

# Automorphisms of Partially Commutative Groups II: Combinatorial Subgroups

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# 1 Introduction

Partially commutative groups are a class of groups widely studied on account both of their intrinsically rich structure and their natural appearance in many diverse branches of mathematics and computer science (see [7] or [21] for example.) It is therefore natural that the pace of study of their automorphism groups should be gaining momentum, as it has been recently.

A *partially commutative group*  $G(\Gamma)$  (also known as a *right-angled Artin group*, a *trace group*, a *semi-free group* or a *graph group*) is a group given by a finite presentation  $\langle X | R \rangle$ , where  $X$  is the vertex set of a simple graph  $\Gamma$  and  $R$  is the set consisting of precisely those commutators  $[x, y]$  of elements of  $X$  such that  $x$  and  $y$  are joined by an edge of  $\Gamma$ . (A simple graph is one without multiple edges or self-incident vertices. Our convention is that  $[x, y] = x^{-1}y^{-1}xy$ .)

Initial work by Servatius [36] and Laurence [29] established a finite generating set for the automorphism group of a partially commutative group. In a resurgence of interest over the last few years considerably more has been discovered: for example, Bux, Charney, Crisp and Vogtmann [8, 10, 5] have shown that these groups are virtually torsion-free and have finite virtual cohomological dimension and Day has shown how peak reduction techniques may be used on certain subsets of the generators and thereby has given a presentation for the automorphism group [13]. Moreover these groups have a very rich subgroup structure: Gutierrez, Piggott and Ruane [27] have constructed a semi-direct product decomposition for the more general case of automorphism groups of graph products of groups. Duncan, Remeslennikov and Kazachkov [19] describe several arithmetic subgroups of the automorphism group of a partially commutative group; while different arithmetic subgroups have been found by Noskov [33]. Under certain conditions on the graph  $\Gamma$ , Charney and Vogtmann have shown [11] that the Tits alternative holds for the outer automorphism group of  $G(\Gamma)$  and moreover Day [15] has shown that in all cases this group contains either a finite-index nilpotent subgroup or a non-Abelian free subgroup. Minasyan has shown [32] that partially commutative groups are conjugacy separable, from which (loc. cit.) it follows that their outer automorphism groups are residually finite. By reduction to the compressed word problem in  $G(\Gamma)$ , Lohrey and Schleimer have shown that the word problem in  $\text{Aut}(G(\Gamma))$  has polynomial time complexity [30]. Charney and Faber [9], and subsequently Day [14], have studied automorphism groups of partially commutative groups associated to random graphs, of Erdős-Rényi type, and found bounds on the edge probabilities so that, with probability tending to one as the number of vertices tends to  $\infty$ , such groups have finite outer automorphism groups.

In this paper we continue the investigation of [19] into the structure of the automorphism group and its subgroups. We introduce several standard automorphisms of a partially commutative group and describe how an arbitrary automorphism may be decomposed as a product of these standard automorphisms. This reduces the study of the automorphism group to the study of subgroups generated by particular types of standard automorphism. We then define subgroups of a geometric character and use these to analyse the group structure. Note that if  $R$  is the ring of integers or a field of characteristic 0 and  $G$  is a partially commutative group in the class of 2-nilpotent  $R$ -groups, the structure of  $\text{Aut}(G)$  has been completely described, by Remeslennikov and Treier [35]; and decomposes as an extension of an Abelian group by a subgroup of  $\text{GL}(n, R)$ .

With this program in mind we define certain automorphisms, based on the combinatorial properties of the graph  $\Gamma$ , and these form our stock of standard automorphisms. The idea is to emulate the theory of automorphisms of algebraic and Chevalley groups. There is extensive literature on abstract isomorphisms of the classical linear groups and algebraic groups, over fields and special classes of rings, in which the fundamental results are theorems on splitting of arbitrary automorphisms into special automorphisms (such as algebraic, semialgebraic, simple, central, etc.) [3, 25] and representations of the group of automorphisms as products of the corresponding subgroups. Similar splitting theorems have also been established for Chevalley groups. Steinberg [37] and Humphreys [28] established such results for Chevalley groups over fields and Bunina [4] has defined several special types of automorphism (Central, Ring, Inner and Graph automorphism) and shown that, if  $G$  is a Chevalley group over a commutative local ring (subject to certain restrictions) then an arbitrary automorphism of  $G$  decomposes as a product of such automorphisms. Moreover similar results have been obtained for Kac-Moody groups (see [6] and the references therein).

In [19] we obtained certain decomposition theorems for the automorphism group of a partially commutative group which we extend in this work. We use the orthogonalisation operator  $Y^\perp$  and a closure operator  $\text{cl}(Y)$  both defined on subsets  $Y \subseteq X$  in [18]. In particular, for  $x \in X$ , the set  $\{x\}^\perp$  consists of all vertices incident to  $x$ , as well as  $x$  itself: so is the “star” of  $x$ ; and the closure  $\text{cl}(\{x\})$  of  $\{x\}$  is the intersection of the stars of all elements of  $\{x\}^\perp$ . The closure operator  $\text{cl}$  defines a lattice of “closed” subsets  $\mathcal{L} = \mathcal{L}(X)$  of  $X$  and the results of [19] were obtained by considering the action of automorphisms on this lattice. In this paper we consider a similar lattice  $\mathcal{K} = \mathcal{K}(X)$  of “admissible” subsets of  $X$  and the action of automorphisms on  $\mathcal{K}$ . In particular there is an admissible set  $\mathfrak{a}(x)$  associated to each element of  $X$ : namely the intersection of the stars of all elements of  $\{x\}^\perp \setminus \{x\}$ . We

consider the following subgroups of the automorphism group  $\text{Aut}(G)$  of the partially commutative group  $G$ .

- The subgroup  $\text{Aut}^\Gamma(G)$  of automorphisms induced by automorphisms of the graph  $\Gamma$  (Definition 3.7).
- The subgroup  $\text{Aut}_{\text{comp}}^\Gamma(G)$  of  $\text{Aut}(G)$ , which is isomorphic to the automorphism group of the graph  $\Gamma^{\text{comp}}$ , the compressed graph of  $\Gamma$  (Definition 3.7).
- The subgroup  $\text{Conj}(G)$  of basis-conjugating automorphisms: those which map each generator  $x$  to  $x^{f_x}$ , for some  $f_x \in G$  (Definition 3.14).
- The subgroup  $\text{Conj}_N(G)$  of  $\text{Conj}(G)$ , of automorphisms such that, for all  $x \in X$ , there exists  $g_x$  in  $G$  with the property that  $z$  maps to  $z^{g_x}$ , for all  $z \in \mathfrak{a}(x)$  (Definition 3.33).
- The subgroup  $\text{St}(\mathcal{K})$ , elements of which stabilise subgroups generated by subsets  $A$ , where  $A$  is an element of the lattice  $\mathcal{K}$  (Definition 4.1).
- The subgroup  $\text{St}^{\text{conj}}(\mathcal{K})$ , elements of which map each subgroup  $\langle A \rangle$ , where  $A \in \mathcal{K}$ , to  $\langle A \rangle^{g_A}$ , for some  $g_A \in G$  (Definition 4.2).
- Various subgroups illustrated in Figure 1.1 below.

(Several of these groups are well-known: some are defined for example in [29] and others in [19].)

The first step in our decomposition of  $\text{Aut}(G)$  is to separate out the automorphisms induced by automorphisms of the compressed graph.

**Theorem 4.4.** *The group  $\text{Aut}(G)$  can be decomposed into the internal semi-direct product of the subgroup  $\text{St}^{\text{conj}}(\mathcal{K})$  and the finite subgroup  $\text{Aut}_{\text{comp}}^\Gamma(G)$ , i.e.*

$$\text{Aut}(G) = \text{St}^{\text{conj}}(\mathcal{K}) \rtimes \text{Aut}_{\text{comp}}^\Gamma(G).$$

This theorem essentially reduces the problem of studying  $\text{Aut}(G(\Gamma))$  to the study of the group  $\text{St}^{\text{conj}}(\mathcal{K})$ .

We may also decompose the automorphism group using the connected components of  $\Gamma$ . If  $\Gamma$  has connected components  $\Gamma_1, \dots, \Gamma_n$  then the partially commutative group determined by  $\Gamma$  is the free product of those determined by the  $\Gamma_i$ . The group of automorphisms of a free product of groups has been completely described (from the point of view of generators and defining relations) in papers [22, 23, 24, 12]. We specialise these results to the case

under consideration to give generators and relations for the full automorphism group in terms of presentations for the automorphism groups of the factors.

However, here we encounter the first of two main obstructions to identifying the structure of  $\text{Aut}(G(\Gamma))$ . The problem arises when there are isolated vertices in the graph  $\Gamma$  (vertices of valency zero). In this case the automorphism group does not have a natural semi-direct product decomposition in terms of the automorphism groups of the factors. Nonetheless, in the special case where there are no isolated vertices the quoted results give the following theorem, where  $\text{LInn}_{\text{ext}}$  is a subset of  $\text{Conj}(G)$ , which is empty unless  $\Gamma$  is disconnected and is defined in Definition 3.24,  $\Gamma$  has connected components  $\Gamma_1, \dots, \Gamma_n$  and  $G_i = G(\Gamma_i)$ .

**Theorem 3.31** (cf. [12], Theorem C)]. *Suppose that no component of  $\Gamma$  is an isolated vertex. Define  $\bar{G} = G_1 \times \dots \times G_n$  and  $\text{FR}(G) = \langle \text{LInn}_{\text{ext}} \rangle$ . Then  $\text{FR}(G)$  is the kernel of the canonical map from  $\text{Aut}(G)$  to  $\text{Aut}(\bar{G})$ . Moreover  $\text{FR}(G)$  has a normal series*

$$1 < P_{n-1} < \dots < P_2 < \text{FR}(G)$$

such that, setting  $\text{FR}_i(G) = \text{FR}(G)/P_i$ ,

- (i)  $\text{FR}(G) = P_i \rtimes \text{FR}_i(G)$ ,
- (ii)  $\text{FR}_i(G) = \text{FR}(G_1 * \dots * G_i)$  and
- (iii) all the  $P_i$  are finitely generated.

The last theorem reduces analysis of the structure of  $\text{Aut}(G)$ , in the case when  $\Gamma$  has no isolated vertices, to analysis of  $\text{Aut}(G(\Gamma_i))$ ,  $i = 1, \dots, n$ , and of the Fouxé-Rabinovitch kernel  $\text{FR}(G)$ .

In the light of these results we may often reduce to the study of  $\text{St}^{\text{conj}}(\mathcal{K})$  where  $\Gamma$  is a connected graph. First of all we have the following theorem.

**Theorem 4.5.** *The subgroup  $\text{Conj}_N(G)$  is a normal subgroup of  $\text{St}^{\text{conj}}(\mathcal{K})$  and therefore of  $\text{Conj}(G)$ .*

The next step might appear to be to give an affirmative answer to the following question.

**Question 4.7.** *Let  $\Gamma$  be a connected graph. Is  $\text{St}^{\text{conj}}(\mathcal{K}) = \text{St}(\mathcal{K}) \text{Conj}_N(G)$ ?*

However as examples show the answer to this question is negative; and this brings us to the second major obstruction to the description of the structure of  $\text{Aut}(G)$ . This is the existence of vertices  $x$  and  $y$  such that  $\{y\}^\perp \setminus \{y\}$  is contained in  $\{x\}^\perp$ . When this occurs we say that  $x$  dominates

$y$  (Definition 3.37). If there are no such vertices  $x$  and  $y$  in  $\Gamma$  then we obtain a clear description of the structure of  $\text{St}^{\text{conj}}(\mathcal{K})$  in terms of  $\text{Conj}(G)$  and the stabiliser  $\text{St}(\mathcal{L})$  of the lattice of closed sets (studied in detail in [19]). In fact, in this case  $\text{Conj}(G) = \text{Conj}_N(G)$  and  $\text{St}(\mathcal{K}) = \text{St}(\mathcal{L})$ .

**Theorem 4.10.** *The following are equivalent for a graph  $\Gamma$ .*

- (i)  $G$  has no dominated vertices.
- (ii)  $\text{St}^{\text{conj}}(\mathcal{K}) = \text{Conj}_N(G) \rtimes \text{St}(\mathcal{L})$ .
- (iii)  $\text{St}^{\text{conj}}(\mathcal{K}) = \text{Conj}(G) \rtimes \text{St}(\mathcal{L})$ .
- (iv)  $\text{St}^{\text{conj}}(\mathcal{K}) = \text{Conj}(G) \rtimes \text{St}(\mathcal{K})$ .

Therefore, in the case where there are no dominated vertices the structure of  $\text{St}^{\text{conj}}(\mathcal{K})$  is determined by the structure of  $\text{Conj}_N(G)$  and  $\text{St}(\mathcal{L})$  and, as we have shown in [19],  $\text{St}(\mathcal{L})$  is an arithmetic group for which we have a complete structural decomposition.

We conclude by establishing conditions under which  $\text{St}^{\text{conj}}(\mathcal{K}) = \text{Conj}(G) \text{St}(\mathcal{K})$ , even though there are dominated vertices (and this product may not be semi-direct). In Section 4.1 we introduce balanced graphs, which include those without dominated vertices, and prove the following theorem.

**Theorem 4.19.** *Let  $\Gamma$  be a connected graph and  $G = G(\Gamma)$ . Then  $\text{St}^{\text{conj}}(\mathcal{K}) = \text{St}(\mathcal{K}) \text{Conj}(G)$  if and only if  $\Gamma$  is a balanced graph.*

Therefore in many cases the structure of  $\text{St}^{\text{conj}}(\mathcal{K})$  is determined by the structure of  $\text{St}(\mathcal{K})$ ,  $\text{Conj}(G)$  and  $\text{Conj}_N(G)$ . In this paper we find generators for (most of) the subgroups discussed above, as well as those appearing in the diagram below, establish some of their basic properties and investigate the decomposition of the automorphism group of  $G$ , in terms of these subgroups, in the simplest cases, leaving the case where there are dominated vertices, and the structure of  $\text{St}(\mathcal{K})$  to later papers.

The structure of the paper is as follows. In Section 2 we introduce partially commutative groups, admissible sets, the lattices  $\mathcal{K}$  and  $\mathcal{L}$  and describe an ordering on the vertex set of a graph induced from the lattice  $\mathcal{K}$ . In Section 3 we turn to the automorphism groups of partially commutative groups, show how they may be decomposed using the subgroups  $\text{Aut}^\Gamma(G)$  and  $\text{Aut}_{\text{comp}}^\Gamma(G)$ , mentioned above, and, using the results of Fousse-Rabinovitch and Gilbert and Collins, show how the connected components of the graph  $\Gamma$  determine generators and relations for the automorphism group of  $G(\Gamma)$ . In Section 3.5 we define a collection of subgroups of the basis-conjugating automorphism subgroup  $\text{Conj}(G)$ , to be used to decompose  $\text{St}^{\text{conj}}(\mathcal{K})$ , and show how these

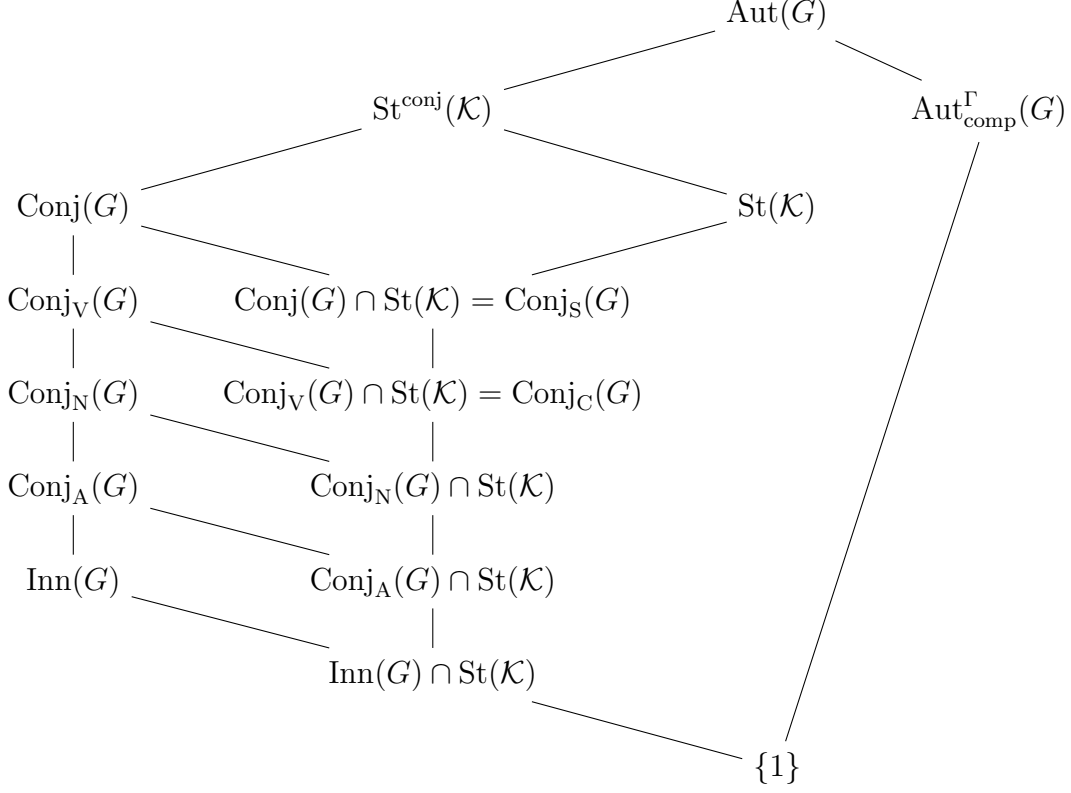


Figure 1.1: Subgroups of  $\text{Aut}(G(\Gamma))$ , where  $\Gamma$  has no isolated vertices

relate to each other (see Figure 1.1 below). In Section 4 we consider the subgroups  $\text{St}(\mathcal{K})$  and  $\text{St}^{\text{conj}}(\mathcal{K})$ , show that  $\text{Conj}_N(G)$  is normal in  $\text{St}^{\text{conj}}(\mathcal{K})$ , describe the intersection  $\text{St}(\mathcal{K}) \cap \text{Conj}_N(G)$  and give an example to show that, in general,  $\text{St}^{\text{conj}}(\mathcal{K}) \neq \text{St}(\mathcal{K}) \text{Conj}(G)$ . Finally in Section 4.1 we define balanced graphs and show that the equality  $\text{St}^{\text{conj}}(\mathcal{K}) = \text{St}(\mathcal{K}) \text{Conj}(G)$  holds for  $G = G(\Gamma)$  if and only if  $\Gamma$  is balanced. In the version of the paper on **arxiv** we include an appendix with details of the construction of a presentation of  $\text{Aut}_{\text{comp}}^{\Gamma}(G)$  and the proof of the Theorem 3.29.

For reference purposes Figure 1.1 shows a diagram of the lattice of the main subgroups which we define in the paper. The figure covers the case where  $\Gamma$  has no isolated vertices (that is vertices of valency zero). If  $\Gamma$  does have isolated vertices then the subgroups  $\text{Conj}_N(G)$  and  $\text{Conj}_N(G) \cap \text{St}(\mathcal{K})$  are removed from the diagram, which otherwise remains the same. (In this case  $\text{Conj}_N(G) = \text{Inn}(G)$ .) Subgroups  $\text{Conj}_A(G)$ ,  $\text{Conj}_V(G)$ ,  $\text{Conj}_S(G)$  and  $\text{Conj}_C(G)$  are defined in definitions 3.32, 3.34, 3.35 and 3.38, respectively.

## 2 Preliminaries

Graph will mean undirected, finite, simple graph throughout this paper. A subgraph  $S$  of a graph  $\Gamma$  is called a *full* subgraph if vertices  $a$  and  $b$  of  $S$  are joined by an edge of  $S$  whenever they are joined by an edge of  $\Gamma$ . If  $S$  is a subset of  $V(\Gamma)$  we shall write  $\Gamma(S)$  for the full subgraph of  $\Gamma$  with vertices  $S$ .

If  $a$  and  $b$  are elements of a group then  $[a, b]$  denotes  $a^{-1}b^{-1}ab$ . If  $A$  and  $B$  are subsets of a group then  $[A, B]$  denotes  $\{[a, b] : a \in A, b \in B\}$ .

For the remainder of the paper let  $\Gamma$  be a finite, undirected, simple graph. Let  $X = V(\Gamma) = \{x_1, \dots, x_n\}$  be the set of vertices of  $\Gamma$  and let  $F(X)$  be the free group on  $X$ . Let

$$R = \{[x_i, x_j] \in F(X) \mid x_i, x_j \in X \text{ and there is an edge of } \Gamma \text{ joining } x_i \text{ to } x_j\}.$$

We define the *partially commutative group* with (*commutation*) *graph*  $\Gamma$  to be the group  $G(\Gamma)$  with presentation  $\langle X \mid R \rangle$ . When the underlying graph is clear from the context we write simply  $G$ .

The subgroup generated by a subset  $Y \subseteq X$  is called a *canonical parabolic subgroup* of  $G$  and denoted  $G(Y)$ . This subgroup is equal to the partially commutative group with commutation graph the full subgraph of  $\Gamma$  with vertices  $Y$  (see [2] or [21]).

By a *word* over  $X$  is meant an element of the free monoid  $(X \cup X^{-1})^*$ . We identify elements of  $F(X)$  with reduced words (that is those have no subwords of the form  $x^\varepsilon x^{-\varepsilon}$ , where  $x \in X$  and  $\varepsilon = \pm 1$ ). The *length* of a word  $w$  is its length as an element of  $(X \cup X^{-1})^*$  and is denoted  $|w|$ . Denote by  $\lg(g)$  the minimum of the lengths of words that represent the element  $g$  of  $G(X)$ . If  $w$  is a word representing  $g$  and  $w$  has length  $\lg(g)$  we call  $w$  a *minimal form* for  $g$ . If  $w$  is a minimal form for some element of  $G$  then we say that  $w$  is a *geodesic word*. When the meaning is clear we shall say that  $w$  is a minimal element of  $G$  when we mean that  $w$  is a minimal form of an element of  $G$ . We say that  $h \in G$  is *cyclically minimal* if and only if

$$\lg(g^{-1}hg) \geq \lg(h)$$

for every  $g \in G$ .

The *support* of a word  $w$  over  $X$  is the set of elements of  $X$  such that  $x$  or  $x^{-1}$  occurs in  $w$ . If  $u$  and  $v$  are minimal forms of an element  $g \in G$  then both  $u$  and  $v$  have the same support (see for example [21]). Therefore we may define the *support*  $\nu(g)$  of an element  $g \in G$  to be the support of a minimal form of  $g$ . If  $w \in G$  define  $A(w) = G(Y)$ , where  $Y$  is the set of elements of  $X \setminus \nu(w)$  which commute with every element of  $\nu(w)$ . If  $S \subseteq G$  then we define  $A(S) = \bigcap_{w \in S} A(w)$ .



From now on we regard words as representing elements of  $G$ , so when we write  $u = v$ , where  $u$  and  $v$  are words, we mean that  $u$  and  $v$  represent the same element of  $G$ . We write  $u \circ w$  to express the fact that  $\lg(uw) = \lg(u) + \lg(w)$ , where  $u, w \in G$ . Let  $u$  and  $w$  be elements of  $G$ . We say that  $u$  is a *left (right) divisor* of  $w$  if there exists  $v \in G$  such that  $w = u \circ v$  ( $w = v \circ u$ ). We partially order the set of all left (right) divisors of a word  $w$  as follows. We say that  $u_2$  is greater than  $u_1$  if and only if  $u_1$  is a left (right) divisor of  $u_2$ . It is shown in [21] that, for any  $w \in G$  and  $Y \subseteq X$ , there exists a unique maximal left divisor of  $w$  which belongs to the subgroup  $G(Y)$  of  $G$ : called the *greatest left divisor*  $\text{gd}_Y^l(w)$  of  $w$  in  $Y$ . The *greatest right divisor* of  $w$  in  $Y$  is defined analogously.

The *non-commutation* graph of the partially commutative group  $G(\Gamma)$  is the graph  $\Delta$ , dual to  $\Gamma$ , with vertex set  $V(\Delta) = X$  and an edge connecting  $x_i$  and  $x_j$  if and only if  $[x_i, x_j] \neq 1$ . The graph  $\Delta$  is the union of its connected components  $\Delta_1, \dots, \Delta_k$  and if  $u$  and  $v$  are words such that  $\nu(u) \subseteq \Delta_i$  and  $\nu(v) \subseteq \Delta_j$ , with  $i \neq j$ , then  $u$  and  $v$  represent commuting elements of  $G$ . Thus, if the vertex set of  $\Delta_j$  is  $I_j$  and  $\Gamma_j = \Gamma(I_j)$ , the full subgraph of  $\Gamma$  on  $I_j$ , then  $G = G(\Gamma_1) \times \dots \times G(\Gamma_k)$ .

Let  $g \in G$  and suppose that the full subgraph  $\Delta(\nu(g))$  of  $\Delta$  with vertices  $\nu(g)$  has connected components  $\Delta'_1, \dots, \Delta'_l$  and let the vertex set of  $\Delta'_j$  be  $I'_j$ . If  $w$  is a minimal form of  $g$  then, since  $[I'_j, I'_k] = 1$ , we can factor  $w$  as a product of commuting words,  $w = w_1 \circ \dots \circ w_l$ , where  $w_j \in G(\Gamma(I'_j))$ , so  $[w_j, w_k] = 1$  for all  $j, k$ . If  $g$  is cyclically minimal then we call this expression for  $g$  a *block decomposition* of  $g$  and say  $w_j$  is a *block* of  $g$ , for  $j = 1, \dots, l$ . Thus  $w$  itself is a block if and only if  $\Delta(\nu(w))$  is connected. Moreover it follows (see [21] for example) that if  $g$  has another minimal form  $u$  with a block decomposition  $u = u_1 \circ \dots \circ u_k$  then  $k = l$  (as  $\Delta(\nu(u)) = \Delta(\nu(g)) = \Delta(\nu(w))$ ) and after reordering the  $u_s$ 's if necessary  $w_s = u_s$ , for  $s = 1, \dots, l$ .

In general let  $v$  be an element of  $G$ , not necessarily cyclically minimal. We may write  $v = u^{-1} \circ w \circ u$ , where  $w$  is cyclically minimal and then  $w$  has a block decomposition  $w = w_1 \cdots w_l$ , say. We call the expression  $v = w_1^u \cdots w_l^u$  a *block decomposition* of  $v$  and say that  $w_j^u$  is a *block* of  $v$ , for  $j = 1, \dots, l$ . Note that this definition is slightly different from that given in [21].

The *centraliser* of a subset  $S$  of  $G$  is

$$C(S) = C_G(S) = \{g \in G : gs = sg, \text{ for all } s \in S\}.$$

An element  $g \in G$  is called a *root element* if  $g$  is not a proper power of any element of  $G$ . If  $h = g^n$ , where  $g$  is a root element and  $n \geq 1$ , then  $g$  is said to be a *root* of  $h$ . As shown in [2] (and also [16]) every element of the partially commutative group  $G$  has a unique root, which we denote  $r(g)$ . Let  $w$  be a

cyclically minimal element of  $G$  with block decomposition  $w = w_1 \cdots w_k$  and let  $v_i = r(w_i)$ . Then from [2] (and also [16, Theorem 3.10]),

$$C(w) = \langle v_1 \rangle \times \cdots \times \langle v_k \rangle \times A(w). \quad (2.1)$$

The following lemma will be useful.

**Lemma 2.1.** *Let  $x, y \in X$  and  $f, g \in G$  such that  $[x, y] = [x^f, y^g] = 1$  and*

$$(i) \ x^f = f^{-1} \circ x \circ f, \ y^g = g^{-1} \circ y \circ g \text{ and}$$

$$(ii) \ \text{gd}_X^r(f, g) = 1.$$

*Then  $[\nu(f), \nu(g)] = \{1\}$  and  $[f, y] = [g, x] = 1$ .*

*Proof.* By hypothesis  $[x^{fg^{-1}}, y] = 1$ . From condition (ii)  $fg^{-1} = f \circ g^{-1}$  and from [17], Lemma 2.3, there exist  $a, b, u \in G$  such that  $fg^{-1} = a \circ b \circ u$ ,  $x^{fg^{-1}} = u^{-1} \circ x \circ u$ ,  $a = x^n$ , for some  $n \in \mathbb{Z}$ , and  $[b, x] = 1$ . From (2.1),  $b = x^m \circ c$ , for some  $c \in A(x)$ ; so  $a \circ b = x^k \circ c$ , for some  $k \in \mathbb{Z}$ . Now  $a \circ b$  is a left divisor of  $f \circ g^{-1}$  and it follows from condition (i) that  $u = u_1 \circ u_2$ , where  $f = u_1$  and  $g^{-1} = a \circ b \circ u_2 = x^k \circ c \circ u_2$ , for some words  $u_1, u_2$  with  $[u_1, x^k \circ c] = 1$ . From condition (i) again,  $k = 0$ ,  $a = 1$  and  $b = c \in A(x)$ .

From [17] Corollary 2.6,  $[x, y] = [x^u, y] = 1$  implies that  $u \in C(y)$ . As  $u$  is a right divisor of  $f \circ g^{-1}$  it follows, from condition (i) again, that  $u_2 = 1$ ; so  $f = u_1 = u$ , and  $g = b$ . It follows, using condition (ii) once more, that  $[\nu(b), \nu(u)] = 1$  and this gives the result.  $\square$

## 2.1 Admissible sets

In this section we establish some properties of graphs which we shall apply to the study of the automorphism group of  $G(\Gamma)$ . If  $x$  and  $y$  are vertices of a graph  $\Gamma$  then we define the *distance*  $d(x, y)$  from  $x$  to  $y$  to be the minimum of the lengths of all paths from  $x$  to  $y$  in  $\Gamma$ . Given a subset  $Y$  of  $X$  the *orthogonal complement* of  $Y$  is defined to be

$$Y^\perp = \{u \in X \mid d(u, y) \leq 1, \text{ for all } y \in Y\}.$$

For a set  $\{x\}$  of one element we write  $x^\perp$  instead of  $\{x\}^\perp$  and in general often write  $x$  in place of  $\{x\}$ . For any set  $Y \subseteq X$  we write  $Y^{\perp\perp}$  for  $(Y^\perp)^\perp$ . By convention we set  $\emptyset^\perp = X$ .

We define the *closure* of  $Y$  to be  $\text{cl}(Y) = Y^{\perp\perp}$ . The closure operator in  $\Gamma$  satisfies, among others, the properties that  $Y \subseteq \text{cl}(Y)$ ,  $\text{cl}(Y^\perp) = Y^\perp$  and  $\text{cl}(\text{cl}(Y)) = \text{cl}(Y)$  [18, Lemma 2.4]. Moreover if  $Y_1 \subseteq Y_2 \subseteq X$  then  $\text{cl}(Y_1) \subseteq \text{cl}(Y_2)$ .

**Definition 2.2.** A subset  $Y$  of  $X$  is called closed (with respect to  $\Gamma$ ) if  $Y = \text{cl}(Y)$ . Denote by  $\mathcal{L} = \mathcal{L}(\Gamma)$  the set of all closed subsets of  $X$ .

For non-empty  $Y \subseteq X$  define  $\mathfrak{a}(Y) = \bigcap_{y \in Y} (y^\perp \setminus y)^\perp$ . Define  $\mathfrak{a}(\emptyset) = X$ . Subsets of the form  $\mathfrak{a}(Y)$ , where  $Y \subseteq X$  are called admissible sets. Let  $\mathcal{K} = \mathcal{K}(\Gamma)$  denote the set of admissible subsets of  $X$ .

Properties of the set  $\mathcal{L}$  are considered in detail in [18] and applied to the study of centralisers and automorphisms of partially commutative groups in [17], [20] and [19]. We shall see, in Section 3, that distinct elements  $x$  and  $y$  of  $X$ , such that  $x^\perp \setminus x \subseteq y^\perp$ , give rise to a particular type of automorphism of  $G$ . The motivation for the definition of an admissible set is then clear from the first part of the following lemma.

**Lemma 2.3.** For all  $x \in X$ ,

- (i) the set  $\mathfrak{a}(x) = \{y \in X : x^\perp \setminus x \subseteq y^\perp\}$  and
- (ii)  $y \in \mathfrak{a}(x)$  if and only if  $\text{cl}(y) \subseteq \mathfrak{a}(x)$ , for all  $y \in X$ .

*Proof.* (i)  $y \in \mathfrak{a}(x)$  if and only if  $[y, v] = 1$ , for all  $v \in x^\perp \setminus x$ , if and only if  $x^\perp \setminus x \subseteq y^\perp$ .

- (ii) For all  $y \in X$  we have  $y \in \text{cl}(y)$ , so the “if” clause follows. On the other hand if  $y \in \mathfrak{a}(x)$  then, from (i),  $x^\perp \setminus x \subseteq y^\perp$ ; so  $y^{\perp\perp} \subseteq (x^\perp \setminus x)^\perp$ , as required.

□

**Example 2.4.** In the graph  $\Gamma$  of Figure 2.1a

- $\mathfrak{a}(a) = \{b, c, d, e, g, h, i\}^\perp = \{a\} = \text{cl}(a)$ ;
- $d^\perp = g^\perp = \{a, c, d, e, g, h\}$  and  $\mathfrak{a}(d) = \mathfrak{a}(g) = \{a, d, g\} = \text{cl}(d) = \text{cl}(g)$ ;
- $\text{cl}(b) = \{a, b, c, h\}^\perp = \{a, b\}$ ,  $\text{cl}(i) = \{a, c, h, i\}^\perp = \{a, i\}$ ,  $i^\perp \setminus i = b^\perp \setminus b$  and  $\mathfrak{a}(i) = \mathfrak{a}(b) = \{a, c, h\}^\perp = \{a, b, d, g, i\} = \text{cl}(b) \cup \text{cl}(d) \cup \text{cl}(i)$ ;
- $\text{cl}(c) = \{a, c\}$ ,  $\text{cl}(h) = \{a, h\}$ ,  $c^\perp \setminus c = h^\perp \setminus h$  and  $\mathfrak{a}(c) = \mathfrak{a}(h) = \{a, c, h\} = \text{cl}(c) \cup \text{cl}(h)$ ;
- $\mathfrak{a}(e) = \{a, d, f, g\}^\perp = \{e\} = \text{cl}(e)$  and
- $\text{cl}(f) = \{e, f\}^\perp = \{e, f\}$  and  $\mathfrak{a}(f) = \{e\}^\perp = \{a, d, e, f, g\} = \text{cl}(d) \cup \text{cl}(f)$ .

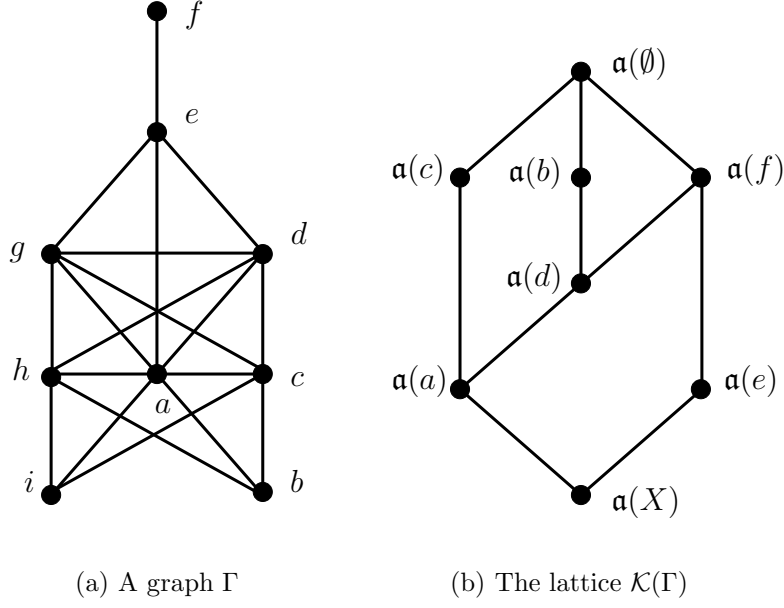


Figure 2.1: A graph and its lattice of admissible sets

For sets  $U, V$  we write  $U < V$  to indicate that  $U \subseteq V$  and  $U \neq V$ . A subset  $Y$  of  $X$  is called a *simplex* if the full subgraph of  $\Gamma$  with vertices  $Y$  is isomorphic to a complete graph.

**Lemma 2.5.** *For  $x \neq z \in X$  and subsets  $U$  and  $V$  of  $X$  the following hold.*

- (i) *If  $U \subseteq V$  then  $\mathfrak{a}(V) \subseteq \mathfrak{a}(U)$ .*
- (ii)  *$\mathfrak{a}(U) \cap \mathfrak{a}(V) = \mathfrak{a}(U \cup V)$ .*
- (iii)  *$\text{cl}(x) = \mathfrak{a}(x) \cap x^\perp$  so  $\mathfrak{a}(x) = \text{cl}(x)$  if and only if  $\mathfrak{a}(x) \subseteq x^\perp$ .*
- (iv)  *$x^\perp \subseteq \mathfrak{a}(x)$  if and only if  $x^\perp$  generates a complete subgraph.*
- (v) *If  $x^\perp \setminus x \subseteq z^\perp \setminus z$  then  $\mathfrak{a}(z) \subseteq \mathfrak{a}(x)$ .*
- (vi) *If  $x^\perp \subseteq z^\perp$  then  $\mathfrak{a}(z) \subseteq \mathfrak{a}(x)$ .*
- (vii)  *$\mathfrak{a}(z) \subseteq \mathfrak{a}(x)$  if and only if  $x^\perp \setminus x \subset z^\perp$ .*
- (viii)  *$\mathfrak{a}(x) = \mathfrak{a}(z)$  if and only if either  $x^\perp = z^\perp$  or  $x^\perp \setminus x = z^\perp \setminus z$ .*
- (ix) *If  $z \in \mathfrak{a}(x)$  then  $\mathfrak{a}(z) \subseteq \mathfrak{a}(x)$ .*
- (x)  *$\mathfrak{a}(U) = \cup_{y \in \mathfrak{a}(U)} \mathfrak{a}(y)$ .*

(xi) If  $\text{cl}(x) = \mathbf{a}(x)$  then  $\text{cl}(y) = \mathbf{a}(y)$ , for all  $y \in \mathbf{a}(x)$ .

(xii) If  $[x, z] = 1$  then  $[G(\mathbf{a}(x)), G(\mathbf{a}(z))] = 1$ .

*Proof.* Statements (i) to (v) follow directly from the definitions and the fact that if  $S \subseteq T$  then  $T^\perp \subseteq S^\perp$ , for all subsets  $S, T$  of  $X$ . For (vi) note that in this case  $z \in x^\perp$ , so as  $x \neq z$ ,  $\mathbf{a}(x) = (x^\perp \setminus x)^\perp = ((x^\perp \setminus \{x, z\}) \cup \{z\})^\perp = (x^\perp \setminus \{x, z\})^\perp \cap z^\perp \supseteq (z^\perp \setminus \{x, z\})^\perp \cap x^\perp = \mathbf{a}(z)$ .

The right to left implication of (vii) is a consequence of (v) and (vi), and the fact that if  $x^\perp \setminus x \subseteq z^\perp$  then  $x^\perp \subseteq z^\perp$  or  $x^\perp \setminus x \subseteq z^\perp \setminus z$ . To see the opposite implication: if  $\mathbf{a}(z) \subseteq \mathbf{a}(x)$  then, as  $z \in \mathbf{a}(z)$ , we have  $z \in \mathbf{a}(x)$ , so  $x^\perp \setminus x \subseteq z^\perp$ , from Lemma 2.3.

To see (viii) suppose first that  $\mathbf{a}(x) = \mathbf{a}(z)$ . Then, from (vii), we have  $x^\perp \setminus x \subseteq z^\perp$  and  $z^\perp \setminus z \subseteq x^\perp$ . If  $x \in z^\perp$  then  $z \in x^\perp$ , and in this case  $x^\perp = z^\perp$ . Otherwise  $x \notin z^\perp$  and  $z \notin x^\perp$  in which case  $x^\perp \setminus x = z^\perp \setminus z$ . Conversely, if either  $x^\perp = z^\perp$  or  $x^\perp \setminus x = z^\perp \setminus z$  then it follows, from (v) and (vi), that  $\mathbf{a}(x) = \mathbf{a}(z)$ .

Statement (ix) follows immediately from (vii) and Lemma 2.3. Statement (x) follows from (ix) as if  $y \in \mathbf{a}(U)$  then  $\mathbf{a}(y) \subseteq \mathbf{a}(U)$ .

To see statement (xi) observe that  $\text{cl}(x)$  is a simplex so if  $\text{cl}(x) = \mathbf{a}(x)$  and  $y \in \mathbf{a}(x)$  then  $\mathbf{a}(y) \subseteq \mathbf{a}(x)$  implies that  $\mathbf{a}(y)$  is a simplex. Therefore  $\mathbf{a}(y) \subseteq y^\perp$  and the result follows from (iii).

For (xii) suppose that  $u \in \mathbf{a}(x)$  and  $v \in \mathbf{a}(z)$ . Since  $z \in x^\perp \setminus x$  we have  $u \in z^\perp$  and similarly  $v \in x^\perp$ . Since  $[u, y] = 1$  for all  $y \in x^\perp$ , except possibly  $x$ , it follows that  $u$  commutes with  $v$ , unless  $v = x$ . However if  $v = x$  then, since  $v \in (z^\perp \setminus z)^\perp$ ,  $v$  commutes with all elements of  $z^\perp$ , including  $u$ .

□

Let  $\sim_\perp$  be the relation on  $X$  given by  $x \sim_\perp y$  if and only if  $x^\perp = y^\perp$  and let  $\sim_\diamond$  be the relation given by  $x \sim_\diamond y$  if and only if  $x^\perp \setminus x = y^\perp \setminus y$ . These are equivalence relations and the equivalence classes of  $x$  under  $\sim_\perp$  and  $\sim_\diamond$  are denoted by  $[x]_\perp$  and  $[x]_\diamond$ , respectively. Note that if  $|[x]_\perp| > 1$  then  $[x]_\diamond = \{x\}$  and the same is true on interchanging  $\perp$  and  $\diamond$ . Therefore the relation  $\sim$ , given by  $x \sim y$  if and only if  $x \sim_\perp y$  or  $x \sim_\diamond y$ , is an equivalence relation. Denote the equivalence class of  $x$  under  $\sim$  by  $[x]$ . Then  $x \sim y$  if and only if  $x \sim_\perp y$  or  $x \sim_\diamond y$ , and  $[x] = [x]_\perp \cup [x]_\diamond$ . It follows that  $x \sim y$  if and only if  $x^\perp \setminus \{x, y\} = y^\perp \setminus \{x, y\}$ .

**Lemma 2.6.** For all  $x, z \in X$ ,

(i)  $\mathbf{a}(x) = \mathbf{a}(z)$  if and only if  $z \in [x]$ ,

(ii)  $[x] = \mathbf{a}(x) \setminus (\cup \{\mathbf{a}(y) \mid y \in \mathbf{a}(x) \text{ and } \mathbf{a}(y) < \mathbf{a}(x)\})$ .

*Proof.* The second statement follows directly from the first, together with (ix) of Lemma 2.5. Therefore it suffices to show that  $\mathbf{a}(x) = \mathbf{a}(z)$  if and only if  $z \in [x]$ . By definition,  $z \in [x]$  if and only if  $x^\perp = z^\perp$  or  $x^\perp \setminus x = z^\perp \setminus z$ . From Lemma 2.5 (viii), this holds if and only if  $\mathbf{a}(x) = \mathbf{a}(z)$ .  $\square$

**Lemma 2.7.**

$$[x] = \begin{cases} [x]_\perp, & \text{if } \mathbf{a}(x) = \text{cl}(x) \\ [x]_\diamond, & \text{if } \mathbf{a}(x) > \text{cl}(x) \end{cases}.$$

*Proof.* First suppose  $\mathbf{a}(x) = \text{cl}(x)$ . Then, from Lemma 2.5 (iii),  $\mathbf{a}(x) \subseteq x^\perp$ . If  $z \in [x]$  then, from Lemma 2.6 (ii),  $z \in [x] \cap x^\perp$  and so  $z \sim_\perp x$ .

Now suppose that  $\text{cl}(x) < \mathbf{a}(x)$ . If  $z \in [x]$ ,  $z \neq x$ , and  $z^\perp = x^\perp$  then  $x \in z^\perp \setminus z$ , so  $\mathbf{a}(x) = \mathbf{a}(z) \subseteq x^\perp$ , contradicting Lemma 2.5 (iii). Hence  $z \in [x]$  implies  $z \sim_\diamond x$ .  $\square$

In view of Lemma 2.5 (ii) above, for an arbitrary subset  $Y$  of  $X$  the  $\mathbf{a}$ -closure of  $Y$  may be defined to be the admissible set

$$\text{cl}_\mathbf{a}(Y) = \cap \{U \subseteq X \mid Y \subseteq U \text{ and } U = \mathbf{a}(V), \text{ for some } V \subseteq X\}.$$

Then  $\text{cl}_\mathbf{a}(Y)$  is the smallest admissible set containing  $Y$  and  $Y$  is admissible if and only if  $Y = \text{cl}_\mathbf{a}(Y)$ .

It is shown in [18] that  $Y \in \mathcal{L}$  if and only if  $Y = U^\perp$ , for some  $U \subseteq X$ . Therefore  $\mathcal{K} \subset \mathcal{L}$ . The set  $\mathcal{K}$ , partially ordered by inclusion, with infimum  $U \wedge V = U \cap V$  and supremum  $U \vee V = \text{cl}_\mathbf{a}(U \cup V)$  forms a lattice. The lattice  $\mathcal{K}$  has maximal element  $X = \text{cl}_\mathbf{a}(X) = \mathbf{a}(\emptyset)$  and minimal element  $\mathbf{a}(X)$ . The lattice of admissible sets for the graph of Example 2.4 is shown in Figure 2.1b.

Although the lattice  $\mathcal{K}(\Gamma)$  of the graph  $\Gamma$  in Example 2.4 consists of  $\mathbf{a}(\emptyset)$ ,  $\mathbf{a}(X)$  and sets of the form  $\mathbf{a}(x)$ , where  $x \in X$ , this is atypical. For example consider the path graph of length three: that is the tree with vertices  $a$ ,  $b$ ,  $c$  and  $d$  and edges  $\{a, b\}$ ,  $\{b, c\}$  and  $\{c, d\}$ . In this case  $\mathbf{a}(a) = \{a, b, c\}$  and  $\mathbf{a}(b) = \{b\}$ . It follows that  $|\mathbf{a}(x)| = 1$  or  $3$ , for all vertices  $x$ . However  $\mathbf{a}(\{a, d\}) = \{b, c\}$ ; so  $\mathbf{a}(\{a, d\}) \neq \mathbf{a}(x)$ , for all vertices  $x$ . Moreover  $\text{cl}_\mathbf{a}(\{a, d\}) = X \neq \mathbf{a}(\{a, d\})$ . We shall be mostly interested here in the admissible sets  $\mathbf{a}(x)$ , for  $x \in X$ ; for which we can say the following.

**Lemma 2.8.** For  $x \in X$ ,  $\text{cl}_\mathbf{a}(x) = \mathbf{a}(x)$ .

*Proof.* By definition  $\text{cl}_\mathbf{a}(x) \subseteq \mathbf{a}(x)$ . If  $U$  is admissible then  $U = \cap_{y \in Y} \mathbf{a}(y)$ , for some  $Y \subseteq X$ . If  $x \in U$  this means that  $x \in \mathbf{a}(y)$ , so  $\mathbf{a}(x) \subseteq \mathbf{a}(y)$ , for all  $y \in Y$ . Hence  $\mathbf{a}(x) \subseteq U$ . It follows that  $\mathbf{a}(x) \subseteq \text{cl}_\mathbf{a}(x)$ .  $\square$

## 2.2 Ordering $X$

The next goal is to define a total ordering on  $X$  which reflects the structure of the lattice  $\mathcal{K}$ . First define a partial order  $<_{\mathcal{K}}$  on  $X$  by  $x <_{\mathcal{K}} y$  if and only if  $\mathfrak{a}(x) < \mathfrak{a}(y)$ . If  $\mathfrak{a}(x) = \mathfrak{a}(y)$  we write  $x =_{\mathcal{K}} y$ . We say  $x$  is  $\mathcal{K}$ -minimal if  $y \leq_{\mathcal{K}} x$  implies  $y =_{\mathcal{K}} x$ . The definition of  $\mathcal{K}$ -maximal is then the obvious one. Recall that in [19] the analogous ordering using  $\mathcal{L}$  instead of  $\mathcal{K}$  was defined, and the definitions of  $\mathcal{L}$ -minimal and  $\mathcal{L}$ -maximal were also defined using the closure operator instead of the  $\mathfrak{a}$  operator.

**Lemma 2.9.** *An element  $x \in X$  is  $\mathcal{K}$ -minimal if and only if  $[x] = \mathfrak{a}(x)$ . If  $x$  is  $\mathcal{K}$ -minimal then*

- (i)  $x$  is  $\mathcal{L}$ -minimal and
- (ii)  $\text{cl}(y) = [y]_{\perp}$ , for all  $y \in \mathfrak{a}(x)$ .

*Proof.* The first statement follows immediately from the definitions and Lemma 2.6. For the second suppose  $x$  is  $\mathcal{K}$ -minimal. If  $[x] = \mathfrak{a}(x) = \text{cl}(x) = [x]_{\perp}$  then  $\text{cl}(y) = \text{cl}(x)$ , for all  $y \in \text{cl}(x)$ , so  $x$  is  $\mathcal{L}$ -minimal and (i) and (ii) hold. If  $[x] = \mathfrak{a}(x) > \text{cl}(x)$  then  $[x]_{\diamond} = \mathfrak{a}(x)$ , so  $\text{cl}(x) = [x]_{\diamond} \cap x^{\perp} = \{x\}$ , from Lemma 2.5(iii), so again  $x$  is  $\mathcal{L}$ -minimal. To see (ii) in this case note that  $y \in \mathfrak{a}(x)$  implies  $y \in [x]_{\diamond} = [y]_{\diamond}$  and  $\text{cl}(y) = y^{\perp} \cap \mathfrak{a}(y) = y^{\perp} \cap \mathfrak{a}(x) = y^{\perp} \cap [y]_{\diamond} = \{y\} = [y]_{\perp}$ .  $\square$

As in Example 2.10 below, there may be elements which are  $\mathcal{L}$ -minimal but are not  $\mathcal{K}$ -minimal.

We now define a total order  $\prec$  on  $X$ , which will have the properties that

1. if  $x <_{\mathcal{K}} y$  then  $y \prec x$  and
2. if  $z \prec y \prec x$  and  $z \in [x]$  then  $y \in [x]$ .

Define  $\mathcal{K}_X$  to be the subset of  $\mathcal{K}$  consisting of sets  $\mathfrak{a}(x)$ , for  $x \in X$ . To begin with let

$$B_0 = \{Y \in \mathcal{K}_X : Y = \mathfrak{a}(x), \text{ where } x \text{ is } \mathcal{K}\text{-minimal}\}.$$

Suppose that  $B_0$  has  $k$  elements and choose an ordering  $Y_1 < \dots < Y_k$  of these elements. If  $i \neq j$  then it follows from Lemma 2.9 that  $Y_i \cap Y_j = \emptyset$ . Therefore we may define the ordering  $\prec$  on  $\cup_{i=1}^k Y_i$  in such a way that if  $x_i \in Y_i$  and  $x_j \in Y_j$  and  $Y_i < Y_j$  then  $x_j \prec x_i$ : merely by choosing an ordering for elements of each  $Y_i$ . (For instance, in Example 2.10 we can choose  $f \prec e \prec d \prec c$ .)

We recursively define sets  $B_i$  of elements of  $\mathcal{K}_X$ , for  $i \geq 0$ , as follows. Assume that we have defined sets  $B_0, \dots, B_i$ , set  $U_i = \cup_{j=0}^i B_j$  and define  $X_i = \{u \in X : u \in Y, \text{ for some } Y \in U_i\}$ . If  $U_i \neq \mathcal{K}_X$  define  $B_{i+1}$  by

$$B_{i+1} = \{Y = \mathbf{a}(x) \in \mathcal{K}_X : Y \notin U_i, \text{ and } y <_{\mathcal{K}} x \text{ implies that } \mathbf{a}(y) \in U_i\}.$$

If  $U_i \neq \mathcal{K}_X$  then  $X_i \neq X$  and  $B_{i+1} \neq \emptyset$ . We assume inductively that we have ordered the set  $X_i$  in such a way that if

$$(i) \quad 0 \leq a < b \leq i,$$

$$(ii) \quad x_a \in Y_a \text{ where } Y_a \in B_a \text{ and}$$

$$(iii) \quad x_b \in Y_b \text{ where } Y_b \in B_b;$$

then  $x_b \prec x_a$ . If  $Y = \mathbf{a}(x) \in B_{i+1}$  then

$$\begin{aligned} [x] &= Y \setminus \{u \in \mathbf{a}(y) : y <_{\mathcal{K}} x\} \\ &= Y \setminus \{u \in X_i\}. \end{aligned}$$

Therefore we have defined  $\prec$  on the set  $Y \setminus [x]$ . Moreover, if  $Y_1 \neq Y_2$  and  $Y_1, Y_2 \in B_{i+1}$  then  $Y_1 \cap Y_2 \in \mathcal{K}$  so  $z \in Y_1 \cap Y_2$  implies  $\mathbf{a}(z) \subseteq Y_1 \cap Y_2$ . As  $Y_1 \neq Y_2$  this implies that  $\mathbf{a}(z)$  is strictly contained in  $Y_i$ ,  $i = 1, 2$ . If  $Y_i = \mathbf{a}(x_i)$  then  $z <_{\mathcal{K}} x_i$  and so  $z \notin [x_i]$ ,  $i = 1, 2$ . That is,  $[x_1] \cap [x_2] = \emptyset$ . Now choose an ordering on the set of elements of  $B_{i+1}$ :  $Z_1 < \dots < Z_k$  say, where  $Z_j = \mathbf{a}(x_j)$ . Then  $Z_j \setminus [x_j] \subseteq X_i$ ,  $j = 1, \dots, k$ . We can extend the total order  $\prec$  on  $X_i$  to

$$X_{i+1} = X_i \cup \cup_{j=1}^k Z_j = X_i \cup \cup_{j=1}^k [x_j]$$

as follows. Assume the order has already been extended to  $X_i \cup_{j=1}^{s-1} [x_j]$ . Extend the order further by choosing the ordering  $\prec$  on the elements of  $[x_s]$  and then defining the greatest element of  $[x_s]$  to be less than the least element of  $X_i \cup_{j=1}^{s-1} [x_j]$  (as in the last step of Example 2.10). At the final stage  $s = k$  and the order on  $X_i$  is extended to  $X_{i+1}$ . We continue until  $U_i = \mathcal{K}_X$ , at which point  $X = X_i$  and we have the required total order on  $X$ . Note that, by construction, if  $x, y \in X$  and  $x <_{\mathcal{K}} y$  then  $y \prec x$ . Also, if  $x \prec y \prec z$  and  $[z] = [x]$  then  $[y] = [x]$ . Thus 1 and 2 above hold. If  $\mathbf{a}(x)$  belongs to  $B_i$  we shall say that  $x$ ,  $\mathbf{a}(x)$  and  $[x]$  have *height*  $i$  and write  $h(x) = h(\mathbf{a}(x)) = h([x]) = i$ .

**Example 2.10.** Let  $\Gamma$  be the graph of Figure 2.2. Then

$$\begin{aligned} \mathbf{a}(a) &= \mathbf{a}(b) = \{a, b, e, f\}, \\ \mathbf{a}(c) &= \mathbf{a}(d) = \{c, d\}, \\ \mathbf{a}(e) &= \mathbf{a}(f) = \{e, f\} \text{ and} \\ \mathbf{a}(g) &= \{c, d, g\}. \end{aligned}$$

In the partial order  $<_{\mathcal{K}}$  we have



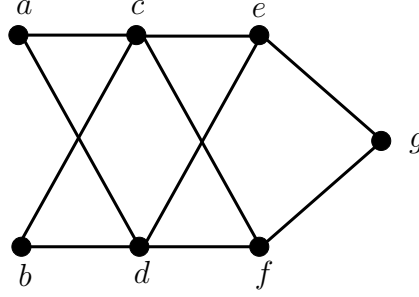


Figure 2.2

- $x <_{\mathcal{K}} a$  and  $x <_{\mathcal{K}} b$ , if  $x \in \{e, f\}$ , and
- $c <_{\mathcal{K}} g$  and  $d <_{\mathcal{K}} g$ .

Elements  $c, d, e$  and  $f$  are  $\mathcal{K}$ -minimal while  $a, b$  and  $g$  are  $\mathcal{K}$ -maximal. (For all  $x \in X$ , we have  $\text{cl}(x) = \{x\}$ , so  $a, b$  and  $g$  are  $\mathcal{L}$ -minimal but not  $\mathcal{K}$ -minimal.)

We have  $B_0 = \{\mathbf{a}(c), \mathbf{a}(e)\}$  and we must choose an order on this set: say  $\mathbf{a}(c) < \mathbf{a}(e)$ . Next choose orders on  $\mathbf{a}(c)$  and  $\mathbf{a}(e)$ : say  $d \prec c$  and  $f \prec e$ . The construction now gives the order

$$f \prec e \prec d \prec c$$

on  $\mathbf{a}(c) \cup \mathbf{a}(e)$ .

Now  $U_0 = B_0$  and  $X_0 = \mathbf{a}(c) \cup \mathbf{a}(e)$ ; so  $B_1 = \{\mathbf{a}(a), \mathbf{a}(g)\}$ . We must choose an ordering on  $B_1$ : say  $\mathbf{a}(a) < \mathbf{a}(g)$ .

$$\begin{aligned} \mathbf{a}(a) &= \{a, b\} \cup \{e, f\} \text{ and} \\ \mathbf{a}(g) &= \{g\} \cup \{e, f\}, \end{aligned}$$

where  $\{a, b\} = [a]$ ,  $\{g\} = [g]$  and  $\{e, f\} \subseteq X_0$ . We must choose an ordering on  $[a]$ : say  $b \prec a$  and then extend the order on  $X_0$  to  $X_0 \cup [a]$  by

$$b \prec a \prec f \prec e \prec d \prec c.$$

Finally extend this order to  $[g]$  to obtain

$$g \prec b \prec a \prec f \prec e \prec d \prec c.$$

### 3 Generators for $\text{Aut}(G)$ and Decomposition over Graph Automorphisms

Recall the convention of Section 2:  $G$  denotes a partially commutative group with commutation graph  $\Gamma$ , and  $X = V(\Gamma)$ .

**Notation.** We shall often abuse notation when discussing elements of  $X^{-1}$  by referring to  $x^{-1}$  as a “vertex” of  $\Gamma$ , when we really mean that  $x$  is a vertex. In particular, for  $z = x^{-1}$  we refer to  $z^\perp$  and  $\mathfrak{a}(z)$  when we mean  $x^\perp$  and  $\mathfrak{a}(x)$ , respectively.

#### 3.1 Graph Automorphisms

For any graph  $\Omega$  let  $\text{Aut}(\Omega)$  denote the group of graph automorphisms of  $\Omega$ . If  $\Omega$  is labelled then by an automorphism of  $\Omega$  we mean a graph automorphism which preserves labels. We shall use the equivalence  $\sim$  of Section 2.1 to define a quotient graph of  $\Gamma$ . Let  $u, v \in X$ . In [17] it is shown that there is an edge of  $\Gamma$  joining  $u$  to  $v$  if and only if there is an edge joining  $a$  to  $b$ , for all  $a \in [u]$  and  $b \in [v]$ . Therefore there is a well-defined graph with vertex set consisting of the equivalence classes of  $\sim$  and an edge joining vertices  $[u]$  and  $[v]$  if and only if there is an edge of  $\Gamma$  joining  $u$  and  $v$ . The resulting graph has no multiple edges but may have loops.

**Definition 3.1.** *The compression of the graph  $\Gamma$  is the labelled graph  $\Gamma^{\text{comp}}$  with vertices  $X^{\text{comp}} = \{[v] : v \in X\}$  and an edge joining  $[u]$  to  $[v]$  if and only if  $u$  is joined to  $v$  by an edge of  $\Gamma$ . Vertices of  $\Gamma^{\text{comp}}$  are labelled as follows. Let  $u \in X$  and  $|[u]| = d$ .*

1. *If  $d = 1$  then  $[u]$  is labelled  $(1, 1)$ .*
2. *If  $d > 1$  and  $[u] = [u]_\perp$  then  $[u]$  is labelled  $(\perp, d)$ .*
3. *If  $d > 1$  and  $[u] = [u]_\diamond$  then  $[u]$  is labelled  $(\diamond, d)$ .*

We shall express each automorphism  $\phi$  of  $\Gamma$  as a product  $\phi = \alpha\beta$ , where  $\alpha$  corresponds to a certain automorphism of  $\Gamma^{\text{comp}}$  and  $\beta$  is an automorphism of  $\Gamma$  which maps  $[u]$  to itself, for all  $u \in X$ . First we make some definitions. If  $\Omega$  and  $\Omega'$  are graphs without multiple edges and  $f$  is a map from  $V(\Omega)$  to  $V(\Omega')$  then we say that  $f$  induces a graph homomorphism if, for all  $u, v \in V(\Omega)$ ,  $uf$  is joined to  $vf$  whenever  $u$  is joined to  $v$ . It is easy to see (for details see [17]) that the map from  $X$  to  $X^{\text{comp}}$  sending  $u$  to  $[u]$  induces a homomorphism  $\text{comp} : \Gamma \rightarrow \Gamma^{\text{comp}}$ . Every automorphism  $\phi \in \text{Aut}(\Gamma)$

induces a label preserving automorphism  $\phi^{\text{comp}} \in \text{Aut}(\Gamma^{\text{comp}})$ : sending  $[u]$  to  $[u\phi] = [u]\phi$ . In fact the map  $\pi_{\text{comp}} : \text{Aut}(\Gamma) \rightarrow \text{Aut}(\Gamma^{\text{comp}})$ , given by  $\phi\pi_{\text{comp}} = \phi^{\text{comp}}$ , is a homomorphism (see [17]).

On the other hand we may fix a total order on the elements of  $[u]$ , for all  $u \in X$ . Then, if  $\phi^{\text{comp}}$  is an automorphism of  $\Gamma^{\text{comp}}$ , the label of  $[u]$  is identical to the label of  $[u]\phi^{\text{comp}}$ , for all  $u \in X$ . Hence there is a unique order preserving bijection from  $[u]$  to  $[u]\phi^{\text{comp}}$ . The union of these bijections is an automorphism  $\phi$  of  $\Gamma$ ; and we may define  $\iota_{\text{comp}}$  to be the map from  $\text{Aut}(\Gamma^{\text{comp}})$  to  $\text{Aut}(\Gamma)$  given by  $\phi^{\text{comp}}\iota_{\text{comp}} = \phi$ . Then  $\iota_{\text{comp}}$  is a homomorphism and  $\iota_{\text{comp}}\pi_{\text{comp}}$  is the identity map of  $\Gamma^{\text{comp}}$ .

**Definition 3.2.** Define the compressed automorphism group  $\text{Aut}_{\text{comp}}(\Gamma)$  of  $\Gamma$  to be the subgroup  $\text{Aut}(\Gamma^{\text{comp}})\iota_{\text{comp}}$  of  $\text{Aut}(\Gamma)$ .

For  $v \in X$  let  $S_{[v]}$  denote the group of permutations of  $[v]$ ; so  $S_{[v]}$  is a subgroup of  $\text{Aut}(\Gamma)$  isomorphic to the symmetric group of degree  $|[v]|$ .

The definition of  $\text{Aut}_{\text{comp}}(\Gamma)$  depends on the choice of ordering of  $[u]$ , which we regard as fixed for the remainder of the paper. We then have the following lemma.

**Lemma 3.3** (cf. [17, Proposition 2.52]).  $\text{Aut}(\Gamma) = \left( \prod_{[v] \in X^{\text{comp}}} S_{[v]} \right) \rtimes \text{Aut}_{\text{comp}}(\Gamma)$ .

Now we focus attention on automorphisms which interchange connected components of  $\Gamma$ . First of all we fix notation for these connected components. In the following definition we adopt the convention that, if  $m$  is a non-negative integer and  $\Omega$  is a graph then  $\Omega^m$  denotes the disjoint union of  $m$  copies of  $\Omega$  (and is empty if  $m = 0$ ).

**Definition 3.4.** Let  $\Omega_0$  denote the graph consisting of a single vertex and no edges. Suppose that there exist pairwise non-isomorphic graphs  $\Omega_1, \dots, \Omega_d$ , such that every connected component of  $\Gamma$  with at least two vertices is isomorphic to  $\Omega_i$ , for some  $i \geq 1$ , and that  $d$  is minimal with this property. Then

$$\Gamma \cong \Omega_0^{m_0} \cup \Omega_1^{m_1} \cup \dots \cup \Omega_d^{m_d}, \quad (3.1)$$

for some  $m_i \in \mathbb{Z}$ , with  $m_0 \geq 0$  and  $m_i \geq 1$ , for  $i \geq 1$ . In this case we say that the right hand side of (3.1) is the isomorphism type of  $\Gamma$ .

Suppose that  $\Gamma$  has isomorphism type  $\Omega_0^{m_0} \cup \Omega_1^{m_1} \cup \dots \cup \Omega_d^{m_d}$ . Identify each connected component of  $\Gamma$  with a particular copy of  $\Omega_j$  (to which it is isomorphic) in the disjoint union  $\Omega_j^{m_j}$ . To be explicit, let

$$\Gamma = \cup_{j=0}^d \cup_{k=1}^{m_j} \Gamma_{j,k},$$

where  $\Gamma_{j,k} \cong \Omega_j$ , for  $k = 1, \dots, m_j$ . Fix an isomorphism from  $\Gamma_{j,k}$  to  $\Omega_j$ , for all  $j$  and  $k$ . For  $0 \leq j \leq d$  and  $1 \leq a < b \leq m_j$  there is an isomorphism of  $\Omega_0^{m_0} \cup \Omega_1^{m_1} \cup \dots \cup \Omega_d^{m_d}$  interchanging the  $a$ th and  $b$ th copy of  $\Omega_j$  and fixing all other connected components pointwise. This induces, via the fixed isomorphisms of  $\Gamma_{j,k}$  and  $\Omega_j$ , an isomorphism  $\omega_{a,b}^j$  of  $\Gamma$ , which interchanges  $\Gamma_{j,a}$  and  $\Gamma_{j,b}$  and leaves all other components fixed. The subgroup of  $\text{Aut}(\Gamma)$  generated by  $\{\omega_{a,b}^j : 1 \leq a < b \leq m_j\}$  is then isomorphic to the symmetric group  $S_{m_j}$  of degree  $m_j$ .

**Definition 3.5.** Denote by  $\text{Aut}_{\text{symm}}(\Gamma_{j,*})$  the subgroup of  $\text{Aut}(\Gamma)$  generated by  $\{\omega_{a,b}^j : 1 \leq a < b \leq m_j\}$ . Denote by  $\text{Aut}_{\text{comp}}(\Gamma_{j,k})$  the subgroup of  $\text{Aut}_{\text{comp}}(\Gamma)$  consisting of compressed automorphisms  $\phi$  such that  $\phi|_{\Gamma_{j,k}}$  is an automorphism of  $\Gamma_{j,k}$  and  $x\phi = x$ , for all  $x \in X \setminus X_{j,k}$ .

Thus  $\text{Aut}_{\text{comp}}(\Gamma_{j,k}) \cong \text{Aut}(\Omega_j^{\text{comp}})$ . Note that  $\Gamma^{\text{comp}}$  has isomorphism type  $\Omega_0^{n_0} \cup \bigcup_{i=1}^d (\Omega_i^{\text{comp}})^{m_i}$ , where  $n_0 = 0$ , if  $m_0 = 0$ , and  $n_0 = 1$ , if  $m_0 > 0$ ; so we obtain the following decomposition.

**Lemma 3.6.** Let  $\Gamma$  have isomorphism type given by (3.1). Then

$$\text{Aut}_{\text{comp}}(\Gamma) = \prod_{j=1}^d \left( \prod_{k=1}^{m_j} \text{Aut}_{\text{comp}}(\Gamma_{j,k}) \rtimes \text{Aut}_{\text{symm}}(\Gamma_{j,*}) \right),$$

with  $\text{Aut}_{\text{symm}}(\Gamma_{j,*})$  isomorphic to the symmetric group of degree  $m_j$  and  $\text{Aut}_{\text{comp}}(\Gamma_{j,k})$  isomorphic to  $\text{Aut}(\Omega_j^{\text{comp}})$ .

Each of the above subgroups of  $\text{Aut}(\Gamma)$  is naturally isomorphic to a subgroup of the automorphism group of  $G$ ; which we shall now name.

**Definition 3.7.** Let  $\text{Aut}(G)$  be the automorphism group of the partially commutative group  $G$  with commutation graph  $\Gamma$ . An element  $\phi \in \text{Aut}(G)$  is

- a graph automorphism if the restriction  $\phi|_X$  of  $\phi$  to  $X$  is an element of  $\text{Aut}(\Gamma)$ ; and
- a compressed graph automorphism if  $\phi|_X$  is an element of  $\text{Aut}_{\text{comp}}(\Gamma)$ .
- Denote by  $\text{Aut}^\Gamma(G)$  and  $\text{Aut}_{\text{comp}}^\Gamma(G)$  the subgroups of  $\text{Aut}(G)$  consisting of graph automorphisms and compressed graph automorphisms, respectively.
- For  $v \in X$ , denote by  $S_{[v]}(G)$  the subgroup of  $\text{Aut}^\Gamma(G)$  consisting of elements  $\phi$  such that  $\phi|_X \in S_{[v]}$ .

- Denote by  $\text{Aut}_{\text{symm}}^{\Gamma}(G_{j,*})$  the subgroup of automorphisms  $\phi$  of  $\text{Aut}(G)$  such that  $\phi|_X$  is an element of  $\text{Aut}_{\text{symm}}(\Gamma_{j,*})$ ; and
- by  $\text{Aut}_{\text{comp}}^{\Gamma}(G_{j,k})$  the subgroup of automorphisms  $\phi$  such that  $\phi|_X$  is an element of  $\text{Aut}_{\text{comp}}(\Gamma_{j,k})$ .

**Remark 3.8.** The subgroup  $\prod_{j=0}^d \text{Aut}_{\text{symm}}^{\Gamma}(G_{j,*})$  of  $\text{Aut}(G)$  is denoted  $\Pi$  and called the group of permutation automorphisms in [24].

The following proposition follows from Lemmas 3.3 and 3.6.

**Proposition 3.9.** Let  $\Gamma$  have isomorphism type given by (3.1). Then

- (i)  $\text{Aut}^{\Gamma}(G) = (\prod_{[v] \in X^{\text{comp}}} S_{[v]}(G)) \rtimes \text{Aut}_{\text{comp}}^{\Gamma}(G)$ , with  $S_{[v]}(G)$  isomorphic to the symmetric group of degree  $|[v]|$ , and

(ii)

$$\text{Aut}_{\text{comp}}^{\Gamma}(G) = \prod_{j=1}^d \left( \prod_{k=1}^{m_j} \text{Aut}_{\text{comp}}^{\Gamma}(G_{j,k}) \rtimes \text{Aut}_{\text{symm}}^{\Gamma}(G_{j,*}) \right).$$

Moreover  $\text{Aut}_{\text{symm}}^{\Gamma}(G_{j,*})$  is isomorphic to the symmetric group of degree  $m_j$  and  $\text{Aut}_{\text{comp}}^{\Gamma}(G_{j,k})$  is isomorphic to  $\text{Aut}_{\text{comp}}^{\Omega_j}(G(\Omega_j))$ .

In the sequel we shall the particular generators for  $\text{Aut}_{\text{comp}}^{\Gamma}(G)$  of the next definition.

**Definition 3.10.** Define the following sets of graph automorphisms.

- (a) For  $1 \leq j \leq d$ , let  $\mathcal{P}_{\text{comp},j}^{\Gamma}$  be a generating set for  $\text{Aut}_{\text{comp}}^{\Gamma}(G_{j,1})$ .
- (b) For  $0 \leq j \leq d$  and  $1 \leq a < b \leq m_j$ , the group automorphism induced by the graph automorphism  $\omega_{a,b}^j$  (defined above) is also called  $\omega_{a,b}^j$  and we define

$$\mathcal{P}_{\text{symm},j}^{\Gamma} = \{\omega_{a,b}^j | 1 \leq a < b \leq m_j\} \subseteq \text{Aut}^{\Gamma}(G).$$

- (c)  $\mathcal{P}_{\text{comp}}^{\Gamma}(G) = \mathcal{P}_{\text{symm},0}^{\Gamma} \cup \bigcup_{j=1}^d (\mathcal{P}_{\text{comp},j}^{\Gamma} \cup \mathcal{P}_{\text{symm},j}^{\Gamma})$ .

When the meaning is clear we write  $\mathcal{P}_{\text{comp}}^{\Gamma}$  instead of  $\mathcal{P}_{\text{comp}}^{\Gamma}(G)$ .

**Lemma 3.11.** (i)  $\text{Aut}_{\text{symm}}^{\Gamma}(G_{j,*})$  is generated by  $\mathcal{P}_{\text{symm},j}^{\Gamma}$ .

- (ii)  $\text{Aut}_{\text{comp}}^{\Gamma}(G)$  is generated by  $\mathcal{P}_{\text{comp}}^{\Gamma}(G)$ .

*Proof.* The lemma follows directly from the definitions and Proposition 3.9(ii).  $\square$

Clearly a presentation for  $\text{Aut}_{\text{comp}}^{\Gamma}(G)$  may be constructed from the decomposition of Proposition 3.9, using the generators  $\mathcal{P}_{\text{comp}}^{\Gamma}(G)$ , but we leave details to the appendix.

### 3.2 Generators for $\text{Aut}(G)$

**Definition 3.12.** Given  $x \in X$ , the automorphism of  $G$  mapping  $x$  to  $x^{-1}$  and fixing all other generators is called an inversion and denoted  $\iota_x$ . The set of all inversions is denoted  $\text{Inv} = \text{Inv}(G)$ .

**Definition 3.13.** For fixed  $x, y \in X^{\pm 1}$  an automorphism sending  $x$  to  $xy$  and fixing all elements of  $X^{\pm 1}$  other than  $x^{\pm 1}$  is denoted  $\tau_{x,y}$  and called a transvection. The set of all transvections  $\tau_{x,y}$  such that  $x \in X^{\pm 1}$  and  $y \in X$  is denoted  $\text{Tr} = \text{Tr}(G)$ .

For distinct  $x, y \in X$ , there exists an element  $\tau_{x^\varepsilon, y^\delta} \in \text{Aut}(G)$  if and only if  $x^\perp \setminus x \subseteq y^\perp$ .

**Definition 3.14.** Let  $\phi \in \text{Aut}(G)$  be an automorphism and assume that, for all  $x \in X$ , there exists  $g_x \in G$  such that  $x\phi = x^{g_x}$ . Then  $\phi$  is called a basis-conjugating automorphism. The subgroup of  $\text{Aut}(G)$  consisting of all basis-conjugating automorphisms is denoted  $\text{Conj}(G)$ .

The group of inner automorphisms  $\text{Inn}(G)$  is a normal subgroup of  $\text{Conj}(G)$ .

**Definition 3.15.** For  $S \subseteq X$  define  $\Gamma_S$  to be  $\Gamma \setminus S$ , the graph obtained from  $\Gamma$  by removing all vertices of  $S$  and all their incident edges.

**Definition 3.16.** Let  $x \in X$ , let  $C$  be the vertex set of a connected component of  $\Gamma_{x^\perp}$  and let  $\varepsilon = \pm 1$ . The automorphism  $\alpha_{C, x^\varepsilon}$  given by

$$y \mapsto \begin{cases} y^{x^\varepsilon}, & \text{if } y \in C \\ y, & \text{otherwise} \end{cases}$$

is called an elementary conjugating automorphism of  $\Gamma$ .

The set of all elementary conjugating automorphisms (over all connected components of  $\Gamma_{x^\perp}$  and all  $x \in X$ ) is denoted  $\text{LInn} = \text{LInn}(G)$ .

**Theorem 3.17** (Laurence [29]). The group of basis-conjugating automorphisms  $\text{Conj}(G)$  is generated by the set  $\text{LInn}(G)$ .

In [29] it is shown that  $\text{Aut}(G)$  is generated by the following automorphisms.

- (i) A fixed choice  $\mathcal{P}^\Gamma$  of generators for the graph automorphisms  $\text{Aut}^\Gamma(G)$ .
- (ii) The set of inversions  $\text{Inv}$ .
- (iii) The set of transvections  $\text{Tr}$ .

(iv) The set of elementary conjugating automorphisms  $\text{LInn}$ .

We shall construct various decompositions of  $\text{Aut}(G)$  and in view of these decompositions we shall reduce to proper generating subsets of Laurence's generators. The first such reduction is the following.

**Proposition 3.18.**  *$\text{Aut}(G)$  is generated by  $\text{Inv} \cup \text{Tr} \cup \text{LInn} \cup \mathcal{P}_{\text{comp}}^\Gamma$ .*

*Proof.* To see that these automorphisms generate  $\text{Aut}(G)$ , it suffices, using Lemma 3.11(ii), to show that every automorphism  $\phi \in \text{Aut}^\Gamma(G)$  belongs to the subgroup generated by  $\text{Inv}$ ,  $\text{Tr}$  and  $\text{Aut}_{\text{comp}}^\Gamma(G)$ . From Proposition 3.9,  $\phi$  may be written as  $\phi = \alpha\beta$ , where  $\alpha \in \prod_{[v] \in X^{\text{comp}}} S_{[v]}(G)$  and  $\beta \in \text{Aut}_{\text{comp}}^\Gamma(G)$ . Hence it is enough to show that  $S_{[v]}(G) \subseteq \langle \text{Inv}, \text{Tr} \rangle$ . As  $[v]$  generates a free or free Abelian subgroup of  $G$ , for all  $x, y \in [v]$  and  $\varepsilon = \pm 1$ , the transvections  $\tau_{x^\varepsilon, y}$  and inversions  $\iota_x$  and  $\iota_y$  are automorphisms of  $G$  and belong to  $\text{Inv} \cup \text{Tr}$ . The permutation  $\sigma_{x, y}$  sending  $x$  to  $y$  and  $y$  to  $x$  and fixing all other generators can be obtained as a word in these generators;  $\sigma_{x, y} = \iota_x \tau_{x, y}^{-1} \tau_{y, x} \tau_{x^{-1}, y}$ . As  $S_{[v]}(G)$  is generated by such elements it follows that  $S_{[v]}(G) \subseteq \langle \text{Inv}, \text{Tr} \rangle$  as required. □

### 3.3 Decomposition of $\text{Aut}(G)$ over Graph Automorphisms

**Definition 3.19.** *Let  $\text{Aut}^*(G)$  denote the subgroup of  $\text{Aut}(G)$  generated by the set  $\mathcal{P}_* = \text{Inv} \cup \text{Tr} \cup \text{LInn}$ .*

Later we shall show that  $\text{Aut}^*(G)$  has a natural description in terms of the stabiliser of the lattice  $\mathcal{K}$ . Here we establish what we need in order to obtain an initial decomposition of  $\text{Aut}(G)$  in terms of  $\text{Aut}_{\text{comp}}^\Gamma(G)$ . It is useful to establish the following fact first.

**Lemma 3.20.** *Let  $x, y \in X$  and let  $C$  be a connected component of  $\Gamma_{y^\perp}$ . If  $\mathfrak{a}(x) \not\subseteq C \cup y^\perp$  and  $\mathfrak{a}(x) \cap C \neq \emptyset$  then  $y \in \mathfrak{a}(x)$ .*

*Proof.* Suppose that  $y \notin \mathfrak{a}(x)$ . Then there exists  $u \in x^\perp \setminus x$  such that  $[y, u] \neq 1$ . Thus  $u \notin y^\perp$  and  $u \in x^\perp \setminus x$ ; so  $[u, v] = 1$  for all  $v \in \mathfrak{a}(x)$ . This means that  $\mathfrak{a}(x) \setminus y^\perp$  is contained in some connected component of  $\Gamma_{y^\perp}$ . If  $\mathfrak{a}(x) \cap C \neq \emptyset$  it follows that  $\mathfrak{a}(x) \subseteq C \cup y^\perp$ , and the result follows. □

**Proposition 3.21.** *Let  $\phi \in \text{Aut}^*(G)$ . Then, for all  $x \in X$ , there exists  $f_x \in G$  such that  $G(\mathfrak{a}(x))\phi = G(\mathfrak{a}(x))^{f_x}$ .*

*Proof.* It suffices to prove the statement in the case where  $\phi$  is a generator of  $\text{Aut}^*(G)$ . First consider the case where  $\phi$  is a elementary conjugating automorphism.

Suppose then that  $y \in X$ ,  $C$  is a connected component of  $\Gamma_{y^\perp}$  and that  $\phi = \alpha_{C,y}$ . Now let  $x \in X$ . If  $C \cap \mathfrak{a}(x) = \emptyset$  then  $G(\mathfrak{a}(x))\phi = G(\mathfrak{a}(x))$ , so we may assume that  $C \cap \mathfrak{a}(x) \neq \emptyset$ . If  $\mathfrak{a}(x) \subseteq C \cup y^\perp$  then  $G(\mathfrak{a}(x))\phi = G(\mathfrak{a}(x))^y$ , as required. This leaves the case where  $C \cap \mathfrak{a}(x) \neq \emptyset$  and  $\mathfrak{a}(x) \not\subseteq C \cup y^\perp$ . In this case  $y \in \mathfrak{a}(x)$ , from Lemma 3.20, and  $z\phi = z$  or  $z\phi = z^y$ , for all  $z \in \mathfrak{a}(x)$ , so  $G(\mathfrak{a}(x))\phi = G(\mathfrak{a}(x))$ .

The case where  $\phi \in \text{Inv}$  is straightforward (and  $f_x = 1$ , for all  $x$ , in this case). Suppose then that  $\phi$  is a transvection; more precisely let  $y, z \in X$  with  $y^\perp \setminus y \subseteq z^\perp$  and  $\phi = \tau_{y^\varepsilon, z}$ , where  $\varepsilon \in \{\pm 1\}$ . Let  $x \in X$ . If  $y \notin \mathfrak{a}(x)$  then  $\phi$  is the identity on  $G(\mathfrak{a}(x))$  so we may assume that  $y \in \mathfrak{a}(x)$ . In this case we have  $z \in \text{cl}(z) \subseteq \mathfrak{a}(y) \subseteq \mathfrak{a}(x)$ . Hence  $G(\mathfrak{a}(x))\phi = G(\mathfrak{a}(x))$ , as required.  $\square$

**Remark 3.22.** Note that the proof of this proposition shows that if  $\phi$  is in the subgroup of  $\text{Aut}^*(G)$  generated by  $\text{Inv}$  and  $\text{Tr}$  then  $G(\mathfrak{a}(x))\phi = G(\mathfrak{a}(x))$ , for all  $x \in X$ .

**Proposition 3.23.**  *$\text{Aut}^*(G)$  is a normal subgroup of  $\text{Aut}(G)$  and the latter decomposes as a semi-direct product  $\text{Aut}(G) \cong \text{Aut}^*(G) \rtimes \text{Aut}_{\text{comp}}^\Gamma(G)$ .*

*Proof.* To see that  $\text{Aut}^*(G)$  is normal in  $\text{Aut}(G)$  it is only necessary to check that  $\theta^{-1}\phi\theta \in \text{Aut}^*(G)$  where  $\theta \in \text{Aut}^\Gamma(G)$  and  $\phi$  is a generator of  $\text{Aut}^*(G)$ . It is straightforward to check from the definitions that if  $\theta \in \text{Aut}^\Gamma(G)$  then  $\theta$  acts by conjugation on the generators of  $\text{Aut}^*(G)$  as follows. If  $\iota_z \in \text{Inv}$  then  $\iota_z^\theta = \iota_{z\theta}$ . If  $x \in X^{\pm 1}$ ,  $y \in Y$  and  $\tau_{x,y} \in \text{Tr}$  then  $\tau_{x,y}^\theta = \tau_{x\theta, y\theta}$ . If  $\alpha_{C,y}$  is an elementary conjugating automorphism then, since  $\theta$  restricts to a graph automorphism of  $\Gamma$ , it follows that  $C\theta$  is a connected component of  $\Gamma_{(y\theta)^\perp}$ . Furthermore  $\alpha_{C,y}^\theta = \alpha_{C\theta, y\theta}$ . Therefore  $\text{Aut}^*(G)$  is normal.

Next we show that  $\text{Aut}^*(G) \cap \langle \mathcal{P}_{\text{comp}}^\Gamma \rangle = \{1\}$ . From Proposition 3.21, for all  $x \in X$  and  $\phi \in \text{Aut}^*(G)$ , we have  $x\phi = w^g$ , for some  $w \in G(\mathfrak{a}(x))$  and  $g \in G$ . This means that the exponent sum of  $y \in X$  in  $x\phi$  is zero unless  $y \in \mathfrak{a}(x)$ .

We claim that if  $\theta \in \text{Aut}^\Gamma(G)$  then, for all  $x \in X$ ,  $x\theta = y$  and  $\mathfrak{a}(x)\theta = \mathfrak{a}(y)\theta$ , for some  $y \in X$  with  $h(y) = h(x)$ . To see this note that  $\theta \in \text{Aut}^\Gamma(G)$  implies  $\theta|_X$  is in  $\text{Aut}(\Gamma)$  so  $\mathfrak{a}(x)\theta = \mathfrak{a}(x\theta)$ . If  $x\theta = y$  then it follows from Lemma 2.6 and induction on the height  $h(x)$  of  $x$  that  $h(y) = h(x)$ , and the claim follows.

Now if  $\theta$  is a non-trivial element of  $\text{Aut}_{\text{comp}}^\Gamma(G)$  there is  $x \in X$  such that  $[x]\theta = [y]$  with  $[y] \neq [x]$ . Without loss of generality we may assume  $x\theta = y$ ; so the exponent sum of  $y$  in  $x\theta$  is non-zero. As  $h(x) = h(y)$  it follows



from Lemma 2.6 that  $[y] \cap \mathfrak{a}(x) = \emptyset$  and so  $y \notin \mathfrak{a}(x)$ . Combining this with the above we have  $\text{Aut}^*(G) \cap \text{Aut}_{\text{comp}}^\Gamma(G) = 1$ . As  $\text{Aut}(G)$  is generated by  $\text{Inv} \cup \text{Tr} \cup \text{LInn} \cup \mathcal{P}_{\text{comp}}^\Gamma$  it follows not only that  $\text{Aut}^*(G)$  is normal but that  $\text{Aut}(G)$  decomposes as a semi-direct product, as claimed.  $\square$

### 3.4 Decomposition over Connected Components

In this section we use the theory of automorphisms of free products developed in [22, 23], [24] and [12] to give a presentation of  $\text{Aut}(G)$ , and describe its structure, in terms of automorphisms of the groups corresponding to the connected components of  $\Gamma$ . A presentation for the automorphism group of a free product in terms of presentations for automorphism groups of the factors is given in [22, 23] and reformulated in [24]. Using the latter we construct a presentation for  $\text{Aut}(G)$ , in terms of presentations of automorphism groups of factors of  $G$ , appropriate to our particular setting.

**Definition 3.24.** *Let  $\Gamma$  have isomorphism type given by (3.1) and, as in Section 3.1, let  $\Gamma_{i,j}$  be the connected components of  $\Gamma$ , where  $\Gamma_{i,j} \cong \Omega_i$ , for  $1 \leq j \leq m_i$  and  $0 \leq i \leq d$ . Let*

- $X_{i,j}$  be the vertex set of  $\Gamma_{i,j}$ ,
- let  $S = \{(0, j) : 1 \leq j \leq m_0\}$ ,
- let  $X_S = \cup_{s \in S} X_s$ , the set of isolated vertices of  $\Gamma$ , and
- let  $J = \{(i, j) : 1 \leq i \leq d, 1 \leq j \leq m_i\}$ .

(i) *Define the following sets of automorphisms which preserve subgroups generated by connected components of  $\Gamma$  and fix all elements of  $X_{i,j}$  when  $j \neq 1$ : let*

- (a)  $\text{Inv}_{\text{int}}(G) = \{\iota_x \in \text{Inv} \mid x \in X_{i,1}, 0 \leq i \leq d\}$ ;
- (b)  $\text{Tr}_{\text{int}}(G) = \{\tau_{x,y} \in \text{Tr} \mid x \in X_{i,1}^{\pm 1}, y \in X_{i,1}^{\pm 1}, 1 \leq i \leq d\}$ ;
- (c)  $\text{LInn}_{\text{int}}(G) = \{\alpha_{C,x} \in \text{LInn} \mid x \in X_{i,1}^{\pm 1}, C \subseteq X_{i,1}, 1 \leq i \leq d\}$ ;
- (d)  $\mathcal{P}_{\text{int}}(G) = \text{Inv}_{\text{int}}(G) \cup \text{Tr}_{\text{int}}(G) \cup \text{LInn}_{\text{int}}(G)$ .

(ii) *Define the following sets of automorphisms which do not preserve subgroups generated by connected components of  $\Gamma$ . Let*

- (a)  $\text{Tr}_{\text{ext}}(G) = \{\tau_{x,y} \in \text{Tr} \mid x \in X_S^{\pm 1}, y \in X^{\pm 1}\}$ ;
- (b)  $\text{LInn}_{\text{ext}}(G) = \{\alpha_{C,y} \in \text{LInn} \mid C = X_j, y \in X_k^{\pm 1}, j \in J, k \in S \cup J, k \neq j\}$ ;

$$(c) \mathcal{P}_{\text{ext}}(G) = \text{Tr}_{\text{ext}}(G) \cup \text{LInn}_{\text{ext}}(G).$$

Finally define  $\mathcal{P}(G) = \mathcal{P}_{\text{comp}}^\Gamma(G) \cup \mathcal{P}_{\text{int}}(G) \cup \mathcal{P}_{\text{ext}}(G)$ .

When the group  $G$  in question is clear from the context we often drop the argument  $G$  from these definitions, writing  $\mathcal{P}$  for  $\mathcal{P}(G)$ , and so on.

- Remark 3.25.** 1. The conditions on  $\text{Tr}$  imply that  $\text{Tr}_{\text{ext}}$  is empty unless there exists an isolated vertex  $x$  of  $\Gamma$ , in which case there is an automorphism  $\tau_{x,y} \in \text{Tr}_{\text{ext}}$ , for all  $y \in X \setminus \{x\}$ .
2. If  $s \in S$  and  $X_s = \{x\}$ ,  $z \in X^{\pm 1}$ ,  $z \neq x^{\pm 1}$ , then  $\text{Aut}(G)$  contains the automorphism  $\alpha_{X_s,z}$ , but also contains  $\tau_{x^{\pm 1},z}$ ; and  $\alpha_{X_s,z} = \tau_{x,z}\tau_{x^{-1},z}$ . Hence we make the restriction  $j \in J$ ; i.e.  $|X_j| \geq 2$ , in Definition 3.24 (ii)b above.
3. In [24] elements of  $\text{Tr}_{\text{ext}} \cup \text{LInn}_{\text{ext}}$  are called Whitehead automorphisms.

**Proposition 3.26.** *The set  $\mathcal{P}$  generates  $\text{Aut}(G)$ .*

*Proof.* In the light of Proposition 3.18, it suffices to show that every automorphism in  $\text{Inv} \cup \text{Tr} \cup \text{LInn}$  belongs to the subgroup generated by  $\mathcal{P}$ . For all  $i, j$  with  $1 \leq i \leq d$  and  $1 < j \leq m_i$  the automorphism  $\omega_{1,j}^i$  belongs to  $\mathcal{P}_{\text{symm},i}^\Gamma \subseteq \mathcal{P}_{\text{comp}}^\Gamma$  and for all  $x_j, y_j \in X_{i,j}$  there are  $x_1, y_1 \in X_{i,1}$  such that  $x_1 = x_j \omega_{1,j}^i$  and  $y_1 = y_j \omega_{1,j}^i$ . Then we have  $\iota_{x_j} = (\omega_{1,j}^i)^{-1} \iota_{x_1} \omega_{1,j}^i$ ,  $\tau_{x_j^\varepsilon, y_j} = (\omega_{1,j}^i)^{-1} \tau_{x_1^\varepsilon, y_1} \omega_{1,j}^i$  and  $\alpha_{C_j, x_j} = (\omega_{1,j}^i)^{-1} \alpha_{C_1, y_1} \omega_{1,j}^i$ , where  $C_j = C_1 \omega_{1,j}^i$ . As  $\iota_{x_1}$ ,  $\tau_{x_1^\varepsilon, y_1}$  and  $\alpha_{C_1, y_1}$  are all elements of  $\mathcal{P}$  it follows that  $\text{Inv} \cup \text{Tr} \cup \text{LInn}$  is contained in the subgroup generated by  $\mathcal{P}$ , as required.  $\square$

We extend the notation of Definition 3.7, for graph automorphisms of the factors of  $G$ , to cover all automorphisms of the factors.

**Definition 3.27.** *For  $i = 0, \dots, d$ , let  $\text{Aut}(G_{i,1})$  denote the subgroup of  $\text{Aut}(G)$  consisting of elements  $\phi \in \text{Aut}(G)$  such that  $x\phi = x$ , for all  $x \in X \setminus X_{i,1}$  and  $G(\Gamma_{i,1})\phi \subseteq G(\Gamma_{i,1})$ .*

Before choosing generators and relators for  $\text{Aut}(G)$  note that

$$\text{Aut}_{\text{comp}}^\Gamma(G_{i,1}) \leq \text{Aut}(G_{i,1}) \cong \text{Aut}(G(\Omega_i))$$

and that, from Proposition 3.18,  $\text{Aut}(G_{i,1})$  is generated by  $(\mathcal{P}_{\text{int}} \cap \text{Aut}(G_{i,1})) \cup \mathcal{P}_{\text{comp},i}^\Gamma$ .

**Definition 3.28.** Choose presentations

$$\langle \mathcal{P}_{\text{symm},i}^\Gamma | \mathcal{R}_{\text{symm},i}^\Gamma \rangle \text{ for } \text{Aut}_{\text{symm}}^\Gamma(G_{i,*}), 0 \leq i \leq d, \text{ and}$$

$$\langle \mathcal{P}_{\text{comp},i}^\Gamma | \mathcal{R}_{\text{comp},i}^\Gamma \rangle \text{ for } \text{Aut}_{\text{comp}}^\Gamma(G_{i,1}), 1 \leq i \leq d.$$

(For notational convenience set  $\mathcal{P}_{\text{comp},0}^\Gamma = \mathcal{R}_{\text{comp},0}^\Gamma = \emptyset$ .) For  $0 \leq i \leq d$ , let

$$\mathcal{P}_i = (\mathcal{P}_{\text{int}} \cap \text{Aut}(G_{i,1})) \cup \mathcal{P}_{\text{comp},i}^\Gamma,$$

so  $\mathcal{P}_i$  is a set of generators for  $\text{Aut}(G_{i,1})$ . Choose a presentation  $\langle \mathcal{P}_i | \mathcal{R}_i \rangle$  for  $\text{Aut}(G_{i,1})$  such that  $\mathcal{R}_i \supseteq \mathcal{R}_{\text{comp},i}^\Gamma$ , the relators chosen (in the appendix) for  $\text{Aut}_{\text{comp}}^\Gamma(G_{i,1})$ .

**Proposition 3.29.**  $\text{Aut}(G)$  has a presentation  $\langle \mathcal{P} | \mathcal{R} \rangle$ , where  $\mathcal{P}$  is given in Definition 3.24 and  $\mathcal{R}$  is defined in Definition 3.30 below.

**Definition 3.30.** Let  $\mathcal{R}$  be the union of the following sets.

(i)  $\mathcal{R}_{\text{symm},i}^\Gamma$ , for  $i = 0, \dots, d$ .

(ii)  $\mathcal{R}_i$ , for  $i = 0, \dots, d$ .

(iii) The sets

$$\begin{aligned} \mathcal{W}_j &= \{[\omega_{a,b}^j, p] : p \in \mathcal{P}_j, 2 \leq a < b \leq m_j\} \\ &\cup \{[p, \omega_{1,a}^j q \omega_{1,a}^j] : p, q \in \mathcal{P}_j, 2 \leq a \leq m_j\} \\ &\cup \{[\omega_{1,a}^j p \omega_{1,a}^j, \omega_{1,b}^j q \omega_{1,b}^j] : p, q \in \mathcal{P}_j, 2 \leq a < b \leq m_j\}, \end{aligned}$$

for  $j = 0, \dots, d$

(iv)  $\mathcal{D} = \{[p, q] | p \in \mathcal{P}_i \cup \mathcal{P}_{\text{symm},i}^\Gamma, q \in \mathcal{P}_j \cup \mathcal{P}_{\text{symm},j}^\Gamma, \text{ with } 0 \leq i < j \leq d\}$ .

(v) The set of relations  $\mathcal{R}_i$ , for  $i=1, \dots, 11$ , below.

Note that if  $\tau_{x,y} \in \text{Tr}_{\text{ext}}$  then necessarily  $x \in X_i^{\pm 1}$ , for some  $i \in S$  and  $y \in X_j^{\pm 1}$ , where  $j \in S \cup J$  with  $j \neq i$ . Similarly, if  $\alpha_{C,x} \in \text{LInn}_{\text{ext}}$  then  $x \in X_i^{\pm 1}$ , for some  $i \in S \cup J$  and  $C = \Gamma_j$ , where  $j \in J$ , with  $j \neq i$ . In the relations below all the transvections  $\tau_{\cdot, \cdot}$  and elementary conjugating automorphisms  $\alpha_{\cdot, \cdot}$ , that are mentioned explicitly, belong to  $\text{Tr}_{\text{ext}}$  or  $\text{LInn}_{\text{ext}}$ , respectively. The relations are defined for all  $u, v, x, y, z \in X^{\pm 1}$  and  $i, j, k, l \in S \cup J$ , for which the preceding conditions hold. To ease the description of conditions placed on such automorphisms, for  $a \in \cup_{j \in J} G(\Gamma_j) \cup X_S^{\pm 1}$ , we define

$$\breve{a} = \begin{cases} j & \text{if } a \in G(\Gamma_j) \\ x & \text{if } a = x^\varepsilon, \text{ where } x \in X_S, \varepsilon = \pm 1 \end{cases}.$$

For  $y \in X_j^{\pm 1}$ , where  $j \in J$ , we write  $\gamma_y(j)$  for the automorphism conjugating every element of  $X_j$  by  $y$  and fixing all elements of  $X \setminus X_j$ . That is

$$x\gamma_y(j) = \begin{cases} x^y & \text{if } x \in X_j \\ x & \text{if } x \in X \setminus X_j \end{cases},$$

so  $\gamma_y(j)$  is equal to the product over all connected components  $C$  of  $(\Gamma_j)_{y^\perp}$  of the automorphisms  $\alpha_{C,y}$ .

$\mathcal{R}1.$   $[\tau_{x,y}, \tau_{u,v}] = 1$ , if either

- (i)  $u = x^{-1}$  or
- (ii)  $\check{x} \neq \check{u}$ ,  $\check{x} \neq \check{v}$  and  $\check{y} \neq \check{u}$ .

$\mathcal{R}2.$   $[\tau_{x,y}^{-1}, \tau_{u,x}^{-1}] = \tau_{u,y}^{-1}$ , if  $\check{x} \neq \check{u}$  and  $\check{y} \neq \check{u}$ .

$\mathcal{R}3.$   $\tau_{x,y}^{-1}\tau_{y,x}\tau_{x^{-1},y} = \omega_{i,j}^0\omega_{1,j}^0\iota_z\omega_{1,j}^0$ , where  $x \in X_{0,i}^{\pm 1} \subseteq X_S^{\pm 1}$  and  $y \in X_{0,j}^{\pm 1} \subseteq X_S^{\pm 1}$ ,  $i \neq j$  and  $X_{0,1} = \{z\}$ . (If  $j = 1$  the right hand side of this relation is replaced by  $\omega_{i,1}^0\iota_z$ .)

$\mathcal{R}4.$   $[\alpha_{X_i,x}, \alpha_{X_j,y}] = 1$ , if  $\check{x}, \check{y} \notin \{i, j\}$ ,  $i \neq j$  and  $i, j \in J$ .

$\mathcal{R}5.$   $[\alpha_{X_j,x}, \alpha_{X_i,y}\alpha_{X_j,y}] = 1$ , if  $i \neq j$  and  $\check{x} = i$ .

$\mathcal{R}6.$   $[\tau_{x,y}, \alpha_{X_l,z}] = 1$ , if  $\check{y} \neq l$  and  $\check{x} \neq \check{z}$ .

$\mathcal{R}7.$   $[\tau_{x,y}^{-1}, \alpha_{X_l,x}^{-1}] = \alpha_{X_l,y}^{-1}$ , if  $\check{y} \neq l$ .

$\mathcal{R}8.$   $[\tau_{x,y}, \alpha_{X_i,z}\tau_{x,z}] = 1$ , if  $\check{y} = i$ .

$\mathcal{R}9.$   $\tau_{x,y}\alpha_{X_i,x} = \alpha_{X_i,x}\tau_{x^{-1},y}^{-1}\gamma_y(i)^{-1}$ , if  $\check{y} = i$ .

$\mathcal{R}10.$  Let  $x \in X_S^{\pm 1}$ ,  $y, z \in X^{\pm 1}$ ,  $i \in S \cup J$  be such that  $\check{y} = \check{z} = i$ , with  $\nu(y) \cap \nu(z) = \emptyset$  and  $[y, z] = 1$ . Let  $u \in X$  and  $j \in J$ , where  $i \neq j$ . Then

- (i)  $\tau_{x,u}^{-1} = \tau_{x,u^{-1}}$ ;
- (ii)  $[\tau_{x,y}, \tau_{x,z}] = 1$ ;
- (iii)  $\alpha_{X_j,u}^{-1} = \alpha_{X_j,u^{-1}}$  and
- (iv)  $[\alpha_{X_j,y}, \alpha_{X_j,z}] = 1$ .

$\mathcal{R}11.$  Let  $y \in X^{\pm 1}$  and  $\theta \in \mathcal{P}_{\text{comp}}^\Gamma \cup \mathcal{P}_{\text{int}}$  and let  $y_1 \cdots y_k$  be a word representing  $y\theta$ , with  $y_i \in X^{\pm 1}$ .

(i) Let  $x, z \in X_S^{\pm 1}$  such that  $z = x\theta$ . Then

$$\tau_{x,y}\theta = \theta\tau_{z,y_k} \cdots \tau_{z,y_1},$$

for all  $\tau_{x,y} \in \text{Tr}_{\text{ext}}$ .

(ii) Let  $i, j \in J$  and  $\Gamma_j\theta = \Gamma_i$ . If  $\check{y} \neq j$  then, with  $C = V(\Gamma_j)$  and  $D = V(\Gamma_i)$ ,

$$\alpha_{C,y}\theta = \theta\alpha_{D,y_k}^{\varepsilon_1} \cdots \alpha_{D,y_1}^{\varepsilon_1},$$

for all  $\alpha_{C,y} \in \text{LInn}_{\text{ext}}$ .

The proof of this theorem is left to the appendix.

In the case where  $m_0 = 0$ , that is, no component of  $\Gamma$  is an isolated vertex, the set  $\text{Tr}_{\text{ext}}$  is empty and the relations of this presentation reduce to the union of the sets  $\cup_{i=1}^d \mathcal{R}_{\text{symm},i}$ ,  $\cup_{i=1}^d \mathcal{R}_i$ ,  $\mathcal{W}$  and  $\mathcal{D}$ , given in (i)–(iv) of Definition 3.30, together with  $\mathcal{R}4$ ,  $\mathcal{R}5$ ,  $\mathcal{R}10(\text{iii})$ , (iv) and  $\mathcal{R}11(\text{ii})$ . In this case  $\text{Aut}(G)$  decomposes as a semi-direct product  $\text{Aut}(G) = \langle \text{LInn}_{\text{ext}} \rangle \rtimes \langle \mathcal{P}_{\text{comp}}^\Gamma \cup \mathcal{P}_{\text{int}} \rangle$ ; and  $\langle \text{LInn}_{\text{ext}} \rangle$  is called the Fouxé-Rabinovitch kernel and denoted  $\text{FR}(G)$  (see [12] for more details). The structure of  $\text{Aut}(G)$  is then given by the following (special case of a) theorem from [12].

**Theorem 3.31** (*cf.* [12], Theorem C). *Suppose that no component of  $\Gamma$  is an isolated vertex. Define  $\bar{G} = G_1 \times \cdots \times G_n$  and  $\text{FR}(G) = \langle \text{LInn}_{\text{ext}} \rangle$ . Then  $\text{FR}(G)$  is the kernel of the canonical map from  $\text{Aut}(G)$  to  $\text{Aut}(\bar{G})$ . Moreover  $\text{FR}(G)$  has a normal series*

$$1 < P_{n-1} < \cdots < P_2 < \text{FR}(G)$$

*such that, setting  $\text{FR}_i(G) = \text{FR}(G)/P_i$ ,*

$$(i) \text{FR}(G) = P_i \rtimes \text{FR}_i(G),$$

$$(ii) \text{FR}_i(G) = \text{FR}(G_1 * \cdots * G_i) \text{ and}$$

*(iii) all the  $P_i$  are finitely generated.*

In the light of the results of this section we may when necessary reduce to the study of  $\text{Aut}^\Gamma(G)$  where  $\Gamma$  is a connected graph. In particular, to give an explicit presentation of  $\text{Aut}(G)$  it remains to determine the sets  $\mathcal{R}_i$  of Definition 3.28.

### 3.5 Conjugating Automorphisms

The subgroup of basis-conjugating automorphisms, which we consider here, plays an important role in the structure of  $\text{Aut}(G)$  and has a rich and complex structure, even in the case of free groups: see for example [31, 26, 34, 1]. We shall consider several subgroups of the basis-conjugating automorphisms  $\text{Conj}(G) = \langle \text{LInn}(G) \rangle$  which we now define.

Let  $x \in X$  and, as usual, denote by  $\Gamma_x$  the full subgraph of  $\Gamma$  generated by  $X \setminus \{x\}$  and note that if  $y \in X$  lies in a connected component  $C$  of  $\Gamma_x$  then  $y^\perp \subseteq C \cup \{x\}$ .

**Definition 3.32.** Let  $x \in X$  and let  $C$  be a connected component of  $\Gamma_x$ . Then the automorphism  $\beta_{C,x}$  given by

$$y\beta_{C,x} = \begin{cases} y^x, & \text{if } y \in C \\ y, & \text{otherwise} \end{cases}$$

is called an aggregate conjugating automorphism. The subgroup of  $\text{Conj}(G)$  generated by all aggregate conjugating automorphisms is denoted  $\text{Conj}_A(G)$ .

**Definition 3.33.** An element  $\phi \in \text{Conj}(G)$  is said to be a normal conjugating automorphism if, for every element  $x \in X$ , there exists  $f_x \in G$  such that  $y\phi = y^{f_x}$ , for all  $y \in \mathfrak{a}(x)$ . The subgroup of all normal conjugating automorphisms is denoted  $\text{Conj}_N(G)$ .

**Definition 3.34.** An element  $\phi \in \text{Conj}(G)$  is said to be a vertex conjugating automorphism if, for every element  $x \in X$  there exists  $f_x \in G$  such that  $y\phi = y^{f_x}$ , for all  $y \in [x]$ . The subgroup of all vertex conjugating automorphisms is denoted  $\text{Conj}_V(G)$ .

If  $\Gamma$  is compressed ( $\Gamma = \Gamma^{\text{comp}}$ ) then  $\text{Conj}_V(G) = \text{Conj}(G)$ .

**Definition 3.35.** An elementary conjugating automorphism  $\alpha_{C,u}$ , where  $u = x^{\pm 1}$ , for some  $x \in X$  is called an elementary singular conjugating automorphism if  $C = \{y\}$ , for some  $y \in X$ , and the set of all such elementary conjugating automorphisms is denoted  $\text{LInn}_S = \text{LInn}_S(G)$ . The subgroup of  $\text{Conj}(G)$  generated by  $\text{LInn}_S(G)$  is called singular and denoted  $\text{Conj}_S(G)$ .

**Definition 3.36.** Let  $\text{Tr}_\perp = \{\tau_{x^\varepsilon, y^\delta} \in \text{Tr} \mid x \in y^\perp, \varepsilon, \delta = \pm 1\}$  and  $\text{Tr}_\diamond = \{\tau_{x^\varepsilon, y^\delta} \in \text{Tr} \mid x \notin y^\perp, \varepsilon, \delta = \pm 1\}$ .

**Definition 3.37.** • If  $x$  and  $y$  are vertices of  $X$  such that  $x^\perp \cap y^\perp = y^\perp \setminus y$  then we say that  $x$  dominates  $y$ .

- The set of all vertices dominated by  $x$  is denoted  $\text{Dom}(x) = \{u \in X \mid x \text{ dominates } u\}$ .
- The set of all dominated vertices is denoted  $\text{Dom}(\Gamma) = \cup_{x \in X} \text{Dom}(x)$ .
- For fixed  $y \in X$  the set of all  $x$  such that  $y \in \text{Dom}(x)$  and  $[y] \neq [x]$  is the outer admissible set of  $y$ , denoted  $\text{out}(y)$ .

From the definition and Lemma 2.5 (vii) it follows that  $x$  dominates  $y$  if and only if  $[x, y] \neq 1$  and  $\mathfrak{a}(x) \subseteq \mathfrak{a}(y)$ . Thus  $\text{out}(y) = \{x \in \mathfrak{a}(y) : x \notin [y] \cup y^\perp\}$ .

If  $\alpha_{C,x} \in \text{LInn}_S(G)$  then  $C = \{y\}$  is a connected component of  $\Gamma_{x^\perp}$  so  $y^\perp \setminus y \subseteq x^\perp$  and  $y \notin x^\perp$ . Therefore  $x$  dominates  $y$  and  $\tau_{y,x} \in \text{Tr}_\diamond$  and  $\alpha_{C,x} = \tau_{y,x} \tau_{y^{-1},x}$ . Hence  $\text{Conj}_S$  is the subgroup of  $\text{Aut}(G)$  generated by the set  $\{\tau_{y,x} \tau_{y^{-1},x} \mid x \text{ dominates } y\} = \text{LInn}_S$ .

**Definition 3.38.** Let  $x, u \in X$  such that  $x$  dominates  $u$  and let  $[u] \setminus \{x\} = \{v_1, \dots, v_n\}$ . The conjugating automorphism

$$\alpha_{[u],x} = \prod_{i=1}^n \alpha_{\{v_i\},x}$$

is called a basic collected conjugating automorphism and the set of all basic collected conjugating automorphisms is denoted  $\text{LInn}_C = \text{LInn}_C(G)$ . The subgroup of  $\text{Conj}(G)$  generated by  $\text{LInn}_C(G)$  is denoted  $\text{Conj}_C = \text{Conj}_C(G)$ .

**Definition 3.39.** • The set of regular elementary conjugating automorphisms is  $\text{LInn}_R = \text{LInn}_R(G) = (\text{LInn}(G) \cap \text{Conj}_V(G)) \setminus \text{LInn}_S(G)$ .

- The set of basic vertex conjugating automorphisms is  $\text{LInn}_V = \text{LInn}_V(G) = \text{LInn}_R(G) \cup \text{LInn}_C(G)$ .

We record some straightforward properties of these definitions in the following lemma.

**Lemma 3.40.** Let  $\Gamma$  be a graph.

- (i) (a) If  $\Gamma$  has an isolated vertex then  $\text{Inn} = \text{Conj}_N$  and
- (b) if  $\Gamma$  has no isolated vertex then  $\text{Conj}_A \leq \text{Conj}_N$ .

In all cases

$$\text{Inn} \leq \text{Conj}_A \leq \text{Conj}_V \leq \text{Conj}$$

and

$$\text{Inn} \leq \text{Conj}_N \leq \text{Conj}_V \leq \text{Conj}.$$

(ii)  $\text{LInn}_V \subseteq \text{Conj}_V$ .

(iii) If  $\phi \in \text{Conj}_S$  then  $x\phi = x^{f_x}$ , where  $\nu(f_x) \subseteq \mathfrak{a}(x)$ , for all  $x \in X$ .

*Proof.* (i) It is immediate from the definitions that  $\text{Inn} \leq \text{Conj}_A$ ,  $\text{Inn} \leq \text{Conj}_N$  and  $\text{Conj}_V \leq \text{Conj}$ . That  $\text{Conj}_A \leq \text{Conj}_V$  follows from the fact that, if  $x, y \in X$  then  $[y] \subseteq C \cup x$ , for some connected component  $C$  of  $\Gamma_x$ . As  $[x] \subseteq \mathfrak{a}(x)$ , for all  $x$ , it follows that  $\text{Conj}_N \leq \text{Conj}_V$ .

If  $x$  is an isolated vertex then  $\mathfrak{a}(x) = X$ , so for  $\phi \in \text{Conj}_N$  there exists  $f_x \in G$  such that  $y\phi = y^{f_x}$ , for all  $y \in X$ . Hence, in this case  $\text{Conj}_N = \text{Inn}$ . Assume then that  $\Gamma$  has no isolated vertex. In this case, for all  $x \in X$ , the connected component of  $\Gamma$  containing  $x$  also contains  $\mathfrak{a}(x)$ . To see that  $\text{Conj}_A \leq \text{Conj}_N$  suppose that  $u \in X$  and consider the aggregate conjugating automorphism  $\beta = \beta_{C,x}$ , where  $x \in X$ . If  $x \in u^\perp \setminus u$  then  $v\beta = v$ , for all  $v \in \mathfrak{a}(u)$ , so assume that this is not the case. If  $x \in \mathfrak{a}(u)$  then  $x \notin u^\perp \setminus u$  implies that  $\mathfrak{a}(u) \subseteq C' \cup \{x\}$ , for some component  $C'$  of  $\Gamma_x$ , so we may also assume that  $x \notin \mathfrak{a}(u)$ .

Now let  $v$  and  $w$  be distinct elements of  $\mathfrak{a}(u)$  and  $r$  be any element of  $u^\perp \setminus u$ . Then the path  $v, r, w$  does not contain  $x$ ; so  $v$  and  $w$  are either both in  $C$  or both outside  $C$ . Hence  $\beta_{C,x}$  either fixes every element of  $\mathfrak{a}(u)$ , or acts as conjugation by  $x$  on every element of  $\mathfrak{a}(u)$ . Thus all elements of  $\text{Conj}_A$  are normal, as required.

(ii) This follows directly from the definitions and the fact that the sets  $[x]$  partition  $X$ , so that  $\text{LInn}_C \subseteq \text{Conj}_V$ .

(iii) An induction on the length of  $\phi$  as a word in the generators  $\text{LInn}_S$  is used. If  $\phi$  is trivial there is nothing to be proved, so assume inductively that the result holds for words of length at most  $n - 1$  and that  $\phi = \phi_0\phi_1$ , where  $\phi_0$  has length  $n - 1$  as a word in  $\text{LInn}_S^{\pm 1}$  and  $\phi_1 \in \text{LInn}_S^{\pm 1}$ , say  $\phi_1 = \alpha_{C,z}$ , for some  $z \in X^{\pm 1}$  and  $C = \{y\}$ . Then  $x\phi_0 = x^{f_x}$ , where  $\nu(f_x) \subseteq \mathfrak{a}(x)$ , for all  $x \in X$ . Let  $x \in X$  and  $u \in \mathfrak{a}(x)^{\pm 1}$ . Then  $u\phi_1 = u$  unless  $u = y^{\pm 1}$ . In the latter case  $y \in \mathfrak{a}(x)$  so  $z \in \mathfrak{a}(y)^{\pm 1} \subseteq \mathfrak{a}(x)^{\pm 1}$  and  $u\phi_1 = u^z$  implies  $\nu(u\phi_1) \subseteq \mathfrak{a}(x)$ . Thus we have  $\nu(f_x\phi_1) \subseteq \mathfrak{a}(x)$ . Now  $x\phi = (x\phi_1)^{f_x\phi_1}$  and since  $x\phi_1 = x^z$  if and only if  $x = y^{\pm 1}$  it follows that  $\nu(x\phi) \subseteq \mathfrak{a}(x)$ , as required.  $\square$

We shall use the following definition of Laurence [29].

**Definition 3.41.** Let  $\phi$  be a conjugating automorphism and for each  $x \in X$  let  $g_x \in G$  be such that  $x\phi = g_x^{-1} \circ x \circ g_x$ . The length  $|\phi|$  of  $\phi$  is  $\sum_{x \in X} \lg(g_x)$ .

We shall prove, in Propositions 3.44 and in a subsequent paper, versions of Theorem 3.17 (i.e. Theorem 2.2 of [29]) appropriate to  $\text{Conj}_V$  and  $\text{Conj}_N$  and to do so make use of Lemma 2.5 and Lemma 2.8 (*loc. cit.*) which we state here for reference.



**Lemma 3.42** ([29][Lemma 2.5 & Lemma 2.8]). *Let  $\phi$  be a non-trivial element of  $\text{Conj}$  and, for each  $x \in X$ , let  $g_x \in G$  such that  $x\phi = g_x^{-1} \circ x \circ g_x$ . Then*

- (i) *there exist  $x, y \in X$  and  $\varepsilon \in \{\pm 1\}$  such that  $x^\varepsilon g_x$  is a right divisor of  $g_y$ , and*
- (ii) *if  $y, z \in X \setminus x^\perp$  such that  $[y, z] = 1$  and  $x^\varepsilon g_x$  is a right divisor of  $g_y$  then  $x^\varepsilon g_x$  is a right divisor of  $g_z$ .*

(As can be seen from the example  $\phi = \alpha_{C,x}^{-1}$  the variable  $\varepsilon$  taking values  $\pm 1$  is a necessary part of this lemma.)

**Lemma 3.43.** *Let  $\phi \in \text{Conj}_V$  and for each  $y \in X$  let  $g_y \in G$  be such that  $y\phi = g_y^{-1} \circ y \circ g_y$ .*

- (i) *If  $[y] = [y]_\perp$  then  $g_u = g_y$ , for all  $u \in [y]$ .*
- (ii) *If  $[y] = [y]_\diamond$  and  $|[y]| \geq 2$  then there exists  $v \in [y]$  and  $m_y \in \mathbb{Z}$  such that  $g_u = v^{m_y} \circ g_v$ , for all  $u \in [y] \setminus \{v\}$ . Moreover if  $m_y \neq 0$  then  $v$  is the unique element of  $[y]$  with this property and, setting  $\varepsilon = -m_y/|m_y|$ ,  $S = [y] \setminus \{v\}$  and  $\alpha = \prod_{u \in S} \alpha_{\{u\}, v^\varepsilon}$  we have  $\alpha \in \text{LInn}_C^{\pm 1}$  and  $|\alpha\phi| < |\phi|$ .*

*Proof.* Since  $\phi \in \text{Conj}_V$ , for all  $y \in X$ , there exists  $f_y \in G$  such that  $u\phi = u^{f_y}$ , for all  $u \in [y]$ , and we may choose an  $f_y$  of minimal length with this property. Fix  $y \in X$ . Then  $u^{f_y} = u\phi = u^{g_u}$  so  $g_u f_y^{-1} \in C_G(u)$ , for all  $u \in [y]$ . Therefore there are  $a, b, c \in G$  such that  $g_u = a \circ b$ ,  $f_y = c \circ b$  and  $g_u f_y^{-1} = a \circ c^{-1} \in C_G(u)$ . As  $g_u$  has no left divisor in  $C_G(u)$  this means that  $a = 1$  and so  $f_y = c_u \circ g_u$ , for  $c = c_u \in C_G(u)$ . If  $[y] = [y]_\perp$  then  $C_G(u) = C_G(y)$ , for all  $u \in [y]$ , so in this case  $g_y = f_y = g_u$ , for all  $u \in [y]$ .

Assume then that  $[y] = [y]_\diamond$ , with  $|[y]| \geq 2$ , and let  $u, v \in [y]$ ,  $v \neq u$ , so  $[u, v] \neq 1$ . Suppose  $v \in \nu(f_y)$ . Then  $f_y = c_v \circ g_v = c'_v \circ v^m \circ g_v$ , where  $c'_v \in G(v^\perp \setminus v)$  and  $m \in \mathbb{Z}$ . Then  $u^{f_y} = u^{v^m g_v}$ , since  $v^\perp \setminus v = u^\perp \setminus u$ . As  $g_v$  has no left divisor in  $C_G(v)$  and  $[v, u] \neq 1$  we have  $u^{v^m g_v} = g_v^{-1} \circ v^{-m} \circ u \circ v^m \circ g_v$ , so  $g_u = v^m \circ g_v$ . By choice of  $f_y$  we have  $c'_v = 1$ , and if  $m \neq 0$  then no element  $u \in [y]$ ,  $u \neq v$ , can be a left divisor of  $v^m \circ g_v$ , so the first statement of (ii) as well as the uniqueness of  $v$  follow. Moreover  $v$  dominates  $u$ , for all  $u \in [y]$ , so the final statement of (ii) also holds.  $\square$

**Proposition 3.44.**  *$\text{Conj}_V$  is generated by  $\text{LInn}_V = \text{LInn}_R \cup \text{LInn}_C$  and  $\text{Conj}_V \cap \text{Conj}_S = \text{Conj}_C$ .*

*Proof.* That  $\langle \text{LInn}_V \rangle \leq \text{Conj}_V$  is Lemma 3.40 (ii). For the opposite inclusion we use induction on the length of an automorphism  $\phi$  in  $\text{Conj}_V$ . If  $|\phi| = 0$  then  $\phi = 1$  and there is nothing to prove. Assume that  $|\phi| > 1$  and that,

for all conjugating automorphisms  $\psi$  of shorter length,  $\psi \in \text{Conj}_V$  implies  $\psi \in \langle \text{LInn}_V \rangle$ . If there exists  $y \in X$  such that,  $[y] = [y]_\diamond$ ,  $|[y]| \geq 2$  and, in the notation of Lemma 3.43,  $m_y \neq 0$ , then it follows from that lemma and induction that  $\phi \in \langle \text{LInn}_V \rangle$ , as claimed. Hence we assume that either  $[y] = [y]_\perp$  or  $m_y = 0$ , and so  $g_y = g_u$ , for all  $u \in [y]$  and for all  $y \in X$ . From Lemma 3.42 (i) there exist  $x, y \in X$ ,  $\varepsilon \in \{\pm 1\}$  such that  $x\phi = g_x^{-1} \circ x \circ g_x$ ,  $y\phi = g_y^{-1} \circ y \circ g_y$  and  $x^\varepsilon g_x$  is a right divisor of  $g_y$ . Suppose that  $[x, y] = 1$ . Then  $[x\phi, y\phi] = 1$ ; that is  $[x^{g_x}, y^{g_y}] = 1$ . If  $g_y = a \circ x^\varepsilon \circ g_x$ , for some  $a \in G$ , then this implies that  $[x, y^{ax^\varepsilon}] = 1$ , from which it follows that  $[x, a] = 1$ . However, in this case  $y^{g_y}$  is not reduced, a contradiction. Therefore  $y \notin x^\perp$ , and so  $u \notin x^\perp$ , for all  $u \in [y]$ .

Let  $[y] = \{v_1, \dots, v_r\}$  and let  $C_1, \dots, C_s$  be the components of  $\Gamma_{x^\perp}$  containing  $v_1, \dots, v_r$ . Then, from Lemma 3.42 (ii),  $x^\varepsilon g_x$  is a right divisor of  $g_c$  for all  $c \in C_1 \cup \dots \cup C_s$ . Let  $\alpha = \prod_{i=1}^s \alpha_{C_i, x^{-\varepsilon}}$ . Then  $|\alpha\phi| < |\phi|$ . We claim that  $\alpha \in \text{Conj}_V$ . Suppose not, so there is some  $z \in X$  and elements  $u, v \in [z]$  such that  $u \in C_i$ , for some  $i$ , but  $v \notin \cup_{i=1}^s C_i \cup \{x^\perp\}$ . This implies that  $u^\perp \setminus u = v^\perp \setminus v \subseteq x^\perp$  and, as  $u \in C_i$  implies  $x \notin u^\perp$ , so  $x$  dominates  $u$ . Then  $C_i = \{u\}$  so  $u \in [y]$  and  $[z] = [y] \subseteq \cup_{i=1}^s C_i$ , a contradiction. Thus no such  $z$  exists and  $\alpha \in \text{Conj}_V$ .

If  $s = 1$  and  $|C_1| \geq 2$  then  $\alpha \in \text{LInn}_R^{\pm 1}$ . If  $s = 1$  and  $|C_1| = 1$  then  $x$  dominates  $y$  and  $\alpha \in \text{LInn}_C^{\pm 1}$ . If  $s > 1$  then  $x^\perp \supseteq y^\perp \setminus y$  and  $x$  dominates every element of  $[y]$ . In this case  $\alpha \in \text{LInn}_C^{\pm 1}$  again. Hence by induction  $\phi \in \langle \text{LInn}_R \cup \text{LInn}_C \rangle$ .

Suppose then that  $\phi \in \text{Conj}_V \cap \text{Conj}_S$ . The first paragraph of the argument above goes through with  $\text{LInn}_C$  in place of  $\text{LInn}_V$  and  $\text{Conj}_V \cap \text{Conj}_S$  in place of  $\text{Conj}_V$ . In the second paragraph, from Lemma 3.40 (iii) it now follows that  $\mathbf{a}(x) \subseteq \mathbf{a}(y) = \mathbf{a}(v_i)$ ; so  $x$  dominates  $v_i$ , for  $i = 1, \dots, r$ . Therefore  $\alpha \in \text{LInn}_C^{\pm 1} \subseteq \text{Conj}_S$  and, by induction on  $|\phi|$  again,  $\phi \in \langle \text{LInn}_C \rangle = \text{Conj}_C$ , as claimed.  $\square$

To describe the structure of  $\text{Conj}_A(G)$  it is convenient to work with outer automorphisms. Denote the group  $\text{Aut}(G)/\text{Inn}(G)$  of outer automorphisms by  $\text{Out}(G)$  as usual and given a subgroup  $B$  of  $\text{Aut}(G)$  let  $\overline{B}$  denote the group  $B \text{Inn}(G)/\text{Inn}(G)$ . We write  $\bar{\beta}$  for the image of  $\beta \in \text{Aut}(G)$  in  $\text{Out}(G)$  and  $\gamma_x$  for the inner automorphism of  $G$  mapping  $g$  to  $g^x$ , for all  $g \in G$ .

**Proposition 3.45.** *Let  $G = G(\Gamma)$ , where  $\Gamma$  is a connected graph. Then  $\text{Conj}_A(G)$  is torsion-free and  $\overline{\text{Conj}}_A(G)$  is free Abelian and a normal subgroup of  $\overline{\text{Aut}}^*(G)$ . Moreover, if  $c(x)$  is the number of connected components of  $\Gamma_x$ , for all  $x \in X$ , then the  $\overline{\text{Conj}}_A(G)$  has rank  $\sum_{x \in X} (c(x) - 1)$ .*

*Proof.* First we show that  $\overline{\text{Conj}}_A(G)$  is a free Abelian group. Let  $x \in X$  and suppose that  $\Gamma_x$  has connected components  $C_1, \dots, C_r$ . If  $y \in X$ ,  $y \neq x$ , then there is some  $i$  such that  $y^\perp \subseteq C_i \cup \{x\}$ . Assume that  $\Gamma_y$  has components  $D_1, \dots, D_s$ . We claim that there is a  $j$  such that

- (i)  $D_j \supseteq C_k \cup \{x\}$ , for all  $k \neq i$ , and
- (ii)  $C_i \supseteq D_k \cup \{y\}$ , for all  $k \neq j$ .

To see this choose  $j$  such that  $x \in D_j$ , so  $x^\perp \subseteq D_j \cup \{y\}$ . Let  $u \in C_k$ ,  $k \neq i$ . Then there exists a path in  $\Gamma$  from  $u$  to  $x$  and, as  $y \in C_i$ , this path may be chosen so that none of its vertices is  $y$ . Hence  $u$  and  $x$  belong to the same component of  $\Gamma_y$ . Thus, if  $D_j$  is the component of  $\Gamma_y$  containing  $x$  then  $D_j \supseteq C_k \cup \{x\}$ , for all  $k \neq i$ . This shows that the first statement of the claim holds and the second follows by symmetry.

The subgroup  $\overline{\text{Conj}}_A(G)$  is generated by the images  $\bar{\beta}_{C,x}$  in  $\text{Out}(G)$  of aggregate conjugating automorphisms  $\beta_{C,x}$ , where  $x$  ranges over  $X$  and  $C$  ranges over the connected components of  $\Gamma_x$ . Let  $\bar{\beta}_{C,x}$  and  $\bar{\beta}_{D,y}$  be generators of  $\overline{\text{Conj}}_A(G)$ . If  $x = y$  then these two generators commute, so we assume  $x \neq y$  and that components  $C_i$  and  $D_j$  of  $\Gamma_x$  and  $\Gamma_y$ , respectively, have been chosen as in the claim above; so  $x \in D_j$  and  $y \in C_i$ . If  $C = C_i$  then let  $\beta_1 = \gamma_{x^{-1}}\beta_{C,x}$ , so  $\bar{\beta}_{C,x} = \bar{\beta}_1$  and  $\beta_1 = \prod_{C_k \neq C} \beta_{C_k,x}^{-1} \in \text{Conj}_A(G)$ . Otherwise let  $\beta_1 = \beta_{C,x}$ . In either case  $u\beta_1 = u$ , for all  $u \in C_i$ . Similarly we may choose a representative  $\beta_2$  of  $\bar{\beta}_{D,y}$  such that  $u\beta_2 = u$ , for all  $u \in D_j$ . Then, from (i) and (ii) above, it follows that  $\beta_1\beta_2 = \beta_2\beta_1$  and so  $\bar{\beta}_{C,x}\bar{\beta}_{D,y} = \bar{\beta}_{D,y}\bar{\beta}_{C,x}$ , and  $\overline{\text{Conj}}_A(G)$  is Abelian as claimed.

Now, for  $i = 1, \dots, r$ , let  $\bar{\beta}_i = \bar{\beta}_{C_i,x} \in \overline{\text{Conj}}_A(G)$ . As  $\prod_{i=1}^r \bar{\beta}_i = 1$ , given any element  $\phi \in \overline{\text{Conj}}_A(G)$  we may write  $\phi = \gamma_0\gamma_1$ , where  $\gamma_0 = \prod_{i=1}^{r-1} \beta_i^{m_i}$ , for some  $m_i \in \mathbb{Z}$ , and  $\gamma_1$  is a product of generators  $\beta_{D,y}$ , with  $y \neq x$ . Let  $y \in C_i$ , where  $1 \leq i \leq r-1$ . Then  $y\gamma_0 = y^{x^{m_i}}$  and  $y\gamma_1 = y^h$ , for some  $h \in G$  such that  $x \notin \nu(h)$ . Also  $x\gamma_1 = x^g$ , for some  $g \in G$  such that  $x \notin \nu(g)$ . Then

$$y\gamma_0\gamma_1 = (y^h)^{(x^{m_i})^g}.$$

If  $w$  is a geodesic word representing  $h(x^{m_i})^g$  then the exponent sum  $|w|_x$  of  $x$  in  $w$  equals  $m_i$ ; so  $y\gamma_0\gamma_1 = v^{-1} \circ y \circ v$ , where  $|v|_x = m_i$ . For  $z \in C_r$  we have  $z\gamma_0\gamma_1 = z\gamma_1 = u^{-1} \circ z \circ u$ , for some  $u \in G$  such that  $|u|_x = 0$ . If  $\phi = 1$  then  $\gamma_0\gamma_1 \in \text{Inn}(G)$  and so it must be that  $m_1 = \dots = m_{r-1} = 0$ . It follows by induction, on the minimal number of generators appearing in a word representing  $\phi$ , that  $\overline{\text{Conj}}_A(G)$  is free Abelian of rank  $\sum_{x \in X} (c(x) - 1)$ , as claimed.

To see that  $\text{Conj}_A(G)$  is torsion free it suffices to note that  $\text{Inn}(G)$  is torsion free; since  $\text{Inn}(G) \cong G/Z(G)$ , which is a partially commutative group.

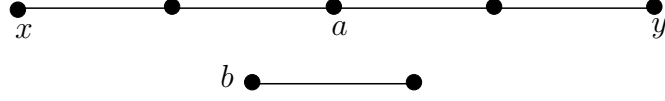


Figure 3.1:  $\overline{\text{Conj}}_A(G)$  is non-Abelian

In fact  $G/Z(G) \cong G(\Gamma_Z)$ , where  $Z$  is the subset of  $X$  consisting of vertices connected to all vertices of  $\Gamma$  and  $\Gamma_Z$  is the full subgraph of  $\Gamma$  on  $X \setminus Z$ . As both  $\text{Inn}(G)$  and  $\overline{\text{Conj}}_A(G) = \text{Conj}_A(G)/\text{Inn}(G)$  are torsion-free, so is  $\text{Conj}_A(G)$ .

To show that  $\overline{\text{Conj}}_A(G)$  is normal in  $\overline{\text{Aut}}^*(G)$  we shall show that if  $\bar{\beta}_{C,x}$  is an arbitrary generator of  $\overline{\text{Conj}}_A(G)$  and  $\phi$  is a generator of  $\overline{\text{Aut}}^*(G)$ , which is not in  $\overline{\text{Conj}}_A(G)$ , then  $\phi^{-1}\bar{\beta}_{C,x}\phi \in \overline{\text{Conj}}_A(G)$ . We consider the cases where  $\phi$  is the image in  $\overline{\text{Aut}}^*(G)$  of an inversion and a transposition separately.

Let  $\phi = \bar{\iota}_z$ ; an inversion. Straightforward checking shows that

- (a) if  $x = z$  then  $\iota_z^{-1}\beta_{C,x}\iota_z = \beta_{C,x}^{-1}$  and
- (b) if  $x \neq z$  then  $\iota_z^{-1}\beta_{C,x}\iota_z = \beta_{C,x}$ .

Hence the result holds in this case.

Next suppose that  $\phi = \bar{\tau}_{v,z}$ , where  $v = y$  or  $y^{-1}$ . If  $y = x$  then we have  $x^\perp \setminus x \subseteq z^\perp$  which implies that  $\Gamma_x$  is connected and so  $\beta_{C,x}$  is an inner automorphism; as are all its conjugates. Thus we assume that  $y \neq x^{\pm 1}$  and  $\Gamma_x$  is not connected. Let  $C_1$  be the component of  $\Gamma_x$  containing  $y$ . Then  $y^\perp \setminus y \subseteq z^\perp$  and  $y^\perp \subseteq C_1 \cup \{x\}$ . If  $z \in C_2$ , for some component  $C_2$  of  $\Gamma_x$  with  $C_2 \neq C_1$  then  $z^\perp \subseteq C_2 \cup \{x\}$  so  $y^\perp \setminus y \subseteq (C_1 \cup \{x\}) \cap (C_2 \cup \{x\}) = \{x\}$ ; in which case  $y^\perp = \{x, y\}$  and  $x \in z^\perp$ . These conditions imply that  $\tau_{v,z}^{-1}\beta_{C,x}\tau_{v,z} = \beta_{C,x}$ , so we may now assume that  $z \in C_1$ . Assume in addition that the connected components of  $\Gamma_x$  are  $C_1, \dots, C_r$ . If  $C = C_i$ , where  $i \neq 1$  then  $\phi^{-1}\beta_{C,x}\phi = \beta_{C,x}$ . If  $C = C_1$  then set  $\beta_1 = \prod_{i=2}^r \beta_{C_i,x}^{-1}$ , so  $\bar{\beta}_{C,x} = \bar{\beta}_1$  and  $\phi^{-1}\beta_1\phi = \beta_1$ . Thus  $\phi^{-1}\bar{\beta}_{C,x}\phi = \bar{\beta}_1 = \bar{\beta}_{C,x}$ , and the result follows.  $\square$

The previous proposition can not be extended to disconnected graphs or to  $\text{Conj}(G)/\text{Inn}(G)$ , in place of  $\text{Conj}_A(G)/\text{Inn}(G)$ , as the following examples show.

**Example 3.46.** In the graph  $\Gamma$  of Figure 3.1, let  $C$  be the component of  $\Gamma_x$  containing  $a$  and let  $D$  be the component of  $\Gamma_y$  containing  $a$ . Then  $\text{Conj}_A$  contains  $\beta_{C,x}$  and  $\beta_{D,y}$  and  $a[\beta_{C,x}, \beta_{D,y}] = a^{[x,y]}$ , while  $b[\beta_{C,x}, \beta_{D,y}] = b$ . Therefore  $[\beta_{C,x}, \beta_{D,y}] \notin \text{Inn}$ , so  $\overline{\text{Conj}}_A$  is non-Abelian.

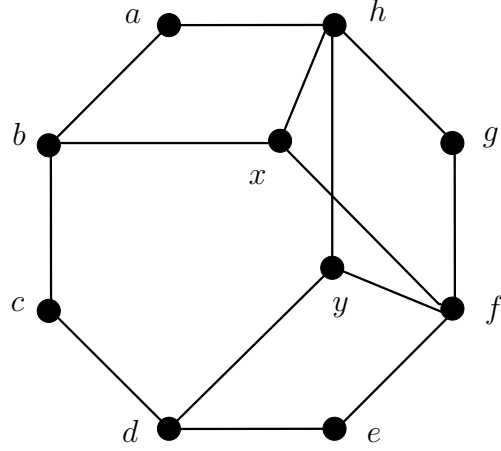


Figure 3.2:  $\text{Conj}(G)/\text{Inn}(G)$  is non-Abelian

**Example 3.47.** Let  $\Gamma$  be the graph of Figure 3.2. Then  $\Gamma_{x^\perp}$  has a component  $C = \{c, d, e, y\}$  and  $\Gamma_{y^\perp}$  has a component  $D = \{a, b, c, x\}$ . Let  $\alpha = \alpha_{C,x}$  and  $\beta = \alpha_{D,y}$ . The images of  $c$  and  $g$  under  $[\alpha, \beta]$  are  $c^{[x,y]}$  and  $g$ , respectively. Therefore  $\alpha\beta \neq \beta\alpha$  modulo  $\text{Inn}(G)$ . In this example  $\text{Inn}(G) = \text{Conj}_A(G)$ , so in general  $\text{Conj}(G)/\text{Conj}_A(G)$  is also non-Abelian.

## 4 The stabilisers $\text{St}(\mathcal{K})$ and $\text{St}^{\text{conj}}(\mathcal{K})$ .

**Definition 4.1.** *Define*

$$\text{St}(\mathcal{K}) = \{\phi \in \text{Aut}(G) \mid G(Y)\phi = G(Y), \text{ for all } Y \in \mathcal{K}\}.$$

Then, from Lemma 2.5 (x) it follows that  $\phi \in \text{St}(\mathcal{K})$  if and only if  $G(\mathbf{a}(x))\phi = G(\mathbf{a}(x))$ , for all  $x \in X$ . Also, from Remark 3.22 the subgroup of  $\text{Aut}^*(G)$  generated by  $\text{Inv}$  and  $\text{Tr}$  is contained in  $\text{St}(\mathcal{K})$ .

**Definition 4.2.** *Define*

$$\text{St}^{\text{conj}}(\mathcal{K}) = \{\phi \in \text{Aut}(G) \mid G(Y)^\phi = G(Y)^{f_Y}, \text{ for some } f_Y \in G, \text{ for all } Y \in \mathcal{K}\}.$$

We shall make use of the following fact in the proof of the next proposition.

**Lemma 4.3.** *Let  $H$  and  $K$  be canonical parabolic subgroups of  $G$  and let  $\theta \in \text{Aut}(G)$  be such that  $H\theta = H^f$  and  $K\theta = K^g$ , for some  $f, g \in G$ . Then there exists  $h \in H$  such that  $(H \cap K)\theta = (H \cap K)^h$ .*

*Proof.* First suppose that  $f = 1$  and that  $g$  has no left divisor in  $K$ . In this case it follows from [17, Corollary 2.4], that if  $u \in K$  then there exist words  $a, b$  (dependent on  $u$ ) such that  $g = a \circ b$  and  $u^g = b^{-1} \circ u \circ b$ . Thus, if  $w \in H \cap K^g$ , then for some  $u \in K$ , we have  $w = u^g = b^{-1} \circ u \circ b$ . This implies that  $u \in H \cap K$ , so  $w = u^g \in (H \cap K)^g$ . That is,  $H \cap K^g \subseteq (H \cap K)^g$ .

Now  $(H \cap K)\theta \subseteq H \cap K^g \subseteq (H \cap K)^g$ . Moreover, from the hypothesis on  $H, K$  and  $\theta$ , we have  $H\theta^{-1} = H$  and  $K\theta^{-1} = K^h$ , where  $h = (g\theta^{-1})^{-1}$ . Applying the previous argument gives  $(H \cap K)\theta^{-1} \subseteq (H \cap K)^h$ , so  $H \cap K \subseteq (H \cap K)\theta^{h\theta}$ , from which we obtain  $(H \cap K)^g \subseteq (H \cap K)\theta$ .

In the general case let  $\gamma_{f^{-1}}$  denote conjugation by  $f^{-1}$  and let  $\phi = \theta\gamma_{f^{-1}}$ . Then  $H\phi = H$  and  $K\phi = K^{gf^{-1}}$ . Let  $gf^{-1}$  have minimal form  $gf^{-1} = a \circ b$ , where  $a \in K$  and  $b$  has no left divisor in  $K$ . Then  $K\phi = K^b$ , so from the first case above  $(H \cap K)\phi = (H \cap K)^b$ . Hence  $(H \cap K)\theta = (H \cap K)^{bf}$ , as required.  $\square$

**Theorem 4.4.**  $\text{St}^{\text{conj}}(\mathcal{K}) = \text{Aut}^*(G)$  and the group  $\text{Aut}(G)$  can be decomposed into the internal semi-direct product of  $\text{St}^{\text{conj}}(\mathcal{K})$  and the finite subgroup  $\text{Aut}_{\text{comp}}^\Gamma(G)$ :

$$\text{Aut}(G) = \text{St}^{\text{conj}}(\mathcal{K}) \rtimes \text{Aut}_{\text{comp}}^\Gamma(G).$$

*Proof.* That  $\text{Aut}(G) = \text{St}^{\text{conj}}(\mathcal{K}) \rtimes \text{Aut}_{\text{comp}}^\Gamma(G)$  follows immediately from Proposition 3.23 once the first statement has been proved. From Lemma

4.3 and the definition of admissible sets,  $\phi \in \text{St}^{\text{conj}}(\mathcal{K})$  if and only if  $G(\mathfrak{a}(x))\phi = G(\mathfrak{a}(x))^{f_x}$ , for some  $f_x \in G$ , for all  $x \in X$ . It follows then from Proposition 3.21 that  $\text{Aut}^*(G) \subseteq \text{St}^{\text{conj}}(\mathcal{K})$ .

For the reverse inclusion note that from Proposition 3.23 every element  $\phi$  of  $\text{Aut}$  can be expressed as  $\phi = \alpha\beta$ , with  $\alpha \in \text{Aut}^*$  and  $\beta \in \text{Aut}_{\text{comp}}^\Gamma(G)$ . If  $\phi \in \text{St}^{\text{conj}}(\mathcal{K})$  then, since  $\text{Aut}^*(G) \subseteq \text{St}^{\text{conj}}(\mathcal{K})$  we have  $\alpha^{-1}\phi \in \text{St}^{\text{conj}}(\mathcal{K}) \cap \text{Aut}_{\text{comp}}^\Gamma(G)$ . However, from the definitions of  $\text{St}^{\text{conj}}(\mathcal{K})$  and  $\text{Aut}_{\text{comp}}^\Gamma(G)$  this means that  $\alpha^{-1}\phi = \beta = 1$ , so  $\phi = \alpha \in \text{Aut}^*(G)$ , as required.  $\square$

**Theorem 4.5.**  $\text{Conj}_N(G)$  is a normal subgroup of  $\text{St}^{\text{conj}}(\mathcal{K})$ .

*Proof.* In the light of Lemma 3.40 we may assume that  $\Gamma$  has no isolated vertex. Let  $\phi \in \text{Conj}_N(G)$  and  $\psi \in \text{St}^{\text{conj}}(\mathcal{K})$ . Let  $x \in X$  and let  $g_x$  and  $h_x$  be elements of  $G$  such that  $u\phi = u^{g_x}$ , for all  $u \in \mathfrak{a}(x)$ , and  $G(\mathfrak{a}(x))\psi^{-1} = G(\mathfrak{a}(x))^{h_x}$ . For each  $u \in \mathfrak{a}(x)$  let  $w_u$  be the minimal form of an element of  $G$  such that  $w_u^{h_x} = u\psi^{-1}$ , so  $\nu(w_u) \subseteq \mathfrak{a}(x)$  and  $w_u\psi = u^{h_x^{-1}\psi}$ . Then  $u\psi^{-1}\phi\psi = u^{f_x}$ , where  $f_x = (h_x^{-1}g_x(h_x\phi))\psi$ , so  $f_x$  is dependent only on  $x$  and  $\psi^{-1}\phi\psi \in \text{Conj}_N(G)$ .  $\square$

**Lemma 4.6.** (i)  $\text{St}(\mathcal{K}) \cap \text{Conj}(G) = \text{Conj}_S(G)$ .

(ii)  $\text{St}(\mathcal{K}) \cap \text{Conj}_V(G) = \text{Conj}_C(G)$ .

(iii) If  $\phi \in \text{Aut}(G)$  then  $\phi \in \text{Conj}_S(G)$  if and only if  $x\phi = x^{f_x}$  where  $\nu(f_x) \subseteq \mathfrak{a}(x)$ , for all  $x \in X$ .

*Proof.* If  $\alpha \in \text{LInn}_S$  then  $G(\mathfrak{a}(x))\alpha = G(\mathfrak{a}(x))$ , for all  $x \in X$ , so  $\text{Conj}_S \subseteq \text{St}(\mathcal{K}) \cap \text{Conj}(G)$ . For the converse use induction on  $|\phi|$ , where  $\phi \in \text{St}(\mathcal{K}) \cap \text{Conj}(G)$ . If  $|\phi| = 0$  then  $\phi = 1$  and so belongs to  $\text{Conj}_S$ . Assume then that  $|\phi| > 0$ . In this case, from Lemma 3.42 (i), there exist  $u_1, u_2 \in X$  such that  $u_i\phi = u_i^{w_i}$ , reduced as written, for some  $w_1, w_2 \in G$ , and  $u_1w_1$  is a right divisor of  $w_2$ . It follows, as in the proof of Proposition 3.44, that  $u_1 \notin u_2^\perp$ . As  $\phi \in \text{St}(\mathcal{K})$  we have  $w_2 \in G(\mathfrak{a}(u_2))$  so  $u_1w_1 \in G(\mathfrak{a}(u_2))$ . In particular  $u_1 \in \mathfrak{a}(u_2)$  so  $u_2^\perp \setminus u_2 \subseteq u_1^\perp$ . Therefore  $\tau_{u_2, u_1} \in \text{Tr}_\diamond$  and  $\beta = \tau_{u_2, u_1} \tau_{u_2^{-1}, u_1} \in \text{Conj}_S \subseteq \text{St}(\mathcal{K}) \cap \text{Conj}(G)$ . Therefore  $\beta\phi \in \text{St}(\mathcal{K}) \cap \text{Conj}(G)$  and  $|\beta\phi| < |\phi|$  so, by induction,  $\beta\phi \in \text{Conj}_S(G)$ . This gives  $\phi \in \text{Conj}_S(G)$ , as required. From this and Lemma 3.40 (iii) the last statement of the lemma follows immediately.

That  $\text{St}(\mathcal{K}) \cap \text{Conj}_V(G) = \text{Conj}_C(G)$  follows immediately from Proposition 3.44  $\square$

The following question now arises naturally.

**Question 4.7.** Let  $\Gamma$  be a connected graph. Is  $\text{St}^{\text{conj}}(\mathcal{K}) = \text{St}(\mathcal{K}) \text{Conj}_N(G)$ ?

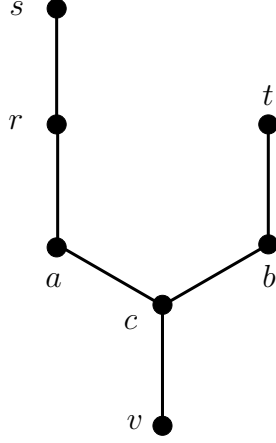


Figure 4.1: Example 4.8

It seems on first sight very plausible that the answer is “yes”, but, as the subsequent example shows, it turns out to be “no” and in fact, in general  $\text{St}(\mathcal{K}) \text{Conj}(G) \subsetneq \text{St}^{\text{conj}}(\mathcal{K})$ .

**Example 4.8.** Take  $G$  to be the group  $G(\Gamma)$  where  $\Gamma$  is the graph of Figure 4.1. Denote the components of  $\Gamma_{v^\perp}$  by  $C = \{a, r, s\}$  and  $D = \{b, t\}$ , let  $\alpha = \alpha_{C,v}$ ,  $\tau = \tau_{v,a}\tau_{v,b}\tau_{v,a^{-1}}$ , and set  $\phi = \alpha\tau$ .

$$z\phi = \begin{cases} z, & \text{if } z = b, c, t \\ vb^a, & \text{if } z = v \\ z^{vb^a} & \text{if } z \in C \end{cases}$$

The answer to question 4.7 above is “no” in this example, as  $\phi$  cannot be written as  $\gamma\delta$ , where  $\delta \in \text{Conj}$  and  $\gamma \in \text{St}(\mathcal{K})$ , as we shall demonstrate. Suppose then that  $\phi = \gamma\delta$ , where  $\delta \in \text{Conj}$  and  $\gamma \in \text{St}(\mathcal{K})$ . The set  $\mathcal{K}$  consists of  $\mathbf{a}(v) = \{a, b, c, v\}$ ,  $\mathbf{a}(s) = \{a, r, s\}$ ,  $\mathbf{a}(t) = \{b, c, t\}$  and four more sets  $\mathbf{a}(z) = \{z\}$ , where  $z = a, b, c$  and  $r$ . As  $\gamma$  maps the subgroup generated by  $\mathbf{a}(a) = \{a\}$  to itself we have  $a\gamma = a^{\pm 1}$ . As  $a\delta = a^g$ , for some  $g \in G$ , it must be that  $a\gamma = a$ . Similarly  $b\gamma = b$ ,  $c\gamma = c$  and  $r\gamma = r$ . Combined with the expression for  $\phi$  above we obtain  $a\delta = a^{vb^a}$ ,  $b\delta = b$  and  $c\delta = c$ . As  $c\delta = c$ , we have  $C_G(c)\delta = C_G(c\delta) = C_G(c)$ , so  $G(c^\perp)\delta = G(c^\perp)$ : that is  $G(a, b, c, v)\delta = G(a, b, c, v)$ . Moreover, as  $\delta$  acts on generators by conjugation,  $\delta$  must map  $G(a, b, v)$  to itself; so  $v\delta = v^g$ , for some  $g \in G(a, b, v)$ , and  $\delta$  restricts to an automorphism of  $G(a, b, v)$ . Applying Lemma 3.42 to the restriction of  $\delta$  to  $G(a, b, v)$  we see that either  $b^\varepsilon$  or  $a^\varepsilon vb^a$  is a right divisor of  $g$ , or that  $vg$  is a right divisor of  $vb^a$ , in which case  $g = b^a$ . In the latter case consider the automorphism  $\alpha_{C,v^{-1}}\delta$ . This maps  $a$  to  $a^{b^a}$ ,  $b$  to itself,  $c$



to itself and  $v$  to  $v^{b^a}$ . Restricting to  $G(a, b, v)$  again gives a contradiction to Lemma 3.42. Thus we may assume that either  $a^\varepsilon v b^a$  or  $b^\varepsilon$  is a right divisor of  $g$ . Suppose that  $m$  is maximal such that  $a^{\varepsilon m} v b^a$  is a right divisor of  $g$ ; say  $g = g_0 \circ a^{\varepsilon m} v b^a$ . If  $m \geq 1$  then  $\delta_0 = \alpha_{v, a^\varepsilon}^{-m} \delta$  maps  $a$  to  $a^{v b^a}$ , fixes  $b$  and  $c$  and maps  $v$  to  $v^{g_0 v b^a}$ . As this contradicts Lemma 3.42 we have  $m = 0$ . Similarly, if  $b^\varepsilon$  is a right divisor of  $g$  then we obtain a contradiction. Hence no such conjugating automorphism  $\delta$  exists.

It is also possible to show that  $\tau\alpha \notin \text{Conj}(G) \text{St}(\mathcal{K})$ . Moreover the example shows that replacing  $\text{St}^{\text{conj}}(\mathcal{K})$ ,  $\text{St}(\mathcal{K})$  and  $\text{Conj}_N(G)$  with their canonical images in  $\text{Out}(G)$  the equality of Question 4.7 still fails.

If there are no dominated vertices in  $\Gamma$ , that is  $\text{Dom}(\Gamma) = \emptyset$ , then following holds. Here

$$\text{St}(\mathcal{L}) = \{\phi \in \text{Aut}(G) \mid G(Y)\phi = G(Y), \text{ for all } Y \in \mathcal{L}\},$$

a subgroup of  $\text{Aut}(G)$  defined originally in [19], where it was shown to be an arithmetic group.

**Lemma 4.9.** *Let  $\Gamma$  be a graph such that  $\text{Dom}(\Gamma) = \emptyset$ . Then*

- (i)  $\text{Conj}(G) \cap \text{St}(\mathcal{K}) = \text{Conj}_S(G) = \{1\}$  and  $\text{Conj}(G) = \text{Conj}_V(G) = \text{Conj}_N(G)$  is normal in  $\text{St}^{\text{conj}}(\mathcal{K})$  and
- (ii)  $\text{St}(\mathcal{L}) = \text{St}(\mathcal{K})$ .

*Proof.* (i) In this case  $\text{Conj}_C(G) = \text{Conj}_S(G) = \{1\}$  so  $\text{Conj}_V \cap \text{St}(\mathcal{K}) = 1$ . To see that  $\text{Conj}_N = \text{Conj}$ , note first that, from Lemma 2.5 (iii), it follows that  $\mathfrak{a}(x) = \text{cl}(x)$ , for all  $x \in X$ . Let  $x, y \in X$  and let  $C$  be a component of  $\Gamma_{y^\perp}$ . If  $\mathfrak{a}(x) \cap C \neq \emptyset$  then, from Lemma 3.20, either  $\mathfrak{a}(x) \subseteq C \cup y^\perp$  or  $y \in \mathfrak{a}(x)$ . If  $y \in \mathfrak{a}(x) = \text{cl}(x)$  then  $\text{cl}(x) \subseteq y^\perp$ ; so either  $\mathfrak{a}(x) \cap C = \emptyset$  or  $\mathfrak{a}(x) \subseteq C \cup y^\perp$ . Therefore, either  $u\alpha_{C,y} = u^y$ , for all  $u \in \mathfrak{a}(x)$ , or  $u\alpha_{C,y} = u$ , for all  $u \in \mathfrak{a}(x)$ ; and it follows that  $\text{Conj}_N = \text{Conj}$ .

(ii) From [18, Lemma 2.4] it follows that if  $Y \in \mathcal{L}$  then  $Y = \bigcup_{y \in Y} \text{cl}(y)$ . Therefore, for all  $\phi \in \text{Aut}(G)$ ,  $\phi \in \text{St}(\mathcal{L})$  if and only if  $G(\text{cl}(y))\phi = G(\text{cl}(y))$ , for all  $y \in Y$ . Given that  $\mathfrak{a}(x) = \text{cl}(x)$ , for all  $x \in X$ , the result follows from the remark following the definition of  $\text{St}(\mathcal{K})$  above.  $\square$

**Theorem 4.10.** *The following are equivalent.*

- (i)  $\text{Dom}(\Gamma) = \emptyset$ .
- (ii)  $\text{St}^{\text{conj}}(\mathcal{K}) = \text{Conj}_N(G) \rtimes \text{St}(\mathcal{L})$ .
- (iii)  $\text{St}^{\text{conj}}(\mathcal{K}) = \text{Conj}(G) \rtimes \text{St}(\mathcal{L})$ .

$$(iv) \text{St}^{\text{conj}}(\mathcal{K}) = \text{Conj}(G) \rtimes \text{St}(\mathcal{K}).$$

*Proof.* In view of Lemma 4.9 it suffices to show that each of the last three statements implies the first. To see that the second or third statement implies the first, suppose  $\text{St}^{\text{conj}}(\mathcal{K})$  decomposes as the given internal semi-direct product. If  $y \in \text{Dom}(x)$ , for some  $x, y \in X$ , then  $\tau = \tau_{y,x} \in \text{St}^{\text{conj}}(\mathcal{K})$ , so  $\tau = \alpha\lambda$ , for some  $\alpha \in \text{Conj}$  and  $\lambda \in \text{St}(\mathcal{L})$ . Then, for  $z \in X \setminus y$  we have  $z = z\tau = z\alpha\lambda = z^g\lambda = z\lambda^{g\lambda}$ , for some  $g \in G$ . As  $z\lambda \in G(\text{cl}(z))$  it follows that  $z\lambda = z \circ w$ , for some  $w \in G(\text{cl}(z))$ , so  $(zw)^{g\lambda} = z$ , from which, counting exponents of letters, we infer that  $w = 1$ . Hence  $g\lambda \in G(z^\perp)$ , so  $g \in G(z^\perp)$ , which implies that  $z\alpha = z$ , and consequently  $z\lambda = z$ . Now  $y\alpha = y^h$  and  $z\alpha = z\lambda = z$ , for some  $h \in G$  and all  $z \in X \setminus y$ . As  $\lambda \in \text{St}(\mathcal{L})$  we have  $y\lambda \in G(\text{cl}(y))$  and, since  $z\lambda = z$  for all  $z \neq y$ , we have  $y\lambda = y^\varepsilon w$ , for some  $w \in G(\text{cl}(y))$ ,  $\varepsilon = \pm 1$ . However this means that  $yx = y\tau = y\alpha\lambda = (y^\varepsilon w)^{h\lambda}$  and, as  $x \notin y^\perp$ , the exponent sum of  $x$  on the left hand side of this expression is zero, while on the right it is one. Hence no such  $x, y$  exist and  $\text{Dom}(\Gamma) = \emptyset$ .

To see that the fourth statement implies the first: from the fourth statement it follows that  $\text{Conj}(G) \cap \text{St}(\mathcal{K}) = \{1\}$ , so  $\text{LInn}_S = \emptyset$  and this implies that  $\text{Dom}(\Gamma) = \emptyset$ .  $\square$

## 4.1 Balanced graphs

Although  $\text{Dom}(\Gamma) = \emptyset$  is a necessary condition for the intersection of  $\text{Conj}(G)$  and  $\text{St}(\mathcal{K})$  to be trivial, the class of graphs for which  $\text{St}^{\text{conj}}(\mathcal{K}) = \text{Conj}(G) \text{St}(\mathcal{K})$  is much wider than those without dominated vertices: it can, as we shall show, be characterised using the following definition.

**Definition 4.11.** *A graph  $\Gamma$  is called balanced if the following condition holds for all  $v \in \text{Dom}(\Gamma)$ . Either*

1.  $\text{out}(v) = \emptyset$ , or
2. *there exists a connected component  $C_v$  of  $\Gamma_{v^\perp}$  such that  $\text{out}(v) \subseteq C_v$ .*

In this section we shall use the following extensions of the terminology for transvections and conjugating automorphisms.

**Definition 4.12.** *If  $\tau_{x,y_i}$  is a transvection, for  $x, y_i \in X^{\pm 1}$ , and  $w = y_1 \cdots y_n$  is a geodesic word in  $G$  then  $\tilde{\tau}_{x,w} = \tau_{x,y_n} \cdots \tau_{x,y_1}$  is called a composite transvection and the set of all composite transvections is denoted  $\tilde{\text{Tr}} = \tilde{\text{Tr}}(G)$ .*

**Definition 4.13.** *• If  $L$  consists of a union  $L = \cup_{i=1}^r C_i$  of connected components  $C_i$  of  $\Gamma_{x^\perp}$  then  $\alpha_{L,x^\varepsilon} = \prod_{i=1}^r \alpha_{C_i,x^\varepsilon}$  is called an extended elementary conjugating automorphism. The set of all extended elementary conjugating automorphisms is denoted  $\text{LInn}_W = \text{LInn}_W(G)$ .*

- Let  $y \in X^{\pm 1}$  and  $\alpha_{L,y} \in \text{LInn}_W(G)$ . If  $\mathfrak{a}(y) \cap L = \emptyset$  then  $\alpha_{L,y}$  is called a tame elementary conjugating automorphism of  $G$ . The set of all tame elementary conjugating automorphisms is denoted  $\text{LInn}_T(G)$ .

**Lemma 4.14.** *Let  $x$  and  $y$  be elements of  $X$  such that  $x$  dominates  $y$  and let  $C$  be a component of  $\Gamma_{y^\perp}$ .*

- (i) *If  $x \notin C$  then  $C$  is a component of  $\Gamma_{x^\perp}$ .*
- (ii) *If  $x \in C$  and the components of  $\Gamma_{x^\perp}$  which meet  $C$  are  $C_1, \dots, C_r$  then  $C = [(C_1 \cup \dots \cup C_r) \cup x^\perp] \setminus y^\perp$ .*

*Proof.* (i) If  $C = \{u\}$ , for some  $u \in X$ , then  $y$  dominates  $u$  so  $u^\perp \setminus u \subseteq y^\perp$ . In this case, if  $x \in u^\perp$  then  $x = u$ , since  $x \notin y^\perp$ , but this contradicts  $x \notin C$ . Thus  $x \notin u^\perp$  and  $u^\perp \cap x^\perp = y^\perp \cap u^\perp \cap x^\perp = u^\perp \setminus u$ ; so  $x$  also dominates  $u$  and  $C$  is a component of  $\Gamma_{x^\perp}$ . If  $C$  contains two elements  $u$  and  $v$  then there is a path  $p$  from  $u$  to  $v$  which does not meet  $y^\perp$ . If  $u$  and  $v$  belong to different components of  $\Gamma_{x^\perp}$  then  $p$  meets  $x^\perp$ , and as  $x \notin y^\perp$  this means that  $x \in C$ , a contradiction. Hence  $C \subseteq C'$ , for some component  $C'$  of  $\Gamma_{x^\perp}$ . As  $y^\perp \setminus y \subseteq x^\perp$ , every component of  $\Gamma_{x^\perp}$  containing at least 2 elements is contained in some component of  $\Gamma_{y^\perp}$ , so  $C = C'$ .

(ii) Suppose that  $u \in C_i$ , for some  $i \in \{1, \dots, r\}$ . Either  $u$  belongs to  $y^\perp$  or to some component of  $\Gamma_{y^\perp}$ . However  $y^\perp \setminus y \subseteq x^\perp$  and  $u \notin x^\perp$ , so  $u \notin y^\perp \setminus y$ . As  $\{y\}$  is a connected component of  $\Gamma_{x^\perp}$ , which does not meet any component of  $\Gamma_{y^\perp}$ , the vertex  $u \neq y$ . Hence  $u$  belongs to some component  $C'$  of  $\Gamma_{y^\perp}$ . If  $x \notin C'$  then, from (i),  $C' = C_i$ , in which case  $C' = C$  and  $x \in C'$ , a contradiction. Hence  $x \in C'$  and  $C = C'$ ; so  $C_i \subseteq C$ , for all  $i$ . By definition  $C \subseteq \cup_{i=1}^r C_i \cup x^\perp$ , and the result follows.  $\square$

**Lemma 4.15.** *Let  $y \in X$ ,  $v, x \in X^{\pm 1}$ ,  $\alpha = \alpha_{L,y} \in \text{LInn}_W$  and  $\tau = \tau_{v,x} \in \text{Tr}$ .*

- (i) *If either  $v \in L$  and  $x \in L \cup y^\perp$  or  $v \notin L$ ,  $v \neq y^{\pm 1}$  and  $x \notin L$  then*

$$\alpha\tau = \tau\alpha.$$

- (ii) *If  $v \in L$  and  $x \notin L \cup y^\perp$  then  $v \in \text{Dom}(y)$  and*

$$\alpha\tau = \tau_{v,y}\tau\tau_{v,y}^{-1}\alpha.$$

- (iii) *If  $v \notin L$ ,  $v \neq y^{\pm 1}$  and  $x \in L$  then  $v \in \text{Dom}(y)$  and*

$$\alpha\tau = \tau_{v,y}^{-1}\tau\tau_{v,y}\alpha.$$

(iv) If  $y = v^{\pm 1}$  and  $x \notin L$  then  $L$  is a union of connected components of  $\Gamma_{x^\perp}$  and, setting  $\beta = \alpha_{L,x}$ ,

$$\alpha\tau = \begin{cases} \tau\beta\alpha, & \text{if } v = y \\ \tau\alpha\beta^{-1}, & \text{if } v = y^{-1} \end{cases}.$$

*Proof.* (i) If  $v \in L$  and  $x \in L \cup y^\perp$  then

$$z\alpha\tau = z\tau\alpha = \begin{cases} z, & \text{if } z \notin L \\ (vx)^y, & \text{if } z = v \\ z^y, & \text{if } z \in L \text{ and } z \neq v^{\pm 1} \end{cases}. \quad (4.1)$$

If  $v \notin L$ ,  $v \neq y^{\pm 1}$  and  $x \notin L$  then

$$z\alpha\tau = z\tau\alpha = \begin{cases} z, & \text{if } z \notin L, z \neq v^{\pm 1} \\ vx, & \text{if } z = v \\ z^y, & \text{if } z \in L \end{cases}. \quad (4.2)$$

(ii) In this case  $[x, v] \neq 1$ , as  $v \in L$  and  $x \notin y^\perp$ ; so  $x \in \text{out}(v)$ . As  $v \in L$  and  $x \notin L \cup y^\perp$ , all paths from  $v$  to  $x$  must intersect  $y^\perp$ , so  $v^\perp \setminus v \subseteq y^\perp$ ; and  $v \notin y^\perp$ , so  $v \in \text{Dom}(y)$ . Then  $z\alpha\tau$  is as given in (4.1), and is equal to  $z\tau_{v,y}\tau\tau_{v,y}^{-1}\alpha$ , for all  $z \in X$ .

(iii) If  $y \in v^\perp$  then, as  $v \neq y^{\pm 1}$ ,  $x \in y^\perp$ , a contradiction. Thus, as in the previous case,  $y$  dominates  $v$ . Then  $z\alpha\tau$  is as given in (4.2), and is equal to  $z\tau_{v,y}^{-1}\tau\tau_{v,y}\alpha$ , for all  $z \in X$ .

(iv) In this case  $y$  is dominated by  $x$  so, from Lemma 4.14,  $L$  is a union of connected components of  $\Gamma_{x^\perp}$ . Suppose  $v = y^\varepsilon$ , where  $\varepsilon = \pm 1$ . Then

$$z\alpha\tau = \begin{cases} z, & \text{if } z \notin L, z \neq v^{\pm 1} \\ vx, & \text{if } z = v \\ z^{(vx)^\varepsilon}, & \text{if } z \in L \end{cases}$$

and this is equal to  $z\tau(\beta\alpha^\varepsilon)^\varepsilon$ , for all  $z \in X$ .  $\square$

**Corollary 4.16.** *Let  $y \in X$  and  $v \in X^{\pm 1}$ ,  $v \neq y^{\pm 1}$ . Let  $\alpha = \alpha_{L,y} \in \text{LInn}_T$  and let  $\tilde{\tau}_{v,a} \in \tilde{\text{Tr}}$ . Then*

$$\alpha\tilde{\tau}_{v,a} = \tilde{\tau}_{v,b}\alpha,$$

*for some  $\tilde{\tau}_{v,b} \in \tilde{\text{Tr}}$ .*

*Proof.* Let  $a = a_1 \cdots a_n$ , where  $a_i \in X^{\pm 1}$ , be a geodesic word representing  $a$ . Then  $\tilde{\tau}_{v,a} = \tau_{v,a_n} \cdots \tau_{v,a_1}$ . As  $v \neq y^{\pm 1}$ ,  $\alpha\tau_{v,a_i} = \tau_{v,y}^{-\varepsilon_i} \tau_{v,a_i} \tau_{v,y}^{\varepsilon_i} \alpha$ , with  $\varepsilon_i = 0$  or  $\pm 1$ , for all  $i$ . The corollary follows on setting  $b$  equal to the word obtained by freely reducing  $\prod_{i=1}^n y^{\varepsilon_i} a_i y^{-\varepsilon_i}$ .  $\square$

**Corollary 4.17.** *Let  $v \in X$ ,  $\alpha = \alpha_{L,v} \in \text{LInn}_T$  and let  $\tau = \tilde{\tau}_{v,a} \in \tilde{\text{Tr}}$ . Then*

$$\alpha\tau = \tau\beta,$$

*for some  $\beta \in \langle \text{LInn}_T \rangle$ .*

*Proof.* Let  $a = a_1 \cdots a_n$ , where  $a_i \in X^{\pm 1}$ , be a geodesic word representing  $a$ . By definition of  $\tilde{\text{Tr}}$  we have  $a_i \in (\mathfrak{a}(v) \setminus \{v\})^{\pm 1}$ , for all  $i$ . Hence, by definition of  $\text{LInn}_T$ ,  $a_i \notin L$ , for all  $i$ . Thus

$$\alpha\tau_{v,a_i} = \tau_{v,a_i}\alpha_{L,a_i}\alpha.$$

Since  $v \neq a_i^{\pm 1}$ ,  $v \notin L$  and  $a_j \notin L$ , also

$$\alpha_{L,a_i}\tau_{v,a_j} = \tau_{v,a_j}\alpha_{L,a_i}$$

when  $i \neq j$ . Therefore

$$\begin{aligned} \alpha\tau &= \alpha\tau_{v,a_n} \cdots \tau_{v,a_1} \\ &= \tau_{v,a_n} \cdots \tau_{v,a_1}\alpha_{L,a_n} \cdots \alpha_{L,a_1}\alpha \\ &= \tau\alpha_{L,a_n} \cdots \alpha_{L,a_1}\alpha. \end{aligned}$$

As  $\alpha \in \text{LInn}_T$ , we have  $\mathfrak{a}(v) \cap L = \emptyset$  and, as  $\tau_{v,a_i} \in \text{Tr}$ , we have  $\mathfrak{a}(a_i) \subseteq \mathfrak{a}(v)$  so  $\mathfrak{a}(a_i) \cap L = \emptyset$ . Hence  $\alpha_{L,a_i} \in \text{LInn}_T$ ; and the result follows.  $\square$

**Proposition 4.18.** *Let  $\Gamma$  be a connected graph. Then  $\langle \text{Tr} \cup \text{LInn}_T \rangle = \langle \text{Tr} \rangle \langle \text{LInn}_T \rangle$ .*

*Proof.* It suffices to prove the proposition holds with  $\tilde{\text{Tr}}$  in place of  $\text{Tr}$ . First suppose that  $u$  is a word on the generators  $\text{LInn}_T$  and their inverses and that  $\tau \in \tilde{\text{Tr}}$ . It follows by a straightforward induction on  $|u|$  and Corollary 4.17 that  $u\tau = \tau u'$  in  $\text{St}^{\text{conj}}(\mathcal{K})$ , for some word  $u'$  over  $\text{LInn}_T^{\pm 1}$ .

Now let  $w$  be a word in the generators of  $\langle \tilde{\text{Tr}} \cup \text{LInn}_T \rangle$  and their inverses. If  $|w| \leq 1$  then  $w \in \langle \tilde{\text{Tr}} \rangle \langle \text{LInn}_T \rangle$ . Assume inductively that for some  $k \geq 1$  all words  $w$  of length at most  $k$  can be expressed as elements of  $\langle \tilde{\text{Tr}} \rangle \langle \text{LInn}_T \rangle$ .

Let  $w$  be a word of length  $k+1$  (in the given generators). Then  $w = w_0\xi$ , for some word  $w_0$  of length  $k$  and generator  $\xi \in (\tilde{\text{Tr}} \cup \text{LInn}_T)^{\pm 1}$ . By induction there exists words  $w_1 \in \langle \tilde{\text{Tr}} \rangle$  and  $w_2 \in \langle \text{LInn}_T \rangle$  such that  $w_0 = w_1w_2$ , in  $\text{St}^{\text{conj}}(\mathcal{K})$ . If  $\xi \in \text{LInn}_T^{\pm 1}$  the proof is complete. Otherwise  $\xi \in \tilde{\text{Tr}}^{\pm 1}$  and, from the first part of the proof we may rewrite  $w_2\xi$  to a word  $\xi'w'_2$ , with  $\xi' \in \tilde{\text{Tr}}^{\pm 1}$  and  $w'_2 \in \langle \text{LInn}_T \rangle$ , such that  $w_2\xi = \xi'w'_2$  in  $\text{St}^{\text{conj}}(\mathcal{K})$ . Then  $w = w_1\xi'w'_2 \in \langle \tilde{\text{Tr}} \rangle \langle \text{LInn}_T \rangle$ , as required.  $\square$

**Theorem 4.19.** *Let  $\Gamma$  be a connected graph and  $G = G(\Gamma)$ . Then  $\text{St}^{\text{conj}}(\mathcal{K}) = \text{St}(\mathcal{K}) \text{Conj}(G)$  if and only if  $\Gamma$  is a balanced graph.*

*Proof.* First suppose that  $\Gamma$  is a balanced graph. Let  $\iota = \iota_y \in \text{Inv}$ , let  $\alpha_{L,x} \in \text{LInn}$  and  $\tau = \tau_{v,x} \in \text{Tr}$ , where  $y \in X$  and  $x, v \in X^{\pm 1}$ . Then  $\alpha\iota = \iota\alpha$  unless  $y = x^{\pm 1}$ , in which case  $\alpha\iota = \iota\alpha^{-1}$ . Also  $\tau\iota = \iota\tau$ , unless  $y = x^{\pm 1}$ , in which case  $\tau\iota = \iota\tau^{-1}$ , or  $v = y^{\pm 1}$ , in which case  $\tau\iota = \iota\tau_{v^{-1},x}$ . It therefore suffices to show that elements of  $\langle \text{Tr} \cup \text{LInn} \rangle$  belong to  $\text{St}(\mathcal{K}) \text{Conj}(G)$ .

First we show that, as  $\Gamma$  is balanced,  $\langle \text{Tr} \cup \text{LInn} \rangle$  is generated by  $\text{Tr} \cup \text{Inn} \cup \text{LInn}_T$ . To see this suppose that  $y \in \text{Dom}(\Gamma)$ ,  $\text{out}(y) \neq \emptyset$  and  $C_y$  is the component of  $\Gamma_{y^\perp}$  meeting  $\text{out}(y)$ . Let  $L = X \setminus (C_y \cup y^\perp \cup [y])$ , so  $L$  is a union of connected components of  $\Gamma_{y^\perp}$  and  $\alpha_{L,y} \in \text{LInn}_T$ . Let  $D_y = [y] \setminus y$  and note that, since  $y \in \text{Dom}(\Gamma)$ , we have  $[y] \setminus y = [y] \setminus y^\perp$  so  $D_y$  is a union of connected components of  $\Gamma_{y^\perp}$ . If  $D_y = \emptyset$  define  $\beta_y = 1$  and otherwise define  $\beta_y = \alpha_{D_y,y}$ . Then  $\alpha_{C_y,y}\beta_y\alpha_{L,y} = \gamma_y \in \text{Inn}(G)$ , so the generators  $\alpha_{C_y,y}$  of this form are contained in the subgroup generated by  $\text{Inn}$ ,  $\text{LInn}_T$  and the set  $\{\beta_y \in \text{LInn}_W : y \in \text{Dom}(\Gamma), \text{out}(y) \neq \emptyset\}$ .

Now suppose that  $y \in \text{Dom}(\Gamma)$  and  $[y] \neq \{y\}$ . Then for all  $v \in [y]$ ,  $v \neq y$ , we have  $\alpha_{v,y} = \tau_{v^{-1},y}\tau_{v,y}$ , so  $\beta_y = \alpha_{D_y,y} = \prod_{v \in D_y} \tau_{v^{-1},y}\tau_{v,y}$ . Thus all generators  $\beta_y$  are contained in the subgroup generated by  $\text{Tr}$ . It follows that every word on generators  $\text{Tr} \cup \text{LInn}$  and their inverses may be replaced by a word on  $\text{Tr} \cup \text{Inn} \cup \text{LInn}_T$  and their inverses. Thus  $\langle \text{Tr} \cup \text{LInn} \rangle = \langle \text{Tr} \cup \text{Inn} \cup \text{LInn}_T \rangle$ . As  $\langle \text{Inn} \rangle$  is normal in  $\text{Aut}(G)$  it suffices to show that elements of  $\langle \text{Tr} \cup \text{LInn}_T \rangle$  belong to  $\text{St}(\mathcal{K}) \text{Conj}(G)$ : and this follows from Proposition 4.18.

For the converse suppose that  $\Gamma$  is not a balanced graph. We shall show that the obstruction of Example 4.8 is also manifested in  $\text{St}^{\text{conj}}(\mathcal{K})$ . Indeed the argument is a generalised version of that example. If  $\Gamma$  is not balanced then there exists a vertex  $v \in \text{Dom}(\Gamma)$  such that  $\text{out}(v) \neq \emptyset$  and there is no component  $C$  of  $\Gamma_{v^\perp}$  such that  $\text{out}(v) \subseteq C$ . Let  $v$  be such a vertex and let  $a, b \in \text{out}(v)$  such that  $a$  and  $b$  are in different components  $C$  and  $B$ , respectively, of  $\Gamma_{v^\perp}$ .

Suppose that  $a_0 \in \text{out}(v)$  and  $a_0 <_{\mathcal{K}} a$ . Then  $a_0 <_{\mathcal{K}} a$  implies that  $a^\perp \setminus a \subseteq a_0^\perp$  and  $a \in \text{out}(v)$  implies that there exists  $u \in a^\perp \setminus a$  such that  $u \notin v^\perp$ . Thus  $a_0 \in u^\perp$ , so  $a_0 \in C$ . We may therefore assume that  $a$  is  $\mathcal{K}$ -minimal among elements of  $\text{out}(v)$ . Similarly we may assume  $b$  is  $\mathcal{K}$ -minimal among elements of  $\text{out}(v)$ .

Define  $\phi = \alpha_{C,v}\tau_{v,a}\tau_{v,b}$  so

$$z\phi = \begin{cases} z, & \text{if } z \notin C \cup \{v\} \\ vba, & \text{if } z = v \\ z^{vba} & \text{if } z \in C \end{cases}.$$

Assume that there exist  $\gamma \in \text{St}(\mathcal{K})$  and  $\delta \in \text{Conj}$  such that  $\phi = \gamma\delta$ .

Note that  $Z(G(\mathfrak{a}(v)))$  is generated by  $\mathfrak{a}(v) \cap (v^\perp \setminus v)$  (which may be empty). Let  $c \in \mathfrak{a}(v) \cap (v^\perp \setminus v)$ . Then  $\mathfrak{a}(c) \subseteq \mathfrak{a}(v) \cap (v^\perp \setminus v)$  and so if  $z \in \mathfrak{a}(c)$  there exists  $w_z \in G(\mathfrak{a}(c))$  such that  $z\gamma = w_z$ . As  $Z(G(\mathfrak{a}(v)))$  is Abelian so is  $G(\mathfrak{a}(c))$  and so  $w_z$  is cyclically reduced. From Lemma 2.1, there exists  $g \in G$  such that  $z\delta = z^g$ , for all  $z \in \mathfrak{a}(c)$ . Hence, if  $z \in \mathfrak{a}(c)$  then  $z = z\phi = z\gamma\delta = w_z\delta = w_z^g$ , with  $w_z \in G(\mathfrak{a}(c))$ , so  $g = 1$  and  $w_z = z$ . Therefore  $z\gamma = z\delta = z$ , for all  $z \in Z(G(\mathfrak{a}(v)))$ .

As  $a$  is  $\mathcal{K}$ -minimal among elements of  $\text{out}(v)$  we have  $\mathfrak{a}(a) \setminus [a] \subseteq \mathfrak{a}(v) \cap (v^\perp \setminus v)$ . As  $\delta \in \text{St}^{\text{conj}}(\mathcal{K})$ , there exists  $g \in G$  such that  $G(\mathfrak{a}(a))\delta = G(\mathfrak{a}(a))^g$  and we may assume that  $g$  has no left divisor in  $G(\mathfrak{a}(a))$  or  $G(\mathfrak{a}(a)^\perp)$ . Let  $z \in [a]$ , so  $z\phi = z^{vba}$ . We have  $z\gamma \in G(\mathfrak{a}(a))$  and so  $z\gamma\delta = u_z^g$ , for some  $u_z \in G(\mathfrak{a}(a))$ . Therefore  $u_z^g = z^{vba}$  and so  $z^{vbag^{-1}} \in G(\mathfrak{a}(a))$ . As neither  $v$  nor  $b$  commute with  $a$  or  $z$  it follows that  $g = g_1 \circ vba$ , and then  $z^{g_1^{-1}} \in G(\mathfrak{a}(a))$ . This holds for all  $z \in [a]$ , and for any  $u \in \mathfrak{a}(a) \setminus [a]$  we have  $u\delta = u$ , from the paragraph above, so  $[u, g] = 1$ . Since  $g$  has no left divisor in  $G(\mathfrak{a}(a) \cup \mathfrak{a}(a)^\perp)$ , [20, Corollary 2.5] implies that  $g_1 = 1$  and  $g = vba$ . Now  $z\delta = z^{g_z}$ , for some  $g_z \in G$ , so we have  $z^{g_z} = w_z^{vba}$ , for some  $w_z \in G(\mathfrak{a}(a))$ . Again  $z = w_z^{vbag_z^{-1}}$ , so  $z \in \alpha(w_z)$  and  $v, b \notin \mathfrak{a}(a)$ , so  $g_z = h_z \circ vba$ , for some  $h_z \in G(\mathfrak{a}(a))$ , and  $w_z = z^{h_z}$ . As elements of  $\mathfrak{a}(a) \setminus [a]$  belong to the centre of  $G(\mathfrak{a}(a))$ , moreover  $h_z \in G[a]$ . Therefore, for all  $z \in [a]$ ,  $z\delta = z^{h_z vba}$ , for some  $h_z \in G[a]$ .

Similarly, we have  $\mathfrak{a}(b) \setminus [b] \subseteq \mathfrak{a}(v) \cap (v^\perp \setminus v)$  and there exists  $g \in G$  such that  $G(\mathfrak{a}(b))\delta = G(\mathfrak{a}(b))^g$  and  $g$  has no left divisor in  $G(\mathfrak{a}(b))$  or  $G(\mathfrak{a}(b)^\perp)$ . Let  $z \in [b]$ , so  $z\phi = z$ . We have  $z\gamma \in G(\mathfrak{a}(b))$  and so  $z\gamma\delta = u_z^g$ , for some  $u_z \in G(\mathfrak{a}(b))$ . Therefore  $u_z^g = z$  and  $z^{g^{-1}} \in G(\mathfrak{a}(b))$ . Thus  $[z, g] = 1$ , which implies  $[[b], g] = 1$ . For any  $u \in \mathfrak{a}(b) \setminus [b]$  we have  $u\delta = u$ , so  $[u, g] = 1$  and therefore  $g = 1$ . Now  $z\delta = z^{g_z}$ , for some  $g_z \in G$ , so we have  $z^{g_z} = w_z$ , for some  $w_z \in G(\mathfrak{a}(b))$ . Thus  $g_z \in G(\mathfrak{a}(b))$ , and as elements of  $\mathfrak{a}(b) \setminus [b]$  belong to the centre of  $G(\mathfrak{a}(b))$ , moreover  $g_z \in G[b]$ . Therefore, for all  $z \in [b]$ ,  $z\delta = z^{g_z}$ , for some  $g_z \in G[b]$ .

Now let  $z \in v^\perp \setminus v$ . Then  $z \in C_G(a, b)$  so  $z\delta \in C_G(a^{h_a vba}) = C_G(a^{h_a})^{vba} \subseteq C_G(a)^{vba} = G(a^\perp)^{vba}$  and  $z\delta \in C_G(b^{g_b}) \subseteq C_G(b) = G(b^\perp)$ . If  $w \in G(b^\perp)$  and  $w = u^{vba}$ , where  $u \in G(a^\perp)$ , then  $a \notin b^\perp$  implies  $a \notin \nu(w)$  so  $a$ , and therefore also  $b$  and  $v$ , cancel in reducing  $u^{vba}$  to  $w$ . Neither  $b$  nor  $v$  belong to  $\nu(u)$ , hence  $[u, b] = [u, v] = 1$  and  $u \in G(v^\perp \setminus v)$ . Thus  $u^{vba} = u$ . It follows that  $G(a^\perp)^{vba} \cap G(b^\perp) = G(v^\perp \setminus v)$  and so  $z\delta \in G(v^\perp \setminus v)$ , for all  $z \in v^\perp \setminus v$ .

As  $G(v^\perp \setminus v)\delta = G(v^\perp \setminus v)$ , for all  $z \in \mathfrak{a}(v)$ , we have  $z\delta \in G(\mathfrak{a}(v))$ , so  $z^\delta = z^{g_z}$ , for some  $g_z \in G(\mathfrak{a}(v))$ . Now  $\delta$  satisfies the following (with  $w = vb$ )

1.  $z\delta = z$ , for all  $z \in \mathfrak{a}(v) \cap v^\perp \setminus v$ .

2.  $z\delta = z^{g_z}$ , with  $g_z \in G[b]$ , for all  $z \in [b]$ .
3.  $z\delta = z^{h_z w a}$ , with  $w = vb$  or  $b$  and  $h_z \in G[a]$ , for all  $z \in [a]$ .
4.  $z\delta = z^{g_z}$ , with  $g_z \in G(\mathfrak{a}(v))$ , for all  $z \in \mathfrak{a}(v)$ .

Let us call an element of  $\text{Conj}$  *v-unlikely* if it satisfies all of these four properties. Amongst all *v-unlikely* basis conjugating automorphisms choose one, which we shall now also call  $\delta$ , of minimal length. As usual, for each  $x \in X$  let  $g_x \in G$  be such that  $x\delta = x^{g_x}$ .

From condition 4,  $\mathfrak{a}(v)\delta \subseteq G(\mathfrak{a}(v))$  and by direct calculation  $\mathfrak{a}(v)\phi^{-1} \subseteq G(\mathfrak{a}(v))$ . As  $\gamma \in \text{St}(\mathcal{K})$  this implies  $\mathfrak{a}(v)\phi^{-1}\gamma = \mathfrak{a}(v)\delta^{-1} \subseteq G(\mathfrak{a}(v))$ . Hence  $\delta$  restricts to an automorphism of  $G(\mathfrak{a}(v))$  and, applying Lemma 3.42 to this restriction, there exist elements  $x, y \in \mathfrak{a}(v)$  such that  $x^\varepsilon g_x$  is a right divisor of  $g_y$ . Moreover,  $x, y \in \text{out}(v) \cup [v]$ , as the centre of  $G(\mathfrak{a}(v))$ , which is pointwise fixed by  $\delta$ , is generated by  $\mathfrak{a}(v) \cap (v^\perp \setminus v)$ . Suppose that  $x, y \in C$  and let  $D$  be the component of  $\Gamma_{x^\perp}$  containing  $y$ . As  $x, y \in \mathfrak{a}(v)$ , Lemma 4.14 (ii) implies that  $D \subseteq C$ . Define  $\delta_0 = \alpha_{D,x}^{-\varepsilon} \delta$ . For all  $z \in X \setminus D$  we have  $z\delta_0 = z\delta$  and (applying Lemma 3.42 again)  $|\delta_0| < |\delta|$ . If  $a \notin D$  then clearly  $\delta_0$  is *v-unlikely*, contrary to the choice of  $\delta$ . If  $a \in D$  then, for all  $z \in [a] \cap D$ ,  $z \neq x$ , it follows that  $x^\varepsilon g_x$  is a right divisor of  $h_z w a$ , which implies  $h_z = h'_z x^\varepsilon h''_z w a$ . Therefore  $z\delta_0 = z^{h'_z h''_z w a}$ , for all  $z \in ([a] \cap D) \setminus \{x\}$ , and again  $\delta_0$  is *v-unlikely*, a contradiction. We may therefore assume that  $\{x, y\} \not\subseteq C$ .

Assume that  $y \notin C$  and that  $D$  is the component of  $\Gamma_{x^\perp}$  containing  $y$ . Then  $D \cap C = \emptyset$  and  $g_z = g'_z x^\varepsilon g_x$ , for all  $z \in D$ . Again set  $\delta_0 = \alpha_{D,x}^{-\varepsilon} \delta$  and  $\delta_0$  is *v-unlikely* with  $|\delta_0| < |\delta|$ . This contradiction shows that we may assume  $y \in C$  and  $x \notin C$ . Then  $C$  is a component of  $\Gamma_{x^\perp}$  and  $x^\varepsilon g_x$  is a right divisor of  $h_z w a$ , for all  $z \in [a]$ , as  $a, y \in C$ . As  $\nu(h_z) \subseteq [a]$  and  $x \notin C$  this implies  $x = v$  or  $b$ . If  $x = b$  then  $g_b = a$ , a contradiction, so we have  $x = v$ ,  $w = vb$  and  $g_v = ba$ .

Let  $\delta_0 = \alpha_{C,v}^{-1} \delta$ , so  $z\delta_0 = z^{h_z b a}$ , for  $z \in [a]$ , and  $z\delta_0 = z\delta$ , for  $z \notin C$ . Again  $\delta_0$  is *v-unlikely*, contrary to minimality of the length of  $\delta$ . In all cases we obtain a contradiction, so there exists no *v-unlikely* automorphism  $\delta$ , completing the proof that  $\phi \notin \text{St}(\mathcal{K}) \text{Conj}$ . □



## 5 Appendix

### 5.1 A presentation for the graph automorphisms of $G$

A presentation for  $\text{Aut}_{\text{comp}}^{\Gamma}(G)$  may be constructed using the wreath product structure, of the factors of the direct sum, in the decomposition of Proposition 3.9(ii). First we establish presentations for these factors.

Recall from Definition 3.10 that  $\text{Aut}_{\text{comp}}^{\Gamma}(G_{j,1})$  has generating set  $\mathcal{P}_{\text{comp},j}^{\Gamma}$ . By definition  $\text{Aut}_{\text{comp}}^{\Gamma}(G_{j,1}) \cong \text{Aut}(\Omega_j)$ , and so we may construct a presentation

$$\langle \mathcal{P}_{\text{comp},j}^{\Gamma} | \mathcal{R}_{\text{comp},j}^{\Gamma} \rangle \text{ for } \text{Aut}_{\text{comp}}^{\Gamma}(G_{j,1}).$$

Also  $\mathcal{P}_{\text{symm},j}^{\Gamma} = \{\omega_{a,b}^j | 1 \leq a < b \leq m_j\}$  is a generating set for  $\text{Aut}_{\text{symm}}^{\Gamma}(G_{j,*})$ , and so we may choose a presentation

$$\langle \mathcal{P}_{\text{symm},j}^{\Gamma} | \mathcal{R}_{\text{symm},j}^{\Gamma} \rangle \text{ for } \text{Aut}_{\text{symm}}^{\Gamma}(G_{j,*}).$$

Let

$$\begin{aligned} \mathcal{W}_j^{\Gamma} = & \{[\omega_{a,b}^j, p] : p \in \mathcal{P}_{\text{comp},j}^{\Gamma}, 2 \leq a < b \leq m_j\} \\ & \cup \{[p, \omega_{1,a}^j q \omega_{1,a}^j] : p, q \in \mathcal{P}_{\text{comp},j}^{\Gamma}, 2 \leq a \leq m_j\} \\ & \cup \{[\omega_{1,a}^j p \omega_{1,a}^j, \omega_{1,b}^j q \omega_{1,b}^j] : p, q \in \mathcal{P}_{\text{comp},j}^{\Gamma}, 2 \leq a < b \leq m_j\}. \end{aligned}$$

Let

$$\mathcal{P}_j^{\Gamma} = \mathcal{P}_{\text{comp},j}^{\Gamma} \cup \mathcal{P}_{\text{symm},j}^{\Gamma} \text{ and } \mathcal{R}_j^{\Gamma} = \mathcal{R}_{\text{comp},j}^{\Gamma} \cup \mathcal{R}_{\text{symm},j}^{\Gamma} \cup \mathcal{W}_j^{\Gamma}.$$

**Proposition 5.1.**  $\prod_{k=1}^{m_j} \text{Aut}_{\text{comp}}^{\Gamma}(G_{j,k}) \rtimes \text{Aut}_{\text{symm}}^{\Gamma}(G_{j,*})$  has presentation  $\langle \mathcal{P}_j^{\Gamma} | \mathcal{R}_j^{\Gamma} \rangle$ .

*Proof.*  $\text{Aut}_{\text{comp}}^{\Gamma}(G_{j,k}) \cong \text{Aut}_{\text{comp}}^{\Gamma}(G_{j,1})$ , for  $k = 2, \dots, m_j$ , and  $\text{Aut}_{\text{symm}}^{\Gamma}(G_{j,*})$  acts on  $\prod_{k=1}^{m_j} \text{Aut}_{\text{comp}}^{\Gamma}(G_{j,k})$  by permuting the factors. Hence the group in question is a wreath product; and the given presentation is obtained from a standard construction.  $\square$

As  $\text{Aut}_{\text{comp}}^{\Gamma}(G)$  is a direct sum of the groups of the previous lemma a presentation can be written down immediately. In order to do so define

$$\mathcal{D}^{\Gamma} = \{[p, q] : p \in \mathcal{P}_i^{\Gamma}, q \in \mathcal{P}_j^{\Gamma}, 1 \leq i < j \leq d\}.$$

From Proposition 3.9(ii) we obtain the next corollary.

**Corollary 5.2.**  $\text{Aut}_{\text{comp}}^G(G)$  has presentation  $\langle \mathcal{P}_{\text{comp}}^{\Gamma} | \mathcal{R}_{\text{comp}}^{\Gamma} \rangle$ , where  $\mathcal{P}_{\text{comp}}^{\Gamma} = \bigcup_{j=1}^d \mathcal{P}_j^{\Gamma}$  and  $\mathcal{R}_{\text{comp}}^{\Gamma} = \bigcup_{j=1}^d \mathcal{R}_j^{\Gamma} \cup \mathcal{D}^{\Gamma}$ .

## Proof of Theorem 3.29

*Proof of Proposition 3.29.* Let  $\mathbb{A}$  be the group with presentation  $\langle \mathcal{P} | \mathcal{R} \rangle$ . Identifying each generator of  $\mathcal{P}$  with the elements of the same name in  $\text{Aut}(G)$  straightforward computation shows that all the relators in  $\mathcal{R}$  hold in  $\text{Aut}(G)$ ; giving a canonical homomorphism  $\Theta$  from  $\mathbb{A}$  to  $\text{Aut}(G)$ . We shall use the presentation of  $\text{Aut}(G)$  given in [24], which we shall call  $\langle \mathcal{Q} | \mathcal{S} \rangle$ , to construct an inverse to  $\Theta$ .

To define  $\langle \mathcal{Q} | \mathcal{S} \rangle$  the automorphisms of a free product are divided into four types, in [24]. The first two types are the permutation and factor automorphisms. The *permutation automorphisms* are those belonging to the subgroup  $\text{Aut}_{\text{symm}}^\Gamma(G)$ . The *factor automorphisms* are those automorphisms  $\alpha$  such that  $\alpha$  restricted to  $G(\Gamma_{j,k})$  is an automorphism of  $G(\Gamma_{j,k})$ , for all  $(j, k) \in S \cup J$ . Let  $\Psi$  be the subgroup generated by the permutation and factor automorphisms.

The first step in the definition of  $\langle \mathcal{Q} | \mathcal{S} \rangle$  is to choose a presentation for  $\Psi$ . In our case we extend the notation of Definition 3.27 to denote by  $\text{Aut}(G_{j,k})$  the subgroup of automorphisms  $\phi$  such that  $x\phi = x$ , if  $x \in X \setminus X_{j,k}$  and  $X_{j,k}\phi \subseteq G(\Gamma_{j,k})$ . Then  $\Psi$  is generated by the subgroups  $\text{Aut}(G_{j,k})$  and  $\text{Aut}_{\text{symm}}^\Gamma(G_{j,*})$ , for  $0 \leq j \leq d$ ,  $1 \leq k \leq m_j$  (see Definition 3.7). For fixed  $j$ , with  $0 \leq j \leq d$ , let  $\Psi_j$  be the subgroup of  $\Psi$  generated by  $\text{Aut}(G_{j,k})$ , for  $1 \leq k \leq m_j$ , and  $\text{Aut}_{\text{symm}}^\Gamma(G_{j,*})$ . Then

$$\Psi_j = \prod_{k=1}^{m_j} \text{Aut}(G_{j,k}) \rtimes \text{Aut}_{\text{symm}}^\Gamma(G_{j,*}).$$

Let

$$\begin{aligned} \mathcal{W}_j = & \{[\omega_{a,b}^j, p] : p \in \mathcal{P}_j, 2 \leq a < b \leq m_j\} \\ & \cup \{[p, \omega_{1,a}^j q \omega_{1,a}^j] : p, q \in \mathcal{P}_j, 2 \leq a \leq m_j\} \\ & \cup \{[\omega_{1,a}^j p \omega_{1,a}^j, \omega_{1,b}^j q \omega_{1,b}^j] : p, q \in \mathcal{P}_j, 2 \leq a < b \leq m_j\}. \end{aligned}$$

(Thus  $\mathcal{W}_j \supseteq \mathcal{W}_j^\Gamma$ .) Then

**Proposition 5.3.**  $\Psi_j$  has a presentation

$$\langle \mathcal{P}_j \cup \mathcal{P}_{\text{symm},j}^\Gamma | \mathcal{R}_j \cup \mathcal{R}_{\text{symm},j}^\Gamma \cup \mathcal{W}_j \rangle.$$

*Proof.*  $\text{Aut}(G_{j,k}) \cong \text{Aut}(G_{j,1})$ , for  $k = 2, \dots, m_j$ , and  $\text{Aut}_{\text{symm}}^\Gamma(G_{j,*})$  acts on  $\prod_{k=1}^{m_j} \text{Aut}(G_{j,k})$  by permuting the factors. The result follows as in the proof of Proposition 5.1.  $\square$

As  $\Psi = \prod_{j=0}^d \Psi_j$  we have

**Corollary 5.4.**  $\Psi$  has presentation  $\langle \mathcal{Q}_\Psi | \mathcal{S}_\Psi \rangle$  where  $\mathcal{Q}_\Psi = \mathcal{P}_{\text{int}} \cup \mathcal{P}_{\text{comp}}^\Gamma$  (see Definitions 3.10 and 3.24) and  $\mathcal{S}_\Psi = \cup_{j=0}^d (\mathcal{R}_j \cup \mathcal{R}_{\text{symm},j}^\Gamma \cup \mathcal{W}_j) \cup \mathcal{D}$ .

*Proof.* This follows from Proposition 5.3 as

$$\begin{aligned} \cup_{j=0}^d (\mathcal{P}_j \cup \mathcal{P}_{\text{symm},j}^\Gamma) &= \cup_{j=0}^d (\mathcal{P}_{\text{int}} \cap \text{Aut}(G_{j,1})) \cup \cup_{j=0}^d \mathcal{P}_{\text{comp},j}^\Gamma \cup \cup_{j=0}^d \mathcal{P}_{\text{symm},j}^\Gamma \\ &= \mathcal{P}_{\text{int}} \cup \mathcal{P}_{\text{comp}}^\Gamma. \end{aligned}$$

□

The generators  $\mathcal{Q}$  consist of  $\mathcal{Q}_\Psi$  together with a set  $\mathcal{Q}_{\text{WH}}$  of elements of  $\langle \text{LInn}_{\text{ext}} \cup \text{Tr}_{\text{ext}} \rangle$ , called *Whitehead automorphisms*, which we now define. First, for  $a \in \cup_{j \in J} G(\Gamma_j) \cup X_S^{\pm 1}$  we define

$$\hat{a} = \begin{cases} j & \text{if } a \in G(\Gamma_j) \\ x & \text{if } a = x^\varepsilon, \text{ where } x \in X_S, \varepsilon = \pm 1 \end{cases}.$$

(Thus, in comparison to the notation of page 27, if  $a \in G(G_j)$  or  $a \in X_S$  then  $\hat{a} = \check{a}$ , whereas if  $a \in X_S^{-1}$  then  $\hat{a} = \check{a}^{-1}$ .) For  $i, j \in J$  with  $i \neq j$ ,  $a \in G(\Gamma_j) \cup X_S^{\pm 1}$  and  $x \in X_S^{\pm 1}$ , with  $x^{\pm 1} \neq a$ , extend the notation for transvections and locally inner automorphisms to denote by

1.  $\tau_{x,a}$  the automorphism  $\tau$  such that  $x\tau = xa$  and  $y\tau = y$ , for all  $y \in X$ ,  $y \neq x$  and
2.  $\alpha_{X_i,a}$  the automorphism  $\alpha$  such that  $u\alpha = u^a$ , for all  $u \in X_i$  and  $z\alpha = z$ , for all  $z \in X \setminus X_i$ .

A Whitehead automorphism is an element of  $\langle \text{LInn}_{\text{ext}} \cup \text{Tr}_{\text{ext}} \rangle$ , determined by an ordered pair  $(A, a)$ , where  $A$  is a subset of  $J \cup X_S \cup X_S^{-1}$  and  $a \in \cup_{j \in J} G(\Gamma_j) \cup X_S^{\pm 1}$ , satisfying the condition that

- $\hat{a} \in A$  and
- if  $a \in X_S^{\pm 1}$  then  $a^{-1} \notin A$ .

Partitioning  $A \setminus \{\hat{a}\}$  as  $A \setminus \{\hat{a}\} = A_J \cup A_S$ , where  $A_J = (A \cap J) \setminus \{\hat{a}\}$  and  $A_S = (A \cap X_S^{\pm 1}) \setminus \{a\}$ , the pair  $(A, a)$  determines the automorphism

$$\prod_{j \in A_J} \alpha_{X_j,a} \prod_{y \in A_S} \tau_{y,a}. \quad (5.1)$$

The set  $\mathcal{Q}_{\text{WH}}$  consists of all Whitehead automorphisms and the set  $\mathcal{Q}$  of generators of  $\text{Aut}(G)$  is the union  $\mathcal{Q} = \mathcal{Q}_\Psi \cup \mathcal{Q}_{\text{WH}}$ .

The relators  $\mathcal{S}$  consist of the relations  $\mathcal{S}_\Psi$  together with relators  $\mathcal{S}1$ – $\mathcal{S}9$  below, for which we need to introduce some terminology. Recall that, for  $h \in G(\Gamma_i)$ , where  $i \in J$ , we have defined  $\gamma_h(i)$  to be the automorphism mapping  $g$  to  $g^h$ , for all  $g \in G(\Gamma_i)$ , and fixing all elements of  $X_j$ , where  $j \neq i$ . Clearly  $\gamma_h(i)$  is a product of elements of  $\text{LInn}_{\text{int}} \subseteq \mathcal{Q}_\Psi$ . We use the same sublabelling, J, S, L, of relators as [24] and these relators apply to all possible Whitehead automorphisms. In particular relators involving elements of  $X_S$  are defined only if  $m_0 > 0$ , in Definition 3.4. Given a set  $W$ , subsets  $U, V$  of  $W$  and  $x \in W$ , we write  $U + V$ ,  $V + x$  and  $V - x$  to denote  $U \cup V$ ,  $V \cup \{x\}$  and  $V \setminus \{x\}$  respectively.

$\mathcal{S}1$

**J**  $(A, a)^{-1} = (A, a^{-1})$ , if  $\hat{a} \in J$ , and

**S**  $(A, a)^{-1} = (A - a + a^{-1}, a^{-1})$ , if  $a \in X_S^{\pm 1}$ .

$\mathcal{S}2$   $(A, a)(B, b) = (B, b)(A, a)$ , if  $A \cap B = \emptyset$  and either

**J**  $\hat{a}, \hat{b} \in J$ , or

**S**  $a, b \in X_S^{\pm 1}$ ,  $a^{-1} \notin B$ ,  $b^{-1} \notin A$ , or

**L**  $\hat{a} \in J$ ,  $b \in X_S^{\pm 1}$ ,  $b^{-1} \notin A$ .

$\mathcal{S}3$   $(A, a)(B, b) = (B, b)(A + B - b, a)$ , if  $A \cap B = \emptyset$  and either

**S**  $a, b \in X_S^{\pm 1}$ ,  $a^{-1} \notin B$ ,  $b^{-1} \in A$ , or

**L**  $\hat{a} \in J$ ,  $b \in X_S^{\pm 1}$ ,  $b^{-1} \in A$ .

$\mathcal{S}4$   $(A, a)(B, a) = (A + B, a)$ , if  $A \cap B = \{a\}$  and  $a \in X_S^{\pm 1}$ .

$\mathcal{S}5$  If  $\hat{a} = \hat{b} \in J$  and  $A \cap B = \{\hat{a}\}$  then

(i)  $(A, a)(B, b) = (B, b)(A, a)$ ,

(ii)  $(A, a)(B, a) = (A + B, a)$  and

(iii)  $(A, a)(A, b) = (A, ba)$ .

$\mathcal{S}6$   $\phi^{-1}(A, a)\phi = (A\phi, a\phi)$ , for all  $\phi \in \Psi$ , with the natural interpretation of  $A\phi$ .

$\mathcal{S}7$  If  $a, b \in X_S^{\pm 1}$ ,  $a \in X_{0,s}$ ,  $b \in X_{0,t}$ ,  $s \neq t$ ,  $b \in A$ ,  $b^{-1} \notin A$ ,  $v$  is the unique element of  $X_{0,1}$  and  $\rho$  denotes the word  $\omega_{1,s}^0 \iota_v \omega_{1,s}^0 \omega_{s,t}^0$  in the generators  $\mathcal{Q}_\Psi$  (so  $\rho$  is the cyclic permutation  $(a, b^{-1}, a^{-1}, b)$ ) then

$$(A, a)(A - a + a^{-1}, b) = \rho(A - b + b^{-1}, a).$$

**S8**  $(A, a)(B, b) = (B, b)(A, a)$ , if  $A \subseteq B$  and  $\hat{b} \notin A$  and either

**J**  $\hat{a} \in J$ , or

**S**  $a \in X_S^{\pm 1}$ ,  $a^{-1} \in B$ .

**S9** If  $A \subseteq B$ ,  $\hat{a} \in J$ ,  $b \in X_S^{\pm 1}$  and  $b \in A$  then

$$(A, a)(B, b) = (B, b)(B - A + \hat{a} + b^{-1}, a^{-1})\gamma_{a^{-1}}(\hat{a}).$$

In fact in [24] a larger set of generators is used involving certain products of Whitehead automorphisms. However these additional generators can all be removed, using Tietze transformations, to give the presentation  $\langle \mathcal{Q} | \mathcal{S} \rangle$  above for  $\text{Aut}(G)$ .

Now let  $\Phi$  be the map from  $\mathcal{Q}$  to  $\mathbb{A}$  defined as follows. Each element of  $\mathcal{Q}_\Psi$  is mapped to the element of the same name in the generators of  $\mathbb{A}$ . For  $i \in J$  and  $x, y \in X^{\pm 1}$ , with  $x \neq y$ ,  $x \notin X_i^{\pm 1}$  and  $y \in X_S^{\pm 1}$ , the Whitehead automorphisms  $(\{i, \hat{x}\}, x)$  and  $(\{\hat{x}, y\}, x)$  map to  $\alpha_{X_i, x} \in \text{LInn}_{\text{ext}}$  and  $\tau_{y, x} \in \text{Tr}_{\text{ext}}$ , respectively. To define  $\Phi$  on general Whitehead automorphisms first choose a geodesic word representing each element of  $G(\Gamma_i)$ , for each  $i \in J$ . If  $g$  is represented by the geodesic word  $a_1 \cdots a_m$ , with  $a_i \in X_j^{\pm 1}$ ,  $j \in J$ , then the Whitehead automorphisms  $(\{i, j\}, g)$  and  $(\{j, y\}, g)$  map to the words  $\alpha_{X_i, a_m} \cdots \alpha_{X_i, a_1}$  and  $\tau_{x, a_m} \cdots \tau_{x, a_1}$ , over  $\mathcal{P}$ , which we write as  $\tilde{\alpha}_{X_i, g}$  and  $\tilde{\tau}_{x, g}$ , respectively, cf. Definition 4.12. (The  $\sim$  indicates that these are words over  $\mathcal{P}$ , in the presentation of  $\mathbb{A}$ , as opposed to elements of  $\text{Aut}(G)$ .) Finally  $\Phi$  maps the Whitehead automorphism  $(A, a)$  to

$$\prod_{j \in A_J} \tilde{\alpha}_{X_j, a} \prod_{y \in A_S} \tilde{\tau}_{y, a} \quad (5.2)$$

(cf. (5.1)). To see that this is a well defined map note that from  $\mathcal{R}1$ ,  $\mathcal{R}4$  and  $\mathcal{R}6$  it follows that all terms of the product (5.2) commute with each other: so the order in which the elements of  $A$  appear in this product does not affect the image  $(A, a)\Phi$  of  $(A, a)$  in  $\mathbb{A}$ .

We claim that the natural extension of this map to  $\text{Aut}(G)$  determines a homomorphism  $\Phi : \text{Aut}(G) \rightarrow \mathbb{A}$ . Clearly all the relators of  $\mathcal{S}_\Psi$  map to the identity of  $\mathbb{A}$ . To prove the claim we need to check that the same is true of the relators  $\mathcal{S}1$ – $\mathcal{S}9$ . First we establish a useful consequence of the relators  $\mathcal{R}10$  of  $\mathbb{A}$ .

**R12.** Let  $i, j \in J$ , with  $i \neq j$ , and let  $x \in X_S^{\pm 1}$ . If  $a_1 \cdots a_m = b_1 \cdots b_n$  are geodesic words in  $G(\Gamma_i)$ , with  $a_i, b_i \in X_i^{\pm 1}$ , then

$$\alpha_{X_j, a_m} \cdots \alpha_{X_j, a_1} = \alpha_{X_j, b_n} \cdots \alpha_{X_j, b_1} \quad \text{and} \quad \tau_{x, a_m} \cdots \tau_{x, a_1} = \tau_{x, b_n} \cdots \tau_{x, b_1},$$

(where elements  $\tau_{x, y}^{-1}$  of  $\text{Tr}_{\text{ext}}^{-1}$  are written as  $\tau_{x, y^{-1}}$ ).

(Therefore the definition of  $\Phi$  is in fact independent of the choice of geodesic word for each element of  $G(\Gamma_i)$ .)

We now check  $\mathcal{S}1$ – $\mathcal{S}9$  in turn to see that they become relations of  $\mathbb{A}$ . To fix notation let us assume that, whenever  $(A, a)$  and  $(B, b)$  are Whitehead automorphisms we have

$$(A, a)\Phi = \prod_{j \in A_J} \tilde{\alpha}_{X_j, a} \prod_{y \in A_S} \tilde{\tau}_{y, a} \text{ and}$$

$$(B, b)\Phi = \prod_{k \in B_J} \tilde{\alpha}_{X_k, b} \prod_{z \in B_S} \tilde{\tau}_{z, b}.$$

As the terms of these products are products  $\tilde{\tau}_{\cdot, \cdot}$  and  $\tilde{\alpha}_{\cdot, \cdot}$  of transvections and locally inner automorphisms, it is useful to establish versions of the relators  $\mathcal{R}1$ – $\mathcal{R}11$  for such automorphisms. With this in mind consider the analogues of these relators where generators  $\tau_{a, s}$  and  $\alpha_{X_n, s}$ , with  $a \in X_S^{\pm 1}$ ,  $n \in J$  and  $s \in X_J^{\pm 1}$ , are replaced by  $\tilde{\tau}_{a, w}$  and  $\tilde{\alpha}_{X_n, w}$ , where  $w$  may be any element of  $G(\Gamma_s)$  (and the conditions on the relators remain otherwise unchanged). This affects  $y$  and  $v$  in  $\mathcal{R}1$ ;  $y$  in  $\mathcal{R}2$ ,  $\mathcal{R}7$ ,  $\mathcal{R}9$  and  $\mathcal{R}11$ ;  $x$  and  $y$  in  $\mathcal{R}4$  and  $\mathcal{R}5$ ;  $y$  and  $z$  in  $\mathcal{R}6$  and  $\mathcal{R}8$ ; and  $u$ ,  $y$  and  $z$  in  $\mathcal{R}10$ .

Denote the  $\sim$ -version of  $\mathcal{R}j$  by  $\mathcal{R}j^\sim$ . Then  $\mathcal{R}1^\sim$ ,  $\mathcal{R}4^\sim$ ,  $\mathcal{R}6^\sim$ ,  $\mathcal{R}10^\sim$  and  $\mathcal{R}11^\sim$  follow directly from the original versions.  $\mathcal{R}2^\sim$  follows using  $\mathcal{R}2$  and  $\mathcal{R}1$ . Similarly,  $\mathcal{R}5^\sim$  follows from  $\mathcal{R}5$  and  $\mathcal{R}4$ ;  $\mathcal{R}7^\sim$  follows from  $\mathcal{R}7$  and  $\mathcal{R}6$ ;  $\mathcal{R}8^\sim$  follows from  $\mathcal{R}8$  and  $\mathcal{R}6$ .  $\mathcal{R}9^\sim$  follows from  $\mathcal{R}9$  and  $\mathcal{R}11$ . Therefore we may now assume each relator  $\mathcal{R}j$  is in fact the relator  $\mathcal{R}j^\sim$  (and drop the  $\sim$ ).

Given the comment following (5.2), we have in  $\mathbb{A}$

$$((A, a)\Phi)^{-1} = \prod_{j \in A_J} \tilde{\alpha}_{X_j, a^{-1}} \prod_{y \in A_S} \tilde{\tau}_{y, a^{-1}}.$$

Therefore the relator  $\mathcal{S}1$  follows from  $\mathcal{R}12$ .

To verify relators  $\mathcal{S}2$  we must check that, for all  $j \in A_J$ ,  $k \in B_J$ ,  $y \in A_S$  and  $z \in B_S$ , we have

$$[\tilde{\alpha}_{X_j, a}, \tilde{\alpha}_{X_k, b}] = [\tilde{\tau}_{y, a}, \tilde{\alpha}_{X_k, b}] = [\tilde{\alpha}_{X_j, a}, \tilde{\tau}_{z, b}] = [\tilde{\tau}_{y, a}, \tilde{\tau}_{z, b}] = 1,$$

in  $\mathbb{A}$ . Assume the conditions of  $\mathcal{S}2$  hold. As  $A \cap B = \emptyset$  we have in all cases  $j \neq k$  and  $y \neq z$ ; so  $y = z^{-1}$  or  $\check{y} \neq \check{z}$ . In case **J** we have  $a \notin G(\Gamma_k)$  and  $b \notin G(\Gamma_j)$ , so  $\check{a}, \check{b} \notin \{k, j\}$ , and  $y, z \notin \{\check{a}, \check{b}\}$ . In case **S** we have again  $\check{a}, \check{b} \notin \{j, k\}$ ,  $y \neq b$ , and  $y \neq b^{-1}$ , as  $b^{-1} \notin A$ , and similarly  $z^{\pm 1} \neq a$ . Hence  $y, z \notin \{\check{a}, \check{b}\}$ . In case **L** we have  $\check{a}, \check{b} \notin \{j, k\}$  and  $y, z \notin \{\check{a}, \check{b}\}$ , as before. Therefore relation  $\mathcal{R}4$  implies that  $[\tilde{\alpha}_{X_j, a}, \tilde{\alpha}_{X_k, b}] = 1$ ; relation  $\mathcal{R}6$  implies that  $[\tilde{\tau}_{y, a}, \tilde{\alpha}_{X_k, b}] = [\tilde{\alpha}_{X_j, a}, \tilde{\tau}_{z, b}] = 1$  and  $\mathcal{R}1(i)$  & (ii) imply that  $[\tilde{\tau}_{y, a}, \tilde{\tau}_{z, b}] = 1$ .

To see  $\mathcal{S}3$  holds let  $A' = A \setminus \{b^{-1}\}$ , so  $(A, a)\Phi = \tilde{\tau}_{b^{-1},a}((A', a)\Phi)$ . Since  $\mathcal{S}2$  maps to the a relation of  $\mathbb{A}$ ,  $(A', a)\Phi(B, b)\Phi = (B, b)\Phi(A', a)\Phi$  and hence it suffices to show that

$$\tilde{\tau}_{b^{-1},a}(B, b)\Phi = (B, b)\Phi(B - b + a, a)\Phi\tilde{\tau}_{b^{-1},a},$$

that is

$$\tilde{\tau}_{b^{-1},a} \left( \prod_{k \in B_J} \tilde{\alpha}_{X_k,b} \prod_{z \in B_S} \tilde{\tau}_{z,b} \right) = \left( \prod_{k \in B_J} \tilde{\alpha}_{X_k,b} \prod_{z \in B_S} \tilde{\tau}_{z,b} \prod_{k \in B_J} \tilde{\alpha}_{X_k,a} \prod_{z \in B_S} \tilde{\tau}_{z,a} \right) \tilde{\tau}_{b^{-1},a}.$$

From  $\mathcal{R}7$  and the conditions of  $\mathcal{S}3$  we have  $\tilde{\tau}_{b^{-1},a}\tilde{\alpha}_{X_k,b} = \tilde{\alpha}_{X_k,b}\tilde{\alpha}_{X_k,a}\tilde{\tau}_{b^{-1},a}$  and using  $\mathcal{R}4$  we obtain

$$\tilde{\tau}_{b^{-1},a} \left( \prod_{k \in B_J} \tilde{\alpha}_{X_k,b} \right) = \left( \prod_{k \in B_J} \tilde{\alpha}_{X_k,b} \prod_{k \in B_J} \tilde{\alpha}_{X_k,a} \right) \tilde{\tau}_{b^{-1},a}.$$

Similarly, using  $\mathcal{R}2$  and  $\mathcal{R}1$  we have

$$\tilde{\tau}_{b^{-1},a} \left( \prod_{z \in B_S} \tilde{\tau}_{z,b} \right) = \left( \prod_{z \in B_S} \tilde{\tau}_{z,b} \prod_{z \in B_S} \tilde{\tau}_{z,a} \right) \tilde{\tau}_{b^{-1},a}.$$

Finally,  $\mathcal{R}6$  may be applied to give the required result.

That  $\mathcal{S}4$  holds after mapping to  $\mathbb{A}$  is another consequence of the the remarks following (5.2).

If the conditions of  $\mathcal{S}5$  hold then, for all  $y \in A_S$ ,  $j \in A_J$  and  $k \in B_J$  we have  $\check{a}, \check{b} \notin \{j, k\}$  and  $\check{y} \neq \check{b}$ ; so from  $\mathcal{R}6$ ,  $[\tilde{\alpha}_{X_k,b}, \tilde{\tau}_{y,a}] = 1$ . As also  $\check{j} \neq \check{k}$ ,  $\mathcal{R}4$  applies to give  $[\tilde{\alpha}_{X_j,a}, \tilde{\alpha}_{X_k,b}] = 1$ . For all  $y \in A_S$  and  $z \in B_S$  we have also  $\check{y}, \check{z} \notin \{\check{b}\}$ , and either  $y = z^{-1}$  or  $\check{y} \neq \check{z}$ . Hence, from  $\mathcal{R}1$ ,  $[\tilde{\tau}_{y,a}, \tilde{\tau}_{z,b}] = 1$ . Therefore  $\mathcal{S}5$  (i) holds after mapping to  $\mathbb{A}$ .  $\mathcal{S}5$  (ii) is dealt with using the remarks following 5.2. From the above the conditions of  $\mathcal{R}6$  also apply to give  $[\tilde{\alpha}_{X_j,b}, \tilde{\tau}_{y,a}] = 1$ , so  $\mathcal{S}5$  (iii) holds after mapping to  $\mathbb{A}$ .

If  $\phi$  is a generator of  $\Psi$  then we have, from  $\mathcal{R}11$ ,  $\phi^{-1}\tilde{\alpha}_{X_j,a}\phi = \tilde{\alpha}_{X_j\phi,a\phi}$  and  $\phi^{-1}\tilde{\tau}_{y,a}\phi = \tilde{\tau}_{y\phi,a\phi}$ . Consequently the equality of  $\mathcal{S}6$  holds on mapping to  $\mathbb{A}$ .

In the case of  $\mathcal{S}7$ , let  $A'_S = A_S \setminus \{b\}$ . Then we must show that

$$\begin{aligned} \left( \prod_{j \in A_J} \tilde{\alpha}_{X_j,a} \prod_{y \in A_S} \tilde{\tau}_{y,a} \right) \left( \prod_{j \in A_J} \tilde{\alpha}_{X_j,b} \prod_{y \in A'_S} \tilde{\tau}_{y,b} \right) \tilde{\tau}_{a^{-1},b} = \\ \rho \left( \prod_{j \in A_J} \tilde{\alpha}_{X_j,a} \prod_{y \in A'_S} \tilde{\tau}_{y,a} \right) \tilde{\tau}_{b^{-1},a}. \end{aligned}$$

If the conditions of  $\mathcal{S}7$  hold then we may apply  $\mathcal{R}7$ ,  $\mathcal{R}6$ ,  $\mathcal{R}4$ ,  $\mathcal{R}2$  and  $\mathcal{R}1$  to the left hand side to obtain

$$\begin{aligned}
& \left( \prod_{j \in A_J} \tilde{\alpha}_{X_j, a} \prod_{y \in A_S} \tilde{\tau}_{y, a} \right) \left( \prod_{j \in A_J} \tilde{\alpha}_{X_j, b} \prod_{y \in A'_S} \tilde{\tau}_{y, b} \right) \tilde{\tau}_{a^{-1}, b} \\
&= \left( \prod_{j \in A_J} \tilde{\alpha}_{X_j, a} \prod_{y \in A'_S} \tilde{\tau}_{y, a} \right) \tilde{\tau}_{b, a} \left( \prod_{j \in A_J} \tilde{\alpha}_{X_j, b} \prod_{y \in A'_S} \tilde{\tau}_{y, b} \right) \tilde{\tau}_{a^{-1}, b} \\
&= \left( \prod_{j \in A_J} \tilde{\alpha}_{X_j, b} \prod_{y \in A'_S} \tilde{\tau}_{y, b} \right) \tilde{\tau}_{b, a} \tilde{\tau}_{a^{-1}, b}.
\end{aligned}$$

From  $\mathcal{R}3$  we have  $\tilde{\tau}_{a, b}^{-1} \tilde{\tau}_{b, a} \tilde{\tau}_{a^{-1}, b} = \rho$ , so  $\tilde{\tau}_{b, a} \tilde{\tau}_{a^{-1}, b} = \tilde{\tau}_{a, b} \rho$ , and from  $\mathcal{R}11$  then  $\tilde{\tau}_{b, a} \tilde{\tau}_{a^{-1}, b} = \rho \tilde{\tau}_{b^{-1}, a}$ . A final application of  $\mathcal{R}11$  then gives the required result.

If the conditions of  $\mathcal{S}8$  hold then  $A \subseteq B$  and from  $\mathcal{S}4$  and  $\mathcal{S}5$  it follows that  $(B, b) = (A + \hat{b}, b)(B \setminus A, b)$ . In case **S** of  $\mathcal{S}8$ , this means that

$$\begin{aligned}
(B, b)(A, a) &= (A + \hat{b}, b)(B \setminus A, b)(A, a) \\
&= (A + \hat{b}, b)(A, a)(B \setminus A + A - a, b), \text{ using } \mathcal{S}3, \\
&= (A + \hat{b}, b)(A, a)(B - a, b).
\end{aligned}$$

Now  $\mathcal{S}1$  implies that  $(A, a)^{-1} = (A - a + a^{-1}, a^{-1})$  and  $\mathcal{S}3$  implies that

$$(\{a, \hat{b}\}, b)(A - a + a^{-1}, a^{-1}) = (A - a + a^{-1}, a^{-1})(A + \hat{b}, b),$$

so

$$(A + \hat{b}, b)(A, a) = (A, a)(\{a, \hat{b}\}, b).$$

Therefore

$$\begin{aligned}
(B, b)(A, a) &= (A, a)(\{a, \hat{b}\}, b)(B - a, b) \\
&= (A, a)(B, b), \text{ using } \mathcal{S}4 \text{ and } \mathcal{S}5.
\end{aligned}$$

Thus, case **S** of  $\mathcal{S}8$  follows from  $\mathcal{S}1$ ,  $\mathcal{S}3$ ,  $\mathcal{S}4$  and  $\mathcal{S}5$ . Since the latter all hold after mapping into  $\mathbb{A}$ , the same is true of  $\mathcal{S}8$ , **S**. Hence it remains to consider  $\mathcal{S}8$ , **J**. In this case we have, from  $\mathcal{S}2$ , that

$$\begin{aligned}
(B, b)(A, a) &= (A + \hat{b})(B \setminus A, b)(A, a) \\
&= (A + \hat{b}, b)(A, a)(B \setminus A, b).
\end{aligned}$$



As  $\mathcal{S}2$  holds in  $\mathbb{A}$  it therefore suffices to check that

$$(A + \hat{b}, b)(A, a) = (A, a)(A + \hat{b}, b)$$

holds after mapping to  $\mathbb{A}$ ; that is

$$\begin{aligned} & \left( \tilde{\alpha}_{X_i, b} \prod_{j \in A_J} \tilde{\alpha}_{X_j, b} \prod_{y \in A_J} \tilde{\tau}_{y, b} \right) \left( \prod_{j \in A_J} \tilde{\alpha}_{X_j, a} \prod_{y \in A_J} \tilde{\tau}_{y, a} \right) \\ &= \left( \prod_{j \in A_J} \tilde{\alpha}_{X_j, a} \prod_{y \in A_J} \tilde{\tau}_{y, a} \right) \left( \tilde{\alpha}_{X_i, b} \prod_{j \in A_J} \tilde{\alpha}_{X_j, b} \prod_{y \in A_J} \tilde{\tau}_{y, b} \right), \end{aligned}$$

where  $i = \hat{a} \in J$  and  $\hat{b} \notin A$ . Let  $w$  denote the left hand side of the above expression. From  $\mathcal{R}6$ , we have  $[\tilde{\tau}_{y, b}, \tilde{\alpha}_{j, a}] = 1$ , for  $y \in A_S$ ,  $j \in A_J$ ; from  $\mathcal{R}4$ , we have  $[\tilde{\alpha}_{j_1, b}, \tilde{\alpha}_{j_2, a}] = 1$ , for  $j_1 \neq j_2 \in A_J$ ; and from  $\mathcal{R}1$ , we have  $[\tilde{\tau}_{y_1, b}, \tilde{\tau}_{y_2, a}] = 1$ , for  $y_1 \neq y_2 \in A_S$ . Hence

$$w = \tilde{\alpha}_{X_i, b} \prod_{j \in A_J} \tilde{\alpha}_{X_j, b} \tilde{\alpha}_{X_j, a} \prod_{y \in A_J} \tilde{\tau}_{y, b} \tilde{\tau}_{y, a}.$$

From  $\mathcal{R}5$ , we have  $\tilde{\alpha}_{X_i, b} \tilde{\alpha}_{X_j, b} \tilde{\alpha}_{X_j, a} = \tilde{\alpha}_{X_j, a} \tilde{\alpha}_{X_i, b} \tilde{\alpha}_{X_j, b}$  and from  $\mathcal{R}4$ , we have  $[\tilde{\alpha}_{X_i, b}, \tilde{\alpha}_{X_j, b}] = 1$ , for  $j \in A_J$ . Thus

$$w = \left( \prod_{j \in A_J} \tilde{\alpha}_{X_j, a} \right) \left( \prod_{j \in A_J} \tilde{\alpha}_{X_j, b} \right) \tilde{\alpha}_{X_i, b} \left( \prod_{y \in A_J} \tilde{\tau}_{y, b} \tilde{\tau}_{y, a} \right).$$

Using  $\mathcal{R}8$  and  $\mathcal{R}6$  in a similar fashion, we finally obtain

$$w = \left( \prod_{j \in A_J} \tilde{\alpha}_{X_j, a} \prod_{y \in A_J} \tilde{\tau}_{y, a} \right) \left( \tilde{\alpha}_{X_i, b} \prod_{j \in A_J} \tilde{\alpha}_{X_j, b} \prod_{y \in A_J} \tilde{\tau}_{y, b} \right),$$

as required.

To establish that  $\mathcal{S}9$  maps to an equality in  $\mathbb{A}$ , first consider the special case where  $\hat{a} \in J$ ,  $b \in X_S^{\pm 1}$ ,  $\hat{a} \in B$  and  $A = \{\hat{a}, b\}$ , in which case  $\mathcal{S}9$  reduces to the statement

$$(\{\hat{a}, b\}, a)(B, b) = (B, b)(B - b + b^{-1}, a^{-1})\gamma_{a^{-1}}(\hat{a}),$$

which we call  $\mathcal{S}9'$ . Then  $\mathcal{S}9$  follows from  $\mathcal{S}9'$  and  $\mathcal{S}1$ – $\mathcal{S}8$ . To see this, let  $(A, a)$  and  $(B, b)$  be such that the conditions of  $\mathcal{S}9$  hold. Then  $(A, a) =$

$(\{\hat{a}, b\}, a)(A - b, a)$ , from  $\mathcal{S}5$ , so

$$\begin{aligned}
(A, a)(B, b) &= (\{\hat{a}, b\}, a)(A - b, a)(B, b) \\
&= (\{\hat{a}, b\}, a)(B, b)(A - b, a), \text{ from } \mathcal{S}8, \\
&= (B, b)(B - b + b^{-1}, a^{-1})\gamma_{a^{-1}}(\hat{a})(A - b, a), \text{ from } \mathcal{S}9', \\
&= (B, b)(B - b + b^{-1}, a^{-1})(A - b, a)\gamma_{a^{-1}}(\hat{a}), \text{ from } \mathcal{S}6, \\
&= (B, b)(B - A + \hat{a} + b^{-1}, a^{-1})(A - b, a^{-1})(A - b, a)\gamma_{a^{-1}}(\hat{a}), \\
&\quad \text{from } \mathcal{S}5, \\
&= (B, b)(B - A + \hat{a} + b^{-1}, a^{-1})\gamma_{a^{-1}}(\hat{a}), \text{ from } \mathcal{S}1.
\end{aligned}$$

Therefore it suffices to check that  $\mathcal{S}9'$  maps to an equality in  $\mathbb{A}$ . Suppose then that  $\hat{a} \in J$ ,  $b \in X_S^{\pm 1}$  and  $\hat{a} \in B$ . From  $\mathcal{R}7$ , for all  $k \in B_J - \hat{a}$ ,

$$\tilde{\tau}_{b^{-1}, a} \tilde{\alpha}_{k, b^{-1}} = \tilde{\alpha}_{k, a}^{-1} \tilde{\alpha}_{k, b^{-1}} \tilde{\tau}_{b^{-1}, a},$$

so

$$\tilde{\alpha}_{k, b^{-1}} \tilde{\tau}_{b^{-1}, a} = \tilde{\alpha}_{k, a} \tilde{\tau}_{b^{-1}, a} \tilde{\alpha}_{k, b^{-1}}$$

and

$$\begin{aligned}
\tilde{\tau}_{b^{-1}, a}^{-1} \tilde{\alpha}_{k, b} &= \tilde{\alpha}_{k, b} \tilde{\tau}_{b^{-1}, a}^{-1} \tilde{\alpha}_{k, a}^{-1} \\
&= \tilde{\alpha}_{k, b} \tilde{\alpha}_{k, a}^{-1} \tilde{\tau}_{b^{-1}, a}^{-1},
\end{aligned}$$

using  $\mathcal{R}10$  and  $\mathcal{R}6$ . Similarly, for all  $z \in B_S$ ,  $\mathcal{R}2$  implies that

$$\tilde{\tau}_{b^{-1}, a} \tilde{\tau}_{z, b^{-1}} = \tilde{\tau}_{z, a}^{-1} \tilde{\tau}_{z, b^{-1}} \tilde{\tau}_{b^{-1}, a},$$

so, using  $\mathcal{R}10$  and  $\mathcal{R}1$ , we obtain

$$\tilde{\tau}_{b^{-1}, a}^{-1} \tilde{\tau}_{z, b} = \tilde{\tau}_{z, b} \tilde{\tau}_{z, a}^{-1} \tilde{\tau}_{b^{-1}, a}^{-1}.$$

Then, setting  $i = \hat{a}$ ,

$$\begin{aligned}
(\{i, b\}, a)\Phi(B, b)\Phi &= \tilde{\tau}_{b, a} \tilde{\alpha}_{i, b} \left( \prod_{k \in B_J - i} \tilde{\alpha}_{k, b} \prod_{z \in B_S} \tilde{\tau}_{z, b} \right) \\
&= \tilde{\alpha}_{i, b} \tilde{\tau}_{b^{-1}, a}^{-1} \left( \prod_{k \in B_J - i} \tilde{\alpha}_{k, b} \prod_{z \in B_S} \tilde{\tau}_{z, b} \right) \gamma_a(i)^{-1}, \text{ from } \mathcal{R}9 \text{ and } \mathcal{R}11 \\
&= \tilde{\alpha}_{i, b} \left( \prod_{k \in B_J - i} \tilde{\alpha}_{k, b} \tilde{\alpha}_{k, a}^{-1} \prod_{z \in B_S} \tilde{\tau}_{z, b} \tilde{\tau}_{z, a}^{-1} \right) \tilde{\tau}_{b^{-1}, a}^{-1} \gamma_a(i)^{-1}, \text{ using the above,} \\
&= \tilde{\alpha}_{i, b} \left( \prod_{k \in B_J - i} \tilde{\alpha}_{k, b} \prod_{z \in B_S} \tilde{\tau}_{z, b} \right) \left( \prod_{k \in B_J - i} \tilde{\alpha}_{k, a}^{-1} \prod_{z \in B_S} \tilde{\tau}_{z, a}^{-1} \right) \tilde{\tau}_{b^{-1}, a}^{-1} \gamma_a(i)^{-1}, \\
&\quad \text{using } \mathcal{R}4 \text{ and } \mathcal{R}6, \\
&= (B, b)\Phi(B + b^{-1} + b, a^{-1})\Phi \gamma_a(i)^{-1}\Phi,
\end{aligned}$$

as required.

This concludes the proof that substitution of  $q\Phi$  for  $q$  in  $s$ , for all  $q \in \mathcal{Q}$  and all  $s \in \mathcal{S}$  results in the trivial element of  $\mathbb{A}$ ; so  $\Phi$  is a homomorphism. From the definitions,  $\Theta\Phi$  is the identity of  $\text{Aut}(G)$  and  $\Phi\Theta$  is the identity of  $\mathbb{A}$ , so  $\mathbb{A} \cong \text{Aut}(G)$  and  $\langle \mathcal{P} | \mathcal{R} \rangle$  is a presentation of  $\text{Aut}(G)$ .  $\square$

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