  $(i, j \in I - I_0, j > i, t \in I)$ 

(c) 
$$h_{i_0 \vdash t} = x_{i_0} \vdash x_t$$
  $(i_0 \in I_0, \ t \in I)$ 

$$(d) f_{t \dashv ji} = x_t \dashv x_j \dashv x_i - x_t \dashv x_i \dashv x_j + x_t \dashv \{x_i, x_j\} (i, j \in I - I_0, \ j > i, t \in I)$$

(e) 
$$h_{t \dashv i_0} = x_t \dashv x_{i_0}$$
  $(i_0 \in I_0, \ t \in I)$ 

Then  $S^{red}$  is a reduced Gröbner-Shirshov basis for D(X|S).

We have the following corollary.

**Corollary 4.4** ([1]) Let the notation be as in Theorem 4.3. Then as linear spaces,  $U(\mathcal{L})$  is isomorphic to  $\mathcal{L} \otimes U(\mathcal{L}/\mathcal{L}_0)$ , where  $U(\mathcal{L}/\mathcal{L}_0)$  is the universal enveloping of the Lie algebra  $\mathcal{L}/\mathcal{L}_0$ .

**Proof.** Clearly,  $\{x_j \mid j \in I - I_0\}$  is a k-basis of the Lie algebra  $\mathcal{L}/\mathcal{L}_0$ . It is well known that the universal enveloping  $U(\mathcal{L}/\mathcal{L}_0)$  of the Lie algebra  $\mathcal{L}/\mathcal{L}_0$  has a k-basis

$$\{x_{i_1}x_{i_2}\dots x_{i_k}\mid i_1\leq \dots \leq i_k,\ i_p\in I-I_0,\ 1\leq p\leq k,\ k\geq 0\}.$$

By using (iii) in Theorem 4.3, the result follows.  $\Box$ 

**Definition 4.5** Let D be a dialgebra. An element  $e \in D$  is called a bar unit of D if  $e \vdash x = x \dashv e = x$  for any  $x \in D$ .

**Theorem 4.6** Each dialgebra has a bar unit extension.

**Proof.** Let  $(D, \vdash, \dashv)$  be an arbitrary dialgebra over a field k and A the ideal of D generated by the set  $\{a \dashv b - a \vdash b \mid a, b \in D\}$ . Let  $X_0 = \{x_{i_0} | i_0 \in I_0\}$  be a k-basis of A and  $X = \{x_i | i \in I\}$  a well ordered k-basis of D such that  $I_0 \subseteq I$ . Then D has a presentation by the multiplication table D = D(X|S), where  $S = \{x_i \vdash x_j - \{x_i \vdash x_j\}, x_i \dashv x_j - \{x_i \dashv x_j\}, i, j \in I\}$ , where  $\{x_i \vdash x_j\}$  and  $\{x_i \dashv x_j\}$  are linear combinations of  $x_t, t \in I$ .

Let  $D_1 = D(X \cup \{e\} | S_1)$ , where  $S_1 = S \cup \{e \vdash y - y, y \dashv e - y, e \dashv x_0, x_0 \vdash e \mid y \in X \cup \{e\}, x_0 \in X_0\}$ . Then  $D_1$  is a dialgebra with a bar unit e.

Denote

1. 
$$f_{i\vdash j} = x_i \vdash x_j - \{x_i \vdash x_j\},\$$

$$2. f_{i\dashv j} = x_i \dashv x_j - \{x_i \dashv x_j\},$$

3. 
$$g_{e \vdash y} = e \vdash y - y$$
,

4. 
$$g_{y\dashv e} = y\dashv e-y,$$

$$5. \qquad h_{x_{i_0} \vdash e} = x_{i_0} \vdash e,$$

6. 
$$h_{e\dashv x_{i_0}} = e \dashv x_{i_0}$$
,

where  $i, j \in I, i_0 \in I_0, y \in X \cup \{e\}.$ 

We show that  $\{x_t \dashv x_{i_0}\} = 0$  and  $\{x_{i_0} \vdash x_t\} = 0$  for any  $t \in I$ ,  $i_0 \in I_0$ .

Since  $x_{i_0} \in A$ , we have  $x_{i_0} = \sum \alpha_i(c_i f_i d_i)$ , where  $f_i = a_i \dashv b_i - a_i \vdash b_i$ ,  $\alpha_i \in k$ ,  $a_i, b_i \in D$  and  $c_i, d_i \in X^*$ .

Since  $x_t \dashv (c_i(a_i \dashv b_i - a_i \vdash b_i)d_i) = 0$ , we have  $\{x_t \dashv \{c_i\{a_i \dashv b_i - a_i \vdash b_i\}d_i\}\} = 0$  for each i. Then  $\{x_t \dashv x_{i_0}\} = 0$ .

By symmetry, we have  $\{x_{i_0} \vdash x_t\} = 0$ .

To prove the theorem, by using our Theorem 3.9, it suffices to prove that with the ordering on  $[(X \cup \{e\})^*]$  as before, where  $x < e, x \in X$ ,  $S_1$  is a Gröbner-Shirshov basis in  $D(X \cup \{e\})$ . Now, we show that all compositions in  $S_1$  are trivial.

All possible compositions of left and right multiplication are:  $z\dashv f_{i\vdash j},\ z\dashv g_{e\vdash y},\ z\dashv h_{x_{i_0}\vdash e},\ f_{i\dashv j}\vdash z,\ g_{y\dashv e}\vdash z,\ h_{e\dashv x_{i_0}}\vdash z,\ z\in X\cup\{e\}.$ 

For  $z \dashv f_{i \vdash j}$ ,  $z = x_t \in X$ , since  $(x_t \dashv x_i) \dashv x_j = x_t \dashv (x_i \vdash x_j)$ , we have  $\{\{x_t \dashv x_i\} \dashv x_j\} = \{x_t \dashv \{x_i \vdash x_j\}\}$  and

$$\begin{array}{ll} x_{t}\dashv f_{i\vdash j} \\ = & x_{t}\dashv x_{i}\dashv x_{j}-x_{t}\dashv \{x_{i}\vdash x_{j}\} \\ = & f_{t\dashv i}\dashv x_{j}+f_{\{t\dashv i\}\dashv j}-f_{t\dashv \{i\vdash j\}}+\{\{x_{t}\dashv x_{i}\}\dashv x_{j}\}-\{x_{t}\dashv \{x_{i}\vdash x_{j}\}\} \\ = & f_{t\dashv i}\dashv x_{j}+f_{\{t\dashv i\}\dashv j}-f_{t\dashv \{i\vdash j\}} \\ \equiv & 0 \ mod(S_{1}). \end{array}$$

For  $z \dashv f_{i \vdash j}, z = e$ , let  $\{x_i \dashv x_j\} - \{x_i \vdash x_j\} = \sum \alpha_{i_0} x_{i_0}$ . Then

$$\begin{array}{rcl} e\dashv f_{i\vdash j} & = & e\dashv x_{i}\dashv x_{j} - e\dashv \{x_{i}\vdash x_{j}\}\\ & = & e\dashv (x_{i}\dashv x_{j} - \{x_{i}\dashv x_{j}\}) + e\dashv \{x_{i}\dashv x_{j}\} - e\dashv \{x_{i}\vdash x_{j}\}\\ & = & e\dashv f_{i\dashv j} + \sum \alpha_{i_{0}}h_{e\dashv x_{i_{0}}}\\ & \equiv & 0 \ mod(S_{1}). \end{array}$$

For  $z \dashv g_{e \vdash y}$ , we have

$$z\dashv g_{e\vdash y}=z\dashv e\dashv y-z\dashv y=(z\dashv e-z)\dashv y=g_{z\dashv e}\dashv y\equiv 0\ mod(S_1).$$

For  $z \dashv h_{x_{i_0} \vdash e}$ , we have

$$z \dashv h_{x_{i_0} \vdash e} = z \dashv x_{i_0} \dashv e = z \dashv g_{x_{i_0} \dashv e} + z \dashv x_{i_0}.$$

It is clear that  $z \dashv x_{i_0} = h_{e \dashv x_{i_0}}$  if z = e and  $z \dashv x_{i_0} = x_t \dashv x_{i_0} - \{x_t \dashv x_{i_0}\} = f_{t \dashv i_0}$  if  $z = x_t \in X$ , since  $\{x_t \dashv x_{i_0}\} = 0$ . This implies that  $z \dashv h_{x_{i_0} \vdash e} \equiv 0 \ mod(S_1)$ .

Thus we show that all compositions of left multiplication in  $S_1$  are trivial modulo  $S_1$ . By symmetry, all compositions of right multiplication in  $S_1$  are trivial modulo  $S_1$ .

Now, all possible ambiguities [w] of compositions of intersection in  $S_1$  are:

 $1 \wedge 1$ ,  $[x_i x_j \dot{x_t}]$ ;  $1 \wedge 2$ ,  $[x_i \dot{x_j} x_t]$ ;  $1 \wedge 4$ ,  $[x_i \dot{x_j} e]$ ;  $1 \wedge 5$ ,  $[x_i x_{i_0} \dot{e}]$ .

 $2 \wedge 2$ ,  $[\dot{x}_i x_j x_t]$ ;  $2 \wedge 4$ ,  $[\dot{x}_i x_j e]$ .

 $3 \wedge 1$ ,  $[ex_i\dot{x}_j]$ ;  $3 \wedge 2$ ,  $[e\dot{x}_ix_j]$ ;  $3 \wedge 3$ ,  $[ee\dot{y}]$ ;  $3 \wedge 4$ ,  $[e\dot{y}e]$ ;  $3 \wedge 5$ ,  $[ex_{i_0}\dot{e}]$ ;  $3 \wedge 6$ ,  $[e\dot{e}x_{i_0}]$ .

 $4 \wedge 4$ ,  $[\dot{y}ee]$ ;  $4 \wedge 6$ ,  $[\dot{y}ex_{i_0}]$ .

 $5 \wedge 3$ ,  $[x_{i_0}e\dot{y}]$ ;  $5 \wedge 4$ ,  $[x_{i_0}\dot{e}e]$ ;  $5 \wedge 6$ ,  $[x_{i_0}\dot{e}x_{j_0}]$ .

 $6 \wedge 2$ ,  $[\dot{e}x_{i_0}x_j]$ ;  $6 \wedge 4$ ,  $[\dot{e}x_{i_0}e]$ .

In the above, all  $i, j, t \in I$ ,  $i_0, j_0 \in I_0$  and  $y \in X \cup \{e\}$ .

There is no composition of inclusion in  $S_1$ .

We will show that all compositions of intersection in  $S_1$  are trivial. We check only the cases of  $1 \wedge 2$ ,  $1 \wedge 5$  and  $4 \wedge 6$ . Others can be similarly proved.

For  $1 \land 2$ ,  $[w] = [x_i \dot{x_j} x_t]$ , since  $(x_i \vdash x_j) \dashv x_t = x_i \vdash (x_j \dashv x_t)$ , we have  $\{\{x_i \vdash x_j\} \dashv x_t\} = \{x_i \vdash \{x_j \dashv x_t\}\}$  and

$$\begin{array}{rcl} (1 \wedge 2)_{[w]} & = & -\{x_i \vdash x_j\} \dashv x_t + x_i \vdash \{x_j \dashv x_t\} \\ & = & -f_{\{i \vdash j\} \dashv t} + f_{i \vdash \{j \dashv t\}} - \{\{x_i \vdash x_j\} \dashv x_t\} + \{x_i \vdash \{x_j \dashv x_t\}\} \\ & = & -f_{\{i \vdash j\} \dashv t} + f_{i \vdash \{j \dashv t\}} \\ & \equiv & 0 \ mod(S_1, [w]). \end{array}$$

For  $1 \wedge 5$ ,  $[w] = [x_i x_{i_0} \dot{e}]$ , since  $x_i \vdash x_{i_0} \in A$ , we have  $\{x_i \vdash x_{i_0}\} = \sum_i \alpha_{j_0} x_{j_0}$  and

$$(1 \wedge 5)_{[w]} = \{x_i \vdash x_{i_0}\} \vdash e = \sum \alpha_{j_0} h_{x_{j_0} \vdash e} \equiv 0 \ mod(S_1, [w]).$$

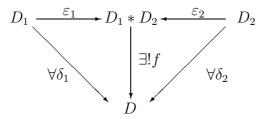
For  $4 \wedge 6$ ,  $[w] = [\dot{y}ex_{i_0}]$ , we have  $(4 \wedge 6)_{[w]} = -h_{e\dashv x_{i_0}}$  if y = e and  $(4 \wedge 6)_{[w]} = -f_{t\dashv i_0}$  if  $y = x_t \in X$  since  $\{x_t \dashv x_{i_0}\} = 0$ . Then  $(4 \wedge 6)_{[w]} \equiv 0 \ mod(S_1, [w])$ .

Then all the compositions in  $S_1$  are trivial.

The proof is complete.  $\square$ 

**Remark:** Let the notation be as in the proof of Theorem 4.6. Let  $D' = D(X \cup \{e_j\}_J | S')$  be a dialgebra, where  $S' = S \cup \{e_j \vdash y - y, y \dashv e_j - y, e_j \dashv x_0, x_0 \vdash e_j \mid y \in X \cup \{e_j\}_J, x_0 \in X_0, j \in J\}$ . Let J be a well ordered set. Then with the ordering on  $[(X \cup \{e_j\}_J)^*]$  as before, where  $x_i < e_j$  for all  $i \in I$ ,  $j \in J$ , by a similar proof of Theorem 4.6, S' is a Gröbner-Shirshov basis in  $D(X \cup \{e_j\}_J)$ . It follows from Theorem 3.9 that D can be embedded into the dialgebra D' while D' has bar units  $\{e_j\}_J$ .

**Definition 4.7** Let  $D_1$ ,  $D_2$  be dialgebras over a field k. The dialgebra  $D_1 * D_2$  with two dialgebra homomorphisms  $\varepsilon_1 : D_1 \to D_1 * D_2$ ,  $\varepsilon_2 : D_2 \to D_1 * D_2$  is called the free product of  $D_1$ ,  $D_2$ , if the following diagram commute:



where D is a dialgebra,  $\delta_1, \delta_2$  are dialgebra homomorphisms and  $f: D_1 * D_2 \rightarrow D$  is a dialgebra homomorphism such that  $f\varepsilon_1 = \delta_1, f\varepsilon_2 = \delta_2$  (i.e.,  $(\varepsilon_1, \varepsilon_2) : (D_1, D_2) \to$  $(D_1 * D_2, D_1 * D_2)$  is a universal arrow in the sense of S. Maclane [13]).

An equivalent definition is as follows: Let  $D_i = D(X_i|S_i)$  be a presentation by generators and defining relations with  $X_1 \cap X_2 = \emptyset$ , i = 1, 2. Then  $D_1 * D_2 = D(X_1 \cup X_2 | S_1 \cup S_2)$ .

Let  $(D_1, \vdash, \dashv)$ ,  $(D_2, \vdash, \dashv)$  be two dialgebras over a field  $k, A_1$  the ideal of  $D_1$  generated by the set  $\{a \dashv b - a \vdash b \mid a, b \in D_1\}$  and  $A_2$  the ideal of  $D_2$  generated by the set  $\{c \dashv d - c \vdash d \mid c, d \in D_2\}$ . Let  $X_0 = \{x_{i_0} | i_0 \in I_0\}$  be a k-basis of  $A_1$  and  $X = \{x_i | i \in I\}$ a well ordered k-basis of  $D_1$  such that  $I_0 \subseteq I$ . Let  $Y_0 = \{y_{l_0} | l_0 \in J_0\}$  be a k-basis of  $A_2$ and  $Y = \{y_l | l \in J\}$  a well ordered k-basis of  $D_2$  such that  $J_0 \subseteq J$ . Then  $D_1$  and  $D_2$  have multiplication tables:

$$D_1 = D(X|S_1), \quad S_1 = \{x_i \vdash x_j - \{x_i \vdash x_j\}, \ x_i \dashv x_j - \{x_i \dashv x_j\}, \ i, j \in I\},$$
  
$$D_2 = D(Y|S_2), \quad S_2 = \{y_l \vdash y_m - \{y_l \vdash y_m\}, \ y_l \dashv y_m - \{y_l \dashv y_m\}, \ l, m \in J\}.$$

The free product  $D_1 * D_2$  of  $D_1$  and  $D_2$  is

$$D_1 * D_2 = D(X \cup Y | S_1 \cup S_2).$$

We order  $X \cup Y$  by  $x_i < y_j$  for any  $i \in I, j \in J$ . Then we have the following theorem.

**Theorem 4.8** (i) S is a Gröbner-Shirshov basis of  $D_1 * D_2 = D(X \cup Y | S_1 \cup S_2)$ , where S consists of the following relations:

- $f_{x_i \vdash x_j} = x_i \vdash x_j \{x_i \vdash x_j\}, \quad i, j \in I,$
- $f_{x_i \dashv x_j} = x_i \dashv x_j \{x_i \dashv x_j\}, \quad i, j \in I,$
- $f_{y_l \vdash y_m} = y_l \vdash y_m \{y_l \vdash y_m\}, \quad l, m \in J,$
- $f_{y_l \dashv y_m} = y_l \dashv y_m \{y_l \dashv y_m\}, \quad l, m \in J,$
- 5.  $h_{x_{i_0} \vdash y_l} = x_{i_0} \vdash y_l, \qquad i_0 \in I_0, l \in J,$
- 6.  $h_{y_l \dashv x_{i_0}} = y_l \dashv x_{i_0},$   $i_0 \in I_0, l \in J,$ 7.  $h_{y_{l_0} \vdash x_i} = y_{l_0} \vdash x_i,$   $i \in I, l_0 \in J_0,$ 8.  $h_{x_i \dashv y_{l_0}} = x_i \dashv y_{l_0},$   $i \in I, l_0 \in J_0.$

(ii) Irr(S), which is a k-linear basis of  $D_1*D_2$ , consists of all elements  $z_{-m}\cdots z_{-1}\dot{z}_0z_1\cdots z_n$ ,  $X \ nor \{z_j, z_{j+1}\} \subseteq Y, -m \le j \le n-1.$ 

**Proof.** By the proof of Theorem 4.6, we have  $\{x_i \dashv x_{i_0}\} = 0$ ,  $\{x_{i_0} \vdash x_i\} = 0$ ,  $\{y_l \dashv y_{l_0}\} = 0$  and  $\{y_{l_0} \vdash y_l\} = 0$  for any  $i \in I$ ,  $i_0 \in I_0$ ,  $l \in J$ ,  $l_0 \in J_0$ .

Firstly, we prove that  $h_{y_l \dashv x_{i_0}} \in Id(S_1 \cup S_2)$  for any  $i_0 \in I_0, l \in J$ .

Since  $y_l \dashv (c_i(\{a_i \dashv b_i\} - \{a_i \vdash b_i\})d_i) = y_l \dashv (c_i((a_i \dashv b_i - \{a_i \dashv b_i\}) - (a_i \vdash b_i - \{a_i \vdash b_i\})d_i) \in Id(S_1 \cup S_2)$ , we have  $y_l \dashv \{c_i\{a_i \dashv b_i - a_i \vdash b_i\}d_i\} \in Id(S_1 \cup S_2)$  for all i, l. Then  $h_{y_l \dashv x_{i_0}} \in Id(S_1 \cup S_2)$ .

Similarly, we have  $h_{x_{i_0} \vdash y_l}$ ,  $h_{y_{l_0} \vdash x_i}$ ,  $h_{x_i \dashv y_{l_0}} \in Id(S_1 \cup S_2)$  for any  $i \in I$ ,  $i_0 \in I_0$ ,  $l \in J$ ,  $l_0 \in J_0$ .

Secondly, we will show that all compositions in S are trivial.

All possible compositions of left and right multiplication are:  $z\dashv f_{x_i\vdash x_j},\ z\dashv f_{y_l\vdash y_m},\ z\dashv h_{x_{i_0}\vdash y_l},\ z\dashv h_{y_{l_0}\vdash x_i},\ f_{x_i\dashv x_j}\vdash z,\ f_{y_l\dashv y_m}\vdash z,\ h_{y_l\dashv x_{i_0}}\vdash z,\ h_{x_i\dashv y_{l_0}}\vdash z,\ \text{where}\ z\in X\cup Y.$ 

By a similar proof in Theorem 4.6, all compositions of left and right multiplication mentioned the above are trivial modulo S.

Now, all possible ambiguities [w] of compositions of intersection in S are:

$$1 \wedge 1, [x_{i}x_{j}\dot{x}_{t}]; 1 \wedge 2, [x_{i}\dot{x}_{j}x_{t}]; 1 \wedge 5, [x_{i}x_{i_{0}}\dot{y}_{t}]; 1 \wedge 8, [x_{i}\dot{x}_{j}y_{l_{0}}].$$

$$2 \wedge 2, [\dot{x}_{i}x_{j}x_{t}]; 2 \wedge 8, [\dot{x}_{i}x_{j}y_{l_{0}}].$$

$$3 \wedge 3, [y_{l}y_{m}\dot{y}_{t}]; 3 \wedge 4, [y_{l}\dot{y}_{m}y_{t}]; 3 \wedge 6, [y_{l}\dot{y}_{m}x_{i_{0}}]; 3 \wedge 7, [y_{m}y_{l_{0}}\dot{x}_{i}].$$

$$4 \wedge 4, [\dot{y}_{l}y_{m}y_{t}]; 4 \wedge 6, [\dot{y}_{l}y_{m}x_{i_{0}}].$$

$$5 \wedge 3, [x_{i_{0}}y_{l}\dot{y}_{t}]; 5 \wedge 4, [x_{i_{0}}\dot{y}_{l}y_{t}]; 5 \wedge 6, [x_{i_{0}}\dot{y}_{l}x_{j_{0}}]; 5 \wedge 7, [x_{i_{0}}y_{l_{0}}\dot{x}_{t}].$$

$$6 \wedge 2, [\dot{y}_{l}x_{i_{0}}x_{t}]; 6 \wedge 8, [\dot{y}_{m}x_{i_{0}}y_{l_{0}}].$$

$$7 \wedge 1, [y_{l_{0}}x_{i}\dot{x}_{j}]; 7 \wedge 2, [y_{l_{0}}\dot{x}_{i}x_{j}]; 7 \wedge 5, [y_{l_{0}}x_{i_{0}}\dot{y}_{m}]; 7 \wedge 8, [y_{l_{0}}\dot{x}_{i}y_{m_{0}}].$$

$$8 \wedge 4, [\dot{x}_{i}y_{l_{0}}y_{t}]; 8 \wedge 6, [\dot{x}_{i}y_{l_{0}}x_{i_{0}}].$$

There is no composition of inclusion in S.

We will show that all compositions of intersection in S are trivial. We check only the cases of  $1 \land 5$  and  $2 \land 8$ . Others can be similarly proved.

For 
$$1 \wedge 5$$
,  $[w] = [x_i x_{i_0} \dot{y}_l]$ , let  $\{x_i \vdash x_{i_0}\} = \sum \alpha_{t_0} x_{t_0}$ . Then 
$$(1 \wedge 5)_{[w]} = -\{x_i \vdash x_{i_0}\} \vdash y_l = -\sum \alpha_{t_0} h_{x_{t_0} \vdash y_l} \equiv 0 \mod(S, [w]).$$
 For  $2 \wedge 8$ ,  $[w] = [\dot{x}_i x_i y_{l_0}]$ , let  $\{x_i \dashv x_i\} = \sum \alpha_t x_t$ . Then

$$(2 \wedge 8)_{[w]} = -\{x_i \dashv x_j\} \dashv y_{l_0} = -\sum_{i=1}^{n} \alpha_i h_{x_t \dashv y_{l_0}} \equiv 0 \mod(S, [w]).$$

Then all the compositions in S are trivial. This show (i).

(ii) follows from our Theorem 3.9.  $\square$ 

**Definition 4.9** Let  $X = \{x_1, \ldots, x_n\}$  be a set, k a field of characteristic  $\neq 2$  and  $(a_{ij})_{n \times n}$  a non-zero symmetric matrix over k. Denote

$$D(X \cup \{e\} \mid x_i \vdash x_j + x_j \dashv x_i - 2a_{ij}e, \ e \vdash y - y, \ y \dashv e - y, \ x_i, x_j \in X, \ y \in X \cup \{e\})$$
  
by  $C(n, f)$ . Then  $C(n, f)$  is called a Clifford dialgebra.

We order  $X \cup \{e\}$  by  $x_1 < \cdots < x_n < e$ .

#### Theorem 4.10 Let the notation be as the above. Then

(i) S is a Gröbner-Shirshov basis of Clifford dialgebra C(n, f), where S consists of the following relations:

1. 
$$f_{x_i x_j} = x_i \vdash x_j + x_j \dashv x_i - 2a_{ij}e$$
,

$$2. g_{e \vdash y} = e \vdash y - y,$$

3. 
$$g_{y\dashv e} = y \dashv e - y$$
,

4. 
$$f_{y \dashv x_i x_i} = y \dashv x_i \dashv x_j + y \dashv x_j \dashv x_i - 2a_{ij}y$$
,  $(i > j)$ ,

5. 
$$f_{y \dashv x_i x_i} = y \dashv x_i \dashv x_i - a_{ii}y$$
,

6. 
$$f_{x_i x_i \vdash y} = x_i \vdash x_j \vdash y + x_j \vdash x_i \vdash y - 2a_{ij}y, \quad (i > j),$$

7. 
$$f_{x_i x_i \vdash y} = x_i \vdash x_i \vdash y - a_{ii}y$$
,

8. 
$$h_{x_ie} = x_i \vdash e - e \dashv x_i$$
,

where  $x_i, x_i \in X, y \in X \cup \{e\}$ .

(ii) A k-linear basis of C(n, f) is a set of all elements of the form  $yx_{i_1} \cdots x_{i_k}$ , where  $y \in X \cup \{e\}, x_{i_j} \in X \text{ and } i_1 < i_2 < \cdots < i_k \ (k \ge 0).$ 

**Proof.** Let  $S_1 = \{f_{x_i x_j}, g_{e \vdash y}, g_{y \dashv e} \mid x_i, x_j \in X, y \in X \cup \{e\}\}.$ 

Firstly, we will show that  $f_{y \dashv x_i x_j}$ ,  $f_{y \dashv x_i x_i}$ ,  $f_{x_i x_j \vdash y}$ ,  $f_{x_i x_i \vdash y}$ ,  $h_{x_i e} \in Id(S_1)$ .

In fact,  $f_{y\dashv x_ix_j} = y\dashv f_{x_ix_j} + 2a_{ij}g_{y\dashv e}$  implies  $f_{y\dashv x_ix_j}, \ f_{y\dashv x_ix_i} \in Id(S_1)$ . By symmetry, we have  $f_{x_ix_j\vdash y}, \ f_{x_ix_i\vdash y} \in Id(S_1)$ .

If there exists t such that  $a_{it} \neq 0$ , then

$$2a_{it}h_{x_ie} = f_{x_ix_i \vdash x_t} - x_i \vdash f_{x_i \vdash x_t} + f_{x_i \vdash x_t} \dashv x_i - f_{x_t \dashv x_i x_i} \in Id(S_1).$$

Otherwise,  $a_{it} = 0$  for any t. Since  $(a_{ij}) \neq 0$ , there exists  $j \neq i$  such that  $a_{jt} \neq 0$  for some t. Then

$$2a_{jt}h_{x_ie} = f_{x_ix_j \vdash x_t} - x_i \vdash f_{x_j \vdash x_t} - x_j \vdash f_{x_i \vdash x_t} + f_{x_i \vdash x_t} \dashv x_j + f_{x_j \vdash x_t} \dashv x_i - f_{x_t \dashv x_ix_j} \in Id(S_1).$$

This shows that  $h_{x_ie} \in Id(S_1)$ .

Secondly, we will show that all compositions in S is trivial.

All possible compositions of left and right multiplication are:  $z \dashv f_{x_i x_j}$ ,  $z \dashv g_{e \vdash y}$ ,  $z \dashv f_{x_i x_j \vdash y}$ ,  $z \dashv f_{x_i x_i \vdash y}$ ,  $z \dashv h_{x_i e}$ ,  $f_{x_i x_j} \vdash z$ ,  $f_{y \dashv x_i x_j} \vdash z$ ,  $f_{y \dashv x_i x_j} \vdash z$ ,  $f_{y \dashv x_i x_i} \vdash z$ , where  $z \in X \cup \{e\}$ . We just check the cases of  $f_{y \dashv x_i x_j} \vdash z$  and  $h_{x_i e} \vdash z$ . Others can be similarly proved.

For  $f_{y \dashv x_i x_i} \vdash z$ , we have

$$f_{y \dashv x_i x_j} \vdash z = y \vdash x_i \vdash x_j \vdash z + y \vdash x_j \vdash x_i \vdash z - 2a_{ij}y \vdash z = y \vdash f_{x_i x_j \vdash z} \equiv 0 \ mod(S).$$

For  $h_{x_ie} \vdash z$ ,

$$h_{x_ie} \vdash z = x_i \vdash e \vdash z - e \vdash x_i \vdash z = x_i \vdash g_{e \vdash z} - g_{e \vdash x_i} \vdash z \equiv 0 \ mod(S).$$

Now, all possible ambiguities [w] of compositions of intersection in S are:

```
1 \wedge 3, [x_i \dot{x}_j e]; 1 \wedge 4, [x_i \dot{x}_j x_m x_n] \ (m > n); 1 \wedge 5, [x_i \dot{x}_j x_n x_n].
2 \wedge 1, [ex_i\dot{x}_i]; 2 \wedge 2, [ee\dot{y}]; 2 \wedge 3, [e\dot{y}e]; 2 \wedge 4, [e\dot{y}x_ix_i] \ (i > j);
      2 \wedge 5, [e\dot{y}x_ix_i]; 2 \wedge 6, [ex_ix_i\dot{y}] (i > j); 2 \wedge 7, [ex_ix_i\dot{y}]; 2 \wedge 8, [ex_i\dot{e}].
3 \wedge 3, [\dot{y}ee]; 3 \wedge 4, [\dot{y}ex_ix_i] \ (i > j); 3 \wedge 5, [\dot{y}ex_ix_i].
4 \wedge 3, [\dot{y}x_ix_je] \ (i > j); 4 \wedge 4, [\dot{y}x_ix_jx_mx_n] \ (i > j, m > n), [\dot{y}x_ix_jx_t] \ (i > j > t);
      4 \wedge 5, [\dot{y}x_ix_jx_tx_t] \ (i > j), [\dot{y}x_ix_jx_j] \ (i > j).
5 \wedge 3, [\dot{y}x_ix_ie]; 5 \wedge 4, [\dot{y}x_ix_ix_mx_n] \ (m > n), [\dot{y}x_ix_ix_j] \ (i > j);
      5 \wedge 5, [\dot{y}x_ix_ix_mx_m], [\dot{y}x_ix_ix_i].
6 \wedge 1, [x_i x_j x_m \dot{x}_n] \ (i > j); 6 \wedge 2, [x_i x_j e \dot{y}] \ (i > j); 6 \wedge 3, [x_i x_j \dot{y}e] \ (i > j);
     6 \wedge 4, [x_i x_i \dot{y} x_m x_n] \ (i > j, m > n); 6 \wedge 5, [x_i x_i \dot{y} x_m x_m] \ (i > j);
     6 \wedge 6, [x_i x_j x_m x_n \dot{y}] \ (i > j, m > n), [x_i x_j x_t \dot{y}] \ (i > j > t);
     6 \wedge 7, [x_i x_j x_m x_m \dot{y}] \ (i > j), [x_i x_j x_j \dot{y}] \ (i > j); 6 \wedge 8, [x_i x_j x_t \dot{e}] \ (i > j).
7 \wedge 1, [x_i x_i x_m \dot{x}_n]; 7 \wedge 2, [x_i x_i e \dot{y}]; 7 \wedge 3, [x_i x_i \dot{y} e]; 7 \wedge 4, [x_i x_i \dot{y} x_m x_n] \ (m > n);
     7 \wedge 5, [x_i x_i \dot{y} x_m x_m]; 7 \wedge 6, [x_i x_i x_m x_n \dot{y}] \ (m > n), [x_i x_i x_t \dot{y}] \ (i > t);
      7 \wedge 7, [x_i x_i x_m x_m \dot{y}], [x_i x_i x_i \dot{y}]; 7 \wedge 8, [x_i x_i x_i \dot{e}].
8 \wedge 3, [x_i \dot{e}e]; 8 \wedge 4, [x_i \dot{e}x_m x_n] \ (m > n); 8 \wedge 5, [x_i \dot{e}x_m x_m].
```

All possible ambiguities [w] of compositions of inclusion in S are:

$$6 \wedge 1, [x_i x_j \dot{x}_t] \ (i > j); \ 6 \wedge 8, [x_i x_j \dot{e}] \ (i > j).$$
  
 $7 \wedge 1, [x_i x_i \dot{x}_j]; \ 7 \wedge 8, [x_i x_i \dot{e}].$ 

We just check the cases of intersection  $1 \land 4, 4 \land 4, 6 \land 4, 6 \land 8, 8 \land 4$  and of inclusion  $6 \land 1, 6 \land 8$ . Others can be similarly proved.

For 
$$1 \wedge 4$$
,  $[w] = [x_i \dot{x}_j x_m x_n] \ (m > n)$ , we have 
$$(1 \wedge 4)_{[w]}$$

$$= x_j \dashv x_i \dashv x_m \dashv x_n - 2a_{ij}e \dashv x_m \dashv x_n - x_i \vdash x_j \dashv x_n \dashv x_m + 2a_{mn}x_i \vdash x_j$$

$$= x_j \dashv f_{x_i \dashv x_m x_n} - 2a_{ij}f_{e \dashv x_m x_n} - f_{x_i x_j} \dashv x_n \dashv x_m + 2a_{mn}f_{x_i x_j}$$

$$\equiv 0 \ mod(S, [w]).$$

For  $4 \wedge 4$ , there are two cases to consider:  $[w_1] = [\dot{y}x_ix_jx_mx_n]$  (i > j, m > n) and  $[w_2] = [\dot{y}x_ix_jx_t]$  (i > j > t). We have

$$\begin{array}{l} (4 \wedge 4)_{[w_1]} \\ = & y \dashv x_j \dashv x_i \dashv x_m \dashv x_n - 2a_{ij}y \dashv x_m \dashv x_n - y \dashv x_i \dashv x_j \dashv x_n \dashv x_m + 2a_{mn}y \dashv x_i \dashv x_j \\ = & y \dashv x_j \dashv f_{x_i \dashv x_m x_n} - 2a_{ij}f_{y \dashv x_m x_n} - f_{y \dashv x_i x_j} \dashv x_n \dashv x_m + 2a_{mn}f_{y \dashv x_i x_j} \\ \equiv & 0 \ mod(S, [w_1]) \quad \text{ and } \\ & (4 \wedge 4)_{[w_2]} \\ = & y \dashv x_j \dashv x_i \dashv x_t - 2a_{ij}y \dashv x_t - y \dashv x_i \dashv x_t \dashv x_j + 2a_{jt}y \dashv x_i \\ = & y \dashv f_{x_j \dashv x_i x_t} - f_{y \dashv x_j x_t} \dashv x_i - f_{y \dashv x_i x_t} \dashv x_j + y \dashv f_{x_t \dashv x_i x_j} \\ \equiv & 0 \ mod(S, [w_2]). \end{array}$$

For  $6 \wedge 4$ ,  $[w] = [x_i x_j \dot{y} x_m x_n]$  (i > j, m > n), we have

$$(6 \wedge 4)_{[w]}$$

$$= x_j \vdash x_i \vdash y \dashv x_m \dashv x_n - 2a_{ij}y \dashv x_m \dashv x_n - x_i \vdash x_j \vdash y \dashv x_n \dashv x_m + 2a_{mn}x_i \vdash x_j \vdash y$$

$$= x_j \vdash x_i \vdash f_{y \dashv x_m x_n} - 2a_{ij}f_{y \dashv x_m x_n} - f_{x_i x_j \vdash y} \dashv x_n \dashv x_m + 2a_{mn}f_{x_i x_j \vdash y}$$

$$\equiv 0 \ mod(S, [w]).$$

For  $6 \wedge 8$ ,  $[w] = [x_i x_j x_t \dot{e}] \ (i > j)$ , we have

$$(6 \wedge 8)_{[w]} = x_j \vdash x_i \vdash x_t \vdash e - 2a_{ij}x_t \vdash e + x_i \vdash x_j \vdash e \dashv x_t$$
$$= x_j \vdash x_i \vdash h_{x_t e} - 2a_{ij}h_{x_t e} + f_{x_i x_j \vdash e} \dashv x_t$$
$$\equiv 0 \ mod(S, [w]).$$

For  $8 \wedge 4$ ,  $[w] = [x_i \dot{e} x_m x_n]$  (m > n), we have

$$(8 \wedge 4)_{[w]} = -e \dashv x_i \dashv x_m \dashv x_n - x_i \vdash e \dashv x_n \dashv x_m + 2a_{mn}x_i \vdash e$$
$$= -e \dashv f_{x_i \dashv x_m x_n} - h_{x_i e} \dashv x_n \dashv x_m + 2a_{mn}h_{x_i e}$$
$$\equiv 0 \ mod(S, [w]).$$

Now, we check the compositions of inclusion  $6 \wedge 1$  and  $6 \wedge 8$ .

For  $6 \wedge 1$ ,  $[w] = [x_i x_j \dot{x}_t]$  (i > j), we have

$$(6 \wedge 1)_{[w]} = x_j \vdash x_i \vdash x_t - 2a_{ij}x_t - x_i \vdash x_t \dashv x_j + 2a_{jt}x_i \vdash e$$
  
=  $x_j \vdash f_{x_ix_t} - f_{x_ix_t} \dashv x_j + 2a_{jt}h_{x_ie} - f_{x_jx_t} \dashv x_i + f_{x_t\dashv x_ix_j} + 2a_{it}h_{x_je}$   
\(\equiv 0 \text{ mod}(S, [w]).

For  $6 \wedge 8$ ,  $[w] = [x_i x_j \dot{e}]$  (i > j), we have

$$(6 \land 8)_{[w]} = x_j \vdash x_i \vdash e - 2a_{ij}e + x_i \vdash e \dashv x_j = x_j \vdash h_{x_ie} + h_{x_ie} \dashv x_j + h_{x_je} \dashv x_i + f_{e \dashv x_i x_j} \equiv 0 \ mod(S, [w]).$$

Then all the compositions in S are trivial. We have proved (i).

For (ii), since the mentioned set is just the set Irr(S), by Theorem 3.9 the result holds. The proof is complete.  $\Box$ 

**Remark:** In the Theorem 4.10, if the matrix  $(a_{ij})_{n \times n} = 0$ , then Clifford dialgebra C(n, f) has a Gröbner-Shirshov basis S' which consists of the relations 1–7.

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### References

- [1] M. Aymon and P.-P. Grivel, Un theoreme de Poincare-Birkhoff-Witt pour les algebres de Leibniz, *Comm. Algebra*, 31(2003), N2, 527-544.
- [2] G.M. Bergman, The diamond lemma for ring theory, Adv. in Math., 29, 178-218(1978).
- [3] L.A. Bokut, Unsolvability of the word problem, and subalgebras of finitely presented Lie algebras, *Izv. Akad. Nauk. SSSR Ser. Mat.*, 36, 1173-1219(1972).
- [4] L.A. Bokut, Imbeddings into simple associative algebras, *Algebra i Logika*, 15, 117-142(1976).
- [5] L.A. Bokut and Yuqun Chen, Gröbner-Shirshov bases for Lie algebras: after A.I. Shirshov, *Southeast Asian Bull. Math.*, 31, 1057-1076(2007).
- [6] L.A. Bokut and K.P. Shum, Gröbner and Gröbner-Shirshov bases in algebra: an elementary approach, *Southeast Asian Bull. Math.*, 29, 227-252(2005).
- [7] B. Buchberger, An algorithm for finding a basis for the residue class ring of a zero-dimensional polynomial ideal [in German], Ph.D. thesis, University of Innsbruck, Austria, (1965).
- [8] B. Buchberger, An algorithmical criteria for the solvability of algebraic systems of equations [in German], Aequationes Math., 4, 374-383(1970).
- [9] P.S. Kolesnikov, Conformal representations of Leibniz algebras, arXiv:math/0611501.
- [10] J.-L. Loday, Une version non commutative des algebres de Lie: les algebres de Leibniz, Ens. Math. 39, 269-293(1993).
- [11] J.-L. Loday, Algebras with two associative operations (dialgebras), C. R. Acad. Sci. Paris 321, 141-146(1995).
- [12] J.-L. Loday, Dialgebras, in: Dialgebras and related operads, Lecture Notes in Mathematics, Vol. 1763. Berlin: Springer Verl., 2001, 7-66.
- [13] S. MacLane, Categories for the Working Mathematician, Springer, 1997.
- [14] A.I. Shirshov, Some algorithmic problem for Lie algebras, *Sibirsk. Mat. Z.*, 3(1962), 292-296(in Russian); English translation in SIGSAM Bull., 33(2), 3-6(1999).