On finite generation of self-similar groups of finite type

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Abstract

A self-similar group of finite type is the profinite group of all automorphisms of a regular rooted tree that locally around every vertex act as elements of a given finite group of allowed actions. We provide criteria for determining when a self-similar group of finite type is finite, level-transitive, or topologically finitely generated. Using these criteria and GAP computations we show that for the binary alphabet there is no infinite topologically finitely generated self-similar group given by patterns of depth 3, and there are 32 such groups for depth 4.

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1 Introduction

There are two important classes of groups acting on regular rooted trees that have arisen as a generalization of the Grigorchuk group: self-similar groups and branch groups. An automorphism group of a regular rooted tree is self-similar if the restriction of the action of every its element onto every subtree can be given again by an element of the group. There are many examples of self-similar groups with numerous extreme properties (like the Grigorchuk group) and this class of groups is very promising for looking different counterexamples. At the same time, self-similar groups appear naturally in many areas of mathematics and have strong connections with fractal geometry, dynamical systems, automata theory (see [8] and the references therein). Branch groups are automorphism groups of a tree whose subgroup lattice is similar to the tree [1]. This class plays an important role in classification of just-infinite groups [5].

Self-similar groups of finite type have arisen as the closure of certain self-similar branch groups in the topology of the tree. It was noticed in [4, Section 7] that the closure of the Grigorchuk group is a profinite self-similar group that can be described by a finite group of allowed local actions on a finite tree (obtained from the binary tree by truncating at some depth). R.I. Grigorchuk used this observation to define a self-similar group of finite type as the group of all tree automorphisms that locally around every vertex act as elements of a given finite group (see precise definition in the next section). The term "group of finite type" comes from the analogy with shifts of finite type in symbolic dynamics [7] (note that a different term, namely finitely constrained group, is used in [9, 10]). Every self-similar group of finite type with transitive action on levels of the tree is a profinite branch group by [4, Proposition 7.5], and conversely, the closure (and profinite completion) of a self-similar regular branch group with congruence subgroup property is a self-similar group of finite type by [9, Theorem 3]. The last observation was the main ingredient to compute the Hausdorff dimension of such branch groups in [9].

Although a self-similar group of finite type is easy to define by a finite group of patterns, it is not clear what are the properties of the group. In particular, R.I. Grigorchuk asked in [4, Problem 7.3(i)] under what conditions a self-similar group of finite type is topologically finitely generated. In this note we address this question and establish certain criterion in Theorem 3 as well as some necessary and sufficient conditions. We also answer such basic questions like how to check whether a self-similar group of finite type is trivial, finite, or acts transitively on levels of the tree.

The closure of the Grigorchuk group is a self-similar group of finite type defined by patters of depth 4 over the binary tree. The closure of groups defined in [9] give examples of infinite finitely generated self-similar groups of finite type defined by patterns of depth d for any $d \ge 4$. For depth 2 and binary tree every self-similar group of finite type is either finite or not finitely generated as shown in [10]. The only unknown case was for depth 3. Using developed criteria and GAP computations we prove that there is no infinite finitely generated self-similar group of finite type defined by patterns of depth 3 over the binary tree. For depth 4 there are 32 such groups (including the closures of the Grigorchuk group and the iterated monodromy group of $z^2 + i$ [6]).

2 Self-similar groups of finite type

In this section we first recall all needed definitions and introduce self-similar groups of finite type (see [8, 4] for more information). After that we study conditions when a self-similar group of finite type is trivial, finite, or level-transitive.

Tree X^{*}. Let X be a finite alphabet with at least two letters. Let X^{*} be the free monoid freely generated by X. The elements of X^{*} are all finite words $x_1x_2...x_n$ over X (including the empty word). We also use notation X^{*} for the tree with the vertex set X^{*} and edges (v, vx) for all $v \in X^*$ and $x \in X$. The set Xⁿ is the *n*-th level of the tree X^{*}. The subtree of X^{*} induced by the set of vertices $\bigcup_{i=0}^{n} X^i$ is denoted by $X^{[n]}$.

Self-similar groups of finite type are defined as special subgroups of the group Aut X^* of all automorphisms of the tree X^* . The group Aut X^* is profinite; it is the inverse limit of the sequence

$$\ldots \to \operatorname{Aut} X^{[3]} \to \operatorname{Aut} X^{[2]} \to \operatorname{Aut} X,$$

where the homomorphisms are given by restriction of the action.

Sections of automorphisms. For every automorphism $g \in \operatorname{Aut} X^*$ and every word $v \in X^*$ define the section $g_{(v)} \in \operatorname{Aut} X^*$ of g at v by the rule: $g_{(v)}(x) = y$ for $x, y \in X^*$ if and only if g(vx) = g(v)y. In other words, the section of g at v is the unique automorphism $g_{(v)}$ of X^* such that $g(vx) = g(v)g_{(v)}(x)$ for all $x \in X^*$. Sections have the following properties:

$$g_{(vu)} = g_{(v)(u)}, \quad (g^{-1})_{(v)} = (g_{(g^{-1}(v))})^{-1}, \quad (gh)_{(v)} = g_{(h(v))}h_{(v)}$$

for all $v, u \in X^*$ (we are using left actions, i.e., (gh)(v) = g(h(v))).

A subgroup $G < \operatorname{Aut} X^*$ is called *self-similar* if $g_{(v)} \in G$ for every $g \in G$ and $v \in X^*$.

The restriction of the action of an automorphism $g \in \operatorname{Aut} X^*$ to the subtree $X^{[d]}$ is denoted by $g|_{X^{[d]}} \in \operatorname{Aut} X^{[d]}$. To every $g \in \operatorname{Aut} X^*$ there corresponds a collection $(g_{(v)}|_{X^{[d]}})_{v \in X^*}$ of automorphisms from $\operatorname{Aut} X^{[d]}$ which completely describe the action of gon the tree X^* .

Self-similar groups of finite type. A subgroup \mathcal{P} of Aut $X^{[d]}$ will be called a group of patterns of depth d (or a pattern group of depth d), $d \geq 1$. We say that an automorphism $g \in \text{Aut } X^*$ agrees with \mathcal{P} if every section $g_{(v)}$, $v \in X^*$, acts on $X^{[d]}$ in the same way as some element in \mathcal{P} , i.e., $g_{(v)}|_{X^{[d]}} \in \mathcal{P}$ for all $v \in X^*$. Since \mathcal{P} is a group, the inverse g^{-1} and all sections $g_{(v)}$ of such an element g agree with \mathcal{P} , the product of two elements that agree with \mathcal{P} also agrees with \mathcal{P} . We obtain the self-similar group $G_{\mathcal{P}}$ of all automorphisms $g \in \text{Aut } X^*$ that agree with \mathcal{P} , i.e., we define the group

$$G_{\mathcal{P}} = \{ g \in \operatorname{Aut} X^* : g_{(v)} |_{X^{[d]}} \in \mathcal{P} \text{ for every } v \in X^* \},\$$

called the *self-similar group of finite type given by the pattern group* \mathcal{P} . Note that Grigorchuk in [4] introduced these groups using finite sets of forbidden patterns, while we are using "allowed" patterns.

Every group $G_{\mathcal{P}}$ is closed in the topology of Aut X^* . Indeed, if for an element $g \in$ Aut X^* the restriction $g|_{X^{[n]}}$ belongs to $G_{\mathcal{P}}|_{X^{[n]}}$ for every $n \in \mathbb{N}$, then $g_{(v)}|_{X^{[d]}} \in \mathcal{P}$ for every $v \in X^*$ and thus $g \in G_{\mathcal{P}}$. Hence $G_{\mathcal{P}}$ is a profinite group.

Let us consider a few simple examples. If \mathcal{P} is trivial then $G_{\mathcal{P}}$ is trivial. If $\mathcal{P} = \operatorname{Aut} X^{[d]}$ then $G_{\mathcal{P}} = \operatorname{Aut} X^*$ (for any $d \in \mathbb{N}$). For every subgroup $\mathcal{P} < \operatorname{Sym}(X)$ the infinitely iterated permutational wreath product $\ldots_X \mathcal{P} \wr_X \mathcal{P}$ is a self-similar group of finite type, where \mathcal{P} is the corresponding group of patterns of depth 1, and every self-similar group of finite type given by patterns of depth 1 is of this form.

Minimal pattern groups. The same self-similar group of finite type may be given by different groups of patterns of depth d and we want to choose a unique pattern group in each class. Let G be a self-similar group of finite type given by a group of patterns of depth d and consider the pattern group $\mathcal{P} = G|_{X^{[d]}}$. Since the group G is self-similar, $g_{(v)}|_{X^{[d]}} \in \mathcal{P}$ for every $g \in G$ and $v \in X^*$, and thus $G < G_{\mathcal{P}}$. On the other hand, it is clear from the definition that every pattern group of depth d that produces G contains \mathcal{P} as a subgroup. Hence $G = G_{\mathcal{P}}$ and \mathcal{P} is the smallest group of patterns of depth d with this property. A pattern group \mathcal{P} of depth d will be called *minimal* if the equality $G_{\mathcal{P}} = G_{\mathcal{Q}}$ for $\mathcal{Q} < \operatorname{Aut} X^{[d]}$ implies $\mathcal{P} < \mathcal{Q}$. It follows from the above arguments that a pattern group \mathcal{P} of depth d is minimal if and only if $\mathcal{P} = G_{\mathcal{P}}|_{X^{[d]}}$, in other words, if every pattern in \mathcal{P} is realized as a restriction of an element of $G_{\mathcal{P}}$. Every self-similar group of finite type given by patterns of depth d is represented by a unique minimal pattern group of depth d.

Pattern graph. Let \mathcal{P} be a group of patterns of depth d. In order to minimize \mathcal{P} one can use a directed labeled graph $\Gamma_{\mathcal{P}}$ which we call the *pattern graph* associated to \mathcal{P} . The vertices of $\Gamma_{\mathcal{P}}$ are the elements of \mathcal{P} and for $a, b \in \mathcal{P}$ and $x \in X$ we put a labeled arrow $a \xrightarrow{x} b$ whenever $a_{(x)}|_{X^{[d-1]}} = b|_{X^{[d-1]}}$. Informally, the arrow $a \xrightarrow{x} b$ shows that we can use the pattern b to extend the action of a on the subtree $xX^{[d]}$ (see Fig. 1). If a vertex $a \in \mathcal{P}$ does not have an outgoing edge labeled by x for some letter $x \in X$, then the action of a cannot be extended to the next level using patterns from \mathcal{P} ; in other words a is not a restriction of an element of $G_{\mathcal{P}}$. Now it is clear how to minimize \mathcal{P} : we just remove every vertex that does not have an outgoing edge labeled by x for some $x \in X$ and repeat this reduction as long as possible. The remaining patterns will form a minimal pattern group for $G_{\mathcal{P}}$. In particular, \mathcal{P} is minimal if and only if every vertex of $\Gamma_{\mathcal{P}}$ has an outgoing edge labeled by x for every $x \in X$.

PSfrag replacements



Figure 1: Coordination between patterns.

The graph $\Gamma_{\mathcal{P}}$ can be used to represent elements of the group $G_{\mathcal{P}}$ by graph homomorphisms as follows. Let us take the tree X^* and add direction and label to every edge by $v \xrightarrow{x} vx$ for every $v \in X^*$ and $x \in X$. Then every element $g \in G_{\mathcal{P}}$ defines a homomorphism $\phi : X^* \to \Gamma_{\mathcal{P}}$ of labeled directed graphs by the rule $\phi(v) = g_{(v)}|_{X^{[d]}}$. Indeed, for every arrow $v \xrightarrow{x} vx$ in the tree X^* the elements $g_{(v)(x)}$ and $g_{(vx)}$ are the same and we have the arrow $\phi(v) \xrightarrow{x} \phi(vx)$ in the graph $\Gamma_{\mathcal{P}}$. And vise versa, every homomorphism $\phi : X^* \to \Gamma_{\mathcal{P}}$ defines an element $g \in G_{\mathcal{P}}$ by its restrictions $g_{(v)}|_{X^{[d]}} = \phi(v)$. This description is an analog of a standard statement in symbolic dynamics that every shift of finite type is sofic (see [7, Theorem 3.1.5]), and pattern graphs play a role of recognition graphs. One can use this observation to introduce the notion of a self-similar group of sofic type which we will discuss elsewhere.

Branching properties. Let us explain the connection between self-similar groups of finite type and branch groups mentioned in Introduction.

Let G be a subgroup of Aut X^{*}. The vertex stabilizer $\operatorname{St}_G(v)$ of a vertex $v \in X^*$ is the subgroup of all $g \in G$ such that g(v) = v. The *n*-th level stabilizer $\operatorname{St}_G(n)$ is the subgroup of all $g \in G$ such that g(v) = v for every $v \in X^n$. Notice that $\operatorname{St}_G(v)$ and $\operatorname{St}_G(n)$ have finite index in G. The rigid vertex stabilizer $\operatorname{RiSt}_G(v)$ of a vertex $v \in X^*$ is the subgroup of all $g \in G$ such that g(u) = u for every vertex $u \in X^* \setminus vX^*$. The set of all sections $g_{(v)}$ for $g \in \operatorname{RiSt}_G(v)$ forms a group which we call the *section group* of $\operatorname{RiSt}_G(v)$ at the vertex v. The group G is called *level-transitive* if it acts transitively on all levels X^n of the tree. The group G is called *regular branch* branching over its subgroup K if G is level-transitive, K is a normal subgroup of finite index, and the group of all automorphism $g \in \operatorname{St}_{\operatorname{Aut} X^*}(1)$ such that the tuple $(g_{(x)})_{x \in X}$ belongs to $\prod_X K$ is a subgroup of finite index in K. Note that the last condition implies that the section group of $\operatorname{RiSt}_K(v)$ at v contains K for every vertex $v \in X^*$.

Every level-transitive self-similar group $G_{\mathcal{P}}$ of finite type given by patterns of depth d is regular branch over its level stabilizer $\operatorname{St}_{G_{\mathcal{P}}}(d-1)$ (see [4, Proposition 7.15]). Indeed, notice that for every element $h \in \operatorname{St}_{G_{\mathcal{P}}}(d-1)$ and any vertex $v \in X^*$ the unique automorphism $g \in \operatorname{RiSt}_{\operatorname{Aut} X^*}(v)$ such that $g_{(v)} = h$ agrees with the pattern group \mathcal{P} and hence belongs to $G_{\mathcal{P}}$. It follows that $\operatorname{St}_{G_{\mathcal{P}}}(n)$ for $n \geq d$ decomposes into the direct product

$$\operatorname{St}_{G_{\mathcal{P}}}(n) \cong \operatorname{St}_{G_{\mathcal{P}}}(d-1) \times \ldots \times \operatorname{St}_{G_{\mathcal{P}}}(d-1)$$

of $|X|^{n-d+1}$ copies of $\operatorname{St}_{G_{\mathcal{P}}}(d-1)$, where each factor acts on the corresponding subtree vX^* for $v \in X^{n-d+1}$. The last condition in the definition of a regular branch group follows. Conversely, if G is a self-similar regular branch group branching over its level stabilizer $\operatorname{St}_G(d-1)$ then the closure of G in Aut X^* is a self-similar group of finite type given by patterns of depth d (see [9, Theorem 3]).

Triviality, finiteness, and level-transitivity of $G_{\mathcal{P}}$. Given a pattern group \mathcal{P} we want to understand whether the group $G_{\mathcal{P}}$ is trivial, finite, or acts transitively on the levels of the tree X^* . The answer to the question about triviality of $G_{\mathcal{P}}$ directly follows from the definition of a minimal pattern group. Namely, the group $G_{\mathcal{P}}$ is trivial if and only if minimizing \mathcal{P} we obtain the trivial group.

The finiteness of $G_{\mathcal{P}}$ can be effectively checked using the next statement.

Proposition 1. Let \mathcal{P} be a minimal pattern group of depth d. The group $G_{\mathcal{P}}$ is finite if and only if the stabilizer $\operatorname{St}_{\mathcal{P}}(d-1)$ is trivial, and in this case $G_{\mathcal{P}}$ is isomorphic to \mathcal{P} .

Proof. Let $\Gamma_{\mathcal{P}}$ be the pattern graph of \mathcal{P} and put $m = |\operatorname{St}_{\mathcal{P}}(d-1)|$. Notice that $(bc)|_{X^{[d-1]}} = b|_{X^{[d-1]}}$ for every $b \in \mathcal{P}$ and $c \in \operatorname{St}_{\mathcal{P}}(d-1)$. Hence if $a \xrightarrow{x} b$ is an arrow in $\Gamma_{\mathcal{P}}$ then $a \xrightarrow{x} bc$ is also an arrow in $\Gamma_{\mathcal{P}}$ for every $c \in \operatorname{St}_{\mathcal{P}}(d-1)$, and every outgoing arrow at a with label x is of this form. Therefore, since \mathcal{P} is minimal, every vertex of $\Gamma_{\mathcal{P}}$ has precisely m outgoing edges labeled by x for every $x \in X$. It follows that for every $a \in \mathcal{P}$ there are precisely $m^{|X|}$ elements $g \in G_{\mathcal{P}}|_{X^{[d+1]}}$ such that $g|_{X^{[d]}} = a$; in other words, every pattern in \mathcal{P} has $m^{|X|}$ extensions to the next level. Then for each level n > d and for every $f \in G_{\mathcal{P}}|_{X^{[n]}}$ there are precisely $m^{|X|^{n-d+1}}$ elements $g \in G_{\mathcal{P}}|_{X^{[n+1]}}$ such that $g|_{X^{[n]}} = f$. Now we can compute the total number of elements in the restriction $G_{\mathcal{P}}|_{X^{[n]}}$:

$$|G_{\mathcal{P}}|_{X^{[n]}}| = |\mathcal{P}| \cdot m^{|X| + |X|^2 + \dots + |X|^{n-d}}, \text{ for } n > d.$$

Therefore the group $G_{\mathcal{P}}$ is finite if and only if m = 1, i.e., when the group $\operatorname{St}_{\mathcal{P}}(d-1)$ is trivial. In this case, $|G_{\mathcal{P}}| = |\mathcal{P}|$ and the restriction $g \mapsto g|_{X^{[d]}}$ is an isomorphism between $G_{\mathcal{P}}$ and \mathcal{P} .

It follows from the proof that we can also use the pattern graph $\Gamma_{\mathcal{P}}$ to check the finiteness of $G_{\mathcal{P}}$. If \mathcal{P} is minimal, then the group $G_{\mathcal{P}}$ is finite if and only if some (equivalently, every) vertex of $\Gamma_{\mathcal{P}}$ has only one outgoing edge labeled by x for each $x \in X$.

Let us treat transitivity on levels. We will use the standard observation that a subgroup $G < \operatorname{Aut} X^*$ acts transitively on X^{n+1} if and only if it acts transitively on X^n and the stabilizer $\operatorname{St}_G(v)$ of some (every) vertex $v \in X^n$ acts transitively on vX.

Let \mathcal{P} be a minimal pattern group of depth d and consider the self-similar group of finite type $G_{\mathcal{P}}$. We fix a letter $x \in X$ and use notation x^n for the word $x \ldots x$ (n times). Let \mathcal{P}_n be the group of all elements $a \in \mathcal{P}$ for which there exists $g \in \operatorname{St}_{G_{\mathcal{P}}}(x^n)$ such that $g_{(x^n)}|_{X^{[d]}} = a$. Then $\operatorname{St}_{G_{\mathcal{P}}}(x^n)$ is transitive on $x^n X$ if and only if \mathcal{P}_n is transitive on X. It follows that $G_{\mathcal{P}}$ is level-transitive if and only if each group \mathcal{P}_n for $n \geq 0$ is transitive on X. Notice that the groups \mathcal{P}_n can be computed recursively by the rule: $\mathcal{P}_0 = \mathcal{P}$ and

$$\mathcal{P}_{n+1} = \left\{ a \in \mathcal{P}_n : \text{ there exists } b \in \operatorname{St}_{\mathcal{P}_n}(x) \text{ such that } b_{(x)} |_{X^{[d-1]}} = a |_{X^{[d-1]}} \right\}.$$

We obtain a decreasing sequence $\mathcal{P} > \mathcal{P}_1 > \ldots$ of finite groups which should stabilize on some subgroup $\mathcal{Q} < \mathcal{P}$, $\mathcal{Q} = \bigcap_{n \geq 0} \mathcal{P}_n$. Moreover, if we take the smallest *n* such that $\mathcal{P}_n = \mathcal{P}_{n+1}$ then $\mathcal{P}_n = \mathcal{P}_{n+k}$ for every $k \in \mathbb{N}$, and thus $\mathcal{Q} = \mathcal{P}_n$. Hence the group \mathcal{Q} can be algorithmically computed. We have proved the following effective criterium.

Proposition 2. The group $G_{\mathcal{P}}$ is level-transitive if and only if the group \mathcal{Q} is transitive on X.

3 Finite generation of groups $G_{\mathcal{P}}$

In this section we study when the group $G_{\mathcal{P}}$ is topologically finitely generated. Further we omit the word "topologically".

Theorem 3. Let G be a level-transitive self-similar group of finite type given by patterns of depth d. The group G is finitely generated if and only if there exists $n \ge d$ such that the commutator of $\operatorname{St}_G(d-1)|_{X^{[n]}}$ contains $\operatorname{St}_G(n-1)|_{X^{[n]}}$.

Proof. Let $G = G_{\mathcal{P}}$ for a minimal pattern group \mathcal{P} of depth d.

First we prove the necessity. The proof will not use transitivity on levels. Assume that the commutator of $\operatorname{St}_G(d-1)|_{X^{[n]}}$ does not contain $\operatorname{St}_G(n-1)|_{X^{[n]}}$ for every $n \ge d$. Let us prove that $\operatorname{St}_G(d-1)$ and thus G are not finitely generated. In the proof we will use notations $S = \operatorname{St}_G(d-1)$ and $S_n = S|_{X^{[n]}}$. For each $m \ge d$ consider the homomorphism

$$\varphi: S \to \prod_{n=d}^{m} S_n / [S_n, S_n], \ \varphi(g) = (g|_{X^{[n]}} [S_n, S_n])_{n=d}^{m}$$

Recall that the stabilizer $\operatorname{St}_G(n-1)|_{X^{[n]}}$ decomposes into the direct product $\operatorname{St}_{\mathcal{P}}(d-1) \times \ldots \times \operatorname{St}_{\mathcal{P}}(d-1)$ of $|X|^{n-d}$ copies of $\operatorname{St}_{\mathcal{P}}(d-1)$. By our assumption there exists an element $g_n = (1, \ldots, a_n, \ldots, 1), a_n \in \operatorname{St}_{\mathcal{P}}(d-1)$, of this product that does not belong to the



Figure 2: The construction of generator g_i and conjugator g.

commutator $[S_n, S_n]$. Let A_n be the group generated by the image of g_n in the quotient of $\operatorname{St}_G(n-1)|_{X^{[n]}}$ by $[S_n, S_n]$. The group A_n is a nontrivial subgroup of the finite abelian group $S_n/[S_n, S_n]$. Hence A_n is also a quotient of $S_n/[S_n, S_n]$. Composing with φ we obtain a homomorphism from S to $\prod_{n=d}^m A_n$. Moreover, for i < n the *i*-th component of the image of g_n in this direct product is trivial. It follows that $\prod_{n=d}^m A_n$ is a homomorphic image of S. Since $|A_n| \leq |\mathcal{P}|$ for all n, the number of generators of $\prod_{n=d}^m A_n$ goes to infinity as mgoes to infinity. Hence S is not finitely generated.

Let us prove the converse. Fix $k \ge d$ such that the commutator of $\operatorname{St}_G(d-1)|_{X^{[k]}}$ contains $\operatorname{St}_G(k-1)|_{X^{[k]}}$. We construct a finitely generated dense subgroup of G using the techniques from branch groups (see [1, 2]). Let f_1, \ldots, f_l and h_1, \ldots, h_m be the elements of G such that

$$\langle f_1, \dots, f_l \rangle |_{X^{[1+d+k]}} = G |_{X^{[1+d+k]}}$$
 and $\langle h_1, \dots, h_m \rangle |_{X^{[k]}} = \operatorname{St}_G(d-1) |_{X^{[k]}}$

The group $\operatorname{St}_{\mathcal{P}}(d-1)$ is nontrivial by Proposition 1, and we can find $v \in X^d$ and $a \in \operatorname{St}_G(d-1)$ such that $a(v) \neq v$ (the element *a* will be used to shift the section of certain automorphisms at the vertex *v*). Fix two letter $x, y \in X, x \neq y$. Define the automorphisms g_1, \ldots, g_m recursively by their sections:

$$g_{i(yv)} = h_i$$
 and $g_{i(x)} = g_i, i = 1, \dots, m,$

and the other sections are trivial (see Fig. 2). Notice that g_1, \ldots, g_m belong to G.

Consider the group $H = \langle f_1, \ldots, f_l, h_1, \ldots, h_m, g_1, \ldots, g_m \rangle$ and let us show that H is dense in G. We need to prove that $H|_{X^{[n]}} = G|_{X^{[n]}}$ for all $n \in \mathbb{N}$. The statement holds for $n \leq 1 + d + k$ by construction. By induction on n assume that we have proved it for all levels $\leq n + d + k$. There exists an element $g \in G$ such that

$$g_{(x^i y)} = a$$
 for $i = 0, \dots, n-1$

and the other sections are trivial (see Fig. 2). By inductive hypothesis there exists $h \in H$ such that $h|_{X^{[n+d+k]}} = g|_{X^{[n+d+k]}}$. Then the commutator $[h^{-1}g_ih, g_j]$ acts trivially on the vertices in $X^{[n+d+k+1]} \setminus x^n y v X^{[k]}$ and at the vertex $x^n y v$ has section

$$[h^{-1}g_ih, g_j]_{(x^nyv)} = [h^{-1}_{(x^nyv)}h_ih_{(x^nyv)}, h_j].$$

Conjugating by generators g_1, \ldots, g_m we obtain that the section group of $\operatorname{RiSt}_H(x^n yv)|_{X^{[n+d+k+1]}}$ at $x^n yv$ contains the commutator of $\operatorname{St}_G(d-1)|_{X^{[k]}}$ and hence $\operatorname{St}_G(k-1)|_{X^{[k]}}$. Since the action is transitive this holds for every vertex of the level X^{n+d+1} . Hence $\operatorname{St}_H(n+d+k)|_{X^{[n+d+k+1]}} = \operatorname{St}_G(n+d+k)|_{X^{[n+d+k+1]}}$ and the statement follows.

Remark 1. An automorphism of the tree X^* is called *finite-state* if it has finitely many sections (the term comes from automata theory); a subgroup is finite-state if it consists of finite-state automorphisms. We can always choose elements f_1, \ldots, f_l and h_1, \ldots, h_m so that they are finite-state. Then the elements g_1, \ldots, g_m and the group H constructed in the proof will be also finite-state. Adding sections of elements we obtain a finitely generated finite-state self-similar dense subgroup in G.

Remark 2. The condition of level-transitivity cannot be dropped in Theorem 3. For example, consider the alternating group A_5 with the natural action on $\{1, 2, 3, 4, 5\}$, extend the action to the alphabet $X = \{0, 1, 2, 3, 4, 5\}$ by putting $\pi(0) = 0$ for every $\pi \in A_5$, and consider the infinitely iterated permutational wreath product $G_{A_5} = \ldots \wr_X A_5 \wr_X A_5$. The group A_5 is perfect, i.e., $[A_5, A_5] = A_5$, hence the condition in Theorem 3 holds for n =d = 1. However the group G is not finitely generated, because the map $g \mapsto (g_{(0^n)}|_X)_{n \in \mathbb{N}}$ is a surjective homomorphism from G to the product $\prod_{\mathbb{N}} A_5$ which is not finitely generated.

Remark 3. It is not difficult to see that for a group $G_{\mathcal{P}}$ given by a transitive pattern group \mathcal{P} of depth 1 the condition in the theorem holds for some n if and only if the group \mathcal{P} is perfect. Hence Theorem 3 generalizes Corollary 3.6 in [2] about finite generation of iterated permutational wreath products $\ldots \wr_X \mathcal{P} \wr_X \mathcal{P}$.

Proposition 4. Let G be a self-similar group of finite type given by patterns of depth d. If there exists $n \ge d$ such that the commutator of $G|_{X^{[n]}}$ does not contain $\operatorname{St}_G(n-1)|_{X^{[n]}}$ then the group G is not finitely generated.

Proof. The proof uses the same arguments as in the first part of the proof above. Fix $n \geq d$ such that the commutator of $G_n := G|_{X^{[n]}}$ does not contain $\operatorname{St}_G(n-1)|_{X^{[n]}}$. For every $k \in \mathbb{N}$ consider the map

$$\varphi_k: G \to G_n/[G_n, G_n], \quad \varphi_k(g) = \prod_{v \in X^k} g_{(v)}|_{X^{[n]}}[G_n, G_n].$$

Since $G_n/[G_n, G_n]$ is abelian every map φ_k is a homomorphism. Now for every $m \in \mathbb{N}$ consider the homomorphism $\varphi: G \to \prod_{k=1}^m G_n/[G_n, G_n], \varphi(g) = (\varphi_k(g))_{k=1}^m$. For every k and every pattern $a \in \operatorname{St}_G(n-1)|_{X^{[n]}}$ there exists g in the rigid stabilizer $\operatorname{RiSt}_G(v)$ of a vertex $v \in X^k$ such that $g_{(v)}|_{X^{[n]}} = a$, and thus $\varphi_k(g) = a$ and $\varphi_i(g) = e$ for i < k. Since $\operatorname{St}_G(n-1)|_{X^{[n]}}/[G_n, G_n]$ is a homomorphic image of G_n , it follows that the abelian group $\prod_{k=1}^m \operatorname{St}_G(n-1)|_{X^{[n]}}/[G_n, G_n]$ is a homomorphic image of G for every m. Hence G is not finitely generated.

The next statement generalizes Proposition 2 in [10].

Corollary 5. Let \mathcal{P} be an abelian pattern group. The group $G_{\mathcal{P}}$ is finitely generated if and only if it is finite.

Proof. The statement follows from Proposition 1 and Proposition 4 with n = d.

Corollary 6. Take a cyclic subgroup C < Sym(X) and consider the group $C \wr_X C$ as a natural subgroup of $\text{Aut } X^{[2]} \cong \text{Sym}(X) \wr_X \text{Sym}(X)$. Then for any nilpotent pattern group $\mathcal{P} < C \wr_X C$ the group $G_{\mathcal{P}}$ is finitely generated if and only if it is finite.

Proof. Since $\mathcal{P}/\operatorname{St}_{\mathcal{P}}(1)$ is cyclic, the commutator $[\mathcal{P}, \mathcal{P}]$ is a subgroup of $\operatorname{St}_{\mathcal{P}}(1)$. If it is a proper subgroup then the group $G_{\mathcal{P}}$ is not finitely generated by Proposition 4. Suppose $[\mathcal{P}, \mathcal{P}] = \operatorname{St}_{\mathcal{P}}(1)$. For any $a, b \in \mathcal{P}$ there exists $k \in \mathbb{N}$ such that $a^k b$ or $b^k a$ belongs to $\operatorname{St}_{\mathcal{P}}(1)$. Using the equality $[a, b] = [a, a^k b] = [b^k a, b]$ we obtain that $[\mathcal{P}, \mathcal{P}] = [\mathcal{P}, \operatorname{St}_{\mathcal{P}}(1)]$. Since \mathcal{P} is nilpotent, the last equality implies that $[\mathcal{P}, \mathcal{P}] = \operatorname{St}_{\mathcal{P}}(1) = \{1\}$ and hence the group $G_{\mathcal{P}}$ is finite by Proposition 1.

4 A few classification results

In this section we classify self-similar groups of finite type for the binary alphabet $X = \{0, 1\}$ and depth ≤ 4 . All computations were made in GAP. Our strategy for classifying self-similar groups of finite type of a given depth d is the following. First we find all subgroups in Aut $X^{[d]}$, then minimize all subgroups and obtain the number of all minimal pattern groups, which is equal to the number of self-similar groups of finite type of a given depth as subgroups in Aut X^* . Further we distinguish all finite groups using Proposition 1. Then we apply Proposition 4 for small values of n to distinguish groups that are not finitely generated. An infinite self-similar group over the binary alphabet is level-transitive (see [3, Lemma 3]), hence the rest of the groups are level-transitive and we can apply Theorem 3. In this way it was possible to obtain the following results.

Depth d = 2. This case was treated in [10]. There are ten subgroups in Aut $X^{[2]}$, six minimal pattern subgroups, and hence six self-similar groups of finite type. Among them there are three finite groups, namely the trivial group and two groups isomorphic to C_2 , and the other three groups are not finitely generated (Proposition 4 works with n = 2).

Depth d = 3. There are 576 subgroups in Aut $X^{[3]}$, 60 minimal pattern subgroups, and hence 60 self-similar groups of finite type. Among them there are 23 finite groups, namely the trivial group, two groups isomorphic to C_2 , four groups isomorphic to $C_2 \times C_2$, 16 groups isomorphic to the dihedral group D_8 . The other 37 groups are not finitely generated (27 groups satisfy Proposition 4 with n = 3 and 10 groups with n = 4).

Corollary 7. A self-similar group of finite type given by patterns of depth $d \leq 3$ over the binary alphabet is either finite or not finitely generated.

Depth d = 4. There are 4544 self-similar groups of finite type. Among them there are 1535 finite groups, namely the trivial group, two groups isomorphic to C_2 , four groups isomorphic to $C_2 \times C_2$, 16 groups isomorphic to D_8 , eight groups isomorphic to $C_2 \times C_2 \times C_2$,

96 groups isomorphic to $C_2 \times D_8$, 128 groups isomorphic to $(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2$, 256 groups isomorphic to $(((C_4 \times C_2) \rtimes C_2) \rtimes C_2) \rtimes C_2$, and 1024 groups isomorphic to Aut $X^{[3]} \cong C_2 \wr_X C_2 \wr_X C_2$. Among the rest of the groups there are 2977 not finitely generated (1235 groups satisfy Proposition 4 with n = 4, 778 groups with n = 5, 508 groups with n = 6, 200 groups with n = 7, and 256 groups with n = 8) and 32 finitely generated groups that satisfy Theorem 3 with n = 6. The pattern groups of these 32 self-similar groups of finite type all have order 4096, their restriction on $X^{[3]}$ is equal to Aut $X^{[3]}$, and among them there are 20 pairwise non-isomorphic groups. These pattern groups can be described as follows. Let us consider the group Aut $X^{[4]}$ as a natural subgroup of the symmetric group Sym(16) on the set $\{1, 2, ..., 16\} \leftrightarrow X^4$ and fix the permutations:

$$a_{1} = (1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16)$$

$$a_{2} = (1,10,2,9)(3,11)(4,12)(5,14,6,13)(7,15)(8,16)$$

$$a_{3} = (1,10)(2,9)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16)$$

$$a_{4} = (1,9,2,10)(3,11)(4,12)(5,14,6,13)(7,15)(8,16)$$

$$b_1 = (1,5)(2,6)(3,7)(4,8)(9,10) \qquad c_1 = (1,3)(2,4) \quad c_3 = (1,3)(2,4)(5,6) \\ b_2 = (1,6)(2,5)(3,7)(4,8)(9,10) \qquad c_2 = (1,4,2,3) \quad c_4 = (1,4,2,3)(5,6)$$

Then the 32 pattern groups mentioned above is the family of groups $\mathcal{P}_{ijk} = \langle a_i, b_j, c_k \rangle$. In this family: the self-similar group of finite type $G_{\mathcal{P}_{123}}$ is the closure of the Grigorchuk group and $G_{\mathcal{P}_{111}}$ is the closure of the iterated monodromy group of $z^2 + i$ [6].

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