# Gröbner-Shirshov bases for dialgebras 

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#### Abstract

In this paper, we define the Gröbner-Shirshov bases for a dialgebra. The composition-diamond lemma for dialgebras is given then. As a result, we obtain a Gröbner-Shirshov basis for the universal enveloping algebra of a Leibniz algebra. Key words: dialgebra, Gröbner-Shirshov bases, composition-diamond lemma, Leibniz algebra


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## 1 Introduction

Recently, J.-L. Loday (1995, [10]) gave the definition of a new class of algebras, dialgebras, which is closely connected to his notion of Leibniz algebras (1993, [9]) and in the same way as associative algebras are connected to Lie algebras. In the manuscript [11], J.-L. Loday found a normal form of elements of a free dialgebra. Here we continue to study free dialgebras and prove the composition-diamond lemma for them. As it is well known, this kind of lemma is the cornerstone of the theory of Gröbner and Gröbner-Shirshov bases (see, for example, [5] and cited literature). In commutative-associative case, this lemma is equivalent to the Main Buchberger's Theorem ([6], [7]). For Lie and associative algebras, this is the Shirshov's lemma [12] (see also L.A. Bokut [3], 4] and G. Bergman [2]). As an application, we get another proof of the Poincare-Birkhoff-Witt theorem for Leibniz algebras, see M. Aymon, P.-P. Grivel [1] and P. Kolesnikov [8].

[^0]
## 2 Preliminaries

Definition 2.1 Let $k$ be a field. A k-linear space $D$ equipped with two bilinear multiplications $\vdash$ and $\dashv$ is called a dialgebra, if both $\vdash$ and $\dashv$ are associative and

$$
\begin{aligned}
a \dashv(b \vdash c) & =a \dashv b \dashv c \\
(a \dashv b) \vdash c & =a \vdash b \vdash c \\
a \vdash(b \dashv c) & =(a \vdash b) \dashv c
\end{aligned}
$$

for any $a, b, c \in D$.
Definition 2.2 Let $D$ be a dialgebra, $B \subset D$. Let us define diwords (dimonomials) of $D$ in the set $B$ by induction:
(i) $b=(b), b \in B$ is a diword in $B$ of length $|b|=1$.
(ii) (u) is called a diword in B of length $n$, if $(u)=((v) \dashv(w))$ or $(u)=((v) \vdash(w))$, where $(v),(w)$ are diwords in $B$ of length $k, l$ respectively and $k+l=n$.

Proposition 2.3 ([11]) Let $D$ be a dialgebra and $B \subset D$. Any diword of $D$ in the set $B$ is equal to a diword in $B$ of the form

$$
\begin{equation*}
(u)=b_{-m} \vdash \cdots \vdash b_{-1} \vdash b_{0} \dashv b_{1} \dashv \cdots \dashv b_{n} \tag{1}
\end{equation*}
$$

where $b_{i} \in B,-m \leq i \leq n, m \geq 0, n \geq 0$. Any bracketing of the right side of (1) gives the same result.

Definition 2.4 Let $X$ be a set. A free dialgebra $D(X)$ generated by $X$ over $k$ is defined in a usual way by the following commutative diagram:

where $D$ is any dialgebra.
In [11], a construction of a free dialgebra is given.
Proposition 2.5 ([11]) Let $D(X)$ be free dialgebra generated by $X$ over $k$. Any diword in $X$ is equal to the unique diword in $X$ of the form

$$
\begin{equation*}
[u]=x_{-m} \vdash \cdots \vdash x_{-1} \vdash x_{0} \dashv x_{1} \dashv \cdots \dashv x_{n}=x_{-m} \cdots x_{-1} \dot{x_{0}} x_{1} \cdots x_{n} \tag{2}
\end{equation*}
$$

where $x_{i} \in X, m \geq 0, n \geq 0$. We call $[u]$ a normal diword (in $X$ ) with the associative word $u, u \in X^{*}$. Clearly, if $[u]=[v]$, then $u=v$. In (园), $x_{0}$ is called the center of the normal diword $[u]$. Let $[u]$, $[v]$ be two normal diwords, then $[u] \vdash[v]$ is the normal diword $[u v]$ with the center at the center of $[v]$. Accordingly, $[u] \dashv[v]$ is the normal diword $[u v]$ with the center at the center of $[u]$.

## Example 2.6

$$
\begin{gathered}
\left(x_{-1} \vdash x_{0} \dashv x_{1}\right) \vdash\left(y_{-1} \vdash y_{0} \dashv y_{1}\right)=x_{-1} \vdash x_{0} \vdash x_{1} \vdash y_{-1} \vdash y_{0} \dashv y_{1}, \\
\left(x_{-1} \vdash x_{0} \dashv x_{1}\right) \dashv\left(y_{-1} \vdash y_{0} \dashv y_{1}\right)=x_{-1} \vdash x_{0} \dashv x_{1} \dashv y_{-1} \dashv y_{0} \dashv y_{1} .
\end{gathered}
$$

Definition 2.7 $A$-linear space $L$ equipped with bilinear multiplication [,] is called a Leibniz algebra if for any $a, b, c \in L$,

$$
[[a, b], c]=[[a, c], b]+[a,[b, c]]
$$

i.e., the Jacobi identity is valid in $L$.

It is clear that if $(D, \dashv, \vdash)$ is a dialgebra then $D^{(-)}=(D,[]$,$) is a Leibniz algebra, where$ $[a, b]=a \dashv b-b \vdash a$ for any $a, b \in D$.

## 3 Composition-Diamond lemma for dialgebras

Let $X$ be a well ordered set, $D(X)$ the free dialgebra over $k, X^{*}$ the free monoid generated by $X$ and $\left[X^{*}\right]$ the set of normal diwords in $X$. Let us define deg-lex order on $\left[X^{*}\right]$ in the following way: for any $[u],[v] \in\left[X^{*}\right]$,

$$
[u]<[v] \Longleftrightarrow w t([u])<w t([v]) \text { lexicographicaly, }
$$

where

$$
w t([u])=\left(n+m+1, m, x_{-m}, \cdots, x_{0}, \cdots, x_{n}\right)
$$

if $[u]=x_{-m} \cdots x_{-1} \dot{x_{0}} x_{1} \cdots x_{n}$. It is easy to see that the order $<$ is monomial in the sense:

$$
[u]<[v] \Longrightarrow x \vdash[u]<x \vdash[v],[u] \dashv x<[v] \dashv x, \text { for any } x \in X .
$$

Any polynomial $f \in D(X)$ has the form

$$
f=\sum_{[u] \in\left[X^{*}\right]} f([u])[u]=\alpha[\bar{f}]+\sum \alpha_{i}\left[u_{i}\right],
$$

where $[\bar{f}],\left[u_{i}\right]$ are normal diwords in $X,[\bar{f}]>\left[u_{i}\right], \alpha, \alpha_{i}, f([u]) \in k$. We call $[\bar{f}]$ the leading term of $f$. Denote by suppf the set $\{[u] \mid f([u]) \neq 0\}$ and $\operatorname{deg}(f)$ by $|[\bar{f}]| . f$ is called monic if $\alpha=1 . f$ is called left (right) normed if $f=\sum \alpha_{i} u_{i} \dot{x_{i}} \quad\left(f=\sum \alpha_{i} \dot{x}_{i} u_{i}\right)$, where each $\alpha_{i} \in k, x_{i} \in X$ and $u_{i} \in X^{*}$. The same terminology will be used for normal diwords.

If $[u],[v]$ are both left normed or both right normed, then it is clear that for any $w \in\left[X^{*}\right]$,

$$
[u]<[v] \Longrightarrow[u] \vdash w<[v] \vdash w, w \vdash[u]<w \vdash[v],[u] \dashv w<[v] \dashv w, w \dashv[u]<w \dashv[v] .
$$

Let $S \subset D(X)$. By an $S$-diword $g$ we will mean $g$ is a diword in $\{X \cup S\}$ with only one occurrence of $s \in S$. If this is the case and $g=(a s b)$ for some $a, b \in X^{*}$ and $s \in S$, we also call $g$ an $s$-diword.

From Proposition 2.3 it follows easily that any $S$-diword is equal to

$$
\begin{equation*}
[a s b]=\left.x_{-m} \vdash \cdots \vdash x_{-1} \vdash x_{0} \dashv x_{1} \dashv \cdots \dashv x_{n}\right|_{x_{k} \mapsto s} \tag{3}
\end{equation*}
$$

where $-m \leq k \leq n, x_{k} \in X, s \in S$. To be more precise, $[a s b]=[a \dot{s} b]$ if $k=0$; $[a s b]=\left[a s b_{1} \dot{x_{0}} b_{2}\right]$ if $k<0$ and $[a s b]=\left[a_{1} \dot{x_{0}} a_{2} s b\right]$ if $k>0$. Note that any bracketing of [asb] gives the same result, for example, $[a s b]=\left[\left(a_{1} a_{2}\right) s b\right]=\left[a_{1}\left(a_{2} s\right) b\right]$ if $a=a_{1} a_{2}$. If the center of the $s$-diword $[a s b]$ is in $a$, then we denote by $[\dot{a} s b]=\left[a_{1} \dot{x_{0}} a_{2} s b\right]$. Similarly, $[a s \dot{b}]=\left[a s b_{1} \dot{x_{0}} b_{2}\right]$ (of course, some $a_{i}, b_{i}$ may be empty).

Definition 3.1 The $S$-diword (3) is called a normal $S$-diword if one of the following conditions holds:
(i) $k=0$.
(ii) $k<0$ and $s$ is left normed.
(iii) $k>0$ and $s$ is right normed.

We call a normal s-diword [asb] a left (right) normed s-diword, if both $s$ and [asb] are left (right) normed. In particulary, $s$ is a left (right) normed $s$-diword, if $s$ is left (right) normed polynomial.

The following lemma follows from the above properties of the order of normal diwords.

Lemma 3.2 For a normal $S$-diword $[a s b]$, the leading term of $[$ asb $]$ is equal to $[a[\bar{s}] b]$, that is, $\overline{[a s b]}=[a[\bar{s}] b]$. More specifically, if

$$
[a s b]=\left.x_{-m} \vdash \cdots \vdash x_{-1} \vdash x_{0} \dashv x_{1} \dashv \cdots \dashv x_{n}\right|_{x_{k} \mapsto s},
$$

then

$$
\begin{aligned}
& \overline{x_{-m} \vdash \cdots \vdash x_{-1} \vdash s \dashv x_{1} \dashv \cdots \dashv x_{n}}=x_{-m} \vdash \cdots \vdash x_{-1} \vdash[\bar{s}] \dashv x_{1} \dashv \cdots \dashv x_{n} \\
& \overline{x_{-m} \vdash \cdots \vdash s \vdash \cdots \vdash x_{0} \dashv \cdots \dashv x_{n}}=x_{-m} \vdash \cdots \vdash[\bar{s}] \vdash \cdots \vdash x_{0} \dashv \cdots \dashv x_{n} \\
& \overline{x_{-m} \vdash \cdots \vdash x_{0} \dashv \cdots \dashv s \dashv \cdots \dashv x_{n}}=x_{-m} \vdash \cdots \vdash x_{0} \dashv \cdots \dashv[\bar{s}] \dashv \cdots \dashv x_{n}
\end{aligned}
$$

For convenience, we denote $[a[\bar{s}] b]$ by $[a \bar{s} b]$ for a normal $S$-diword [asb].
Now, we define compositions of dipolynomials in $D(X)$.
Definition 3.3 Let the order $<$ be as before and $f, g \in D(X)$ with $f, g$ monic.

1) Composition of left (right) multiplication.

Let $f$ be a not right normed polynomial and $x \in X$. Then $x \dashv f$ is called the composition of left multiplication. Clearly, $x \dashv f$ is a right normed polynomial (or $0)$.
Let $f$ be a not left normed polynomial and $x \in X$. Then $f \vdash x$ is called the composition of right multiplication. Clearly, $f \vdash x$ is a left normed polynomial (or $0)$.
2) Composition of including.

Let

$$
[w]=[\bar{f}]=[a \bar{g} b],
$$

where $[a g b]$ is a normal $g$-diword. Then

$$
(f, g)_{[w]}=f-[a g b]
$$

is called the composition of including. The transformation $f \mapsto f-[a g b]$ is called the elimination of leading diword (ELW) of $g$ in $f$.
3) Composition of intersection.

Let

$$
[w]=[\bar{f} b]=[a \bar{g}],|\bar{f}|+|\bar{g}|>|w|,
$$

where $[f b]$ is a normal $f$-diword and $[a g]$ a normal $g$-diword. Then

$$
(f, g)_{[w]}=[f b]-[a g]
$$

is called the composition of intersection.
Remark In the Definition 3.3, for the case of 2) or 3), we have $\overline{(f, g)_{[w]}}<[w]$. For the case of 1$)$, $\operatorname{deg}(x \dashv f) \leq \operatorname{deg}(f)+1$ and $\operatorname{deg}(f \vdash x) \leq \operatorname{deg}(f)+1$.

Definition 3.4 Let the order $<$ be as before, $S \subset D(X)$ a monic set and $f, g \in S$.

1) Let $x \dashv f$ be a composition of left multiplication. Then $x \dashv f$ is called trivial modulo $S$, denoted by $\quad x \dashv f \equiv 0 \bmod (S), \quad i f$

$$
x \dashv f=\sum \alpha_{i}\left[a_{i} s_{i} b_{i}\right],
$$

where each $\alpha_{i} \in k, a_{i}, b_{i} \in\left[X^{*}\right], s_{i} \in S,\left[a_{i} s_{i} b_{i}\right]$ right normed $s_{i}$-diword and $\left|\left[a_{i} \bar{S}_{i} b_{i}\right]\right| \leq \operatorname{deg}(x \dashv f)$.

Let $f \vdash x$ be a composition of right multiplication. Then $f \vdash x$ is called trivial modulo $S$, denoted by $\quad f \vdash x \equiv 0 \bmod (S)$, if

$$
f \vdash x=\sum \alpha_{i}\left[a_{i} s_{i} b_{i}\right],
$$

where each $\alpha_{i} \in k, a_{i}, b_{i} \in\left[X^{*}\right], s_{i} \in S,\left[a_{i} s_{i} b_{i}\right]$ left normed $s_{i}$-diword and $\left|\left[a_{i} \overline{s_{i}} b_{i}\right]\right| \leq$ $\operatorname{deg}(f \vdash x)$.
2) Composition $(f, g)_{[w]}$ of including (intersection) is called trivial modulo $(S,[w])$, denoted by $\quad(f, g)_{[w]} \equiv 0 \bmod (S,[w]), \quad$ if

$$
(f, g)_{[w]}=\sum \alpha_{i}\left[a_{i} s_{i} b_{i}\right],
$$

where each $\alpha_{i} \in k, a_{i}, b_{i} \in\left[X^{*}\right], s_{i} \in S,\left[a_{i} s_{i} b_{i}\right]$ normal $s_{i}$-diword, $\left[a_{i} \overline{s_{i}} b_{i}\right]<[w]$ and each $\left[a_{i} s_{i} b_{i}\right]$ is right (left) normed $s_{i}$-diword whenever both $f$ and $[a g b]$ ( $[f b]$ and [ag]) are right (left) normed $S$-diwords.

The following proposition is useful when one checks the compositions of left and right multiplications.

Proposition 3.5 Let the order $<$ be as before, $S \subset D(X)$ a monic set and $f \in S$. Let $x \dashv f$ be a composition of left multiplication. Then $x \dashv f \equiv 0 \bmod (S)$ if and only if

$$
x \dashv f=\sum \alpha_{i}\left[a_{i} s_{i} b_{i}\right],
$$

where each $\alpha_{i} \in k, a_{i}, b_{i} \in X^{*}, s_{i} \in S$ is right normed, $\left[a_{i} s_{i} b_{i}\right]=\left[\dot{a}_{i} s_{i} b_{i}\right]$ and $\left|\left[a_{i} \overline{S_{i}} b_{i}\right]\right| \leq$ $\operatorname{deg}(x \dashv f)$.

Accordingly, for the composition of right multiplication, we have a similar conclusion.

Proof Assume that $x \dashv f=\sum \alpha_{i}\left[a_{i} s_{i} b_{i}\right]$, where each $\alpha_{i} \in k, a_{i}, b_{i} \in\left[X^{*}\right],\left[a_{i} s_{i} b_{i}\right]=$ $\left[\dot{a}_{i} s_{i} b_{i}\right], s_{i} \in S$ right normed and $\left|\left[a_{i} \overline{s_{i}} b_{i}\right]\right| \leq \operatorname{deg}(x \dashv f)$. Then, we have the expression

$$
x \dashv f=[\dot{x} f]=\sum_{I_{1}} \alpha_{p}\left[\dot{x_{p}} a_{p} s_{p} b_{p}\right]+\sum_{I_{2}} \beta_{q}\left[a_{q} \dot{x_{q}} a_{q}^{\prime} s_{q} b_{q}\right],
$$

where each $\alpha_{p}, \beta_{q} \in k, x_{p}, x_{q} \in X, a_{p}, a_{q}, a_{q}^{\prime}, b_{p}, b_{q} \in X^{*}, a_{q} \neq 1, s_{p}, s_{q} \in S$ are right normed. From this it follows that $\sum_{I_{2}} \beta_{q}\left[a_{q} \dot{x_{q}} a_{q}^{\prime} s_{q} b_{p}\right]=0$. Now, the results follow.

Definition 3.6 Let $S \subset D(X)$ be a monic set and the order $<$ as before. We call the set $S$ a Gröbner-Shirshov set (basis) in $D(X)$ if any composition of polynomials in $S$ is trivial modulo $S$ (and $[w]$ ).

The following two lemmas play key role in the proof of Theorem 3.9,

Lemma 3.7 Let $S \subset D(X)$ and $[a s b]$ an $S$-diword. Assume that each composition of right or left multiplication is trivial modulo $S$. Then, $[$ asb $]$ has a presentation:

$$
[a s b]=\sum \alpha_{i}\left[a_{i} s_{i} b_{i}\right]
$$

where each $\alpha_{i} \in k, s_{i} \in S, a_{i}, b_{i} \in\left[X^{*}\right]$ and each $\left[a_{i} s_{i} b_{i}\right]$ is normal $s_{i}$-diword.
Proof Following Proposition 2.3, we assume that

$$
[a s b]=\left.x_{-m} \vdash \cdots \vdash x_{-1} \vdash x_{0} \dashv x_{1} \dashv \cdots \dashv x_{n}\right|_{x_{k} \mapsto s} .
$$

There are three cases to consider.
Case 1. $k=0$. Then $[a s b]$ is a normal $S$-diword.
Case 2. $k<0$. Then $[a s b]=a \vdash\left(s \vdash x_{k+1}\right) \vdash b, k<-1$ or $[a s b]=a \vdash\left(s \vdash x_{0}\right) \dashv b$. If $s$ is left normed then $[a s b]$ is a normal $S$-diword. If $s$ is not left normed then for the composition $s \vdash x_{k+1} \quad(k<0)$ of right multiplication, we have

$$
s \vdash x_{k+1}=\sum \alpha_{i}\left[a_{i} s_{i} b_{i}\right],
$$

where each $\alpha_{i} \in k, a_{i}, b_{i} \in\left[X^{*}\right], s_{i} \in S$ and $\left[a_{i} s_{i} b_{i}\right]$ is left normed $s_{i}$-diword. Then

$$
[a s b]=\sum \alpha_{i}\left(a \vdash\left[a_{i} s_{i} b_{i}\right] \vdash b\right)
$$

or

$$
[a s b]=\sum \alpha_{i}\left(a \vdash\left[a_{i} s_{i} b_{i}\right] \dashv b\right)
$$

is a linear combination of normal $S$-diwords.
Case 3. $k>0$ is similar to the Case 2.
Lemma 3.8 Let $S \subset D(X)$ and each composition $(f, g)_{[w]}$ in $S$ of including (intersection) trivial modulo $(S,[w])$. Let $\left[a_{1} s_{1} b_{1}\right]$ and $\left[a_{2} s_{2} b_{2}\right]$ be normal $S$-diwords such that $[w]=$ $\left[a_{1} \overline{s_{1}} b_{1}\right]=\left[a_{2} \overline{s_{2}} b_{2}\right]$. Then,

$$
\left[a_{1} s_{1} b_{1}\right] \equiv\left[a_{2} s_{2} b_{2}\right] \bmod (S,[w]) .
$$

Proof Because $a_{1} \overline{s_{1}} b_{1}=a_{2} \overline{s_{2}} b_{2}$ as words, there are three cases to consider.
Case 1. Subwords $\overline{s_{1}}, \overline{s_{2}}$ have empty intersection. Assume, for example, that $b_{1}=b \overline{s_{2}} b_{2}$ and $a_{2}=a_{1} \overline{s_{1}} b$. Because any normal $S$-diword may be bracketing in any way, we have

$$
\left[a_{2} s_{2} b_{2}\right]-\left[a_{1} s_{1} b_{1}\right]=\left(a_{1} s_{1}\left(b\left(s_{2}-\left[\overline{s_{2}}\right]\right) b_{2}\right)\right)-\left(\left(a_{1}\left(s_{1}-\left[\overline{s_{1}}\right]\right) b\right) s_{2} b_{2}\right) .
$$

For any $t \in \operatorname{supp}\left(s_{2}-\overline{s_{2}}\right) \quad\left(t \in \operatorname{supp}\left(s_{1}-\overline{s_{1}}\right)\right)$, we prove that $\left(a_{1} s_{1} b t b_{2}\right)\left(\left(a_{1} t b s_{2} b_{2}\right)\right)$ is a normal $s_{1}$-diword ( $s_{2}$-diword ). There are five cases to consider.
$1.1[w]=\left[a_{1} \overline{s_{1}} b \overline{s_{2}} b_{2}\right] ;$
$1.2[w]=\left[a_{1} \dot{\overline{s_{1}}} b \overline{s_{2}} b_{2}\right] ;$
$1.3[w]=\left[a_{1} \overline{s_{1}} \dot{b} \overline{s_{2}} b_{2}\right] ;$
$1.4[w]=\left[a_{1} \overline{s_{1}} b \dot{\overline{s_{2}}} b_{2}\right] ;$
$1.5[w]=\left[a_{1} \overline{s_{1}} b \overline{s_{2}} \dot{b_{2}}\right]$.
For 1.1, since $\left[a_{1} s_{1} b_{1}\right]$ and $\left[a_{2} s_{2} b_{2}\right]$ are normal $S$-diwords, both $s_{1}$ and $s_{2}$ are right normed by the definition, in particular, $t$ is right normed. It follows that $\left(a_{1} s_{1} b t b_{2}\right)=\left[a_{1} s_{1} b t b_{2}\right]$ is a normal $s_{1}$-diword.

For 1.2, it is clear that $\left(a_{1} s_{1} b t b_{2}\right)$ is a normal $s_{1}$-diword and $t$ is right normed.
For 1.3, 1.4 and 1.5 , since $\left[a_{1} s_{1} b_{1}\right]$ is normal $s_{1}$-diword, $s_{1}$ is left normed by the definition, which implies that $\left(a_{1} s_{1} b t b_{2}\right)$ is a normal $s_{1}$-diword. Moreover, $t$ is right normed, if 1.3, and left normed, if 1.5.

Thus, for all cases, we have $\left.\overline{\left[a_{1} s_{1} b t b_{2}\right.}\right]=\left[a_{1} \overline{s_{1}} b t b_{2}\right]<\left[a_{1} \overline{s_{1}} b \overline{s_{2}} b_{2}\right]=[w]$.
Similarly, for any $t \in \operatorname{supp}\left(s_{1}-\overline{s_{1}}\right),\left(a_{1} t b s_{2} b_{2}\right)$ is a normal $s_{2}$-diword and $\left[a_{1} t b \overline{s_{2}} b_{2}\right]<[w]$.
Case 2. Subwords $\overline{s_{1}}$ and $\overline{s_{2}}$ have non-empty intersection $c$. Assume, for example, that $b_{1}=b b_{2}, a_{2}=a_{1} a, w_{1}=\overline{s_{1}} b=a \overline{s_{2}}=a c b$.

There are following five cases to consider:
$2.1[w]=\left[a_{1} \overline{s_{1}} b b_{2}\right]$;
$2.2[w]=\left[a_{1} \overline{s_{1}} b \dot{b_{2}}\right] ;$
$2.3[w]=\left[a_{1} \dot{a} c b b_{2}\right] ;$
$2.4[w]=\left[a_{1} a \dot{c} b b_{2}\right] ;$
$2.5[w]=\left[a_{1} a c \dot{b} b_{2}\right]$.
Then

$$
\left[a_{2} s_{2} b_{2}\right]-\left[a_{1} s_{1} b_{1}\right]=\left(a_{1}\left(\left[a s_{2}\right]-\left[s_{1} b\right]\right) b_{2}\right)=\left(a_{1}\left(s_{1}, s_{2}\right)_{\left[w_{1}\right]} b_{2}\right)
$$

where $\left[w_{1}\right]=[a c b]$ is as follows:
$2.1\left[w_{1}\right]$ is right normed;
$2.2\left[w_{1}\right]$ is left normed;
$2.3\left[w_{1}\right]=[\dot{a} c b]$;
$2.4\left[w_{1}\right]=[a \dot{c} b] ;$
$2.5\left[w_{1}\right]=[a c \dot{b}]$.
Since $S$ is a Gröbner-Shirshov basis, there exist $\beta_{j} \in k, u_{j}, v_{j} \in\left[X^{*}\right], s_{j} \in S$ such that $\left[s_{1} b\right]-\left[a s_{2}\right]=\sum_{j} \beta_{j}\left[u_{j} s_{j} v_{j}\right]$, where each $\left[u_{j} s_{j} v_{j}\right]$ is normal $S$-diword and $\left[u_{j} \overline{s_{j}} v_{j}\right]<\left[w_{1}\right]=$ [acb]. Therefore,

$$
\left[a_{2} s_{2} b_{2}\right]-\left[a_{1} s_{1} b_{1}\right]=\sum_{j} \beta_{j}\left(a_{1}\left[u_{j} s_{j} v_{j}\right] b_{2}\right)
$$

Now, we prove that each $\left(a_{1}\left[u_{j} s_{j} v_{j}\right] b_{2}\right)$ is normal $s_{j}$-diword and $\overline{\left(a_{1}\left[u_{j} s_{j} v_{j}\right] b_{2}\right)}<[w]=$ [ $\left.a_{1} \overline{s_{1}} b b_{2}\right]$.

For 2.1, since $\left[\dot{a}_{1} s_{1} b b_{2}\right]$ and $\left[\dot{a}_{1} a s_{2} b_{2}\right]$ are normal $S$-diwords, both $\left[s_{1} b\right]$ and $\left[a s_{2}\right]$ are right normed $S$-diwords. Then, by the definition, each $\left[u_{j} s_{j} v_{j}\right]$ is right normed $S$-diword, and so each $\left(a_{1}\left[u_{j} s_{j} v_{j}\right] b_{2}\right)=\left[\dot{1}_{1} u_{j} s_{j} v_{j} b_{2}\right]$ is a normal $S$-diword.

For 2.2, both $\left[s_{1} b\right]$ and $\left[a s_{2}\right]$ must be left normed $S$-diwords. Then, by the definition, each $\left[u_{j} s_{j} v_{j}\right]$ is left normed $S$-diword, and so each $\left(a_{1}\left[u_{j} s_{j} v_{j}\right] b_{2}\right)=\left[a_{1} u_{j} s_{j} v_{j} \dot{b}_{2}\right]$ is a normal $S$-diword.

For 2.3, 2.4 or 2.5, by noting that $\left(a_{1}\left[u_{j} s_{j} v_{j}\right] b_{2}\right)=\left(\left(a_{1}\right) \vdash\left[u_{j} s_{j} v_{j}\right] \dashv\left(b_{2}\right)\right)$ and $\left[u_{j} s_{j} v_{j}\right]$ is normal $S$-diword, $\left(a_{1}\left[u_{j} s_{j} v_{j}\right] b_{2}\right)$ is also normal $S$-diword.

Now, for all cases, we have $\overline{\left[a_{1} u_{j} s_{j} v_{j} b_{2}\right]}=\left[a_{1} u_{j} \overline{s_{j}} v_{j} b_{2}\right]<[w]=\left[a_{1} a c b b_{2}\right]$.
Case 3. One of the subwords $\overline{s_{1}}$ and $\overline{s_{2}}$ contains another as a subword. Assume, for example, that $b_{2}=b b_{1}, a_{2}=a_{1} a, w_{1}=\overline{s_{1}}=a \overline{s_{2}} b$.

Again there are following five cases to consider:
$2.1[w]=\left[a_{1} a \overline{s_{2}} b b_{1}\right] ;$
$2.2[w]=\left[a_{1} a \overline{a_{2}} b \dot{b_{1}}\right] ;$
$2.3[w]=\left[a_{1} \dot{a} \overline{s_{2}} b b_{1}\right] ;$
$2.4[w]=\left[a_{1} a \dot{\overline{s_{2}}} b b_{1}\right] ;$
$2.5[w]=\left[a_{1} a \overline{a_{2}} \dot{b} b_{1}\right]$.
Then

$$
\left[a_{1} s_{1} b_{1}\right]-\left[a_{2} s_{2} b_{2}\right]=\left(a_{1}\left(s_{1}-a s_{2} b\right) b_{1}\right)=\left(a_{1}\left(s_{1}, s_{2}\right)_{\left[w_{1}\right]} b_{1}\right)
$$

It is similar to the proof of the Case 2, that we have $\left[a_{1} s_{1} b_{1}\right] \equiv\left[a_{2} s_{2} b_{2}\right] \bmod (S,[w])$.

The following theorem is the main result.

Theorem 3.9 (Composition-Diamond Lemma) Let $S \subset D(X)$ be a monic set and the order $<$ as before. Then $(i) \Rightarrow(i i) \Leftrightarrow(i i)^{\prime} \Leftrightarrow(i i i) \Rightarrow(i v)$, where
(i) $S$ is a Gröbner-Shirshov basis.
(ii) For any $f \in D(X), 0 \neq f \in I d(S) \Rightarrow \bar{f}=[a \bar{s} b]$ for some $s \in S, a, b \in\left[X^{*}\right]$ and [asb] a normal $S$-diword.
(ii) $)^{\prime}$ For any $f \in D(X)$, if $0 \neq f \in \operatorname{Id}(S)$, then $f=\alpha_{1}\left[a_{1} s_{1} b_{1}\right]+\alpha_{2}\left[a_{2} s_{2} b_{2}\right]+\cdots+\alpha_{n}\left[a_{n} s_{n} b_{n}\right]$ with $\left[a_{1} \overline{s_{1}} b_{1}\right]>\left[a_{2} \overline{s_{2}} b_{2}\right]>\cdots>\left[a_{n} \overline{s_{n}} b_{n}\right]$, where $\left[a_{i} s_{i} b_{i}\right]$ is normal $S$-diword, $i=1,2, \cdots, n$.
(iii) The set

$$
\operatorname{Irr}(S)=\left\{u \in\left[X^{*}\right] \mid u \neq[a \bar{s} b], s \in S, a, b \in\left[X^{*}\right],[a s b] \text { is normal } S \text {-diword }\right\}
$$ is a linear basis of the dialgebra $D(X \mid S)$.

(iv) For each composition $(f, g)_{[w]}$ of including (intersection), we have

$$
(f, g)_{[w]}=\sum \alpha_{i}\left[a_{i} s_{i} b_{i}\right],
$$

where each $\alpha_{i} \in k, a_{i}, b_{i} \in\left[X^{*}\right], s_{i} \in S,\left[a_{i} s_{i} b_{i}\right]$ normal $S$-diword and $\left[a_{i} \bar{S}_{i} b_{i}\right]<[w]$.
Proof $(i) \Rightarrow(i i)$. Let $S$ be a Gröbner-Shirshov basis and $0 \neq f \in \operatorname{Id}(S)$. We can assume, by Lemma 3.7, that

$$
f=\sum_{i=1}^{n} \alpha_{i}\left[a_{i} s_{i} b_{i}\right]
$$

where each $\alpha_{i} \in k, a_{i}, b_{i} \in\left[X^{*}\right], s_{i} \in S$ and $\left[a_{i} s_{i} b_{i}\right]$ normal $S$-diword. Let

$$
\left[w_{i}\right]=\left[a_{i} \bar{S}_{i} b_{i}\right],\left[w_{1}\right]=\left[w_{2}\right]=\cdots=\left[w_{l}\right]>\left[w_{l+1}\right] \geq \cdots
$$

We will use the induction on $l$ and $\left[w_{1}\right]$ to prove that $\bar{f}=[a \bar{s} b]$, for some $s \in S$ and $a, b \in$ $\left[X^{*}\right]$. If $l=1$, then $\bar{f}=\overline{\left[a_{1} s_{1} b_{1}\right]}=\left[a_{1} \overline{s_{1}} b_{1}\right]$ and hence the result holds. Assume that $l \geq 2$. Then, by Lemma 3.8, we have $\left[a_{1} s_{1} b_{1}\right] \equiv\left[a_{2} s_{2} b_{2}\right] \bmod \left(S,\left[w_{1}\right]\right)$.

Thus, if $\alpha_{1}+\alpha_{2} \neq 0$ or $l>2$, then the result holds. For the case $\alpha_{1}+\alpha_{2}=0$ and $l=2$, we use the induction on $\left[w_{1}\right]$. Now, the result follows.
$(i i) \Rightarrow(i i)^{\prime}$. Assume (ii) and $f \in \operatorname{Id}(S)$. Let $f=\alpha_{1} \bar{f}+\sum_{\left[u_{i}\right]<\bar{f}} \alpha_{i}\left[u_{i}\right]$. Then, by (ii), $\bar{f}=\left[a_{1} \overline{s_{1}} b_{1}\right]$, where $\left[a_{1} s_{1} b_{1}\right]$ is a normal $S$-diword. Therefore,

$$
f_{1}=f-\alpha_{1}\left[a_{1} s_{1} b_{1}\right], \overline{f_{1}}<\bar{f}, f_{1} \in I d(S)
$$

Now, by using induction on $\bar{f}$, we have $(i i)^{\prime}$.
$(i i)^{\prime} \Rightarrow(i i)$. This part is clear.
$(i i)^{\prime} \Rightarrow(i i i)$. Assume $(i i)^{\prime}$. We firstly prove that, for any $h \in D(X)$, we have

$$
\begin{equation*}
h=\sum_{I_{1}} \alpha_{i}\left[u_{i}\right]+\sum_{I_{2}} \beta_{j}\left[a_{j} s_{j} b_{j}\right] \tag{4}
\end{equation*}
$$

where $\left[u_{i}\right] \in \operatorname{Irr}(S), i \in I_{1},\left[a_{j} s_{j} b_{j}\right]$ normal $S$-diwords, $j \in I_{2}$.

Let $h=\alpha_{1} \bar{h}+\cdots$. We use the induction on $\bar{h}$.
If $\bar{h} \in \operatorname{Irr}(S)$, then take $\left[u_{1}\right]=\bar{h}$ and $h_{1}=h-\alpha_{1}\left[u_{1}\right]$. Clearly, $\overline{h_{1}}<\bar{h}$.
If $\bar{h} \notin \operatorname{Irr}(S)$, then $\bar{h}=\left[a_{1} \overline{s_{1}} b_{1}\right]$ with $\left[a_{1} s_{1} b_{1}\right]$ a normal $S$-diword. Let $h_{1}=h-\beta_{1}\left[a_{1} s_{1} b_{1}\right]$. Then $\overline{h_{1}}<\bar{h}$.
Suppose that $0 \neq \sum \alpha_{i}\left[u_{i}\right]=\sum \beta_{j}\left[a_{j} s_{j} b_{j}\right]$, where $\left[u_{1}\right]>\left[u_{2}\right]>\cdots,\left[u_{i}\right] \in \operatorname{Irr}(S)$ and $\left[a_{1} \overline{\bar{s}_{1}} b_{1}\right]>\left[a_{2} \overline{\bar{s}_{2}} b_{2}\right]>\cdots$. Then, $\left[u_{1}\right]=\left[a_{1} \overline{s_{1}} b_{1}\right]$, a contradiction.

Now, (iii) follows.
$(i i i) \Rightarrow(i i)$ and (iv). Assume (iii). For any $0 \neq f \in \operatorname{Id}(S), \bar{f} \notin \operatorname{Irr}(S)$ implies that $\bar{f}=[a \bar{s} b]$, where $[a s b]$ is a normal $S$-diword. This shows (ii).

By noting that $(f, g)_{[w]} \in I d(S)$ and by using (4) and ELW, we have

$$
(f, g)_{[w]}=\sum \alpha_{i}\left[a_{i} s_{i} b_{i}\right]
$$

where each $\alpha_{i} \in k, a_{i}, b_{i} \in\left[X^{*}\right], s_{i} \in S,\left[a_{i} s_{i} b_{i}\right]$ normal $S$-diword and $\left[a_{i} \overline{s_{i}} b_{i}\right]<[w]$.

## 4 Applications

Now, by using Theorem [3.9, we obtain a Gröbner-Shirshov basis for the universal enveloping algebra of a Leibniz algebra.

Theorem 4.1 Let $\mathcal{L}$ be a Leibniz algebra over a field $k$ with the product $\{$,$\} . Let \mathcal{L}_{0}$ be the subspace of $\mathcal{L}$ generated by the set $\{\{a, a\},\{a, b\}+\{b, a\} \mid a, b \in \mathcal{L}\}$. Let $\left\{x_{i} \mid i \in I_{0}\right\}$ be a basis of $\mathcal{L}_{0}$ and $X=\left\{x_{i} \mid i \in I\right\}$ a linearly ordered basis of $\mathcal{L}$ such that $I_{0} \subseteq I$. Let $D\left(X \mid x_{i} \dashv x_{j}-x_{j} \vdash x_{i}-\left\{x_{i}, x_{j}\right\}\right)$ be the dialgebra and the order $<$ on $\left[X^{*}\right]$ as before. Then
(i) $D\left\langle X \mid x_{i} \dashv x_{j}-x_{j} \vdash x_{i}-\left\{x_{i}, x_{j}\right\}\right\rangle=D(X \mid S)$, where $S$ consists of the following polynomials:

1. $f_{j i}=x_{j} \vdash x_{i}-x_{i} \dashv x_{j}+\left\{x_{i}, x_{j}\right\} \quad(i, j \in I)$
2. $f_{j i \vdash t}=x_{j} \vdash x_{i} \vdash x_{t}-x_{i} \vdash x_{j} \vdash x_{t}+\left\{x_{i}, x_{j}\right\} \vdash x_{t} \quad(i, j, t \in I, j>i)$
3. $h_{i_{0} \vdash t}=x_{i_{0}} \vdash x_{t} \quad\left(i_{0} \in I_{0}, t \in I\right)$
4. $f_{t \dashv j i}=x_{t} \dashv x_{j} \dashv x_{i}-x_{t} \dashv x_{i} \dashv x_{j}+x_{t} \dashv\left\{x_{i}, x_{j}\right\} \quad(i, j, t \in I, j>i)$
5. $h_{t \dashv i_{0}}=x_{t} \dashv x_{i_{0}} \quad\left(i_{0} \in I_{0}, t \in I\right)$
(ii) $S$ is a Gröbner-Shirshov basis.
(iii) The set

$$
\left\{x_{j} \dashv x_{i_{1}} \dashv \cdots \dashv x_{i_{k}} \mid j \in I, i_{p} \in I-I_{0}, 1 \leq p \leq k, i_{1} \leq \cdots \leq i_{k}, k \geq 0\right\}
$$

is a linear basis of the universal enveloping algebra $U(\mathcal{L})=D(X \mid S)$. In particular, $\mathcal{L}$ can be embedded into $U(\mathcal{L})$.

Proof (i) By using the following

$$
f_{j i \vdash t}=f_{j i} \vdash x_{t} \text { and } f_{j i} \vdash x_{t}+f_{i j} \vdash x_{t}=\left(\left\{x_{i}, x_{j}\right\}+\left\{x_{j}, x_{i}\right\}\right) \vdash x_{t},
$$

we have 2 and 3 are in $\operatorname{Id}\left(f_{j i}\right)$. By symmetry, 4 and 5 are in $\operatorname{Id}\left(f_{j i}\right)$. This shows (i).
(ii) We will prove that all compositions in $S$ are trivial modulo $S$. We denote by $(i \wedge j)$ the composition of the polynomials of type $i$ and type $j$. For convenience, we extend linearly the functions $f_{j i}, f_{j i \vdash t}, f_{t \dashv j i}, h_{i_{0} \vdash t}$ and $h_{t \dashv i_{0}}$ to $f_{j\{p, q\}}\left(f_{\{p, q\} i}\right), f_{j i \vdash\{p, q\}}$ and $h_{\{p, q\}-i_{0}}$, etc respectively, where, for example, if $\left\{x_{p}, x_{q}\right\}=\sum \alpha_{p q}^{s} x_{s}$, then

$$
\begin{aligned}
f_{j\{p, q\}} & =x_{j} \vdash\left\{x_{p}, x_{q}\right\}-\left\{x_{p}, x_{q}\right\} \dashv x_{j}+\left\{\left\{x_{p}, x_{q}\right\}, x_{j}\right\}=\sum \alpha_{p q}^{s} f_{j s}, \\
f_{j i \vdash\{p, q\}} & =\sum \alpha_{p q}^{s}\left(x_{j} \vdash x_{i} \vdash x_{s}-x_{i} \vdash x_{j} \vdash x_{s}+\left\{x_{i}, x_{j}\right\} \vdash x_{s}\right)=f_{j i} \vdash\left\{x_{p}, x_{q}\right\}, \\
h_{\{p, q\} \dashv i_{0}} & =\sum \alpha_{p q}^{s} h_{s \dashv i_{0}} .
\end{aligned}
$$

By using the Jacobi identity in $\mathcal{L}$, for any $a, b, c \in \mathcal{L}$,

$$
\begin{equation*}
\{\{a, b\}, c\}=\{a,\{b, c\}\}+\{\{a, c\}, b\} \tag{5}
\end{equation*}
$$

we have

$$
\{a,\{b, b\}\}=0 \text { and }\{a,\{b, c\}+\{c, b\}\}=0
$$

and in particular, for any $i_{0} \in I_{0}, j \in I$,

$$
\begin{equation*}
\left\{x_{j}, x_{i_{0}}\right\}=0 \tag{6}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\{x_{i_{0}}, x_{j}\right\} \in \mathcal{L}_{0} \tag{7}
\end{equation*}
$$

which implies that $\mathcal{L}_{0}$ is an ideal of $\mathcal{L}$. Clearly, $\mathcal{L} / \mathcal{L}_{0}$ is a Lie algebra.
Since $\left\{x_{i_{0}}, x_{j}\right\}=\left\{x_{i_{0}}, x_{j}\right\}+\left\{x_{j}, x_{i_{0}}\right\} \in \mathcal{L}_{0}$, the (7) follows.
The formulas (5), (6) and (7) are useful in the sequel.
In $S$, all the compositions are as follows.

1) Compositions of left or right multiplication.

All possible compositions in $S$ of left multiplication are ones related to 1,2 and 3 .
By noting that for any $s, i, j, t \in I$, we have

$$
\begin{aligned}
x_{s} \dashv f_{j i} & =f_{s \dashv j i} \quad(j>i), \\
x_{s} \dashv f_{j i} & =-f_{s \dashv i j}+x_{s} \dashv\left(\left\{x_{i}, x_{j}\right\}+\left\{x_{j}, x_{i}\right\}\right) \quad(j<i), \\
x_{s} \dashv f_{i i} & =x_{s} \dashv\left\{x_{i}, x_{i}\right\}, \\
x_{s} \dashv f_{j i \vdash t} & =f_{s \dashv j i} \dashv x_{t} \quad(j>i) \quad \text { and } \\
x_{s} \dashv h_{i_{0} \vdash t} & =h_{s \dashv i_{0}} \dashv x_{t},
\end{aligned}
$$

it is clear that all cases are trivial modulo $S$.
By symmetry, all compositions in $S$ of right multiplication are trivial modulo $S$.
2) Compositions of including or intersection.

All possible compositions of including or intersection are as follows.
$(1 \wedge 3) \quad w=x_{i_{0}} \vdash x_{i}\left(i_{0} \in I_{0}\right)$. We have, by (6),

$$
\left(f_{i_{0} i}, h_{i_{0} \vdash i}\right)_{w}=-x_{i} \dashv x_{i_{0}}+\left\{x_{i}, x_{i_{0}}\right\}=-h_{i \dashv i_{0}} .
$$

$(1 \wedge 4) \quad w=x_{j} \vdash x_{i} \dashv x_{q} \dashv x_{p} \quad(q>p)$. We have

$$
\begin{aligned}
& \left(f_{j i}, f_{i \dashv q p}\right)_{w} \\
= & -x_{i} \dashv x_{j} \dashv x_{q} \dashv x_{p}+\left\{x_{i}, x_{j}\right\} \dashv x_{q} \dashv x_{p}+x_{j} \vdash x_{i} \dashv x_{p} \dashv x_{p}-x_{j} \vdash x_{i} \dashv\left\{x_{p}, x_{q}\right\} \\
= & -x_{i} \dashv f_{j \dashv q p}+f_{\{i, j\} \dashv q p}+f_{j i} \dashv x_{p} \dashv x_{q}-f_{j i} \dashv\left\{x_{p}, x_{q}\right\} .
\end{aligned}
$$

$(1 \wedge 5) \quad w=x_{j} \vdash x_{i} \dashv x_{i_{0}}\left(i_{0} \in I_{0}\right)$. We have

$$
\left(f_{j i}, h_{i \dashv i_{0}}\right)_{w}=-x_{i} \dashv x_{j} \dashv x_{i_{0}}+\left\{x_{i}, x_{j}\right\} \dashv x_{i_{0}}=-x_{i} \dashv h_{j \dashv i_{0}}+h_{\{i, j\} \dashv i_{0}} .
$$

$(2 \wedge 1)$ There are two cases to consider: $w=x_{j} \vdash x_{i} \vdash x_{t}$ and $w=x_{j} \vdash x_{i} \vdash x_{t} \vdash x_{p}$.
For $w=x_{j} \vdash x_{i} \vdash x_{t}(j>i)$, by (5), we have

$$
\begin{aligned}
\left(f_{j i \vdash t}, f_{i t}\right)_{w} & =-x_{i} \vdash x_{j} \vdash x_{t}+\left\{x_{i}, x_{j}\right\} \vdash x_{t}+x_{j} \vdash x_{t} \dashv x_{i}-x_{j} \vdash\left\{x_{t}, x_{i}\right\} \\
& =-x_{i} \vdash f_{j t}+f_{\{i, j\} t}+f_{j t} \dashv x_{i}-f_{j\{t, i\}}+f_{i\{t, j\}}-f_{i t} \dashv x_{j}+f_{t \dashv j i} .
\end{aligned}
$$

For $w=x_{j} \vdash x_{i} \vdash x_{t} \vdash x_{p} \quad(j>i)$, we have

$$
\begin{aligned}
& \left(f_{j \vdash \vdash t}, f_{t p}\right)_{w} \\
= & -x_{i} \vdash x_{j} \vdash x_{t} \vdash x_{p}+\left\{x_{i}, x_{j}\right\} \vdash x_{t} \vdash x_{p}+x_{j} \vdash x_{i} \vdash x_{p} \dashv x_{t}-x_{j} \vdash x_{i} \vdash\left\{x_{p}, x_{t}\right\} \\
= & -x_{i} \vdash x_{j} \vdash f_{t p}+\left\{x_{i}, x_{j}\right\} \vdash f_{t p}+f_{j i \vdash p} \dashv x_{t}-f_{j i \vdash\{p, t\}} .
\end{aligned}
$$

$(2 \wedge 2) \quad$ There are two cases to consider: $w=x_{j} \vdash x_{i} \vdash x_{t} \vdash x_{s} \vdash x_{p}$ and $w=x_{j} \vdash x_{i} \vdash x_{t} \vdash$ $x_{p}$.
For $w=x_{j} \vdash x_{i} \vdash x_{t} \vdash x_{s} \vdash x_{p} \quad(j>i, t>s)$, we have

$$
\begin{aligned}
& \left(f_{j i \vdash t}, f_{t s \vdash p}\right)_{w} \\
= & -x_{i} \vdash x_{j} \vdash x_{t} \vdash x_{s} \vdash x_{p}+\left\{x_{i}, x_{j}\right\} \vdash x_{t} \vdash x_{s} \vdash x_{p}+x_{j} \vdash x_{i} \vdash x_{s} \vdash x_{t} \vdash x_{p} \\
& -x_{j} \vdash x_{i} \vdash\left\{x_{s}, x_{t}\right\} \vdash x_{p} \\
= & -x_{i} \vdash x_{j} \vdash f_{t s \vdash p}+\left\{x_{i}, x_{j}\right\} \vdash f_{t s \vdash p}+f_{j i \vdash s} \vdash x_{t} \vdash x_{p}-f_{j i \vdash\{s, t\}} \vdash x_{p} .
\end{aligned}
$$

For $w=x_{j} \vdash x_{i} \vdash x_{t} \vdash x_{p} \quad(j>i>t)$, suppose that

$$
\left\{x_{i}, x_{j}\right\}=\sum_{m \in I_{1}} \alpha_{i j}^{m} x_{m}+\alpha_{i j}^{t} x_{t}+\sum_{n \in I_{2}} \alpha_{i j}^{n} x_{n}(m<t<n) .
$$

Denote by

$$
B_{t \vdash\{i, j\} \vdash p}=x_{t} \vdash\left\{x_{i}, x_{j}\right\} \vdash x_{p}-\left\{x_{i}, x_{j}\right\} \vdash x_{t} \vdash x_{p}-\left\{x_{t},\left\{x_{i}, x_{j}\right\}\right\} \vdash x_{p} .
$$

Then

$$
B_{t \vdash\{i, j\} \vdash p}=\sum_{m \in I_{1}} \alpha_{i j}^{m} f_{t m \vdash p}-\sum_{n \in I_{2}} \alpha_{i j}^{n} f_{n t \vdash p}-\sum_{q \in I_{0}} \beta_{q} h_{q \vdash p}
$$

is a linear combination of normal $s$-diwords of length 2 or 3 , where

$$
\sum_{q \in I_{0}} \beta_{q} x_{q}=\sum_{m \in I_{1}} \alpha_{i j}^{m}\left(\left\{x_{t}, x_{m}\right\}+\left\{x_{m}, x_{t}\right\}\right)+\alpha_{i j}^{t}\left\{x_{t}, x_{t}\right\} .
$$

Now, by (5), we have

$$
\begin{aligned}
& \left(f_{j i \vdash t}, f_{i \vdash \vdash p}\right)_{w} \\
= & -x_{i} \vdash x_{j} \vdash x_{t} \vdash x_{p}+\left\{x_{i}, x_{j}\right\} \vdash x_{t} \vdash x_{p}+x_{j} \vdash x_{t} \vdash x_{i} \vdash x_{p}-x_{j} \vdash\left\{x_{t}, x_{i}\right\} \vdash x_{p} \\
= & -x_{i} \vdash f_{j \vdash \vdash p}-B_{t \vdash\{i, j\} \vdash p}+f_{j \vdash \vdash i} \vdash x_{p}-B_{j \vdash\{t, i\} \vdash p}+\sum_{l \in I_{0}} \gamma_{l} h_{l \vdash p} \\
& +B_{i \vdash\{t, j\} \vdash p}-f_{i t \vdash j} \vdash x_{p}+x_{t} \vdash f_{j \vdash \vdash p},
\end{aligned}
$$

where $\sum_{l \in I_{0}} \gamma_{l} x_{l}=-\left(\left\{x_{j},\left\{x_{t}, x_{i}\right\}\right\}+\left\{\left\{x_{t}, x_{i}\right\}, x_{j}\right\}\right)+\left(\left\{x_{i},\left\{x_{t}, x_{j}\right\}\right\}+\left\{\left\{x_{t}, x_{j}\right\}, x_{i}\right\}\right)$.
$(2 \wedge 3)$ There are three cases to consider: $w=x_{j} \vdash x_{i_{0}} \vdash x_{t}\left(i_{0} \in I_{0}\right), w=x_{j_{0}} \vdash x_{i} \vdash$ $x_{t}\left(j_{0} \in I_{0}\right)$ and $w=x_{j} \vdash x_{i} \vdash x_{t_{0}} \vdash x_{n}\left(t_{0} \in I_{0}\right)$.
Case 1. $w=x_{j} \vdash x_{i_{0}} \vdash x_{t} \quad\left(j>i_{0}, i_{0} \in I_{0}\right)$. By (7), we can assume that $\left\{x_{i_{0}}, x_{j}\right\}=\sum_{l \in I_{0}} \gamma_{l} x_{l}$. Then, we have

$$
\left(f_{j i_{0} \vdash t}, h_{i_{0} \vdash t}\right)_{w}=-x_{i_{0}} \vdash x_{j} \vdash x_{t}+\left\{x_{i_{0}}, x_{j}\right\} \vdash x_{t}=-h_{i_{0} \vdash j} \vdash x_{t}+\sum_{l \in I_{0}} \gamma_{l} h_{l \vdash t} .
$$

Case 2. $w=x_{j_{0}} \vdash x_{i} \vdash x_{t} \quad\left(j_{0}>i, j_{0} \in I_{0}\right)$. By (6), we have

$$
\left(f_{j_{0} i \vdash t}, h_{j_{0} \vdash i}\right)_{w}=-x_{i} \vdash x_{j_{0}} \vdash x_{t}+\left\{x_{i}, x_{j_{0}}\right\} \vdash x_{t}=-x_{i} \vdash h_{j_{0} \vdash t} .
$$

Case 3. $w=x_{j} \vdash x_{i} \vdash x_{t_{0}} \vdash x_{n} \quad\left(j>i, t_{0} \in I_{0}\right)$. We have

$$
\begin{aligned}
\left(f_{j i \vdash t_{0}}, h_{t_{0} \vdash n}\right)_{w} & =-x_{i} \vdash x_{j} \vdash x_{t_{0}} \vdash x_{n}+\left\{x_{i}, x_{j}\right\} \vdash x_{t_{0}} \vdash x_{n} \\
& =\left(-x_{i} \vdash x_{j}+\left\{x_{i}, x_{j}\right\}\right) \vdash h_{t_{0} \vdash n} .
\end{aligned}
$$

$(2 \wedge 4) \quad w=x_{j} \vdash x_{i} \vdash x_{t} \dashv x_{q} \dashv x_{p} \quad(j>i, q>p)$. We have

$$
\begin{aligned}
& \left(f_{j i \vdash t}, f_{t \dashv q \mathrm{q}}\right)_{w} \\
= & -x_{i} \vdash x_{j} \vdash x_{t} \dashv x_{q} \dashv x_{p}+\left\{x_{i}, x_{j}\right\} \vdash x_{t} \dashv x_{q} \dashv x_{p} \\
& +x_{j} \vdash x_{i} \vdash x_{t} \dashv x_{p} \dashv x_{q}-x_{j} \vdash x_{i} \vdash x_{t} \dashv\left\{x_{p}, x_{q}\right\} \\
= & -x_{i} \vdash x_{j} \vdash f_{t \dashv q p}+\left\{x_{i}, x_{j}\right\} \vdash f_{t \dashv q p}+f_{j i \vdash t} \dashv x_{p} \dashv x_{q}-f_{j i \vdash t} \dashv\left\{x_{p}, x_{q}\right\} .
\end{aligned}
$$

$(2 \wedge 5) \quad w=x_{j} \vdash x_{i} \vdash x_{t} \dashv x_{n_{0}}\left(j>i, n_{0} \in I_{0}\right)$. We have

$$
\begin{aligned}
\left(f_{j \vdash \vdash t}, h_{t \dashv n_{0}}\right)_{w} & =-x_{i} \vdash x_{j} \vdash x_{t} \dashv x_{n_{0}}+\left\{x_{i}, x_{j}\right\} \vdash x_{t} \dashv x_{n_{0}} \\
& =\left(-x_{i} \vdash x_{j}+\left\{x_{i}, x_{j}\right\}\right) \vdash h_{t \dashv n_{0}} .
\end{aligned}
$$

$(3 \wedge 1)$ There are two cases to consider: $w=x_{n_{0}} \vdash x_{t}\left(n_{0} \in I_{0}\right)$ and $w=x_{n_{0}} \vdash x_{t} \vdash$ $x_{s}\left(n_{0} \in I_{0}\right)$.
For $w=x_{n_{0}} \vdash x_{t}\left(n_{0} \in I_{0}\right)$, we have

$$
\left(h_{n_{0} \vdash t}, f_{n_{0} t}\right)_{w}=x_{t} \dashv x_{n_{0}}-\left\{x_{t}, x_{n_{0}}\right\}=h_{t \dashv n_{0}} .
$$

For $w=x_{n_{0}} \vdash x_{t} \vdash x_{s}\left(n_{0} \in I_{0}\right)$, we have

$$
\left(h_{n_{0} \vdash t}, f_{t s}\right)_{w}=x_{n_{0}} \vdash x_{s} \dashv x_{t}-x_{n_{0}} \vdash\left\{x_{s}, x_{t}\right\}=h_{n_{0} \vdash s} \dashv x_{t}-h_{n_{0} \vdash\{s, t\}} .
$$

$(3 \wedge 2) \quad w=x_{n_{0}} \vdash x_{t} \vdash x_{s} \vdash x_{p} \quad\left(t>s, n_{0} \in I_{0}\right)$. We have

$$
\begin{aligned}
\left(h_{n_{0} \vdash t}, f_{t s \vdash p}\right)_{w} & =x_{n_{0}} \vdash x_{s} \vdash x_{t} \vdash x_{p}-x_{n_{0}} \vdash\left\{x_{s}, x_{t}\right\} \vdash x_{p} \\
& =h_{n_{0} \vdash s} \vdash x_{t} \vdash x_{p}-h_{n_{0} \vdash\{s, t\}} \vdash x_{p} .
\end{aligned}
$$

$(3 \wedge 3) \quad w=x_{n_{0}} \vdash x_{t_{0}} \vdash x_{r}\left(n_{0}, t_{0} \in I_{0}\right)$. We have

$$
\left(h_{n_{0} \vdash t_{0}}, h_{t_{0} \vdash r}\right)_{w}=0 .
$$

$(3 \wedge 4) \quad w=x_{n_{0}} \vdash x_{t} \dashv x_{q} \dashv x_{p} \quad\left(q>p, n_{0} \in I_{0}\right)$. We have

$$
\begin{aligned}
\left(h_{n_{0} \vdash t}, f_{t \dashv q p}\right)_{w} & =x_{n_{0}} \vdash x_{t} \dashv x_{p} \dashv x_{q}-x_{n_{0}} \vdash x_{t} \dashv\left\{x_{p}, x_{q}\right\} \\
& =h_{n_{0} \vdash t} \dashv\left(x_{p} \dashv x_{q}-\left\{x_{p}, x_{q}\right\}\right) .
\end{aligned}
$$

$(3 \wedge 5) \quad w=x_{n_{0}} \vdash x_{t} \dashv x_{s_{0}}\left(n_{0}, s_{0} \in I_{0}\right)$. We have

$$
\left(h_{n_{0} \vdash t}, h_{t \dashv s_{0}}\right)_{w}=0 .
$$

Since $(4 \wedge 4),(4 \wedge 5),(5 \wedge 4),(5 \wedge 5)$ are symmetric with $(2 \wedge 2),(2 \wedge 3),(3 \wedge 2)$, $(3 \wedge 3)$ respectively, they have the similar representations. We omit the details.

From the above representations, we know that all compositions in $S$ are trivial modulo $S$. So, $S$ is a Gröbner-Shirshov basis.
(iii) Clearly, the mentioned set is just the set $\operatorname{Irr}(S)$. Now, the results follow from Theorem 3.9,

By using the Theorem 4.1, we have the following corollary.
Corollary 4.2 ([1], [8]) Let the notations be as in Theorem 4.1. Then $U(\mathcal{L})$ is isomorphic to $\mathcal{L} \otimes U\left(\mathcal{L} / \mathcal{L}_{0}\right)$, where $U\left(\mathcal{L} / \mathcal{L}_{0}\right)$ is the universal enveloping of the Lie algebra $\mathcal{L} / \mathcal{L}_{0}$.

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# Gröbner-Shirshov bases for dialgebras* 

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#### Abstract

In this paper, we define the Gröbner-Shirshov basis for a dialgebra. The Composition-Diamond lemma for dialgebras is given then. As results, we give GröbnerShirshov bases for the universal enveloping algebra of a Leibniz algebra, the bar extension of a dialgebra, the free product of two dialgebras, and Clifford dialgebra. We obtain some normal forms for algebras mentioned the above.


Key words: dialgebra; Gröbner-Shirshov basis; Leibniz algebra; Clifford dialgebra.
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## 1 Introduction

J.-L. Loday (1995, [1]) gave the definition of a new class of algebras, dialgebras, which is closely connected to his notion of Leibniz algebras (1993, [10]) in the same way as associative algebras connected to Lie algebras. In the manuscript [12], J.-L. Loday found a normal form of elements of a free dialgebra. Here we continue to study free dialgebras and prove the Composition-Diamond lemma for dialgebras. As it is well known, this kind of lemma is the cornerstone of the theory of Gröbner and Gröbner-Shirshov bases (see, for example, [6] and cited literature). In commutative-associative case, this lemma

[^1]is equivalent to the Main Buchberger's Theorem ([7, 8]). For Lie and associative algebras, this is the Shirshov's lemma [14] (see also L.A. Bokut [3, 4], G. Bergman [2], L.A. Bokut and Y. Chen [5]). As results, we obtain Gröbner-Shirshov bases for the universal enveloping algebra of a Leibniz algebra, the bar extension of a dialgebra, the free product of two dialgebras, and Clifford dialgebra. By using our Composition-Diamond lemma for dialgebras (Theorem 3.9), we obtain some normal forms for algebras mentioned the above. Moreover, we get another proof of the M. Aymon, P.-P. Grivel's result (1]) on the Poincare-Birkhoff-Witt theorem for Leibniz algebras (see P. Kolesnikov [9] for other proof).

## 2 Preliminaries

Definition 2.1 Let $k$ be a field. A $k$-linear space $D$ equipped with two bilinear multiplications $\vdash$ and $\dashv$ is called a dialgebra, if both $\vdash$ and $\dashv$ are associative and

$$
\begin{aligned}
a \dashv(b \vdash c) & =a \dashv b \dashv c \\
(a \dashv b) \vdash c & =a \vdash b \vdash c \\
a \vdash(b \dashv c) & =(a \vdash b) \dashv c
\end{aligned}
$$

for any $a, b, c \in D$.
Definition 2.2 Let $D$ be a dialgebra, $B \subset D$. Let us define diwords of $D$ in the set $B$ by induction:
(i) $b=(b), b \in B$ is a diword in $B$ of length $|b|=1$.
(ii) (u) is called a diword in B of length $|(u)|=n$, if $(u)=((v) \dashv(w))$ or $(u)=((v) \vdash$ $(w))$, where $(v),(w)$ are diwords in $B$ of length $k, l$ respectively and $k+l=n$.

Proposition 2.3 ([12]) Let $D$ be a dialgebra and $B \subset D$. Any diword of $D$ in the set $B$ is equal to a diword in $B$ of the form

$$
\begin{equation*}
(u)=b_{-m} \vdash \cdots \vdash b_{-1} \vdash b_{0} \dashv b_{1} \dashv \cdots \dashv b_{n} \tag{1}
\end{equation*}
$$

where $b_{i} \in B,-m \leq i \leq n, m \geq 0, n \geq 0$. Any bracketing of the right side of (1) gives the same result.

Definition 2.4 Let $X$ be a set. A free dialgebra $D(X)$ generated by $X$ over $k$ is defined in a usual way by the following commutative diagram:

where $D$ is any dialgebra.

In［12］，a construction of a free dialgebra is given．
Proposition 2.5 （［12］）Let $D(X)$ be a free dialgebra over $k$ generated by $X$ ．Any diword in $D(X)$ is equal to the unique diword of the form

$$
\begin{equation*}
[u]=x_{-m} \vdash \cdots \vdash x_{-1} \vdash x_{0} \dashv x_{1} \dashv \cdots \dashv x_{n} \triangleq x_{-m} \cdots x_{-1} \dot{x_{0}} x_{1} \cdots x_{n} \tag{2}
\end{equation*}
$$

where $x_{i} \in X, m \geq 0, n \geq 0$ ，and $x_{0}$ is called the center of the normal diword $[u]$ ．We call $[u]$ a normal diword（in $X$ ）with the associative word $u, u \in X^{*}$ ．Clearly，if $[u]=[v]$ ， then $u=v$ ．In（⿴囗⿱一𧰨丶（）．Let $[u],[v]$ be two normal diwords．Then $[u] \vdash[v]$ is the normal diword $[u v]$ with the center at the center of $[v]$ ．Accordingly，$[u] \dashv[v]$ is the normal diword ［uv］with the center at the center of $[u]$ ．

## Example 2.6

$$
\begin{gathered}
\left(x_{-1} \vdash x_{0} \dashv x_{1}\right) \vdash\left(y_{-1} \vdash y_{0} \dashv y_{1}\right)=x_{-1} \vdash x_{0} \vdash x_{1} \vdash y_{-1} \vdash y_{0} \dashv y_{1}, \\
\left(x_{-1} \vdash x_{0} \dashv x_{1}\right) \dashv\left(y_{-1} \vdash y_{0} \dashv y_{1}\right)=x_{-1} \vdash x_{0} \dashv x_{1} \dashv y_{-1} \dashv y_{0} \dashv y_{1} .
\end{gathered}
$$

## 3 Composition－Diamond lemma for dialgebras

Let $X$ be a well ordered set，$D(X)$ the free dialgebra over $k, X^{*}$ the free monoid generated by $X$ and $\left[X^{*}\right]$ the set of normal diwords in $X$ ．Let us define the deg－lex ordering on $\left[X^{*}\right]$ in the following way：for any $[u],[v] \in\left[X^{*}\right]$ ，

$$
[u]<[v] \Longleftrightarrow w t([u])<w t([v]) \text { lexicographicaly, }
$$

where

$$
w t([u])=\left(n+m+1, m, x_{-m}, \cdots, x_{0}, \cdots, x_{n}\right)
$$

if $[u]=x_{-m} \cdots x_{-1} \dot{x_{0}} x_{1} \cdots x_{n}$ ．
Throughout the paper，we will use this ordering．
It is easy to see that the ordering＜is satisfied the following properties：

$$
[u]<[v] \Longrightarrow x \vdash[u]<x \vdash[v],[u] \dashv x<[v] \dashv x, \text { for any } x \in X .
$$

Any polynomial $f \in D(X)$ has the form

$$
f=\sum_{[u] \in\left[X^{*}\right]} f([u])[u]=\alpha[\bar{f}]+\sum \alpha_{i}\left[u_{i}\right],
$$

where $[\bar{f}],\left[u_{i}\right]$ are normal diwords in $X,[\bar{f}]>\left[u_{i}\right], \alpha, \alpha_{i}, f([u]) \in k, \alpha \neq 0$ ．We call $[\bar{f}]$ the leading term of $f$ ．Denote suppf by the set $\{[u] \mid f([u]) \neq 0\}$ and $\operatorname{deg}(f)$ by $|[\bar{f}]| . f$ is called monic if $\alpha=1 . f$ is called left（right）normed if $f=\sum \alpha_{i} u_{i} \dot{x_{i}} \quad\left(f=\sum \alpha_{i} \dot{x_{i}} u_{i}\right)$ ， where each $\alpha_{i} \in k, x_{i} \in X$ and $u_{i} \in X^{*}$ ．

If $[u],[v]$ are both left normed or both right normed，then it is clear that for any $[w] \in\left[X^{*}\right]$,

$$
\begin{aligned}
{[u]<[v] \Longrightarrow } & {[u] \vdash[w]<[v] \vdash[w],[w] \vdash[u]<[w] \vdash[v], } \\
& {[u] \dashv[w]<[v] \dashv[w],[w] \dashv[u]<[w] \dashv[v] . }
\end{aligned}
$$

Let $S \subset D(X)$. By an $S$-diword $g$ we will mean a diword in $\{X \cup S\}$ with only one occurrence of $s \in S$. If this is the case and $g=(a s b)$ for some $a, b \in X^{*}, s \in S$, we also call $g$ an $s$-diword.

From Proposition 2.3 it follows that any $s$-diword is equal to

$$
\begin{equation*}
[a s b]=\left.x_{-m} \vdash \cdots \vdash x_{-1} \vdash x_{0} \dashv x_{1} \dashv \cdots \dashv x_{n}\right|_{x_{k} \mapsto s} \tag{3}
\end{equation*}
$$

where $-m \leq k \leq n, s \in S, x_{i} \in X,-m \leq i \leq n$. To be more precise, $[a s b]=[a \dot{s} b]$ if $k=0 ;[a s b]=\left[a s b_{1} \dot{x_{0}} b_{2}\right]$ if $k<0$ and $[a s b]=\left[a_{1} \dot{x_{0}} a_{2} s b\right]$ if $k>0$. If the center of the $s$-diword $[a s b]$ is in $a$, then we denote it by $[\dot{a} s b]=\left[a_{1} \dot{x_{0}} a_{2} s b\right]$. Similarly, $[a s \dot{b}]=\left[a s b_{1} \dot{x_{0}} b_{2}\right]$ (of course, either $a_{i}$ or $b_{i}$ may be empty).

Definition 3.1 The s-diword (3) is called a normal s-diword if one of the following conditions holds:
(i) $k=0$,
(ii) $k<0$ and $s$ is left normed,
(iii) $k>0$ and $s$ is right normed.

We call a normal s-diword [asb] a left (right) normed s-diword if both $s$ and [asb] are left (right) normed. In particulary, s is a left (right) normed s-diword if $s$ is left (right) normed polynomial.

The following lemma follows from the above properties of the ordering $<$.
Lemma 3.2 For a normal s-diword $[a s b]$, the leading term of $[a s b]$ is equal to $[a[\bar{s}] b]$, that is, $\overline{[a s b]}=[a[\bar{s}] b]$. More specifically, if

$$
[a s b]=\left.x_{-m} \vdash \cdots \vdash x_{-1} \vdash x_{0} \dashv x_{1} \dashv \cdots \dashv x_{n}\right|_{x_{k} \mapsto s},
$$

then corresponding to $k=0, k<0, k>0$, respectively, we have

$$
\begin{aligned}
& \overline{x_{-m} \vdash \cdots \vdash x_{-1} \vdash s \dashv x_{1} \dashv \cdots \dashv x_{n}}=x_{-m} \vdash \cdots \vdash x_{-1} \vdash[\bar{s}] \dashv x_{1} \dashv \cdots \dashv x_{n}, \\
& \overline{x_{-m} \vdash \cdots \vdash s \vdash \cdots \vdash x_{0} \dashv \cdots \dashv x_{n}}=x_{-m} \vdash \cdots \vdash[\bar{s}] \vdash \cdots \vdash x_{0} \dashv \cdots \dashv x_{n}, \\
& \overline{x_{-m} \vdash \cdots \vdash x_{0} \dashv \cdots \dashv s \dashv \cdots \dashv x_{n}}=x_{-m} \vdash \cdots \vdash x_{0} \dashv \cdots \dashv[\bar{s}] \dashv \cdots \dashv x_{n} .
\end{aligned}
$$

Now, we define compositions of polynomials in $D(X)$.
Definition 3.3 Let the ordering $<$ be as before and $f, g \in D(X)$ with $f, g$ monic.

1) Composition of left (right) multiplication.

Let $f$ be not a right normed polynomial and $x \in X$. Then $x \dashv f$ is called the composition of left multiplication. Clearly, $x \dashv f$ is a right normed polynomial (or 0).

Let $f$ be not a left normed polynomial and $x \in X$. Then $f \vdash x$ is called the composition of right multiplication. Clearly, $f \vdash x$ is a left normed polynomial (or $0)$.
2) Composition of inclusion.

Let

$$
[w]=[\bar{f}]=[a[\bar{g}] b],
$$

where $[a g b]$ is a normal $g$-diword. Then

$$
(f, g)_{[w]}=f-[a g b]
$$

is called the composition of inclusion. The transformation $f \mapsto f-[a g b]$ is called the elimination of leading diword (ELW) of $g$ in $f$, and $[w]$ is called the ambiguity of $f$ and $g$.
3) Composition of intersection.

Let

$$
[w]=[[\bar{f}] b]=[a[\bar{g}]],|\bar{f}|+|\bar{g}|>|w|,
$$

where $[f b]$ is a normal $f$-diword and $[a g]$ a normal $g$-diword. Then

$$
(f, g)_{[w]}=[f b]-[a g]
$$

is called the composition of intersection, and $[w]$ is called the ambiguity of $f$ and $g$.
Remark In the Definition [3.3, for the case of 2) or 3), we have $\overline{(f, g)_{[w]}}<[w]$. For the case of 1$), \operatorname{deg}(x \dashv f) \leq \operatorname{deg}(f)+1$ and $\operatorname{deg}(f \vdash x) \leq \operatorname{deg}(f)+1$.

Definition 3.4 Let the ordering $<$ be as before, $S \subset D(X)$ a monic set and $f, g \in S$.

1) Let $x \dashv f$ be a composition of left multiplication. Then $x \dashv f$ is called trivial modulo $S$, denoted by $\quad x \dashv f \equiv 0 \bmod (S), \quad i f$

$$
x \dashv f=\sum \alpha_{i}\left[a_{i} s_{i} b_{i}\right],
$$

where each $\alpha_{i} \in k, a_{i}, b_{i} \in X^{*}, s_{i} \in S,\left[a_{i} s_{i} b_{i}\right]$ right normed $s_{i}$-diword and $\left|\left[a_{i}\left[\overline{s_{i}}\right] b_{i}\right]\right| \leq \operatorname{deg}(x \dashv f)$.

Let $f \vdash x$ be a composition of right multiplication. Then $f \vdash x$ is called trivial modulo $S$, denoted by $f \vdash x \equiv 0 \bmod (S)$, if

$$
f \vdash x=\sum \alpha_{i}\left[a_{i} s_{i} b_{i}\right],
$$

where each $\alpha_{i} \in k, a_{i}, b_{i} \in X^{*}, s_{i} \in S,\left[a_{i} s_{i} b_{i}\right]$ left normed $s_{i}$-diword and $\left|\left[a_{i}\left[\overline{s_{i}}\right] b_{i}\right]\right| \leq$ $\operatorname{deg}(f \vdash x)$.
2) Composition $(f, g)_{[w]}$ of inclusion (intersection) is called trivial modulo $(S,[w])$, denoted by $\quad(f, g)_{[w]} \equiv 0 \bmod (S,[w]), \quad$ if

$$
(f, g)_{[w]}=\sum \alpha_{i}\left[a_{i} s_{i} b_{i}\right],
$$

where each $\alpha_{i} \in k, a_{i}, b_{i} \in X^{*}, s_{i} \in S,\left[a_{i} s_{i} b_{i}\right]$ normal $s_{i}$-diword, $\left[a_{i}\left[\overline{s_{i}}\right] b_{i}\right]<[w]$ and each $\left[a_{i} s_{i} b_{i}\right]$ is right (left) normed $s_{i}$-diword whenever either both $f$ and $[a g b]$ or both $[f b]$ and $[a g]$ are right (left) normed $S$-diwords.

We call the set $S$ a Gröbner-Shirshov basis in $D(X)$ if any composition of polynomials in $S$ is trivial modulo $S$ (and $[w]$ ).

The following lemmas play key role in the proof of Theorem 3.9.

Lemma 3.5 Let $S \subset D(X)$ and $[a s b]$ an $s$-diword, $s \in S$. Assume that each composition of right and left multiplication is trivial modulo $S$. Then, $[a s b]$ has a presentation:

$$
[a s b]=\sum \alpha_{i}\left[a_{i} s_{i} b_{i}\right]
$$

where each $\alpha_{i} \in k, s_{i} \in S, a_{i}, b_{i} \in X^{*}$ and each $\left[a_{i} s_{i} b_{i}\right]$ is normal $s_{i}$-diword.
Proof. Following Proposition 2.3, we assume that

$$
[a s b]=\left.x_{-m} \vdash \cdots \vdash x_{-1} \vdash x_{0} \dashv x_{1} \dashv \cdots \dashv x_{n}\right|_{x_{k} \rightarrow s}
$$

There are three cases to consider.
Case 1. $k=0$. Then $[a s b]$ is a normal $s$-diword.
Case 2. $k<0$. Then $[a s b]=a \vdash\left(s \vdash x_{k+1}\right) \vdash b, k<-1$ or $[a s b]=a \vdash\left(s \vdash x_{0}\right) \dashv b$. If $s$ is left normed then $[a s b]$ is a normal $s$-diword. If $s$ is not left normed then for the composition $s \vdash x_{k+1} \quad(k<0)$ of right multiplication, we have

$$
s \vdash x_{k+1}=\sum \alpha_{i}\left[a_{i} s_{i} b_{i}\right],
$$

where each $\alpha_{i} \in k, a_{i}, b_{i} \in X^{*}, s_{i} \in S$ and $\left[a_{i} s_{i} b_{i}\right]$ is left normed $s_{i}$-diword. Then

$$
[a s b]=\sum \alpha_{i}\left(a \vdash\left[a_{i} s_{i} b_{i}\right] \vdash b\right)
$$

or

$$
[a s b]=\sum \alpha_{i}\left(a \vdash\left[a_{i} s_{i} b_{i}\right] \dashv b\right)
$$

is a linear combination of normal $s_{i}$-diwords.
Case 3. $k>0$ is similar to the Case 2.
Lemma 3.6 Let $S \subset D(X)$ and each composition $(f, g)_{[w]}$ in $S$ of inclusion (intersection) trivial modulo $(S,[w])$. Let $\left[a_{1} s_{1} b_{1}\right]$ and $\left[a_{2} s_{2} b_{2}\right]$ be normal $S$-diwords such that $[w]=$ $\left[a_{1}\left[\bar{s}_{1}\right] b_{1}\right]=\left[a_{2}\left[\overline{s_{2}}\right] b_{2}\right]$, where $s_{1}, s_{2} \in S, a_{1}, a_{2}, b_{1}, b_{2} \in X^{*}$. Then,

$$
\left[a_{1} s_{1} b_{1}\right] \equiv\left[a_{2} s_{2} b_{2}\right] \quad \bmod (S,[w])
$$

i.e., $\left[a_{1} s_{1} b_{1}\right]-\left[a_{2} s_{2} b_{2}\right]=\sum \alpha_{i}\left[a_{i} s_{i} b_{i}\right]$, where each $\alpha_{i} \in k, a_{i}, b_{i} \in X^{*}, s_{i} \in S,\left[a_{i} s_{i} b_{i}\right]$ normal $s_{i}$-diword and $\left[a_{i}\left[\overline{s_{i}}\right] b_{i}\right]<[w]$.

Proof. In the following, all letters $a, b, c$ with indexis are words and $s_{1}, s_{2}, s_{j} \in S$.
Because $a_{1} \overline{s_{1}} b_{1}=a_{2} \overline{s_{2}} b_{2}$ as ordinary words, there are three cases to consider.

Case 1. Subwords $\overline{s_{1}}, \overline{s_{2}}$ have empty intersection. Assume, for example, that $b_{1}=b \overline{s_{2}} b_{2}$ and $a_{2}=a_{1} \overline{s_{1}} b$. Because any normal $S$-diword may be bracketing in any way, we have

$$
\left[a_{2} s_{2} b_{2}\right]-\left[a_{1} s_{1} b_{1}\right]=\left(a_{1} s_{1}\left(b\left(s_{2}-\left[\overline{s_{2}}\right]\right) b_{2}\right)\right)-\left(\left(a_{1}\left(s_{1}-\left[\overline{s_{1}}\right]\right) b\right) s_{2} b_{2}\right) .
$$

For any $[t] \in \operatorname{supp}\left(s_{2}-\left[\overline{s_{2}}\right]\right)$, we prove that $\left(a_{1} s_{1} b[t] b_{2}\right)$ is a normal $s_{1}$-diword. There are five cases to consider.
$1.1[w]=\left[\dot{a}_{1}\left[\overline{s_{1}}\right] b\left[\overline{s_{2}}\right] b_{2}\right] ;$
$1.2[w]=\left[a_{1}\left[\dot{s_{1}}\right] b\left[\overline{s_{2}}\right] b_{2}\right] ;$
$1.3[w]=\left[a_{1}\left[\overline{s_{1}}\right] b\left[\overline{s_{2}}\right] b_{2}\right] ;$
$1.4[w]=\left[a_{1}\left[\overline{s_{1}}\right] b\left[\dot{s_{2}}\right] b_{2}\right] ;$
$1.5[w]=\left[a_{1}\left[\overline{s_{1}}\right] b\left[\overline{s_{2}}\right] \dot{b_{2}}\right]$.
For 1.1, since $\left[a_{1} s_{1} b_{1}\right]$ and $\left[a_{2} s_{2} b_{2}\right]$ are normal $S$-diwords, both $s_{1}$ and $s_{2}$ are right normed by the definition, in particular, $[t]$ is right normed. It follows that $\left(a_{1} s_{1} b[t] b_{2}\right)=\left[\dot{a}_{1} s_{1} b[t] b_{2}\right]$ is a normal $s_{1}$-diword.

For 1.2, it is clear that $\left(a_{1} s_{1} b[t] b_{2}\right)$ is a normal $s_{1}$-diword and $[t]$ is right normed.
For 1.3, 1.4 and 1.5 , since $\left[a_{1} s_{1} b_{1}\right]$ is normal $s_{1}$-diword, $s_{1}$ is left normed by the definition, which implies that $\left(a_{1} s_{1} b[t] b_{2}\right)$ is a normal $s_{1}$-diword. Moreover, $[t]$ is right normed, if 1.3 , and left normed, if 1.5.

Clearly, for all cases, we have $\overline{\left[a_{1} s_{1} b[t] b_{2}\right]}=\left[a_{1}\left[\overline{s_{1}}\right] b[t] b_{2}\right]<\left[a_{1}\left[\overline{s_{1}}\right] b\left[\overline{s_{2}}\right] b_{2}\right]=[w]$.
Similarly, for any $[t] \in \operatorname{supp}\left(s_{1}-\left[\overline{s_{1}}\right]\right),\left(a_{1}[t] b s_{2} b_{2}\right)$ is a normal $s_{2}$-diword and $\left[a_{1}[t] b\left[\overline{s_{2}}\right] b_{2}\right]<$ [ $w$ ].

Case 2. Subwords $\overline{s_{1}}$ and $\overline{s_{2}}$ have non-empty intersection $c$. Assume, for example, that $b_{1}=b b_{2}, a_{2}=a_{1} a, w_{1}=\overline{s_{1}} b=a \overline{s_{2}}=a c b$.

There are following five cases to consider:
$2.1[w]=\left[a_{1}\left[\overline{s_{1}}\right] b b_{2}\right] ;$
$2.2[w]=\left[a_{1}\left[\overline{s_{1}}\right] b \dot{b_{2}}\right] ;$
$2.3[w]=\left[a_{1} \dot{a} c b b_{2}\right] ;$
$2.4[w]=\left[a_{1} a \dot{c} b b_{2}\right] ;$
$2.5[w]=\left[a_{1} a c \dot{b} b_{2}\right]$.
Then

$$
\left[a_{2} s_{2} b_{2}\right]-\left[a_{1} s_{1} b_{1}\right]=\left(a_{1}\left(\left[a s_{2}\right]-\left[s_{1} b\right]\right) b_{2}\right)=\left(a_{1}\left(s_{1}, s_{2}\right)_{\left[w_{1}\right]} b_{2}\right),
$$

where $\left[w_{1}\right]=[a c b]=\left[\left[\overline{s_{1}}\right] b\right]=\left[a\left[\overline{s_{2}}\right]\right]$ is as follows:
$2.1\left[w_{1}\right]$ is right normed;
$2.2\left[w_{1}\right]$ is left normed;
$2.3\left[w_{1}\right]=[\dot{a} c b]$;
$2.4\left[w_{1}\right]=[a \dot{c} b] ;$
$2.5\left[w_{1}\right]=[a c b]$.
Since each composition $(f, g)_{[w]}$ in $S$ is trivial modulo ( $S,[w]$ ), there exist $\beta_{j} \in k, u_{j}, v_{j} \in$ $X^{*}, s_{j} \in S$ such that $\left[s_{1} b\right]-\left[a s_{2}\right]=\sum_{j} \beta_{j}\left[u_{j} s_{j} v_{j}\right]$, where each $\left[u_{j} s_{j} v_{j}\right]$ is normal $S$-diword and $\left[u_{j}\left[\overline{s_{j}}\right] v_{j}\right]<\left[w_{1}\right]=[a c b]$. Therefore,

$$
\left[a_{2} s_{2} b_{2}\right]-\left[a_{1} s_{1} b_{1}\right]=\sum_{j} \beta_{j}\left(a_{1}\left[u_{j} s_{j} v_{j}\right] b_{2}\right) .
$$

Now, we prove that each $\left(a_{1}\left[u_{j} s_{j} v_{j}\right] b_{2}\right)$ is normal $s_{j}$-diword and $\overline{\left(a_{1}\left[u_{j} s_{j} v_{j}\right] b_{2}\right)}<[w]=$ $\left[a_{1}\left[\left[\overline{s_{1}}\right] b\right] b_{2}\right]$.

For 2.1, since $\left[\dot{a}_{1} s_{1} b b_{2}\right]$ and $\left[\dot{a}_{1} a s_{2} b_{2}\right]$ are normal $S$-diwords, both $\left[s_{1} b\right]$ and $\left[a s_{2}\right]$ are right normed $S$-diwords. Then, by definition, each $\left[u_{j} s_{j} v_{j}\right]$ is right normed $S$-diword, and so each $\left(a_{1}\left[u_{j} s_{j} v_{j}\right] b_{2}\right)=\left[\dot{a}_{1} u_{j} s_{j} v_{j} b_{2}\right]$ is normal $S$-diword.

For 2.2, both $\left[s_{1} b\right]$ and $\left[a s_{2}\right]$ must be left normed $S$-diwords. Then, by definition, each $\left[u_{j} s_{j} v_{j}\right]$ is left normed $S$-diword, and so each $\left(a_{1}\left[u_{j} s_{j} v_{j}\right] b_{2}\right)=\left[a_{1} u_{j} s_{j} v_{j} \dot{b}_{2}\right]$ is normal $S$-diword.

For 2.3, 2.4 or 2.5 , by noting that $\left(a_{1}\left[u_{j} s_{j} v_{j}\right] b_{2}\right)=\left(\left(a_{1}\right) \vdash\left[u_{j} s_{j} v_{j}\right] \dashv\left(b_{2}\right)\right)$ and $\left[u_{j} s_{j} v_{j}\right]$ is normal $S$-diword, $\left(a_{1}\left[u_{j} s_{j} v_{j}\right] b_{2}\right)$ is also normal $S$-diword.

Now, for all cases, we have $\overline{\left[a_{1} u_{j} s_{j} v_{j} b_{2}\right]}=\left[a_{1} u_{j}\left[\overline{s_{j}}\right] v_{j} b_{2}\right]<[w]=\left[a_{1}[a c b] b_{2}\right]$.
Case 3. One of the subwords $\overline{s_{1}}$ and $\overline{s_{2}}$ contains another as a subword. Assume, for example, that $b_{2}=b b_{1}, a_{2}=a_{1} a, w_{1}=\overline{s_{1}}=a \overline{s_{2}} b$.

Again there are following five cases to consider:
$2.1[w]=\left[\dot{a}_{1} a\left[\overline{s_{2}}\right] b b_{1}\right] ;$
$2.2[w]=\left[a_{1} a\left[\overline{s_{2}}\right] b \dot{b_{1}}\right] ;$
$2.3[w]=\left[a_{1} \dot{a}\left[\overline{s_{2}}\right] b b_{1}\right] ;$
$2.4[w]=\left[a_{1} a\left[\dot{\overline{s_{2}}}\right] b b_{1}\right] ;$
$2.5[w]=\left[a_{1} a\left[\bar{s}_{2}\right] \dot{b} b_{1}\right]$.
Then

$$
\left[a_{1} s_{1} b_{1}\right]-\left[a_{2} s_{2} b_{2}\right]=\left(a_{1}\left(s_{1}-a s_{2} b\right) b_{1}\right)=\left(a_{1}\left(s_{1}, s_{2}\right)_{\left[w_{1}\right]} b_{1}\right)
$$

It is similar to the proof of the Case 2 that we have $\left[a_{1} s_{1} b_{1}\right] \equiv\left[a_{2} s_{2} b_{2}\right] \bmod (S,[w])$.

Definition 3.7 Let $S \subset D(X)$. Then

$$
\operatorname{Irr}(S) \triangleq\left\{u \in\left[X^{*}\right] \mid u \neq[a[\bar{s}] b], s \in S, a, b \in X^{*},[a s b] \text { is normal s-diword }\right\} .
$$

Lemma 3.8 Let $S \subset D(X)$ and $h \in D(X)$. Then $h$ has a representation

$$
h=\sum_{I_{1}} \alpha_{i}\left[u_{i}\right]+\sum_{I_{2}} \beta_{j}\left[a_{j} s_{j} b_{j}\right]
$$

where $\left[u_{i}\right] \in \operatorname{Irr}(S), i \in I_{1},\left[a_{j} s_{j} b_{j}\right]$ normal $s_{j}$-diwords, $s_{j} \in S, j \in I_{2}$ with $\left[a_{1}\left[\overline{s_{1}}\right] b_{1}\right]>$ $\left[a_{2}\left[\overline{s_{2}}\right] b_{2}\right]>\cdots>\left[a_{n}\left[\overline{s_{n}}\right] b_{n}\right]$.

Proof. Let $h=\alpha_{1}[\bar{h}]+\cdots$. We prove the result by induction on $[\bar{h}]$.
If $[\bar{h}] \in \operatorname{Irr}(S)$, then take $\left[u_{1}\right]=[\bar{h}]$ and $h_{1}=h-\alpha_{1}\left[u_{1}\right]$. Clearly, $\left[\overline{h_{1}}\right]<[\bar{h}]$ or $h_{1}=0$.
If $[\bar{h}] \notin \operatorname{Irr}(S)$, then $[\bar{h}]=\left[a_{1}\left[\overline{s_{1}}\right] b_{1}\right]$ with $\left[a_{1} s_{1} b_{1}\right]$ a normal $s_{1}$-diword. Let $h_{1}=$ $h-\beta_{1}\left[a_{1} s_{1} b_{1}\right]$. Then $\left[\overline{h_{1}}\right]<[\bar{h}]$ or $h_{1}=0$.

The following theorem is the main result.

Theorem 3.9 (Composition-Diamond lemma) Let $S \subset D(X)$ be a monic set and the ordering $<$ as before, $\operatorname{Id}(S)$ is the ideal generated by $S$. Then $(i) \Rightarrow(i i) \Leftrightarrow(i i)^{\prime} \Leftrightarrow(i i i)$, where
(i) $S$ is a Gröbner-Shirshov basis in $D(X)$.
(ii) $f \in I d(S) \Rightarrow[\bar{f}]=[a[\bar{s}] b]$ for some $s \in S, a, b \in X^{*}$ and [asb] a normal $S$-diword.
$(i i)^{\prime} f \in \operatorname{Id}(S) \Rightarrow f=\alpha_{1}\left[a_{1} s_{1} b_{1}\right]+\alpha_{2}\left[a_{2} s_{2} b_{2}\right]+\cdots+\alpha_{n}\left[a_{n} s_{n} b_{n}\right]$ with $\left[a_{1}\left[\overline{s_{1}}\right] b_{1}\right]>$ $\left[a_{2}\left[\overline{s_{2}}\right] b_{2}\right]>\cdots>\left[a_{n}\left[\overline{s_{n}}\right] b_{n}\right]$, where $\left[a_{i} s_{i} b_{i}\right]$ is normal $s_{i}$-diword, $i=1,2, \cdots, n$.
(iii) The set $\operatorname{Irr}(S)$ is a linear basis of the dialgebra $D(X \mid S)=D(X) / \operatorname{Id}(S)$ generated by $X$ with defining relations $S$.

Proof. $\quad(i) \Rightarrow(i i)$. Let $S$ be a Gröbner-Shirshov basis and $0 \neq f \in I d(S)$. We may assume, by Lemma 3.5, that

$$
f=\sum_{i=1}^{n} \alpha_{i}\left[a_{i} s_{i} b_{i}\right]
$$

where each $\alpha_{i} \in k, a_{i}, b_{i} \in X^{*}, s_{i} \in S$ and $\left[a_{i} s_{i} b_{i}\right]$ normal $S$-diword. Let

$$
\left[w_{i}\right]=\left[a_{i}\left[\overline{s_{i}}\right] b_{i}\right], \quad\left[w_{1}\right]=\left[w_{2}\right]=\cdots=\left[w_{l}\right]>\left[w_{l+1}\right] \geq \cdots, l \geq 1 .
$$

We will use induction on $l$ and $\left[w_{1}\right]$ to prove that $[\bar{f}]=[a[\bar{s}] b]$ for some $s \in S$ and $a, b \in X^{*}$. If $l=1$, then $[\bar{f}]=\overline{\left[a_{1} s_{1} b_{1}\right]}=\left[a_{1}\left[\overline{s_{1}}\right] b_{1}\right]$ and hence the result holds. Assume that $l \geq 2$. Then, by Lemma 3.6, we have $\left[a_{1} s_{1} b_{1}\right] \equiv\left[a_{2} s_{2} b_{2}\right] \bmod \left(S,\left[w_{1}\right]\right)$.

Thus, if $\alpha_{1}+\alpha_{2} \neq 0$ or $l>2$, then the result follows from induction on $l$. For the case $\alpha_{1}+\alpha_{2}=0$ and $l=2$, we use induction on $\left[w_{1}\right]$. Now, the result follows.
(ii) $\Rightarrow(i i)^{\prime}$. Assume (ii) and $0 \neq f \in \operatorname{Id}(S)$. Let $f=\alpha_{1}[\bar{f}]+\sum_{\left[u_{i}\right]<[\bar{f}]} \alpha_{i}\left[u_{i}\right]$. Then, by (ii), $[\bar{f}]=\left[a_{1}\left[\overline{s_{1}}\right] b_{1}\right]$, where $\left[a_{1} s_{1} b_{1}\right]$ is a normal $S$-diword. Therefore,

$$
f_{1}=f-\alpha_{1}\left[a_{1} s_{1} b_{1}\right],\left[\overline{f_{1}}\right]<[\bar{f}] \text { or } f_{1}=0, f_{1} \in I d(S)
$$

Now, by using induction on $[\bar{f}]$, we have $(i i)^{\prime}$.
$(i i)^{\prime} \Rightarrow(i i)$. This part is clear.
$(i i) \Rightarrow(i i i)$. Assume $(i i)$. Then by Lemma [3.8, $\operatorname{Irr}(S)$ spans $D(X \mid S)$ as $k$-space.
Suppose that $0 \neq \sum \alpha_{i}\left[u_{i}\right] \in \operatorname{Id}(S)$ where $\left[u_{1}\right]>\left[u_{2}\right]>\cdots,\left[u_{i}\right] \in \operatorname{Irr}(S)$. Then by (ii), $\left[u_{1}\right]=\left[a_{1}\left[\overline{s_{1}}\right] b_{1}\right]$ where $\left[a_{1} s_{1} b_{1}\right]$ is a normal $S$-diword, a contradiction.

This shows (iii).
(iii) $\Rightarrow(i i)$. Assume (iii). Let $0 \neq f \in \operatorname{Id}(S)$. Since the elements in $\operatorname{Irr}(S)$ are linearly independent in $D(X \mid S)$, by Lemma $3.8,[\bar{f}]=[a[\bar{s}] b]$, where $[a s b]$ is a normal $S$-diword. Thus, (ii) follows.

Remark: In general, $(i i i) \nRightarrow(i)$. For example, it is noted that

$$
\operatorname{Irr}(S)=\left\{x_{j} \dashv x_{i_{1}} \dashv \cdots \dashv x_{i_{k}} \mid j \in I, i_{p} \in I-I_{0}, 1 \leq p \leq k, i_{1} \leq \cdots \leq i_{k}, k \geq 0\right\}
$$

is a linear basis of $D(X \mid S)$ in Theorem 4.3, Let

$$
S_{1}=\left\{x_{j} \vdash x_{i}-x_{i} \dashv x_{j}+\left\{x_{i}, x_{j}\right\}, x_{t} \dashv x_{i_{0}}, i, j, t \in I, i_{0} \in I_{0}\right\} .
$$

Then $\operatorname{Irr}\left(S_{1}\right)=\operatorname{Irr}(S)$ is a linear basis of $D(X \mid S)$. But in the proof of Theorem 4.3, we know that $S_{1}$ is not a Gröbner-Shirshov basis of $D(X \mid S)$.

## 4 Applications

In this section, we give Gröbner-Shirshov bases for the universal enveloping dialgebra of a Leibniz algebra, the bar extension of a dialgebra, the free product of two dialgebras, and the Clifford dialgebra. By using our Theorem 3.9, we obtain some normal forms for dialgebras mentioned the above.
Definition 4.1 ([10]) A $k$-linear space $L$ equipped with bilinear multiplication [,] is called a Leibniz algebra if for any $a, b, c \in L$,

$$
[[a, b], c]=[[a, c], b]+[a,[b, c]]
$$

i.e., the Leibniz identity is valid in $L$.

It is clear that if $(D, \dashv, \vdash)$ is a dialgebra then $D^{(-)}=(D,[]$,$) is a Leibniz algebra, where$ $[a, b]=a \dashv b-b \vdash a$ for any $a, b \in D$.

If $f$ is a Leibniz polynomial in variables $X$, then by $f^{(-)}$we mean a dialgebra polynomial in $X$ obtained from $f$ by transformation $[a, b] \mapsto a \dashv b-b \vdash a$.

Definition 4.2 Let $L$ be a Leibniz algebra. A dialgebra $U(L)$ together with a Leibniz homomorphism $\varepsilon: L \rightarrow U(L)$ is called the universal enveloping dialgebra for $L$, if the following diagram commute:

where $D$ is a dialgebra, $\delta$ is a Leibniz homomorphism and $f: U(L) \rightarrow D$ is a dialgebra homomorphism such that $f \varepsilon=\delta$ (i.e., $\varepsilon: L \rightarrow U(L)$ is a universal arrow in the sense of S. MacLane [13], p55).

An equivalent definition is as follows: Let $L=\operatorname{Lei}(X \mid S)$ is a Leibniz algebra presented by generators $X$ and definition relations $S$. Then $U(L)=D\left(X \mid S^{(-)}\right)$is the dialgebra with generators $X$ and definition relations $S^{(-)}=\left\{s^{(-)} \mid s \in S\right\}$.

Theorem 4.3 Let $\mathcal{L}$ be a Leibniz algebra over a field $k$ with the product $\{$,$\} . Let \mathcal{L}_{0}$ be the subspace of $\mathcal{L}$ generated by the set $\{\{a, a\},\{a, b\}+\{b, a\} \mid a, b \in \mathcal{L}\}$. Let $\left\{x_{i} \mid i \in I_{0}\right\}$ be a basis of $\mathcal{L}_{0}$ and $X=\left\{x_{i} \mid i \in I\right\}$ a well ordered basis of $\mathcal{L}$ such that $I_{0} \subseteq I$. Let $U(L)=D\left(X \mid x_{i} \dashv x_{j}-x_{j} \vdash x_{i}-\left\{x_{i}, x_{j}\right\}\right)$ be the universal enveloping dialgebra for $L$ and the ordering $<$ on $\left[X^{*}\right]$ as before. Then
(i) $D\left(X \mid x_{i} \dashv x_{j}-x_{j} \vdash x_{i}-\left\{x_{i}, x_{j}\right\}\right)=D(X \mid S)$, where $S$ consists of the following polynomials:
(a) $f_{j i}=x_{j} \vdash x_{i}-x_{i} \dashv x_{j}+\left\{x_{i}, x_{j}\right\}$

$$
\text { (b) } \quad f_{j i \vdash t}=x_{j} \vdash x_{i} \vdash x_{t}-x_{i} \vdash x_{j} \vdash x_{t}+\left\{x_{i}, x_{j}\right\} \vdash x_{t}
$$

$$
\text { (e) } \quad h_{t \dashv i_{0}}=x_{t} \dashv x_{i_{0}}
$$

$$
\begin{aligned}
& (i, j \in I) \\
& (i, j, t \in I, j>i) \\
& \left(i_{0} \in I_{0}, t \in I\right) \\
& (i, j, t \in I, j>i) \\
& \left(i_{0} \in I_{0}, t \in I\right)
\end{aligned}
$$

(c) $h_{i_{0} \vdash t}=x_{i_{0}} \vdash x_{t}$
(d) $\quad f_{t \dashv j i}=x_{t} \dashv x_{j} \dashv x_{i}-x_{t} \dashv x_{i} \dashv x_{j}+x_{t} \dashv\left\{x_{i}, x_{j}\right\}$
(ii) $S$ is a Gröbner-Shirshov basis in $D(X)$.
(iii) The set

$$
\left\{x_{j} \dashv x_{i_{1}} \dashv \cdots \dashv x_{i_{k}} \mid j \in I, i_{p} \in I-I_{0}, 1 \leq p \leq k, i_{1} \leq \cdots \leq i_{k}, k \geq 0\right\}
$$ is a linear basis of the universal enveloping algebra $U(\mathcal{L})$. In particular, $\mathcal{L}$ is a Leibniz subalgebra of $U(\mathcal{L})$.

Proof. (i) By using the following

$$
f_{j i \vdash t}=f_{j i} \vdash x_{t} \text { and } f_{j i} \vdash x_{t}+f_{i j} \vdash x_{t}=\left(\left\{x_{i}, x_{j}\right\}+\left\{x_{j}, x_{i}\right\}\right) \vdash x_{t}
$$

we have (b) and (c) are in $\operatorname{Id}\left(f_{j i}\right)$. By symmetry, (d) and (e) are in $\operatorname{Id}\left(f_{j i}\right)$. This shows (i).
(ii) We will prove that all compositions in $S$ are trivial modulo $S$ (and [w]). For convenience, we extend linearly the functions $f_{j i}, f_{j i \vdash t}, f_{t \dashv j i}, h_{i_{0} \vdash t}$ and $h_{t \dashv i_{0}}$ to $f_{j\{p, q\}}\left(f_{\{p, q\} i}\right), f_{j i \vdash\{p, q\}}$ and $h_{\{p, q\} \dashv i_{0}}$, etc respectively. For example, if $\left\{x_{p}, x_{q}\right\}=\sum \alpha_{p q}^{s} x_{s}$, then

$$
\begin{aligned}
f_{j\{p, q\}} & =x_{j} \vdash\left\{x_{p}, x_{q}\right\}-\left\{x_{p}, x_{q}\right\} \dashv x_{j}+\left\{\left\{x_{p}, x_{q}\right\}, x_{j}\right\}=\sum \alpha_{p q}^{s} f_{j s}, \\
f_{j i \vdash\{p, q\}} & =\sum \alpha_{p q}^{s}\left(x_{j} \vdash x_{i} \vdash x_{s}-x_{i} \vdash x_{j} \vdash x_{s}+\left\{x_{i}, x_{j}\right\} \vdash x_{s}\right)=f_{j i} \vdash\left\{x_{p}, x_{q}\right\}, \\
h_{\{p, q\} \dashv i_{0}} & =\sum \alpha_{p q}^{s} h_{s \dashv i_{0}} .
\end{aligned}
$$

By using the Leibniz identity,

$$
\begin{equation*}
\{\{a, b\}, c\}=\{a,\{b, c\}\}+\{\{a, c\}, b\} \tag{4}
\end{equation*}
$$

we have

$$
\{a,\{b, b\}\}=0 \text { and }\{a,\{b, c\}+\{c, b\}\}=0
$$

for any $a, b, c \in \mathcal{L}$. It means that for any $i_{0} \in I_{0}, j \in I$,

$$
\begin{equation*}
\left\{x_{j}, x_{i_{0}}\right\}=0 \tag{5}
\end{equation*}
$$

and by noting that $\left\{x_{i_{0}}, x_{j}\right\}=\left\{x_{j}, x_{i_{0}}\right\}+\left\{x_{i_{0}}, x_{j}\right\}$, we have

$$
\begin{equation*}
\left\{x_{i_{0}}, x_{j}\right\} \in \mathcal{L}_{0} . \tag{6}
\end{equation*}
$$

This implies that $\mathcal{L}_{0}$ is an ideal of $\mathcal{L}$. Clearly, $\mathcal{L} / \mathcal{L}_{0}$ is a Lie algebra.
The formulas (4), (5) and (6) are useful in the sequel.
In $S$, all the compositions are as follows.

1) Compositions of left or right multiplication.

All possible compositions in $S$ of left multiplication are ones related to (a), (b) and (c).
By noting that for any $s, i, j, t \in I$, we have

$$
\begin{aligned}
x_{s} \dashv f_{j i} & =f_{s \dashv j i} \quad(j>i), \\
x_{s} \dashv f_{j i} & =-f_{s \dashv i j}+x_{s} \dashv\left(\left\{x_{i}, x_{j}\right\}+\left\{x_{j}, x_{i}\right\}\right) \quad(j<i), \\
x_{s} \dashv f_{i i} & =x_{s} \dashv\left\{x_{i}, x_{i}\right\}, \\
x_{s} \dashv f_{j i \vdash t} & =f_{s \dashv j i} \dashv x_{t} \quad(j>i) \quad \text { and } \\
x_{s} \dashv h_{i_{0} \vdash t} & =h_{s \dashv \imath_{0}} \dashv x_{t},
\end{aligned}
$$

it is clear that all cases are trivial modulo $S$.
By symmetry, all compositions in $S$ of right multiplication are trivial modulo $S$.
2) Compositions of inclusion and intersection.

We denote, for example, $(a \wedge b)$ the composition of the polynomials of type $(a)$ and type (b). It is noted that since (b) and (c) are both left normed, we have to prove that the corresponding compositions of the cases of $(b \wedge b),(b \wedge c),(c \wedge c)$ and $(c \wedge b)$ must be a linear combination of left normed $S$-diwords in which the leading term of each $S$-diword is less than $w$. Symmetrically, we consider the cases for the right normed (d) and (e).

All possible compositions of inclusion and intersection are as follows.
$(a \wedge c) \quad[w]=x_{i_{0}} \vdash x_{i}\left(i_{0} \in I_{0}\right)$. We have, by (5),

$$
\left(f_{i_{0} i}, h_{i_{0} \vdash i}\right)_{[w]}=-x_{i} \dashv x_{i_{0}}+\left\{x_{i}, x_{i_{0}}\right\}=-h_{i \dashv i_{0}} \equiv 0 \bmod (S,[w]) .
$$

$(a \wedge d) \quad[w]=x_{j} \vdash x_{i} \dashv x_{q} \dashv x_{p} \quad(q>p)$. We have

$$
\begin{aligned}
& \left(f_{j i}, f_{i \dashv q p}\right)_{[w]} \\
= & -x_{i} \dashv x_{j} \dashv x_{q} \dashv x_{p}+\left\{x_{i}, x_{j}\right\} \dashv x_{q} \dashv x_{p}+x_{j} \vdash x_{i} \dashv x_{p} \dashv x_{p}-x_{j} \vdash x_{i} \dashv\left\{x_{p}, x_{q}\right\} \\
= & -x_{i} \dashv f_{j \dashv q p}+f_{\{i, j\} \dashv q p}+f_{j i} \dashv x_{p} \dashv x_{q}-f_{j i} \dashv\left\{x_{p}, x_{q}\right\} \\
\equiv & 0 \bmod (S,[w]) .
\end{aligned}
$$

$(a \wedge e) \quad[w]=x_{j} \vdash x_{i} \dashv x_{i_{0}}\left(i_{0} \in I_{0}\right)$. We have

$$
\left(f_{j i}, h_{i \dashv i_{0}}\right)_{[w]}=-x_{i} \dashv x_{j} \dashv x_{i_{0}}+\left\{x_{i}, x_{j}\right\} \dashv x_{i_{0}}=-x_{i} \dashv h_{j \dashv i_{0}}+h_{\{i, j\} \dashv i_{0}} \equiv 0 \bmod (S,[w]) .
$$

$(b \wedge a)$ There are two cases to consider: $[w]=x_{j} \vdash x_{i} \vdash x_{t}$ and $[w]=x_{j} \vdash x_{i} \vdash x_{t} \vdash x_{p}$.
For $[w]=x_{j} \vdash x_{i} \vdash x_{t} \quad(j>i)$, by (4), we have

$$
\begin{aligned}
\left(f_{j i \vdash t}, f_{i t}\right)_{[w]} & =-x_{i} \vdash x_{j} \vdash x_{t}+\left\{x_{i}, x_{j}\right\} \vdash x_{t}+x_{j} \vdash x_{t} \dashv x_{i}-x_{j} \vdash\left\{x_{t}, x_{i}\right\} \\
& =-x_{i} \vdash f_{j t}+f_{\{i, j\} t}+f_{j t} \dashv x_{i}-f_{j\{t, i\}}+f_{i\{t, j\}}-f_{i t} \dashv x_{j}+f_{t \dashv j i} \\
& \equiv 0 \bmod (S,[w]) .
\end{aligned}
$$

For $[w]=x_{j} \vdash x_{i} \vdash x_{t} \vdash x_{p} \quad(j>i)$, we have

$$
\begin{aligned}
& \left(f_{j i \vdash t}, f_{t p}\right)_{[w]} \\
= & -x_{i} \vdash x_{j} \vdash x_{t} \vdash x_{p}+\left\{x_{i}, x_{j}\right\} \vdash x_{t} \vdash x_{p}+x_{j} \vdash x_{i} \vdash x_{p} \dashv x_{t}-x_{j} \vdash x_{i} \vdash\left\{x_{p}, x_{t}\right\} \\
= & -x_{i} \vdash x_{j} \vdash f_{t p}+\left\{x_{i}, x_{j}\right\} \vdash f_{t p}+f_{j i \vdash p} \dashv x_{t}-f_{j i \vdash\{p, t\}} \\
\equiv & 0 \bmod (S,[w]) .
\end{aligned}
$$

$(b \wedge b) \quad$ There are two cases to consider: $[w]=x_{j} \vdash x_{i} \vdash x_{t} \vdash x_{s} \vdash x_{p}$ and $[w]=x_{j} \vdash x_{i} \vdash$ $x_{t} \vdash x_{p}$.
For $[w]=x_{j} \vdash x_{i} \vdash x_{t} \vdash x_{s} \vdash x_{p} \quad(j>i, t>s)$, we have

$$
\begin{aligned}
& \left(f_{j i \vdash t}, f_{t s \vdash p}\right)_{[w]} \\
= & -x_{i} \vdash x_{j} \vdash x_{t} \vdash x_{s} \vdash x_{p}+\left\{x_{i}, x_{j}\right\} \vdash x_{t} \vdash x_{s} \vdash x_{p}+x_{j} \vdash x_{i} \vdash x_{s} \vdash x_{t} \vdash x_{p} \\
& -x_{j} \vdash x_{i} \vdash\left\{x_{s}, x_{t}\right\} \vdash x_{p} \\
= & -x_{i} \vdash x_{j} \vdash f_{t s \vdash p}+\left\{x_{i}, x_{j}\right\} \vdash f_{t s \vdash p}+f_{j i \vdash s} \vdash x_{t} \vdash x_{p}-f_{j i \vdash\{s, t\}} \vdash x_{p} \\
\equiv & 0 \bmod (S,[w])
\end{aligned}
$$

since it is a combination of left normed $S$-diwords in which the leading term of each $S$-diword is less than $w$.
For $[w]=x_{j} \vdash x_{i} \vdash x_{t} \vdash x_{p} \quad(j>i>t)$, suppose that

$$
\left\{x_{i}, x_{j}\right\}=\sum_{m \in I_{1}} \alpha_{i j}^{m} x_{m}+\alpha_{i j}^{t} x_{t}+\sum_{n \in I_{2}} \alpha_{i j}^{n} x_{n}(m<t<n) .
$$

Denote

$$
B_{t \vdash\{i, j\} \vdash p}=x_{t} \vdash\left\{x_{i}, x_{j}\right\} \vdash x_{p}-\left\{x_{i}, x_{j}\right\} \vdash x_{t} \vdash x_{p}-\left\{x_{t},\left\{x_{i}, x_{j}\right\}\right\} \vdash x_{p} .
$$

Then

$$
B_{t \vdash\{i, j\} \vdash p}=\sum_{m \in I_{1}} \alpha_{i j}^{m} f_{t m \vdash p}-\sum_{n \in I_{2}} \alpha_{i j}^{n} f_{n \vdash \vdash p}-\sum_{q \in I_{0}} \beta_{q} h_{q \vdash p}
$$

is a linear combination of left normed $S$-diwords of length 2 or 3 , where

$$
\sum_{q \in I_{0}} \beta_{q} x_{q}=\sum_{m \in I_{1}} \alpha_{i j}^{m}\left(\left\{x_{t}, x_{m}\right\}+\left\{x_{m}, x_{t}\right\}\right)+\alpha_{i j}^{t}\left\{x_{t}, x_{t}\right\} .
$$

Denote

$$
\sum_{l \in I_{0}} \gamma_{l} x_{l}=-\left(\left\{x_{j},\left\{x_{t}, x_{i}\right\}\right\}+\left\{\left\{x_{t}, x_{i}\right\}, x_{j}\right\}\right)+\left(\left\{x_{i},\left\{x_{t}, x_{j}\right\}\right\}+\left\{\left\{x_{t}, x_{j}\right\}, x_{i}\right\}\right)
$$

Now, by (4), we have

$$
\begin{aligned}
& \left(f_{j i \vdash t}, f_{i \vdash \vdash p}\right)_{[w]} \\
= & -x_{i} \vdash x_{j} \vdash x_{t} \vdash x_{p}+\left\{x_{i}, x_{j}\right\} \vdash x_{t} \vdash x_{p}+x_{j} \vdash x_{t} \vdash x_{i} \vdash x_{p}-x_{j} \vdash\left\{x_{t}, x_{i}\right\} \vdash x_{p} \\
= & -x_{i} \vdash f_{j \nvdash \vdash p}-B_{t \vdash\{i, j\} \vdash p}+f_{j \vdash \vdash i} \vdash x_{p}-B_{j \vdash\{t, i\} \vdash p}+\sum_{l \in I_{0}} \gamma_{l} h_{l \vdash p} \\
& +B_{i \vdash\{t, j\} \vdash p}-f_{i \vdash \vdash j} \vdash x_{p}+x_{t} \vdash f_{j i \vdash p} \\
\equiv & 0 \bmod (S,[w])
\end{aligned}
$$

since it is a combination of left normed $S$-diwords in which the leading term of each $S$-diword is less than $w$.
$(b \wedge c)$ There are three cases to consider: $[w]=x_{j} \vdash x_{i_{0}} \vdash x_{t}\left(i_{0} \in I_{0}\right), \quad[w]=x_{j_{0}} \vdash x_{i} \vdash$ $x_{t}\left(j_{0} \in I_{0}\right)$ and $[w]=x_{j} \vdash x_{i} \vdash x_{t_{0}} \vdash x_{n}\left(t_{0} \in I_{0}\right)$.
Case 1. $[w]=x_{j} \vdash x_{i_{0}} \vdash x_{t} \quad\left(j>i_{0}, i_{0} \in I_{0}\right)$. By (6), we can assume that $\left\{x_{i_{0}}, x_{j}\right\}=\sum_{l \in I_{0}} \gamma_{l} x_{l}$. Then, we have

$$
\left(f_{j i_{0} \vdash t}, h_{i_{0} \vdash t}\right)_{[w]}=-x_{i_{0}} \vdash x_{j} \vdash x_{t}+\left\{x_{i_{0}}, x_{j}\right\} \vdash x_{t}=-h_{i_{0} \vdash j} \vdash x_{t}+\sum_{l \in I_{0}} \gamma_{l} h_{l \vdash t} \equiv 0 \bmod (S,[w]) .
$$

Case 2. $[w]=x_{j_{0}} \vdash x_{i} \vdash x_{t}\left(j_{0}>i, j_{0} \in I_{0}\right)$. By (5), we have

$$
\left(f_{j_{0} i \vdash t}, h_{j_{0} \vdash i}\right)_{[w]}=-x_{i} \vdash x_{j_{0}} \vdash x_{t}+\left\{x_{i}, x_{j_{0}}\right\} \vdash x_{t}=-x_{i} \vdash h_{j_{0} \vdash t} \equiv 0 \quad \bmod (S,[w]) .
$$

Case 3. $[w]=x_{j} \vdash x_{i} \vdash x_{t_{0}} \vdash x_{n} \quad\left(j>i, t_{0} \in I_{0}\right)$. We have

$$
\begin{aligned}
\left(f_{j i \vdash t_{0}}, h_{t_{0} \vdash n}\right)_{[w]} & =-x_{i} \vdash x_{j} \vdash x_{t_{0}} \vdash x_{n}+\left\{x_{i}, x_{j}\right\} \vdash x_{t_{0}} \vdash x_{n} \\
& =\left(-x_{i} \vdash x_{j}+\left\{x_{i}, x_{j}\right\}\right) \vdash h_{t_{0} \vdash n} \\
& \equiv 0 \bmod (S,[w]) .
\end{aligned}
$$

$(b \wedge d) \quad[w]=x_{j} \vdash x_{i} \vdash x_{t} \dashv x_{q} \dashv x_{p} \quad(j>i, q>p)$. We have

$$
\begin{aligned}
& \left(f_{j \vdash \vdash t}, f_{t \dashv q p}\right)_{[w]} \\
= & -x_{i} \vdash x_{j} \vdash x_{t} \dashv x_{q} \dashv x_{p}+\left\{x_{i}, x_{j}\right\} \vdash x_{t} \dashv x_{q} \dashv x_{p} \\
& +x_{j} \vdash x_{i} \vdash x_{t} \dashv x_{p} \dashv x_{q}-x_{j} \vdash x_{i} \vdash x_{t} \dashv\left\{x_{p}, x_{q}\right\} \\
= & -x_{i} \vdash x_{j} \vdash f_{t \dashv q p}+\left\{x_{i}, x_{j}\right\} \vdash f_{t \dashv q p}+f_{j i \vdash t} \dashv x_{p} \dashv x_{q}-f_{j i \vdash t} \dashv\left\{x_{p}, x_{q}\right\} \\
\equiv & 0 \bmod (S,[w]) .
\end{aligned}
$$

$(b \wedge e) \quad[w]=x_{j} \vdash x_{i} \vdash x_{t} \dashv x_{n_{0}} \quad\left(j>i, n_{0} \in I_{0}\right)$. We have

$$
\begin{aligned}
\left(f_{j i \vdash t}, h_{t \dashv n_{0}}\right)_{[w]} & =-x_{i} \vdash x_{j} \vdash x_{t} \dashv x_{n_{0}}+\left\{x_{i}, x_{j}\right\} \vdash x_{t} \dashv x_{n_{0}} \\
& =\left(-x_{i} \vdash x_{j}+\left\{x_{i}, x_{j}\right\}\right) \vdash h_{t \dashv n_{0}} \\
& \equiv 0 \bmod (S,[w]) .
\end{aligned}
$$

$(c \wedge a) \quad$ There are two cases to consider: $[w]=x_{n_{0}} \vdash x_{t}\left(n_{0} \in I_{0}\right)$ and $[w]=x_{n_{0}} \vdash x_{t} \vdash$ $x_{s}\left(n_{0} \in I_{0}\right)$.
For $[w]=x_{n_{0}} \vdash x_{t}\left(n_{0} \in I_{0}\right)$, we have

$$
\left(h_{n_{0} \vdash t}, f_{n_{0} t}\right)_{[w]}=x_{t} \dashv x_{n_{0}}-\left\{x_{t}, x_{n_{0}}\right\}=h_{t \dashv n_{0}} \equiv 0 \bmod (S,[w]) .
$$

For $[w]=x_{n_{0}} \vdash x_{t} \vdash x_{s}\left(n_{0} \in I_{0}\right)$, we have

$$
\left(h_{n_{0} \vdash t}, f_{t s}\right)_{[w]}=x_{n_{0}} \vdash x_{s} \dashv x_{t}-x_{n_{0}} \vdash\left\{x_{s}, x_{t}\right\}=h_{n_{0} \vdash s} \dashv x_{t}-h_{n_{0} \vdash\{s, t\}} \equiv 0 \quad \bmod (S,[w]) .
$$

$(c \wedge b) \quad[w]=x_{n_{0}} \vdash x_{t} \vdash x_{s} \vdash x_{p} \quad\left(t>s, n_{0} \in I_{0}\right)$. We have

$$
\begin{aligned}
\left(h_{n_{0} \vdash t}, f_{t s \vdash p}\right)_{[w]} & =x_{n_{0}} \vdash x_{s} \vdash x_{t} \vdash x_{p}-x_{n_{0}} \vdash\left\{x_{s}, x_{t}\right\} \vdash x_{p} \\
& =h_{n_{0} \vdash s} \vdash x_{t} \vdash x_{p}-h_{n_{0} \vdash\{s, t\}} \vdash x_{p} \\
& \equiv 0 \bmod (S,[w]) .
\end{aligned}
$$

$(c \wedge c) \quad[w]=x_{n_{0}} \vdash x_{t_{0}} \vdash x_{r}\left(n_{0}, t_{0} \in I_{0}\right)$. We have

$$
\left(h_{n_{0} \vdash t_{0}}, h_{t_{0} \vdash r r}\right)_{[w]}=0 .
$$

$(c \wedge d) \quad[w]=x_{n_{0}} \vdash x_{t} \dashv x_{q} \dashv x_{p} \quad\left(q>p, n_{0} \in I_{0}\right)$. We have

$$
\begin{aligned}
\left(h_{n_{0} \vdash t}, f_{t \dashv q p}\right)_{[w]} & =x_{n_{0}} \vdash x_{t} \dashv x_{p} \dashv x_{q}-x_{n_{0}} \vdash x_{t} \dashv\left\{x_{p}, x_{q}\right\} \\
& =h_{n_{0} \vdash t} \dashv\left(x_{p} \dashv x_{q}-\left\{x_{p}, x_{q}\right\}\right) \\
& \equiv 0 \bmod (S,[w]) .
\end{aligned}
$$

$(c \wedge e) \quad[w]=x_{n_{0}} \vdash x_{t} \dashv x_{s_{0}}\left(n_{0}, s_{0} \in I_{0}\right)$. We have

$$
\left(h_{n_{0} \vdash t}, h_{t \dashv s_{0}}\right)_{[w]}=0 .
$$

Since $(d \wedge d),(d \wedge e),(e \wedge d),(e \wedge e)$ are symmetric with $(b \wedge b),(b \wedge c),(c \wedge b),(c \wedge c)$ respectively, they have the similar representations. We omit the details.

So, we show that $S$ is a Gröbner-Shirshov basis.
(iii) Clearly, the mentioned set is just the set $\operatorname{Ir}(S)$. Now, the results follow from Theorem 3.9,

A Gröbner-Shirshov basis $S$ is called reduced if $S$ is a monic set and no monomial in any element of the basis contains the leading words of the other elements of the basis as subwords.

Remark: Let the notation be in Theorem 4.3. Let $S^{\text {red }}$ consist of the following polynomials:
(a) $\quad f_{j i}=x_{j} \vdash x_{i}-x_{i} \dashv x_{j}+\left\{x_{i}, x_{j}\right\}$
(b) $\quad f_{j i \vdash t}=x_{j} \vdash x_{i} \vdash x_{t}-x_{i} \vdash x_{j} \vdash x_{t}+\left\{x_{i}, x_{j}\right\} \vdash x_{t}$
(c) $\quad h_{i_{0} \vdash t}=x_{i_{0}} \vdash x_{t}$
(d) $\quad f_{t \dashv j i}=x_{t} \dashv x_{j} \dashv x_{i}-x_{t} \dashv x_{i} \dashv x_{j}+x_{t} \dashv\left\{x_{i}, x_{j}\right\}$
(e) $\quad h_{t \dashv i_{0}}=x_{t} \dashv x_{i_{0}}$

$$
\begin{aligned}
& \left(i \in I, j \in I-I_{0}\right) \\
& \left(i, j \in I-I_{0}, j>i, t \in I\right) \\
& \left(i_{0} \in I_{0}, t \in I\right) \\
& \left(i, j \in I-I_{0}, j>i, t \in I\right) \\
& \left(i_{0} \in I_{0}, t \in I\right)
\end{aligned}
$$

Then $S^{r e d}$ is a reduced Gröbner-Shirshov basis for $D(X \mid S)$.
We have the following corollary.

Corollary 4.4 ([1]) Let the notation be as in Theorem 4.3. Then as linear spaces, $U(\mathcal{L})$ is isomorphic to $\mathcal{L} \otimes U\left(\mathcal{L} / \mathcal{L}_{0}\right)$, where $U\left(\mathcal{L} / \mathcal{L}_{0}\right)$ is the universal enveloping of the Lie algebra $\mathcal{L} / \mathcal{L}_{0}$.

Proof. Clearly, $\left\{x_{j} \mid j \in I-I_{0}\right\}$ is a $k$-basis of the Lie algebra $\mathcal{L} / \mathcal{L}_{0}$. It is well known that the universal enveloping $U\left(\mathcal{L} / \mathcal{L}_{0}\right)$ of the Lie algebra $\mathcal{L} / \mathcal{L}_{0}$ has a $k$-basis

$$
\left\{x_{i_{1}} x_{i_{2}} \ldots x_{i_{k}} \mid i_{1} \leq \cdots \leq i_{k}, \quad i_{p} \in I-I_{0}, 1 \leq p \leq k, \quad k \geq 0\right\}
$$

By using (iii) in Theorem 4.3, the result follows.

Definition 4.5 Let $D$ be a dialgebra. An element $e \in D$ is called a bar unit of $D$ if $e \vdash x=x \dashv e=x$ for any $x \in D$.

Theorem 4.6 Each dialgebra has a bar unit extension.

Proof. Let $(D, \vdash, \dashv)$ be an arbitrary dialgebra over a field $k$ and $A$ the ideal of $D$ generated by the set $\{a \dashv b-a \vdash b \mid a, b \in D\}$. Let $X_{0}=\left\{x_{i_{0}} \mid i_{0} \in I_{0}\right\}$ be a $k$-basis of $A$ and $X=\left\{x_{i} \mid i \in I\right\}$ a well ordered $k$-basis of $D$ such that $I_{0} \subseteq I$. Then $D$ has a presentation by the multiplication table $D=D(X \mid S)$, where $S=\left\{x_{i} \vdash x_{j}-\left\{x_{i} \vdash\right.\right.$ $\left.\left.x_{j}\right\}, x_{i} \dashv x_{j}-\left\{x_{i} \dashv x_{j}\right\}, i, j \in I\right\}$, where $\left\{x_{i} \vdash x_{j}\right\}$ and $\left\{x_{i} \dashv x_{j}\right\}$ are linear combinations of $x_{t}, t \in I$.

Let $D_{1}=D\left(X \cup\{e\} \mid S_{1}\right)$, where $S_{1}=S \cup\left\{e \vdash y-y, y \dashv e-y, e \dashv x_{0}, x_{0} \vdash e \mid y \in\right.$ $\left.X \cup\{e\}, x_{0} \in X_{0}\right\}$. Then $D_{1}$ is a dialgebra with a bar unit $e$.

Denote

1. $f_{i \vdash j}=x_{i} \vdash x_{j}-\left\{x_{i} \vdash x_{j}\right\}$,
2. $f_{i \dashv j}=x_{i} \dashv x_{j}-\left\{x_{i} \dashv x_{j}\right\}$,
3. $g_{e \vdash y}=e \vdash y-y$,
4. $g_{y \dashv e}=y \dashv e-y$,
5. $\quad h_{x_{i_{0}} \vdash e}=x_{i_{0}} \vdash e$,
6. $h_{e \dashv x_{i_{0}}}=e \dashv x_{i_{0}}$,
where $i, j \in I, i_{0} \in I_{0}, y \in X \cup\{e\}$.
We show that $\left\{x_{t} \dashv x_{i_{0}}\right\}=0$ and $\left\{x_{i_{0}} \vdash x_{t}\right\}=0$ for any $t \in I, i_{0} \in I_{0}$.
Since $x_{i_{0}} \in A$, we have $x_{i_{0}}=\sum \alpha_{i}\left(c_{i} f_{i} d_{i}\right)$, where $f_{i}=a_{i} \dashv b_{i}-a_{i} \vdash b_{i}, \alpha_{i} \in k, a_{i}, b_{i} \in D$ and $c_{i}, d_{i} \in X^{*}$.

Since $x_{t} \dashv\left(c_{i}\left(a_{i} \dashv b_{i}-a_{i} \vdash b_{i}\right) d_{i}\right)=0$, we have $\left\{x_{t} \dashv\left\{c_{i}\left\{a_{i} \dashv b_{i}-a_{i} \vdash b_{i}\right\} d_{i}\right\}\right\}=0$ for each $i$. Then $\left\{x_{t} \dashv x_{i_{0}}\right\}=0$.

By symmetry, we have $\left\{x_{i_{0}} \vdash x_{t}\right\}=0$.
To prove the theorem, by using our Theorem [3.9, it suffices to prove that with the ordering on $\left[(X \cup\{e\})^{*}\right]$ as before, where $x<e, x \in X, S_{1}$ is a Gröbner-Shirshov basis in $D(X \cup\{e\})$. Now, we show that all compositions in $S_{1}$ are trivial.

All possible compositions of left and right multiplication are: $z \dashv f_{i \vdash j}, z \dashv g_{e \vdash y}$, $z \dashv$ $h_{x_{i_{0}} \vdash e}, f_{i \dashv j} \vdash z, g_{y \dashv e} \vdash z, h_{e \dashv x_{i_{0}}} \vdash z, z \in X \cup\{e\}$.

For $z \dashv f_{i \vdash j}, z=x_{t} \in X$, since $\left(x_{t} \dashv x_{i}\right) \dashv x_{j}=x_{t} \dashv\left(x_{i} \vdash x_{j}\right)$, we have $\left\{\left\{x_{t} \dashv x_{i}\right\} \dashv\right.$ $\left.x_{j}\right\}=\left\{x_{t} \dashv\left\{x_{i} \vdash x_{j}\right\}\right\}$ and

$$
\begin{aligned}
& x_{t} \dashv f_{i \vdash j} \\
= & x_{t} \dashv x_{i} \dashv x_{j}-x_{t} \dashv\left\{x_{i} \vdash x_{j}\right\} \\
= & f_{t \dashv i} \dashv x_{j}+f_{\{t \dashv i\} \dashv j}-f_{t \dashv\{i \vdash j\}}+\left\{\left\{x_{t} \dashv x_{i}\right\} \dashv x_{j}\right\}-\left\{x_{t} \dashv\left\{x_{i} \vdash x_{j}\right\}\right\} \\
= & f_{t \dashv i} \dashv x_{j}+f_{\{t \dashv i\} \dashv j}-f_{t \dashv\{i \vdash j\}} \\
\equiv & 0 \bmod \left(S_{1}\right) .
\end{aligned}
$$

For $z \dashv f_{i \vdash j}, z=e$, let $\left\{x_{i} \dashv x_{j}\right\}-\left\{x_{i} \vdash x_{j}\right\}=\sum \alpha_{i 0} x_{i_{0}}$. Then

$$
\begin{aligned}
e \dashv f_{i \vdash j} & =e \dashv x_{i} \dashv x_{j}-e \dashv\left\{x_{i} \vdash x_{j}\right\} \\
& =e \dashv\left(x_{i} \dashv x_{j}-\left\{x_{i} \dashv x_{j}\right\}\right)+e \dashv\left\{x_{i} \dashv x_{j}\right\}-e \dashv\left\{x_{i} \vdash x_{j}\right\} \\
& =e \dashv f_{i \dashv j}+\sum \alpha_{i_{0}} h_{e \dashv x_{i_{0}}} \\
& \equiv 0 \bmod \left(S_{1}\right) .
\end{aligned}
$$

For $z \dashv g_{\text {e卜 } y}$, we have

$$
z \dashv g_{e \vdash y}=z \dashv e \dashv y-z \dashv y=(z \dashv e-z) \dashv y=g_{z \dashv e} \dashv y \equiv 0 \bmod \left(S_{1}\right)
$$

For $z \dashv h_{x_{i_{0}} \vdash e}$, we have

$$
z \dashv h_{x_{i_{0}} \dashv e}=z \dashv x_{i_{0}} \dashv e=z \dashv g_{x_{i_{0}} \dashv e}+z \dashv x_{i_{0}} .
$$

It is clear that $z \dashv x_{i_{0}}=h_{e \dashv x_{i_{0}}}$ if $z=e$ and $z \dashv x_{i_{0}}=x_{t} \dashv x_{i_{0}}-\left\{x_{t} \dashv x_{i_{0}}\right\}=f_{t-i_{0}}$ if $z=x_{t} \in X$, since $\left\{x_{t} \dashv x_{i_{0}}\right\}=0$. This implies that $z \dashv h_{x_{i_{0}} \vdash e} \equiv 0 \bmod \left(S_{1}\right)$.

Thus we show that all compositions of left multiplication in $S_{1}$ are trivial modulo $S_{1}$. By symmetry, all compositions of right multiplication in $S_{1}$ are trivial modulo $S_{1}$.

Now, all possible ambiguities [ $w$ ] of compositions of intersection in $S_{1}$ are:
$1 \wedge 1,\left[x_{i} x_{j} \dot{x}_{t}\right] ; 1 \wedge 2,\left[x_{i} \dot{x_{j}} x_{t}\right] ; 1 \wedge 4,\left[x_{i} \dot{x}_{j} e\right] ; 1 \wedge 5,\left[x_{i} x_{i_{0}} \dot{e}\right]$.
$2 \wedge 2,\left[\dot{x}_{i} x_{j} x_{t}\right] ; 2 \wedge 4,\left[\dot{x}_{i} x_{j} e\right]$.
$3 \wedge 1,\left[e x_{i} \dot{x}_{j}\right] ; 3 \wedge 2,\left[e \dot{x}_{i} x_{j}\right] ; 3 \wedge 3,[e e \dot{y}] ; 3 \wedge 4,[e \dot{y} e] ; 3 \wedge 5,\left[e x_{i_{0}} \dot{e}\right] ; 3 \wedge 6,\left[e \dot{e} x_{i_{0}}\right]$.
$4 \wedge 4,[$ yee $] ; 4 \wedge 6,\left[\right.$ yex $\left._{i_{0}}\right]$.
$5 \wedge 3,\left[x_{i_{0}} e \dot{y}\right] ; 5 \wedge 4,\left[x_{i_{0}} \dot{e} e\right] ; 5 \wedge 6,\left[x_{i_{0}} \dot{e} x_{j_{0}}\right]$.
$6 \wedge 2,\left[\dot{e} x_{i_{0}} x_{j}\right] ; 6 \wedge 4,\left[\dot{e} x_{i_{0}} e\right]$.
In the above, all $i, j, t \in I, i_{0}, j_{0} \in I_{0}$ and $y \in X \cup\{e\}$.
There is no composition of inclusion in $S_{1}$.
We will show that all compositions of intersection in $S_{1}$ are trivial. We check only the cases of $1 \wedge 2,1 \wedge 5$ and $4 \wedge 6$. Others can be similarly proved.

For $1 \wedge 2,[w]=\left[x_{i} \dot{x_{j}} x_{t}\right]$, since $\left(x_{i} \vdash x_{j}\right) \dashv x_{t}=x_{i} \vdash\left(x_{j} \dashv x_{t}\right)$, we have $\left\{\left\{x_{i} \vdash x_{j}\right\} \dashv\right.$ $\left.x_{t}\right\}=\left\{x_{i} \vdash\left\{x_{j} \dashv x_{t}\right\}\right\}$ and

$$
\begin{aligned}
(1 \wedge 2)_{[w]} & =-\left\{x_{i} \vdash x_{j}\right\} \dashv x_{t}+x_{i} \vdash\left\{x_{j} \dashv x_{t}\right\} \\
& =-f_{\{i \vdash j\} \dashv t}+f_{i \vdash\{j \dashv t t\}}-\left\{\left\{x_{i} \vdash x_{j}\right\} \dashv x_{t}\right\}+\left\{x_{i} \vdash\left\{x_{j} \dashv x_{t}\right\}\right\} \\
& =-f_{\{i \vdash j\} \dashv t}+f_{i \vdash 〔 j \dashv-t\}} \\
& \equiv 0 \bmod \left(S_{1},[w]\right) .
\end{aligned}
$$

For $1 \wedge 5,[w]=\left[x_{i} x_{i_{0}} \dot{e}\right]$, since $x_{i} \vdash x_{i_{0}} \in A$, we have $\left\{x_{i} \vdash x_{i_{0}}\right\}=\sum \alpha_{j_{0}} x_{j_{0}}$ and

$$
(1 \wedge 5)_{[w]}=\left\{x_{i} \vdash x_{i_{0}}\right\} \vdash e=\sum \alpha_{j_{0}} h_{x_{j_{0}} \vdash e} \equiv 0 \bmod \left(S_{1},[w]\right) .
$$

For $4 \wedge 6,[w]=\left[\right.$ yex $\left._{i_{0}}\right]$, we have $(4 \wedge 6)_{[w]}=-h_{e \dashv x_{i_{0}}}$ if $y=e$ and $(4 \wedge 6)_{[w]}=-f_{t-i_{0}}$ if $y=x_{t} \in X$ since $\left\{x_{t} \dashv x_{i_{0}}\right\}=0$. Then $(4 \wedge 6)_{[w]} \equiv 0 \bmod \left(S_{1},[w]\right)$.

Then all the compositions in $S_{1}$ are trivial.
The proof is complete.

Remark: Let the notation be as in the proof of Theorem4.6. Let $D^{\prime}=D\left(X \cup\left\{e_{j}\right\}_{J} \mid S^{\prime}\right)$ be a dialgebra, where $S^{\prime}=S \cup\left\{e_{j} \vdash y-y, y \dashv e_{j}-y, e_{j} \dashv x_{0}, x_{0} \vdash e_{j} \mid y \in X \cup\left\{e_{j}\right\}_{J}, x_{0} \in\right.$ $\left.X_{0}, j \in J\right\}$. Let $J$ be a well ordered set. Then with the ordering on $\left[\left(X \cup\left\{e_{j}\right\}_{J}\right)^{*}\right]$ as before, where $x_{i}<e_{j}$ for all $i \in I, j \in J$, by a similar proof of Theorem 4.6, $S^{\prime}$ is a Gröbner-Shirshov basis in $D\left(X \cup\left\{e_{j}\right\}_{J}\right)$. It follows from Theorem 3.9 that $D$ can be embedded into the dialgebra $D^{\prime}$ while $D^{\prime}$ has bar units $\left\{e_{j}\right\}_{J}$.

Definition 4.7 Let $D_{1}, D_{2}$ be dialgebras over a field $k$. The dialgebra $D_{1} * D_{2}$ with two dialgebra homomorphisms $\varepsilon_{1}: D_{1} \rightarrow D_{1} * D_{2}, \varepsilon_{2}: D_{2} \rightarrow D_{1} * D_{2}$ is called the free product of $D_{1}, D_{2}$, if the following diagram commute:

where $D$ is a dialgebra, $\delta_{1}, \delta_{2}$ are dialgebra homomorphisms and $f: D_{1} * D_{2} \rightarrow D$ is a dialgebra homomorphism such that $f \varepsilon_{1}=\delta_{1}, f \varepsilon_{2}=\delta_{2}$ (i.e., $\left(\varepsilon_{1}, \varepsilon_{2}\right):\left(D_{1}, D_{2}\right) \rightarrow$ $\left(D_{1} * D_{2}, D_{1} * D_{2}\right)$ is a universal arrow in the sense of $S$. Maclane [13]).

An equivalent definition is as follows: Let $D_{i}=D\left(X_{i} \mid S_{i}\right)$ be a presentation by generators and defining relations with $X_{1} \cap X_{2}=\varnothing, i=1,2$. Then $D_{1} * D_{2}=D\left(X_{1} \cup X_{2} \mid S_{1} \cup S_{2}\right)$.

Let $\left(D_{1}, \vdash, \dashv\right),\left(D_{2}, \vdash, \dashv\right)$ be two dialgebras over a field $k, A_{1}$ the ideal of $D_{1}$ generated by the set $\left\{a \dashv b-a \vdash b \mid a, b \in D_{1}\right\}$ and $A_{2}$ the ideal of $D_{2}$ generated by the set $\left\{c \dashv d-c \vdash d \mid c, d \in D_{2}\right\}$. Let $X_{0}=\left\{x_{i_{0}} \mid i_{0} \in I_{0}\right\}$ be a $k$-basis of $A_{1}$ and $X=\left\{x_{i} \mid i \in I\right\}$ a well ordered $k$-basis of $D_{1}$ such that $I_{0} \subseteq I$. Let $Y_{0}=\left\{y_{l_{0}} \mid l_{0} \in J_{0}\right\}$ be a $k$-basis of $A_{2}$ and $Y=\left\{y_{l} \mid l \in J\right\}$ a well ordered $k$-basis of $D_{2}$ such that $J_{0} \subseteq J$. Then $D_{1}$ and $D_{2}$ have multiplication tables:

$$
\begin{array}{ll}
D_{1}=D\left(X \mid S_{1}\right), & S_{1}=\left\{x_{i} \vdash x_{j}-\left\{x_{i} \vdash x_{j}\right\}, x_{i} \dashv x_{j}-\left\{x_{i} \dashv x_{j}\right\}, i, j \in I\right\}, \\
D_{2}=D\left(Y \mid S_{2}\right), & S_{2}=\left\{y_{l} \vdash y_{m}-\left\{y_{l} \vdash y_{m}\right\}, y_{l} \dashv y_{m}-\left\{y_{l} \dashv y_{m}\right\}, l, m \in J\right\} .
\end{array}
$$

The free product $D_{1} * D_{2}$ of $D_{1}$ and $D_{2}$ is

$$
D_{1} * D_{2}=D\left(X \cup Y \mid S_{1} \cup S_{2}\right)
$$

We order $X \cup Y$ by $x_{i}<y_{j}$ for any $i \in I, j \in J$. Then we have the following theorem.
Theorem 4.8 (i) $S$ is a Gröbner-Shirshov basis of $D_{1} * D_{2}=D\left(X \cup Y \mid S_{1} \cup S_{2}\right)$, where $S$ consists of the following relations:

1. $f_{x_{i} \vdash x_{j}}=x_{i} \vdash x_{j}-\left\{x_{i} \vdash x_{j}\right\}, \quad i, j \in I$,
2. $f_{x_{i} \dashv x_{j}}=x_{i} \dashv x_{j}-\left\{x_{i} \dashv x_{j}\right\}, \quad i, j \in I$,
3. $\quad f_{y_{l} \vdash y_{m}}=y_{l} \vdash y_{m}-\left\{y_{l} \vdash y_{m}\right\}, \quad l, m \in J$,
4. $\quad f_{y_{l} \dashv y_{m}}=y_{l} \dashv y_{m}-\left\{y_{l} \dashv y_{m}\right\}, \quad l, m \in J$,
5. $\quad h_{x_{i_{0}} \vdash y_{l}}=x_{i_{0}} \vdash y_{l}, \quad i_{0} \in I_{0}, l \in J$,
6. $\quad h_{y_{l} \dashv x_{i_{0}}}=y_{l} \dashv x_{i_{0}}, \quad i_{0} \in I_{0}, l \in J$,
7. $h_{y_{l_{0}}+x_{i}}=y_{l_{0}} \vdash x_{i}, \quad i \in I, l_{0} \in J_{0}$,
8. $h_{x_{i} \dashv y_{l_{0}}}=x_{i} \dashv y_{l_{0}}, \quad i \in I, l_{0} \in J_{0}$.
(ii) $\operatorname{Irr}(S)$, which is a $k$-linear basis of $D_{1} * D_{2}$, consists of all elements $z_{-m} \cdots z_{-1} \dot{z}_{0} z_{1} \cdots z_{n}$, where $m, n \geq 0, z_{0} \in X \cup Y, z_{i} \in\left(X \backslash X_{0}\right) \cup\left(Y \backslash Y_{0}\right),-m \leq i \leq n, i \neq 0$, neither $\left\{z_{j}, z_{j+1}\right\} \subseteq$ $X$ nor $\left\{z_{j}, z_{j+1}\right\} \subseteq Y,-m \leq j \leq n-1$.

Proof. By the proof of Theorem 4.6, we have $\left\{x_{i} \dashv x_{i_{0}}\right\}=0,\left\{x_{i_{0}} \vdash x_{i}\right\}=0,\left\{y_{l} \dashv\right.$ $\left.y_{l_{0}}\right\}=0$ and $\left\{y_{l_{0}} \vdash y_{l}\right\}=0$ for any $i \in I, i_{0} \in I_{0}, l \in J, l_{0} \in J_{0}$.

Firstly, we prove that $h_{y_{l} \dashv x_{i_{0}}} \in I d\left(S_{1} \cup S_{2}\right)$ for any $i_{0} \in I_{0}, l \in J$.
Since $y_{l} \dashv\left(c_{i}\left(\left\{a_{i} \dashv b_{i}\right\}-\left\{a_{i} \vdash b_{i}\right\}\right) d_{i}\right)=y_{l} \dashv\left(c_{i}\left(\left(a_{i} \dashv b_{i}-\left\{a_{i} \dashv b_{i}\right\}\right)-\left(a_{i} \vdash b_{i}-\left\{a_{i} \vdash\right.\right.\right.\right.$ $\left.\left.\left.b_{i}\right\}\right) d_{i}\right) \in \operatorname{Id}\left(S_{1} \cup S_{2}\right)$, we have $y_{l} \dashv\left\{c_{i}\left\{a_{i} \dashv b_{i}-a_{i} \vdash b_{i}\right\} d_{i}\right\} \in \operatorname{Id}\left(S_{1} \cup S_{2}\right)$ for all $i, l$. Then $h_{y_{l} \dashv x_{i_{0}}} \in \operatorname{Id}\left(S_{1} \cup S_{2}\right)$.

Similarly, we have $h_{x_{i_{0}} \vdash y_{l}}, h_{y_{l_{0}} \vdash x_{i}}, h_{x_{i} \dashv y_{l_{0}}} \in I d\left(S_{1} \cup S_{2}\right)$ for any $i \in I, i_{0} \in I_{0}, l \in$ $J, l_{0} \in J_{0}$.

Secondly, we will show that all compositions in $S$ are trivial.
All possible compositions of left and right multiplication are: $z \dashv f_{x_{i} \vdash x_{j}}, z \dashv f_{y_{l} \vdash y_{m}}, z \dashv$ $h_{x_{i_{0}} \vdash y_{l}}, z \dashv h_{y_{l_{0}} \vdash x_{i}}, f_{x_{i} \dashv x_{j}} \vdash z, f_{y_{l} \dashv y_{m}} \vdash z, h_{y_{l} \dashv x_{i_{0}}} \vdash z, h_{x_{i} \dashv y_{l_{0}}} \vdash z$, where $z \in X \cup Y$.

By a similar proof in Theorem 4.6, all compositions of left and right multiplication mentioned the above are trivial modulo $S$.

Now, all possible ambiguities $[w]$ of compositions of intersection in $S$ are:

$$
\begin{aligned}
& 1 \wedge 1,\left[x_{i} x_{j} \dot{x}_{t}\right] ; 1 \wedge 2,\left[x_{i} \dot{x}_{j} x_{t}\right] ; 1 \wedge 5,\left[x_{i} x_{i_{0}} \dot{y}_{l}\right] ; 1 \wedge 8,\left[x_{i} \dot{x}_{j} y_{l_{0}}\right] . \\
& 2 \wedge 2,\left[\dot{x}_{i} x_{j} x_{t}\right] ; 2 \wedge 8,\left[\dot{x}_{i} x_{j} y_{l_{0}}\right] . \\
& 3 \wedge 3,\left[y_{l} y_{m} \dot{y}_{t}\right] ; 3 \wedge 4,\left[y_{l} \dot{y}_{m} y_{t}\right] ; 3 \wedge 6,\left[y_{l} \dot{y}_{m} x_{i_{0}}\right] ; 3 \wedge 7,\left[y_{m} y_{l_{0}} \dot{x}_{i}\right] . \\
& 4 \wedge 4,\left[\dot{y}_{l} y_{m} y_{t}\right] ; 4 \wedge 6,\left[\dot{y}_{l} y_{m} x_{i_{0}}\right] . \\
& 5 \wedge 3,\left[x_{i_{0}} y_{l} \dot{y}_{t}\right] ; 5 \wedge 4,\left[x_{i_{0}} \dot{y}_{l} y_{t}\right] ; 5 \wedge 6,\left[x_{i_{0}} \dot{y}_{j_{l}}\right] ; 5 \wedge 7,\left[x_{i_{0}} y_{l_{0}} \dot{x}_{t}\right] . \\
& 6 \wedge 2,\left[\dot{y}_{l} x_{i_{0}} x_{t}\right] ; 6 \wedge 8,\left[\dot{y}_{m} x_{i_{0}} y_{l_{0}}\right] . \\
& 7 \wedge 1,\left[y_{l_{0}} x_{i} \dot{x}_{j}\right] ; 7 \wedge 2,\left[y_{l_{0}} \dot{x}_{i} x_{j}\right] ; 7 \wedge 5,\left[y_{l_{0}} x_{i_{0}} \dot{y}_{m}\right] ; 7 \wedge 8,\left[{l_{0}} y_{m_{0}}\right] . \\
& 8 \wedge 4,\left[\dot{x} i_{i} y_{l_{0}} y_{t}\right] ; 8 \wedge 6,\left[\dot{x}_{i} y_{l_{0}} x_{i_{0}}\right] .
\end{aligned}
$$

There is no composition of inclusion in $S$.
We will show that all compositions of intersection in $S$ are trivial. We check only the cases of $1 \wedge 5$ and $2 \wedge 8$. Others can be similarly proved.

For $1 \wedge 5,[w]=\left[x_{i} x_{i_{0}} \dot{y}_{l}\right]$, let $\left\{x_{i} \vdash x_{i_{0}}\right\}=\sum \alpha_{t_{0}} x_{t_{0}}$. Then

$$
(1 \wedge 5)_{[w]}=-\left\{x_{i} \vdash x_{i_{0}}\right\} \vdash y_{l}=-\sum \alpha_{t_{0}} h_{x_{t_{0}} \vdash y_{l}} \equiv 0 \quad \bmod (S,[w]) .
$$

For $2 \wedge 8,[w]=\left[\dot{x}_{i} x_{j} y_{l_{0}}\right]$, let $\left\{x_{i} \dashv x_{j}\right\}=\sum \alpha_{t} x_{t}$. Then

$$
(2 \wedge 8)_{[w]}=-\left\{x_{i} \dashv x_{j}\right\} \dashv y_{l_{0}}=-\sum \alpha_{t} h_{x_{t} \dashv y_{l_{0}}} \equiv 0 \bmod (S,[w]) .
$$

Then all the compositions in $S$ are trivial. This show (i).
(ii) follows from our Theorem 3.9,

Definition 4.9 Let $X=\left\{x_{1}, \ldots, x_{n}\right\}$ be a set, $k$ a field of characteristic $\neq 2$ and $\left(a_{i j}\right)_{n \times n}$ a non-zero symmetric matrix over $k$. Denote

$$
D\left(X \cup\{e\} \mid x_{i} \vdash x_{j}+x_{j} \dashv x_{i}-2 a_{i j} e, e \vdash y-y, y \dashv e-y, x_{i}, x_{j} \in X, y \in X \cup\{e\}\right)
$$

by $C(n, f)$. Then $C(n, f)$ is called a Clifford dialgebra.

We order $X \cup\{e\}$ by $x_{1}<\cdots<x_{n}<e$.
Theorem 4.10 Let the notation be as the above. Then
(i) $S$ is a Gröbner-Shirshov basis of Clifford dialgebra $C(n, f)$, where $S$ consists of the following relations:

$$
\begin{aligned}
& \text { 1. } f_{x_{i} x_{j}}=x_{i} \vdash x_{j}+x_{j} \dashv x_{i}-2 a_{i j} e, \\
& \text { 2. } g_{e \vdash y}=e \vdash y-y, \\
& \text { 3. } g_{y \dashv e}=y \dashv e-y, \\
& \text { 4. } f_{y \dashv x_{i} x_{j}}=y \dashv x_{i} \dashv x_{j}+y \dashv x_{j} \dashv x_{i}-2 a_{i j} y, \quad(i>j) \text {, } \\
& \text { 5. } f_{y \dashv x_{i} x_{i}}=y \dashv x_{i} \dashv x_{i}-a_{i i} y, \\
& \text { 6. } f_{x_{i} x_{j} \vdash y}=x_{i} \vdash x_{j} \vdash y+x_{j} \vdash x_{i} \vdash y-2 a_{i j} y, \quad(i>j) \text {, } \\
& \text { 7. } f_{x_{i} x_{i} \vdash y}=x_{i} \vdash x_{i} \vdash y-a_{i i} y, \\
& \text { 8. } h_{x_{i} e}=x_{i} \vdash e-e \dashv x_{i} \text {, }
\end{aligned}
$$

where $x_{i}, x_{j} \in X, y \in X \cup\{e\}$.
(ii) A $k$-linear basis of $C(n, f)$ is a set of all elements of the form $\dot{y} x_{i_{1}} \cdots x_{i k}$, where $y \in X \cup\{e\}, x_{i j} \in X$ and $i_{1}<i_{2}<\cdots<i_{k} \quad(k \geq 0)$.

Proof. Let $S_{1}=\left\{f_{x_{i} x_{j}}, g_{e \vdash y}, g_{y \dashv e} \mid x_{i}, x_{j} \in X, y \in X \cup\{e\}\right\}$.
Firstly, we will show that $f_{y \dashv x_{i} x_{j}}, f_{y \dashv x_{i} x_{i}}, f_{x_{i} x_{j} \vdash y}, f_{x_{i} x_{i} \vdash y}, h_{x_{i} e} \in \operatorname{Id}\left(S_{1}\right)$.
In fact, $f_{y \dashv x_{i} x_{j}}=y \dashv f_{x_{i} x_{j}}+2 a_{i j} g_{y \dashv e}$ implies $f_{y \dashv x_{i} x_{j}}, f_{y \dashv x_{i} x_{i}} \in I d\left(S_{1}\right)$. By symmetry, we have $f_{x_{i} x_{j} \vdash y}, f_{x_{i} x_{i} \vdash y} \in \operatorname{Id}\left(S_{1}\right)$.

If there exists $t$ such that $a_{i t} \neq 0$, then

$$
2 a_{i t} h_{x_{i} e}=f_{x_{i} x_{i} \vdash x_{t}}-x_{i} \vdash f_{x_{i} \vdash x_{t}}+f_{x_{i} \vdash x_{t}} \dashv x_{i}-f_{x_{t} \dashv x_{i} x_{i}} \in I d\left(S_{1}\right) .
$$

Otherwise, $a_{i t}=0$ for any $t$. Since $\left(a_{i j}\right) \neq 0$, there exists $j \neq i$ such that $a_{j t} \neq 0$ for some $t$. Then

$$
\begin{aligned}
& 2 a_{j t} h_{x_{i} e} \\
= & f_{x_{i} x_{j} \vdash x_{t}}-x_{i} \vdash f_{x_{j} \vdash x_{t}}-x_{j} \vdash f_{x_{i} \vdash x_{t}}+f_{x_{i} \vdash x_{t}} \dashv x_{j}+f_{x_{j} \vdash x_{t}} \dashv x_{i}-f_{x_{t} \dashv x_{i} x_{j}} \in \operatorname{Id}\left(S_{1}\right) .
\end{aligned}
$$

This shows that $h_{x_{i} e} \in \operatorname{Id}\left(S_{1}\right)$.
Secondly, we will show that all compositions in $S$ is trivial.
All possible compositions of left and right multiplication are: $z \dashv f_{x_{i} x_{j}}, z \dashv g_{\text {eคy }}, z \dashv$ $f_{x_{i} x_{j} \vdash y}, z \dashv f_{x_{i} x_{i} \vdash y}, z \dashv h_{x_{i} e}, f_{x_{i} x_{j}} \vdash z, g_{y \dashv e} \vdash z, f_{y \dashv x_{i} x_{j}} \vdash z, f_{y \dashv x_{i} x_{i}} \vdash z, h_{x_{i} e} \vdash z$, where $z \in X \cup\{e\}$. We just check the cases of $f_{y \dashv x_{i} x_{j}} \vdash z$ and $h_{x_{i} e} \vdash z$. Others can be similarly proved.

For $f_{y \dashv x_{i} x_{j}} \vdash z$, we have

$$
f_{y \dashv x_{i} x_{j}} \vdash z=y \vdash x_{i} \vdash x_{j} \vdash z+y \vdash x_{j} \vdash x_{i} \vdash z-2 a_{i j} y \vdash z=y \vdash f_{x_{i} x_{j} \vdash z} \equiv 0 \bmod (S) .
$$

For $h_{x_{i} e} \vdash z$,

$$
h_{x_{i} e} \vdash z=x_{i} \vdash e \vdash z-e \vdash x_{i} \vdash z=x_{i} \vdash g_{e \vdash z}-g_{e \vdash x_{i}} \vdash z \equiv 0 \bmod (S) .
$$

Now, all possible ambiguities $[w]$ of compositions of intersection in $S$ are:

$$
\begin{aligned}
& 1 \wedge 3,\left[x_{i} \dot{x}_{j} e\right] ; 1 \wedge 4,\left[x_{i} \dot{x}_{j} x_{m} x_{n}\right](m>n) ; 1 \wedge 5,\left[x_{i} \dot{x}_{j} x_{n} x_{n}\right] . \\
& 2 \wedge 1,\left[e x_{i} \dot{x}_{j}\right] ; 2 \wedge 2,[e e \dot{y}] ; 2 \wedge 3,[e \dot{y} e] ; 2 \wedge 4,\left[e \dot{y} x_{i} x_{j}\right](i>j) ; \\
& 2 \wedge 5,\left[e \dot{y} x_{i} x_{i}\right] ; 2 \wedge 6,\left[e x_{i} x_{j} \dot{y}\right](i>j) ; 2 \wedge 7,\left[e x_{i} x_{i} \dot{y}\right] ; 2 \wedge 8,\left[e x_{i} \dot{e}\right] . \\
& 3 \wedge 3,[\dot{y} e e] ; 3 \wedge 4,\left[\dot{\text { y }} e x_{i} x_{j}\right](i>j) ; 3 \wedge 5 \text {, }\left[\dot{\text { y }} e x_{i} x_{i}\right] \text {. } \\
& 4 \wedge 3,\left[\dot{y} x_{i} x_{j} e\right](i>j) ; 4 \wedge 4,\left[\dot{y} x_{i} x_{j} x_{m} x_{n}\right](i>j, m>n),\left[\dot{y} x_{i} x_{j} x_{t}\right](i>j>t) ; \\
& 4 \wedge 5,\left[\dot{y} x_{i} x_{j} x_{t} x_{t}\right](i>j),\left[\dot{y} x_{i} x_{j} x_{j}\right](i>j) . \\
& 5 \wedge 3,\left[\dot{y} x_{i} x_{i} e\right] ; 5 \wedge 4,\left[\dot{y} x_{i} x_{i} x_{m} x_{n}\right](m>n),\left[\dot{y} x_{i} x_{i} x_{j}\right](i>j) \text {; } \\
& 5 \wedge 5,\left[\dot{y} x_{i} x_{i} x_{m} x_{m}\right],\left[\dot{y} x_{i} x_{i} x_{i}\right] . \\
& 6 \wedge 1,\left[x_{i} x_{j} x_{m} \dot{x}_{n}\right](i>j) ; 6 \wedge 2,\left[x_{i} x_{j} e \dot{y}\right](i>j) ; 6 \wedge 3,\left[x_{i} x_{j} \dot{y} e\right](i>j) ; \\
& 6 \wedge 4,\left[x_{i} x_{j} \dot{y} x_{m} x_{n}\right](i>j, m>n) ; 6 \wedge 5,\left[x_{i} x_{j} \dot{y} x_{m} x_{m}\right](i>j) ; \\
& 6 \wedge 6,\left[x_{i} x_{j} x_{m} x_{n} \dot{y}\right](i>j, m>n),\left[x_{i} x_{j} x_{t} \dot{y}\right](i>j>t) \text {; } \\
& 6 \wedge 7,\left[x_{i} x_{j} x_{m} x_{m} \dot{y}\right](i>j),\left[x_{i} x_{j} x_{j} \dot{y}\right](i>j) ; 6 \wedge 8,\left[x_{i} x_{j} x_{t} \dot{e}\right](i>j) . \\
& 7 \wedge 1,\left[x_{i} x_{i} x_{m} \dot{x}_{n}\right] ; 7 \wedge 2,\left[x_{i} x_{i} e \dot{y}\right] ; 7 \wedge 3,\left[x_{i} x_{i} \dot{y} e\right] ; 7 \wedge 4,\left[x_{i} x_{i} \dot{y} x_{m} x_{n}\right](m>n) ; \\
& 7 \wedge 5,\left[x_{i} x_{i} \dot{y} x_{m} x_{m}\right] ; 7 \wedge 6,\left[x_{i} x_{i} x_{m} x_{n} \dot{y}\right](m>n),\left[x_{i} x_{i} x_{t} \dot{y}\right](i>t) ; \\
& 7 \wedge 7,\left[x_{i} x_{i} x_{m} x_{m} \dot{y}\right],\left[x_{i} x_{i} x_{i} \dot{y}\right] ; 7 \wedge 8,\left[x_{i} x_{i} x_{j} \dot{e}\right] . \\
& 8 \wedge 3,\left[x_{i} \dot{e} e\right] ; 8 \wedge 4,\left[x_{i} \dot{e} x_{m} x_{n}\right](m>n) ; 8 \wedge 5,\left[x_{i} \dot{e} x_{m} x_{m}\right] \text {. }
\end{aligned}
$$

All possible ambiguities [ $w$ ] of compositions of inclusion in $S$ are:

$$
\begin{aligned}
& 6 \wedge 1,\left[x_{i} x_{j} \dot{x}_{t}\right](i>j) ; 6 \wedge 8,\left[x_{i} x_{j} \dot{e}\right](i>j) . \\
& 7 \wedge 1,\left[x_{i} x_{i} \dot{x}_{j}\right] ; 7 \wedge 8,\left[x_{i} x_{i} \dot{e}\right] .
\end{aligned}
$$

We just check the cases of intersection $1 \wedge 4,4 \wedge 4,6 \wedge 4,6 \wedge 8,8 \wedge 4$ and of inclusion $6 \wedge 1,6 \wedge 8$. Others can be similarly proved.

For $1 \wedge 4,[w]=\left[x_{i} \dot{x}_{j} x_{m} x_{n}\right](m>n)$, we have

$$
\begin{aligned}
& (1 \wedge 4)_{[w]} \\
= & x_{j} \dashv x_{i} \dashv x_{m} \dashv x_{n}-2 a_{i j} e \dashv x_{m} \dashv x_{n}-x_{i} \vdash x_{j} \dashv x_{n} \dashv x_{m}+2 a_{m n} x_{i} \vdash x_{j} \\
= & x_{j} \dashv f_{x_{i} \dashv x_{m} x_{n}}-2 a_{i j} f_{e \dashv x_{m} x_{n}}-f_{x_{i} x_{j}} \dashv x_{n} \dashv x_{m}+2 a_{m n} f_{x_{i} x_{j}} \\
\equiv & 0 \bmod (S,[w]) .
\end{aligned}
$$

For $4 \wedge 4$, there are two cases to consider: $\left[w_{1}\right]=\left[\dot{y} x_{i} x_{j} x_{m} x_{n}\right](i>j, m>n)$ and $\left[w_{2}\right]=\left[\dot{y} x_{i} x_{j} x_{t}\right](i>j>t)$. We have
$(4 \wedge 4)_{\left[w_{1}\right]}$
$=y \dashv x_{j} \dashv x_{i} \dashv x_{m} \dashv x_{n}-2 a_{i j} y \dashv x_{m} \dashv x_{n}-y \dashv x_{i} \dashv x_{j} \dashv x_{n} \dashv x_{m}+2 a_{m n} y \dashv x_{i} \dashv x_{j}$
$=y \dashv x_{j} \dashv f_{x_{i} \dashv x_{m} x_{n}}-2 a_{i j} f_{y \dashv x_{m} x_{n}}-f_{y \dashv x_{i} x_{j}} \dashv x_{n} \dashv x_{m}+2 a_{m n} f_{y \dashv x_{i} x_{j}}$
$\equiv 0 \bmod \left(S,\left[w_{1}\right]\right) \quad$ and
$(4 \wedge 4)_{\left[w_{2}\right]}$
$=y \dashv x_{j} \dashv x_{i} \dashv x_{t}-2 a_{i j} y \dashv x_{t}-y \dashv x_{i} \dashv x_{t} \dashv x_{j}+2 a_{j t} y \dashv x_{i}$
$=y \dashv f_{x_{j} \dashv x_{i} x_{t}}-f_{y \dashv x_{j} x_{t}} \dashv x_{i}-f_{y \dashv x_{i} x_{t}} \dashv x_{j}+y \dashv f_{x_{t} \dashv x_{i} x_{j}}$
$\equiv 0 \bmod \left(S,\left[w_{2}\right]\right)$.

For $6 \wedge 4,[w]=\left[x_{i} x_{j} \dot{y} x_{m} x_{n}\right](i>j, m>n)$, we have

$$
\begin{aligned}
& (6 \wedge 4)_{[w]} \\
= & x_{j} \vdash x_{i} \vdash y \dashv x_{m} \dashv x_{n}-2 a_{i j} y \dashv x_{m} \dashv x_{n}-x_{i} \vdash x_{j} \vdash y \dashv x_{n} \dashv x_{m}+2 a_{m n} x_{i} \vdash x_{j} \vdash y \\
= & x_{j} \vdash x_{i} \vdash f_{y \dashv x_{m} x_{n}}-2 a_{i j} f_{y \dashv x_{m} x_{n}}-f_{x_{i} x_{j} \vdash y} \dashv x_{n} \dashv x_{m}+2 a_{m n} f_{x_{i} x_{j} \vdash y} \\
\equiv & 0 \bmod (S,[w]) .
\end{aligned}
$$

For $6 \wedge 8,[w]=\left[x_{i} x_{j} x_{t} \dot{e}\right](i>j)$, we have

$$
\begin{aligned}
(6 \wedge 8)_{[w]} & =x_{j} \vdash x_{i} \vdash x_{t} \vdash e-2 a_{i j} x_{t} \vdash e+x_{i} \vdash x_{j} \vdash e \dashv x_{t} \\
& =x_{j} \vdash x_{i} \vdash h_{x_{t} e}-2 a_{i j} h_{x_{t} e}+f_{x_{i} x_{j} \vdash e} \dashv x_{t} \\
& \equiv 0 \bmod (S,[w]) .
\end{aligned}
$$

For $8 \wedge 4,[w]=\left[x_{i} \dot{e} x_{m} x_{n}\right](m>n)$, we have

$$
\begin{aligned}
(8 \wedge 4)_{[w]} & =-e \dashv x_{i} \dashv x_{m} \dashv x_{n}-x_{i} \vdash e \dashv x_{n} \dashv x_{m}+2 a_{m n} x_{i} \vdash e \\
& =-e \dashv f_{x_{i} \dashv x_{m} x_{n}}-h_{x_{i} e} \dashv x_{n} \dashv x_{m}+2 a_{m n} h_{x_{i} e} \\
& \equiv 0 \bmod (S,[w]) .
\end{aligned}
$$

Now, we check the compositions of inclusion $6 \wedge 1$ and $6 \wedge 8$.
For $6 \wedge 1,[w]=\left[x_{i} x_{j} \dot{x}_{t}\right](i>j)$, we have

$$
\begin{aligned}
(6 \wedge 1)_{[w]} & =x_{j} \vdash x_{i} \vdash x_{t}-2 a_{i j} x_{t}-x_{i} \vdash x_{t} \dashv x_{j}+2 a_{j t} x_{i} \vdash e \\
& =x_{j} \vdash f_{x_{i} x_{t}}-f_{x_{i} x_{t}} \dashv x_{j}+2 a_{j t} h_{x_{i} e}-f_{x_{j} x_{t}} \dashv x_{i}+f_{x_{t} \dashv x_{i} x_{j}}+2 a_{i t} h_{x_{j} e} \\
& \equiv 0 \bmod (S,[w]) .
\end{aligned}
$$

For $6 \wedge 8,[w]=\left[x_{i} x_{j} \dot{e}\right](i>j)$, we have

$$
\begin{aligned}
(6 \wedge 8)_{[w]} & =x_{j} \vdash x_{i} \vdash e-2 a_{i j} e+x_{i} \vdash e \dashv x_{j} \\
& =x_{j} \vdash h_{x_{i} e}+h_{x_{i} e} \dashv x_{j}+h_{x_{j} e} \dashv x_{i}+f_{e \dashv x_{i} x_{j}} \\
& \equiv 0 \bmod (S,[w]) .
\end{aligned}
$$

Then all the compositions in $S$ are trivial. We have proved (i).
For (ii), since the mentioned set is just the set $\operatorname{Irr}(S)$, by Theorem 3.9 the result holds. The proof is complete.

Remark: In the Theorem4.10, if the matrix $\left(a_{i j}\right)_{n \times n}=0$, then Clifford dialgebra $C(n, f)$ has a Gröbner-Shirshov basis $S^{\prime}$ which consists of the relations 1-7.

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