

# Boolean Factor Congruences and Property (\*)

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## Abstract

A variety  $\mathcal{V}$  has *Boolean factor congruences (BFC)* if the set of factor congruences of every algebra in  $\mathcal{V}$  is a distributive sublattice of its congruence lattice; this property holds in rings with unit and in every variety which has a semilattice operation. BFC has a prominent role in the study of uniqueness of direct product representations of algebras, since it is a strengthening of the *refinement property*.

We provide an explicit Mal'cev condition for BFC. With the aid of this condition, it is shown that BFC is equivalent to a variant of the definability *property (\*)*, an open problem in R. Willard's work [8].

## 1 Introduction

There is an extensive research concerning uniqueness of direct product representations (the book of McKenzie, McNulty and Taylor [3] is an excellent reference in the subject). We may start mentioning the classical theorem of Wedderburn and R. Remak, afterwards generalized by Krull and Schmidt, about direct representations of groups.

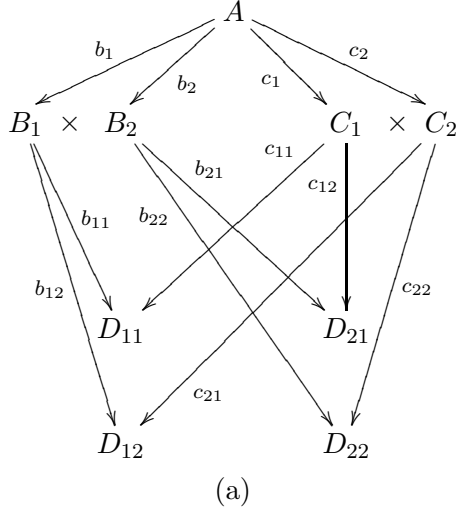
It is convenient to adopt the language of universal algebra at this point. An *algebra* is a nonempty set together with an arbitrary but fixed collection of finitary operations. A *variety* is an equationally-definable class of algebras over the same language. One fruitful approach to the problem of uniqueness is given by several notions of refinement. We say that an algebra  $A$  has the *refinement property* if for every two direct product decompositions  $A \cong \prod_i B_i \cong \prod_j C_j$ , there exist  $D_{ij}$  such that  $B_i \cong \prod_j D_{ij}$  and  $C_j \cong \prod_i D_{ij}$ . In Figure 1 (a) we have pictured this situation in the case  $I = J = \{1, 2\}$ , where every arrow correspond to a canonical projection onto a direct factor.

In [2], C. C. Chang, Jónsson and Tarski defined Boolean factor congruences in its full generality and proved it equivalent to a *strict* version of the refinement property. A variety  $\mathcal{V}$  has *Boolean factor congruences (BFC)* if the set of factor congruences of any algebra in  $\mathcal{V}$  is a distributive sublattice of its congruence lattice. Equivalently, if every algebra in  $\mathcal{V}$  satisfy the refinement property with the extra requirement that the diagram in Figure 1 is commutative, as in (b) (see [2, Theorem 5.6]).

Several years later, D. Bigelow and S. Burris [1] proved that BFC is a *Mal'cev property*, and hence one can assign to every variety  $\mathcal{V}$  with BFC a family of terms and identities (a *Mal'cev condition*) that “link” this property to the syntax of the defining identities of  $\mathcal{V}$ . In

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$$\begin{aligned}
 b_{11} \circ b_1 &= c_{11} \circ c_1 \\
 b_{21} \circ b_2 &= c_{12} \circ c_1 \\
 b_{12} \circ b_1 &= c_{21} \circ c_1 \\
 &\dots
 \end{aligned} \tag{1}$$

for BFC

(b)

Figure 1: An instance of refinement and its strict version.

our experience, having an explicit Mal'cev condition may be very helpful in the search of first-order-logic characterizations of algebraic concepts. But the result of Bigelow and Burris was based on Theorem 4.2 of Taylor [5], which gives a proof using preservation techniques but does not provide an explicit Mal'cev condition.

The next step in this direction was taken by Ross Willard. In his work [8], he found a very nice definability property (\*) and he proved that it was equivalent to BFC in a broad class of varieties. A variety  $\mathcal{V}$  satisfies *property (\*)* if and only if there exists a factorable<sup>1</sup> first-order formula  $\pi(x, y, z, w)$  in the language of  $\mathcal{V}$  such that:

- $\mathcal{V} \models \pi(x, y, x, y)$
- $\mathcal{V} \models \pi(x, x, z, w)$
- $\mathcal{V} \models \pi(x, y, z, z) \rightarrow x = y$

That work aimed to obtain a Mal'cev condition for BFC, but only in 2000 Willard found a way to achieve this. He presented his result at the AMS Spring Southeastern Section Meeting (Columbia, SC). In a personal communication, Willard informed D. Vaggione and the author about this result. He starts at a property of (not necessarily factor) congruences which is equivalent to BFC and then explains a syntactic procedure in order to produce an explicit Mal'cev condition. However, it appears that a condition thus generated would be very complicated.

Here begins the story of this paper. Vaggione and the author were studying the definability of factor congruences and the center [7, 4] and proved that the former implies BFC. In the search of an explicit definition, the author pursued the Mal'cev condition indicated by Willard. From this, a very similar condition for “definable factor congruences” was found. As a confirmation

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<sup>1</sup>The definition of *factorable* formulas is given in [8]; the main feature of these formulas is that they are preserved by taking direct products and factors.

of our early remark about the role of Mal'cev properties, we were able to construct a first-order definition  $\Phi$  of factor congruences using *central elements* (introduced in [6]) as parameters.

This result was presented in the “Conference in Universal Algebra and Lattice Theory” at Szeged in 2005. During this conference, Willard asserted that BFC is equivalent property (\*) in general, arguing on the finiteness of the set of terms involved in witnessing BFC. Soon after that, we realized that a construction line-by-line analog to that of the formula  $\Phi$  provides a formula  $\pi$  and proves this converse.

In this work we prove:

**Theorem 1.** *Let  $\mathcal{V}$  be a variety. The following are equivalent:*

1. *There exists a first-order formula  $\pi(x, y, z, w)$  in the language of  $\mathcal{V}$  which is preserved by direct factors and direct products, and such that:*

- (a)  $\mathcal{V} \models \pi(x, y, x, y)$
- (b)  $\mathcal{V} \models \pi(x, x, z, w)$
- (c)  $\mathcal{V} \models \pi(x, y, z, z) \rightarrow x = y$

2.  $\mathcal{V}$  has BFC.

Strictly speaking, statement (1) in the theorem is not property (\*) as stated in [8]. It remains to be checked if every sentence having these preservation properties is factorable. In any case, this definition captures the true essence of BFC, concerning its relation to preservation by taking direct factors (see [7, 4]), and we will keep that name.

The proof of this theorem will be an application of the results in [4]. In order to do this we will have to restate several results in that work for the case of BFC. We will do this in Section 2, where the Mal'cev condition for BFC is obtained. The terms of this condition are the building blocks for our definition of  $\pi$ , carried out in Section 3. Finally, we consider some (counter)examples in Section 4.

Throughout this paper, the following notation will be used. For  $A \in \mathcal{V}$  and  $\vec{a}, \vec{b} \in A^n$ ,  $\text{Cg}^A(\vec{a}, \vec{b})$  will denote the congruence generated by the set  $\{(a_k, b_k) : 1 \leq k \leq n\}$ . The symbols  $\nabla$  and  $\Delta$  will stand for the universal and trivial congruence, respectively. We will use  $\theta \times \theta^* = \Delta$  in place of “ $\theta$  and  $\theta^*$  are complementary factor congruences”. The term algebra (in the language of  $\mathcal{V}$ ) and the  $\mathcal{V}$ -free algebra on  $X$  will be denoted by  $T(X)$  and  $F(X)$ , respectively. The  $i$ -th component of an element  $a$  in a direct product  $\prod_i A_i$  will be called  $a^i = \text{pj}_i(a)$ ; hence, if  $a \in A_0 \times A_1$ ,  $a = \langle a^0, a^1 \rangle$ . If elements  $a, b$  of an algebra  $A$  are related by a congruence  $\theta \in \text{Con}(A)$ , we will write interchangeably  $(a, b) \in \theta$ ,  $a \theta b$  or  $a \stackrel{\theta}{\equiv} b$ . This notation generalizes to tuples, viz.,  $\vec{a} \theta \vec{b}$  means  $(a_i, b_i) \in \theta$  for all  $i$ .

## 2 A Mal'cev Condition for BFC

In this section we will rewrite several combinatorial lemmas from [4] for the case of BFC. In the first place, we need new definitions of our former functions  $\sigma$ ,  $\sigma^*$ ,  $\rho$  and  $\rho^*$ .

Let  $s_i, t_i$  be  $(2i + 2)$ -ary terms (in the language of  $\mathcal{V}$ ) for each  $i = 1, \dots, n$  and let  $A \in \mathcal{V}$ . Let  $(a, b, c, d, a_1, b_1, \dots, a_n, b_n) \in A^{4+2n}$ ; we define  $\sigma(a, b, c, d, a_1, b_1, \dots, a_n, b_n)$  to be the tuple  $(x, y, z, w, x_1, y_1, \dots, x_n, y_n)$  given by the following recursion:

$$\begin{aligned} x &:= a & w &:= b \\ y &:= b & x_j &:= s_j(x, y, z, w, x_1, y_1, \dots, x_{j-1}, y_{j-1}) \\ z &:= a & y_j &:= b_j \end{aligned}$$

We define  $\sigma^*, \rho, \rho^*$  analogously.

- $\sigma^*(a, b, c, d, a_1, b_1, \dots, a_n, b_n) = (x, y, z, w, x_1, y_1, \dots, x_n, y_n)$  where:

$$\begin{aligned} x &:= a & w &:= d \\ y &:= a & x_j &:= t_j(x, y, z, w, x_1, y_1, \dots, x_{j-1}, y_{j-1}) \\ z &:= c & y_j &:= b_j \end{aligned}$$

- $\rho(a, b, c, d, a_1, b_1, \dots, a_n, b_n) = (x, y, z, w, x_1, y_1, \dots, x_n, y_n)$  where:

$$\begin{aligned} x &:= a & w &:= c \\ y &:= b & x_j &:= a_j \\ z &:= c & y_j &:= s_j(x, y, z, w, x_1, y_1, \dots, x_{j-1}, y_{j-1}) \end{aligned}$$

- $\rho^*(a, b, c, d, a_1, b_1, \dots, a_n, b_n) = (x, y, z, w, x_1, y_1, \dots, x_n, y_n)$  where:

$$\begin{aligned} x &:= a & w &:= d \\ y &:= b & x_j &:= a_j \\ z &:= c & y_j &:= t_j(x, y, z, w, x_1, y_1, \dots, x_{j-1}, y_{j-1}) \end{aligned}$$

In the following we restate the first lemmas in [4] for these new functions:

**Lemma 2.** *For every  $(a, b, c, d, a_1, b_1, \dots, a_n, b_n) \in A^{4+2n}$ , we have the following identities:*

$$\begin{aligned} \text{Cg}(a, c) \vee \text{Cg}(b, d) \vee \bigvee_i \text{Cg}(a_i, s_i(a, b, c, d, a_1, b_1, \dots, a_{i-1}, b_{i-1})) &= \\ &= \text{Cg}((a, b, c, d, f, a_1, b_1, \dots, a_n, b_n), \sigma(a, b, c, d, f, a_1, b_1, \dots, a_n, b_n)) \end{aligned}$$

$$\begin{aligned} \text{Cg}(a, b) \vee \bigvee_i \text{Cg}(a_i, t_i(a, b, c, d, a_1, b_1, \dots, a_{i-1}, b_{i-1})) &= \\ &= \text{Cg}((a, b, c, d, a_1, b_1, \dots, a_n, b_n), \sigma^*(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) \end{aligned}$$

$$\begin{aligned} \text{Cg}(c, d) \vee \bigvee_i \text{Cg}(b_i, s_i(a, b, c, d, a_1, b_1, \dots, a_{i-1}, b_{i-1})) &= \\ &= \text{Cg}((a, b, c, d, \dots, a_n, b_n), \rho(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) \end{aligned}$$

$$\bigvee_i \text{Cg}(b_i, t_i(a, b, c, d, a_1, b_1, \dots, a_{i-1}, b_{i-1})) = \\ = \text{Cg}((a, b, c, d, a_1, b_1, \dots, a_n, b_n), \rho^*(a, b, c, d, a_1, b_1, \dots, a_n, b_n))$$

**Corollary 3.** *Given  $a, b, c, d \in A$  and  $\theta, \theta^* \in \text{Con}(A)$  such that  $c\theta a\theta^* b\theta d$  and for every  $a_i$  and  $b_i$  with  $i = 1, \dots, n$  such that*

$$\begin{aligned} s_1(a, b, c, d) &\stackrel{\theta}{\equiv} a_1 \stackrel{\theta^*}{\equiv} t_1(a, b, c, d) \\ s_2(a, b, c, d, a_1, b_1) &\stackrel{\theta}{\equiv} a_2 \stackrel{\theta^*}{\equiv} t_2(a, b, c, d, a_1, b_1) \\ &\dots \\ s_{j+1}(a, b, c, d, a_1, b_1, \dots, a_j, b_j) &\stackrel{\theta}{\equiv} a_{j+1} \stackrel{\theta^*}{\equiv} t_{j+1}(a, b, c, d, a_1, b_1, \dots, a_j, b_j) \end{aligned} \quad (2)$$

we have

$$t(\sigma(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) \stackrel{\theta}{\equiv} t(a, b, c, d, a_1, b_1, \dots, a_n, b_n) \stackrel{\theta^*}{\equiv} t(\sigma^*(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) \quad (3)$$

for every  $(2n+4)$ -ary term  $t$  in the language of  $\mathcal{V}$ .

**Corollary 4.** *Suppose  $a, b, c, d \in A$ ,  $\varphi, \varphi^* \in \text{Con}(A)$  such that  $c\varphi d$ . If  $a_i$  and  $b_i$  satisfy*

$$\begin{aligned} s_1(a, b, c, d) &\stackrel{\varphi}{\equiv} b_1 \stackrel{\varphi^*}{\equiv} t_1(a, b, c, d) \\ &\dots \\ s_{j+1}(a, b, c, d, a_1, b_1, \dots, a_j, b_j) &\stackrel{\varphi}{\equiv} b_{j+1} \stackrel{\varphi^*}{\equiv} t_{j+1}(a, b, c, d, a_1, b_1, \dots, a_j, b_j) \end{aligned} \quad (4)$$

we obtain

$$t(\rho(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) \stackrel{\varphi}{\equiv} t(a, b, c, d, a_1, b_1, \dots, a_n, b_n) \stackrel{\varphi^*}{\equiv} t(\rho^*(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) \quad (5)$$

for every  $(2n+4)$ -ary term  $t$  in the language of  $\mathcal{V}$ .

We will also need the following (Grätzer's) version of Mal'cev's key observation on principal congruences.

**Lemma 5.** *Let  $A$  be any algebra and let  $a, b \in A$ ,  $\vec{a}, \vec{b} \in A^n$ . Then  $(a, b) \in \text{Cg}^A(\vec{a}, \vec{b})$  if and only if there exist  $(n+m)$ -ary terms  $p_1(\vec{x}, \vec{u}), \dots, p_k(\vec{x}, \vec{u})$ , with  $k$  odd and,  $\vec{u} \in A^m$  such that:*

$$\begin{aligned} a &= p_1(\vec{a}, \vec{u}) \\ p_i(\vec{b}, \vec{u}) &= p_{i+1}(\vec{b}, \vec{u}), \quad i \text{ odd} \\ p_i(\vec{a}, \vec{u}) &= p_{i+1}(\vec{a}, \vec{u}), \quad i \text{ even} \\ p_k(\vec{b}, \vec{u}) &= b \end{aligned}$$

The formula  $\xi(x, y, \vec{x}, \vec{y}, \vec{u})$  given by

$$x = p_1(\vec{x}, \vec{u}) \wedge \bigwedge_{i \text{ odd}} p_i(\vec{y}, \vec{u}) = p_{i+1}(\vec{y}, \vec{u}) \wedge \bigwedge_{i \text{ even}} p_i(\vec{x}, \vec{u}) = p_{i+1}(\vec{x}, \vec{u}) \wedge p_k(\vec{y}, \vec{u}) = y$$

is called a principal congruence formula<sup>2</sup>.

**Corollary 6.** *For every homomorphism  $F : A \rightarrow B$ , if  $(a, b) \in \text{Cg}^A(\vec{a}, \vec{b})$ , then  $(F(a), F(b)) \in \text{Cg}^B(F(\vec{a}), F(\vec{b}))$ .*

The following theorem gives a Mal'cev condition for BFC. We will use  $|\alpha|$  to denote the length of a word  $\alpha$  and  $\varepsilon$  will denote the empty word.

**Theorem 7.** *A variety  $\mathcal{V}$  has BFC if and only if there exist integers  $N = 2k$  and  $n$ ,  $(2i+2)$ -ary terms  $s_i$  and  $t_i$  for each  $i = 1, \dots, n$ , and for every word  $\alpha$  in the alphabet  $\{1, \dots, N\}$  of length no greater than  $N$  there are terms  $L_\alpha, R_\alpha$  such that*

$$\boxed{|\alpha| = N}$$

$$\begin{aligned} L_\alpha(\rho(\vec{X})) &\approx R_\alpha(\rho(\vec{X})) \\ L_\alpha(\rho^*(\vec{X})) &\approx R_\alpha(\rho^*(\vec{X})) \end{aligned} \tag{6}$$

$$\boxed{|\alpha| = 0}$$

$$\begin{aligned} x &\approx L_\varepsilon(\vec{X}) \\ y &\approx R_\varepsilon(\vec{X}) \end{aligned} \tag{7}$$

$$L_\varepsilon(\rho(\vec{X})) \approx L_1(\rho(\vec{X})) \tag{8}$$

$$R_j(\rho(\vec{X})) \approx L_{j+1}(\rho(\vec{X})) \quad \text{if } 1 \leq j \leq N-1 \tag{9}$$

$$R_N(\rho(\vec{X})) \approx R_\varepsilon(\rho(\vec{X})) \tag{10}$$

$$\boxed{0 < |\alpha| < N}$$

If  $|\alpha|$  is even then

$$L_\alpha(\rho(\vec{X})) \approx L_{\alpha 1}(\rho(\vec{X})) \tag{11}$$

$$R_{\alpha j}(\rho(\vec{X})) \approx L_{\alpha(j+1)}(\rho(\vec{X})) \quad \text{if } 1 \leq j \leq k-1 \tag{12}$$

$$R_{\alpha k}(\rho(\vec{X})) \approx R_\alpha(\rho(\vec{X})) \tag{13}$$

$$L_\alpha(\rho^*(\vec{X})) \approx L_{\alpha(k+1)}(\rho^*(\vec{X}))$$

$$R_{\alpha j}(\rho^*(\vec{X})) \approx L_{\alpha(j+1)}(\rho^*(\vec{X})) \quad \text{if } k+1 \leq j \leq N-1 \tag{14}$$

$$R_{\alpha N}(\rho^*(\vec{X})) \approx R_\alpha(\rho^*(\vec{X}))$$

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<sup>2</sup>It is customary to call “principal congruence formula” the existential formula  $\exists \vec{u} \xi(x, y, \vec{x}, \vec{y}, \vec{u})$ , but we took this license here for technical reasons (see the comments after Corollary 9).

If  $|\alpha|$  is odd then

$$\begin{aligned} L_\alpha(\sigma(\vec{X})) &\approx L_{\alpha 1}(\sigma(\vec{X})) \\ R_{\alpha j}(\sigma(\vec{X})) &\approx L_{\alpha(j+1)}(\sigma(\vec{X})) \quad \text{if } 1 \leq j \leq k-1 \end{aligned} \quad (15)$$

$$\begin{aligned} R_{\alpha k}(\sigma(\vec{X})) &\approx R_\alpha(\sigma(\vec{X})) \\ L_\alpha(\sigma^*(\vec{X})) &\approx L_{\alpha(k+1)}(\sigma^*(\vec{X})) \\ R_{\alpha j}(\sigma^*(\vec{X})) &\approx L_{\alpha(j+1)}(\sigma^*(\vec{X})) \quad \text{if } k+1 \leq j \leq N-1 \\ R_{\alpha N}(\sigma^*(\vec{X})) &\approx R_\alpha(\sigma^*(\vec{X})) \end{aligned} \quad (16)$$

where  $\vec{X} = (x, y, z, w, x_1, y_1, \dots, x_n, y_n)$  and  $\sigma, \sigma^*, \rho$  and  $\rho^*$  are defined relative to  $s_i, t_i$ , on  $T_V(\vec{X})$ .

*Proof.* ( $\Leftarrow$ ) Assume the existence of the terms, and suppose  $\varphi \times \varphi^* = \Delta$ ,  $\theta \times \theta^* = \Delta$ , and  $a \theta c \varphi d \theta b \theta^* a$ . By [8, Lemma 0.2], we will prove BFC in the moment we see  $a \varphi b$ . There exist unique  $a_i, b_i$  satisfying the following relations:

$$\begin{aligned} s_1(a, b, c, d) &\stackrel{\theta}{\equiv} a_1 \stackrel{\theta^*}{\equiv} t_1(a, b, c, d) \\ s_1(a, b, c, d) &\stackrel{\varphi}{\equiv} b_1 \stackrel{\varphi^*}{\equiv} t_1(a, b, c, d) \\ &\dots \\ s_{j+1}(a, b, c, d, a_1, b_1, \dots, a_j, b_j) &\stackrel{\theta}{\equiv} a_{j+1} \stackrel{\theta^*}{\equiv} t_{j+1}(a, b, c, d, a_1, b_1, \dots, a_j, b_j) \\ s_{j+1}(a, b, c, d, a_1, b_1, \dots, a_j, b_j) &\stackrel{\varphi}{\equiv} b_{j+1} \stackrel{\varphi^*}{\equiv} t_{j+1}(a, b, c, d, a_1, b_1, \dots, a_j, b_j) \end{aligned} \quad (17)$$

Note that their definition combines schemes in Corollaries 3 and 4. So, by equations (3) and (5) we have, taking  $t := L_\alpha, R_\alpha$ :

$$\begin{aligned} L_\alpha(\sigma(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) &\stackrel{\theta}{\equiv} L_\alpha(a, b, c, d, a_1, b_1, \dots) \stackrel{\theta^*}{\equiv} L_\alpha(\sigma^*(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) \\ L_\alpha(\rho(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) &\stackrel{\varphi}{\equiv} L_\alpha(a, b, c, d, a_1, b_1, \dots) \stackrel{\varphi^*}{\equiv} L_\alpha(\rho^*(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) \end{aligned} \quad (18)$$

$$\begin{aligned} R_\alpha(\sigma(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) &\stackrel{\theta}{\equiv} R_\alpha(a, b, c, d, a_1, b_1, \dots) \stackrel{\theta^*}{\equiv} R_\alpha(\sigma^*(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) \\ R_\alpha(\rho(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) &\stackrel{\varphi}{\equiv} R_\alpha(a, b, c, d, a_1, b_1, \dots) \stackrel{\varphi^*}{\equiv} R_\alpha(\rho^*(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) \end{aligned} \quad (19)$$

for every  $\alpha$ . It can be proved by an inductive argument that  $L_\alpha(a, b, c, d, a_1, b_1, \dots, a_n, b_n) = R_\alpha(a, b, c, d, a_1, b_1, \dots, a_n, b_n)$  for all  $\alpha \neq \varepsilon$ , and the proof in [4] carries over *mutatis mutandis*. The reader may find very similar arguments to the those needed to fulfill this part of the proof in Corollary 8.

( $\Rightarrow$ ) For each set of variables  $Y$ , define

$$\begin{aligned} Y^* &:= Y \cup \{x_{p,q} : p, q \in T(Y)\} \cup \{y_{p,q} : p, q \in T(Y)\} \\ Y^{0*} &:= Y \\ Y^{(n+1)*} &:= (Y^{n*})^* \\ Y^\infty &:= \bigcup_{n \geq 1} Y^{n*} \end{aligned}$$

where  $x_{p,q}$  and  $y_{p,q}$  are new variables. Take  $Z := \{x, y, z, w\}$  and  $F := F(Z^\infty)$ . Define the *index* of  $p \in T(Z^\infty)$  as  $\text{ind}(p) = \min\{j : p \in T(Z^{j*})\}$ ; it is evident that if  $\text{ind}(x_{p,q}) \leq \text{ind}(x_{r,s})$ , neither  $p$  nor  $q$  can be terms depending on  $x_{r,s}$ . The same holds for  $\text{ind}(x_{p,q}) \leq \text{ind}(y_{r,s})$  and symmetrically, and for  $\text{ind}(y_{p,q}) \leq \text{ind}(y_{r,s})$ .

Take the following congruences on  $F$ :

$$\begin{aligned} \theta &:= \text{Cg}(x, z) \vee \text{Cg}(y, w) \vee \bigvee \{\text{Cg}(p, x_{p,q}) : p, q \in F\} & \delta_0 &= \epsilon_0 := \Delta^F \\ \theta^* &:= \text{Cg}(x, y) \vee \bigvee \{\text{Cg}(x_{p,q}, q) : p, q \in F\} & \delta_{n+1} &:= (\theta \vee \epsilon_n) \cap (\theta^* \vee \epsilon_n) \\ \varphi &:= \text{Cg}(z, w) \vee \bigvee \{\text{Cg}(p, y_{p,q}) : p, q \in F\} & \epsilon_{n+1} &:= (\varphi \vee \delta_n) \cap (\varphi^* \vee \delta_n) \\ \varphi^* &:= \bigvee \{\text{Cg}(y_{p,q}, q) : p, q \in F\} & \delta_\infty &:= \bigvee_{n \geq 0} \delta_n = \bigvee_{n \geq 0} \epsilon_n. \end{aligned}$$

By construction,  $\varphi \circ \varphi^* = \theta \circ \theta^* = \nabla^F$ ,  $x \theta z \varphi w \theta y \theta^* x$ . Observe that if  $(a, b) \in (\varphi \vee \delta_\infty) \cap (\varphi^* \vee \delta_\infty)$  then there exists an  $n \geq 0$  such that  $(a, b) \in (\varphi \vee \delta_n) \cap (\varphi^* \vee \delta_n)$ . But this congruence is exactly  $\epsilon_{n+1}$ , hence  $(a, b) \in \epsilon_{n+1} \subseteq \delta_\infty$ . We may conclude  $(\varphi \vee \delta_\infty) \cap (\varphi^* \vee \delta_\infty) = \delta_\infty$ . The same happens with  $\theta$  and  $\theta^*$ , hence

$$(\varphi \vee \delta_\infty) / \delta_\infty \times (\varphi^* \vee \delta_\infty) / \delta_\infty = \Delta \quad (\theta \vee \delta_\infty) / \delta_\infty \times (\theta^* \vee \delta_\infty) / \delta_\infty = \Delta$$

in  $F / \delta_\infty$ . Then, by BFC we have  $(x / \delta_\infty, y / \delta_\infty) \in (\varphi \vee \delta_\infty) / \delta_\infty$  and hence  $(x, y) \in \varphi \vee \delta_\infty$ . We may find an even integer  $N = 2k$  such that  $(x, y) \in \varphi \circ^{2N} \delta_N^N$ , where  $\delta_N^N$  is the result of replacing each occurrence of “ $\vee$ ” in the definition of  $\delta_N$  by  $\circ^N$ , the  $n$ -fold relational product. Now the terms  $L_\alpha$  and  $R_\alpha$ , for  $\alpha$  a word of length at most  $N$  in the alphabet  $\{1, \dots, N\}$ , can be defined recursively by using this last congruential equation. Details are analogous to those in [4].  $\square$

In the next results, we keep the notation of Theorem 7.

**Corollary 8.** *A variety has BFC if and only if there exist integers  $N$  and  $n$ ,  $(2i+2)$ -ary terms  $s_i$  and  $t_i$  for each  $i = 1, \dots, n$  such that for all  $A \in \mathcal{V}$  and all  $\theta, \theta^*, \varphi, \varphi^* \in \text{Con}(A)$  the following holds*

$$\left. \begin{aligned} \text{Cg}(\vec{X}, \sigma(\vec{X})) &\subseteq \theta \\ \text{Cg}(\vec{X}, \sigma^*(\vec{X})) &\subseteq \theta^* \\ \text{Cg}(\vec{X}, \rho(\vec{X})) &\subseteq \varphi \\ \text{Cg}(\vec{X}, \rho^*(\vec{X})) &\subseteq \varphi^* \end{aligned} \right\} \Rightarrow (x, y) \in \varphi \vee \delta_N. \quad (20)$$

for all  $x, y, z, w, x_1, y_1, \dots, x_n, y_n$  in  $A$ .



*Proof.* ( $\Leftarrow$ ) Suppose  $\theta, \theta^*, \varphi, \varphi^* \in \text{Con}(A)$  satisfy

$$\begin{array}{ll} \theta \times \theta^* = \Delta & x \theta z \varphi w \theta y \\ \varphi \times \varphi^* = \Delta & x \theta^* y \end{array}$$

and  $(x, y) \in \theta$ . As we saw in the first part of the proof of Theorem 7, the congruential equations in the antecedent of (20) have (unique) solution for the indeterminates  $x_1, y_1, \dots, x_n, y_n$ . The construction is given by equations (17), and Lemma 2 says that these equations are the same as those above.

Since  $\theta \cap \theta^* = \varphi \cap \varphi^* = \Delta$ , we have  $\delta_N = \Delta$  and we conclude  $(x, y) \in \varphi$ . Hence we proved that the variety has BFC.

( $\Rightarrow$ ) Suppose  $\mathcal{V}$  has BFC. The integers  $N$  and  $n$  and the terms are provided by Theorem 7. Thanks to Corollary 6, it suffices to verify the result in the instance given by  $A = F(\vec{X}) = F(x, y, z, w, x_1, y_1, \dots, x_n, y_n)$  and the congruences

$$\begin{array}{ll} \theta = \text{Cg}(\vec{X}, \sigma(\vec{X})) & \varphi = \text{Cg}(\vec{X}, \rho(\vec{X})) \\ \theta^* = \text{Cg}(\vec{X}, \sigma^*(\vec{X})) & \varphi^* = \text{Cg}(\vec{X}, \rho^*(\vec{X})). \end{array}$$

In this context, we will run an inductive argument to show that the terms  $L_\alpha, R_\alpha$  witness that  $(x, y) \in \varphi \vee \delta_N$ . (This argument is similar to the ( $\Leftarrow$ )-part of the proof of Theorem 7.)

Take  $\alpha$  such that  $|\alpha| = N$ , then

$$\begin{aligned} L_\alpha(x, y, z, w, x_1, y_1, \dots, x_n, y_n) &\stackrel{\varphi}{\equiv} L_\alpha(\rho(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) && \text{by definition of } \varphi \\ &= R_\alpha(\rho(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) && \text{using identities (6)} \\ &\stackrel{\varphi}{\equiv} R_\alpha(x, y, z, w, x_1, y_1, \dots, x_n, y_n) && \text{by definition of } \varphi \end{aligned}$$

And,

$$\begin{aligned} L_\alpha(x, y, z, w, x_1, y_1, \dots, x_n, y_n) &\stackrel{\varphi^*}{\equiv} L_\alpha(\rho^*(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) && \text{by definition of } \varphi^* \\ &= R_\alpha(\rho^*(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) && \text{using identities (6)} \\ &\stackrel{\varphi^*}{\equiv} R_\alpha(x, y, z, w, x_1, y_1, \dots, x_n, y_n) && \text{by definition of } \varphi^* \end{aligned}$$

Hence  $(L_\alpha(x, y, z, w, x_1, y_1, \dots, x_n, y_n), R_\alpha(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) \in \varphi \cap \varphi^* = \epsilon_1$  (recall the definition of  $\epsilon_n$  in page 8).

Suppose  $\alpha \neq \varepsilon$  has odd length  $|\alpha| < N$  and assume

$$L_{\alpha j}(x, y, z, w, x_1, y_1, \dots, x_n, y_n) \stackrel{\epsilon_{N-|\alpha|}}{\equiv} R_{\alpha j}(x, y, z, w, x_1, y_1, \dots, x_n, y_n)$$

for every  $j = 1, \dots, N$ . We check that

$$L_\alpha(x, y, z, w, x_1, y_1, \dots, x_n, y_n) \stackrel{\theta \vee \epsilon_{N-|\alpha|}}{\equiv} R_\alpha(x, y, z, w, x_1, y_1, \dots, x_n, y_n) :$$

$$\begin{aligned}
L_\alpha(x, y, z, w, x_1, y_1, \dots, x_n, y_n) &\stackrel{\theta}{=} L_\alpha(\sigma(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) && \text{by definition of } \theta \\
&= L_{\alpha 1}(\sigma(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) && \text{by identities (15)} \\
&\stackrel{\theta}{=} L_{\alpha 1}(x, y, z, w, x_1, y_1, \dots, x_n, y_n) && \text{by definition of } \theta \\
&\stackrel{\epsilon_{N-|\alpha|}}{=} R_{\alpha 1}(x, y, z, w, x_1, y_1, \dots, x_n, y_n) && \text{by inductive hypothesis} \\
&\stackrel{\theta}{=} R_{\alpha 1}(\sigma(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) && \text{by definition of } \theta \\
&\stackrel{\theta}{=} \dots && \text{using (15)} \\
&\stackrel{\epsilon_{N-|\alpha|}}{=} \dots && \text{and iterating...} \\
&= R_{\alpha k}(\sigma(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) \\
&= R_\alpha(\sigma(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) && \text{using identities (15)} \\
&\stackrel{\theta}{=} R_\alpha(x, y, z, w, x_1, y_1, \dots, x_n, y_n),
\end{aligned}$$

In the same way we show

$$L_\alpha(x, y, z, w, x_1, y_1, \dots, x_n, y_n) \stackrel{\theta^* \vee \epsilon_{N-|\alpha|}}{=} R_\alpha(x, y, z, w, x_1, y_1, \dots, x_n, y_n) :$$

$$\begin{aligned}
L_\alpha(x, y, z, w, x_1, y_1, \dots, x_n, y_n) &\stackrel{\theta^*}{=} L_\alpha(\sigma^*(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) && \text{by definition of } \theta^* \\
&= L_{\alpha(k+1)}(\sigma^*(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) && \text{by identities (16)} \\
&\stackrel{\theta^*}{=} L_{\alpha(k+1)}(x, y, z, w, x_1, y_1, \dots, x_n, y_n) && \text{by definition of } \theta^* \\
&\stackrel{\epsilon_{N-|\alpha|}}{=} R_{\alpha(k+1)}(x, y, z, w, x_1, y_1, \dots, x_n, y_n) && \text{by ind. hypothesis} \\
&\stackrel{\theta^*}{=} R_{\alpha(k+1)}(\sigma^*(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) && \text{by definition of } \theta^* \\
&\stackrel{\theta^*}{=} \dots && \text{using (16)} \\
&\stackrel{\epsilon_{N-|\alpha|}}{=} \dots && \text{and iterating...} \\
&= R_{\alpha N}(\sigma^*(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) \\
&= R_\alpha(\sigma^*(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) && \text{using identities (16)} \\
&\stackrel{\theta^*}{=} R_\alpha(x, y, z, w, x_1, y_1, \dots, x_n, y_n),
\end{aligned}$$

and hence we obtain

$$(L_\alpha(x, y, z, w, x_1, y_1, \dots), R_\alpha(x, y, z, w, x_1, y_1, \dots)) \in (\theta \vee \epsilon_{N-|\alpha|}) \cap (\theta^* \vee \epsilon_{N-|\alpha|}) = \delta_{N-|\alpha|+1}$$

Now suppose  $\alpha \neq \varepsilon$  has even length  $|\alpha| < N$  and assume

$$L_{\alpha j}(x, y, z, w, x_1, y_1, \dots, x_n, y_n) \stackrel{\delta_{N-|\alpha|}}{=} R_{\alpha j}(x, y, z, w, x_1, y_1, \dots, x_n, y_n)$$

for every  $j = 1, \dots, N$ . Then

$$\begin{aligned}
L_\alpha(x, y, z, w, x_1, y_1, \dots, x_n, y_n) &\stackrel{\varphi}{=} L_\alpha(\rho(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) && \text{by definition of } \varphi \\
&= L_{\alpha 1}(\rho(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) && \text{by identity (11)} \\
&\stackrel{\varphi}{=} L_{\alpha 1}(x, y, z, w, x_1, y_1, \dots, x_n, y_n) && \text{by definition of } \varphi \\
&\stackrel{\delta_{N-|\alpha|}}{=} R_{\alpha 1}(x, y, z, w, x_1, y_1, \dots, x_n, y_n) && \text{by inductive hypothesis} \\
&\stackrel{\varphi}{=} R_{\alpha 1}(\rho(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) && \text{by definition of } \varphi \\
&\stackrel{\varphi}{=} \dots && \text{using (12)} \\
&\stackrel{\delta_{N-|\alpha|}}{=} \dots && \text{and iterating...} \\
&= R_{\alpha k}(\rho(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) \\
&= R_\alpha(\rho(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) && \text{using identity (13)} \\
&\stackrel{\varphi}{=} R_\alpha(x, y, z, w, x_1, y_1, \dots, x_n, y_n) && \text{by definition of } \varphi
\end{aligned}$$

proves  $(L_\alpha(x, y, z, w, x_1, y_1, \dots, x_n, y_n), R_\alpha(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) \in \varphi \vee \delta_{N-|\alpha|}$ . We can see analogously (using  $\rho^*$  and identities (14)) that

$$(L_\alpha(x, y, z, w, x_1, y_1, \dots, x_n, y_n), R_\alpha(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) \in \varphi^* \vee \delta_{N-|\alpha|},$$

therefore

$$(L_\alpha(x, y, z, w, x_1, y_1, \dots), R_\alpha(x, y, z, w, x_1, y_1, \dots)) \in (\varphi \vee \delta_{N-|\alpha|}) \cap (\varphi^* \vee \delta_{N-|\alpha|}) = \epsilon_{N-|\alpha|+1}$$

Finally, for  $\alpha = \varepsilon$ , and noting that  $\delta_{N-|\alpha|} = \delta_N$ , we have:

$$\begin{aligned}
x &= L_\varepsilon(\rho(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) && \text{using identities (7)} \\
&= L_1(\rho(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) && \text{by identity (8)} \\
&\stackrel{\varphi}{=} L_1(x, y, z, w, x_1, y_1, \dots, x_n, y_n) && \text{by definition of } \varphi \\
&\stackrel{\delta_N}{=} R_1(x, y, z, w, x_1, y_1, \dots, x_n, y_n) && \text{by inductive hypothesis} \\
&\stackrel{\varphi}{=} R_1(\rho(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) && \text{by definition of } \varphi \\
&\stackrel{\varphi}{=} \dots && \text{using identities (9)} \\
&\stackrel{\delta_N}{=} \dots && \text{and iterating...} \\
&= R_N(\rho(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) \\
&= R_\varepsilon(\rho(x, y, z, w, x_1, y_1, \dots, x_n, y_n)) && \text{using identity (10)} \\
&= y && \text{using identities (7)}
\end{aligned}$$

This proves  $(x, y) \in \varphi \vee \delta_N$ . □

This corollary is a variant of Willard's original condition. He states that a variety has BFC if there exist  $n \geq 0$ , and terms

$$\begin{aligned} & s_1(x, y, z, w), t_1(x, y, z, w) \\ & s_2(x, y, z, w, u_1), t_2(x, y, z, w, u_1) \\ & s_3(x, y, z, w, u_1, u_2), t_2(x, y, z, w, u_1, u_2) \\ & \vdots \\ & s_n(x, y, z, w, u_1, \dots, u_{n-1}), t_n(x, y, z, w, u_1, \dots, u_{n-1}) \end{aligned}$$

such that  $\forall A \in \mathcal{V}, \forall \theta, \theta^*, \varphi, \varphi^* \in \text{Con}(A), \forall a, b, c, d, e_1, \dots, e_n \in A$ , if  $a \stackrel{\theta}{\equiv} c \stackrel{\varphi}{\equiv} d \stackrel{\theta}{\equiv} b \stackrel{\theta^*}{\equiv} a$  and

$$\begin{aligned} & s_i(a, b, c, d, e_1, \dots, e_{i-1}) \stackrel{\theta}{\equiv} e_i \stackrel{\theta^*}{\equiv} t_i(a, b, c, d, e_1, \dots, e_{i-1}) \quad (1 \leq i \leq n, i \text{ odd}) \\ & s_i(a, b, c, d, e_1, \dots, e_{i-1}) \stackrel{\varphi}{\equiv} e_i \stackrel{\varphi^*}{\equiv} t_i(a, b, c, d, e_1, \dots, e_{i-1}) \quad (2 \leq i \leq n, i \text{ even}) \end{aligned}$$

then  $(a, b) \in \varphi \vee \delta_\infty$ .

The procedure of using  $\delta_\infty$ , to force a pair of congruences in a free algebra freely generated by an infinite set to be factor complementary, already appears as part of Vaggione's work on Boolean-representable varieties [6].

In the next corollary, we obtain an infinitary "formula" which is our first approximation to  $\pi$ .

**Corollary 9.** *Let  $A = A_0 \times A_1$  be an algebra in a variety with BFC, and let  $\Pi(x, y, z, w)$  be the following predicate:*

$$\exists x_1 \forall y_1 \dots \exists x_n \forall y_n \quad \text{Cg}^A(\vec{X}, \sigma(\vec{X})) \cap \text{Cg}^A(\vec{X}, \sigma^*(\vec{X})) = \Delta^A \quad (21)$$

*Then, for all  $a, b \in A_0$  and  $a', b', c' \in A_1$ ,  $\Pi(\langle a, a' \rangle, \langle b, b' \rangle, \langle a, c' \rangle, \langle b, c' \rangle)$  holds in  $A$  if and only if  $a' = b'$ .*

*Proof.* We will need to do the following definitions:

$$\begin{aligned} x &:= \langle a, a' \rangle & y &:= \langle b, b' \rangle \\ z &:= \langle a, c' \rangle & w &:= \langle b, c' \rangle, \end{aligned}$$

hence we have  $\text{Cg}(x, z) \vee \text{Cg}(y, w) \subseteq \ker \text{pj}_0$  and  $\text{Cg}(z, w) \subseteq \ker \text{pj}_1$ .

( $\Leftarrow$ ) Suppose  $(x, y) \in \ker \text{pj}_1$ . Take  $x_1$  such that

$$s_1(x, y, z, w) \stackrel{\ker \text{pj}_0}{\equiv} x_1 \stackrel{\ker \text{pj}_1}{\equiv} t_1(x, y, z, w)$$

and assuming  $x_i$  has already been chosen and  $y_i$  is given, let

$$s_{i+1}(x, y, z, w, x_1, y_1, \dots, x_i, y_i) \stackrel{\ker \text{pj}_0}{\equiv} x_{i+1} \stackrel{\ker \text{pj}_1}{\equiv} t_{i+1}(x, y, z, w, x_1, y_1, \dots, x_i, y_i).$$

By means of this procedure, and taking into account Corollary 3, we may conclude that  $\text{Cg}^A(\vec{X}, \sigma(\vec{X})) \subseteq \ker \text{pj}_0$  and  $\text{Cg}^A(\vec{X}, \sigma^*(\vec{X})) \subseteq \ker \text{pj}_1$ . Since  $\ker \text{pj}_0 \cap \ker \text{pj}_1 = \Delta^A$ , we have (21).

( $\Rightarrow$ ) Suppose (21) holds. Take  $y_1$  such that

$$s_1(x, y, z, w) \stackrel{\ker \text{pj}_1}{\equiv} y_1 \stackrel{\ker \text{pj}_0}{\equiv} t_1(x, y, z, w).$$

(Note: the order of congruences is reversed.) Let  $x_1$  given by the outer existential quantifier of (21). Assuming  $y_i$  is already chosen and  $x_i$  is the corresponding witness for (21), let

$$s_{i+1}(x, y, z, w, x_1, y_1, \dots, x_i, y_i) \stackrel{\ker \text{pj}_1}{\equiv} y_{i+1} \stackrel{\ker \text{pj}_0}{\equiv} t_{i+1}(x, y, z, w, x_1, y_1, \dots, x_i, y_i).$$

Corollary 4 ensures that  $\text{Cg}^A(\vec{X}, \rho(\vec{X})) \subseteq \ker \text{pj}_1$  and  $\text{Cg}^A(\vec{X}, \rho^*(\vec{X})) \subseteq \ker \text{pj}_0$ .

Take in Corollary 8

$$\begin{aligned} \theta &:= \text{Cg}(\vec{X}, \sigma(\vec{X})) & \varphi &:= \ker \text{pj}_1 \\ \theta^* &:= \text{Cg}(\vec{X}, \sigma^*(\vec{X})) & \varphi^* &:= \ker \text{pj}_0 \end{aligned}$$

We thus obtain  $(x, y) \in \varphi \vee \delta_N$ . Since  $\varphi \cap \varphi^* = \ker \text{pj}_1 \cap \ker \text{pj}_0 = \Delta^A$  and the same holds for  $\theta, \theta^*$ , we have  $\delta_N = \Delta^A$  and hence  $(x, y) \in \varphi = \ker \text{pj}_1$ . This is the same to say  $a^1 = b^1$ .  $\square$

Though “formula” (21) is not in first-order logic, it corresponds to a formula of the infinitary logic  $L_{\kappa+\omega}$  (here  $\kappa$  is the cardinal of the language of  $\mathcal{V}$  plus  $\omega$ ), since its “matrix”  $\text{Cg}^A(\vec{X}, \sigma(\vec{X})) \cap \text{Cg}^A(\vec{X}, \sigma^*(\vec{X})) = \Delta^A$  can be replaced by an infinite conjunction of quasi-identities. This can be seen by considering principal congruence formulas (recall Lemma 5). We may write “ $\text{Cg}(\vec{a}, \vec{b}) = \Delta$ ” in the following fashion:

$$\bigwedge_{\xi \text{ PCF}} \forall x, y \forall \vec{u}_\xi \xi(x, y, \vec{a}, \vec{b}, \vec{u}_\xi) \rightarrow x = y.$$

In the same way,

$$\bigwedge_{\xi, \zeta \text{ PCF}} \forall x, y \forall \vec{u}_\xi, \vec{v}_\zeta : \xi(x, y, \vec{X}, \sigma(\vec{X}), \vec{u}_\xi) \wedge \zeta(x, y, \vec{X}, \sigma^*(\vec{X}), \vec{v}_\zeta) \rightarrow x = y,$$

is equivalent to “ $\text{Cg}^A(\vec{X}, \sigma(\vec{X})) \cap \text{Cg}^A(\vec{X}, \sigma^*(\vec{X})) = \Delta^A$ ”.

In the next Section we will see that it is indeed possible to find a first-order formula with a similar syntactic structure that satisfies property (\*).

### 3 Property (\*) and BFC

Let  $\mathcal{V}$  be a variety with BFC. By Theorem 7, we may define the following formulas in the language of  $\mathcal{V}$ :

$$\Psi_m := \bigwedge_{|\alpha|=m} \left( \left( \bigwedge_{\gamma \neq \varepsilon} L_{\alpha\gamma}(\vec{X}) = R_{\alpha\gamma}(\vec{X}) \right) \rightarrow L_\alpha(\vec{X}) = R_\alpha(\vec{X}) \right).$$

where every word-subindex moves over words of length less than or equal to  $N$ ; so, any expression of the form “ $\bigwedge_{\gamma \neq \varepsilon} L_{\alpha\gamma} = R_{\alpha\gamma}$ ” should be read as “ $\bigwedge \{L_{\alpha\gamma} = R_{\alpha\gamma} : \gamma \neq \varepsilon \text{ and } |\alpha\gamma| \leq N\}$ ”. Thus, if  $m > N$ ,  $\Psi_m = \text{true}$  (empty conjunction) and  $\Psi_N = (\bigwedge_{|\beta|=N} L_{\beta}(\vec{X}) = R_{\beta}(\vec{X}))$  (the antecedent “vanishes”).

The formulas  $\Psi_m$  will be the building blocks for constructing a formula  $\Phi_2$  that satisfies the elementary requirements of property (\*). But it is not immediate that  $\Phi_2$  will satisfy the necessary preservation property. Nevertheless, in the context of  $\mathcal{V}$  we may prove this. Readily, there is a formula  $\Phi_1(x, y, z, w)$  valid in  $\mathcal{V}$  such that  $\Phi_1 \wedge \Phi_2$  is preserved by direct products and direct factors.

The following lemma defines  $\Phi_1$  and proves its validity over  $\mathcal{V}$ .

**Lemma 10.** *Let  $\mathcal{V}$  be a variety with BFC. Then*

$$\mathcal{V} \models \Phi_1(x, y, z, w) := \exists y_1 \forall x_1 \dots \exists y_n \forall x_n \bigwedge_{m=1}^k \Psi_{2m} \quad (22)$$

with  $n, k$  as in Theorem 7.

*Proof.* Suppose  $a, b, c, d \in A \in \mathcal{V}$ . Take  $b_1 := t_1(a, b, c, d)$ . Assuming  $b_i$  is already chosen and  $a_i$  is given, define

$$b_{i+1} := t_{i+1}(a, b, c, d, a_1, b_1, \dots, a_i, b_i).$$

The construction of  $b_i$ ’s ensures

$$(a, b, c, d, a_1, b_1, \dots, a_n, b_n) = \rho^*(a, b, c, d, a_1, b_1, \dots, a_n, b_n). \quad (23)$$

Hence we have that for each  $\beta$  with  $|\beta| = N$ ,

$$L_{\beta}(a, b, c, d, a_1, b_1, \dots, a_n, b_n) = R_{\beta}(a, b, c, d, a_1, b_1, \dots, a_n, b_n)$$

by equations (6), and we conclude  $A \models \Psi_N(a, b, c, d, a_1, b_1, \dots, a_n, b_n)$ .

Take nonempty  $\alpha$  with  $0 < |\alpha| < N$  even. We will prove that  $\Psi_{\alpha}$  holds. Suppose

$$A \models \bigwedge_{\gamma \neq \varepsilon} L_{\alpha\gamma}(a, b, c, d, a_1, b_1, \dots, a_n, b_n) = R_{\alpha\gamma}(a, b, c, d, a_1, b_1, \dots, a_n, b_n).$$

or, equivalently,

$$A \models \bigwedge_{\gamma \neq \varepsilon} L_{\alpha\gamma}(\rho^*(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) = R_{\alpha\gamma}(\rho^*(a, b, c, d, a_1, b_1, \dots, a_n, b_n)). \quad (24)$$

We then have:

$$\begin{aligned}
L_\alpha(a, b, c, d, a_1, b_1, \dots) &= L_\alpha(\rho^*(a, b, c, d, a_1, b_1, \dots)) && \text{by equation (23)} \\
&= L_{\alpha(k+1)}(\rho^*(a, b, c, d, a_1, b_1, \dots)) && \text{by identities (14)} \\
&= R_{\alpha(k+1)}(\rho^*(a, b, c, d, a_1, b_1, \dots)) && \text{by (24)} \\
&= \dots && \text{using (14), (24)} \\
&= \dots && \text{and iterating...} \\
&= R_{\alpha N}(\rho^*(a, b, c, d, a_1, b_1, \dots)) \\
&= R_\alpha(\rho^*(a, b, c, d, a_1, b_1, \dots)) && \text{using identities (14)} \\
&= R_\alpha(a, b, c, d, a_1, b_1, \dots) && \text{by equation (23).}
\end{aligned}$$

Hence we have

$$A \models L_\alpha(a, b, c, d, a_1, b_1, \dots) = R_\alpha(a, b, c, d, a_1, b_1, \dots),$$

and we have proved the Lemma.  $\square$

**Lemma 11.** *Let  $\mathcal{V}$  be a variety with BFC. Define:*

$$\Phi_2(x, y, z, w) := \exists x_1 \forall y_1 \dots \exists x_n \forall y_n \bigwedge_{m=1}^k \Psi_{2m-1} \quad (25)$$

Then  $\mathcal{V} \models \Phi_2(x, y, x, y)$  and  $\mathcal{V} \models \Phi_2(x, x, z, w)$ .

*Proof.* We only prove the first one, since the proofs are analogous to that of the previous lemma. Suppose  $a, b \in A \in \mathcal{V}$ . Take  $a_1 := s_1(a, b, a, b)$ . Assuming  $a_i$  is already chosen and  $b_i$  is given, define

$$a_{i+1} := s_{i+1}(a, b, a, b, a_1, b_1, \dots, a_i, b_i).$$

The construction of  $b_i$ 's ensures

$$(a, b, a, b, a_1, b_1, \dots, a_n, b_n) = \sigma(a, b, a, b, a_1, b_1, \dots, a_n, b_n). \quad (26)$$

Take nonempty  $\alpha$  with  $|\alpha| < N$  odd. We will prove that  $\Psi_\alpha$  holds. Suppose

$$A \models \bigwedge_{\gamma \neq \varepsilon} L_{\alpha\gamma}(a, b, a, b, a_1, b_1, \dots, a_n, b_n) = R_{\alpha\gamma}(a, b, a, b, a_1, b_1, \dots, a_n, b_n).$$

or, equivalently,

$$A \models \bigwedge_{\gamma \neq \varepsilon} L_{\alpha\gamma}(\sigma(a, b, a, b, a_1, b_1, \dots, a_n, b_n)) = R_{\alpha\gamma}(\sigma(a, b, a, b, a_1, b_1, \dots, a_n, b_n)). \quad (27)$$

We then have:

$$\begin{aligned}
L_\alpha(a, b, a, b, a_1, b_1, \dots) &= L_\alpha(\sigma(a, b, a, b, a_1, b_1, \dots)) && \text{by equation (26)} \\
&= L_{\alpha 1}(\sigma(a, b, a, b, a_1, b_1, \dots)) && \text{by identities (15)} \\
&= R_{\alpha 1}(\sigma(a, b, a, b, a_1, b_1, \dots)) && \text{by (27)} \\
&= \dots && \text{using (15), (24)} \\
&= \dots && \text{and iterating. . .} \\
&= R_{\alpha k}(\sigma(a, b, a, b, a_1, b_1, \dots)) \\
&= R_\alpha(\sigma(a, b, a, b, a_1, b_1, \dots)) && \text{using identities (15)} \\
&= R_\alpha(a, b, a, b, a_1, b_1, \dots) && \text{by equation (26)}.
\end{aligned}$$

Hence we have

$$A \models L_\alpha(a, b, a, b, a_1, b_1, \dots, a_n, b_n) = R_\alpha(a, b, a, b, a_1, b_1, \dots, a_n, b_n).$$

The proof that  $\mathcal{V} \models \Phi_2(x, x, z, w)$  is similar, but using  $\sigma^*$  and  $t_i$ 's in place of  $\sigma$  and  $s_i$ 's, respectively.  $\square$

**Lemma 12.** *Let  $a, b, c \in A \in \mathcal{V}$  with BFC. If  $A$  satisfies  $\Phi_2(a, b, c, c)$ , then  $a = b$ .*

*Proof.* Assume  $A \models \Phi_2(a, b, c, c)$ . Take  $b_1 := s_1(a, b, c, c)$ . Let  $a_1$  be given by the outermost existential quantifier of  $\Phi_2$ .

Assuming  $b_i$  is already chosen and  $a_i$  is the corresponding witness for  $\Phi_2$ , let

$$b_{i+1} := s_{i+1}(a, b, c, c, a_1, b_1, \dots, a_i, b_i) \quad (28)$$

This selection satisfies

$$(a, b, c, c, a_1, b_1, \dots, a_n, b_n) = \rho(a, b, c, d, a_1, b_1, \dots, a_n, b_n). \quad (29)$$

Using an analogous reasoning to that in the proof of Lemma 10 (replacing there  $t_i$ 's and  $\rho^*$  by  $s_i$ 's and  $\rho$ , respectively), the reader may check that this choice of  $a_i, b_i$  satisfies the matrix of  $\Phi_1(a, b, c, c)$ . We hence obtain

$$A \models \left( \bigwedge_{m=1}^N \Psi_m \right) (a, b, c, c, a_1, b_1, \dots, a_n, b_n)$$

From an easy inspection of the form of  $\Psi_m$ , it can be deduced that

$$A \models \bigwedge_{j=1}^N L_j(a, b, c, c, a_1, b_1, \dots, a_n, b_n) = R_j(a, b, c, c, a_1, b_1, \dots, a_n, b_n),$$

and using (29),

$$A \models \bigwedge_{j=1}^N L_j(\rho(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) = R_j(\rho(a, b, c, d, a_1, b_1, \dots, a_n, b_n)). \quad (30)$$



Therefore,

$$\begin{aligned}
a &= L_\varepsilon(a, b, c, c, a_1, b_1, \dots, a_n, b_n) && \text{by identities (7)} \\
&= L_\varepsilon(\rho(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) && \text{by equation (29)} \\
&= L_1(\rho(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) && \text{by identities (8), with } \alpha = \varepsilon \\
&= R_1(\rho(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) && \text{by equations (30)} \\
&= L_2(\rho(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) && \text{by identities (12)} \\
&= \dots && \text{using equations (12), (30)} \\
&= \dots && \text{and iterating...} \\
&= R_N(\rho(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) && \text{using equations (13) once more:} \\
&= R_\varepsilon(\rho(a, b, c, d, a_1, b_1, \dots, a_n, b_n)) \\
&= R_\varepsilon(a, b, c, c, a_1, b_1, \dots, a_n, b_n) && \text{by equation (29)} \\
&= b && \text{by identities (7)}
\end{aligned}$$

Hence  $a = b$ . □

*Proof of Theorem 1.* ( $\Leftarrow$ ) The formula  $\pi(x, y, z, w) := \Phi_1(x, y, z, w) \wedge \Phi_2(x, y, z, w)$  satisfies (a), (b) and (c) in Theorem 1(1) by the previous lemmas. It is also preserved by taking direct factors and direct products: this is an immediate application of [4, Theorem 22], where we take  $\vec{z} = (z, w)$  and  $\tau_\alpha(\vec{X})$  to be “ $L_\alpha(\vec{X}) = R_\alpha(\vec{X})$ ”.

( $\Rightarrow$ ) This is easy to show; for details see [8, Theorem 1.5]. □

## 4 Some (Counter)examples

One of our main interests was to find an algebraic counterpart of the formula  $\pi$  witnessing property (\*). The first approach is the characterization in Corollary 9. A second one is given by the following semantic consequence of  $\pi$ : every time one has  $A \models \pi(a, b, c, d)$ , one obtains

$$\text{for every } \theta \in FC(A), (c, d) \in \theta \text{ implies } (a, b) \in \theta. \quad (31)$$

where  $FC(A)$  is the set of factor congruences of  $A$ . This can be immediately seen by noting that for all  $\theta \in FC(A)$  we have  $A/\theta \models \pi(a/\theta, b/\theta, c/\theta, d/\theta)$  since  $\pi$  is preserved by direct factors, and if  $c/\theta = d/\theta$  we must have  $a/\theta = b/\theta$ .

Now call  $\Gamma(a, b, c, d)$  the assertion (31). In spite this predicate might not be expressible in first-order logic, it can be proved that it satisfies all conditions for property (\*):

**Proposition 1.** *For all,  $a, b, c, d \in A \in \mathcal{V}$ , where  $\mathcal{V}$  has BFC, we have:*

1.  $\Gamma(a, b, c, d)$  is equivalent to “ $(a, b) \in \bigcap \{ \theta \in FC(A) : (c, d) \in \theta \}$ ”, and hence  $\Gamma(a_i, b_i, c, d)$  for all  $i = 1, \dots, l$  implies  $\Gamma(F(\vec{a}), F(\vec{b}), c, d)$ , for every  $l$ -ary basic operation  $F$  in the language of  $\mathcal{V}$ .
2.  $A \models \Gamma(a, a, b, c)$ .

3.  $A \models \Gamma(a, b, a, b)$ .
4.  $A \models \Gamma(a, b, c, c) \rightarrow a = b$ .
5. If  $a', b', c', d' \in B \in \mathcal{V}$ ,  $A \times B \models \Gamma(\langle a, a' \rangle, \langle b, b' \rangle, \langle c, c' \rangle, \langle d, d' \rangle)$  if and only if  $A \models \Gamma(a, b, c, d)$  and  $B \models \Gamma(a', b', c', d')$ .

*Proof.* The first four are obvious. To check  $\Gamma$  is preserved by direct products, suppose  $A \models \Gamma(a, b, c, d)$  and  $B \models \Gamma(a', b', c', d')$ . Now take  $\theta \in FC(A \times B)$  and assume  $(\langle c, c' \rangle, \langle d, d' \rangle) \in \theta$ . By BFC, there exist factor congruences  $\theta^0 \in FC(A)$  and  $\theta^1 \in FC(B)$  such that  $\theta = \{(\langle x, x' \rangle, \langle y, y' \rangle) : (x, x') \in \theta^0, (y, y') \in \theta^1\}$ . This yields  $(c, d) \in \theta^0$  and  $(c', d') \in \theta^1$ , and then we have  $(a, b) \in \theta^0$  and  $(a', b') \in \theta^1$  by hypothesis. Hence  $(\langle a, a' \rangle, \langle b, b' \rangle) \in \theta$  and we have showed that  $A \times B \models \Gamma(\langle a, a' \rangle, \langle b, b' \rangle, \langle c, c' \rangle, \langle d, d' \rangle)$ . Preservation of  $\Gamma$  by direct factors is similar.  $\square$

It turns out that if  $\Gamma$  is a first-order formula, it is the *weakest* witness for (\*). In the case of finite languages, it can be proved that if no nontrivial algebra of  $\mathcal{V}$  has a trivial subalgebra, then  $\Gamma$  is a first-order formula. This is an easy consequence of [4].

If one replaces  $FC(A)$  in the definition of  $\Gamma$  by some other set of congruences that contains  $\Delta$ , Proposition 1 will still hold with the possible exception of (5). One nice conjecture would be that one may obtain some first-order formula by replacing  $FC(A)$  in the definition of  $\Gamma$  by some bigger set of congruences. While this is indeed the case for semilattices, we cannot expect to obtain in such manner every formula witnessing (\*), even not one that results from our construction, as the following counterexample shows.

Take the variety  $\mathcal{V}$  in the language  $\{0, \cdot\}$  defined by the following identities:

$$\begin{aligned} (x \cdot y) \cdot z &\approx x \cdot (y \cdot z) \\ x \cdot x &\approx x \\ x \cdot 0 &\approx 0 \cdot x \approx 0. \end{aligned}$$

We will calculate the terms  $s_i, t_i$  and  $L_\alpha, R_\alpha$ . For this particular case,  $N = n = 2$ . Define:

$$\begin{aligned} s_1 &:= x & s_2 &:= y \\ t_1 &:= 0 & t_2 &:= 0 \end{aligned}$$

$$\begin{array}{llll} L_1 := x \cdot y_1 & L_{11} := z \cdot y_1 & L_{12} := y \cdot y_1 & R_1 := y \cdot y_1 \\ & R_{11} := w \cdot y_1 & R_{12} := y \cdot y_1 & \\ \\ L_2 := y_2 \cdot x & L_{21} := y_2 \cdot z & L_{22} := y_2 \cdot y & R_2 := y_2 \cdot y \\ & R_{21} := y_2 \cdot w & R_{22} := y_2 \cdot y & \end{array}$$

Then the formula  $\pi(x, y, z, w)$  obtained for these terms is the conjunction of  $\Phi_1$  and  $\Phi_2$ :

$$\begin{aligned} \Phi_1 &:= \exists y_1 \forall x_1 \exists y_2 \forall x_2 : z \cdot y_1 = w \cdot y_1 \wedge y \cdot y_1 = y \cdot y_1 \wedge \\ &\quad \wedge y_2 \cdot z = y_2 \cdot w \wedge y_2 \cdot y = y_2 \cdot y. \end{aligned}$$

$$\Phi_2 := \exists x_1 \forall y_1 \exists x_2 \forall y_2 : ((z \cdot y_1 = w \cdot y_1 \wedge y \cdot y_1 = y \cdot y_1) \rightarrow x \cdot y_1 = y \cdot y_1) \wedge \\ \wedge ((y_2 \cdot z = y_2 \cdot w \wedge y_2 \cdot y = y_2 \cdot y) \rightarrow y_2 \cdot x = y_2 \cdot y).$$

Formula  $\Phi_1$  holds trivially in  $\mathcal{V}$  (take  $y_1, y_2 = 0$ ) and  $\Phi_2$  may be simplified to:

$$\forall u (z \cdot u = w \cdot u \rightarrow x \cdot u = y \cdot u) \wedge (u \cdot z = u \cdot w \rightarrow u \cdot x = u \cdot y).$$

Now, the algebra  $A$  given by the table on the right is in  $\mathcal{V}$ . We have  $A \models \pi(a, b, a, b)$  and  $A \models \pi(c, c, a, b)$ . If  $\pi(x, y, z, w)$  were of the form

$$\forall \theta \in FC^*(A) : (z, w) \in \theta \Rightarrow (x, y) \in \theta,$$

$\cdot^A$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	c
c	0	c	c	c

for some set of congruences  $FC^*(A)$ , we should also have  $A \models \pi(a \cdot c, b \cdot c, a, b)$  by Proposition 1 (1). But that's not the case since  $a \cdot a = b \cdot a$  and  $(a \cdot c) \cdot a \neq (b \cdot c) \cdot a$ .

For the case of semilattices, the terms  $L_\alpha$ ,  $R_\alpha$  and  $s_i$  are the same and we have to take  $t_1 = t_2 := z \cdot w$ . We obtain the simpler formula  $\pi_s(x, y, z, w)$ :

$$\forall u (z \cdot u = w \cdot u \rightarrow x \cdot u = y \cdot u),$$

which is equivalent to  $\forall \theta \in FC^*(A) : (z, w) \in \theta \Rightarrow (x, y) \in \theta$  for every semilattice  $A$ , where we take

$$FC^*(A) := \{\theta_{z,w} \in \text{Con}(A) : z, w \in A\} \text{ and } \theta_{z,w} := \{(x, y) \in A^2 : \pi_s(x, y, z, w)\}.$$

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