

Actions, length functions, and non-archimedean words

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Abstract

In this paper we survey recent developments in the theory of groups acting on Λ -trees. We are trying to unify all significant methods and techniques, both classical and recently developed, in an attempt to present various faces of the theory and to show how these methods can be used to solve major problems about finitely presented Λ -free groups. Besides surveying results known up to date we draw many new corollaries concerning structural and algorithmic properties of such groups.

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1 Introduction

In 2013 the theory of group actions on Λ -trees will be half a century old. There are three stages in the development of the theory: the initial period when the basic concepts, methods and open problem were laid down; the period of concentration on actions on \mathbb{R} -trees; and the recent phase when the focus is mostly on non-Archimedean actions.

The initial stage takes its roots in several, seemingly independent, areas of group theory and topology: the study of abstract length functions in groups, Bass-Serre theory of actions on simplicial trees, Stallings pregroups and free constructions, and the theory of \mathbb{R} -trees with its connections with Thurston's Geometrisation Theorem.

In 1963 Lyndon introduced groups G equipped with an abstract length function $l : G \rightarrow \mathbb{Z}$ as a tool to carry over Nielsen cancellation theory from free groups and free products, to a much more general setting [77]. The main idea was to measure the amount of cancellation that occurs in passing to the reduced form of a product of two reduced forms in a free group (or a free product of groups) and describe it in an abstract axiomatic way. To this end Lyndon introduced a new quantity $c(g, h) = \frac{1}{2}(l(g) + l(h) - l(g^{-1}h))$ for elements $g, h \in G$ (to be precise, he considered $(c(h^{-1}, g^{-1}))$, which gives equivalent axioms), thus anticipating the Gromov product in metric spaces, and gave some rather natural and simple axioms on the length function l , which now could be interpreted as axioms of 0-hyperbolic Λ -metric spaces. He completely described groups with *free* \mathbb{Z} -valued length functions (such length functions correspond to actions on simplicial trees without fixed points and edge inversion), as subgroups of ambient free groups with induced length. Some initial results for groups with length functions in \mathbb{Z} or \mathbb{R} were obtained in 1970s (see [51, 21, 52, 53]), which led to the general notion of a Lyndon length function $l : G \rightarrow \Lambda$ with values in an ordered abelian group Λ . We refer to the book [78] on the early history of the subject.

At about the same time Serre laid down fundamentals of the theory of groups acting on simplicial trees. Serre and Bass described the structure of such groups

via fundamental groups of graphs of groups. In the following decade their approach unified several geometric and combinatorial methods of group theory into a unique tool, known today as Bass-Serre theory. This was one of the major achievements of combinatorial group theory in 1970s. We refer here to Serre's seminal book [120] and more recent books (for example, in [34]).

On the other hand, in 1960s Stallings introduced a notion of a *pregroup* P and its *universal group* $U(P)$ [123]. Pregarps P provide a very convenient tool to describe reduced forms of elements of their universal groups $U(P)$. Precise connections between pregroups and free constructions (the fundamental groups of graphs of groups) were established by Rimlinger in [111, 110]. Recently this technique was invaluable in dealing with infinite non-Archimedean words (see below).

The first definition of an \mathbb{R} -tree appeared in the work of Tits [125] in connection with Bruhat-Tits buildings. One year earlier Chiswell [21] came up with a crucial construction that shows that a group G with a Lyndon length function $l : G \rightarrow \mathbb{R}$ has a natural action by isometries on an \mathbb{R} -tree (even though he did not refer to the space as an \mathbb{R} -tree), and vice versa. This was an important result which showed that free (no fixed points) group actions and free Lyndon length functions are just two equivalent languages describing essentially the same objects. We refer to the book [24] for a detailed discussion on the subject. In 1984, in their very influential paper [84], Morgan and Shalen linked group actions on \mathbb{R} -trees with Thurston's Geometrization Theorem. In the same paper they introduced Λ -trees for an arbitrary ordered abelian group Λ and established a general form of Chiswell's construction. Thus, it became clear that abstract length functions with values in Λ and group actions on Λ -trees are just two equivalent approaches to the same realm of group theory questions. In the subsequent papers Morgan and Shalen, and Culler and Morgan further developed the theory of \mathbb{R} -trees and group actions [84, 29, 85, 86], we refer to surveys [122, 121, 83] for more details and a discussion on these developments.

Alperin and Bass [1] developed much of the initial framework of the theory of group actions on Λ -trees and stated the fundamental research goal (see also [4]):

Fundamental Problem: *Find the group-theoretic information carried by a Λ -tree action, analogous to the Bass-Serre theory for the case $\Lambda = \mathbb{Z}$.*

It is not surprising, from the view-point of Bass-Serre theory, that the following problem (from [1]) became of crucial importance:

The Main Problem: *Describe groups acting freely on Λ -trees.*

Following Bass [4] we refer to such groups as Λ -free groups. They are the main object of this survey.

Recall that Lyndon himself completely characterized arbitrary \mathbb{Z} -free groups [77]. He showed also that free products of subgroups of \mathbb{R} are \mathbb{R} -free and conjectured (in the language of length functions) that the converse is also true. It was shown by Alperin and Moss [2], and by Promislow [103] that there are infinitely generated \mathbb{R} -free groups whose length functions are not induced by

the natural ones on free products of subgroups of \mathbb{R} . Furthermore, examples by Dunwoody[39] and Zastrow [130] show that there are infinitely generated \mathbb{R} -free groups that cannot be decomposed as free products of surface groups and subgroups of \mathbb{R} (see Section 12.1).

It became clear that infinitely generated Λ -free groups are so diverse that any reasonable description of them was (and still is) out of reach. The problem was modified by imposing some natural restrictions:

The Main Problem (Restricted): *Describe finitely generated (finitely presented) groups acting freely on Λ -trees.*

This ends the first and starts the second stage of the development of the theory of group actions, where the main effort is focused on group actions on \mathbb{R} -trees.

In 1991 Morgan and Shalen [87] proved that all surface groups (with exception of non-orientable ones of genus 1, 2 and 3) are \mathbb{R} -free. In the same paper they conjectured the following, thus enlarging the original Lyndon's statement.

The Morgan-Shalen Conjecture: *Show that finitely generated \mathbb{R} -free groups are free products of free abelian groups of finite rank and non-exceptional surface groups.*

This conjecture turned out to be extremely influential and in the years after much of the research in this area was devoted to proving it. In 1991 Rips came out with the idea of a solution of the conjecture in the affirmative. Gaboriau, Levitt and Paulin [40], and independently Bestvina and Feighn [14] published the solution. The final result that completely characterizes finitely generated \mathbb{R} -free groups is now known as Rips Theorem. Notice, that free actions on \mathbb{R} -trees cover all free Archimedean actions, since every group acting freely on a Λ -tree for an Archimedean ordered abelian group Λ also acts freely on an \mathbb{R} -tree.

In fact, Bestvina and Feighn proved much more, they showed that if a finitely presented group G admits a nontrivial, stable and minimal action on an \mathbb{R} -tree then it splits into a free construction of a special type. The key ingredient of their proof is what they called a "Rips machine", which is a geometric version of the Makanin process (algorithm) developed for solving equations in free groups [80, 105]. The machine (either Rips or Makanin) takes a band complex or a generalized equation related to the action as an input and then outputs again a band complex or a generalized equation but in a very specific form, which allows one to see the splitting of the group. Nowadays, as we will see below, there are many different variations of the Makanin-Razborov processes (besides the Rips machine), we will refer to them in all their incarnations as *Machines* or *Elimination Processes*. In fact, there is no any published description of the original Rips machine, and the machine described by Bestvina and Feighn in [14] (we refer to it as Bestvina-Feighn machine) seems much more powerful then was envisioned by Rips. The range of applications of Bestvina-Feighn machines is quite broad due to a particular construction introduced by Bestvina [12] and Paulin [101]. This construction shows how a sequence of isometric actions of a

group G on some “negatively curved spaces” gives rise to an isometric action of G on the “limit space” (in the Gromov-Hausdorff topology) of the sequence, which is an \mathbb{R} -tree, so the machine applies. We refer to Bestvina’s preprint [13] for various applications of Rips-Bestvina-Feighn machine.

There are other interesting results on non-free actions on \mathbb{R} -trees which we do not discuss here, since our main focus is on Λ -free groups, but there is one which we would like to mention now. In [44] Gromov introduced a notion of an asymptotic cone of a metric space. He showed that an asymptotic cone of a hyperbolic group G is an \mathbb{R} -tree, on which G acts non-trivially. This gives a powerful tool to study hyperbolic groups.

In 1991 Bass in his pioneering work [4] made inroads into the realm of non-Archimedean trees (Λ -trees with non-Archimedean Λ) and obtained some structural results on λ -free groups for specific Λ , thus began the third stage in study of Λ -trees. To explain, let Λ_0 be an ordered abelian group and $\Lambda = \Lambda_0 \oplus \mathbb{Z}$ with the right lexicographical order. It was shown in [4] that if G is Λ -free then G is the fundamental group of a graph of groups where the vertex groups are Λ_0 -free, the edge groups are either trivial or maximal abelian in the adjacent vertex groups, and some other “compatibility conditions” are satisfied. In particular, since $\mathbb{Z}^n = \mathbb{Z}^{n-1} \oplus \mathbb{Z}$ one gets by induction on n structural results on \mathbb{Z}^n -free groups. This result gave a standard on how to describe algebraic structure of groups acting freely on Λ -trees.

In 1995 the first two authors of this survey together with Remeslennikov began a detailed study of \mathbb{Z}^n -free groups. The starting point was to show that every finitely generated fully residually free group G is \mathbb{Z}^n -free for a suitable n . These groups play an important part in several areas of group theory, where they occur under different names: freely discriminated groups (see, for example, [95, 91, 7]) E -free groups (that is, groups with the same existential theory as free non-abelian groups [108]), limit groups in terms of Sela [117], and limits of free groups in the Gromov-Hausdorff topology (see [20]). Though the first two names (for the same definition) occurred much earlier, we use here the term “limit” as a much shorter one. We describe the main points of our approach to limit groups here since these techniques became the corner stones for the further development of the general theory of groups acting on Λ -trees and hyperbolic Λ -metric spaces.

In 1960 Lyndon introduced a free $\mathbb{Z}[t]$ -exponential group $F^{\mathbb{Z}[t]}$, where F is a free non-abelian group, and showed that $F^{\mathbb{Z}[t]}$ is freely discriminated by F , so all finitely generated subgroups of $F^{\mathbb{Z}[t]}$ are limit groups [76]. In 1995 Myasnikov and Remeslennikov constructed a free Lyndon length function on $F^{\mathbb{Z}[t]}$ with values in \mathbb{Z}^ω (the direct sum of countably many copies of \mathbb{Z} with the lexicographical order) [89]. To do this we proved first that $F^{\mathbb{Z}[t]}$ is the union of a countable chain of subgroups $F = G_1 < G_2 < \dots < G_n < \dots$, where each group G_{n+1} is obtained from G_n by an extension of a centralizer [88, 90]; and then showed that if G_n is a \mathbb{Z}^n -free group then G_{n+1} is \mathbb{Z}^{n+1} -free [89]. Two years later Kharlampovich and Myasnikov showed that every finitely generated fully residually free group embeds into G_n (from the sequence above) for a suitable n , hence every limit groups is a \mathbb{Z}^n -free group for a suitable n

[59, 64]. These results became crucial in the study of limit groups and their solution to the Tarski problems [62]. A direct application of Bass-Serre theory to subgroups of the groups G_n gives an algebraic structure of limits groups as the fundamental groups of graphs of groups induced from the corresponding graphs for G_n . In [59, 64] a new version of Makanin-Razborov process was introduced, termed the *Elimination process*, a.k.a. a general \mathbb{Z} -machine (it works over arbitrary finite sets of usual words in a finite alphabet, that is, \mathbb{Z} -words). Notice that we will further denote systems of equations by $S = 1$ meaning that S is a set of words, and each of these words is equal to the identity. Application of such a machine to a finite system $S = 1$ of equations (with constants) over F results in finitely many systems $U_1 = 1, \dots, U_n = 1$ over F such that the solution set of $S = 1$ is the union (up to a change of coordinates) of the solution sets of $U_i, i \in [1, n]$ and each U_i is given in the standard NTQ (non-degenerate quasi-quadratic) form [64]. This is a precise analog of the elimination and parametrization theorems from the classical algebraic geometry, thus \mathbb{Z} -machines play the role of the classical resolution method in the case of algebraic geometry over free groups [91, 7, 59, 64]. At the level of groups (here limit groups occur as the coordinate groups of systems of equations in F) \mathbb{Z} -machines give embeddings of limit groups into the coordinate groups of NTQs, which were later also called fully residually free towers [117]. There are very interesting corollaries from this result along for limit groups (see Section 9.7 for details). Observe, that these \mathbb{Z} -machines (accordingly modified) were successfully used for other classes of groups in similar situations. For example, Kazachkov and Casals-Ruiz used it to solve equations in free product of groups [19] and right-angled Artin groups [18].

To study algorithmic properties of limit groups Kharlampovich and Myasnikov developed an elimination process (a \mathbb{Z}^n -machine) which works for arbitrary free Lyndon length functions with values in \mathbb{Z}^n (see [61]), thus giving a further generalization of the original \mathbb{Z} -machines mentioned above. These \mathbb{Z}^n -machines can be described in several (equivalent) forms: algorithmic, geometric, group-theoretic, or as dynamical systems. In the group-theoretic language the machines provide various splittings of a group as the fundamental group of a graph of groups with abelian edge groups (abelian splittings). Since \mathbb{Z}^n -machines (unlike \mathbb{R} -machines mentioned above) are algorithms - one can use them in solving algorithmic problems. A direct application of such machines (precisely like in the case of free groups) allows one to solve arbitrary equations and describe their solution sets in arbitrary limit groups [61]. In [61] Kharlampovich and Myasnikov used these machines to effectively construct JSJ-decompositions of limit groups given by their finite presentations (or almost any other effective way), which allowed them together with Bumagin to solve the isomorphism problem for limit groups (see [17]). Later these results (following basically the same line of argument) were generalized to arbitrary total relatively hyperbolic groups [31].

\mathbb{Z}^n -machines, while working with \mathbb{Z}^n -free groups in fact manipulate with the so-called *infinite non-Archimedean words*, in this case \mathbb{Z}^n -words (so the ordinary words are just \mathbb{Z} -words). Discovery of non-Archimedean words turned out to be

one of the major recent developments in the theory of group actions, it is worthwhile to say a few words about it here. In [92] Myasnikov, Remeslennikov and Serbin introduced infinite Λ -words for arbitrary Λ and showed that groups admitting faithful representations by Λ -words act freely on Λ -trees, while Chiswell proved the converse [25]. This gives another, equivalent, approach to free actions on Λ -trees, so now one can replace the axiomatic viewpoint of length functions along with many geometric arguments coming from Λ -trees by the standard combinatorics on Λ -words. In particular, this approach allows one to naturally generalize powerful techniques of Nielsen's method, Stallings' graph approach to subgroups, and Makanin-Razborov type of elimination processes (the machines) from free groups to Λ -free groups (see [92, 93, 61, 62, 63, 67, 35, 69, 68, 97, 98, 119]). In the case when Λ is equal to either \mathbb{Z}^n or \mathbb{Z}^∞ all these techniques are effective, so, for example, one can solve many algorithmic problems for limit groups or \mathbb{Z}^n -free groups using these methods.

The technique of infinite \mathbb{Z}^n -words allowed us to solve many algorithmic problems for limit groups precisely in the same manner as it was done for free groups by the standard Stallings graph techniques (we refer to [56] for a survey on the related results in free groups). We discuss these results in Section 9.7.

In our approach to limit groups (or arbitrary \mathbb{Z}^n -groups) one more important concept transpired. To carry over Nielsen cancellation argument or apply the machines the length function (or the action) has to satisfy some natural "completeness" conditions. To this end, given a group G acting on a Λ -tree Γ , we say that the action is *regular with respect to* $x \in \Gamma$ (see [67] for details) if for any $g, h \in G$ there exists $f \in G$ such that $[x, fx] = [x, gx] \cap [x, hx]$. In fact, the definition above does not depend on x and there exist equivalent formulations for length functions and Λ -words (see [103, 92]). Roughly speaking, regularity of action implies that all branch-points of Γ belong to the same G -orbit and it tells a lot about the structure of G in the case of free actions (see [68, 67] or Section 8.2). In the language of Λ -words this condition means that for any given two infinite words representing elements in the group G their common longest initial segment (which always exists) represents an element in G . The importance of the regularity condition was not recognized earlier simply because in the case when Λ is \mathbb{Z} , or \mathbb{R} every Λ -free group has a regular action. The regularity axiom appeared first in [89, 92] as a tool to deal with length functions in \mathbb{Z}^n (with respect to the lexicographic order). The outcome of this research is that if a finitely generated group G has a regular free action on a \mathbb{Z}^n -tree, then the Nielsen method and \mathbb{Z}^n -machines work in G . This, as expected, implies a lot of interesting results for \mathbb{Z}^n -free groups.

In the paper [67] we described finitely generated groups which admit free regular actions on \mathbb{Z}^n -trees, we call such groups finitely generated *complete \mathbb{Z}^n -free groups*.

Suppose G_i is an \mathbb{Z}^{n_i} -free group, $i = 1, 2$ with a maximal abelian subgroups $A \leq G_1$, $B \leq G_2$ such that

- (a) A and B are cyclically reduced with respect to the corresponding embeddings of G_1 and G_2 into infinite words,

- (b) there exists an isomorphism $\phi : A \rightarrow B$ such that $|\phi(a)| = |a|$ for any $a \in A$.

Then we call the amalgamated free product

$$\langle G_1, G_2 \mid A \stackrel{\phi}{=} B \rangle$$

the *length-preserving amalgam* of G_1 and G_2 .

Given a \mathbb{Z}^n -free group H and non-conjugate maximal abelian subgroups $A, B \leq H$ such that

- (a) A and B are cyclically reduced with respect to the embedding of H into infinite words,
- (b) there exists an isomorphism $\phi : A \rightarrow B$ such that $|\phi(a)| = |a|$ and a is not conjugate to $\phi(a)^{-1}$ in H for any $a \in A$.

then we call the HNN extension

$$\langle H, t \mid t^{-1}At = B \rangle$$

the *length-preserving separated HNN extension* of H .

Given a \mathbb{Z}^n -free group H and a maximal abelian subgroup A , we call the HNN extension

$$\langle H, t \mid t^{-1}At = A \rangle$$

(here the isomorphism $\phi : A \rightarrow A$ is identical) the *centralizer extension* of H .

The following result describes complete \mathbb{Z}^n -free groups.

Theorem 50. *A finitely generated group G is complete \mathbb{Z}^n -free if and only if it can be obtained from free groups by finitely many length-preserving separated HNN extensions and centralizer extensions.*

Notice that this is "if and only if" characterization of finitely generated complete \mathbb{Z}^n -free groups.

The following principle theorem allows one to transfer results from complete \mathbb{Z}^n -free groups to arbitrary \mathbb{Z}^n -free groups.

Theorem 49. [70] *Every finitely generated \mathbb{Z}^n -free group G has a length-preserving embedding into a finitely generated complete \mathbb{Z}^n -free group H .*

As in the case of limit groups, Bass-Serre theory, applied to subgroups of complete \mathbb{Z}^n -free groups, immediately gives algebraic structure of arbitrary finitely generated \mathbb{Z}^n -free groups:

Theorem 54. *A finitely generated group G is \mathbb{Z}^n -free if and only if it can be obtained from free groups by a finite sequence of length-preserving amalgams, length-preserving separated HNN extensions, and centralizer extensions.*

Again, we would like to emphasize here that the description above is "if and only if".

Using these techniques we generalized many algorithmic results from limit groups to arbitrary finitely generated \mathbb{Z}^n -free groups. We refer to the Section 9.7 for details.

In 2004 Guirardel described the structure of finitely generated \mathbb{R}^n -free groups in a similar fashion [49] (see Theorem 56 in Section 12.2). Since \mathbb{Z}^n -free groups are also \mathbb{R}^n -free this, of course, generalizes the Bass' result on \mathbb{Z}^n -free groups. Observe that given description of the algebraic structure of finitely generated \mathbb{R}^n -free groups does not “characterize” such groups completely, that is, the converse of the theorem does not hold. Nevertheless, the result is strong, it implies several important corollaries: firstly, it shows that finitely generated \mathbb{R}^n -free groups are finitely presented; and secondly, taken together with Dahmani's combination theorem [30], it implies that every finitely generated \mathbb{R}^n -free group is hyperbolic relative to its non-cyclic abelian subgroups.

New techniques. The recent developments in the theory of groups acting freely on Λ -trees are based on several new techniques that occurred after 1995: infinite Λ -words as an equivalent language for free Λ -actions [92]; regular actions and regular completions; general Machines (Elimination Processes) for regular Λ -actions that generalize Rips and Bestvina-Feighn Machines.

The elimination process techniques developed in Section 11 allow one to prove the following theorems.

Theorem 57. [The Main Structure Theorem [68]] *Any finitely presented group G with a regular free length function in an ordered abelian group Λ can be represented as a union of a finite series of groups*

$$G_1 < G_2 < \cdots < G_n = G,$$

where

1. G_1 is a free group,
2. G_{i+1} is obtained from G_i by finitely many HNN-extensions in which associated subgroups are maximal abelian, finitely generated, and length isomorphic as subgroups of Λ .

Theorem 58. [68] *Any finitely presented Λ -free group is \mathbb{R}^n -free.*

In his book [24] Chiswell (see also [108]) asked the following very important question (Question 1, p. 250): If G is a finitely generated Λ -free group, is G Λ_0 -free for some finitely generated abelian ordered group Λ_0 ? The following result answers this question in the affirmative in the strongest form. It comes from the proof of Theorem 58 (not the statement of the theorem itself).

Theorem 59 *Let G be a finitely presented group with a free Lyndon length function $l : G \rightarrow \Lambda$. Then the subgroup Λ_0 generated by $l(G)$ in Λ is finitely generated.*

Theorem 60. [68] *Any finitely presented group \tilde{G} with a free length function in an ordered abelian group Λ can be isometrically embedded into a finitely presented group G that has a free regular length function in Λ .*

The following result automatically follows from Theorem 57 and Theorem 60 by simple application of Bass-Serre Theory.

Theorem 61. *Any finitely presented Λ -free group G can be obtained from free groups by a finite sequence of amalgamated free products and HNN extensions with maximal abelian associated subgroups, which are free abelian groups of finite rank.*

The following result is about abelian subgroups of Λ -free groups. For $\Lambda = \mathbb{Z}^n$ it follows from the main structural result for \mathbb{Z}^n -free groups and [66], for $\Lambda = \mathbb{R}^n$ it was proved in [49]. The statement (1) below answers Question 2, p. 250 in [24] in the affirmative for finitely presented Λ -free groups.

Theorem 62. *Let G be a finitely presented Λ -free group. Then*

- (1) *every abelian subgroup of G is a free abelian group of finite rank uniformly bounded from above by the rank of the abelianization of G .*
- (2) *G has only finitely many conjugacy classes of maximal non-cyclic abelian subgroups,*
- (3) *G has a finite classifying space and the cohomological dimension of G is at most $\max\{2, r\}$ where r is the maximal rank of an abelian subgroup of G .*

Theorem 63. *Every finitely presented Λ -free group is hyperbolic relative to its non-cyclic abelian subgroups.*

The following results answers affirmatively in the strongest form to the Problem (GO3) from the Magnus list of open problems [10] in the case of finitely presented groups.

Corollary 18. *Every finitely presented Λ -free group is biautomatic.*

Theorem 64. *Every finitely presented Λ -free group G has a quasi-convex hierarchy.*

As a corollary one gets the following result.

Theorem 65. *Every finitely presented Λ -free group G is locally undistorted, that is, every finitely generated subgroup of G is quasi-isometrically embedded into G .*

Since a finitely generated \mathbb{R}^n -free group G is hyperbolic relative to its non-cyclic abelian subgroups and G admits a quasi-convex hierarchy then recent results of Wise [129] imply the following.

Corollary 19. *Every finitely presented Λ -free group G is virtually special, that is, some subgroup of finite index in G embeds into a right-angled Artin group.*

In his book [24] Chiswell posted Question 3 (p. 250): Is every Λ -free group orderable, or at least right-orderable? The following result answers this question in the affirmative in the case of finitely presented groups.

Theorem 66. *Every finitely presented Λ -free group is right orderable.*

The following addresses Chiswell's question whether Λ -free groups are orderable or not.

Theorem 67. *Every finitely presented Λ -free group is virtually orderable, that is, it contains an orderable subgroup of finite index.*

Since right-angled Artin groups are linear (see [55, 54, 32] and the class of linear groups is closed under finite extension we get the following

Theorem 68. *Every finitely presented Λ -free group is linear.*

Since every linear group is residually finite we get the following.

Corollary 20. *Every finitely presented Λ -free group is residually finite.*

It is known that linear groups are equationally Noetherian (see [7] for discussion on equationally Noetherian groups), therefore the following result holds.

Corollary 21. *Every finitely presented Λ -free group is equationally Noetherian.*

The structural results of the previous section give solution to many algorithmic problems on finitely presented Λ -free groups.

Theorem 69. [68] *Let G be a finitely presented Λ -free group. Then the following algorithmic problems are decidable in G :*

- *the Word Problem;*
- *the Conjugacy Problems;*
- *the Diophantine Problem (decidability of arbitrary equations in G).*

Theorem 63 combined with results of Dahmani and Groves [31] immediately implies the following two corollaries.

Corollary 22. *Let G be a finitely presented Λ -free group. Then:*

- *G has a non-trivial abelian splitting and one can find such a splitting effectively,*
- *G has a non-trivial abelian JSJ-decomposition and one can find such a decomposition effectively.*

Corollary 23. *The Isomorphism Problem is decidable in the class of finitely presented groups that act freely on some Λ -tree.*

Theorem 70. *The Subgroup Membership Problem is decidable in every finitely presented Λ -free group.*

2 Bass-Serre Theory

In his seminal book [120] J. P. Serre laid down fundamentals of the theory of groups acting on simplicial trees. In the following decade Serre's novel approach unified several geometric, algebraic, and combinatorial methods of group theory into a unique powerful tool, known today as Bass-Serre Theory. This tool allows

one to obtain a lot of structural information about the group from its action on a simplicial tree in terms of stabilizers of vertices and edges. One of the most important consequences of this approach is the structure of subgroups of the initial group which inherit the action on the tree and hence can be dealt with in the same manner as the ambient group.

In this section, following [34] we give basic treatment of Bass-Serre Theory. The original ideas and results can be found in [120], a more topological treatment of the theory - in [115].

2.1 G -sets and G -graphs

Let G be a group. X is called a G -set if there is a function $G \times X \rightarrow X$ (*left action of G on X*) given by $(g, x) \rightarrow gx$ such that $1x = x$ for all $x \in X$ and $f(gx) = (fg)x$ for any $f, g \in G$, $x \in X$.

A function $\alpha : X \rightarrow Y$ between G -sets is a G -map if $\alpha(gx) = g\alpha(x)$ for any $g \in G$, $x \in X$. X and Y are G -isomorphic if α is a bijection.

If X is a G -set, then the G -stabilizer of $x \in X$ is a subgroup $G_x = \{g \in G \mid gx = x\}$ of G . X is G -free if $G_x = 1$ for any $x \in X$. For $x \in X$, the G -orbit of x is $Gx = \{gx \mid g \in G\}$, a G -subset of X which is G -isomorphic to G/G_x with $gx \in Gx$ corresponding to $gG_x \in G/G_x$. The *quotient set* for the G -set X is defined as $G \backslash X = \{Gx \mid x \in X\}$, the set of G -orbits. A G -transversal in X is a subset S of X which meets each G -orbit exactly once, so $S \rightarrow G \backslash X$ is bijective.

A G -graph $(X, V(X), E(X), \sigma, \tau)$ is a non-empty G -set X with a non-empty G -subset $V(X)$, its complement $E(X) = X - V(X)$, and three maps

$$\sigma : E(X) \rightarrow V(X), \quad \tau : E(X) \rightarrow V(X), \quad \bar{\cdot} : E(X) \rightarrow E(X),$$

which satisfy the following conditions:

$$\sigma(\bar{e}) = \tau(e), \quad \tau(\bar{e}) = \sigma(e), \quad \bar{\bar{e}} = e, \quad \bar{e} \neq e.$$

σ and τ are called *incidence maps*.

For G -graphs X, Y , a G -graph map $\alpha : X \rightarrow Y$ is a G -map such that $\alpha(V(X)) \subseteq V(Y)$, $\alpha(E(X)) \subseteq E(Y)$, and $\alpha(\sigma(e)) = \sigma(\alpha(e))$, $\alpha(\tau(e)) = \tau(\alpha(e))$ for any $e \in E(X)$.

A *path* p in a G -graph X is a sequence $e_1 \cdots e_n$ of edges such that $\tau(e_i) = \sigma(e_{i+1})$, $i \in [1, n-1]$. In this case, $\sigma(e_1)$ is the origin of p and $\tau(e_n)$ is its terminus. p is *closed* if $\sigma(e_1) = \tau(e_n)$.

A G -graph X is a G -tree if for any $x, y \in V(X)$ there exists a unique path from x to y , this path is called in this case the X -geodesic from x to y .

Proposition 1 ([34], Prop 2.6). *If X is a G -graph and $G \backslash X$ is connected then there exist subsets $Y_0 \subseteq Y \subseteq X$ such that Y is a G -transversal in X , Y_0 is a subtree of X , $V(Y) = V(Y_0)$, and for each $e \in E(Y)$, $\sigma(e) \in V(Y) = V(Y_0)$.*

The subset Y from Proposition 1 is called a *fundamental G -transversal* in X , with subtree Y_0 .

2.2 Graphs of groups

Let \mathcal{G} be a class of groups. A *graph of groups* (\mathcal{G}, X) consists of a connected graph X and an assignment of $G(x) \in \mathcal{G}$ to every $x \in V(X) \cup E(X)$, such that for every $e \in E(X)$, $G(e) = G(\bar{e})$, and there exists a boundary monomorphism $i_e : G(e) \rightarrow G(\sigma(e))$. $G(v)$, $v \in V(X)$ and $G(e)$, $e \in E(X)$ are called respectively *vertex* and *edge groups*.

Let (\mathcal{G}, X) be a graph of groups with a maximal subtree Y_0 . The *fundamental group* $\pi(\mathcal{G}, X, Y_0)$ of the graph of groups (\mathcal{G}, X) with respect to Y_0 is the group with the following presentation:

$$\begin{aligned} \langle G(v) \ (v \in V(X)), \ t_e \ (e \in E(X)) \mid \text{rel}(G(v)), \ t_e i_e(g) t_e^{-1} = i_{\bar{e}}(g) \ (g \in G(e)), \\ t_e t_{\bar{e}} = 1, \ (e \in E(X)), \ t_e = 1 \ (e \in Y_0) \rangle. \end{aligned}$$

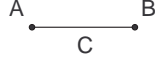
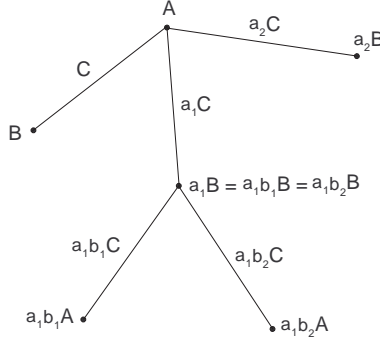
Let X be a G -graph such that $G \backslash X$ is connected, and let Y be a fundamental G -transversal for X with subtree Y_0 . For each $e \in E(Y)$ there are unique $\tilde{\sigma}(e), \tilde{\tau}(e) \in V(Y)$ which belong to the same G -orbits as $\sigma(e)$ and $\tau(e)$ respectively, and we can assume $\tilde{\sigma}(e) = \sigma(e)$. Y equipped with incidence functions $\tilde{\sigma}, \tilde{\tau} : E(Y) \rightarrow V(Y)$ becomes a graph G -isomorphic to $G \backslash X$, and Y_0 is its maximal subtree. Observe that for each $e \in E(Y)$, $\tau(e)$ and $\tilde{\tau}(e)$ are in the same G -orbit, so we can choose $t_e \in G$ such that $t_e \tilde{\tau}(e) = \tau(e)$. It is easy to see that $t_e = 1$ if $e \in E(Y_0)$ since Y_0 is a subtree of X and $\tilde{\tau}(e) = \tau(e)$. The set $\{t_e \mid e \in E(Y)\}$ is called a *family of connecting elements*. Now, $G_e \subseteq G_{\sigma(e)}$ and $G_e \subseteq G_{\tau(e)} = t_e G_{\tilde{\tau}(e)} t_e^{-1}$, so there is an embedding $i_e : G_e \rightarrow G_{\tilde{\tau}(e)}$ defined by $g \rightarrow t_e g t_e^{-1}$. This data defines the graph of groups *associated to* X with respect to the fundamental G -transversal Y , the maximal subtree Y_0 , and the family of connecting elements t_e . Denote $\mathcal{G} = \{G_v \mid v \in V(Y)\} \cup \{G_e \mid e \in E(Y)\}$.

Theorem 1. [120, Theorem I.13] *If X is a G -tree then G is naturally isomorphic to $\pi(\mathcal{G}, Y, Y_0)$.*

Remark 1. *From Theorem 1 it follows that if X is G -free then G is isomorphic to a free group.*

On the other hand, given a graph of groups (\mathcal{G}, X) with the fundamental group $G = \pi(\mathcal{G}, X, X_0)$, one can construct a G -tree Y which is a universal cover of X .

Example 1. Let $G = A *_C B$ be a free product of groups A and B with amalgamation along a subgroup C . Observe, that G is isomorphic to the fundamental group $\pi(\mathcal{G}, X, X_0)$ of the graph of groups X (see Figure 1), where $X_0 = X$. Define Y as follows: $V(Y)$ consists of all cosets gA and gB ($g \in G$), $E(Y)$ consists of all cosets gC ($g \in G$), the maps σ and τ which give the endpoints of the edge are defined as $\sigma(gC) = gA$, $\tau(gC) = gB$. It is easy to check that Y is a tree and that G acts on Y without inversions by the left multiplication (see Figure 2). All vertices gA , $g \in G$ are in the same orbit, the same is true about all vertices gB , $g \in G$ and edges gC , $g \in G$, and $G \backslash Y = X$.


 Figure 1: The graph of groups for $G = A *_C B$

 Figure 2: The tree Y for $A *_C B$: $a_1, a_2 \in A \setminus B$, $b_1, b_2 \in B \setminus A$

Example 2. Let $G = A *_C = \langle A, t | t^{-1}ct = \phi(c) \rangle$ be an HNN extension of a group A with associated subgroups C and $\phi(C)$. Here G is isomorphic to the fundamental group $\pi(\mathcal{G}, X, X_0)$ of the graph of groups X (see Figure 3), where X_0 consists of a single vertex with no edges. Define Y as follows:


 Figure 3: The graph of groups for $G = A *_C$

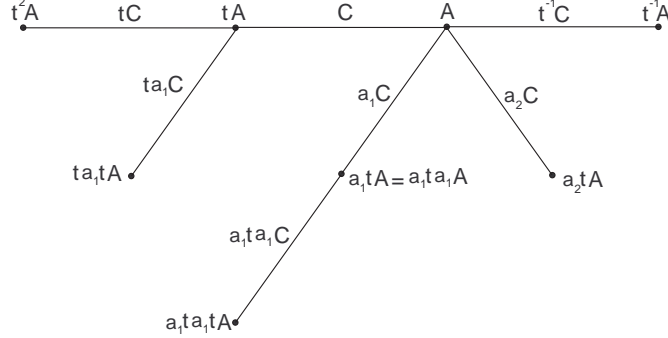
$V(Y) = \{gA | g \in G\}$, $E(Y) = \{gC | g \in G\}$, and $\sigma(gC) = gA$, $\tau(gC) = (gt)A$. Again, it is easy to check that Y is a tree, G acts on Y without inversions by the left multiplication, and $G \backslash Y = X$ (see Figure 4).

The idea of constructing a covering tree for a given graph of groups given in the examples above can be generalized as follows.

Given a graph of groups (\mathcal{G}, X) , A (\mathcal{G}, X) -path of length $k \geq 0$ from $v \in V(X)$ to $v' \in V(X)$ is a sequence

$$p = g_0, e_1, g_1, \dots, e_k, g_k,$$

where $k \geq 0$ is an integer, $e_1 \cdots e_k$ is a path in X from $v \in V(X)$ to $v' \in V(X)$, $g_0 \in G(v)$, $g_k \in G(v')$ and $g_i \in G(\tau(e_i)) = G(\sigma(e_{i+1}))$, $i \in [1, k-1]$. If p is

Figure 4: The tree Y for $A*_C : a_1, a_2 \in A \setminus C$

a (\mathcal{G}, X) -path from v to v' and q is a (\mathcal{G}, X) -path from v' to v'' then one can define the concatenation pq of p and q in the obvious way.

One can introduce the equivalence relation on the set of all (\mathcal{G}, X) -paths generated by

$$g, e, i_{\bar{e}}(c), \bar{e}, f \sim g i_e(c) f,$$

where $e \in E(X)$, $c \in G(e)$, $g, f \in G(\sigma(e))$. Observe that if $p \sim q$ then p and q have the same initial and terminal vertices in $V(X)$.

Lemma 1. [57] *Let (\mathcal{G}, X) be a graph of groups and let $v_0 \in V(X)$. Then*

1. *the set $P(\mathcal{G}, X, v_0)$ of “ \sim ”-equivalence classes of (\mathcal{G}, X) -loops at v_0 is a group with respect to concatenation of paths,*
2. *for any spanning tree T of X , $P(\mathcal{G}, X, v_0)$ is naturally isomorphic to $\pi(\mathcal{G}, X, T)$.*

Let (\mathcal{G}, X) be a graph of groups and let $v_0 \in V(X)$. For (\mathcal{G}, X) -paths p, q originating at v_0 we write $p \approx q$ if

1. $t(p) = t(q)$,
2. $p \sim qa$ for some $a \in G(t(p))$.

For (\mathcal{G}, X) -path p from v_0 to v we denote the “ \approx ”-equivalence class of p by $\bar{p} G(v)$.

Now one can define the *universal Bass-Serre covering tree* T_X associated with (\mathcal{G}, X) as follows. The vertices of T_X are “ \approx ”-equivalence classes of (\mathcal{G}, X) -paths originating at v_0 . Two vertices $x, x' \in T_X$ are connected by an edge if and only if $x = \bar{p} G(v)$, $x' = \bar{p} a \bar{e} G(v')$, where p is a (\mathcal{G}, X) -path from v_0 to v and $a \in G(v)$, $e \in E(X)$ with $\sigma(e) = v$, $\tau(e) = v'$.

It is easy to see that T_X is indeed a tree with a natural base vertex $x_0 = \bar{1} G(v_0)$ and $G = P(\mathcal{G}, X, v_0)$ has a natural simplicial action on T_X defined as follows: if $g = \bar{q} \in G$, where q is a (\mathcal{G}, X) -loop at v_0 and $u = \bar{p} G(v)$, where p is a (\mathcal{G}, X) -path from v_0 to v , then $g \cdot u = \bar{qp} G(v)$.

2.3 Induced splittings

Let (\mathcal{G}, X) be a graph of groups with a base-vertex $v_0 \in V(X)$. Let $G = P(\mathcal{G}, X, v_0)$ and T_X be the universal Bass-Serre covering tree associated with (\mathcal{G}, X) . If $H \leq G$ then the action of G on T_X induces an action of H on T_X and H can be represented as the fundamental group of a graph of groups by Theorem 1. More precisely, the following result holds.

Theorem 2. [120] *Let x_0 be the base-vertex of T_X mapping to v_0 under the natural quotient map and let $T_H \subset T_X$ be an H -invariant subtree containing x_0 . Then H is isomorphic to the fundamental group of a graph of groups (\mathcal{H}, Y) , where $Y = H \backslash T_H$ and $\mathcal{H} = \{K \cap H \mid K \in \mathcal{G}\}$.*

The graph of groups (\mathcal{H}, Y) from the theorem above is called the *induced splitting of H with respect to T_H* .

Remark 2. *If T_X is G -free and $H \leq G$, then H also acts on T_X and T_X is H -free. Now it follows that any subgroup of a free group is free, which is a well-known result (see [96, 114])*

Example 3. *Let $G = A * B$ be a free product of A and B . The amalgamated subgroup is trivial and G is isomorphic to the fundamental graph of groups X (see Figure 5). As in Example 1, one can construct a tree Y on which G acts*

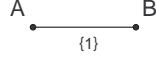


Figure 5: The graph of groups for $G = A * B$

so that $X = G \backslash Y$. Since the amalgamated subgroup is trivial, $G(e)$ is trivial for every $e \in E(Y)$ and for each $v \in V(Y)$, $G(v)$ is either $g^{-1}Ag$, or $g^{-1}Bg$ for some $g \in G$. Now, if $H \leq G$ then by Theorem 2, H is isomorphic to the fundamental group of a graph of groups (\mathcal{H}, X') , where $X' = H \backslash Y_H$, Y_H is an H -invariant subtree of Y , and $\mathcal{H} = \{K \cap H \mid K \in \mathcal{G}\}$. In other words, for each $e \in E(Y_H)$, $H(e) = H \cap G(e)$ is trivial and for each $v \in V(Y_H)$, $H(v) = H \cap G(v)$ is either $H \cap g^{-1}Ag$, or $H \cap g^{-1}Bg$ for some $g \in G$. It follows that if H is finitely generated then

$$H \simeq H_1 * \cdots * H_k * F,$$

where each H_i is conjugate into either A , or B , and F is a finitely generated free group. This result is known as the Kurosh subgroup theorem (see [72]).

In some situations it is possible to construct an induced splitting of H effectively. Below is an algorithmic version of Theorem 2 which can be applied when the splitting of the ambient group is “nice”.

Theorem 3. [57, Theorem 1.1] *Let (\mathcal{G}, X) be a graph of groups such that*

1. $G(v)$ is either locally quasiconvex word-hyperbolic or polycyclic-by-finite for every $v \in V(X)$,
2. $G(e)$ is polycyclic-by-finite for every $e \in E(X)$.

Then there is an algorithm which, given a finite subset $S \subset G$, constructs the induced splitting and a finite presentation for $H = \langle S \rangle \leq G = \pi(\mathcal{G}, X, T)$, where T is a maximal subtree of X .

Recall that the *Uniform Membership Problem* is solvable in a group G with a finite generating set S if there is an algorithm which, for any finite family of words u, w_1, w_2, \dots, w_n over $S^{\pm 1}$ decides whether or not the element of G represented by u belongs to the subgroup of G generated by the elements of G corresponding to w_1, w_2, \dots, w_n . The definition does not depend on the choice of a finite generating set for G .

In particular, Theorem 3 implies the following result.

Corollary 1. [57, Theorem 1.1] *Let (\mathcal{G}, X) be a graph of groups with the properties listed in Theorem 3, and let $G = \pi(\mathcal{G}, X, T)$, where T is a maximal subtree of X . Then the uniform membership problem for G is solvable.*

The proof of Theorem 3 given in [57] involves the notion of *folding*, which are particular transformations of graphs of groups.

3 Stallings' pregroups and their universal groups

The notions of a *pregroup* and its *universal group* were first introduced by J. Stallings in [123], but the ideas behind these notions go back to B. L. van der Waerden [128] and R. Baer [3]. A pregroup P provides a convenient tool to introduce reduced forms for elements of $U(P)$ and in some cases gives rise to an integer-valued length function on $U(P)$ which can be connected with an action of $U(P)$ on a simplicial tree. Connections between pregroups and free constructions in groups were established by F. Rimlinger in [111, 110]. Some generalizations of pregroups were obtained in [73, 75, 74].

3.1 Definitions and examples

A *pregroup* P is a set P , with a distinguished element 1, equipped with a partial multiplication, that is, a function $D \rightarrow P$, $(x, y) \rightarrow xy$, where $D \subset P \times P$, and an inversion, that is, a function $^{-1} : P \rightarrow P$, $x \rightarrow x^{-1}$, satisfying the following axioms (below xy is *defined* if $(x, y) \in D$):

- (P1) for all $u \in P$, the products $u1$ and $1u$ are defined and $u1 = 1u = u$,
- (P2) for all $u \in P$, the products $u^{-1}u$ and uu^{-1} are defined and $u^{-1}u = uu^{-1} = 1$,
- (P3) for all $u, v \in P$, if uv is defined, then so is $v^{-1}u^{-1}$ and $(uv)^{-1} = v^{-1}u^{-1}$,

- (P4) for all $u, v, w \in P$, if uv and vw are defined, then $(uv)w$ is defined if and only if $u(vw)$ is defined, in which case

$$(uv)w = u(vw),$$

- (P5) for all $u, v, w, z \in P$, if uv, vw , and wz are all defined then either uvw , or vwz is defined.

It was noticed (see [53]) that (P3) follows from (P1), (P2), and (P4), hence, it can be omitted.

A finite sequence u_1, \dots, u_n of elements from P is termed a P -product and it is denoted by $u_1 \cdots u_n$ (one may view it as a word in the alphabet P). The P -length of $u_1 \cdots u_n$ is equal to n . A P -product $u_1 \cdots u_n$ is called *reduced* if for every $i \in [1, n-1]$ the product $u_i u_{i+1}$ is not defined in P .

The following lemma lists some simple implications from the axioms (P1) – (P5).

Lemma 2. [123] *Let P be a pregroup. Then*

- (1) $(x^{-1})^{-1} = x$ for every $x \in P$,
- (2) if ax is defined, then $a^{-1}(ax)$ is defined and $a^{-1}(ax) = x$,
- (3) if xa is defined, then $(xa)a^{-1}$ is defined and $(xa)a^{-1} = x$,
- (4) if ax and $a^{-1}y$ are defined, then xy is defined if and only if $(xa)(a^{-1}y)$ is defined, in which case $xy = (xa)(a^{-1}y)$,
- (5) if xa and $a^{-1}y$ are defined, then xyz is a reduced P -product if and only if $(xa)(a^{-1}y)z$ is reduced; similarly, zxy is reduced if and only if $z(xa)(a^{-1}y)$ is reduced,
- (6) if xy is a reduced P -product and if $xa, a^{-1}y, yb$ are defined then $(a^{-1}y)b$ is defined,
- (7) if xy is a reduced P -product and $xa, a^{-1}y, (xa)b$ and $b^{-1}(a^{-1}y)$ are defined then ab is defined.

Now, one can define the universal group $U(P)$ of a pregroup P as follows. Consider all reduced P -products. Observe that if a product $u_1 \cdots u_n$ is not reduced, that is, the product $u_i u_{i+1}$ is defined in P for some i then $u_i u_{i+1} = v \in P$ and one can *reduce* $u_1 \cdots u_n$ by replacing the pair $u_i u_{i+1}$ by v . Now, given two reduced P -products $u_1 \cdots u_n$ and $v_1 \cdots v_m$, we write

$$u_1 \cdots u_n \sim v_1 \cdots v_m$$

if and only if $m = n$ and there exist elements $a_1, \dots, a_{n-1} \in P$ such that $a_{i-1}^{-1} u_i a_i$ are defined and $v_i = a_{i-1}^{-1} u_i a_i$ for $i \in [1, n]$ (here $a_0 = a_n = 1$). In this case we also say that $v_1 \cdots v_m$ can be obtained from $u_1 \cdots u_n$ by *interleaving*. From Lemma 2 it follows that “ \sim ” is an equivalence relation on the set of all

reduced P -products. Now, the group $U(P)$ can be described as the set $U(P)$ of “ \sim ”-equivalence classes of reduced P -products, where multiplication is given by concatenation of representatives and consecutive reduction of the resulting product. Obviously, P embeds into $U(P)$ via the canonical map $u \rightarrow u$.

Recall that a mapping $\phi : P \rightarrow Q$ of pregroups is a *morphism* if for any $x, y \in P$ whenever xy is defined in P , $\phi(x)\phi(y)$ is defined in Q and equal to $\phi(xy)$. Now the group $U(P)$ can be characterized by the following universal property: there is a morphism of pregroups $\lambda : P \rightarrow U(P)$, such that, for any morphism $\phi : P \rightarrow G$ of P into a group G , there is a unique group homomorphism $\psi : U(P) \rightarrow G$ for which $\psi\lambda = \phi$. This shows that $U(P)$ is a group with a generating set P and a set of relations $xy = z$, where $x, y \in P$, xy is defined in P , and equal to z . If the map $\psi : U(P) \rightarrow G$ above is an isomorphism then we say that P is a *pregroup structure* for G .

Given a group G one can try to find a pregroup structure for G as a subset of G . In this case the following lemma helps (this result was used implicitly in [92, 67] to find certain pregroup structures).

Lemma 3. *Let G be a group and let $P \subseteq G$ be a generating set for G such that $P^{-1} = P$. Let $D \subseteq (P, P)$ be such that $(x, y) \in D$ implies $xy \in P$, and assume that multiplication and inversion are induced on P from G . Then P is a pregroup structure for G if P satisfies (P5).*

Proof. The result follows immediately from the fact that P is a subset of G , because in this case the axioms (P1) - (P4) are satisfied automatically for P . \square

There is another way to check if $P \subseteq G$ is a pregroup structure. Again, we assume that P is a generating set for G such that $P^{-1} = P$, $D \subseteq (P, P)$ is such that $(x, y) \in D$ implies $xy \in P$, and the multiplication and inversion are induced on P from G . If all reduced P -products representing the same group element have equal P -length then we say that (P, D) is a *reduced word structure* for G .

Theorem 4. [110] *If P is a reduced word structure for G then P is a pregroup structure for G .*

The principal examples of pregroups and their universal groups are shown below.

Example 4. *For any group G define $P = G$, $D = (P, P)$, and let the multiplication and inversion on P be induced from G . Then $U(P) \simeq G$.*

Example 5. *Let X be a set. Define $P = X \cup \overline{X} \cup \{1\}$, where $\overline{X} = \{\overline{x} | x \in X\}$ and $1 \notin X$. Define the inversion function $^{-1} : P \rightarrow P$ as follows: $x^{-1} = \overline{x}$, $\overline{x}^{-1} = x$ for every $x \in X$ and the corresponding $\overline{x} \in \overline{X}$, and $1^{-1} = 1$. Without loss of generality we identify \overline{X} with X^{-1} , the image of X under $^{-1}$. Next, $(x, y) \in D$ if either $y = x^{-1}$ (hence, $xy = 1 \in P$), or either $x = 1$ (hence, $xy = y \in P$), or $y = 1$ (so, $xy = x \in P$). It is easy to check that P with the inversion and the set D is a pregroup, and $U(P) \simeq F(X)$, a free group on X . The pregroup P is called the free pregroup on X .*

Example 6. [123] Let A, B and C be groups, and $\phi : C \rightarrow A$, $\psi : C \rightarrow B$ monomorphisms. Identify $\phi(C)$ with $\psi(C)$, then $A \cap B = C$. Let $P = A \cup B$. The identity 1 and inversion function are obvious. For, $x, y \in P$, the product xy is defined only if x and y both belong either to A , or to B . One can verify that P is a pregroup and $U(P) \simeq A *_C B$.

Example 7. [123] Consider $A *_C B$. Let P be the subset of all elements that can be written as the product bab' for some $b, b' \in B$, $a \in A$. In particular, P contains A and B . For, $x, y \in P$, the product xy is defined if $xy \in P$. Using the structure of $A *_C B$, one can prove that P is a pregroup. The universal group $U(P)$ is isomorphic to $A *_C B$, but observe that P is not the same as in Example 6.

Example 8. [123] Let G be a group, H a subgroup of G , and $\phi : H \rightarrow G$ a monomorphism. For $t \notin G$ construct four sets in one-to-one correspondence with G :

$$G, t^{-1}G, Gt, t^{-1}Gt.$$

Identify $h \in H \subseteq G$, with $t^{-1}\phi(h)t \in t^{-1}Gt$. The multiplication is naturally defined between G and G , G and Gt , $t^{-1}G$ and G , $t^{-1}G$ and Gt , Gt and $t^{-1}G$, Gt and $t^{-1}Gt$, $t^{-1}Gt$ and $t^{-1}G$, $t^{-1}Gt$ and $t^{-1}Gt$, by cancelation of tt^{-1} and multiplication in G . By the formulas $ht^{-1} = t^{-1}\phi(h)$ and $th = \phi(h)t$, the multiplication is defined in all cases when one factor belongs to H . Hence,

$$P = G \cup t^{-1}G \cup Gt \cup t^{-1}Gt$$

is a pregroup and $U(P) \simeq \langle G, t | t^{-1}ht = \phi(h), h \in H \rangle$.

From the examples above it follows that a group which splits into an amalgamated free product or an HNN-extension has a non-trivial pregroup structure. The same holds in general: it was proved in [111, Theorem B] that a group isomorphic to the fundamental group of a finite graph of groups has a pregroup structure which arises from the graph of groups. The converse, namely, that the universal group of a pregroup P is isomorphic to the fundamental group of a graph of groups also holds provided P is of *finite height*. This property is explained below.

For $x, y \in P$ we write $x \preceq y$ if and only if for any $z \in P$, zx is defined whenever zy is defined. The relation “ \preceq ” is a *tree ordering* on P (see [123]), that is, there exists a smallest element 1 and

$$\forall x, y, z \in P : (x \preceq z \text{ and } y \preceq z) \Rightarrow (x \preceq y \text{ or } y \preceq x)$$

Elements $x, y \in P$ are comparable if either $x \preceq y$, or $y \preceq x$, or both. If both $x \preceq y$ and $y \preceq x$ then we write $x \approx y$, and if $x \preceq y$ but not $y \preceq x$ then we write $x \prec y$.

Lemma 4. [111, Lemma I.2.6] $x, y \in P$ are comparable if and only if $x^{-1}y \in P$.

An element $x \in P$ is of finite height $n \in \mathbb{N}$ if there exist $x_0, x_1, \dots, x_n \in P$ such that $1 = x_0 \prec x_1 \prec \dots \prec x_n = x$ and for each $0 \leq i \leq n-1$, if $z \in P$ and $x_i \preceq z \preceq x_{i+1}$ then $z \approx x_i$ and $z \approx x_{i+1}$. P is of finite height if there exists a natural number N which bounds heights of all elements of P .

Now, if P is of finite height then $U(P)$ is isomorphic to the fundamental group of a graph of groups whose vertex and edge groups can be obtained as the universal groups of certain subpregroups of P (see [111, Theorem A]).

3.2 Connection with length functions

Following the notation from [22] for a pregroup P define

$$B = \{a \in P \mid za \text{ and } az \text{ are defined for all } z \in P\}.$$

Obviously, B is a subgroup of P . Furthermore, if a reduced P -product contains an element from B then it consists of a single element.

Below we use the following notation:

- (1) if $x, y \in P$ then we write $xy = x \circ y$ if xy is not defined,
- (2) if a P -product $u = u_1 \cdots u_n$ is reduced then we put $|u| = n$. Notice, that the function $u \rightarrow |u|$ induces a well-defined function on $U(P)$.

Theorem 5. [22] *Let P be a pregroup and let $|\cdot| : U(P) \rightarrow \mathbb{Z}$ be defined by $u \rightarrow |u|$ for each $u \in U(P)$. Then “ $|\cdot|$ ” is a Lyndon length function (see Section 5) if and only if P satisfies an additional axiom (P6):*

(P6) *for any $x, y \in P$, if xy is not defined but xa and $a^{-1}y$ are both defined for some $a \in P$ then $a \in B$.*

It is known (Theorem 2.7 in [94]) that the axiom (P6) is equivalent in a pregroup P to the following one:

(P6') *for any $x, y \in P$, if xy is not defined and $(ax)y$ is defined for some $a \in P$ then $ax \in B$.*

Remark 3. *Suppose P satisfies (P6). If a reduced P -product $v_1 \cdots v_n$ is obtained from $u_1 \cdots u_n$ by interleaving $v_i = a_{i-1}^{-1}u_i a_i$ for $i \in [1, n]$, where $a_0 = a_n = 1$ then $a_1, \dots, a_{n-1} \in B$.*

More on the connection of pregroups with length functions and Bass-Serre theory can be found in [22], [53] and [111].

4 Λ -trees

The theory of Λ -trees (where $\Lambda = \mathbb{R}$) has its origins in the papers by I. Chiswell [21] and J. Tits [125]. The first paper contains a construction of a \mathbb{Z} -tree starting from a Lyndon length function on a group (see Section 5), an idea considered earlier by R. Lyndon in [77].

Later, in their very influential paper [87] J. Morgan and P. Shalen linked group actions on \mathbb{R} -trees with topology and generalized parts of Thurston's Geometrization Theorem. Next, they introduced Λ -trees for an arbitrary ordered abelian group Λ and the general form of Chiswell's construction. Thus, it became clear that abstract length functions with values in Λ and group actions on Λ -trees are just two equivalent approaches to the same realm of group theory questions (more on this equivalence can be found in Section 7). The unified theory was further developed in the important paper by R. Alperin and H. Bass [1], where authors state a fundamental problem in the theory of group actions on Λ -trees: find the group theoretic information carried by a Λ -tree action (analogous to Bass-Serre theory), in particular, describe finitely generated groups acting freely on Λ -trees (Λ -free groups).

Here we introduce basics of the theory of Λ -trees, which can be found in more detail in [1] and [24].

4.1 Ordered abelian groups

In this section some well-known results on ordered abelian groups are collected. For proofs and details we refer to the books [43] and [71].

A set A equipped with addition “+” and a partial order “ \leq ” is called a *partially ordered abelian group* if:

- (1) $\langle A, + \rangle$ is an abelian group,
- (2) $\langle A, \leq \rangle$ is a partially ordered set,
- (3) for all $a, b, c \in A$, $a \leq b$ implies $a + c \leq b + c$.

An abelian group A is called *orderable* if there exists a linear order “ \leq ” on A , satisfying the condition (3) above. In general, the ordering on A is not unique.

Let A and B be ordered abelian groups. Then the direct sum $A \oplus B$ is orderable with respect to the *right lexicographic order*, defined as follows:

$$(a_1, b_1) < (a_2, b_2) \Leftrightarrow b_1 < b_2 \text{ or } b_1 = b_2 \text{ and } a_1 < a_2.$$

Similarly, one can define the right lexicographic order on finite direct sums of ordered abelian groups or even on infinite direct sums if the set of indices is linearly ordered.

For elements a, b of an ordered group A the *closed segment* $[a, b]$ is defined by

$$[a, b] = \{c \in A \mid a \leq c \leq b\}.$$

A subset $C \subset A$ is called *convex*, if for every $a, b \in C$ the set C contains $[a, b]$. In particular, a subgroup B of A is convex if $[0, b] \subset B$ for every positive $b \in B$. In this event, the quotient A/B is an ordered abelian group with respect to the order induced from A .

A group A is called *archimedean* if it has no non-trivial proper convex subgroups. It is known that A is archimedean if and only if A can be embedded

into the ordered abelian group of real numbers \mathbb{R}_+ , or equivalently, for any $0 < a \in A$ and any $b \in A$ there exists an integer n such that $na > b$.

It is not hard to see that the set of convex subgroups of an ordered abelian group A is linearly ordered by inclusion (see, for example, [43]), it is called *the complete chain of convex subgroups* in A . Notice that

$$E_n = \{f(t) \in \mathbb{Z}[t] \mid \deg(f(t)) \leq n\}$$

is a convex subgroup of $\mathbb{Z}[t]$ (here $\deg(f(t))$ is the degree of $f(t)$) and

$$0 < E_0 < E_1 < \cdots < E_n < \cdots$$

is the complete chain of convex subgroups of $\mathbb{Z}[t]$.

If A is finitely generated then the complete chain of convex subgroups of A

$$0 = A_0 < A_1 < \cdots < A_n = A$$

is finite. The following result (see, for example, [24]) shows that this chain completely determines the order on A , as well as the structure of A . Namely, the groups A_i/A_{i-1} are archimedean (with respect to the induced order) and A is isomorphic (as an ordered group) to the direct sum

$$A_1 \oplus A_2/A_1 \oplus \cdots \oplus A_n/A_{n-1} \tag{1}$$

with the right lexicographic order.

An ordered abelian group A is called *discretely ordered* if A has a non-trivial minimal positive element (we denote it by 1_A). In this event, for any $a \in A$ the following hold:

- (1) $a + 1_A = \min\{b \mid b > a\}$,
- (2) $a - 1_A = \max\{b \mid b < a\}$.

For example, $A = \mathbb{Z}^n$ with the right lexicographic order is discretely ordered with $1_{\mathbb{Z}^n} = (1, 0, \dots, 0)$. The additive group of integer polynomials $\mathbb{Z}[t]$ is discretely ordered with $1_{\mathbb{Z}[t]} = 1$.

Lemma 5. [92] *A finitely generated discretely ordered archimedean abelian group is infinite cyclic.*

Recall that an ordered abelian group A is *hereditary discrete* if for any convex subgroup $E \leq A$ the quotient A/E is discrete with respect to the induced order.

Corollary 2. [92] *Let A be a finitely generated hereditary discrete ordered abelian group. Then A is isomorphic to the direct product of finitely many copies of \mathbb{Z} with the lexicographic order.*

4.2 Λ -metric spaces

Let X be a non-empty set, Λ an ordered abelian group. A Λ -metric on X is a mapping $d : X \times X \rightarrow \Lambda$ such that for all $x, y, z \in X$:

- (M1) $d(x, y) \geq 0$,
- (M2) $d(x, y) = 0$ if and only if $x = y$,
- (M3) $d(x, y) = d(y, x)$,
- (M4) $d(x, y) \leq d(x, z) + d(y, z)$.

So a Λ -metric space is a pair (X, d) , where X is a non-empty set and d is a Λ -metric on X . If (X, d) and (X', d') are Λ -metric spaces, an *isometry* from (X, d) to (X', d') is a mapping $f : X \rightarrow X'$ such that $d(x, y) = d'(f(x), f(y))$ for all $x, y \in X$.

A *segment* in a Λ -metric space is the image of an isometry $\alpha : [a, b]_\Lambda \rightarrow X$ for some $a, b \in \Lambda$ and $[a, b]_\Lambda$ is a segment in Λ . The endpoints of the segment are $\alpha(a), \alpha(b)$.

We call a Λ -metric space (X, d) *geodesic* if for all $x, y \in X$, there is a segment in X with endpoints x, y and (X, d) is *geodesically linear* if for all $x, y \in X$, there is a unique segment in X whose set of endpoints is $\{x, y\}$.

It is not hard to see, for example, that (Λ, d) is a geodesically linear Λ -metric space, where $d(a, b) = |a - b|$, and the segment with endpoints a, b is $[a, b]_\Lambda$.

Let (X, d) be a Λ -metric space. Choose a point $v \in X$, and for $x, y \in X$, define

$$(x \cdot y)_v = \frac{1}{2}(d(x, v) + d(y, v) - d(x, y)).$$

Observe, that in general $(x \cdot y)_v \in \frac{1}{2}\Lambda$.

The following simple result follows immediately

Lemma 6. [24] *If (X, d) is a Λ -metric space then the following are equivalent:*

1. *for some $v \in X$ and all $x, y \in X$, $(x \cdot y)_v \in \Lambda$,*
2. *for all $v, x, y \in X$, $(x \cdot y)_v \in \Lambda$.*

Let $\delta \in \Lambda$ with $\delta \geq 0$. Then (X, p) is δ -hyperbolic with respect to v if, for all $x, y, z \in X$,

$$(x \cdot y)_v \geq \min\{(x \cdot z)_v, (z \cdot y)_v\} - \delta.$$

Lemma 7. [24] *If (X, d) is δ -hyperbolic with respect to v , and t is any other point of X , then (X, d) is 2δ -hyperbolic with respect to t .*

A Λ -tree is a Λ -metric space (X, d) such that:

- (T1) (X, d) is geodesic,
- (T2) if two segments of (X, d) intersect in a single point, which is an endpoint of both, then their union is a segment,

(T3) the intersection of two segments with a common endpoint is also a segment.

Example 9. Λ together with the usual metric $d(a, b) = |a - b|$ is a Λ -tree. Moreover, any convex set of Λ is a Λ -tree.

Example 10. A \mathbb{Z} -metric space (X, d) is a \mathbb{Z} -tree if and only if there is a simplicial tree Γ such that $X = V(\Gamma)$ and p is the path metric of Γ .

Observe that in general a Λ -tree can not be viewed as a simplicial tree with the path metric like in Example 10.

Lemma 8. [24] Let (X, d) be Λ -tree. Then (X, d) is 0-hyperbolic, and for all $x, y, v \in X$ we have $(x \cdot y)_v \in \Lambda$.

Eventually, we say that a group G acts on a Λ -tree X if any element $g \in G$ defines an isometry $g : X \rightarrow X$. An action on X is *non-trivial* if there is no point in X fixed by all elements of G . Note, that every group has a *trivial action* on any Λ -tree, when all group elements act as identity. An action of G on X is *minimal* if X does not contain a non-trivial G -invariant subtree X_0 .

Let a group G act as isometries on a Λ -tree X . $g \in G$ is called *elliptic* if it has a fixed point. $g \in G$ is called an *inversion* if it does not have a fixed point, but g^2 does. If g is not elliptic and not an inversion then it is called *hyperbolic*.

A group G acts *freely* and *without inversions* on a Λ -tree X if for all $1 \neq g \in G$, g acts as a hyperbolic isometry. In this case we also say that G is Λ -free.

4.3 Theory of a single isometry

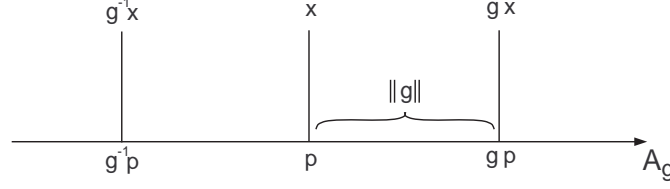
Let (X, d) be a Λ -tree, where Λ is an arbitrary ordered abelian group. Recall that an isometry g of X is called *elliptic* if it has a fixed point.

Lemma 9. [24, Lemma 3.1.1] Let g be an elliptic isometry of X and let X^g denote the set of fixed points of g . Then X^g is a closed non-empty $\langle g \rangle$ -invariant subtree of X . If $x \in X$ and $[x, p]$ is the bridge between x and X^g , then for any $a \in X^g$, $p = Y(x, a, ga)$ is the midpoint of $[x, gx]$.

Next, an isometry g of a Λ -tree (X, d) is called an *inversion* if g^2 has a fixed point, but g does not.

Lemma 10. [24, Lemma 3.1.2] Let g be an isometry of a Λ -tree (X, d) . The following are equivalent:

1. g is an inversion,
2. there is a segment of X invariant under g , and the restriction of g to this segment has no fixed points,
3. there is a segment $[x, y]$ in X such that $gx = y$, $gy = x$ and $d(x, y) \notin 2\Lambda$,
4. g^2 has a fixed point, and for all $x \in X$, $d(x, gx) \notin 2\Lambda$.


 Figure 6: The characteristic set of g

An isometry g of a Λ -tree (X, d) is called *hyperbolic* if it is not an inversion and not elliptic. It follows that an isometry g is hyperbolic if and only if g^2 has no fixed point.

Suppose g is an isometry of a Λ -tree (X, d) . The *characteristic set* of g is the subset $A_g \subseteq X$ defined by

$$A_g = \{p \in X \mid [g^{-1}p, p] \cap [p, gp] = \{p\}\}.$$

By Lemma 9, if g is elliptic then $A_g = X^g$, and by Lemma 10, if g is an inversion then $A_g = \emptyset$.

Theorem 6. [24, Theorem 3.1.4] *Let g be a hyperbolic isometry of a Λ -tree (X, d) . Then A_g is a non-empty closed $\langle g \rangle$ -invariant subtree of X . Further, A_g is a linear tree, and g restricted to A_g is equivalent to a translation $a \rightarrow a + \|g\|$ for some $\|g\| \in \Lambda$ with $\|g\| > 0$. If $x \in X$ and $[x, p]$ is the bridge between x and A_g , then $p = Y(g^{-1}x, x, gx)$, $[x, gx] \cap A_g = [p, gp]$, $[x, gx] = [x, p, gp, gx]$ and $d(x, gx) = \|g\| + 2d(x, p)$.*

If g is hyperbolic then A_g is called the *axis* and $\|g\|$ the *translation length* of g which can be defined as follows

$$\|g\| = \begin{cases} \min\{d(x, gx) \mid x \in X\} & \text{if } g \text{ is not an inversion} \\ 0 & \text{otherwise} \end{cases}$$

It can be shown that this minimum is always realized. If g is elliptic or an inversion, then $\|g\| = 0$.

Corollary 3. [24, Corollary 3.1.5] *Let g be an isometry of a Λ -tree (X, d) . Then g is an inversion if and only if $A_g = \emptyset$. If g is not an inversion then $\|g\| = \min\{d(x, gx) \mid x \in X\}$ and $A_g = \{p \in X \mid d(p, gp) = \|g\|\}$.*

Corollary 4. [24, Corollary 3.1.6] *Let g be an isometry of a Λ -tree (X, d) which is not an inversion. Then A_g meets every $\langle g \rangle$ -invariant subtree of X , and A_g is contained in every $\langle g \rangle$ -invariant subtree of X with the property that it meets every $\langle g \rangle$ -invariant subtree of X .*

Lemma 11. [24, Lemma 3.1.7] *If g, h are both isometries of a Λ -tree (X, d) , then*

1. $A_{hgh^{-1}} = hA_g$ and $\|hgh^{-1}\| = \|g\|$,
2. $A_{g^{-1}} = A_g$,
3. if $n \in \mathbb{Z}$ then $\|g^n\| = |n|\|g\|$, and $A_g \subseteq A_{g^n}$. If $\|g\| > 0$ and $n \neq 0$ then $A_g = A_{g^n}$,
4. if Y is a $\langle g \rangle$ -invariant subtree of X , then $\|g\| = \|g|_Y\|$ and $A_{g|_Y} = A_g \cap Y$.

4.4 Λ -free groups

Recall that a group G is called Λ -free if for all $1 \neq g \in G$, g acts as a hyperbolic isometry. Here we list some known results about Λ -free groups for an arbitrary ordered abelian group Λ . For all these results the reader can be referred to [1, 4, 24, 82]

Theorem 7. (a) The class of Λ -free groups is closed under taking subgroups.

(b) If G is Λ -free and Λ embeds (as an ordered abelian group) in Λ' then G is Λ' -free.

(c) Any Λ -free group is torsion-free.

(d) Λ -free groups have the CSA property. That is, every maximal abelian subgroup A is malnormal: $A^g \cap A = 1$ for all $g \notin A$.

(e) Commutativity is a transitive relation on the set of non-trivial elements of a Λ -free group.

(f) Solvable subgroups of Λ -free groups are abelian.

(g) If G is Λ -free then any abelian subgroup of G can be embedded in Λ .

(h) Λ -free groups cannot contain Baumslag-Solitar subgroups other than $\mathbb{Z} \times \mathbb{Z}$. That is, no group of the form $\langle a, t \mid t^{-1}a^p t = a^q \rangle$ can be a subgroup of a Λ -free group unless $p = q = \pm 1$.

(i) Any two generator subgroup of a Λ -free group is either free, or free abelian.

(j) The class of Λ -free groups is closed under taking free products.

The following result was originally proved in [51] in the case of finitely many factors and $\Lambda = \mathbb{R}$. A proof of the result in the general formulation given below can be found in [24, Proposition 5.1.1].

Theorem 8. If $\{G_i \mid i \in I\}$ is a collection of Λ -free groups then the free product $*_{i \in I} G_i$ is Λ -free.

The following result gives a lot of information about the group structure in the case when $\Lambda = \mathbb{Z} \times \Lambda_0$ with the left lexicographic order.

Theorem 9. [4, Theorem 4.9] *Let a group G act freely and without inversions on a Λ -tree, where $\Lambda = \mathbb{Z} \times \Lambda_0$. Then there is a graph of groups (Γ, Y^*) such that:*

- (1) $G = \pi_1(\Gamma, Y^*)$,
- (2) *for every vertex $x^* \in Y^*$, a vertex group Γ_{x^*} acts freely and without inversions on a Λ_0 -tree,*
- (3) *for every edge $e \in Y^*$ with an endpoint x^* an edge group Γ_e is either maximal abelian subgroup in Γ_{x^*} or is trivial and Γ_{x^*} is not abelian,*
- (4) *if $e_1, e_2, e_3 \in Y^*$ are edges with an endpoint x^* then $\Gamma_{e_1}, \Gamma_{e_2}, \Gamma_{e_3}$ are not all conjugate in Γ_{x^*} .*

Conversely, from the existence of a graph (Γ, Y^) satisfying conditions (1)–(4) it follows that G acts freely and without inversions on a $\mathbb{Z} \times \Lambda_0$ -tree in the following cases: Y^* is a tree, $\Lambda_0 \subset Q$ and either $\Lambda_0 = Q$ or Y^* is finite.*

4.5 \mathbb{R} -trees

The case when $\Lambda = \mathbb{R}$ in the theory of groups acting on Λ -trees appears to be the most well-studied (other than $\Lambda = \mathbb{Z}$, of course). \mathbb{R} -trees are usual metric spaces with nice properties which makes them very attractive from geometric point of view. Lots of results were obtained in the last two decades about group actions on these objects. The most celebrated one is Rips' Theorem about free actions and a more general result of M. Bestvina and M. Feighn about stable actions on \mathbb{R} -trees (see [40, 14]). In particular, the main result of Bestvina and Feighn together with the idea of obtaining a stable action on an \mathbb{R} -tree as a limit of actions on an infinite sequence of \mathbb{Z} -trees gives a very powerful tool in obtaining non-trivial splittings of groups into fundamental groups of graphs of groups which is known as *Rips machine*.

An \mathbb{R} -tree (X, d) is a Λ -metric space which satisfies the axioms (T1) – (T3) listed in Subsection 4.2 for $\Lambda = \mathbb{R}$ with usual order. Hence, all the definitions and notions given in Section 4 hold for \mathbb{R} -trees.

The definition of an \mathbb{R} -tree was first given by Tits in [125].

Proposition 2. [24, Proposition 2.2.3] *Let (X, d) be an \mathbb{R} -metric space. Then the following are equivalent:*

1. (X, d) is an \mathbb{R} -tree,
2. *given two point of X , there is a unique arc having them as endpoints, and it is a segment,*
3. (X, d) is geodesic and it contains no subspace homeomorphic to the circle.

Example 11. Let $Y = \mathbb{R}^2$ be the plane, but with metric p defined by

$$p((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1| + |y_2| + |x_1 - x_2| & \text{if } x_1 \neq x_2 \\ |y_1 - y_2| & \text{if } x_1 = x_2 \end{cases}$$

That is, to measure the distance between two points not on the same vertical line, we take their projections onto the horizontal axis, and add their distances to these projections and the distance between the projections (distance in the usual Euclidean sense).

Example 12. [24, Proposition 2.2.5] Given a simplicial tree Γ , one can construct its realization $\text{real}(\Gamma)$ by identifying each non-oriented edge of Γ with the unit interval. The metric on $\text{real}(\Gamma)$ is induced from Γ .

Example 13. Let G be a δ -hyperbolic group. Then its Cayley graph with respect to any finite generating set S is a δ -hyperbolic metric space (X, d) (where d is a word metric) on which G acts by isometries. Now, the asymptotic cone $\text{Cone}_\omega(X)$ of G is a real tree (see [45, 127, 36]) on which G acts by isometries.

An \mathbb{R} -tree is called *polyhedral* if the set of all branch points and endpoints is closed and discrete. Polyhedral \mathbb{R} -trees have strong connection with simplicial trees as shown below.

Theorem 10. [24, Theorem 2.2.10] An \mathbb{R} -tree (X, d) is polyhedral if and only if it is homeomorphic to $\text{real}(\Gamma)$ (with metric topology) for some simplicial tree Γ .

Now we briefly recall some known results related to group actions on \mathbb{R} -trees. The first result shows that an action on a Λ -tree always implies an action on an \mathbb{R} -tree.

Theorem 11. [24, Theorem 4.1.2] If a finitely generated group G has a non-trivial action on a Λ -tree for some ordered abelian group Λ then it has a non-trivial action on some \mathbb{R} -tree.

Observe that in general nice properties of the action on a Λ -tree are not preserved when passing to the corresponding action on an \mathbb{R} -tree above.

Next result was one of the first in the theory of group actions on \mathbb{R} -trees. Later it was generalized to the case of an arbitrary Λ in [126, 23].

Theorem 12. [51] Let G be a group acting freely and without inversions on an \mathbb{R} -tree X , and suppose $g, h \in G \setminus \{1\}$. Then $\langle g, h \rangle$ is either free of rank two or abelian.

It is not hard to define an action of a free abelian group on an \mathbb{R} -tree.

Example 14. Let $A = \langle a, b \rangle$ be a free abelian group. Define an action of A on \mathbb{R} (which is an \mathbb{R} -tree) by embedding A into $\text{Isom}(\mathbb{R})$ as follows

$$a \rightarrow t_1, \quad b \rightarrow t_{\sqrt{2}},$$

where $t_\alpha(x) = x + \alpha$ is a translation. It is easy to see that

$$a^n b^m \rightarrow t_{n+m\sqrt{2}},$$

and since 1 and $\sqrt{2}$ are rationally independent it follows that the action is free.

The following result was very important in the direction of classifying finitely generated \mathbb{R} -free groups.

Theorem 13. [87] *The fundamental group of a closed surface is \mathbb{R} -free, except for the non-orientable surfaces of genus 1, 2 and 3.*

Then, in 1991 E. Rips completely classified finitely generated \mathbb{R} -free groups. The ideas outlined by Rips were further developed by Gaboriau, Levitt and Paulin who gave a complete proof of this classification in [40].

Theorem 14 (Rips' Theorem). *Let G be a finitely generated group acting freely and without inversions on an \mathbb{R} -tree. Then G can be written as a free product $G = G_1 * \cdots * G_n$ for some integer $n \geq 1$, where each G_i is either a finitely generated free abelian group, or the fundamental group of a closed surface.*

4.6 Rips-Bestvina-Feighn machine

Suppose G is a finitely presented group acting isometrically on an \mathbb{R} -tree Γ . We assume the action to be non-trivial and minimal. Since G is finitely presented there is a finite simplicial complex K of dimension at most 2 such that $\pi(K) \simeq G$. Moreover, one can assume that K is a *band complex* with underlying *union of bands* which is a finite simplicial \mathbb{R} -tree X with finitely many *bands* of the type $[0, 1] \times \alpha$, where α is an arc of the real line, glued to X so that $\{0\} \times \alpha$ and $\{1\} \times \alpha$ are identified with sub-arcs of edges of X . Following [14] (the construction originally appears in [85]) one can construct a transversely measured lamination L on K and an equivariant map $\phi : \tilde{K} \rightarrow \Gamma$, where \tilde{K} is the universal cover of K , which sends leaves of the induced lamination on \tilde{K} to points in Γ . The complex K together with the lamination L is called a *band complex with \tilde{K} resolving the action of G on Γ* .

Now, *Rips-Bestvina-Feighn machine* is a procedure which given a band complex K , transforms it into another band complex K' (we still have $\pi(K') \simeq G$), whose lamination splits into a disjoint union of finitely many sub-laminations of several types - simplicial, surface, toral, thin - and these sub-laminations induce a splitting of K' into sub-complexes containing them. K' can be thought of as the “normal form” of the band complex K . Analyzing the structure of K' and its sub-complexes one can obtain some information about the structure of the group G .

In particular, in the case when the original action of G on Γ is *stable* one can obtain a splitting of G . Recall that a non-degenerate (that is, containing more than one point) subtree S of Γ is *stable* if for every non-degenerate subtree S' of S , we have $\text{Fix}(S') = \text{Fix}(S)$ (here, $\text{Fix}(I) \leq G$ consists of all elements which fix I point-wise). The action of G on Γ is *stable* if every non-degenerate subtree of Γ contains a stable subtree.

Theorem 15. [14, Theorem 9.5] *Let G be a finitely presented group with a nontrivial, stable, and minimal action on an \mathbb{R} -tree Γ . Then either*

- (1) G splits over an extension E -by-cyclic, where E fixes an arc of Γ , or
- (2) Γ is a line. In this case, G splits over an extension of the kernel of the action by a finitely generated free abelian group.

The key ingredient of the Rips-Bestvina-Feighn machine is a set of particular operations, called *moves*, on band complexes applied in a certain order. These operations originate from the work of Makanin [80] and Razborov [105] that ideas of Rips are built upon.

Observe that the group G in Theorem 15 must be finitely presented. To obtain a similar result about finitely generated groups acting on \mathbb{R} -trees one has to further restrict the action. An action of a group G on an \mathbb{R} -tree Γ satisfies the *ascending chain condition* if for every decreasing sequence

$$I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$$

of arcs in Γ which converge into a single point, the corresponding sequence

$$\text{Fix}(I_1) \subset \text{Fix}(I_2) \subset \cdots \subset \text{Fix}(I_n) \subset \cdots$$

stabilizes.

Theorem 16. [50] *Let G be a finitely generated group with a nontrivial minimal action on an \mathbb{R} -tree Γ . If*

- (1) Γ satisfies the ascending chain condition,
- (2) for any unstable arc J of Γ ,
 - (a) $\text{Fix}(J)$ is finitely generated,
 - (b) $\text{Fix}(J)$ is not a proper subgroup of any conjugate of itself, that is, if $\text{Fix}(J)^g \subset \text{Fix}(J)$ for some $g \in G$ then $\text{Fix}(J)^g = \text{Fix}(J)$.

Then either

- (1) G splits over a subgroup H which is an extension of the stabilizer of an arc of Γ by a cyclic group, or
- (2) Γ is a line.

Now, we would like to discuss some applications of the above results which are based on the construction outlined in [12] and [101] making possible to obtain isometric group actions on \mathbb{R} -trees as Gromov-Hausdorff limits of actions on hyperbolic spaces. All the details can be found in [13].

Let (X, d_X) be a metric space equipped with an isometric action of a group G which can be viewed as a homomorphism $\rho : G \rightarrow \text{Isom}(X)$. Assume that X contains a point ε which is not fixed by G . In this case, we call the triple (X, ε, ρ) a *based G -space*.

Observe that given a based G -space (X, ε, ρ) one can define a pseudometric $d = d_{(X, \varepsilon, \rho)}$ on G as follows

$$d(g, h) = d_X(\rho(g) \cdot \varepsilon, \rho(h) \cdot \varepsilon).$$

Now, the set \mathcal{D} of all non-trivial pseudometrics on G taken up to rescaling by positive reals forms a topological space and we say that a sequence $(X_i, \varepsilon_i, \rho_i)$, $i \in \mathbb{N}$ of based G -spaces *converges* to the based G -space (X, ε, ρ) and write

$$\lim_{i \rightarrow \infty} (X_i, \varepsilon_i, \rho_i) = (X, \varepsilon, \rho)$$

if $d_{(X_i, \varepsilon_i, \rho_i)} \rightarrow d_{(X, \varepsilon, \rho)}$ in \mathcal{D} . Now, the following result is the main tool in obtaining isometric group actions on \mathbb{R} -trees from actions on Gromov-hyperbolic spaces.

Theorem 17. [13, Theorem 3.3] *Let $(X_i, \varepsilon_i, \rho_i)$, $i \in \mathbb{N}$ be a convergent sequence of based G -spaces. Assume that*

- (1) *there exists $\delta > 0$ such that every X_i is δ -hyperbolic,*
- (2) *there exists $g \in G$ such that the sequence $d_{X_i}(\varepsilon_i, \rho_i(g) \cdot \varepsilon_i)$ is unbounded.*

Then there is a based G -space (Γ, ε) which is an \mathbb{R} -tree and an isometric action $\rho : G \rightarrow \text{Isom}(\Gamma)$ such that $(X_i, \varepsilon_i, \rho_i) \rightarrow (\Gamma, \varepsilon, \rho)$.

In fact, the above theorem does not guarantee that the limiting action of G on Γ has no global fixed points. But in the case when G is finitely generated and each X_i is proper (closed metric balls are compact), it is possible to choose base-points in $\varepsilon_i \in X_i$ to make the action of G on Γ non-trivial (see [13, Proposition 3.8, Theorem 3.9]). Moreover, one can retrieve some information about stabilizers of arcs in Γ (see [13, Proposition 3.10]).

Note that Theorem 17 can also be interpreted in terms of asymptotic cones (see [36, 37] for details).

The power of Theorem 17 becomes obvious in particular when a finitely generated group G has infinitely many pairwise non-conjugate homomorphisms $\phi_i : G \rightarrow H$ into a word-hyperbolic group H . In this case, each ϕ_i defines an action of G on the Cayley graph X of H with respect to some finite generating set. Now, one can define X_i to be X with a word metric rescaled so that the sequence of $(X_i, \varepsilon_i, \rho_i)$, $i \in \mathbb{N}$ satisfies the requirements of Theorem 17 and thus obtain a non-trivial isometric action of G on an \mathbb{R} -tree. Many results about word-hyperbolic groups were obtained according to this scheme, for example, the following classical result.

Theorem 18. [102] *Let G be a word-hyperbolic group such that the group of its outer automorphisms $\text{Out}(G)$ is infinite. Then G splits over a virtually cyclic group.*

Combined with the *shortening argument* due to Rips and Sela [112] this scheme gives many other results about word-hyperbolic groups, for example, the theorems below.

Theorem 19. [112] *Let G be a torsion-free freely indecomposable word-hyperbolic group. Then the internal automorphism group $\text{Inn}(G)$ of G has finite index in $\text{Aut}(G)$.*

Theorem 20. [44, 116] *Let G be a finitely presented torsion-free freely indecomposable group and let H be a word-hyperbolic group. Then there are only finitely many conjugacy classes of subgroups of G isomorphic to H .*

For more detailed account of applications of the Rips-Bestvina-Feighn machine please refer to [13].

5 Lyndon length functions

In 1963, R. Lyndon (see [77]) introduced a notion of *length function on a group* in an attempt to axiomatize cancellation arguments in free groups as well as free products with amalgamation and HNN extensions, and to generalize them to a wider class of groups. The main idea was to measure the amount of cancellation in passing to the reduced form of a product of reduced words in a free group and free constructions, and it turned out that the cancellation process could be described by rather simple axioms. Using simple combinatorial techniques Lyndon described groups with *free* \mathbb{Z} -valued length functions and conjectured (see [78]) that any finitely generated group with a free \mathbb{R} -valued length function can be embedded into a free product of finitely many copies of \mathbb{R} . The conjectures eventually was proved wrong (counterexamples were initially given in [2] and [103]) but the idea of using length functions became quite popular (see, for example, [51, 21, 52]), and then it turned out that the language of length functions described the same class of groups as the language of actions on trees (see Section 7 for more details).

Below we give the axioms of (Lyndon) length function and recall the main results in this field.

Let G be a group and Λ be an ordered abelian group. Then a function $l : G \rightarrow \Lambda$ is called a (*Lyndon*) *length function* on G if the following conditions hold:

- (L1) $\forall g \in G : l(g) \geq 0$ and $l(1) = 0$,
- (L2) $\forall g \in G : l(g) = l(g^{-1})$,
- (L3) $\forall f, g, h \in G : c(f, g) > c(f, h) \rightarrow c(f, h) = c(g, h)$,
 where $c(f, g) = \frac{1}{2}(l(f) + l(g) - l(f^{-1}g))$.

Observe that in general $c(f, g) \notin \Lambda$, but $c(f, g) \in \Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$, where \mathbb{Q} is the additive group of rational numbers, so, in the axiom (L3) we view Λ as a subgroup of $\Lambda_{\mathbb{Q}}$. But in some cases the requirement $c(f, g) \in \Lambda$ is crucial so we state it as a separate axiom

- (L4) $\forall f, g \in G : c(f, g) \in \Lambda$.

It is not difficult to derive the following two properties of Lyndon length functions from the axioms (L1) – (L3):

- $\forall f, g \in G : l(fg) \leq l(f) + l(g),$
- $\forall f, g \in G : 0 \leq c(f, g) \leq \min\{l(f), l(g)\}.$

The following examples motivated the whole theory of groups with length functions.

Example 15. *Given a free group $F(X)$ on the set X one can define a (Lyndon) length function on F as follows*

$$w(X) \rightarrow |w(X)|,$$

where $|\cdot|$ is the length of the reduced word in $X \cup X^{\pm 1}$ representing w .

Example 16. *Given two groups G_1 and G_2 with length functions $L_1 : G_1 \rightarrow \Lambda$ and $L_2 : G_2 \rightarrow \Lambda$ for some ordered abelian group Λ one can construct a length function on $G_1 * G_2$ as follows (see [24, Proposition 5.1.1]). For any $g \in G_1 * G_2$ such that*

$$g = f_1 g_1 \cdots f_k g_k f_{k+1},$$

where $f_i \in G_1$, $i \in [1, k+1]$, $f_i \neq 1$, $i \in [2, k]$ and $1 \neq g_i \in G_2$, $i \in [1, k]$, define

$$L(g) = \sum_{i=1}^{k+1} L_1(f_i) + \sum_{j=1}^k L_2(g_j) \in \Lambda.$$

A length function $l : G \rightarrow \Lambda$ is called *free*, if it satisfies

$$(L5) \quad \forall g \in G : g \neq 1 \rightarrow l(g^2) > l(g).$$

Obviously, the \mathbb{Z} -valued length function constructed in Example 15 is free. The converse is shown below (see also [52] for another proof of this result).

Theorem 21. [77] *Any group G with a length function $L : G \rightarrow \mathbb{Z}$ can be embedded into a free group F of finite rank whose natural length function extends L .*

Example 17. *Given two groups G_1 and G_2 with free length functions $L_1 : G_1 \rightarrow \Lambda$ and $L_2 : G_2 \rightarrow \Lambda$ for some ordered abelian group Λ , the length function on $G_1 * G_2$ constructed in Example 16 is free.*

Observe that if a group G acts on a Λ -tree (X, d) then we can fix a point $x \in X$ and consider a function $l_x : G \rightarrow \Lambda$ defined as $l_x(g) = d(x, gx)$. Such a function l_x on G we call a *length function based at x* . It is easy to check that l_x satisfies all the axioms (L1) – (L4) of Lyndon length function. Now if $\|\cdot\|$ is the translation length function associated with the action of G on (X, d) then the following axioms show the connection between l_x and $\|\cdot\|$.

- (i) $l_x(g) = \|g\| + 2d(x, A_g)$ if g is not an inversion.

$$(ii) \|g\| = \max\{0, l_x(g^2) - l_x(g)\}.$$

Here, it should be noted that for points $x \notin A_g$, there is a unique closest point of A_g to x . The distance between these points is the one referred to in (i). While $A_g = A_{g^n}$ for all $n \neq 0$ in the case where g is hyperbolic, if g fixes a point, it is possible that $A_g \subset A_{g^2}$. We may have $l_x(g^2) - l_x(g) < 0$ in this case. Free actions are characterized, in the language of length functions, by the facts (a) $\|g\| > 0$ for all $g \neq 1$, and (b) $l_x(g^2) > l_x(g)$ for all $g \neq 1$. The latter follows from the fact that $\|g\| = n\|g\|$ for all g . We note that there are axioms for the translation length function which were shown to essentially characterize actions on Λ -trees, up to equivariant isometry, by W. Parry, [100].

6 Infinite words

The notion of Lyndon length function provides a very nice tool to study properties of groups but unfortunately its applications are quite limited due to very abstract axiomatic approach. Even in the case of free length functions methods are close in a sense to free group cancelation techniques but necessity to use axioms makes everything cumbersome. Introduction of *infinite words* was first of all motivated by the idea that working with elements of group with Lyndon length function should be exactly as in free group. It is not surprising that the first group which the method of infinite words was applied to was Lyndon's free group $F^{\mathbb{Z}[t]}$ (see [92]) which shares many properties with free group. Later the infinite words techniques were extensively applied in [93, 65, 58, 97, 69, 70, 68, 67, 98, 81].

Below we follow the construction given in [92].

6.1 Definition and preliminaries

Let Λ be a discretely ordered abelian group with the minimal positive element 1. It is going to be clear from the context if we are using 1 as an element of Λ , or as an integer. Let $X = \{x_i \mid i \in I\}$ be a set. Put $X^{-1} = \{x_i^{-1} \mid i \in I\}$ and $X^\pm = X \cup X^{-1}$. A Λ -word is a function of the type

$$w : [1, \alpha_w] \rightarrow X^\pm,$$

where $\alpha_w \in \Lambda$, $\alpha_w \geq 0$. The element α_w is called the *length* $|w|$ of w .

By $W(\Lambda, X)$ we denote the set of all Λ -words. Observe, that $W(\Lambda, X)$ contains an empty Λ -word which we denote by ε .

Concatenation uv of two Λ -words $u, v \in W(\Lambda, X)$ is an Λ -word of length $|u| + |v|$ and such that:

$$(uv)(a) = \begin{cases} u(a) & \text{if } 1 \leq a \leq |u| \\ v(a - |u|) & \text{if } |u| < a \leq |u| + |v| \end{cases}$$

Next, for any Λ -word w we define an *inverse* w^{-1} as an Λ -word of the length $|w|$ and such that

$$w^{-1}(\beta) = w(|w| + 1 - \beta)^{-1} \quad (\beta \in [1, |w|]).$$

A Λ -word w is *reduced* if $w(\beta + 1) \neq w(\beta)^{-1}$ for each $1 \leq \beta < |w|$. We denote by $R(\Lambda, X)$ the set of all reduced Λ -words. Clearly, $\varepsilon \in R(\Lambda, X)$. If the concatenation uv of two reduced Λ -words u and v is also reduced then we write $uv = u \circ v$.

For $u \in W(\Lambda, X)$ and $\beta \in [1, \alpha_u]$ by u_β we denote the restriction of u on $[1, \beta]$. If $u \in R(\Lambda, X)$ and $\beta \in [1, \alpha_u]$ then

$$u = u_\beta \circ \tilde{u}_\beta,$$

for some uniquely defined \tilde{u}_β .

An element $\text{com}(u, v) \in R(\Lambda, X)$ is called the (*longest*) *common initial segment* of Λ -words u and v if

$$u = \text{com}(u, v) \circ \tilde{u}, \quad v = \text{com}(u, v) \circ \tilde{v}$$

for some (uniquely defined) Λ -words \tilde{u}, \tilde{v} such that $\tilde{u}(1) \neq \tilde{v}(1)$.

Now, we can define the product of two Λ -words. Let $u, v \in R(\Lambda, X)$. If $\text{com}(u^{-1}, v)$ is defined then

$$u^{-1} = \text{com}(u^{-1}, v) \circ \tilde{u}, \quad v = \text{com}(u^{-1}, v) \circ \tilde{v},$$

for some uniquely defined \tilde{u} and \tilde{v} . In this event put

$$u * v = \tilde{u}^{-1} \circ \tilde{v}.$$

The product $*$ is a partial binary operation on $R(\Lambda, X)$.

Example 18. Let $\Lambda = \mathbb{Z}^2$ with the right lexicographic order (in this case $1 = (1, 0)$). Put

$$w(\beta) = \begin{cases} x & \text{if } \beta = (s, 0) \text{ and } s \geq 1 \\ x^{-1} & \text{if } \beta = (s, 1) \text{ and } s \leq 0 \end{cases}$$

Then

$$w : [1, (0, 1)] \rightarrow X^\pm$$

is a reduced Λ -word. Clearly, $w^{-1} = w$ so $w * w = \varepsilon$. In particular, $R(\Lambda, X)$ has 2-torsion with respect to $*$.

Theorem 22. [92, Theorem 3.4] Let Λ be a discretely ordered abelian group and X be a set. Then the set of reduced Λ -words $R(\Lambda, X)$ with the partial binary operation $*$ satisfies the axioms (P1) – (P4) of a pregroup.

An element $v \in R(\Lambda, X)$ is termed *cyclically reduced* if $v(1)^{-1} \neq v(|v|)$. We say that an element $v \in R(\Lambda, X)$ admits a *cyclic decomposition* if $v = c^{-1} \circ u \circ c$, where $c, u \in R(\Lambda, X)$ and u is cyclically reduced. Observe that a

cyclic decomposition is unique (whenever it exists). We denote by $CR(\Lambda, X)$ the set of all cyclically reduced words in $R(\Lambda, X)$ and by $CDR(\Lambda, X)$ the set of all words from $R(\Lambda, X)$ which admit a cyclic decomposition.

Below we refer to Λ -words as *infinite words* usually omitting Λ whenever it does not produce any ambiguity.

A subset $G \leq R(\Lambda, X)$ is called a *subgroup* of $R(\Lambda, X)$ if G is a group with respect to $*$. We say that a subset $Y \subset R(\Lambda, X)$ *generates a subgroup* $\langle Y \rangle$ in $R(\Lambda, X)$ if the product $y_1 * \dots * y_n$ is defined for any finite sequence of elements $y_1, \dots, y_n \in Y^{\pm 1}$.

Example 19. *Let Λ be a direct sum of copies of \mathbb{Z} with the right lexicographic order. Then the set of all elements of finite length in $R(\Lambda, X)$ forms a subgroup which is isomorphic to a free group with basis X .*

6.2 Commutation in infinite words

Let G be a subgroup of $CDR(\Lambda, X)$, where Λ is a discretely ordered abelian group. We fix G for the rest of the subsection.

Let I_Λ index the set of all convex subgroups of Λ . I_Λ is linearly ordered (see, for example, [24]): $i < j$ if and only if $\Lambda_i < \Lambda_j$, and

$$\Lambda = \bigcup_{i \in I_\Lambda} \Lambda_i.$$

We say that $g \in G$ has the height $i \in I_\Lambda$ and denote $ht(g) = i$ if $|g| \in \Lambda_i$ and $|g| \notin \Lambda_j$ for any $j < i$. Observe that this definition depends only on G since the complete chain of convex subgroups of Λ is unique.

It is easy to see that

$$ht(g_1 g_2) \leq \max\{ht(g_1), ht(g_2)\},$$

hence, if $G = \langle g_1, \dots, g_k \rangle$ then we define

$$ht(G) = \max\{ht(g_1), \dots, ht(g_k)\}.$$

Using the characteristics of elements of G introduced above we prove several technical results we are used, for example, in Sections 8 and 10.

The first result is an analog of Harrison's Theorem (see [51]) in the case of cyclically reduced elements.

Lemma 12. [67, Lemma 5] *Let $f, h \in G$ be cyclically reduced. If $c(f^m, h^n) \geq |f| + |h|$ for some $m, n > 0$ then $[f, h] = \varepsilon$.*

The next lemma shows that commutation implies similar cyclic decompositions.

Lemma 13. [67, Lemma 6] *For any two $g_1, g_2 \in G$ if $[g_1, g_2] = \varepsilon$ and $g_1 = c^{-1} \circ h_1 \circ c$, $g_2 = d^{-1} \circ h_2 \circ d$ are their cyclic decompositions then $c = d$.*

In particular, it follows that if $g \in G$ is cyclically reduced then all elements of $C_G(g)$ are cyclically reduced as well.

Lemma 14. [67, Lemma 7] *Let $f, h \in G$ be such that h is cyclically reduced and $ht(f) > ht(h)$. If $ht(f^{-1} * h * f) < ht(f)$ then for every $n \in \mathbb{N}$ either $f = h^n \circ f_n$, or $f = h^{-n} \circ f_n$ for some $f_n \in G$.*

Here is another special case of Harrison's Theorem.

Lemma 15. [67, Lemma 8] *Let $f, h_1, h_2 \in G$ be such that $ht(h_1), ht(h_2) < ht(f)$ and $ht(f^{-1} * h_1 * f), ht(f^{-1} * h_2 * f) < ht(f)$. Then $[h_1, h_2] = \varepsilon$.*

Lemma 16. [67, Lemma 9] *Let $f, h_1 \neq \varepsilon \in G$ be such that f is cyclically reduced and $ht(h_1) < ht(f)$. If $ht(f^{-1} * h_1 * f) < ht(f)$ and $[h_1, h_2] = \varepsilon$, where $ht(h_2) < ht(f)$, then $ht(f^{-1} * h_2 * f) < ht(f)$.*

Using the lemmas above one can easily prove the following well-known result.

Proposition 3. [1] *If $G < CDR(\Lambda, X)$ then for any $g \in G$, its centralizer $C_G(g)$ is a subgroup of A . In particular, if $\Lambda = \mathbb{Z}^n$ then $C_G(g)$ is a free abelian group of rank not more than n .*

7 Equivalence

Here we show that all three approaches, namely, free actions on Λ -trees (see Section 4), free Lyndon Λ -valued length functions (see Section 5), and Λ -words (see Section 6) describe the same class of groups which is called Λ -free groups.

7.1 From actions to length functions

The following theorem is one of the most important results in the theory of length functions.

Theorem 23. [21] *Let G be a group and $l : G \rightarrow \Lambda$ a Lyndon length function satisfying the following condition:*

$$(L4) \quad \forall f, g \in G : c(f, g) \in \Lambda.$$

Then there are a Λ -tree (X, d) , an action of G on X , and a point $x \in X$ such that $l = l_x$.

The proof is constructive, that is, one can define a Λ -metric space out of G and l , and then prove that this space is in fact a Λ -tree on which G acts by isometries (see [24, Subsection 2.4] for details).

7.2 From length functions to infinite words

The following results show the connection between groups with Lyndon length functions and groups of infinite words.

Theorem 24. *[92, Theorem 4.1] Let Λ be a discretely ordered abelian group and X be a set. Then any subgroup G of $CDR(\Lambda, X)$ has a free Lyndon length function with values in Λ – the restriction $|\cdot|_G$ on G of the standard length function $|\cdot|$ on $CDR(\Lambda, X)$.*

The converse of Theorem 24 was obtained by Chiswell in [25].

Theorem 25. *[25, Theorem 3.9] Let G have a free Lyndon length function $L : G \rightarrow \Lambda$, where Λ is a discretely ordered abelian group. Then there exists a length preserving embedding $\phi : G \rightarrow CDR(\Lambda, X)$, that is, $|\phi(g)| = L(g)$ for any $g \in G$.*

Corollary 5. *[25, Corollary 3.10] Let G have a free Lyndon length function $L : G \rightarrow \Lambda$, where Λ is an arbitrary ordered abelian group. Then there exists an embedding $\phi : G \rightarrow CDR(\Lambda', X)$, where $\Lambda' = \mathbb{Z} \oplus \Lambda$ is discretely ordered with respect to the right lexicographic order and X is some set, such that, $|\phi(g)| = (0, L(g))$ for any $g \in G$.*

7.3 From infinite words to actions

Below we follow the construction given in [69].

7.3.1 Universal trees

Let G be a subgroup of $CDR(\Lambda, X)$ for some discretely ordered abelian group Λ and a set X . We assume G , Λ , and X to be fixed for the rest of this section.

Every element $g \in G$ is a function

$$g : [1, |g|] \rightarrow X^\pm,$$

with the domain $[1, |g|]$ which is a closed segment in Λ . Since Λ can be viewed as a Λ -metric space then $[1, |g|]$ is a geodesic connecting 1 and $|g|$, and every $\alpha \in [1, |g|]$ we view as a pair (α, g) . We would like to identify initial subsegments of the geodesics corresponding to all elements of G as follows.

Let

$$S_G = \{(\alpha, g) \mid g \in G, \alpha \in [0, |g|]\}.$$

Since for every $f, g \in G$ the word $com(f, g)$ is defined, we can introduce an equivalence relation on S_G as follows: $(\alpha, f) \sim (\beta, g)$ if and only if $\alpha = \beta \in [0, c(f, g)]$. Obviously, it is symmetric and reflexive. For transitivity observe that if $(\alpha, f) \sim (\beta, g)$ and $(\beta, g) \sim (\gamma, h)$ then $0 \leq \alpha = \beta = \gamma \leq c(f, g), c(g, h)$. Since $c(f, h) \geq \min\{c(f, g), c(g, h)\}$ then $\alpha = \gamma \leq c(f, h)$.

Let $\Gamma_G = S_G / \sim$ and $\epsilon = \langle 0, 1 \rangle$, where $\langle \alpha, f \rangle$ is the equivalence class of (α, f) .

Proposition 4. [69] Γ_G is a Λ -tree,

Proof. At first we show that Γ_G is a Λ -metric space. Define the metric by

$$d(\langle \alpha, f \rangle, \langle \beta, g \rangle) = \alpha + \beta - 2 \min\{\alpha, \beta, c(f, g)\}.$$

Let us check if it is well-defined. Indeed, $c(f, g) \in \Lambda$ is defined for every $f, g \in G$. Moreover, let $(\alpha, f) \sim (\gamma, u)$ and $(\beta, g) \sim (\delta, v)$, we want to prove

$$d(\langle \alpha, f \rangle, \langle \beta, g \rangle) = d(\langle \gamma, u \rangle, \langle \delta, v \rangle)$$

which is equivalent to

$$\min\{\alpha, \beta, c(f, g)\} = \min\{\alpha, \beta, c(u, v)\}$$

since $\alpha = \gamma$, $\beta = \delta$. Consider the following cases.

(a) $\min\{\alpha, \beta\} \leq c(u, v)$

Hence, $\min\{\alpha, \beta, c(u, v)\} = \min\{\alpha, \beta\}$ and it is enough to prove $\min\{\alpha, \beta\} = \min\{\alpha, \beta, c(f, g)\}$. From length function axioms for G we have

$$c(f, g) \geq \min\{c(u, f), c(u, g)\}, \quad c(u, g) \geq \min\{c(u, v), c(v, g)\}.$$

Hence,

$$\begin{aligned} c(f, g) &\geq \min\{c(u, f), c(u, g)\} \geq \min\{c(u, f), \min\{c(u, v), c(v, g)\}\} \\ &= \min\{c(u, f), c(u, v), c(v, g)\}. \end{aligned}$$

Now, from $(\alpha, f) \sim (\gamma, u)$, $(\beta, g) \sim (\delta, v)$ it follows that $\alpha \leq c(u, f)$, $\beta \leq c(v, g)$ and combining it with the assumption $\min\{\alpha, \beta\} \leq c(u, v)$ we have

$$c(f, g) \geq \min\{c(u, f), c(u, v), c(v, g)\} \geq \min\{\alpha, \beta\},$$

or, in other words,

$$\min\{\alpha, \beta, c(f, g)\} = \min\{\alpha, \beta\}.$$

(b) $\min\{\alpha, \beta\} > c(u, v)$

Hence, $\min\{\alpha, \beta, c(u, v)\} = c(u, v)$ and it is enough to prove $c(f, g) = c(u, v)$.

Since

$$c(u, f) \geq \alpha > c(u, v), \quad c(v, g) \geq \beta > c(u, v),$$

then $\min\{c(u, f), c(u, v), c(v, g)\} = c(u, v)$ and

$$c(f, g) \geq \min\{c(u, f), c(u, v), c(v, g)\} = c(u, v).$$

Now we prove that $c(f, g) \leq c(u, v)$. From length function axioms for G we have

$$c(u, v) \geq \min\{c(v, g), c(u, g)\} = c(u, g) \geq \min\{c(v, g), c(u, v)\} = c(u, v),$$

that is, $c(u, v) = c(u, g)$. Now,

$$c(u, v) = c(u, g) \geq \min\{c(u, f), c(f, g)\},$$

where $\min\{c(u, f), c(f, g)\} = c(f, g)$ since otherwise we have $c(u, v) \geq c(u, f) \geq \alpha$ - a contradiction. Hence, $c(u, v) \geq c(f, g)$ and we have $c(f, g) = c(u, v)$.

By definition of d , for any $\langle \alpha, f \rangle, \langle \beta, g \rangle$ we have

$$d(\langle \alpha, f \rangle, \langle \beta, g \rangle) = d(\langle \beta, g \rangle, \langle \alpha, f \rangle) \geq 0,$$

$$d(\langle \alpha, f \rangle, \langle \alpha, f \rangle) = 0.$$

If

$$d(\langle \alpha, f \rangle, \langle \beta, g \rangle) = \alpha + \beta - 2 \min\{\alpha, \beta, c(f, g)\} = 0$$

then $\alpha + \beta = 2 \min\{\alpha, \beta, c(f, g)\}$. It is possible only if $\alpha = \beta \leq c(f, g)$ which implies $\langle \alpha, f \rangle = \langle \beta, g \rangle$. Finally, we have to prove the triangle inequality

$$d(\langle \alpha, f \rangle, \langle \beta, g \rangle) \leq d(\langle \alpha, f \rangle, \langle \gamma, h \rangle) + d(\langle \beta, g \rangle, \langle \gamma, h \rangle)$$

for every $\langle \alpha, f \rangle, \langle \beta, g \rangle, \langle \gamma, h \rangle \in \Gamma_G$. The inequality above is equivalent to

$$\alpha + \beta - 2 \min\{\alpha, \beta, c(f, g)\} \leq \alpha + \gamma$$

$$-2 \min\{\alpha, \gamma, c(f, h)\} + \beta + \gamma - 2 \min\{\beta, \gamma, c(g, h)\}$$

which comes down to

$$\min\{\alpha, \gamma, c(f, h)\} + \min\{\beta, \gamma, c(g, h)\} \leq \min\{\alpha, \beta, c(f, g)\} + \gamma.$$

First of all, observe that for any $\alpha, \beta, \gamma \in \Lambda$ the triple $(\min\{\alpha, \beta\}, \min\{\alpha, \gamma\}, \min\{\beta, \gamma\})$ is isosceles. Hence, by Lemma 1.2.7(1) [24], the triple

$$(\min\{\alpha, \beta, c(f, g)\}, \min\{\alpha, \gamma, c(f, h)\}, \min\{\beta, \gamma, c(g, h)\})$$

is isosceles too. In particular,

$$\begin{aligned} \min\{\alpha, \beta, c(f, g)\} &\geq \min\{\min\{\alpha, \gamma, c(f, h)\}, \min\{\beta, \gamma, c(g, h)\}\} \\ &= \min\{\alpha, \beta, \gamma, c(f, h), c(g, h)\}. \end{aligned}$$

Now, if

$$\min\{\alpha, \beta, \gamma, c(f, h), c(g, h)\} = \min\{\alpha, \gamma, c(f, h)\}$$

then $\min\{\beta, \gamma, c(g, h)\} = \gamma$ and

$$\min\{\alpha, \gamma, c(f, h)\} + \min\{\beta, \gamma, c(g, h)\} \leq \min\{\alpha, \beta, c(f, g)\} + \gamma$$

holds. If

$$\min\{\alpha, \beta, \gamma, c(f, h), c(g, h)\} = \min\{\beta, \gamma, c(g, h)\}$$

then $\min\{\alpha, \gamma, c(f, h)\} = \gamma$ and

$$\min\{\alpha, \gamma, c(f, h)\} + \min\{\beta, \gamma, c(g, h)\} \leq \min\{\alpha, \beta, c(f, g)\} + \gamma$$

holds again. So, d is a Λ -metric.

Finally, we want to prove that Γ_G is 0-hyperbolic with respect to $\epsilon = \langle 0, 1 \rangle$ (and, hence, with respect to any other point in Γ_G). It is enough to prove that the triple

$$((\langle \alpha, f \rangle \cdot \langle \beta, g \rangle)_\epsilon, (\langle \alpha, f \rangle \cdot \langle \gamma, h \rangle)_\epsilon, (\langle \beta, g \rangle \cdot \langle \gamma, h \rangle)_\epsilon)$$

is isosceles for every $\langle \alpha, f \rangle, \langle \beta, g \rangle, \langle \gamma, h \rangle \in \Gamma_G$. But by definition of d the above triple is isosceles if and only if

$$(\min\{\alpha, \beta, c(f, g)\}, \min\{\alpha, \gamma, c(f, h)\}, \min\{\beta, \gamma, c(g, h)\})$$

is isosceles which holds.

So, Γ_G is a Λ -tree. □

Since G is a subset of $CDR(\Lambda, X)$ and every element $g \in G$ is a function defined on $[1_A, |g|]$ with values in X^\pm then we can define a function

$$\xi : (\Gamma_G - \{\epsilon\}) \rightarrow X^\pm, \quad \xi(\langle \alpha, g \rangle) = g(\alpha).$$

It is easy to see that ξ is well-defined. Indeed, if $(\alpha, g) \sim (\alpha_1, g_1)$ then $\alpha = \alpha_1 \leq c(g, g_1)$, so $g(\alpha) = g_1(\alpha_1)$. Moreover, since every $g \in G$ is reduced then $\xi(p) \neq \xi(q)^{-1}$ whenever $d(p, q) = 1$.

ξ can be extended to a function

$$\Xi : \text{geod}(\Gamma_G)_\epsilon \rightarrow R(\Lambda, X),$$

where $\text{geod}(\Gamma_G)_\epsilon = \{(\epsilon, p) \mid p \in \Gamma_G\}$, so that

$$\Xi(\epsilon, \langle \alpha, g \rangle)(t) = g(t), \quad t \in [1_A, \alpha].$$

That is, $\Xi(\epsilon, \langle \alpha, g \rangle)$ is the initial subword of g of length α , and

$$\Xi(\epsilon, \langle |g|, g \rangle) = g.$$

On the other hand, if $g \in G$ and $\alpha \in [1_A, |g|]$ then the initial subword of g of length α uniquely corresponds to $\Xi(\epsilon, \langle \alpha, g \rangle)$. If $(\alpha, g) \sim (\alpha_1, g_1)$ then $\alpha = \alpha_1 \leq c(g, g_1)$, and since $g(t) = g_1(t)$ for any $t \in [1_A, c(g, g_1)]$ then

$$\Xi(\epsilon, \langle \alpha, g \rangle) = \Xi(\epsilon, \langle \alpha_1, g_1 \rangle).$$

Lemma 17. [69] *Let $u, v \in R(\Lambda, X)$. If $u * v$ is defined then $u * a$ is also defined, where $v = a \circ b$. Moreover, $u * a$ is an initial subword of either u or $u * v$.*

Proof. The proof follows from Figure 7. □

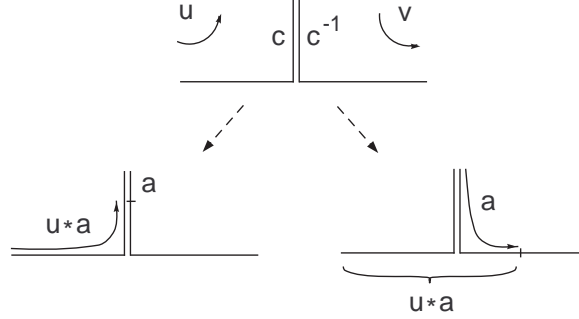


Figure 7: Possible cancelation diagrams in Lemma 17.

Now, since for every $\langle \alpha, g \rangle \in \Gamma_G$, $\Xi(\langle \epsilon, \langle \alpha, g \rangle \rangle)$ is an initial subword of $g \in G$ then by Lemma 17, $f * \Xi(\langle \epsilon, \langle \alpha, g \rangle \rangle)$ is defined for any $f \in G$. Moreover, again by Lemma 17, $f * \Xi(\langle \epsilon, \langle \alpha, g \rangle \rangle)$ is an initial subword of either f or $f * g$. More precisely,

$$f * \Xi(\langle \epsilon, \langle \alpha, g \rangle \rangle) = \Xi(\langle \epsilon, \langle |f| - \alpha, f \rangle \rangle)$$

if $f * \Xi(\langle \epsilon, \langle \alpha, g \rangle \rangle)$ is an initial subword of f , and

$$f * \Xi(\langle \epsilon, \langle \alpha, g \rangle \rangle) = \Xi(\langle \epsilon, \langle |f| + \alpha - 2c(f^{-1}, g), f * g \rangle \rangle)$$

if $f * \Xi(\langle \epsilon, \langle \alpha, g \rangle \rangle)$ is an initial subword of $f * g$.

Hence, we define a (left) action of G on Γ_G as follows:

$$f \cdot \langle \alpha, g \rangle = \langle |f| + \alpha - 2 \min\{\alpha, c(f^{-1}, g)\}, f \rangle$$

if $\alpha \leq c(f^{-1}, g)$, and

$$f \cdot \langle \alpha, g \rangle = \langle |f| + \alpha - 2 \min\{\alpha, c(f^{-1}, g)\}, f * g \rangle$$

if $\alpha > c(f^{-1}, g)$.

The action is well-defined. Indeed, it is easy to see that $f \cdot \langle \alpha, g \rangle = f \cdot \langle \alpha_1, g_1 \rangle$ whenever $(\alpha, g) \sim (\alpha_1, g_1)$.

Lemma 18. [69] *The action of G on Γ_G defined above is isometric.*

Proof. Observe that it is enough to prove

$$d(\epsilon, \langle \alpha, g \rangle) = d(f \cdot \epsilon, f \cdot \langle \alpha, g \rangle)$$

for every $f, g \in G$. Indeed, from the statement above it is going to follow that the geodesic tripod $(\epsilon, \langle |g|, g \rangle, \langle |h|, h \rangle)$ is isometrically mapped to the geodesic tripod $(\langle |f|, f \rangle, f \cdot \langle |g|, g \rangle, f \cdot \langle |h|, h \rangle)$ and isometricity follows.

We have

$$\begin{aligned} d(\epsilon, \langle \alpha, g \rangle) &= d(\langle 0, 1 \rangle, \langle \alpha, g \rangle) = 0 + \alpha - 2 \min\{0, \alpha, c(1, g)\} = \alpha, \\ d(f \cdot \epsilon, f \cdot \langle \alpha, g \rangle) &= d(\langle |f|, f \rangle, f \cdot \langle \alpha, g \rangle). \end{aligned}$$

Consider two cases.

(a) $\alpha \leq c(f^{-1}, g)$

Hence,

$$\begin{aligned} d(\langle |f|, f \rangle, f \cdot \langle \alpha, g \rangle) &= d(\langle |f|, f \rangle, \langle |f| - \alpha, f \rangle) \\ &= |f| + |f| - \alpha - 2 \min\{|f|, |f| - \alpha, c(f, f)\} = |f| + |f| - \alpha - 2(|f| - \alpha) = \alpha. \end{aligned}$$

(b) $\alpha > c(f^{-1}, g)$

Hence,

$$\begin{aligned} d(\langle |f|, f \rangle, f \cdot \langle \alpha, g \rangle) &= d(\langle |f|, f \rangle, \langle |f| + \alpha - 2c(f^{-1}, g), f * g \rangle) \\ &= |f| + |f| + \alpha - 2c(f^{-1}, g) - 2 \min\{|f|, |f| + \alpha - 2c(f^{-1}, g), c(f, f * g)\} \\ &= 2|f| + \alpha - 2c(f^{-1}, g) - 2 \min\{|f| + \alpha - 2c(f^{-1}, g), c(f, f * g)\}. \end{aligned}$$

Let $f = f_1 \circ c^{-1}$, $g = c \circ g_1$, $|c| = c(f^{-1}, g)$. Then $|f| + \alpha - 2c(f^{-1}, g) = |f_1| + \alpha - c(f^{-1}, g) > |f_1|$. At the same time, $c(f, f * g) = |f_1|$, so $\min\{|f| + \alpha - 2c(f^{-1}, g), c(f, f * g)\} = |f_1|$ and

$$d(\langle |f|, f \rangle, f \cdot \langle \alpha, g \rangle) = 2|f| + \alpha - 2c(f^{-1}, g) - 2|f_1| = 2|f| + \alpha - 2|c| - 2|f_1| = \alpha$$

□

Proposition 5. [69] *The action of G on Γ_G defined above is free and $L_\epsilon(g) = |g|$. Moreover, Γ_G is minimal with respect to this action if and only if G contains a cyclically reduced element $h \in G$, that is, $|h^2| = 2|h|$.*

Proof. **Cialm 1.** The stabilizer of every $x \in \Gamma_G$ is trivial.

Next, suppose $f \cdot \langle \alpha, g \rangle = \langle \alpha, g \rangle$. First of all, if $\alpha = 0$ then $|f| + \alpha - 2 \min\{\alpha, c(f^{-1}, g)\} = |f|$ then $|f| = \alpha = 0$. Also, if $c(f^{-1}, g) = 0$ then $|f| + \alpha - 2 \min\{\alpha, c(f^{-1}, g)\} = |f| + \alpha$ which has to be equal to α from our assumption. In both cases $f = 1$ follows.

Assume $f \neq 1$ (which implies $\alpha, c(f^{-1}, g) \neq 0$) and consider the following cases.

(a) $\alpha < c(f^{-1}, g)$

Hence, from

$$\langle \alpha, g \rangle = \langle |f| - \alpha, f \rangle$$

we get $\alpha = |f| - \alpha \leq c(f, g)$. In particular, $|f| = 2\alpha$.

Consider the product $f * g$. We have

$$f = f_1 \circ \text{com}(f^{-1}, g)^{-1}, \quad g = \text{com}(f^{-1}, g) \circ g_1.$$

Since $\alpha < c(f^{-1}, g)$ then we have $\text{com}(f^{-1}, g) = c_\alpha \circ c$, $|c_\alpha| = \alpha$. Hence,

$$f = f_1 \circ c^{-1} \circ c_\alpha^{-1}, \quad g = c_\alpha \circ c \circ g_1.$$

On the other hand, from $|f| = 2\alpha$ we get $|f_1| + |c| = \alpha \leq c(f, g)$, so, $\text{com}(f, g)$ has $f_1 \circ c$ as initial subword. That is, $g = f_1 \circ c \circ g_2$, but now comparing two representations of g above we get $c_\alpha = f_1 \circ c^{-1}$ and $c_\alpha * c \neq c_\alpha \circ c$ - a contradiction.

(b) $\alpha = c(f^{-1}, g)$

We have $f = f_1 \circ c_\alpha^{-1}$, $g = c_\alpha \circ g_1$, $|c_\alpha| = \alpha$. From $\langle \alpha, g \rangle = \langle |f| - \alpha, f \rangle$ we get $\alpha = |f| - \alpha \leq c(f, g)$, so $|f| = 2\alpha$ and $|f_1| = \alpha$. Since $|f_1| = \alpha \leq c(f, g)$ then $g = f_1 \circ g_2$ from which it follows that $f_1 = c_\alpha$. But then $f_1 * c_\alpha^{-1} \neq f_1 \circ c_\alpha^{-1}$ - contradiction.

(c) $\alpha > c(f^{-1}, g)$

Hence, from

$$\langle \alpha, g \rangle = \langle |f| + \alpha - 2c(f^{-1}, g), f * g \rangle$$

we get $\alpha = |f| + \alpha - 2c(f^{-1}, g) \leq c(g, f * g)$. In particular, $|f| = 2c(f^{-1}, g)$.

Consider the product $f * g$. We have

$$f = f_1 \circ c^{-1}, \quad g = c \circ g_1,$$

where $c = \text{com}(f^{-1}, g)$. Hence, $|f_1| = |c| < \alpha \leq c(g, f * g) = c(g, f_1 \circ g_1)$. It follows that $g = f_1 \circ g_2$ and, hence, $c = f_1$ which is impossible.

Cialm 2. $L_\epsilon(g) = |g|$

We have $L_\epsilon(g) = d(\epsilon, g \cdot \epsilon)$. Hence, by definition of d

$$d(\langle 0, 1 \rangle, g \cdot \langle 0, 1 \rangle) = d(\langle 0, 1 \rangle, \langle |g|, g \rangle) = 0 + |g| - 2 \min\{0, |g|, c(1, g)\} = |g|.$$

Cialm 3. Γ_G is minimal with respect to the action if and only if G contains a cyclically reduced element $h \in G$, that is, $|h^2| = 2|h|$.

Suppose there exists a cyclically reduced element $h \in G$. Let $\Delta \subset \Gamma_G$ be a G -invariant subtree.

First of all, observe that $\epsilon \notin \Delta$. Indeed, if $\epsilon \in \Delta$ then $f \cdot \epsilon \in \Delta$ for every $f \in G$ and since Δ is a tree then $[\epsilon, f \cdot \epsilon] \in \Delta$ for every $f \in G$. At the same time, Γ_G is spanned by $[\epsilon, f \cdot \epsilon]$, $f \in G$, so, $\Delta = \Gamma_G$ - a contradiction.

Let $u \in \Delta$. By definition of Γ_G there exists $g \in G$ such that $u \in [\epsilon, g \cdot \epsilon]$. Observe that $A_g \subseteq \Delta$. Indeed, for example by Theorem 1.4 [24], if $[u, p]$ is the bridge between u and A_g then $p = Y(g^{-1} \cdot u, u, g \cdot u)$. In particular, $p \in \Delta$

and since for every $v \in A_g$ there exist $g_1, g_2 \in C_G(g)$ such that $v \in [g_1 \cdot p, g_2 \cdot p]$ then $A_g \subseteq \Delta$.

Observe that if g is cyclically reduced then $\epsilon \in A_g$, that is, $\epsilon \in \Delta$ - a contradiction. More generally, $\Delta \cap A_f = \emptyset$ for every cyclically reduced $f \in G$. Hence, let $[p, q]$ be the bridge between A_g and A_h so that $p \in A_g$, $q \in A_h$. Then by Lemma 2.2 [24], $[p, q] \subset A_{gh}$, in particular, $p, q \in A_{gh}$. It follows that $A_{gh} \subseteq \Delta$, $q \in A_{gh} \cap A_h$, and $\Delta \cap A_h \neq \emptyset$ - a contradiction.

Hence, there can be no proper G -invariant subtree Δ .

Now, suppose G contains no cyclically reduced element. Hence, $\epsilon \notin A_f$ for every $f \in G$. Let Δ be spanned by A_f , $f \in G$. Obviously, Δ is G -invariant. Indeed, let $u \in [p, q]$, where $p \in A_f$, $q \in A_g$ for some $f, g \in G$. Then $h \cdot u \in [h \cdot p, h \cdot q]$, where $h \cdot p \in h \cdot A_f = A_{hfh^{-1}}$, $h \cdot q \in h \cdot A_g = A_{hgh^{-1}}$, that is, $h \in \Delta$.

Finally, $\epsilon \in \Gamma_G - \Delta$.

□

Proposition 6. [69] *If (Z, d') is a Λ -tree on which G acts freely as isometries, and $w \in Z$ is such that $L_w(g) = |g|$, $g \in G$ then there is a unique G -equivariant isometry $\mu : \Gamma_G \rightarrow Z$ such that $\mu(\epsilon) = w$, whose image is the subtree of Z spanned by the orbit $G \cdot w$ of w .*

Proof. Define a mapping $\mu : \Gamma_G \rightarrow Z$ as follows

$$\mu(\langle \alpha, f \rangle) = x \text{ if } d'(w, x) = \alpha, \quad d'(f \cdot w, x) = |f| - \alpha.$$

Observe that $\mu(\epsilon) = \mu(\langle 0, 1 \rangle) = w$

Claim 1. μ is an isometry.

Let $\langle \alpha, f \rangle, \langle \beta, g \rangle \in \Gamma_G$. Then by definition of d we have

$$d(\langle \alpha, f \rangle, \langle \beta, g \rangle) = \alpha + \beta - 2 \min\{\alpha, \beta, c(f, g)\}.$$

Let $x = \mu(\langle \alpha, f \rangle)$, $y = \mu(\langle \beta, g \rangle)$. Then By Lemma 1.2 [24] in (Z, d') we have

$$d'(x, y) = d(w, x) + d(w, y) - 2 \min\{d(w, x), d(w, y), d(w, z)\},$$

where $z = Y(w, f \cdot w, g \cdot w)$. Observe that $d(w, x) = \alpha$, $d(w, y) = \beta$. At the same time, since $L_w(g) = |g|$, $g \in G$ then

$$d(w, z) = \frac{1}{2}(d(w, f \cdot w) + d(w, g \cdot w) - d(f \cdot w, g \cdot w)) = \frac{1}{2}(|f| + |g| - |f^{-1}g|) = c(f, g),$$

and

$$d(\mu(\langle \alpha, f \rangle), \mu(\langle \beta, g \rangle)) = d'(x, y) = \alpha + \beta - 2 \min\{\alpha, \beta, c(f, g)\} = d(\langle \alpha, f \rangle, \langle \beta, g \rangle).$$

Claim 1. μ is equivariant.

We have to prove

$$\mu(f \cdot \langle \alpha, g \rangle) = f \cdot \mu(\langle \alpha, g \rangle).$$

Let $x = \mu(\langle \alpha, g \rangle)$, $y = \mu(f \cdot \langle \alpha, g \rangle)$. By definition of μ we have $d'(w, x) = \alpha$, $d'(g \cdot w, x) = |g| - \alpha$.

(a) $\alpha \leq c(f^{-1}, g)$

Hence,

$$f \cdot \langle \alpha, g \rangle = \langle |f| - \alpha, f \rangle.$$

and to prove $y = f \cdot x$ it is enough to show that $d'(w, f \cdot x) = |f| - \alpha$ and $d'(f \cdot w, f \cdot x) = \alpha$.

Observe that the latter equality holds since $d'(f \cdot w, f \cdot x) = d'(w, x) = \alpha$. To prove the former one, by Lemma 1.2 [24] we have

$$\begin{aligned} d(w, f \cdot x) &= d'(w, f \cdot w) + d'(f \cdot x, f \cdot w) \\ &\quad - 2 \min\{d'(w, f \cdot w), d'(f \cdot x, f \cdot w), d'(f \cdot w, z)\}, \end{aligned}$$

where $z = Y(w, f \cdot w, (fg) \cdot w)$. Also,

$$\begin{aligned} d'(f \cdot w, z) &= \frac{1}{2}(d'(f \cdot w, w) + d'(f \cdot w, (fg) \cdot w) - d'(w, (fg) \cdot w)) \\ &= \frac{1}{2}(|f| + |g| - |f^{-1}g|) = c(f^{-1}, g). \end{aligned}$$

Since, $d'(w, f \cdot w) = |f|$, $d'(f \cdot x, f \cdot w) = \alpha$ then $\min\{d'(w, f \cdot w), d'(f \cdot x, f \cdot w), d'(f \cdot w, z)\} = \alpha$, and

$$d'(w, f \cdot x) = |f| + \alpha - 2\alpha = |f| - \alpha.$$

(b) $\alpha > c(f^{-1}, g)$

Hence,

$$f \cdot \langle \alpha, g \rangle = \langle |f| + \alpha - 2c(f^{-1}, g), f * g \rangle.$$

and to prove $y = f \cdot x$ it is enough to show that $d'(w, f \cdot x) = |f| + \alpha - 2c(f^{-1}, g)$ and $d'(f \cdot x, (fg) \cdot w) = |fg| - (|f| + \alpha - 2c(f^{-1}, g))$.

Observe that $d'(f \cdot x, (fg) \cdot w) = d'(x, gw) = |g| - \alpha = |fg| - (|f| + \alpha - 2c(f^{-1}, g))$, so the latter equality holds.

By Lemma 1.2 [24] we have

$$\begin{aligned} d(w, f \cdot x) &= d'(w, f \cdot w) + d'(f \cdot x, f \cdot w) \\ &\quad - 2 \min\{d'(w, f \cdot w), d'(f \cdot x, f \cdot w), d'(f \cdot w, z)\}, \end{aligned}$$

where $z = Y(w, f \cdot w, (fg) \cdot w)$. Also,

$$\begin{aligned} d'(f \cdot w, z) &= \frac{1}{2}(d'(f \cdot w, w) + d'(f \cdot w, (fg) \cdot w) - d'(w, (fg) \cdot w)) \\ &= \frac{1}{2}(|f| + |g| - |f^{-1}g|) = c(f^{-1}, g). \end{aligned}$$

$d'(w, f \cdot w) = |f|$, $d'(f \cdot x, f \cdot w) = \alpha$, so $\min\{d'(w, f \cdot w), d'(f \cdot x, f \cdot w), d'(f \cdot w, z)\} = d'(f \cdot w, z) = c(f^{-1}, g)$, and

$$d(w, f \cdot x) = |f| + \alpha - 2c(f^{-1}, g).$$

Claim 1. μ is unique.

Observe that if $\mu' : \Gamma_G \rightarrow Z$ is another equivariant isometry such that $\mu'(\epsilon) = w$ then for every $g \in G$ we have

$$\mu'(\langle |g|, g \rangle) = \mu'(g \cdot \langle 0, 1 \rangle) = g \cdot \mu'(\langle 0, 1 \rangle) = g \cdot w.$$

That is, μ' agrees with μ on $G \cdot \epsilon$, hence $\mu = \mu'$ because isometries preserve geodesic segments.

Thus, μ is unique. Moreover, $\mu(\Gamma_G)$ is the subtree of Z spanned by $G \cdot w$.

□

The discussion above can be summarized in the following theorem.

Theorem 26. [69] *Let G be a subgroup of $CDR(\Lambda, X)$ for some discretely ordered abelian group Λ and a set X . Let $|\cdot| : G \rightarrow \Lambda$ be the length function on G induced from $CDR(\Lambda, X)$. Then there are a Λ -tree (Γ_G, p) , an action of G on Γ_G and a point $x \in \Gamma_G$ such that $|g| = l_x(g)$ for any $g \in G$, where $l_x(g) = p(x, g \cdot x)$. Moreover, If (Y, d) is a Λ -tree on which G acts freely by isometries, and $y \in Y$ is such that $l_y(g) = |g|$, $g \in G$, then there is a unique G -equivariant isometry $\mu : \Gamma_G \rightarrow Y$ such that $\mu(x) = y$, whose image is the subtree of Y spanned by the orbit $G \cdot y$ of y .*

7.3.2 Examples

Below we consider two examples of subgroups of $CDR(\Lambda, X)$, where $\Lambda = \mathbb{Z}^2$ and X an arbitrary alphabet, and explicitly construct the corresponding universal trees for these groups.

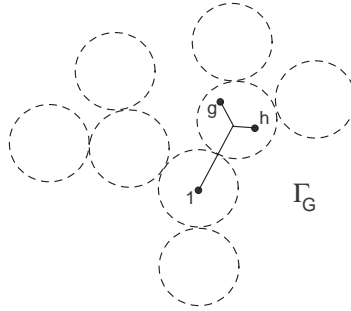


Figure 8: Γ_G as a \mathbb{Z} -tree of \mathbb{Z} -trees.

Example 20. *Let $F = F(X)$ be a free group with basis X and the standard length function $|\cdot|$, and let $u \in F$ a cyclically reduced element which is not*

a proper power. If we assume that $\mathbb{Z}^2 = \langle 1, t \rangle$ is the additive group of linear polynomials in t ordered lexicographically then the HNN-extension

$$G = \langle F, s \mid u^s = u \rangle$$

embeds into $CDR(\mathbb{Z}^2, X)$ under the following map ϕ :

$$\phi(x) = x, \quad \forall x \in X,$$

$$\phi(s)(\beta) = \begin{cases} u(\alpha), & \text{if } \beta = m|u| + \alpha, m \geq 0, 1 \leq \alpha \leq |u|, \\ u(\alpha), & \text{if } \beta = t - m|u| + \alpha, m > 0, 1 \leq \alpha \leq |u|. \end{cases}$$

It is easy to see that $|\phi(s)| = t$ and $\phi(s)$ commutes with u in $CDR(\mathbb{Z}^2, X)$. To simplify the notation we identify G with its image $\phi(G)$.

Every element g of G can be represented as the following reduced \mathbb{Z}^2 -word

$$g = g_1 \circ s^{\delta_1} \circ g_2 \circ \cdots \circ g_k \circ s^{\delta_k} \circ g_{k+1},$$

where $[g_i, u] \neq 1$. Now, according to the construction described in Subsection 7.3.1, the universal tree Γ_G consists of the segments in \mathbb{Z}^2 labeled by elements from G which are glued together along their common initial subwords.

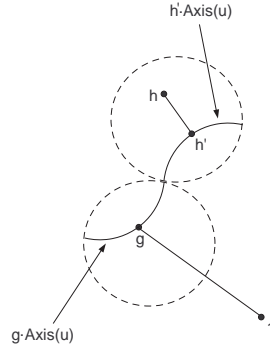
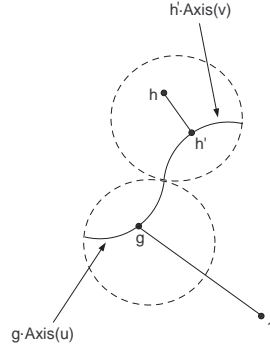


Figure 9: Adjacent \mathbb{Z} -subtrees in Γ_G .

Thus, Γ_G can be viewed as a \mathbb{Z} -tree of \mathbb{Z} -trees which are Cayley graphs of $F(X)$ and every vertex \mathbb{Z} -subtree can be associated with a right representative in G by F . The end-points of the segments $[1, |g|]$ and $[1, |h|]$ labeled respectively by g and h belong to the same vertex \mathbb{Z} -subtree if and only if $h^{-1}g \in F$ (see Figure 8).

In other words, Γ_G is a “more detailed” version of the Bass-Serre tree T for G , in which every vertex is replaced by the Cayley graph of the base group F and the adjacent \mathbb{Z} -subtrees of Γ_G corresponding to the representatives g and h are “connected” by means of s^\pm which extends $g \cdot \text{Axis}(u)$ to $h' \cdot \text{Axis}(u)$, where $h'^{-1}h \in F$ and $g^{-1}h \in s^\pm F$ (see Figure 9).


 Figure 10: Adjacent \mathbb{Z} -subtrees in Γ_H .

The following example is a generalization of the previous one.

Example 21. Let $F = F(X)$ be a free group with basis X and the standard length function $|\cdot|$, and let $u, v \in F$ be cyclically reduced elements which is not a proper powers and such that $|u| = |v|$. The HNN-extension

$$H = \langle F, s \mid u^s = v \rangle$$

embeds into $CDR(\mathbb{Z}^2, X)$ under the following map ψ :

$$\psi(x) = x, \quad \forall x \in X,$$

$$\psi(s)(\beta) = \begin{cases} u(\alpha), & \text{if } \beta = m|u| + \alpha, m \geq 0, 1 \leq \alpha \leq |u|, \\ v(\alpha), & \text{if } \beta = t - m|v| + \alpha, m > 0, 1 \leq \alpha \leq |v|. \end{cases}$$

It is easy to see that $|\psi(s)| = t$ and $u \circ \psi(s) = \psi(s) \circ v$ in $CDR(\mathbb{Z}^2, X)$. Again, to simplify the notation we identify H with its image $\psi(H)$.

The structure of Γ_H is basically the same as the structure of Γ_G in Example 20. The only difference is that the adjacent \mathbb{Z} -subtrees of Γ_H corresponding to the representatives g and h are “connected” by means of s^\pm which extends $g \cdot \text{Axis}(u)$ to $h' \cdot \text{Axis}(v)$, where $h'^{-1}h \in F$ and $g^{-1}h \in s^\pm F$ (see Figure 10).

7.3.3 Labeling of “edges” and “paths” in universal trees

Let G be a subgroup of $CDR(\Lambda, X)$ for some discretely ordered abelian group Λ and a set X , and let Γ_G be its universal Λ -tree constructed in Subsection 7.3.1. Recall that there exists a labeling function

$$\xi : (\Gamma_G - \{\epsilon\}) \rightarrow X^\pm, \quad \xi(\langle \alpha, g \rangle) = g(\alpha)$$

on all points of Γ_G except the base-point ϵ .

It is easy to see that the labeling ξ is not equivariant, that is, $\xi(v) \neq \xi(g \cdot v)$ in general (even if both v and $g \cdot v$ are in $\Gamma_G - \{\epsilon\}$, which is not stable under the

action of G). In the present paper we are going to introduce another labeling function for Γ_G defined not on vertices but on “edges”, stable under the action of G . With this new labeling Γ_G becomes an extremely useful combinatorial object in the case $\Lambda = \mathbb{Z}^n$, but in general such a labeling can be defined for every discretely ordered Λ .

First of all, for every $v_0, v_1 \in \Gamma_G$ such that $d(v_0, v_1) = 1$ we call the ordered pair (v_0, v_1) the *edge* from v_0 to v_1 . Here, if $e = (v_0, v_1)$ then denote $v_0 = o(e)$, $v_1 = t(e)$ which are respectively the *origin* and *terminus* of e . Now, if the vertex $v_1 \in \Gamma_G - \{\varepsilon\}$ is fixed then, since Γ_G is a Λ -tree, there is exactly one point v_0 such that $d(\varepsilon, v_1) = d(\varepsilon, v_0) + 1$. Hence, there exists a natural orientation, with respect to ε , of edges in Γ_G , where an edge (v_0, v_1) is *positive* if $d(\varepsilon, v_1) = d(\varepsilon, v_0) + 1$, and *negative* otherwise. Denote by $E(\Gamma_G)$ the set of edges in Γ_G . If $e \in E(\Gamma_G)$ and $e = (v_0, v_1)$ then the pair (v_1, v_0) is also an edge and denote $e^{-1} = (v_1, v_0)$. Obviously, $o(e) = t(e^{-1})$. Because of the orientation, we have a natural splitting

$$E(\Gamma_G) = E(\Gamma_G)^+ \cup E(\Gamma_G)^-,$$

where $E(\Gamma_G)^+$ and $E(\Gamma_G)^-$ denote respectively the sets of positive and negative edges. Now, we can define a function $\mu : E(\Gamma_G)^+ \rightarrow X^\pm$ as follows: if $e = (v_0, v_1) \in E(\Gamma_G)^+$ then $\mu(e) = \xi(v_1)$. Next, μ can be extended to $E(\Gamma_G)^-$ (and hence to $E(\Gamma_G)$) by setting $\mu(f) = \mu(f^{-1})^{-1}$ for every $f \in E(\Gamma_G)^-$.

Example 22. Let $F = F(X)$ be a free group on X . Hence, F embeds into (coincides with) $CDR(\mathbb{Z}, X)$ and Γ_F with the labeling μ defined above is just a Cayley graph of F with respect to X . That is, Γ_F is a labeled simplicial tree.

The action of G on Γ_G induces the action on $E(\Gamma_G)$ as follows $g \cdot (v_0, v_1) = (g \cdot v_0, g \cdot v_1)$ for each $g \in G$ and $(v_0, v_1) \in E(\Gamma_G)$. It is easy to see that $E(\Gamma_G)^+$ is not closed under the action of G but the labeling is equivariant as the following lemma shows.

Lemma 19. If $e, f \in E(\Gamma_G)$ belong to one G -orbit then $\mu(e) = \mu(f)$.

Proof. Let $e = (v_0, v_1) \in E(\Gamma_G)^+$. Hence, there exists $g \in G$ such that $v_0 = \langle \alpha, g \rangle$, $v_1 = \langle \alpha + 1, g \rangle$. Let $f \in G$ and consider the following cases.

Case 1. $c(f^{-1}, g) = 0$

Then $f * g = f \circ g$. If $\alpha = 0$ then $f \cdot v_0 = \langle |f|, f \rangle = \langle |f|, f \circ g \rangle$, and $f \cdot v_1 = \langle |f| + 1, f \circ g \rangle$. Hence, $f \cdot e \in E(\Gamma_G)^+$ and $\mu(f \cdot e) = \xi(f \cdot v_1) = g(1) = \xi(v_1) = \mu(e)$.

Case 2. $c(f^{-1}, g) > 0$

(a) $\alpha + 1 \leq c(f^{-1}, g)$

Then $f \cdot v_0 = \langle |f| + \alpha - 2\alpha, f \rangle = \langle |f| - \alpha, f \rangle$ and $f \cdot v_1 = \langle |f| - (\alpha + 1), f \rangle$. So, $d(\varepsilon, f \cdot v_1) < d(\varepsilon, f \cdot v_0)$ and $f \cdot e \in E(\Gamma_G)^-$. Now,

$$\begin{aligned} \mu(f \cdot e) &= \mu((f \cdot e)^{-1})^{-1} = \mu((f \cdot v_1, f \cdot v_0))^{-1} = \xi(f \cdot v_0)^{-1} = f(|f| - \alpha)^{-1} \\ &= g(\alpha + 1) = \xi(v_1) = \mu(e). \end{aligned}$$

(b) $\alpha = c(f^{-1}, g)$

We have $f \cdot v_0 = \langle |f| - \alpha, f \rangle$ and $f \cdot v_1 = \langle |f| + (\alpha + 1) - 2c(f^{-1}, g), f * g \rangle = \langle |f| - \alpha + 1, f * g \rangle$. It follows that $f \cdot e \in E(\Gamma_G)^+$ and $\mu(f \cdot e) = \xi(f \cdot v_1) = (f * g)(|f| - \alpha + 1)$. At the same time, $f * g = f_1 \circ g_1$, where $|f_1| = |f| - c(f^{-1}, g) = |f| - \alpha$, $g = g_0 \circ g_1$, $|g_0| = \alpha$, so, $(f * g)(|f| - \alpha + 1) = g_1(1) = g(\alpha + 1)$ and $\mu(f \cdot e) = g(\alpha + 1) = \xi(\langle \alpha + 1, g \rangle) = \xi(v_1) = \mu(e)$.

(c) $\alpha > c(f^{-1}, g)$

Hence, $f \cdot v_0 = \langle |f| + \alpha - 2c(f^{-1}, g), f * g \rangle$ and $f \cdot v_1 = \langle |f| + \alpha + 1 - 2c(f^{-1}, g), f * g \rangle$. Obviously, $f \cdot e \in E(\Gamma_G)^+$ and

$$\begin{aligned} \mu(f \cdot e) &= \xi(f \cdot v_1) = (f * g)(|f| + \alpha + 1 - 2c(f^{-1}, g)) = g_1(\alpha + 1 - c(f^{-1}, g)) \\ &= g(\alpha + 1) = \xi(v_1) = \mu(e), \end{aligned}$$

where $f * g = f_1 \circ g_1$, $|f_1| = |f| - c(f^{-1}, g) = |f| - \alpha$, $g = g_0 \circ g_1$, $|g_0| = \alpha$.

Thus, in all possible cases we got $\mu(f \cdot e) = \mu(e)$ and the required statement follows. \square

Let v, w be two points of Γ_G . Since Γ_G is a Λ -tree there exists a unique geodesic connecting v to w , which can be viewed as a “path” in the following sense. A *path from v to w* is a sequence of edges $p = \{e_\alpha\}$, $\alpha \in [1, d(v, w)]$ such that $o(e_1) = v$, $t(e_{d(v, w)}) = w$ and $t(e_\alpha) = o(e_{\alpha+1})$ for every $\alpha \in [1, d(v, w) - 1]$. In other words, a path is an “edge” counter-part of a geodesic and usually, for the path from v to w (which is unique since Γ_G is a Λ -tree) we are going to use the same notation as for the geodesic between these points, that is, $p = [v, w]$. In the case when $v = w$ the path p is empty. The *length* of p we denote by $|p|$ and set $|p| = d(v, w)$. Now, the *path label* $\mu(p)$ for a path $p = \{e_\alpha\}$ is the function $\mu : \{e_\alpha\} \rightarrow X^\pm$, where $\mu(e_\alpha)$ is the label of the edge e_α .

Lemma 20. *Let v, w be points of Γ_G and p the path from v to w . Then $\mu(p) \in R(\Lambda, X)$.*

Proof. From the definition of Γ_G it follows that the statement is true when $v = \varepsilon$. Let $v_0 = Y(\varepsilon, v, w)$ and let p_v and p_w be the paths from ε respectively to v and w . Also, let p_1 and p_2 be the paths from v_0 to v and w . Since $\mu(p_v), \mu(p_w) \in R(\Lambda, X)$ then $\mu(p_1), \mu(p_2) \in R(\Lambda, X)$ as subwords. Hence, $\mu(p) \notin R(\Lambda, X)$ implies that the first edges e_1 and e_2 correspondingly of p_1 and p_2 have the same label. But this contradicts the definition of Γ_G because in this case $t(e_1) \sim t(e_2)$, but $t(e_1) \neq t(e_2)$. \square

As usual, if p is a path from v to w then its *inverse* denoted p^{-1} is a path from w back to v . In this case, the label of p^{-1} is $\mu(p)^{-1}$, which is again an element of $R(\Lambda, X)$.

Define

$$V_G = \{v \in \Gamma_G \mid \exists g \in G : v = \langle |g|, g \rangle\},$$

which is a subset of points in Γ_G corresponding to the elements of G . Also, for every $v \in \Gamma_G$ let

$$\text{path}_G(v) = \{\mu(p) \mid p = [v, w] \text{ where } w \in V_G\}.$$

The following lemma follows immediately.

Lemma 21. *Let $v \in V_G$. Then $\text{path}_G(v) = G \subset CDR(\Lambda, X)$.*

The action of G on $E(\Gamma_G)$ extends to the action on all paths in Γ_G , hence, Lemma 19 extends to the case when e and f are two G -equivalent paths in Γ_G .

8 Regular length functions and actions

Regularity of Lyndon length function, or of the underlying action turns out to be an important property in the theory of Λ -free groups. By imposing this restriction on the length function or action one gains a lot of information about the group through inner combinatorics of group elements viewed as Λ -words.

8.1 Regular length functions

In this section we define regular length functions and show some examples of groups with regular length functions.

A length function $l : G \rightarrow \Lambda$ is called *regular* if it satisfies the *regularity* axiom:

$$(L6) \quad \forall g, f \in G, \exists u, g_1, f_1 \in G :$$

$$g = u \circ g_1 \text{ \& } f = u \circ f_1 \text{ \& } l(u) = c(g, f).$$

Observe that a regular length function does not have to be free, as well as, freeness does not imply regularity.

Here are several examples of groups with regular free length functions.

Example 23. *Let $F = F(X)$ be a free group on X . The length function*

$$|\cdot| : F \rightarrow \mathbb{Z},$$

where $|f|$ is a the length of $f \in F$ as a word in $X^{\pm 1}$, is regular.

The following is a more general example.

Example 24. [67] *Let $F = F(X)$ be a free group on X , H a finitely generated subgroup of F , and l_H the restriction to H of the length function in F relative to X . Then l_H is a regular length function on H if and only if there exists a basis U of H such that every two non-equal elements from $U^{\pm 1}$ have different initial letters.*

Example 25. [92] *Lyndon's free $\mathbb{Z}[t]$ -group $F^{\mathbb{Z}[t]}$ has a regular free length function with values in $\mathbb{Z}[t]$.*

Example 26. [67] Let $F = F(X)$ be a free group with basis X , $|\cdot|$ the standard length function on F relative to X , and $u, v \in F$ such that $|u| = |v|$ and u is not conjugate to v^{-1} . Then the HNN-extension

$$G = \langle F, s \mid u^s = v \rangle,$$

has a regular free length function $l : G \rightarrow \mathbb{Z}^2$ which extends $|\cdot|$.

Example 27. [67] For any $n \geq 1$ the orientable surface group

$$G = \langle x_1, x_2, \dots, x_{2n-1}, x_{2n} \mid [x_1, x_2] \cdots [x_{2n-1}, x_{2n}] = 1 \rangle$$

has a regular free length function $l : G \rightarrow \mathbb{Z}^2$.

Proof. It suffices to represent G as an HNN extension from Example 26. The word

$$R(X) = x_1 \cdots x_{2n} x_1^{-1} \cdots x_{2n}^{-1}.$$

is quadratic, so there exists an automorphism ϕ of $F = F(x_1, \dots, x_{2n})$ such that

$$R(X)^\phi = [x_1, x_2] \cdots [x_{2n-1}, x_{2n}]$$

(see, for example, Proposition 7.6 [78]). It follows that G is isomorphic to

$$G' = \langle x_1, \dots, x_{2n} \mid x_1 \cdots x_{2n} x_1^{-1} \cdots x_{2n}^{-1} = 1 \rangle,$$

which can be represented as an HNN-extension of the required form

$$G' = \langle F(x_2, \dots, x_{2n}), x_1 \mid x_1(x_2 \cdots x_{2n})x_1^{-1} = x_{2n}x_{2n-1} \cdots x_2 \rangle,$$

since $|x_2 \cdots x_{2n}| = |x_{2n}x_{2n-1} \cdots x_2|$. □

Example 28. [67] For any n , $n \geq 3$ the non-orientable surface group

$$G = \langle x_1, x_2, \dots, x_n \mid x_1^2 x_2^2 \cdots x_n^2 = 1 \rangle$$

has a regular free length function $l : G \rightarrow \mathbb{Z}^2$.

Proof. Again, it suffices to represent G as an HNN extension from Example 26. An argument similar to the one in the proof of Example 27 shows that the group G is isomorphic to

$$G' = \langle x_1, x_2, \dots, x_n \mid x_1 \cdots x_{n-1} x_n x_1^{-1} \cdots x_{n-1}^{-1} x_n \rangle$$

and the result follows, since the presentation above can be written as

$$G' = \langle x_1, x_2, \dots, x_n \mid x_1(x_2 \cdots x_{n-1} x_n)x_1^{-1} = x_n^{-1} x_{n-1} \cdots x_2 \rangle$$

□

Example 29. [67] A free abelian group of rank n has a free regular length function in \mathbb{Z}^n .

Proof. Let $G = \mathbb{Z}^n$ be a free abelian group of rank n . Then G is an ordered abelian group relative to the right lexicographic order " \leq ". The absolute value $|u|$ of an element $u \in G$, defined as $|u| = \max\{u, -u\}$, gives a free length function $l : G \rightarrow \mathbb{Z}^n$. It is easy to see that l is regular. \square

Example 30. [67] Let G_i , $i = 1, 2$ be a group with a free regular length function $l_i : G_i \rightarrow \mathbb{Z}^n$. Then the free product $G = G_1 * G_2$ has a free regular length function in $l : G \rightarrow \mathbb{Z}^n$ that extends the functions l_1 and l_2 .

Proof. Let $g \in G$ given in the reduced form $g = u_1 v_1 \dots u_k v_k$, where $u_1, \dots, u_k \in G_1$ and $v_1, \dots, v_k \in G_2$. Define $l : G \rightarrow \mathbb{Z}^n$ by

$$l(g) = \sum_{i=1}^k (l_1(u_i) + l_2(v_i)).$$

l is a free length function (see, for example, [24]). Also, note that $c(g, h) = 0$ if $g \in G_1$ and $h \in G_2$. Next, l is regular. Indeed, let $g = g_1 \dots g_k$, $h = h_1 \dots h_n \in G$, where the products $g_1 \dots g_k$, $h_1 \dots h_n$ are reduced and $k \leq n$. Let $m = \max\{i \mid g_i = h_i\}$. If $m = k$ then $h = g \circ (h_{m+1} \dots h_n)$ and $c(g, h) = l(g)$, so the regularity axiom holds for the pair g, h . If $m = n$ then necessarily $m = k$ and $h = g$, so the the axiom holds again. Now assume that $m < k$. If we denote $c = g_1 \dots g_m$ then $g = c \circ g_{m+1} \circ g'$, $h = c \circ h_{m+1} \circ h'$, where $g' = g_{m+2} \dots g_k$, $h' = h_{m+2} \dots h_n$ (g' is trivial if $k = m + 1$, and h' is trivial if $n = m + 1$). Hence, $g^{-1}h = (g')^{-1} \circ (g_{m+1}^{-1} h_{m+1}) \circ h'$ and

$$\begin{aligned} c(g, h) &= \frac{1}{2}(l(g) + l(h) - l(g^{-1}h)) = \frac{1}{2}(l(c) + l(g_{m+1}) + l(g') + l(c) + l(h_{m+1}) + l(h') \\ &\quad - (l(g') + l(g_{m+1}^{-1} h_{m+1}) + l(h'))) = l(c) + c(g_{m+1}, h_{m+1}). \end{aligned}$$

Since both g_{m+1} and h_{m+1} belong to the same factor G_i and the length function on G_i is regular, it follows that there exists $u \in G_i$ such that $l(u) = c(g_{m+1}, h_{m+1})$, $g_{m+1} = u \circ v$, $h_{m+1} = u \circ w$ for some $v, w \in G_i$. Hence, $g = (c u) \circ v \circ g'$, $h = (c u) \circ w \circ h'$, where $c(g, h) = l(c u)$ and $c u \in G$. \square

Example 31. Let G be a finitely generated \mathbb{R} -free group. Then G has a free regular length function in \mathbb{Z}^n , where n is the maximal rank of free abelian subgroups (centralizers) of G .

Proof. By Rips' Theorem every finitely generated \mathbb{R} -free group is a free product of groups described in Examples 27, 28, 29, hence the result. \square

Theorem 27. Let G have a free regular Lyndon length function $L : G \rightarrow \Lambda$, where Λ is an arbitrary ordered abelian group. Then there exists an embedding $\phi : G \rightarrow R(\Lambda', X)$, where Λ' is a discretely ordered abelian group and X is some set, such that, the Lyndon length function on $\phi(G)$ induced from $R(\Lambda', X)$ is regular.

Proof. Since L is regular then for any $g, f \in G$ there exist $u, g_1, f_1 \in G$ such that

$$g = u \circ g_1 \text{ \& } f = u \circ f_1 \text{ \& } L(u) = c(g, f)$$

By Corollary 5 it follows that

$$|\phi(g)| = |\phi(u)| + |\phi(g_1)|, \quad |\phi(f)| = |\phi(u)| + |\phi(f_1)|.$$

Indeed, if for example $|\phi(g)| < |\phi(u)| + |\phi(g_1)|$ then $L(g) < L(u) + L(g_1)$ – a contradiction. So, we have

$$2c(\phi(g), \phi(f)) = |\phi(g)| + |\phi(f)| - |\phi(g^{-1}f)| = |\phi(g)| + |\phi(f)| - |\phi(g_1^{-1} \circ f_1)| = 2|\phi(u)|.$$

Hence, $c(\phi(g), \phi(f)) = |\phi(u)|$ and the length function on $\phi(G)$ induced from $R(A, X)$ is regular. \square

Notice that the converse of the theorem above is obviously true.

8.2 Regular actions

In this section we give a geometric characterization of group actions that come from regular length functions.

Let G act on a Λ -tree Γ . The action is *regular with respect to* $x \in \Gamma$ if for any $g, h \in G$ there exists $f \in G$ such that $[x, fx] = [x, gx] \cap [x, hx]$.

The next lemma shows that regular actions exactly correspond to regular length functions (hence the term).

Lemma 22. [67] *Let G act on a Λ -tree Γ . Then the action of G is regular with respect to $x \in \Gamma$ if and only if the length function $L_x : G \rightarrow \Lambda$ based at x is regular.*

Proof. Let d be the Λ -metric on Γ . By definition, the length function L_x is regular if for every $g, h \in G$ there exists $f \in G$ such that $g = fg_1$, $h = fh_1$, where $L_x(f) = c(g, h)$ and $L_x(g) = L_x(f) + L_x(g_1)$, $L_x(h) = L_x(f) + L_x(h_1)$.

Suppose the action of G is regular with respect to x . Then for $g, h \in G$ there exists $f \in G$ such that $[x, fx] = [x, gx] \cap [x, hx]$. We have $[x, gx] = [x, fx] \cup [fx, gx]$, $[x, hx] = [x, fx] \cup [fx, hx]$ and $d(fx, gx) = d(x, (f^{-1}g)x) = L_x(f^{-1}g)$, $d(fx, hx) = d(x, (f^{-1}h)x) = L_x(f^{-1}h)$. Taking $g_1 = f^{-1}g$, $h_1 = f^{-1}h$ we have $L_x(g) = L_x(f) + L_x(g_1)$, $L_x(h) = L_x(f) + L_x(h_1)$. Finally, since $c(g, h) = \frac{1}{2}(L_x(g) + L_x(h) - L_x(g^{-1}h))$ and $L_x(g^{-1}h) = d(x, (g^{-1}h)x) = d(gx, hx) = d(fx, gx) + d(fx, hx)$ we get $L_x(f) = c(g, h)$.

Suppose that L_x is regular. Then from $g = fg_1$, $h = fh_1$, where $L_x(g) = L_x(f) + L_x(g_1)$, $L_x(h) = L_x(f) + L_x(h_1)$ it follows that $[x, gx] = [x, fx] \cup [fx, gx]$, $[x, hx] = [x, fx] \cup [fx, hx]$. Now, $L_x(f) = c(g, h) = \frac{1}{2}(L_x(g) + L_x(h) - L_x(g^{-1}h))$, so $2d(x, fx) = d(x, gx) + d(x, hx) - d(x, (g^{-1}h)x) = d(x, gx) + d(x, hx) - d(gx, hx)$. In other words,

$$d(gx, hx) = d(x, gx) + d(x, hx) - 2d(x, fx) = (d(x, gx) - d(x, fx)) +$$

$$(d(x, hx) - d(x, fx)) = d(fx, gx) + d(fx, hx)$$

which is equivalent to $[x, fx] = [x, gx] \cap [x, hx]$. \square

Lemma 23. [67] *Let G act minimally on a Λ -tree Γ . If the action of G is regular with respect to $x \in \Gamma$ then all branch points of Γ are G -equivalent.*

Proof. From minimality of the action it follows that Γ is spanned by the set of points $Gx = \{gx \mid g \in G\}$.

Now let y be a branch point in Γ . It follows that there exist (not unique in general) $g, h \in G$ such that $[x, y] = [x, gx] \cap [x, hx]$. From regularity of the action it follows that there exists $f \in G$ such that $y = fx$. Hence, every branch point is G -equivalent to x and the statement of the lemma follows. \square

Lemma 24. [67] *Let G act on a Λ -tree Γ . If the action of G is regular with respect to $x \in \Gamma$ then it is regular with respect to any $y \in Gx$.*

Proof. We have to show that for every $g, h \in G$ there exists $f \in G$ such that $[y, fy] = [y, gy] \cap [y, hy]$. Since $y = tx$ for some $t \in G$ then we have to prove that $[tx, (ft)x] = [tx, (gt)x] \cap [tx, (ht)x]$. The latter equality is equivalent to $[x, (t^{-1}ft)x] = [x, (t^{-1}gt)x] \cap [x, (t^{-1}ht)x]$ which follows from regularity of the action with respect to x . \square

Lemma 25. [67] *Let G act freely on a Λ -tree Γ so that all branch points of Γ are G -equivalent. Then the action of G is regular with respect to any branch point in Γ .*

Proof. Let x be a branch point in Γ and $g, h \in G$. If $g = h$ then $[x, gx] \cap [x, hx] = [x, gx]$ and g is the required element. Suppose $g \neq h$. Since the action is free then $gx \neq hx$ and we consider the tripod formed by x, gx, hx . Hence, $y = Y(x, gx, hx)$ is a branch point in Γ and by the assumption there exists $f \in G$ such that $y = fx$. \square

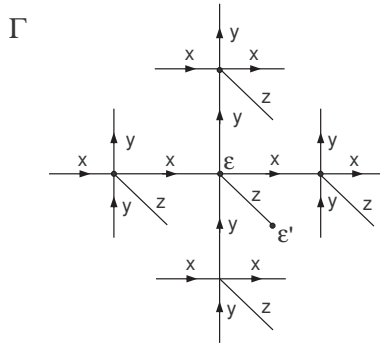


Figure 11: Γ in Example 32.

Example 32. Let Γ' be the Cayley graph of a free group $F(x, y)$ with the base-point ε . Let Γ be obtained from Γ' by adding an edge labeled by $z \neq x^{\pm 1}, y^{\pm 1}$ at every vertex of Γ' . $F(x, y)$ has a natural action on Γ' which we can extend to the action on Γ . The edge at ε labeled by z has an endpoint not equal to ε and we denote it by ε' . Observe that the action of $F(x, y)$ on Γ is regular with respect to ε but is not regular with respect to ε' .

8.3 Merzlyakov's Theorem for Λ -free groups with regular actions

Here are the results which show that groups with regular free Lyndon length functions generalize free groups in the following sense.

Theorem 28. [58] *Every finitely generated non-abelian group G with a regular free Lyndon length function freely lifts every positive sentence of the language L_G which holds in G , that is, G freely lifts its positive theory $Th^+(G)$ (the set of all positive sentences that are true in G).*

Theorem 29. [58] *Let G be a finitely generated non-abelian group with a regular free Lyndon length function. Then every positive sentence in the language L_G which holds in G has term-definable Skolem functions in G . Moreover, if G has a decidable word problem then such Skolem functions can be found effectively.*

9 Limit groups

These numerous characterizations of limit groups make them into a very robust tool linking group theory, topology and logic.

Limit groups play an important part in modern group theory. They appear in many different situations: in combinatorial group theory as groups discriminated by G (ω -residually G -groups or fully residually G -groups) [6, 5, 91, 8, 9], in the algebraic geometry over groups as the coordinate groups of irreducible varieties over G [7, 59, 64, 60, 117], groups universally equivalent to G [108, 41, 91], limit groups of G in the Gromov-Hausdorff topology [20], in the theory of equations in groups [76, 104, 106, 59, 64, 60, 46], in group actions [14, 40, 92, 47, 49], in the solutions of Tarski problems [62, 118], etc.

Recall, that a group G is called *fully residually free* if for any non-trivial $g_1, \dots, g_n \in G$ there exists a homomorphism ϕ of G into a free group such that $\phi(g_1), \dots, \phi(g_n)$ are non-trivial.

It is a crucial result that every limit group admits a free action on a \mathbb{Z}^n -tree for an appropriate $n \in \mathbb{N}$, where \mathbb{Z}^n is ordered lexicographically (see [64]). The proof comes in several steps. The initial breakthrough is due to Lyndon, who introduced a construction of the free $\mathbb{Z}[t]$ -completion $F^{\mathbb{Z}[t]}$ of a free group F (nowadays it is called Lyndon's free $\mathbb{Z}[t]$ -group) and showed that this group, as well as all its subgroups, is fully residually free [76]. Much later Remeslennikov proved that every finitely generated fully residually free group has a free Lyndon length function with values in \mathbb{Z}^n (but not necessarily ordered lexicographically)

[108]. That was a first link between limit groups and free actions on \mathbb{Z}^n -trees. In 1995 Myasnikov and Remeslennikov showed that Lyndon free exponential group $F^{\mathbb{Z}[t]}$ has a free Lyndon length function with values in \mathbb{Z}^n with lexicographical ordering [89] and stated a conjecture that every limit group embeds into $F^{\mathbb{Z}[t]}$. Finally, Kharlampovich and Myasnikov proved that every limit group G embeds into $F^{\mathbb{Z}[t]}$ [64].

Below, following [92] we construct a free $\mathbb{Z}[t]$ -valued length function on $F^{\mathbb{Z}[t]}$ which combined with the result of Kharlampovich and Myasnikov mentioned above gives a free \mathbb{Z}^n -valued length function on a given limit group G . Then we discuss various algorithmic applications of these results which are based on the technique of infinite words and Stallings foldings techniques for subgroups of $F^{\mathbb{Z}[t]}$ (see [93, 65, 97]).

We fix a set X , a free group $F = F(X)$, and consider the additive group of the polynomial ring $\mathbb{Z}[t]$ as a free abelian group (with basis $1, t, t^2, \dots$) ordered lexicographically.

9.1 Lyndon's free group $F^{\mathbb{Z}[t]}$

Let A be an associative unitary ring. A group G is termed an A -group if it is equipped with a function (*exponentiation*) $G \times A \rightarrow G$:

$$(g, \alpha) \rightarrow g^\alpha$$

satisfying the following conditions for arbitrary $g, h \in G$ and $\alpha, \beta \in A$:

$$\text{(Exp1)} \quad g^1 = g, \quad g^{\alpha+\beta} = g^\alpha g^\beta, \quad g^{\alpha\beta} = (g^\alpha)^\beta,$$

$$\text{(Exp2)} \quad g^{-1} h^\alpha g = (g^{-1} h g)^\alpha,$$

$$\text{(Exp3)} \quad \text{if } g \text{ and } h \text{ commute, then } (gh)^\alpha = g^\alpha h^\alpha.$$

The axioms (Exp1) and (Exp2) were introduced originally by R. Lyndon in [76], the axiom (Exp3) was added later in [88]. A homomorphism $\phi : G \rightarrow H$ between two A -groups is termed an A -homomorphism if $\phi(g^\alpha) = \phi(g)^\alpha$ for every $g \in G$ and $\alpha \in A$. It is not hard to prove (see, [88]) that for every group G there exists an A -group H (which is unique up to an A -isomorphism) and a homomorphism $\mu : G \rightarrow H$ such that for every A -group K and every A -homomorphism $\theta : G \rightarrow K$, there exists a unique A -homomorphism $\phi : H \rightarrow K$ such that $\phi\mu = \theta$. We denote H by G^A and call it the A -completion of G .

In [90] an effective construction of $F^{\mathbb{Z}[t]}$ was given in terms of extensions of centralizers. For a group G let $S = \{C_i \mid i \in I\}$ be a set of representatives of conjugacy classes of proper cyclic centralizers in G , that is, every proper cyclic centralizer in G is conjugate to one from S , and no two centralizers from S are conjugate. Then the HNN-extension

$$H = \langle G, s_{i,j} \ (i \in I, j \in \mathbb{N}) \mid [s_{i,j}, u_i] = [s_{i,j}, s_{i,k}] = 1 \ (u_i \in C_i, i \in I, j, k \in \mathbb{N}) \rangle,$$

is termed an *extension of cyclic centralizers* in G . Now the group $F^{\mathbb{Z}[t]}$ is isomorphic to the direct limit of the following infinite chain of groups:

$$F = G_0 < G_1 < \cdots < G_n < \cdots < \cdots, \quad (2)$$

where G_{i+1} is obtained from G_i by extension of all cyclic centralizers in G_i .

9.2 $\mathbb{Z}[t]$ -exponentiation on $CR(\mathbb{Z}[t], X)$

Define t -exponentiation on $CR(\mathbb{Z}[t], X)$ as follows.

- (1) Let $u \in CR(\mathbb{Z}[t], X)$ be not a proper power and such that

$$|u| = f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0.$$

Observe that for every $\beta \in [1, t^{n+1}]$ there exists $m \geq 0$ such that either $\beta \in [m|u| + 1, (m+1)|u|]$, or $\beta \in [t|u| - (m+1)|u| + 1, t|u| - m|u|]$. Hence, define u^t to be an element of $CR(\mathbb{Z}[t], X)$ with length t^{n+1} as follows

$$u^t(\beta) = \begin{cases} u(\alpha), & \text{if } \beta = m|u| + \alpha, m \geq 0, 1 \leq \alpha \leq |u|, \\ u(\alpha), & \text{if } \beta = t|u| - m|u| + \alpha, m > 0, 1 \leq \alpha \leq |u|. \end{cases}$$

- (2) If $v \in CR(\mathbb{Z}[t], X)$ is such that $v = u^k$ for some $u \in CR(\mathbb{Z}[t], X)$ then we set $v^t = (u^t)^k$.

Thus we have defined an exponent v^t for a given $v \in CR(\mathbb{Z}[t], X)$. Notice that it follows from the construction that v^t starts with v and ends with v . In particular, $v^t \in CR(\mathbb{Z}[t], X)$. It follows that $v^t * v = v^t \circ v = v \circ v^t = v * v^t$, hence, $[v^t, v] = \varepsilon$.

- (3) Now for $v \in CR(\mathbb{Z}[t], X)$ we define exponents v^{t^k} by induction. Since $v^t \in CR(\mathbb{Z}[t], X)$ one can repeat the construction from (1) and define

$$v^{t^{k+1}} = (v^{t^k})^t.$$

- (4) Now we define $v^{f(t)}$, where $f(t) \in \mathbb{Z}[t]$, by linearity, that is, if $f(t) = m_0 + m_1 t + \cdots + m_k t^k$ then

$$v^{f(t)} = v^{m_0} * (v^t)^{m_1} * \cdots * (v^{t^k})^{m_k}.$$

Observe that the product above is defined because $v^{t^{m+1}}$ is cyclically reduced, and starts and ends with v^{t^m} .

Lemma 26. *Let $v \in CR(\mathbb{Z}[t], X)$, $f(t) \in \mathbb{Z}[t]$. Then $v^{f(t)} \in CR(\mathbb{Z}[t], X)$ and $[v^{f(t)}, v] = \varepsilon$, $v^{-f(t)} = (v^{-1})^{f(t)}$.*

Proof. Follows directly from the definition of $\mathbb{Z}[t]$ -exponentiation. \square

Remark 4. Observe that under the above definition we loose the property $|u^{f(t)}| = |u||f(t)|$ for $u \in F^{\mathbb{Z}[t]}$, $f(t) \in \mathbb{Z}[t]$. But, at the same time, we obtain computational advantages which will be clear later.

Lemma 27. Let $u, v \in CR(\mathbb{Z}[t], X)$ and $u = c^{-1} * v * c$ for some $c \in R(\mathbb{Z}[t], X)$. Then for every $f(t) \in \mathbb{Z}[t]$ we have

$$u^{f(t)} = c^{-1} * v^{f(t)} * c.$$

Proof. Since u and v are cyclically reduced and $u = c^{-1} * v * c$ then $v = v_1 \circ v_2$, $u = v_2 \circ v_1$, $c = v_1$.

In view of (4) in the definition of $\mathbb{Z}[t]$ -exponentiation above it suffices to prove the lemma for $f(t) = t^n$. For $f(t) = t$ we immediately get

$$(v_2 \circ v_1)^t = v_1^{-1} * (v_1 \circ v_2)^t * v_1$$

from the definition. This implies that v^t and u^t are cyclic permutations of each other and both belong to $CR(\mathbb{Z}[t], X)$, therefore one can apply the induction on $\deg f(t)$ and the lemma follows. \square

Lemma 28. Let $u, v \in CR(\mathbb{Z}[t], X)$ and $f(t), g(t) \in \mathbb{Z}[t]$ be such that $u^{f(t)} = v^{g(t)}$. Then $[u, v]$ is defined and is equal to ε .

Proof. Since $[u, u^{f(t)}] = \varepsilon$ and $[v, v^{g(t)}] = \varepsilon$ then $[u, v^{g(t)}] = \varepsilon$ and $[v, u^{f(t)}] = \varepsilon$. Now we are going to derive the required statement from these equalities.

Observe that if $|u| = |v|$ then it follows automatically that $u = v^{\pm 1}$. Indeed, by the definition of exponents $u^{f(t)}$ and $v^{g(t)}$ have correspondingly $u^{\pm 1}$ and $v^{\pm 1}$ as initial segments. Since $u^{f(t)} = v^{g(t)}$ then initial segments of length $|u|$ in both coincide.

We can assume $|u| < |v|$ and consider $[u, v^{g(t)}] = \varepsilon$ (if $|u| > |v|$ then we consider $[v, u^{f(t)}] = \varepsilon$ and apply the same arguments). Also, $g(t) > 1$, otherwise we have nothing to prove.

Thus we have $u * v^{g(t)} = v^{g(t)} * u$. Since u and v are cyclically reduced and equal $\mathbb{Z}[t]$ -words have equal initial and terminal segments of the same length then $[u, v]$ is defined and we have two cases.

(a) Suppose $u * v = u \circ v$.

Thus, automatically we have $v * u = v \circ u$. Next, $u \circ v^{g(t)}$ and $v^{g(t)} \circ u$ have the same initial segment of length $2|v|$. So $v = u \circ v_1 = v_1 \circ v_2$ and $|u| = |v_2|$. Comparing terminal segments of $u \circ v^{g(t)}$ and $v^{g(t)} \circ u$ of length $|u|$ we have $u = v_2$ and from $u \circ v_1 = v_1 \circ u$ it follows that $[u, v] = \varepsilon$.

(b) Suppose there is a cancellation in $u * v$.

Then, from $u^{f(t)} = v^{g(t)}$ it follows that $v^{-1} = v_1^{-1} \circ u$ and so $v = u^{-1} \circ v_1$. Using the same arguments as in (a) we obtain $v = u^{-1} \circ v_1 = v_1 \circ v_2$, $|u| = |v_2|$ and $u^{-1} = v_2$. It follows immediately that $[u, v] = \varepsilon$.

\square

9.3 $\mathbb{Z}[t]$ -exponentiation on $CDR(\mathbb{Z}[t], X)$

Let $v \in CDR(\mathbb{Z}[t], X)$ have a cyclic decomposition $v = c^{-1} \circ u \circ c$ and $f(t) \in \mathbb{Z}[t]$. We define $v^{f(t)}$ as follows

$$v^{f(t)} = c^{-1} \circ u^{f(t)} \circ c. \quad (3)$$

Observe that the product above is well defined since $u^{f(t)}$ starts and ends on u if $f(t) > 0$, and starts and ends on u^{-1} if $f(t) < 0$.

Thus we have defined $\mathbb{Z}[t]$ -exponentiation function

$$\exp : CDR(\mathbb{Z}[t], X) \times \mathbb{Z}[t] \rightarrow CDR(\mathbb{Z}[t], X)$$

on the whole set $CDR(\mathbb{Z}[t], X)$.

There are other ways of defining $\mathbb{Z}[t]$ -exponentiation on $CDR(\mathbb{Z}[t], X)$ but from now on we fix the exponentiation described above.

Lemma 29. *Let $u, v \in CDR(\mathbb{Z}[t], X)$ be such that $h(u) = h(v)$ and $[u, v] = \varepsilon$. Then $[u^{f(t)}, v] = \varepsilon$ for any $f(t) \in \mathbb{Z}[t]$ provided $[u^{f(t)}, v]$ is defined.*

Proof. We can assume that either u or v is cyclically reduced. This is always possible because both elements belong to $CDR(\mathbb{Z}[t], X)$. Suppose we have $v^{-1} * u * v = u$, where u is cyclically reduced.

(a) Suppose $|u| < |v|$.

Since u is cyclically reduced either $v^{-1} * u = v^{-1} \circ u$ or $u * v = u \circ v$. Assume the former. Then v has to cancel completely in $v^{-1} * u * v$ because this product is equal to u which is cyclically reduced. So v has the form $v = u^k \circ w$, where $k < 0$ is the smallest possible and w does not have u as an initial segment. We have then

$$v^{-1} * u * v = w^{-1} * u * w = w^{-1} * (u \circ w) = u.$$

and w^{-1} cancels completely. In this case the only possibility is that $|w| < |u|$ (otherwise we have a contradiction with the choice of k) and $[u, w] = \varepsilon$. So now we reduced everything to the case (b) because clearly $[u^{f(t)}, u^k] = \varepsilon$ for any $f(t) \in \mathbb{Z}[t]$.

(b) Suppose $|u| > |v|$.

We have $v^{-1} * u * v = u$. u is cyclically reduced, moreover, u is a cyclic permutation of itself that is $v^{-1} * u * v = u$. Finally, since $[u^{f(t)}, v]$ is defined then

$$v^{-1} * u^{f(t)} * v = u^{f(t)}$$

follows from Lemma 27.

□

We summarize the properties of the exponentiation \exp in the following theorem.

Theorem 30. *The $\mathbb{Z}[t]$ -exponentiation function*

$$\exp : (u, f(t)) \mapsto u^{f(t)}$$

defined in (3) satisfies the following axioms:

- (E1) $u^1 = u, \quad u^{fg} = (u^f)^g, \quad u^{f+g} = u^f * u^g,$
- (E2) $(v^{-1} * u * v)^f = v^{-1} * u^f * v$ provided $[u, v] = \varepsilon$ and $h(u) = h(v)$, or $u = v \circ w$, or $u = w^\alpha, v = w^\beta$ for some $w \in CDR(\mathbb{Z}[t], X)$ and $\alpha, \beta \in \mathbb{Z}[t]$,
- (E3) if $[u, v] = \varepsilon$ and $u = w^\alpha, v = w^\beta$ for some $w \in CDR(\mathbb{Z}[t], X)$ and $\alpha, \beta \in \mathbb{Z}[t]$ then

$$(u * v)^f = u^f * v^f$$

Proof. Let $u \in CDR(\mathbb{Z}[t], X)$ and $\alpha, \beta \in \mathbb{Z}[t]$.

- (E1) The equalities $u^1 = u$ and $(u^f)^g = u^{fg}$ follow directly from the definition of exponentiation. We need to prove only that $u^{f+g} = u^f * u^g$. Let

$$u = c^{-1} \circ u_1^k \circ c$$

be a cyclic decomposition of u . Then

$$u^f = c^{-1} \circ (u_1^f)^k \circ c, \quad u^g = c^{-1} \circ (u_1^g)^k \circ c.$$

Now

$$u^{f+g} = c^{-1} \circ (u_1^{f+g})^k \circ c = (c^{-1} \circ (u_1^f)^k \circ c) * (c^{-1} \circ (u_1^g)^k \circ c),$$

as required.

- (E2) If $u = w^\alpha, v = w^\beta$ for some $w \in CDR(\mathbb{Z}[t], X)$ and $\alpha, \beta \in \mathbb{Z}[t]$, then the result follows from the definition of exponentiation. If $[u, v] = \varepsilon$ and $h(u) = h(v)$ then result follows from Lemma 29. If $u = v \circ w$ then the result follows from Lemma 4.
- (E3) We have $(u * v)^f = (w^{\alpha+\beta})^f = w^{(\alpha+\beta)f} = w^{\alpha f} * w^{\beta f} = (w^\alpha)^f * (w^\beta)^f = u^f * v^f$.

□

9.4 Extension of centralizers in $CDR(\mathbb{Z}[t], X)$

Below we recall some of the definitions given in [92] and state a few lemmas and theorems needed for extension of centralizers in $CDR(\mathbb{Z}[t], X)$. The proofs are exactly the same as in the case of $\mathbb{Z}[t]$ -exponentiation introduced in [92], so we only give references.

$u, v \in CDR(\mathbb{Z}[t], X)$ are called *separated* if $u^m * v^n$ is defined for any $n, m \in \mathbb{N}$ and there exists $r = r(u, v) \in \mathbb{N}$ such that for all $m, n > r$

$$u^m * v^n = u^{m-r} \circ_\delta (u^r * v^r) \circ_\delta v^{n-r}.$$

A subset $M \subseteq CDR(\mathbb{Z}[t], X)$ is called an *S-set* if any two non-commuting elements of M with cyclic centralizers are separated. For example, it is easy to see that the free group F is an *S*-subgroup of $CDR(\mathbb{Z}[t], X)$.

Next, let $M \subseteq CDR(\mathbb{Z}[t], X)$. A subset $R_M \subseteq CR(\mathbb{Z}[t], X)$ is called a set of *representatives of M* if R_M satisfies the following conditions:

- (1) R_M does not contain proper powers,
- (2) for any $u, v \in R_M$, $u \neq v^{-1}$,
- (3) for each $u \in M$ there exist $v \in R_M$, $k \in \mathbb{Z}$, $c \in R(\mathbb{Z}[t], X)$, and a cyclic permutation $\pi(v)$ of v such that

$$u = c^{-1} \circ \pi(v)^k \circ c,$$

moreover, such $v, c, k, \pi(v)$ are unique.

In [92] it was shown that a set of representatives R_M exists for any $M \subseteq CDR(\mathbb{Z}[t], X)$. Observe that R_M does not have to be a subset of M .

The next definition we also borrow from [92] but here we restate it with respect to the $\mathbb{Z}[t]$ -exponentiation we introduced in Subsection 9.3.

Let G be a subgroup of $CDR(\mathbb{Z}[t], X)$ and let

$$K(G) = \{v \in G \mid C_G(v) = \langle v \rangle\}.$$

A *Lyndon's set* of G is a set $R = R_{K(G)}$ of representatives of $K(G)$ which satisfies the following conditions:

- (1) $R \subset G$,
- (2) for any $g \in G$, $u \in R$, and $\alpha \in \mathbb{Z}[t]$ the inner product $c(u^\alpha, g)$ exists and $c(u^\alpha, g) < k|u|$ for some $k \in \mathbb{N}$,
- (3) no word from G contains a subword u^α , where $u \in R$ and $\alpha \in \mathbb{Z}[t]$ with $\deg(\alpha) > 0$.

Lemma 30. *Let G be an *S*-subgroup of $CDR(\mathbb{Z}[t], X)$ and let R be a Lyndon's set of G . If $u, v \in R^{\pm 1}$ and $g \in G$ are such that either $[u, v] \neq \varepsilon$ or $[u, g] \neq \varepsilon$ then there exists $r \in \mathbb{N}$ such that for all $m, n \geq r$ the following holds:*

$$u^m * g * v^n = u^{m-r} \circ (u^r * g * v^r) \circ v^{n-r}.$$

Proof. See the proof of Lemma 6.9 in [92]. □

Lemma 31. *Let G be an S -subgroup of $CDR(\mathbb{Z}[t], X)$ and R a Lyndon's set of G . If $u_1, \dots, u_n \in R^{\pm 1}$ and $g_1, \dots, g_{n+1} \in G$ are such that for any $i = 2, \dots, n$ either $[u_{i-1}, u_i] \neq \varepsilon$, or $[u_i, g_i] \neq \varepsilon$ then there exists $r \in \mathbb{N}$ such that*

$$\begin{aligned} & g_1 * u_1^{m_1} * g_2 * \dots * u_n^{m_n} * g_{n+1} \\ &= (g_1 * u_1^r) \circ u_1^{m_1-2r} \circ (u_1^r * g_2 * u_2^r) \circ u_2^{m_2-2r} \circ \dots \circ u_n^{m_n-2r} \circ (u_n^r * g_{n+1}) \end{aligned}$$

for all $m_i \in \mathbb{N}$, $m_i > 2r$, $i \in [1, n]$.

Proof. See the proof of Lemma 6.10 in [92]. \square

Let G be a subgroup of $CDR(\mathbb{Z}[t], X)$ with a Lyndon's set R . A sequence

$$p = (g_1, u_1^{\alpha_1}, g_2, \dots, g_n, u_n^{\alpha_n}, g_{n+1}), \quad (4)$$

where $g_i \in G$, $u_i \in R$, $\alpha_i \in \mathbb{Z}[t]$, $n \geq 1$ is called an R -form over G . An R -form (4) is *reduced* if $\deg(\alpha_i) > 0$, $i \in [1, n]$, and if $u_i = u_{i-1}$ then $[u_i, g_i] \neq \varepsilon$.

Denote by $\mathcal{P}(G, R)$ the set of all R -forms over G . We define a partial function $w : \mathcal{P}(G, R) \rightarrow R(\mathbb{Z}[t], X)$ as follows. If

$$p = (g_1, u_1^{\alpha_1}, g_2, \dots, g_n, u_n^{\alpha_n}, g_{n+1})$$

then

$$w(p) = (\dots (g_1 * u_1^{\alpha_1}) * g_2) * \dots * g_n * u_n^{\alpha_n} * g_{n+1}$$

if it is defined.

An R -form $p = (g_1, u_1^{\alpha_1}, g_2, \dots, g_n, u_n^{\alpha_n}, g_{n+1})$ over G is called *normal* if it is reduced and the following conditions hold:

- (1) $w(p) = g_1 \circ u_1^{\alpha_1} \circ g_2 \circ \dots \circ g_n \circ u_n^{\alpha_n} \circ g_{n+1}$,
- (2) g_i does not have $u_i^{\pm 1}$ as a terminal segment for any $i \in [1, n]$ and $g_i \circ u_i^{\alpha_i}$ does not have $u_{i-1}^{\pm 1}$ as an initial segment for any $i \in [2, n]$.

Lemma 32. *Let G be an S -subgroup of $CDR(\mathbb{Z}[t], X)$ with a Lyndon's set R . Then for every R -form p over G the following holds:*

- (1) the product $w(p)$ is defined and it does not depend on the placement of parentheses,
- (2) there exists a reduced R -form q over G such that $w(q) = w(p)$,
- (3) there exists a unique normal R -form q over G such that $w(p) = w(q)$,
- (4) $w(p) \in CDR(\mathbb{Z}[t], X)$.

Proof. See the proof of Lemma 6.13 in [92]. \square

Let G and R be as above. By Lemma 32 for every $g, h \in G$, $u \in R$, $\alpha \in \mathbb{Z}[t]$ the product $g * u^\alpha * h$ is defined and belongs to $CDR(\mathbb{Z}[t], X)$. Put

$$P = P(G, R) = \{g * u^\alpha * h \mid g, h \in G, u \in R, \alpha \in \mathbb{Z}[t]\}.$$

Multiplication $*$ induces a partial multiplication (which we again denote by $*$) on P so that for $p, q \in P$ the product $p * q$ is defined in P if and only if $p * q$ is defined in $R(\mathbb{Z}[t], X)$ and $p * q \in P$. Now we are ready to prove the main technical result of this subsection.

Lemma 33. *Let G be an S -subgroup of $CDR(\mathbb{Z}[t], X)$ and let R be a Lyndon's set for G . Then the set P forms a pregroup with respect to the multiplication $*$.*

Proof. See the proof of Proposition 6.14 in [92]. \square

The next two results reveal the structure of the universal group $U(P)$ of P .

Theorem 31. *P generates a subgroup $\langle P \rangle$ in $CDR(\mathbb{Z}[t], X)$, which is isomorphic to $U(P)$.*

Proof. See the proof of Theorem 6.15 in [92]. \square

Let $R = \{c_i \mid i \in I\}$. Put $S = \{s_{i,j} \mid i \in I, j \in \mathbb{N}\}$. Then the group

$$G(R, S) = \langle G, S \mid [c_i, s_{i,j}] = [s_{i,j}, s_{k,j}] = 1, i \in I, j, k \in \mathbb{N} \rangle$$

is an extension of all cyclic centralizers of G by a direct sum of countably many copies of an infinite cyclic group. Sometimes, we will refer to $G(R, S)$ as an extension of all cyclic centralizers of G by $\mathbb{Z}[t]$.

Theorem 32. $\langle P \rangle \simeq G(R, S)$.

Proof. Define $\phi : P \rightarrow G(R, S)$ as follows. Let $g_i * c_i^\alpha * h_i \in P$ and $\alpha = a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0$. Put

$$g_i * c_i^\alpha * h_i \xrightarrow{\phi} g_i s_{i,n}^{a_n} s_{i,n-1}^{a_{n-1}} \dots s_{i,1}^{a_1} c_i^{a_0} h_i.$$

It is easy to see that ϕ is a morphism of pregroups. Since $\langle P \rangle \simeq U(P)$, the morphism ϕ extends to a unique homomorphism $\psi : \langle P \rangle \rightarrow G(R, S)$. We claim that ψ is bijective. Indeed, observe first that $G(R, S)$ is generated by $G \cup S$. Now, since $\psi(c_i^{t^j}) = s_{i,j}$ and ψ is identical on G , it follows that ψ is onto. To see that ψ is one-to-one it suffices to notice that if

$$y = (g_1 * c_1^{\alpha_1} * h_1, g_2 * c_2^{\alpha_2} * h_2, \dots, g_m * c_m^{\alpha_m} * h_m).$$

is a reduced R -form then $y^\psi \neq 1$ by Britton's Lemma (see, for example, [79]). This proves that ψ is an isomorphism, as required. \square

Lemma 34. *If G is subwords-closed then so is $\langle P \rangle$.*

Proof. See the proof of Lemma 6.18 in [92]. \square

Lemma 35. *If G is subwords-closed then H is an S -subgroup.*

Proof. See the proof of Lemma 6.19 in [92]. \square

9.5 Embedding of $F^{\mathbb{Z}[t]}$ into $CDR(\mathbb{Z}[t], X)$

Let F be a free non-abelian group. Recall that one can view the group $F^{\mathbb{Z}[t]}$ as a union of the following infinite chain of groups:

$$F = G_0 < G_1 < G_2 < \cdots < G_n < \cdots, \quad (5)$$

where G_n is obtained from G_{n-1} by extension of all cyclic centralizers of G_{n-1} .

For each $n \in \mathbb{N}$ we construct by induction an embedding

$$\psi_n : G_n \rightarrow CDR(\mathbb{Z}[t], X)$$

such that ψ_{n-1} is the restriction of ψ_n to G_{n-1} . To this end, let H_0 be the set of all words of finite length in $CDR(\mathbb{Z}[t], X)$. Clearly, $F = H_0$. We denote by $\psi_0 : F \rightarrow H_0$ the identity isomorphism. It is obvious that H_0 is subwords-closed. Moreover, H_0 is an S -subgroup and it has a Lyndon's set.

Suppose by induction that there exists an embedding

$$\psi_{n-1} : G_{n-1} \rightarrow CDR(\mathbb{Z}[t], X)$$

such that the image $H_{n-1} = \psi_{n-1}(G_{n-1})$ is an S -subgroup, it is subwords-closed, and there exists a Lyndon's set, say R_{n-1} , in H_{n-1} . Then by Proposition 33 and Theorem 31, there exists an embedding $\psi_n : G_n \rightarrow CDR(\mathbb{Z}[t], X)$. Moreover, in this case, the image $H_n = \psi_n(G_n)$ is the subgroup of $CDR(\mathbb{Z}[t], X)$ generated by the pregroup

$$P(H_{n-1}, R_{n-1}) = \{f * u^\alpha * h \mid f, h \in H_{n-1}, u \in R_{n-1}, \alpha \in \mathbb{Z}[t]\}.$$

Notice that by Lemma 35, the group H_n is an S -subgroup of $CDR(\mathbb{Z}[t], X)$, and by Lemma 34, H_n is subwords-closed. So to finish the proof one needs to show that H_n has a Lyndon's set.

Lemma 36. *Let H_{n-1} from the series (5) be a subwords-closed S -subgroup of $CDR(\mathbb{Z}[t], X)$ with a Lyndon's set R_{n-1} . Then there exists a Lyndon's set R_n in H_n .*

Proof. Recall that $K = K(H_n) \subset H_n$ is the subset consisting of all elements $v \in H_n$ such that $C_{H_n}(v) = \langle v \rangle$. Denote by R a set of representatives for K .

Since H_n is subwords-closed then we may assume that $R \subset H_n$. The same argument shows that an element $f \in H_n$ does not contain a subword u^α , where $u \in R$ and $\alpha \in \mathbb{Z}[t]$ is infinite. Indeed, in this case it would imply that $u^\alpha \in H_n$, hence, $[u^\alpha, u] = \varepsilon$, so the centralizer of u in H_n is not cyclic – a contradiction with $u \in R$. Finally, let $u \in R$, $g \in H_n$. Observe that $u \notin H_{n-1}$, so u has a unique normal form

$$u = f_1 \circ u_1^{\alpha_1} \circ f_2 \circ \cdots \circ u_k^{\alpha_k} \circ f_{k+1},$$

where $f_i \in H_{n-1}$, $u_i \in R_{n-1}$, and $\alpha_i \in \mathbb{Z}[t]$ is infinite for any $i \in [1, k]$. If $g \in H_{n-1}$ then

$$(g * u^m) * u = (g * u^m) \circ u, \quad u * (u^m * g) = u \circ (u^m * g) \quad (6)$$

holds for $m = 1$, since R_{n-1} is a Lyndon's set for H_{n-1} . If $g \notin H_{n-1}$ then

$$g = g_1 \circ v_1^{\beta_1} \circ g_2 \circ \dots \circ v_l^{\beta_l} \circ g_{p+1},$$

where $g_j \in H_{n-1}$, $v_j \in R_{n-1}$ and $\beta_j \in \mathbb{Z}[t]$ is infinite for any $j \in [1, p]$. In this case (6) holds for any $m > p$.

It follows that the set $R_n = R$ is a Lyndon's set for H_n . \square

9.6 Limit groups embed into $F^{\mathbb{Z}[t]}$

The following results illustrate the connection of limit groups and finitely generated subgroups of $F^{\mathbb{Z}[t]}$.

Theorem 33. [64] *Given a finite presentation of a finitely generated fully residually free group G one can effectively construct an embedding $\phi : G \rightarrow F^{\mathbb{Z}[t]}$ (by specifying the images of the generators of G).*

Combining Theorem 33 with the result on the representation of $F^{\mathbb{Z}[t]}$ as a union of a sequence of extensions of centralizers one can get the following theorem.

Theorem 34. [61] *Given a finite presentation of a finitely generated fully residually free group G one can effectively construct a finite sequence of extension of centralizers*

$$F < G_1 < \dots < G_n,$$

where G_{i+1} is an extension of the centralizer of some element $u_i \in G_i$ by an infinite cyclic group \mathbb{Z} , and an embedding $\psi^* : G \rightarrow G_n$ (by specifying the images of the generators of G).

Now Theorem 34 implies the following important corollaries.

Corollary 6. [61] *For every freely indecomposable non-abelian finitely generated fully residually free group one can effectively find a non-trivial splitting (as an amalgamated product or HNN extension) over a cyclic subgroup.*

Corollary 7. [61] *Every finitely generated fully residually free group is finitely presented. There is an algorithm that, given a presentation of a finitely generated fully residually free group G and generators of the subgroup H , finds a finite presentation for H .*

Corollary 8. [61] *Every finitely generated residually free group G is a subgroup of a direct product of finitely many fully residually free groups; hence, G is embeddable into $F^{\mathbb{Z}[t]} \times \dots \times F^{\mathbb{Z}[t]}$. If G is given as a coordinate group of a finite system of equations, then this embedding can be found effectively.*

Let K be an HNN-extension of a group G with associated subgroups A and B . K is called a separated HNN-extension if for any $g \in G$, $A^g \cap B = 1$.

Corollary 9. [61] *Let a group G be obtained from a free group F by finitely many centralizer extensions. Then every finitely generated subgroup H of G can be obtained from free abelian groups of finite rank by finitely many operations of the following type: free products, free products with abelian amalgamated subgroups at least one of which is a maximal abelian subgroup in its factor, free extensions of centralizers, separated HNN-extensions with abelian associated subgroups at least one of which is maximal.*

Corollary 10. [61, 48] *One can enumerate all finite presentations of finitely generated fully residually free groups.*

Corollary 11. [64] *Every finitely generated fully residually free group acts freely on some \mathbb{Z}^n -tree with lexicographic order for a suitable n .*

Finally, combining Theorem 33 with the result on the effective embedding of $F^{\mathbb{Z}[t]}$ into $R(\mathbb{Z}[t], X)$ obtained in [92] one can get the following theorem.

Theorem 35. *Given a finite presentation of a finitely generated fully residually free group G one can effectively construct an embedding $\psi : G \rightarrow R(\mathbb{Z}[t], X)$ (by specifying the images of the generators of G).*

9.7 Algorithmic problems for limit groups

Existence of a free length function on $F^{\mathbb{Z}[t]}$ becomes a very powerful tool in solving various algorithmic problems for subgroups of $F^{\mathbb{Z}[t]}$ (which are precisely limit groups). Using the length function one can introduce unique normal forms for elements of $F^{\mathbb{Z}[t]}$ and then work with them pretty much in the same way as in free groups. In the seminal paper [124] J. Stallings introduced an extremely useful notion of a folding of graphs and initiated the study of subgroups (and automorphisms) of free groups via folded directed labeled graphs. This approach turned out to be very influential and allowed researches to prove many new results and simplify old proofs (see [56]). In [93] Stallings techniques were generalized in order to effectively solve the Membership Problem for finitely generated subgroups of $F^{\mathbb{Z}[t]}$ and this result can be reformulated for limit groups.

Theorem 36. [93] *Let G be a limit group and $G \hookrightarrow F^{\mathbb{Z}[t]}$ the effective embedding. For any f.g. subgroup $H \leq G$ one can effectively construct a finite labeled graph Γ_H that in the group $F^{\mathbb{Z}[t]}$ accepts precisely the normal forms of elements from H .*

Now, that the graph Γ_H is constructed for H , one can use it to solve a lot of algorithmic problems almost as in free groups. Recall that limit groups are finitely presented.

Theorem 37. *Let G be a limit group given by a finite presentation with additional information that it is a limit group or given as a subgroup of $F^{\mathbb{Z}[t]}$, and H, K f.g. subgroups of G given by their generators. Then*

- $H \cap K$ is f.g (Howson Property) and can be found effectively ([65]),

- up to conjugation by elements from K there are only finitely many subgroups of G of the type $H^g \cap K$, where $g \in G$ ([65]),
- it can be decided effectively if H is malnormal in G ([65]),
- it can be decided if $H^g = K$ (and $H^g \leq K$) for some $g \in G$ and if yes such g can be found effectively ([65]),
- for any $g \in G$, its centralizer $C_G(g)$ in G can be found effectively ([65]),
- homological and cohomological dimensions of G can be computed effectively ([66]),
- it can be decided if $|G : H| < \infty$ ([97]),
- $\text{Comm}_G(H)$ can be found effectively, hence it is possible to find effectively $n(H) \in \mathbb{N}$ such that for any $P \leq G$ if $|P : H| < \infty$ then $|P : H| < n(H)$ ([97]).

The following theorems use the elimination process for limit groups. For a limit group G , two homomorphisms $\phi_i : G \rightarrow F$, $i = 1, 2$ are automorphically equivalent if ϕ_1 is obtained by pre-composing ϕ_2 with an automorphism of G and post-composing with conjugation. The quotient group of G over the intersection of the kernels of all homomorphisms minimal in their equivalence classes is called the maximal standard quotient of G (or the shortening quotient).

Theorem 38. ([61, Theorem 13.1], [62, Theorem 35]) *Let G be a freely indecomposable limit group and K_1, \dots, K_m be finitely generated subgroups of G . There exists an algorithm to obtain an abelian JSJ-decomposition of G with subgroups K_1, \dots, K_m being elliptic, and to find the maximal standard quotient (or shortening quotient) with respect to this decomposition.*

Theorem 39. [17] *Let $G \cong \langle S_G \mid \mathcal{R}_G \rangle$ and $H \cong \langle S_H \mid \mathcal{R}_H \rangle$ be finite presentations of limit groups. There exists an algorithm that determines whether or not G and H are isomorphic. If the groups are isomorphic, then the algorithm finds an isomorphism $G \rightarrow H$.*

Theorem 40. [61] *The universal theory of a limit group G in the language with coefficients from G is decidable (in the language without coefficients the universal theory of G is the same as the universal theory of a free group [108]).*

10 \mathbb{Z}^n -free groups

In this section we consider in detail the situation when $\Lambda = \mathbb{Z}^n$ with the right lexicographic order. The structure of \mathbb{Z}^n -free groups is very clear and the machinery of infinite words plays a significant role in all proofs.

10.1 Complete \mathbb{Z}^n -free groups

In this section we fix a finitely generated group G which has a free regular length function with values in \mathbb{Z}^n , $n \in \mathbb{Z}$ (with the right lexicographic order). In other words, G is a *complete* \mathbb{Z}^n -free group (see [67]). Due to Theorem 27 we may and will view G as a subgroup of $CDR(\mathbb{Z}^n, X)$ for an appropriate set X . Therefore, elements of G are infinite words from $CDR(\mathbb{Z}^n, X)$, multiplication in G is the multiplication “ $*$ ” of infinite words, and the regular length function is the standard length $|\cdot|$ of infinite words. That is, G is complete in the sense that it is closed under the operation of taking common initial subwords of its elements.

In the ordered group $\mathbb{Z}^n = \langle a_1 \rangle \oplus \dots \oplus \langle a_n \rangle$ with basis a_1, \dots, a_n the subgroups $E_k = \langle a_1, \dots, a_k \rangle$ are convex, and every non-trivial convex subgroup is equal to E_k for some k , so

$$0 = E_0 < E_1 < \dots < E_n,$$

is the complete chain of convex subgroups of \mathbb{Z}^n . Recall, that the height $ht(g)$ of a word $g \in G$ is equal to k if $|g| \in E_k - E_{k-1}$ (see [58]). Since $|g * h| \leq |g| + |h|$ and $|g^{-1}| = |g|$ one has for any $f, g \in G$:

1. $ht(f * g) \leq \max\{ht(f), ht(g)\}$,
2. $ht(g) = ht(g^{-1})$.

We will assume that there is an element $g \in G$ with $ht(g) = n$, otherwise, the length function on G has values in \mathbb{Z}^{n-1} , in which case we replace \mathbb{Z}^n with \mathbb{Z}^{n-1} . For any $k \in [1, n]$

$$G_k = \{g \in G \mid ht(g) \leq k\}$$

is a subgroup of G and

$$1 = G_0 < G_1 < \dots < G_n = G.$$

Observe, that if $l : G \rightarrow \Lambda$ is a Lyndon length function with values in some ordered abelian group Λ and $\mu : \Lambda \rightarrow \Lambda'$ is a homomorphism of ordered abelian groups then the composition $l' = \mu \circ l$ gives a Lyndon length function $l' : G \rightarrow \Lambda'$. In particular, since E_{n-1} is a convex subgroup of \mathbb{Z}^n then the canonical projection $\pi_n : \mathbb{Z}^n \rightarrow \mathbb{Z}$ such that $\pi_n(x_1, x_2, \dots, x_n) = x_n$ is an ordered homomorphism, so the composition $\pi_n \circ |\cdot|$ gives a Lyndon length function $\lambda : G \rightarrow \mathbb{Z}$ such that $\lambda(g) = \pi_n(|g|)$. Notice also that if $u = g \circ h$ then $\lambda(u) = \lambda(g) + \lambda(h)$ for any $g, h, u \in G$.

All the proofs and details can be found in [67].

10.1.1 Elementary transformations of infinite words

In this section we describe an analog of Nielsen reduction in the group G . Since G is complete the Nielsen reduced sets have much stronger non-cancelation properties than usual and the transformations are simpler. On the other hand, since

\mathbb{Z}^n is non-Archimedean (for $n > 1$) the reduction process is more cumbersome, it goes in stages along the complete series of convex subgroups in \mathbb{Z}^n .

For a finite subset Y of G define its λ -length as

$$|Y|_\lambda = \sum_{g \in Y} \lambda(g).$$

If Y is a generating set of G then $|Y|_\lambda > 0$, otherwise $G = G_{n-1}$. It follows that

$$Y = Y_+ \cup Y_0,$$

where

$$Y_+ = \{g \in Y \mid \lambda(g) > 0\}, \quad Y_0 = \{g \in Y \mid \lambda(g) = 0\}.$$

Obviously, $|Y|_\lambda = |Y_+|_\lambda$ and $\langle Y_0 \rangle$ is a finitely generated subgroup of G_{n-1} .

Let Y be a finite generating set for G . Assuming $Y = Y^{-1}$ we define three types of elementary transformations of Y .

Transformation μ . Let $f, g \in Y_+$, $f \neq g$, $h \in \langle Y_0 \rangle$, $u = \text{com}(f, h * g)$, and $\lambda(u) > 0$. Then $f = u \circ w_1$, $h * g = u \circ w_2$ for some u, w_1, w_2 from G (since G is complete). Put

$$\begin{aligned} \mu_{f,g,h}(Y) &= (Y - \{f^{\pm 1}, g^{\pm 1}\}) \bigcup \{w_1^{\pm 1}, w_2^{\pm 1}, u^{\pm 1}\} \\ &\quad \bigcup \{(f^{-1} * h * g)^{\pm 1} \mid \text{if } \lambda(f^{-1} * h * g) = 0\} \end{aligned}$$

if $f \neq g^{-1}$, and

$$\mu_{f,g,h}(Y) = (Y - \{g^{\pm 1}\}) \bigcup \{u^{\pm 1}, (w_2 * u)^{\pm 1}\}$$

if $f = g^{-1}$.

Lemma 37. [67] *In the notation above*

$$(1) \quad \langle Y \rangle = \langle \mu_{f,g,h}(Y) \rangle,$$

$$(2) \quad |\mu_{f,g,h}(Y)|_\lambda < |Y|_\lambda.$$

Transformation η . Let $f \in Y_+$ be such that $\lambda(f) > \lambda(\text{com}(f, h * f)) > 0$ for some $h \in \langle Y_0 \rangle$. Then $f = u \circ f_1$, $h * f = u \circ f_2$, $\lambda(u) > 0$. Define

$$\eta_{f,h}(Y) = (Y - \{f^{\pm 1}\}) \bigcup \{f_1^{\pm 1}, u^{\pm 1}, (u^{-1} * h * u)^{\pm 1}\}.$$

Notice, that $f = (h^{-1} * u) \circ f_2 = u \circ f_1$ hence $\lambda(f_2) = \lambda(f_1) > 0$. On the other hand $f_2 = (u^{-1} * h * u) * f_1$ and it follows that $\lambda(u^{-1} * h * u) = 0$.

Lemma 38. [67] *In the notation above*

$$(1) \quad \langle Y \rangle = \langle \eta_{f,h}(Y) \rangle,$$

(2) either $|\eta_{f,h}(Y)|_\lambda < |Y|_\lambda$, or $|\eta_{f,h}(Y)|_\lambda = |Y|_\lambda$ but then $|\eta_{f,h}(Y)_+| > |Y_+|$.

Transformation ν . Let $f \in Y_+$ be not cyclically reduced. Then $f = c^{-1} \circ \bar{f} \circ c$, where $c \neq 1$ and \bar{f} is cyclically reduced. In this case $c^{-1} = \text{com}(f, f^{-1})$, hence (since G is complete) $c, \bar{f} \in G$. Put

$$\nu_f(Y) = (Y - \{f^{\pm 1}\}) \bigcup \{c^{\pm 1}, \bar{f}^{\pm 1}\}.$$

Lemma 39. [67] *In the notation above*

(1) $\langle Y \rangle = \langle \nu_f(Y) \rangle$,

(2) either $|\nu_f(Y)|_\lambda < |Y|_\lambda$, or $|\nu_f(Y)|_\lambda = |Y|_\lambda$ but then $|\nu_f(Y)_+| = |Y_+|$.

We write $Y \rightarrow Y'$ ($Y \rightarrow^* Y'$) if Y' is obtained from Y by a single (finitely many) elementary transformation, that is, \rightarrow^* is the transitive closure of the relation \rightarrow . We call a generating set Y of G *transformation-reduced* if none of the transformations μ, η, ν can be applied to Y . Recall that the binary relation \rightarrow^* is called *terminating* if there is no an infinite sequence of finite subsets $Y_i, i \in \mathbb{N}$, of G such that $Y_i \rightarrow Y_{i+1}$ for every $i \in \mathbb{N}$, i.e., every rewriting system $Y_1 \rightarrow Y_2 \rightarrow \dots$ is finite. We say that \rightarrow^* is *uniformly terminating* if for every finite set Y of G there is a natural number n_Y such that every rewriting system starting at Y terminates in at most n_Y steps.

Proposition 7. [67] *The following hold:*

(1) *The relation \rightarrow^* is uniformly terminating. Moreover, for any finite subset Y of G one has $n_Y \leq (|Y|_\lambda)^3$. In particular, there exists a finite transformation-reduced $Z \subset G$ which can be obtained from Y in not more than $(|Y|_\lambda)^3$ steps.*

(2) *If Z is a transformation-reduced finite subset of G then:*

- (a) *all elements of Z_+ are cyclically reduced;*
- (b) *if $f, g \in Z_+^{\pm 1}$, $f \neq g$ then $\lambda(\text{com}(f, h * g)) = 0$ for any $h \in \langle Z_0 \rangle$;*
- (c) *if $f \in Z_+^{\pm 1}$ and $\lambda(\text{com}(f, h * f)) > 0$ for some $h \in \langle Z_0 \rangle$ then $\lambda(\text{com}(f, h * f)) = \lambda(f)$.*

(3) *If Z is a transformation-reduced finite subset of G then one can add to Z finitely many elements $h_1, \dots, h_m \in G_{n-1}$ such that $T = Z \cup \{h_1, \dots, h_m\}$ is transformation-reduced and satisfies the following condition*

- (d) *if $f \in T_+^{\pm 1}$ and $\lambda(\text{com}(f, h * f)) > 0$ for some $h \in \langle T_0 \rangle$ then $\lambda(\text{com}(f, h * f)) = \lambda(f)$ and $f^{-1} * h * f \in \langle T_0 \rangle$.*

A finite set Y of G is called *reduced* if it satisfies the conditions (a) - (d) from Proposition 7.

10.1.2 Minimal sets of generators and pregroups

Let Z be a finite reduced generating set of G . Put

$$P_Z = \{g * f * h \mid f \in Z_+^{\pm 1}, g, h \in \langle Z_0 \rangle\} \cup \langle Z_0 \rangle.$$

Multiplication $*$ induces a partial multiplication (which we again denote by $*$) on P_Z so that for $p, q \in P_Z$ the product $p * q$ is defined in P_Z if and only if $p * q \in P_Z$. Notice, that P_Z is closed under inversion.

Lemma 40. [67] *Let $x = h_1(x) * f_x * h_2(x)$, $y = h_1(y) * f_y * h_2(y) \in P_Z$, where $h_i(x), h_i(y) \in \langle Z_0 \rangle$, $i = 1, 2$ and $f_x, f_y \in Z_+^{\pm 1}$. Then $x * y \in P_Z$ if and only if $f_x = f_y^{-1}$ and $f_x * (h_2(x) * h_1(y)) * f_x^{-1} \in \langle Z_0 \rangle$.*

Now we are ready to prove the main technical result of this section.

Theorem 41. [67] *Let G be a finitely generated complete \mathbb{Z}^n -free group. Then:*

- (1) P_Z forms a pregroup with respect to the multiplication $*$ and inversion;
- (2) the inclusion $P_Z \rightarrow G$ extends to the group isomorphism $U(P_Z) \rightarrow G$, where $U(P)$ is the universal group of P_Z ;
- (3) if (g_1, \dots, g_k) is a reduced P_Z -sequence for an element $g \in G$ then

$$\lambda(g) = \sum_{i=1}^k \lambda(g_i).$$

Proof. Observe that $P_Z = P_Z^{-1} \subset G$ generates G and every $g \in G$ corresponds to a finite reduced P_Z -sequence

$$(u_1, u_2, \dots, u_k),$$

where $u_i \in P_Z$, $i \in [1, k]$, $u_i * u_{i+1} \notin P_Z$, $i \in [1, k-1]$ and $g = u_1 * u_2 * \dots * u_k$ in G . By Theorem 2, [110], to prove that P_Z is a pregroup and the inclusion $P_Z \rightarrow G$ extends to the isomorphism $U(P_Z) \rightarrow G$ it is enough to show that all reduced P_Z -sequences representing the same element have the same P_Z -length.

Suppose two reduced P_Z -sequences

$$(u_1, u_2, \dots, u_k), (v_1, v_2, \dots, v_n)$$

represent the same element $g \in G$. That is,

$$(u_1 * \dots * u_k) * (v_1 * \dots * v_n)^{-1} = \varepsilon.$$

We use induction on $k + n$ to show that $k = n$. If the P_Z -sequence

$$(u_1, \dots, u_k, v_n^{-1}, \dots, v_1^{-1})$$

is reduced then

$$u_1 * \dots * u_k * v_n^{-1} * \dots * v_1^{-1} \neq \varepsilon$$

because Z is a reduced set. Hence,

$$(u_1, \dots, u_k, v_n^{-1}, \dots, v_1^{-1})$$

is not reduced and $u_k * v_n^{-1} \in P_Z$. If $u_k = h_1 * f_1 * g_1$, $v_n = h_2 * f_2 * g_2$, where $h_i, g_i \in \langle Z_0 \rangle$ and $f_i \in Z_+^{\pm 1}$, $i = 1, 2$ then by Lemma 40 $f_1 = f_2$ and $f_1 * (g_1 * g_2^{-1}) * f_2^{-1} = c \in \langle Z_0 \rangle$. It follows that

$$(u_1, u_2, \dots, u_{k-1} * (h_1 * c * h_2^{-1})), (v_1, v_2, \dots, v_{n-1})$$

represent the same element $g * v_n^{-1} \in G$ and the sum of their lengths is less than $k + n$, so the result follows by induction. Hence, (1) and (2) follow.

Finally we prove (3).

If $g_i = h_1(g_i) * f_{g_i} * h_2(g_i)$, $i \in [1, k]$ then $\lambda(g_i) = \lambda(f_{g_i})$ because $\lambda(h_1(g_i)) = \lambda(h_2(g_i)) = 0$. On the other hand, since Z is reduced and (g_1, \dots, g_k) is a reduced P_Z -sequence then $\lambda(\text{com}(g_i^{-1}, g_{i+1})) = 0$ for $i \in [1, k-1]$. In other words $\lambda(g_i * g_{i+1}) = \lambda(g_i) + \lambda(g_{i+1})$ and the result follows. \square

Corollary 12. [67] $G_{n-1} = \langle Z_0 \rangle$.

10.1.3 Algebraic structure of complete \mathbb{Z}^n -free groups

Theorem 42. [67] Let G be a finitely generated complete \mathbb{Z}^n -free group and let Z be a reduced generating set for G . Then G has the following presentation

$$G = \langle H, Y \mid t_i^{-1} C_H(u_{t_i}) t_i = C_H(v_{t_i}), t_i \in Y^{\pm 1} \rangle,$$

where $Y = Z_+$ is finite, $H = G_{n-1} = \langle Z_0 \rangle$ is finitely generated and $C_H(u_{t_i})$ $C_H(v_{t_i})$ are either trivial or finitely generated free abelian subgroups of H . Moreover, H has a regular free Lyndon length function in \mathbb{Z}^{n-1} .

Proof. From Theorem 41 it follows that $G = U(P_Z)$, where

$$P_Z = \{g * f * h \mid f \in Z_+^{\pm 1}, g, h \in \langle Z_0 \rangle\} \cup \langle Z_0 \rangle.$$

It follows that every element g of G can be represented as a reduced P_Z -sequence $g = (g_1, \dots, g_k)$, where $g_i \in P_Z - \langle Z_0 \rangle$ for any $i \in [1, k]$ and $g_i * g_{i+1} \notin P_Z$ for any $i \in [1, k-1]$ if $k > 1$ (if $k = 1$ then g_1 may be in $\langle Z_0 \rangle$). In fact (see [111]), we have

$$G = U(G) = \langle P_Z \mid xy = z, (x, y, z \in P_Z \text{ and } x * y = z) \rangle.$$

Denote $H = \langle Z_0 \rangle$ and $Y = Z_+$. By Corollary 12 we have $H = G_{n-1}$.

At first observe that P_Z is infinite but for each $p \in P_Z$ either $p \in H$ or $p = h_1(p) * f_p * h_2(p)$, where $f_p \in Y^{\pm 1}$ and $h_i(p) \in H$, $i = 1, 2$. Hence, every $p \in P_Z$ can be rewritten in terms of $Y^{\pm 1}$ and finitely many generators of H . On the other hand, if $x, y, z \in P_Z$ and $x * y = z$ then one of the three of them is in H and without loss of generality we can assume $z \in H$. Hence, either x, y are in H too, or $x, y \notin H$ and assuming $x = h_1(x) * f_x * h_2(x)$, $y = h_1(y) * f_y * h_2(y)$, where

$h_i(x), h_i(y) \in H$, $i = 1, 2$, $f_x, f_y \in Y^{\pm 1}$ by Lemma 40 we get $f_x = f_y^{-1}$ and $f_x * (h_2(x) * h_1(y)) * f_x^{-1} \in H$. Hence, every relator $xy = z$, where $x, y, z \in P_Z$ and $x * y = z$ can be rewritten as

$$f_x * u_{x,y} * f_x^{-1} = v_{x,y,z},$$

where $f_x \in Y^{\pm 1}$ and $u_{x,y}, v_{x,y,z} \in H$. By Lemma 16, for each $q \in Y^{\pm 1}$ there exists $u_q \in H$ such that for each $u \in H$ we have $q * u * q^{-1} \in H$ if and only if $u \in C_H(u_q)$. Since Z is reduced it follows that for each $q \in Y^{\pm 1}$ both $C_H(u_q)$ and $q * C_H(u_q) * q^{-1}$ are in H and also note that $q * C_H(u_q) * q^{-1}$ is a centralizer of some element in H . Hence, every

$$f_x * u_{x,y} * f_x^{-1} = v_{x,y,z},$$

is a consequence of

$$f_x * C_H(u_x) * f_x^{-1} = C_H(v_x),$$

where u_x, v_x depend only on f_x . Thus,

$$G = \langle Y, H \mid t_i^{-1} C_H(u_{t_i}) t_i = C_H(v_{t_i}), t_i \in Y^{\pm 1} \rangle,$$

where Y is finite, H is finitely generated and $C_H(u_y), C_H(v_y)$ are finitely generated abelian (see Proposition 3).

Finally, we have to show that H has a regular free Lyndon length function in \mathbb{Z}^{n-1} . Indeed, since $H = G_{n-1} < G$ then the free Lyndon length function with values in \mathbb{Z}^{n-1} is automatically induced on H . We just have to check if it is regular.

Take $g, h \in H$ and consider $\text{com}(g, h)$. Since the length function on G is regular then $\text{com}(g, h) \in G = U(P_Z)$ and $\text{com}(g, h)$ can be represented by the reduced P_Z -sequence (g_1, \dots, g_k) . By Theorem 41, (3) it follows that

$$\lambda(\text{com}(g, h)) = \sum_{i=1}^k \lambda(g_i).$$

But if $\lambda(\text{com}(g, h)) > 0$ then $\lambda(g), \lambda(h) > 0$ - contradiction with the choice of g and h . Hence, $\lambda(g_i) = 0$, $i \in [1, k]$ and it follows that $k = 1$. Thus, $\text{com}(g, h) = g_1 \in H$. This completes the proof of the theorem. \square

Theorem 43. [67] *Let G be a finitely generated complete \mathbb{Z}^n -free group. Then G can be represented as a union of a finite series of groups*

$$G_1 < G_2 < \dots < G_n = G,$$

where G_1 is a free group of finite rank, and

$$G_{i+1} = \langle G_i, s_{i,1}, \dots, s_{i,k_i} \mid s_{i,j}^{-1} C_{i,j} s_{i,j} = \phi_{i,j}(C_{i,j}) \rangle,$$

where for each $j \in [1, k_i]$, $C_{i,j}$ and $\phi_{i,j}(C_{i,j})$ are cyclically reduced centralizers of G_i , $\phi_{i,j}$ is an isomorphism, and the following conditions are satisfied:

- (1) $C_{i,j} = \langle c_1^{(i,j)}, \dots, c_{m_{i,j}}^{(i,j)} \rangle$, $\phi_{i,j}(C_{i,j}) = \langle d_1^{(i,j)}, \dots, d_{m_{i,j}}^{(i,j)} \rangle$, where $\phi_{i,j}(c_k^{(i,j)}) = d_k^{(i,j)}$, $k \in [1, m_{i,j}]$ and

$$ht(c_k^{(i,j)}) = ht(d_k^{(i,j)}) < ht(d_{k+1}^{(i,j)}) = ht(c_{k+1}^{(i,j)}), \quad k \in [1, m_{i,j} - 1],$$

$$ht(s_{i,j}) > ht(c_k^{(i,j)}),$$

- (2) $|\phi_{i,j}(w)| = |w|$ for any $w \in C_{i,j}$,
 (3) w is not conjugate to $\phi_{i,j}(w)^{-1}$ in G_i for any non-trivial $w \in C_{i,j}$,
 (4) if $A, B \in \{C_{i,1}, \phi_{i,1}(C_{i,1}), \dots, C_{i,k_i}, \phi_{i,k_i}(C_{i,k_i})\}$ then either $A = B$, or A and B are not conjugate in G_i ,
 (5) $C_{i,j}$ can appear in the list

$$\{C_{i,k}, \phi_{i,k}(C_{i,k}) \mid k \neq j\}$$

not more than twice.

Proof. Existence of the series

$$G_1 < G_2 < \dots < G_n = G,$$

where G_{i+1} , $i \in [1, n-1]$ can be obtained from G_i by finitely many HNN-extensions in which associated subgroups are maximal abelian of finite rank follows by induction applying Theorem 42. Also, observe that G_1 has a free length function with values in \mathbb{Z} , hence, by the result of Lyndon [77] it follows that G_1 is a free group. Moreover, G_1 is of finite rank by Theorem 42.

Now, consider G_{i+1} . By Theorem 42 we can assume that

$$G_{i+1} = \langle G_i, t_1, t_2, \dots, t_p \mid t_j^{-1} C_{G_i}(u_{t_j}) t_j = C_{G_i}(v_{t_j}) \rangle, \quad (7)$$

where

- (a) all t_j are cyclically reduced,
 (b) $G_i = \langle Y \rangle$, $ht(t_j) > ht(G_i)$ and

$$Y \cup \{t_1, t_2, \dots, t_p\}$$

is a reduced generating set for G_{i+1} .

In particular, $Y \cup \{t_1, t_2, \dots, t_p\}$ is reduced, that is, it has the properties listed in Proposition 7.

At first, we can assume that all $C_{G_i}(u_{t_j})$, $C_{G_i}(v_{t_j})$ are cyclically reduced. Indeed, if not then by Lemma 13 we have $C_{G_i}(u_{t_j}) = c^{-1} \circ B \circ c$, where B is cyclically reduced, $c \in G_i$ by regularity of the length function on G_i , and

$$(t_j^{-1} * c^{-1}) * B * (c * t_j) = C_{G_i}(v_{t_j}).$$

Thus, we can substitute t_j by $c * t_j$, $C_{G_i}(u_{t_j})$ by B , and the same can be done for $C_{G_i}(v_{t_j})$.

Observe that conjugation by t_j induces an isomorphism between $C_{G_i}(u_{t_j})$ and $C_{G_i}(v_{t_j})$, and since we can assume both centralizers to be cyclically reduced then from

$$t_j^{-1} * C_{G_i}(u_{t_j}) * t_j = C_{G_i}(v_{t_j})$$

it follows that for $a \in C_{G_i}(u_{t_j})$, $b \in C_{G_i}(v_{t_j})$ if $t_j^{-1} * a * t_j = b$ then $|a| = |b|$. In particular, if

$$C_{G_i}(u_{t_j}) = \langle c_1^{(i,j)}, \dots, c_{m_{i,j}}^{(i,j)} \rangle,$$

where we can assume $ht(c_k^{(i,j)}) < ht(c_{k+1}^{(i,j)})$ for $k \in [1, m_{i,j} - 1]$, then all $d_k^{(i,j)} = t_j^{-1} * c_k^{(i,j)} * t_j$ generate $C_{G_i}(v_{t_j})$ and $|c_k^{(i,j)}| = |d_k^{(i,j)}|$. This proves (1) and (2).

Suppose there exist $w_1 \in C_{G_i}(u_{t_j})$ and $g \in G_i$ such that $g^{-1} * w_1 * g = w_2^{-1}$, where $w_2 = \phi_i(w_1) \in C_{G_i}(v_{t_j})$. Observe that either $ht(g) \leq ht(w_1) = ht(w_2)$ and in this case w_1 is a cyclic permutation of w_2^{-1} , or $ht(g) > ht(w_1)$. In the latter case, g has any positive power of w_1^δ , $\delta \in \{1, -1\}$ as an initial subword and any positive power of $w_2^{-\delta}$ as a terminal subword. Without loss of generality we can assume $\delta = 1$. Hence, $t_j * g^{-1} = t_j \circ g^{-1}$ and

$$(t_j \circ g^{-1})^{-1} * w_1 * (t_j \circ g^{-1}) = w_1^{-1}.$$

Consider $h = com(t_j \circ g^{-1}, (t_j \circ g^{-1})^{-1})$. Observe that $h \in G_i$ and $|h^{-1} * w_1 * h| = |w_1|$. Indeed, w_1 is cyclically reduced, so either $(t_j \circ g^{-1})^{-1} * w_1 = (t_j \circ g^{-1})^{-1} \circ w_1$, or $w_1 * (t_j \circ g^{-1}) = w_1 \circ (t_j \circ g^{-1})$. Assuming the latter (the other case is similar) we have

$$(t_j \circ g^{-1})^{-1} * w_1 * (t_j \circ g^{-1}) = (t_j \circ g^{-1})^{-1} * (w_1 \circ (t_j \circ g^{-1}))$$

and from $|(t_j \circ g^{-1})^{-1} * w_1 * (t_j \circ g^{-1})| = |w_1|$ it follows that $(t_j \circ g^{-1})^{-1}$ cancels completely in the product $(t_j \circ g^{-1})^{-1} * w_1 * (t_j \circ g^{-1})$. Eventually, since h is an initial subword of $t_j \circ g^{-1}$, it follows that h^{-1} cancels completely in the product $h^{-1} * (w_1 \circ h)$, so $|h^{-1} * w_1 * h| = |w_1|$. Thus, if $w_3 = h^{-1} * w_1 * h$ then h ends with any positive power of w_3 . We have $t_j \circ g^{-1} = h \circ f \circ h^{-1}$, where f is cyclically reduced. But at the same time we have $f^{-1} * w_3 * f = w_3^{-1}$ and this produces a contradiction. Indeed, if $ht(f) \leq ht(w_3)$ then w_3 is a cyclic permutation of w_3^{-1} which is impossible. On the other hand, if $ht(f) > ht(w_3)$ then f has any power of w_3^α , $\alpha \in \{1, -1\}$ as an initial subword, and any power of $w_3^{-\alpha}$ as a terminal subword - a contradiction with the fact that f is cyclically reduced. This proves (3).

To prove (4), assume that two centralizers from the list

$$C_{G_i}(u_{t_1}), \dots, C_{G_i}(u_{t_p})$$

are conjugate in G_i . Denote $C_1 = C_{G_i}(u_{t_1})$, $C_2 = C_{G_i}(u_{t_2})$ and let $C_1 = h^{-1} * C_2 * h$ for some $h \in G_i$. Hence, in (7) every entry of C_1 can be substituted by $h^{-1} * C_2 * h$ and some of the elements t_1, t_2, \dots, t_p can be changed accordingly.

Finally, assume that there exists $t_k \neq t_j$ such that $t_k^{-1} * C_{G_i}(u_{t_j}) * t_k \leq G_i$. Suppose $c(t_j, t_k) > 0$ and denote $z = \text{com}(t_j, t_k)$. Observe that $ht(z) > ht(C_{G_i}(u_{t_j}))$. If $ht(z) < ht(t_j)$ then z conjugates $C_{G_i}(u_{t_j})$ into a cyclically reduced centralizer A of G_i and z has any positive power of some $a \in A$, $ht(a) = ht(A)$ as a terminal subword. But then $ht(z^{-1} * t_j) = ht(z^{-1} * t_k) = ht(t_j)$ and since both $z^{-1} * t_j$ and $z^{-1} * t_k$ conjugate A into a cyclically reduced centralizer of G_i it follows that $z^{-1} * t_j$ and $z^{-1} * t_k$ have $a^{\pm 1}$ as an initial subword. If $z^{-1} * t_j$ has a as an initial subword and $z^{-1} * t_k$ has a^{-1} as an initial subword then $z * (z^{-1} * t_k) \neq z \circ (z^{-1} * t_k)$, and we have a contradiction. If both $z^{-1} * t_j$ and $z^{-1} * t_k$ have a as an initial subword then z cannot be $\text{com}(t_j, t_k)$ and again we have a contradiction. Thus, $ht(z) = ht(t_j)$, but it is possible only if $t_j = t_k$ since $Y \cup \{t_1, t_2, \dots, t_p\}$ is a minimal generating set, and again we have a contradiction with our choice of t_k . It follows that $c(t_j, t_k) = 0$ and if t_j begins with $c \in C_{G_i}(u_{t_j})$ then t_k begins with c^{-1} . It also follows that there can be only one $t_k \neq t_j$ such that $t_k^{-1} * C_{G_i}(u_{t_j}) * t_k \leq G_i$.

This completes the proof of the theorem. \square

10.2 HNN-extensions of complete \mathbb{Z}^n -free groups

Let H be a finitely generated complete \mathbb{Z}^n -free group. Observe that $\mathbb{Z}^n \simeq \bigoplus_{i=0}^{n-1} \langle t^i \rangle$ which is a subgroup of $\mathbb{Z}[t]$, so we can always assume that H has a regular free length function with values in $\mathbb{Z}[t]$. On the other hand, observe that for a finitely generated complete \mathbb{Z}^n -free group there exists $n \in \mathbb{N}$ such that the length function takes values in \mathbb{Z}^n .

The main goal of this section is to prove the following result.

Theorem 44. [67] *Let H be a finitely generated complete \mathbb{Z}^n -free group. Let A and B be centralizers in H whose elements are cyclically reduced and such that there exists an isomorphism $\phi : A \rightarrow B$ with the following properties*

1. *a is not conjugate to $\phi(a)^{-1}$ in H for any $a \in A$,*
2. *$|\phi(a)| = |a|$ for any $a \in A$.*

Then the group

$$G = \langle H, z \mid z^{-1}Az = B \rangle, \quad (8)$$

is a finitely generated complete $\mathbb{Z}[t]$ -free group and the length function on G extends the one on H .

10.2.1 Cyclically reduced centralizers and attached elements

From Theorem 43, H is union of the chain

$$F(X) = H_1 < H_2 < \dots < H_n = H,$$

where

$$H_{i+1} = \langle H_i, s_{i,1}, \dots, s_{i,k_i} \mid s_{i,j}^{-1} C_{i,j} s_{i,j} = D_{i,j} \rangle,$$

$C_{i,j}$, $D_{i,j}$ are maximal abelian subgroups of H_i , and $ht(s_{i,j}) > ht(H_i)$ for any $i \in [1, n-1]$, $j \in [1, k_i]$.

Let K be a cyclically reduced centralizer in H . It is easy to see that either $ht(K) = ht(H) = n$, or $ht(K) < ht(H)$ and K is a centralizer from H_{n-1} .

For a cyclically reduced centralizer K of H we define

$$\mathcal{C}(K) = \{C_{i,j}, D_{i,j}\} \cap \{\text{cyclically reduced centralizers conjugate to } K \text{ in } H\}.$$

Lemma 41. [67] *Let K be a cyclically reduced centralizer of H , and let $\mathcal{C}(K)$ be empty. Let a be a generator of K of maximal height. Then there is no element in H which has any positive power of $a^{\pm 1}$ as an initial subword.*

If $\mathcal{C}(K) \neq \emptyset$ then for $C \in \mathcal{C}(K)$ we call w from the list $s_{i,j}$, $i \in [1, n-1]$, $j \in [1, k_i]$ attached to C if $ht(w) > ht(C)$ and $ht(w^{-1} * C * w) = ht(C)$. Observe that by Theorem 43, C can have at most two attached elements of the same height, and if w_1, w_2 , $w_1 \neq w_2$ are attached to C and $ht(w_1) = ht(w_2)$ then $w_1^{-1} * w_2 = w_1^{-1} \circ w_2$.

Below we are going to distinguish attached elements in the following way. Suppose $C \in \mathcal{C}(K)$, and let w be an element attached to C . If c is a generator of C of maximal height then we call w *left-attached to C with respect to c* if $c^{-1} * w = c^{-1} \circ w$, and we call w *right-attached to C with respect to c* if $c * w = c \circ w$.

Lemma 42. [67] *Let K be a cyclically reduced centralizer of H , and let $C \in \mathcal{C}(K)$. Let c be a generator of C of maximal height. If there exists a right(left)-attached to C with respect to c element, then there exists $D \in \mathcal{C}(K)$ and its generator d of maximal height such that c is conjugate to d in H and D does not have right(left)-attached with respect to d elements.*

Lemma 43. [67] *Let K be a cyclically reduced centralizer of H , and let $C \in \mathcal{C}(K)$. Let c be a generator of C of maximal height. If there exists no right(left)-attached to C with respect to c element, then there is no element $g \in H$ which has any positive power of c as an initial (terminal) subword.*

10.2.2 Connecting elements

We call a pair of elements $u, v \in CDR(\mathbb{Z}[t], X)$ an *admissible pair* if

1. u, v are cyclically reduced,
2. u, v are not proper powers,
3. $|u| = |v|$,
4. u is not conjugate to v^{-1} (in particular, $u \neq v^{-1}$).

For an admissible pair $\{u, v\}$ we define an infinite word $s_{u,v} \in R(\mathbb{Z}[t], X)$, which we call the *connecting element* for the pair $\{u, v\}$, in the following way

$$s_{u,v}(\beta) = \begin{cases} u(\alpha) & \text{if } \beta = (k|u| + \alpha, 0), k \geq 0, 1 \leq \alpha \leq |u|, \\ v(\alpha) & \text{if } \beta = (-k|v| + \alpha, 1), k \geq 1, 1 \leq \alpha \leq |v|. \end{cases}$$

Since there exists $m > 0$ such that $u, v \in CDR(\mathbb{Z}^m, X) - CDR(\mathbb{Z}^{m-1}, X)$ then it is easy to see that $s_{u,v} \in R(\mathbb{Z}^{m+1}, X) - R(\mathbb{Z}^m, X)$. Also $s_{u,v}^{-1} = s_{v^{-1}, u^{-1}}$ and $u \circ s_{u,v} = s_{u,v} \circ v$ - both follow directly from the definition.

Notice that any two connecting elements $s_{u_1, v_1}, s_{u_2, v_2}$ have the same length whenever $u_1, v_1, u_2, v_2 \in CDR(\mathbb{Z}^m, X) - CDR(\mathbb{Z}^{m-1}, X)$. In this event we have

$$|s_{u_1, v_1}| = |s_{u_2, v_2}| = (0, \dots, 0, 1) \in \mathbb{Z}^{m+1}.$$

Lemma 44. [67] *Let u, v be elements of a group $H \subset CDR(\mathbb{Z}[t], X)$. If the pair $\{u, v\}$ is admissible then $s_{u,v} \in CDR(\mathbb{Z}[t], X)$.*

10.3 Main construction

Now, let A, B be cyclically reduced centralizers in H such that there exists an isomorphism $\phi : A \rightarrow B$ satisfying the following conditions

1. a is not conjugate to $\phi_i(a)^{-1}$ in H for any $a \in A$,
2. $|\phi(a)| = |a|$ for any $a \in A$.

In particular, it follows that $ht(A) = ht(B)$.

Remark 5. *Observe that if C is conjugate to A and D is conjugate to B then*

$$\langle H, z \mid z^{-1}Az = B \rangle \simeq \langle H, z' \mid z'^{-1}Cz' = D \rangle.$$

Hence, it is always possible to consider A and B up to taking conjugates.

Let u be a generator of A of maximal height, and let $v = \phi(u) \in B$. Then v is a generator of B of maximal height and $|u| = |v|$. Observe that from the conditions imposed on ϕ it follows that the pair u, v is admissible. We fix u and v for the rest of the paper.

Remark 6. *Observe that if $\mathcal{C}(A) = \emptyset$ then by Lemma 41, H does not contain an element which has any positive power of $u^{\pm 1}$ as an initial subword (similar statement for B and v if $\mathcal{C}(B) = \emptyset$). If $\mathcal{C}(A) \neq \emptyset$ then by Lemma 42 we can assume A to have no right-attached elements with respect to u . Hence, by Lemma 43, H does not contain an element which has any positive power of u as an initial subword. Similarly, if $\mathcal{C}(B) \neq \emptyset$ then by Lemma 42 we can assume B to have no left-attached elements with respect to v . Again, by Lemma 43, H does not contain an element which has any positive power of v as a terminal subword.*

Now, we are in position to define $s \in R(\mathbb{Z}^{n+1}, X)$ which is going to be an infinite word representing z from the presentation (8). Since $\mathbb{Z}[t]$ -exponentiation is defined on $CR(\mathbb{Z}[t], X)$ (see [92] for details) then for any $f(t) \in \mathbb{Z}[t]$ we can define $v^{f(t)}, u^{f(t)} \in CR(\mathbb{Z}[t], X)$ so that

$$|v^{f(t)}| = |v||f(t)|, \quad |u^{f(t)}| = |u||f(t)|$$

and

$$[v^{f(t)}, v] = \varepsilon, [u^{f(t)}, u] = \varepsilon.$$

Thus, if $\alpha = t^{n-ht(A)}$ then $|u^\alpha| = |v^\alpha| = |u||\alpha|$ and $ht(u^\alpha) = ht(v^\alpha) = ht(u) + (n - ht(A)) = n$. Hence, we define

$$s = s_{u^\alpha, v^\alpha} \in CDR(\mathbb{Z}^{n+1}, X).$$

Observe that $ht(s) = n + 1 = ht(G) + 1$.

Remark 7. *It is easy to see that no element of H has $s^{\pm 1}$ as a subword.*

Lemma 45. *[67] For any $h \in A$ we have $s^{-1} * h * s = \phi(h) \in B$.*

Now, our goal is to prove that a pair H, s generates a group in $CDR(\mathbb{Z}[t], X)$.

Lemma 46. *[67] For any $g \in H$ there exists $N = N(g) > 0$ such that*

$$g * u^k = (g * u^N) \circ u^{k-N}, \quad v^k * g = v^{k-N} \circ (v^N * g).$$

for any $k > N$.

Lemma 47. *[67]*

(i) *For any $g \in H - A$ there exists $N = N(g) > 0$ such that for any $k > N$*

$$u^{-k} * g * u^k = u^{-k+N} \circ (u^{-N} * g * u^N) \circ u^{k-N}.$$

(ii) *For any $g \in H$ there exists $N = N(g) > 0$ such that for any $k > N$*

$$v^k * g * u^k = v^{k-N} \circ (v^N * g * u^N) \circ u^{k-N}.$$

(iii) *For any $g \in H - B$ there exists $N = N(g) > 0$ such that for any $k > N$*

$$v^k * g * v^{-k} = v^{k-N} \circ (v^N * g * v^{-N}) \circ u^{-k+N}.$$

A sequence

$$p = (g_1, s^{\epsilon_1}, g_2, \dots, g_k, s^{\epsilon_k}, g_{k+1}), \quad (9)$$

where $g_j \in H$, $\epsilon_j \in \{-1, 1\}$, $k \geq 1$, is called an s -form over H .

An s -form (9) is *reduced* if subsequences

$$\{s^{-1}, c, s\}, \quad \{s, d, s^{-1}\}$$

where $c \in A$, $d \in B$, do not occur in it.

Denote by $\mathcal{P}(H, s)$ the set of all s -forms over H . We define a partial function $w : \mathcal{P}(H, s) \rightarrow R(\mathbb{Z}[t], X)$ as follows. If

$$p = (g_1, s^{\epsilon_1}, g_2, \dots, g_k, s^{\epsilon_k}, g_{k+1})$$

then

$$w(p) = (\dots (g_1 * s^{\epsilon_1}) * g_2) * \dots * g_k * s^{\epsilon_k} * g_{k+1})$$

if it is defined.

Lemma 48. [67] *Let $p = (g_1, s^{\epsilon_1}, g_2, \dots, g_k, s^{\epsilon_k}, g_{k+1})$ be an s -form over H . Then the following hold.*

- (1) *The product $w(p)$ is defined and it does not depend on the placement of parentheses.*
- (2) *There exists a reduced s -form q over H such that $w(q) = w(p)$.*
- (3) *If p is reduced then there exists a unique representation for $w(p)$ of the following type*

$$w(p) = (g_1 * u_1^{N_1}) \circ (u_1^{-N_1} * s^{\epsilon_1} * v_1^{-M_1}) \circ (v_1^{M_1} * g_2 * u_2^{N_2}) \circ \dots \\ \dots \circ (u_k^{-N_k} * s^{\epsilon_k} * v_k^{-M_k}) \circ (v_k^{M_k} * g_{k+1}),$$

where $N_j, M_j \geq 0$, $u_j = u, v_j = v$ if $\epsilon_j = 1$, and $N_j, M_j \leq 0$, $u_j = v, v_j = u$ if $\epsilon_j = -1$ for $j \in [1, k]$. Moreover, $g_1 * u_1^{N_1}$ does not have $u_1^{\pm 1}$ as a terminal subword, $v_{j-1}^{M_{j-1}} * g_j * u_j^{N_j}$ does not have $u_j^{\pm 1}$ as a terminal subword for every $j \in [2, k]$, and $v_{j-1}^{M_{j-1}} * g_j * s_i^{\epsilon_j}$ does not have $v_{j-1}^{\pm 1}$ as an initial subword for every $j \in [2, k]$, $v_k^{M_k} * g_{k+1}$ does not have $v_k^{\pm 1}$ as an initial subword.

- (4) $w(p) \in CDR(\mathbb{Z}[t], X)$.

Proof. Let

$$p = (g_1, s^{\epsilon_1}, g_2, \dots, g_k, s^{\epsilon_k}, g_{k+1})$$

be an s -form over H .

We show first that (1) implies (2). Suppose that $w(p)$ is defined for every placement of parentheses and all such products are equal. If p is not reduced then there exists $j \in [2, k]$ such that either $g_j \in A$, $\epsilon_{j-1} = -1$, $\epsilon_j = 1$, or $g_j \in B$, $\epsilon_{j-1} = 1$, $\epsilon_j = -1$. Without loss of generality we can assume the former. Thus, we have

$$s^{-1} * g_j * s = g'_j \in B \subseteq H$$

and we obtain a new s -form

$$p_1 = (g_1, s^{\epsilon_1}, g_2, \dots, g_{j-1} * g'_j * g_{j+1}, s^{\epsilon_{j+1}}, \dots, g_k, s^{\epsilon_k}, g_{k+1})$$

which is shorter than p and $w(p) = w(p_1)$. Proceeding this way (or by induction) in a finite number of steps we obtain a reduced s -form

$$q = (f_1, s^{\delta_1}, f_2, \dots, s^{\delta_l}, f_{l+1}),$$

such that $w(q) = w(p)$, as required.

Now we show that (1) implies (3). Assume that

$$p = (g_1, s^{\epsilon_1}, g_2, \dots, g_k, s^{\epsilon_k}, g_{k+1})$$

is reduced.

By Lemma 46 and Lemma 47 there exists $r \in \mathbb{N}$ such that for any $\alpha > r$

- (a) $g_1 * u^\alpha = (g_1 * u^r) \circ u^{\alpha-r}$, $g_1 * v^{-\alpha} = (g_1 * u^{-r}) \circ u^{-\alpha+r}$,
- (b) $v^\alpha * g_{k+1} = v^{\alpha-r} \circ (v^r * g_{k+1})$, $u^{-\alpha} * g_{k+1} = u^{-\alpha+r} \circ (u^{-r} * g_{k+1})$,
- (c) $u^{-\alpha} * g_j * u^\alpha = u^{-(\alpha-r)} \circ (u^{-r} * g_j * u^r) \circ u^{\alpha-r}$ for all $j \in [2, k]$ such that $g_j \notin A$,
- (d) $v^\alpha * g_j * u^\alpha = v^{\alpha-r} \circ (v^r * g_j * u^r) \circ u^{\alpha-r}$ for all $j \in [2, k]$,
- (e) $v^\alpha * g_j * v^{-\alpha} = v^{\alpha-r} \circ (v^r * g_j * v^{-r}) \circ u^{-(\alpha-r)}$ for all $j \in [2, k]$ such that $g_j \notin B$.

Since p is reduced, that is, it does not contain neither a subsequence $\{s^{-1}, g_j, s\}$, where $g_j \in A$, nor a subsequence $\{s, g_j, s^{-1}\}$, where $g_j \in B$, and s has any power of u as an initial subword and any power of v as a terminal subword then we have

$$\begin{aligned} w(p) &= g_1 * s^{\epsilon_1} * g_2 * \cdots * g_k * s^{\epsilon_k} * g_{k+1} = \\ &= (g_1 * u_1^r) \circ (u_1^{-r} * s^{\epsilon_1} * v_1^{-r}) \circ (v_1^r * g_2 * u_2^r) \circ \cdots \circ (u_k^{-r} * s^{\epsilon_k} * v_k^{-r}) \circ (v_k^r * g_{k+1}), \end{aligned}$$

where $u_j = u, v_j = v$ if $\epsilon_j = 1$ and $u_j = v^{-1}, v_j = u^{-1}$ if $\epsilon_j = -1$ for every $j \in [1, k]$.

Now, if $g_1 * u_1^r$ has $u_1^{\gamma_1}$, $\gamma_1 \in \mathbb{Z}$ (with γ_1 maximal possible) as a terminal subword then we denote $N_1 = r - \gamma_1$ and rewrite $w(p)$ as follows

$$\begin{aligned} w(p) &= (g_1 * u_1^{N_1}) \circ (u_1^{-N_1} * s^{\epsilon_1} * v_1^{-r}) \circ (v_1^r * g_2 * u_2^r) \circ \cdots \\ &\quad \cdots \circ (u_k^{-r} * s^{\epsilon_k} * v_k^{-r}) \circ (v_k^r * g_{k+1}). \end{aligned}$$

Now, if $v_1^r * g_2 * u_2^r$ contains $v_1^{\delta_1}$, $\delta_1 \in \mathbb{Z}$ (with δ_1 maximal possible) as an initial subword then we denote $M_1 = r - \delta_1$ and again rewrite $w(p)$

$$\begin{aligned} w(p) &= (g_1 * u_1^{N_1}) \circ (u_1^{-N_1} * s^{\epsilon_1} * v_1^{-M_1}) \circ (v_1^{M_1} * g_2 * u_2^r) \circ \cdots \\ &\quad \cdots \circ (u_k^{-r} * s^{\epsilon_k} * v_k^{-r}) \circ (v_k^r * g_{k+1}). \end{aligned}$$

In a finite number of steps we obtain the required result. Observe that by the choice of N_i, M_i the representation of $w(p)$ is unique.

Now we prove (1) by induction on k . If $k = 1$ then by Lemma 46 there exists $r \in \mathbb{N}$ such that

$$\begin{aligned} g_1 * u^\alpha &= (g_1 * u^r) \circ u^{\alpha-r}, \quad g_1 * v^{-\alpha} = (g_1 * v^{-r}) \circ v^{-\alpha+r}, \\ v^\beta * g_2 &= u^{\beta-r} \circ (u^r * g_2), \quad u^{-\beta} * g_2 = u^{-\beta+r} \circ (v^{-r} * g_2) \end{aligned}$$

for any $\alpha, \beta > r$. Hence,

$$\begin{aligned} (g_1 * s^{\epsilon_1}) * g_2 &= ((g_1 * u_1^r) \circ (u_1^{-r} * s^{\epsilon_1})) * g_2 = \\ &= ((g_1 * u_1^r) \circ (u_1^{-r} * s^{\epsilon_1} * v_1^{-r})) \circ (v_1^r * g_2), \end{aligned}$$

where $u_1 = u, v_1 = v$ if $\epsilon_1 = 1$ and $u_1 = v^{-1}, v_1 = u^{-1}$ if $\epsilon_1 = -1$. By Theorem 3.4 [92], the product $(g_1 * s^{\epsilon_1}) * g_2$ does not depend on the placement of parentheses. So (1) holds for $k = 1$.

Now, consider an initial s -subsequence of p

$$p_1 = (g_1, s^{\epsilon_1}, g_2, \dots, g_{k-1}, s^{\epsilon_{k-1}}, g_k).$$

By induction $w(p_1)$ is defined and it does not depend on the placement of parentheses. By the argument above there exists a unique representation of $w(p_1)$

$$\begin{aligned} w(p_1) &= (g_1 * u_1^{N_1}) \circ (u_1^{-N_1} * s^{\epsilon_1} * v_1^{-M_1}) \circ (v_1^{M_1} * g_2 * u_2^{N_2}) \circ \dots \\ &\dots \circ (u_{k-1}^{-N_{k-1}} * s^{\epsilon_{k-1}} * v_{k-1}^{-M_{k-1}}) \circ (v_{k-1}^{M_{k-1}} * g_k), \end{aligned}$$

where $N_j, M_j \geq 0$, $u_j = u, v_j = v$ if $\epsilon_j = 1$, and $N_j, M_j \leq 0$, $u_j = v, v_j = u$ if $\epsilon_j = -1$ for $j \in [1, k-1]$. To prove that p satisfies (1) it suffices to show that

$$w(p_1) * (s^{\epsilon_k} * g_{k+1})$$

is defined and does not depend on the placement of parentheses.

Without loss of generality we assume $\epsilon_{k-1} = 1, \epsilon_k = 1$ - other combinations of ϵ_{k-1} and ϵ_k are considered similarly.

By Lemma 47

$$\begin{aligned} ((u^{-N_{k-1}} * s * v^{-M_{k-1}}) \circ (v^{M_{k-1}} * g_k)) * (s * g_{k+1}) &= (u^{-N_{k-1}} * s * v^{-M_{k-1}-r}) \\ &\circ (v^{M_{k-1}+r} * g_k * u^{m_1}) \circ (u^{-m_1} * s * v^{-m_2}) \circ (v^{m_2} * g_{k+1}) \end{aligned}$$

for some $m_1, m_2, r \in \mathbb{N}$. Thus $w(p_1) * (s * g_{k+1})$ is defined and does not depend on the placement of parentheses.

Now we prove (4). By (3) there exists a unique representation of $w(p)$

$$\begin{aligned} w(p) &= (g_1 * u_1^{N_1}) \circ (u_1^{-N_1} * s^{\epsilon_1} * v_1^{-M_1}) \circ (v_1^{M_1} * g_2 * u_2^{N_2}) \circ \dots \\ &\dots \circ (u_k^{-N_k} * s^{\epsilon_k} * v_k^{-M_k}) \circ (v_k^{M_k} * g_{k+1}), \end{aligned}$$

where $N_j, M_j \geq 0$, $u_j = u, v_j = v$ if $\epsilon_j = 1$, and $N_j, M_j \leq 0$, $u_j = v, v_j = u$ if $\epsilon_j = -1$ for $j \in [1, k]$. By Lemma 3.8 [92], to prove that $w(p) \in CDR(\mathbb{Z}[t], X)$ it suffices to show that

$$g^{-1} * w(p) * g \in CDR(\mathbb{Z}[t], X)$$

for some $g \in R(\mathbb{Z}[t], X)$.

Without loss of generality we assume $\epsilon_k = 1$ and consider two cases.

- (i) $g_{k+1} * g_1 * u_1^{N_1} \notin B$ or $g_{k+1} * g_1 * u_1^{N_1} \in B$ but $\epsilon_1 = 1$.

Without loss of generality we assume the former and $\epsilon_1 = 1$. That is, $u_1 = u, v_1 = v$. By Lemma 47 there exists $N \in \mathbb{N}$ such that

$$(u^{-m} * (g_1 * u^{N_1})^{-1}) * w(p) * ((g_1 * u^{N_1}) * u^m) = (u^{-N_1-m} * s * v^{-M_1})$$

$$\circ(v^{M_1} * g_2 * u_2^{N_2}) \circ \dots \circ(u^{-N_k} * s * v^{-M_k-m}) \circ(v^{M_k+m} * g_{k+1} * g_1 * u^{N_1+N}) \circ u^{m-N}$$

for any $m > N$. Thus,

$$(u^{-m} * (g_1 * u^{N_1})^{-1}) * w(p) * ((g_1 * u^{N_1}) * u^m) \in CR(\mathbb{Z}[t], X) \subset CDR(\mathbb{Z}[t], X).$$

(ii) $g_{k+1} * g_1 \in B$ and $\epsilon_1 = -1$.

Thus, $a = s * (g_{k+1} * g_1) * s^{-1} \in A$ and we have

$$(g_1 * s)^{-1} * w(p) * (g_1 * s) = \left((g_2 * u_2^{N_2}) \circ (u_2^{-N_2} * s^{\epsilon_2} * v_2^{-M_2}) \circ \dots \right.$$

$$\left. \dots \circ (u_{k-1}^{-N_{k-1}} * s^{\epsilon_{k-1}} * v_{k-1}^{-M_{k-1}}) \circ (v_{k-1}^{M_{k-1}} * g_k) \right) * a = (g_2 * u_2^{N_2})$$

$$\circ(u_2^{-N_2} * s^{\epsilon_2} * v_2^{-M_2}) \circ \dots \circ (u_{k-1}^{-N_{k-1}} * s^{\epsilon_{k-1}} * v_{k-1}^{-M'_{k-1}}) \circ (v_{k-1}^{M'_{k-1}} * (g_k * a)),$$

where $M'_{k-1} \in \mathbb{Z}$ is the power which works for $g_k * a$. So the number of $s^{\pm 1}$ is reduced by two and we can use induction. □

Now we are ready to prove the main results of this subsection from which Theorem 44 follows.

Theorem 45. [67] *Put*

$$P = P(H, s) = \{g * s^\epsilon * h \mid g, h \in H, \epsilon \in \{-1, 0, 1\}\} \subseteq CDR(\mathbb{Z}[t], X).$$

Then the following hold.

- (1) P generates a subgroup H^* in $CDR(\mathbb{Z}[t], X)$.
- (2) P , with the multiplication $*$ induced from $R(\mathbb{Z}[t], X)$, is a pregroup and H^* is isomorphic to $U(P)$.
- (3) H^* is isomorphic to $G = \langle H, z \mid z^{-1}Az = B \rangle$.

Proof. We need the following claims.

Claim 1. Let $g_j * s^{\epsilon_j} * h_j \in P$ $j = 1, 2$. If

$$g_1 * s^{\epsilon_1} * h_1 = g_2 * s^{\epsilon_2} * h_2$$

then $\epsilon_1 = \epsilon_2$ and $h_1 * h_2^{-1} \in A$ if $\epsilon_1 = -1$, and $h_1 * h_2^{-1} \in B$ if $\epsilon_1 = 1$.

To prove the claim consider an s -form

$$a = (g_1, s^{\epsilon_1}, h_1 * h_2^{-1}, s^{-\epsilon_2}, g_2^{-1}).$$

By Lemma 48, $w(a)$ is defined and

$$g_1 * s^{\epsilon_1} * h_1 * h_2^{-1} * s^{-\epsilon_2} * g_2^{-1} = \varepsilon.$$

Hence, a is not reduced and the claim follows.

For every $p \in P$ we fix now a representation $p = g_p * s^{\epsilon_p} * h_p$, where $g_p, h_p \in H$, $\epsilon_p \in \{-1, 0, 1\}$.

Claim 2. Let $p = g_p * s^{\epsilon_p} * h_p$, $q = g_q * s^{\epsilon_q} * h_q$ be in P . If $p * q \in P$ then either $\epsilon_p \epsilon_q = 0$, or $\epsilon_p = -\epsilon_q \neq 0$ and $h_p * g_q \in A$ if $\epsilon_p = -1$, and $h_p * g_q \in B$ if $\epsilon_q = 1$.

Let $x^{-1} = p * q \in P$ and $x = g_x * s^{\epsilon_x} * h_x$. Assume that $\epsilon_p \epsilon_q \neq 0$.

(a) $\epsilon_x \neq 0$

Consider an s -form

$$a = (g_p, s^{\epsilon_p}, h_p * g_q, s^{\epsilon_q}, h_q * g_x, s^{\epsilon_x}, h_x).$$

By Lemma 48, $w(a)$ is defined and

$$w(a) = g_p * s^{\epsilon_p} * h_p * g_q * s^{\epsilon_q} * h_q * g_x * s^{\epsilon_x} * h_x = \varepsilon.$$

Hence, a is not reduced and either a subsequence

$$\{s^{\epsilon_p}, h_p * g_q, s^{\epsilon_q}\},$$

or a subsequence

$$\{s^{\epsilon_q}, h_q * g_x, s^{\epsilon_x}\}$$

is reducible. In the former case we are done, so assume that $\{s^{\epsilon_q}, h_q * g_x, s^{\epsilon_x}\}$ can be reduced. Without loss of generality we can assume that $\epsilon_q = -1$, $\epsilon_x = 1$, $h_q * g_x \in A$. Hence,

$$s^{\epsilon_q} * h_q * g_x * s^{\epsilon_x} = g \in B$$

and we have

$$w(a) = g_p * s^{\epsilon_p} * h_p * g_q * g * h_x = \varepsilon.$$

Now, it follows $\epsilon_p = 0$ - a contradiction with our assumption.

(b) $\epsilon_x = 0$

Hence, $x = g \in H$ and we consider an s -form

$$a = (g_p, s^{\epsilon_p}, h_p * g_q, s^{\epsilon_q}, h_q * g).$$

By Lemma 48, $w(a)$ is defined and

$$w(a) = g_p * s^{\epsilon_p} * h_p * g_q * s^{\epsilon_q} * h_q * g = \varepsilon.$$

Now, the claim follows automatically.

Below we call a tuple $y = (y_1, \dots, y_k) \in P^k$ a *reduced P -sequence* if $y_j * y_{j+1} \notin P$ for $j \in [1, k-1]$. Observe, that if $y = (y_1, \dots, y_k)$ is a reduced P -sequence and $y_j = g_j * s_i^{\epsilon_j} * h_j$ then either $k \leq 1$ or y has the following properties which follow from Claim 2:

- (a) $\epsilon_j \neq 0$ for all $j \in [1, k]$,
- (b) if $\epsilon_j = -1$, $\epsilon_{j+1} = 1$ then $h_j * g_{j+1} \notin A$ for $j \in [1, k-1]$,
- (c) if $\epsilon_j = 1$, $\epsilon_{j+1} = -1$ then $h_j * g_{j+1} \notin B$ for $j \in [1, k-1]$.

In particular, the s -form over H

$$p_y = (g_1, s^{\epsilon_1}, h_1 * g_2, s^{\epsilon_2}, \dots, h_{n-1} * g_n, s^{\epsilon_k}, h_n),$$

is reduced.

To prove (1) observe first that $P^{-1} = P$. Now if $y_1, \dots, y_k \in P$ then $y_1 * \dots * y_k = w(p_y)$, where $y = (y_1, \dots, y_k)$. Hence, by Lemma 48, the product $y_1 * y_2 * \dots * y_k$ is defined in $CDR(\mathbb{Z}[t], X)$ and it belongs to $CDR(\mathbb{Z}[t], X)$. It follows that $H^* = \langle P \rangle$ is a subgroup of $CDR(\mathbb{Z}[t], X)$ which consists of all words $w(p)$, where p ranges through all possible s -forms over H . Hence, (1) is proved.

Now we prove (2). By Theorem 2, [110], to prove that P is a pregroup and the inclusion $P \rightarrow H^*$ extends to an isomorphism $U(P) \simeq H^*$ it is enough to show that all reduced P -sequences representing the same element have the same P -length.

Suppose two reduced P -sequences

$$(u_1, u_2, \dots, u_k), (v_1, v_2, \dots, v_n)$$

represent the same element $g \in H^*$. That is,

$$(u_1 * \dots * u_k) * (v_1 * \dots * v_n)^{-1} = \varepsilon.$$

We use induction on $k + n$ to show that $k = n$. Observe that $k = 0$ implies $n = 0$, otherwise we get a contradiction with Lemma 48 (3). Hence, we can assume $k, n > 0$, that is, $k + n \geq 2$. If the P -sequence

$$a = (u_1, \dots, u_k, v_n^{-1}, \dots, v_1^{-1})$$

is reduced then the underlying s -form is reduced and hence, by Lemma 48 (3)

$$w(a) = u_1 * \dots * u_k * v_n^{-1} * \dots * v_1^{-1} \neq \varepsilon.$$

Hence,

$$(u_1, \dots, u_k, v_n^{-1}, \dots, v_1^{-1})$$

is not reduced and $u_k * v_n^{-1} \in P$. If $u_k = g_1 * s^{\epsilon_1} * h_1$, $v_n = g_2 * s^{\epsilon_2} * h_2$, where $g_i, h_i \in H$ and $\epsilon_i \in \{-1, 0, 1\}$, $i = 1, 2$ then by Claim 2 either $\epsilon_1 \epsilon_2 = 0$, or $\epsilon_1 = \epsilon_2 \neq 0$ and $h_1 * h_2^{-1} \in A$ if $\epsilon_1 = -1$, and $h_1 * h_2^{-1} \in B$ if $\epsilon_1 = 1$. In the former case, for example, if $\epsilon_2 = 0$ then $n = 1$, $v_n \in H$ and $b = (u_1, \dots, u_k * v_n^{-1})$ is a reduced P -sequence such that $w(b) = \varepsilon$ - a contradiction with Lemma 48 (3) unless $k = 1$, $u_1 \in H$. In the latter case, $u_k * v_n^{-1} \in H$ and it follows that

$$(u_1, u_2, \dots, u_{k-1} * (u_k * v_n^{-1})), (v_1, v_2, \dots, v_{n-1})$$

represent the same element in H^* while the sum of their lengths is less than $k + n$, so the result follows by induction.

Finally, to prove (3) observe first that H embeds into G . We denote this embedding by θ . Now we define a map $\phi : P \rightarrow G$ as follows. For $g * s^\epsilon * h \in P$ put

$$g * s^\epsilon * h \xrightarrow{\phi} \theta(g) z^\epsilon \theta(h).$$

It follows from Claim 2 that ϕ is a morphism of pregroups. Since $H^* \simeq U(P)$, the morphism ϕ extends to a unique homomorphism $\psi : H^* \rightarrow G$. We claim that ψ is bijective. Indeed, observe first that $G = \langle H, z \rangle$. Now, since $\psi(s^\epsilon) = z^\epsilon$ and $\psi = \phi = \theta$ on H , it follows that ψ is onto. To see that ψ is one-to-one it suffices to notice that if

$$(g_1 * s^{\epsilon_1} * h_1, g_2 * s^{\epsilon_2} * h_2, \dots, g_m * s^{\epsilon_m} * h_m)$$

is a reduced P -sequence then

$$y = (g_1, s^{\epsilon_1}, h_1 * g_2, s^{\epsilon_2}, \dots, s^{\epsilon_m}, h_m)$$

is a reduced s -form and $w(y)^\psi \neq 1$ by Britton's Lemma (see, for example, [79]). This proves that ψ is an isomorphism, as required. \square

Theorem 46. [67] *Let $G = \langle H, z \mid z^{-1}Az = B \rangle$. Then, in the notation above, the free length function on $L : G \rightarrow \mathbb{Z}^{n+1}$ induced by the isomorphism $\psi : H^* \rightarrow G$ is regular.*

Proof. Observe that it is enough to show that the length function induced on $H^* = \langle P \rangle$ from $CDR(\mathbb{Z}[t], X)$ is regular.

Let $g, h \in H^*$. Then g and h can be written in the unique normal forms

$$g = g_1 \circ (u_1^{-N_1} * s^{\epsilon_1} * v_1^{-M_1}) \circ g_2 \circ \dots \circ (u_k^{-N_k} * s^{\epsilon_k} * v_k^{-M_k}) \circ g_{k+1},$$

$$h = h_1 \circ (w_1^{-L_1} * s^{\delta_1} * x_1^{-P_1}) \circ h_2 \circ \dots \circ (w_m^{-L_m} * s^{\delta_m} * x_m^{-P_m}) \circ h_{m+1},$$

where $N_j, M_j \geq 0$, $u_j = u, v_j = v$ if $\epsilon_j = 1$, and $N_j, M_j \leq 0$, $u_j = v, v_j = u$ if $\epsilon_j = -1$ for $j \in [1, k]$; $L_i, P_i \geq 0$, $w_i = u, x_i = v$ if $\delta_i = 1$, and $L_i, P_i \leq 0$, $w_i = v, x_i = u$ if $\delta_i = -1$ for $i \in [1, m]$. Moreover, g_1 does not have $u_1^{\pm 1}$ as a terminal subword, g_j does not have $u_j^{\pm 1}$ as a terminal subword for every $j \in [2, k]$, and g_j does not have $v_{j-1}^{\pm 1}$ as an initial subword for every $j \in [2, k]$, g_{k+1} does not have $v_k^{\pm 1}$ as an initial subword; h_1 does not have $w_1^{\pm 1}$ as a terminal subword, h_i does not have $w_i^{\pm 1}$ as a terminal subword for every $i \in [2, m]$, and h_i does not have $x_{i-1}^{\pm 1}$ as an initial subword for every $i \in [2, m]$, h_{m+1} does not have $x_m^{\pm 1}$ as an initial subword.

If there exist $k_1, k_2 > 0$ such that

$$c = c(g, h) \leq \min\{|g_1 \circ u_1^{k_1}|, |h_1 \circ w_1^{k_2}|\}$$

then $\text{com}(g, h) = \text{com}(g_1 \circ u_1^{k_1}, h_1 \circ w_1^{k_2}) \in H$. Now, assume that $r \in [1, k]$ is the minimal natural number such that $\text{com}(g, h)$ is an initial subword of

$$f_1 = g_1 \circ (u_1^{-N_1} * s^{\epsilon_1} * v_1^{-M_1}) \circ g_2 \circ \dots \circ (u_r^{-N_r} * s^{\epsilon_r} * v_r^{-M_r}) \circ g_{r+1} \circ u_r^{p_1},$$

where $p_1 \in \mathbb{Z}$. Similarly, assume that $q \in [1, m]$ is the minimal natural number such that $\text{com}(g, h)$ is an initial subword of

$$f_2 = h_1 \circ (w_1^{-L_1} * s^{\delta_1} * x_1^{-P_1}) \circ h_2 \circ \dots \circ (w_q^{-L_q} * s^{\delta_q} * x_q^{-P_q}) \circ h_{q+1} \circ w_q^{p_2},$$

where $p_2 \in \mathbb{Z}$. From uniqueness of normal forms it follows that $r = q$ and we have $g_i = h_i$, $u_i = w_i$, $v_i = x_i$, $N_i = L_i$, $\epsilon_i = \delta_i$, $i \in [1, r]$ and $M_i = P_i$, $i \in [1, r-1]$.

Without loss of generality we can assume $\epsilon_r = 1$. Hence, $v_r = x_r = v$.

Observe that $\text{com}(g, h)$ can be represented as a concatenation $\text{com}(g, h) = c_1 \circ c_2$, where

$$c_1 = g_1 \circ (u_1^{-N_1} * s^{\epsilon_1} * v_1^{-M_1}) \circ g_2 \circ \dots \circ (u^{-N_r} * s * v^{-l})$$

and $l \geq \max\{M_r, P_r\}$, and

$$c_2 = \text{com}(v^{l-M_r} \circ g_{r+1} \circ u_r^{p_1}, v^{l-P_r} \circ h_{r+1} \circ w_r^{p_2}).$$

Obviously, $c_1 \in H^*$. Also, $c_2 \in H$ since $v^{l-M_r} \circ g_{r+1} \circ u_r^{p_1}, v^{l-P_r} \circ h_{r+1} \circ w_r^{p_2} \in H$ and the length function on H is regular. Hence, $\text{com}(g, h) \in H^*$. \square

10.4 Completions of \mathbb{Z}^n -free groups

Let G be a finitely generated subgroup of $CDR(\mathbb{Z}^n, X)$, where \mathbb{Z}^n is ordered with respect to the right lexicographic order. Here we do not assume X to be finite. We are going to construct a finite alphabet Y and a finitely generated group H which is subgroup of $CDR(\mathbb{Z}^n, Y)$ such that the length function on H induced from $CDR(\mathbb{Z}^n, Y)$ is regular and G embeds into H so that the length is preserved by the embedding. In other words, we are going to construct a finitely generated \mathbb{Z}^n -completion of G (see [69]). All the proofs and details can be found in [70].

Consider a finitely generated \mathbb{Z}^n -free group G , where $n \in \mathbb{N}$. Suppose $n > 1$ and consider the \mathbb{Z}^n -tree (Γ_G, d) which arises from the embedding of G into $CDR(\mathbb{Z}^n, X)$.

We say that $p, q \in \Gamma_G$ are \mathbb{Z}^{n-1} -equivalent ($p \sim q$) if $d(p, q) \in \mathbb{Z}^{n-1}$, that is, $d(p, q) = (a_1, \dots, a_n)$, $a_n = 0$. From metric axioms it follows that “ \sim ” is an equivalence relation and every equivalence class defines a \mathbb{Z}^{n-1} -subtree of Γ_G .

Let $\Delta_G = \Gamma_G / \sim$ and denote by ρ the mapping $\Gamma_G \rightarrow \Gamma_G / \sim$. It is easy to see that Δ_G is a simplicial tree. Indeed, define $\tilde{d} : \Delta_G \rightarrow \mathbb{Z}$ as follows:

$$\forall \tilde{p}, \tilde{q} \in \Delta_G : \tilde{d}(\tilde{p}, \tilde{q}) = k \text{ iff } d(p, q) = (a_1, \dots, a_n) \text{ and } a_n = k. \quad (10)$$

From metric properties of d it follows that \tilde{d} is a well-defined metric.

Since G acts on Γ_G by isometries then $p \sim q$ implies $g \cdot p \sim g \cdot q$ for every $g \in G$. Moreover, if $d(p, q) = (a_1, \dots, a_n)$ then $d(g \cdot p, g \cdot q) = (a_1, \dots, a_n)$. Hence, $\tilde{d}(g \cdot \tilde{p}, g \cdot \tilde{q}) = \tilde{d}(\tilde{p}, \tilde{q})$, that is, G acts on Δ_G by isometries, but the action is not free in general. From Bass-Serre theory it follows that $\Psi_G = \Delta_G/G$ is a graph in which vertices and edges correspond to G -orbits of vertices and edges in Δ_G .

Lemma 49. [70] Ψ_G is a finite graph.

From Lemma 49 it follows that the number of G -orbits of \mathbb{Z}^{n-1} -subtrees in Γ_G is finite and equal to $|V(\Psi_G)|$. So, let $|V(\Psi_G)| = m$ and $\mathcal{T}_1, \dots, \mathcal{T}_m$ be these G -orbits.

Consider Ψ_G . The set of vertices and edges of Ψ_G we denote correspondingly by $V(\Psi_G)$ and $E(\Psi_G)$ so that

$$\sigma : E(\Psi_G) \rightarrow V(\Psi_G), \quad \tau : E(\Psi_G) \rightarrow V(\Psi_G), \quad \bar{\cdot} : E(\Psi_G) \rightarrow E(\Psi_G)$$

satisfy the following conditions:

$$\sigma(\bar{e}) = \tau(e), \quad \tau(\bar{e}) = \sigma(e), \quad \bar{\bar{e}} = e, \quad \bar{e} \neq e.$$

Let \mathcal{T} be a maximal subtree of Ψ_G and let $\pi : \Delta_G \rightarrow \Delta_G/G = \Psi_G$ be the canonical projection of Δ_G onto its quotient, so $\pi(v) = Gv$ and $\pi(e) = Ge$ for every $v \in V(\Delta_G)$, $e \in E(\Delta_G)$. There exists an injective morphism of graphs $\eta : \mathcal{T} \rightarrow \Delta_G$ such that $\pi \circ \eta = id_{\mathcal{T}}$ (see Section 8.4 of [28]), in particular $\eta(\mathcal{T})$ is a subtree of Δ_G . One can extend η to a map (which we again denote by η) $\eta : \Psi_G \rightarrow \Delta_G$ such that η maps vertices to vertices, edges to edges, and so that $\pi \circ \eta = id_{\Psi_G}$. Notice, that in general η is not a graph morphism. To this end choose an orientation O of the graph Ψ_G . Let $e \in O - \mathcal{T}$. Then there exists an edge $e' \in \Delta_G$ such that $\pi(e') = e$. Clearly, $\sigma(e')$ and $\eta(\sigma(e))$ are in the same G -orbit. Hence $g \cdot \sigma(e') = \eta(\sigma(e))$ for some $g \in G$. Define $\eta(e) = g \cdot e'$ and $\eta(\bar{e}) = \eta(e)$. Notice that vertices $\eta(\tau(e))$ and $\tau(\eta(e))$ are in the same G -orbit. Hence there exists an element $\gamma_e \in G$ such that $\gamma_e \cdot \tau(\eta(e)) = \eta(\tau(e))$.

Put

$$G_v = \text{Stab}_G(\eta(v)), \quad G_e = \text{Stab}_G(\eta(e))$$

and define boundary monomorphisms as inclusion maps $i_e : G_e \hookrightarrow G_{\sigma(e)}$ for edges $e \in \mathcal{T} \cup O$ and as conjugations by $\gamma_{\bar{e}}$ for edges $e \notin \mathcal{T} \cup O$, that is,

$$i_e(g) = \begin{cases} g, & \text{if } e \in \mathcal{T} \cup O, \\ \gamma_{\bar{e}} g \gamma_{\bar{e}}^{-1}, & \text{if } e \notin \mathcal{T} \cup O. \end{cases}$$

According to the Bass-Serre structure theorem we have

$$G \simeq \pi(G, \Psi_G, \mathcal{T}) = \langle G_v \ (v \in V(\Psi_G)), \ \gamma_e \ (e \in E(\Psi_G)) \mid \text{rel}(G_v), \quad (11)$$

$$\gamma_e i_e(g) \gamma_e^{-1} = i_{\bar{e}}(g) \ (g \in G_e), \ \gamma_e \gamma_{\bar{e}} = 1, \ \gamma_e = 1 \ (e \in \mathcal{T}).$$

Let $\mathcal{K} = \rho^{-1}(\eta(\mathcal{T}))$, $\overline{\mathcal{K}} = \rho^{-1}(\eta(\Psi_G))$, hence, \mathcal{K} , $\overline{\mathcal{K}}$ are subtrees of Γ_G such that $\mathcal{K} \subseteq \overline{\mathcal{K}}$. Obviously $T_0 \subseteq \mathcal{K}$. Moreover, both \mathcal{K} and $\overline{\mathcal{K}}$ contain finitely many \mathbb{Z}^{n-1} -subtrees, and meet every G -orbit of \mathbb{Z}^{n-1} -subtrees of Γ_G .

For every $v \in \Psi_G$ we have $\text{Stab}_G(\eta(v)) = \text{Stab}_G(T_{\eta(v)})$, where $\eta(v) = \rho(T_{\eta(v)})$. Denote by T_0 the \mathbb{Z}^{n-1} -subtree containing ε . Obviously, $\text{Stab}_G(T_0)$ is a subgroup of $\text{CDR}(\mathbb{Z}^{n-1}, X)$.

Lemma 50. [70] *Let T be a \mathbb{Z}^{n-1} -subtree of \mathcal{K} . Then*

$$\text{Stab}_G(T) = f_T * K_T * f_T^{-1},$$

where K_T is a subgroup of $\text{CDR}(\mathbb{Z}^{n-1}, X)$ (possibly trivial) and $f_T = \mu([\varepsilon, x_T]) \in \text{CDR}(\mathbb{Z}^n, X)$. Moreover, is $\text{Stab}_G(T)$ is not trivial then $x_T \in \text{Axis}(g) \cap T$ for some $g \in \text{Stab}_G(T)$.

Let e be an edge of Ψ_G such that $e \in O$, $e \notin \mathcal{T}$. Let $v = \sigma(\eta(e)) = \eta(\sigma(e))$, $w = \tau(\eta(e))$ and $u = \eta(\tau(e)) = \gamma_e \cdot w$. We have $u, v \in \eta(\mathcal{T})$, $w \notin \eta(\mathcal{T})$. Hence,

$$\gamma_e \text{Stab}_G(w) \gamma_e^{-1} = \text{Stab}_G(u).$$

By definition we have $i_e(G_e) \subseteq G_v = \text{Stab}_G(T)$, where $T = \rho^{-1}(v)$ and $i_e(G_e) = \gamma_e G_e \gamma_e^{-1} \subseteq G_u = \text{Stab}_G(S)$, where $S = \rho^{-1}(u)$. Thus, we have $i_e(G_e) = f_T * A * f_T^{-1}$, $i_e(G_e) = f_S * B * f_S^{-1}$, where $A \leq K_T$ and $B \leq K_S$ are isomorphic abelian subgroups of $\text{CDR}(\mathbb{Z}^{n-1}, X)$. So,

$$\gamma_e * (f_T * A * f_T^{-1}) * \gamma_e^{-1} = f_S * B * f_S^{-1}$$

and it follows that $f_S^{-1} * \gamma_e * f_T = r_e \in \text{CDR}(\mathbb{Z}^n, X)$ so that $r_e * A * r_e^{-1} = B$. Thus, we have

$$\gamma_e = f_S * r_e * f_T^{-1}.$$

Observe that $r_e \in \text{CDR}(\mathbb{Z}^n, X) - \text{CDR}(\mathbb{Z}^{n-1}, X)$ because otherwise $\gamma_e \cdot T = S$, that is, $u = v$, $S = T$ and thus $\gamma_e \in \text{Stab}_G(T)$ - a contradiction.

10.4.1 Simplicial case

Let G be a finitely generated subgroup of $\text{CDR}(\mathbb{Z}, X)$. Hence, Γ_G is a simplicial tree and $\Delta = \Gamma_G/G$ is a folded X -labeled digraph (see [56]) with labeling induced from Γ_G . Δ is finite which follows from the fact that G is finitely generated and from the construction of Γ_G . Moreover, Δ recognizes G with respect to some vertex v (the image of ε) in the sense that $g \in \text{CDR}(\mathbb{Z}, X)$ belongs to G if and only if there exists a loop in Δ at v such that its label is exactly g .

The following lemma provides the required result.

Lemma 51. [70] *Let G be a finitely generated subgroup of $\text{CDR}(\mathbb{Z}, X)$. Then there exists a finite alphabet Y and an embedding $\phi : G \rightarrow H$, where $H = F(Y)$, inducing an embedding $\psi : \Gamma_G \rightarrow \Gamma_H$ such that*

$$(i) \quad |g|_G = |\phi(g)|_H \text{ for every } g \in G,$$

- (ii) if A is a maximal abelian subgroup of G then $\phi(A)$ is a maximal abelian subgroup of H ,
- (iii) if a and b are non- G -equivalent ends of Γ_G then $\psi(a)$ and $\psi(b)$ are non- H -equivalent ends of Γ_H ,
- (iv) if A and B are maximal abelian subgroups of G which are not conjugate in G then $\phi(A)$ and $\phi(B)$ are not conjugate in H .

Lemma 51 can be generalized to the following result.

Corollary 13. [70] *Let G be a finitely generated subgroup of $CDR(\mathbb{Z}, X)$. Assume that Γ_G is embedded into a \mathbb{Z} -tree T whose edges are labeled by X^\pm so that the action of G on Γ_G extends to an action of G on T , and there are only finitely many G -orbits of ends of T which belong to $T - \Gamma_G$. Then there exists a finite alphabet Y , a \mathbb{Z} -tree T' whose edges are labeled by Y^\pm , and a finitely generated subgroup $H \subseteq CDR(\mathbb{Z}, Y)$ such that Γ_H is embedded into T' so that the action of H on Γ_H extends to a regular action of H on T' . Also, there is an embedding $\theta : T \rightarrow T'$, where $\theta(\Gamma_G) \subseteq \Gamma_H$, which induces an embedding $\phi : G \rightarrow H$ such that*

- (i) $|g|_G = |\phi(g)|_H$ for every $g \in G$,
- (ii) if A is a maximal abelian subgroup of G then $\phi(A)$ is a maximal abelian subgroup of H ,
- (iii) if a and b are non- G -equivalent ends of T then $\theta(a)$ and $\theta(b)$ are non- H -equivalent ends of T' .

10.4.2 General case

Let G be a finitely generated subgroup of $CDR(\mathbb{Z}^n, X)$ for some alphabet X . We are going to use the notations introduced in Subsection 10.4, that is, we assume that \mathcal{K} , Ψ_G , Δ_G etc. are defined for G as well as the presentation (11).

First of all, we relabel Γ_G so that non- G -equivalent \mathbb{Z}^{n-1} -subtrees are labeled by disjoint alphabets.

Recall that every edge e in Γ_G is labeled by a letter $\mu(e) \in X^\pm$. Let T be a \mathbb{Z}^{n-1} -subtree of \mathcal{K} and X_T a copy of X (disjoint from X) so that we have a bijection $\pi_T : X \rightarrow X_T$, where $\pi_T(x^{-1}) = \pi_T(x)^{-1}$ for every $x \in X$. We assume $X_S \cap X_T = \emptyset$ for distinct $S, T \in \mathcal{K}$. Let Γ' be a copy of Γ_G and $\nu : \Gamma' \rightarrow \Gamma_G$ a natural bijection (the bijection on points naturally induces the bijection on edges). Denote $\varepsilon' = \nu^{-1}(\varepsilon)$.

Let $X' = \bigcup \{X_T \mid T \in \mathcal{K}\}$. We introduce a labeling function $\mu' : E(\Gamma') \rightarrow X'^\pm$ as follows: $\mu'(e) = \pi_T(\mu(\nu(e)))$ if $\nu(e) \in T$. μ' naturally extends to the labeling of paths in Γ' . Now, if $V' = \nu^{-1}(V_G)$ then define

$$G' = \{\mu'(p) \mid p = [\varepsilon', v'] \text{ for some } v' \in V'\}.$$

Lemma 52. [70] G' is a subgroup of $CDR(\mathbb{Z}^n, X')$ which acts freely on Γ' and there exists an isomorphism $\phi : G \rightarrow G'$ such that $L_\varepsilon(g) = L_{\varepsilon'}(\phi(g))$.

According to Lemma 52 we have $\Gamma' = \Gamma_{G'}$. Observe that the structure of \mathbb{Z}^{n-1} -trees in $\Gamma_{G'}$ is the same as in Γ_G . Hence, if “ \sim ” is a \mathbb{Z}^{n-1} -equivalence of points of $\Gamma_{G'}$ then $\Delta_{G'} = \Gamma_{G'}/\sim$ and $\Psi_{G'} = \Delta_{G'}/G'$ are naturally isomorphic respectively to $\Delta_G = \Gamma_G/\sim$ and $\Psi_G = \Delta_G/G$. So, with a slight abuse of notation let $X = X'$, $G = G'$.

Next, we would like to refine the labeling so as to make the alphabet X finite. To do this we have to analyze the structure of the \mathbb{Z}^{n-1} -subtrees of \mathcal{K} .

Lemma 53. [70] Let T be a \mathbb{Z}^{n-1} -subtree of \mathcal{K} such that $\text{Stab}_G(T)$ is trivial. Then T contains only finitely many branch-points and each branch-point of T is of the form $Y(\varepsilon, x, y)$, where $x, y \in \{x_S \mid S \in \mathcal{K}\}$, $\gamma_e^{\pm 1} \cdot \varepsilon$ ($e \in \Psi_G$).

In particular, from Lemma 53 it follows that every \mathbb{Z}^{n-1} -subtree T of \mathcal{K} with trivial stabilizer can be relabeled by a finite alphabet. Indeed, T may be cut at its branch-points into finitely many closed segments and half-open rays which do not contain any branch-points. Then all these segments and rays can be labeled by different letters (all points in each piece is labeled by one letter).

In the case of non-trivial stabilizer the situation is a little more complicated.

Lemma 54. [70] Let T be a \mathbb{Z}^{n-1} -subtree of \mathcal{K} such that $\text{Stab}_G(T) = f_T * K_T * f_T^{-1}$ is non-trivial. Then Γ_{K_T} embeds into T (the base-point of Γ_{K_T} is identified with x_T), the action of K_T on Γ_{K_T} extends to the action of K_T on T and the following hold

- (a) every end of T which does not belong to Γ_{K_T} is K_T -equivalent to one of the ends of a finite subtree which is the intersection of T and the segments $[\varepsilon, x_S]$, $S \in \mathcal{K}$,
- (b) every end a of T which does not belong to Γ_{K_T} extends the axis of some centralizer C_a of K_T ,
- (c) there are only finitely many K_T -orbits of branch-points of T which do not belong to Γ_{K_T} ,
- (d) if $K_T \subset CDR(\mathbb{Z}^{n-1}, Y)$ for some finite alphabet Y then the labeling of Γ_{K_T} by Y can be K_T -equivariantly extended to a labeling of T by a finite extension Y' of Y .

Corollary 14. [70] If G is a finitely generated subgroup of $CDR(\mathbb{Z}^n, X)$ then X can be taken to be finite.

Proof. Follows from Lemma 53 and Lemma 54. □

For a non-linear \mathbb{Z}^{n-1} -subtree T of \mathcal{K} with a non-trivial stabilizer let $\mathcal{B}(T)$ be the set of representatives of branch-points of $T - \Gamma_{K_T}$. By Lemma 54,

$\mathcal{B}(\mathcal{T})$ is finite and every branch-point of T which does not belong to Γ_{K_T} is K_T -equivalent to a branch-point from $\mathcal{B}(\mathcal{T})$. Let

$$\mathcal{D}(T) = \{\mu([x_T, y]) \mid y \in \mathcal{B}(\mathcal{T})\}.$$

Observe that $\mathcal{D}(T)$ is a finite subset of $CDR(\mathbb{Z}^{n-1}, X)$.

Let $g \in G$. Hence, $[\varepsilon, g \cdot \varepsilon]$ meets finitely many \mathbb{Z}^{n-1} -subtrees T_0, T_1, \dots, T_k , where $T(g)_0 = T_0$ and T_i is adjacent to T_{i-1} for each $i \in [1, k]$. Observe that T_0 is \mathbb{Z}^{n-1} -subtree of \mathcal{K} . We have

$$[\varepsilon, g \cdot \varepsilon] \subseteq [x_{T_0}, x_{T_1}] \cup \dots \cup [x_{T_{k-1}}, x_{T_k}] \cup [x_{T_k}, g \cdot \varepsilon].$$

Now, there exists $g_0 \in \text{Stab}_G(T_0)$ and a \mathbb{Z}^{n-1} -subtree S_1 of \mathcal{K} adjacent to T_0 such that $T_1 = g_0 \cdot S_1$. Next, there exists $g_1 \in \text{Stab}_G(T_1)$ and a \mathbb{Z}^{n-1} -subtree S_2 of \mathcal{K} adjacent to S_1 such that $T_2 = (g_1 g_0) \cdot S_2$, and so on. After k steps we find a sequence of \mathbb{Z}^{n-1} -subtrees S_0, S_1, \dots, S_k from \mathcal{K} , where $S_0 = T_0$, S_i is adjacent to S_{i-1} , $i \in [1, k]$ and $T_i = (g_{i-1} \dots g_0) \cdot S_i$, where $g_i \in \text{Stab}_G(T_i)$. Hence,

$$\begin{aligned} [\varepsilon, g \cdot \varepsilon] &\subseteq [x_{T_0}, g_0 \cdot x_{T_0}] \cup [g_0 \cdot x_{T_0}, g_0 \cdot x_{S_1}] \cup [g_0 \cdot x_{S_1}, x_{T_1}] \cup [x_{T_1}, (g_1 g_0) \cdot x_{S_1}] \\ &\cup [(g_1 g_0) \cdot x_{S_1}, (g_1 g_0) \cdot x_{S_2}] \cup \dots \cup [(g_{k-1} \dots g_0) \cdot x_{S_{k-1}}, (g_{k-1} \dots g_0) \cdot x_{S_k}] \\ &\cup [(g_{k-1} \dots g_0) \cdot x_{S_k}, x_{T_k}] \cup [x_{T_k}, (g_k \dots g_0) \cdot x_{S_k}], \end{aligned}$$

where $(g_k \dots g_0) \cdot x_{S_k} = g \cdot \varepsilon$.

Since

$$\mu([p, q]) = \mu(g \cdot [p, q]) = \mu([g \cdot p, g \cdot q])$$

and

$$[(g_{i-1} \dots g_0) \cdot x_{S_{i-1}}, (g_{i-1} \dots g_0) \cdot x_{S_i}] = (g_{i-1} \dots g_0) \cdot [x_{S_{i-1}}, x_{S_i}]$$

for $i \in [1, k]$, then

$$\mu([(g_{i-1} \dots g_0) \cdot x_{S_{i-1}}, (g_{i-1} \dots g_0) \cdot x_{S_i}]) = \mu([x_{S_{i-1}}, x_{S_i}]).$$

Also, observe that for any $i \in [1, k]$

$$[(g_{i-1} \dots g_0) \cdot x_{S_i}, x_{T_i}] \cup [x_{T_i}, (g_i \dots g_0) \cdot x_{S_i}]$$

is a path in T_i , where $(g_{i-1} \dots g_0) \cdot x_{S_i}$ and $(g_i \dots g_0) \cdot x_{S_i}$ are $\text{Stab}_G(T_i)$ -equivalent to x_{T_i} . So, it follows that

$$\mu([x_{T_i}, (g_{i-1} \dots g_0) \cdot x_{S_i}]) = f_i \in K_{T_i}, \quad \mu([x_{T_i}, (g_i \dots g_0) \cdot x_{S_i}]) = h_i \in K_{T_i}.$$

Also, observe that $g_0 = \mu([x_{T_0}, g_0 \cdot x_{T_0}])$. Eventually, we have

$$g = g_0 * c_{S_0, S_1} * (f_1^{-1} * h_1) * c_{S_1, S_2} * \dots * c_{S_{k-1}, S_k} * (f_k^{-1} * h_k),$$

where c_{S_{i-1}, S_i} is the label of the path $[x_{S_{i-1}}, x_{S_i}]$ and the product on the right-hand side is defined in $CDR(\mathbb{Z}^n, X)$.

Now we are ready to perform the inductive step.

Theorem 47. [70] *Let G be a finitely generated subgroup of $CDR(\mathbb{Z}^n, X)$ (assume that \mathcal{K} , Ψ_G , Δ_G etc. are defined for G as above). Suppose that for every non-linear \mathbb{Z}^{n-1} -subtree T of \mathcal{K} with a non-trivial stabilizer there exists*

- (a) *an alphabet $Y(T)$,*
- (b) *a \mathbb{Z}^{n-1} -tree T' whose edges are labeled by $Y(T)$,*
- (c) *a finitely generated group $H_T \subset CDR(\mathbb{Z}^{n-1}, Y(T))$*

such that Γ_{H_T} is embedded into T' and the action of H_T on Γ_{H_T} extends to a regular action of H_T on T' . Moreover, assume that there is an embedding $\psi_T : T \rightarrow T'$, where $\psi_T(\Gamma_{K_T}) \subseteq \Gamma_{H_T}$, which induces an embedding $\phi_T : K_T \rightarrow H_T$, and such that if a and b are non- K_T -equivalent ends of T then $\psi_T(a)$ and $\psi_T(b)$ are non- H_T -equivalent ends of $\psi_T(T)$.

Then there exists an embedding of $\mathcal{D}(T)$, $T \in \mathcal{K}$ into $CDR(\mathbb{Z}^n, Y)$, where Y is a finite alphabet containing $\bigcup_{T \in \mathcal{K}} Y(T)$ such that

- (i) *$\{H(T), \mathcal{D}(T), \{c_{x_T, x_S} \mid S \text{ is adjacent to } T \text{ in } \mathcal{K}\} \mid T \in \mathcal{K}\}$ generates a group H of $CDR(\mathbb{Z}^n, Y)$ which acts regularly on Γ_H with respect to ε_H ,*
- (ii) *there exists an embedding $\psi : \Gamma_G \rightarrow \Gamma_H$, $\psi(\varepsilon_G) = \varepsilon_H$ which induces an embedding $\phi : G \rightarrow H$, such that if a and b are non- G -equivalent ends of Γ_G then $\psi(a)$ and $\psi(b)$ are non- H -equivalent ends of $\psi(\Gamma_G)$.*

Proof. First of all, by Corollary 14 we can assume X to be finite. Hence, we can assume that any two distinct \mathbb{Z}^{n-1} -subtrees S and T of \mathcal{K} are labeled distinct alphabets $X(S)$ and $X(T)$. Next, by Lemma 53, in each \mathbb{Z}^{n-1} -subtree S of \mathcal{K} with trivial stabilizer there are only finitely many branch-points, so we can cut S along these branch-points, obtain finitely many closed and half-open segments, and relabel them by a finite alphabet. Thus we can assume all this to be done already.

Let T be a non-linear \mathbb{Z}^{n-1} -subtree of \mathcal{K} with a non-trivial stabilizer. Observe that by Lemma 54 every end a of T either is an end of Γ_{K_T} , or $a = g \cdot b$, where b is from a finite list of representatives of orbits of ends of $T - \Gamma_{K_T}$.

By the assumption, T embeds into T' labeled by $Y(T)$, while Γ_{K_T} embeds into $\Gamma_{H(T)}$, where $H(T)$ acts regularly on T' . It follows that for every branch-point b of T the label of $\psi_T([x_T, b])$ defines an element of $H(T)$. In particular, the label of $\psi_T(d)$ belongs to $H(T)$ for every $d \in \mathcal{D}(T)$. Moreover, if S_1, S_2 are \mathbb{Z}^{n-1} -subtrees of \mathcal{K} adjacent to T and a_{S_1}, a_{S_2} are the corresponding ends of T then a_{S_1} is not $H(T)$ -equivalent to a_{S_2} . So, by the assumption, a_{S_1} is not $H(T)$ -equivalent to a_{S_2} and it follows that

$$(h_1 \cdot \theta([x_T, x_{S_1}] \cap T)) \cap (h_2 \cdot \theta([x_T, x_{S_2}] \cap T))$$

is a closed segment of T' , hence,

$$\text{com}(h_1 * c_{x_T, x_{S_1}}, h_2 * c_{x_T, x_{S_2}}),$$

is defined in $CDR(\mathbb{Z}^{n-1}, Y(T))$. Since $X(T) \cap X(S) = \emptyset$ then $h * c_{x_T, x_S}^{-1} = h \circ c_{x_T, x_S}^{-1}$ for every \mathbb{Z}^{n-1} -subtree S of \mathcal{K} adjacent to T . Thus,

$$\{H(T), \mathcal{D}(T), \{c_{x_T, x_S} \mid S \text{ is adjacent to } T \text{ in } \mathcal{K}\}\},$$

which is finite, generates a subgroup $H'(T)$ in $CDR(\mathbb{Z}^n, Q)$, where

$$Q = \bigcup_{T \in \mathcal{K}} Y(T),$$

so that T embeds into $\Gamma_{H'(T)}$. Moreover, $H'(T)$ acts regularly on $\Gamma_{H'(T)}$.

Now, from the fact that alphabet $X(T)$ is disjoint from $X(S)$ if T is not G -equivalent to S it follows that $\{H'(T) \mid T \in \mathcal{K}\}$ generates a subgroup H of $CDR(\mathbb{Z}^n, Y)$, where Y is a finite alphabet containing Q . Observe that $\Gamma_{H'(T)}$ embeds into Γ_H for each $T \in \mathcal{K}$. Moreover, for every $f, g \in H$ we have $w = Y(\varepsilon_H, f \cdot \varepsilon_H, g \cdot \varepsilon_H)$ belongs to one of the subtrees $\Gamma_{H'(T)}$, hence $[\varepsilon_H, w]$ defines an element of $H'(T) \subset H$. That is, H acts regularly on Γ_H .

Next, since

$$G \leq \langle K_T, \{\mathcal{D}(T) \mid T \in \mathcal{K}\} \rangle \leq H$$

then G embeds into H .

Finally, every end a of Γ_G uniquely corresponds to an end in Δ_G . Every end of Δ_G can be viewed as a reduced infinite path p_a in Δ_G originating at $v \in \Delta_G$ which is the image of $\varepsilon \in \Gamma_G$. Observe that two ends a and b of Γ_G are G -equivalent if and only if $\pi(p_a) = \pi(p_b)$ in Ψ_G .

Denote $\Delta_H = \Gamma_H / \sim$, where “ \sim ” is the equivalence of \mathbb{Z}^{n-1} -close points. Since $\psi : \Gamma_G \rightarrow \Gamma_H$ is an embedding then Δ_G embeds into Δ_H and with an abuse of notation we are going to denote this embedding by ψ . Let $w = \psi(v)$.

Let a and b be non- G -equivalent ends of Γ_G and let

$$p_a = v \ v_1 \ v_2 \ \cdots, \quad p_b = v \ u_1 \ u_2 \ \cdots.$$

Assume that $\psi(a)$ and $\psi(b)$ are H -equivalent in Γ_H , that is, there exists $h \in H$ such that $h \cdot p_{\psi(a)} = p_{\psi(b)}$. Since $p_{\psi(a)}$ and $p_{\psi(b)}$ have the same origin w then $h \cdot w = w$, that is, $h \in \text{Stab}_H(T'_0)$, where T'_0 is a \mathbb{Z}^{n-1} -subtree of Γ_H containing $\psi(T_0)$. Moreover, if $e_1 = (w, \psi(v_1))$, $f_1 = (w, \psi(u_1))$ then $h \cdot e_1 = f_1$ and it follows that $h \cdot a_1 = b_1$, where a_1 and b_1 are ends of $\psi(T_0)$ corresponding to e_1 and f_1 . By the assumption of the theorem there exists $\phi(g_1) \in \text{Stab}_{\phi(G)}(\psi(T_0))$ such that $\phi(g_1) \cdot a_1 = b_1$, so, $\phi(g_1) \cdot \psi(v_1) = \psi(u_1)$. Since $\phi : G \rightarrow H$ and $\psi : \Gamma_G \rightarrow \Gamma_H$ are embeddings, it follows that $g_1 \cdot v_1 = u_1$ and the images of $\pi(u_1) = \pi(v_1)$ in Δ_G .

Continuing in the same way we obtain $\pi(u_i) = \pi(v_i)$, $i \geq 1$ in Δ_G , so, a and b are G -equivalent which gives a contradiction with the assumption that $\psi(a)$ and $\psi(b)$ are H -equivalent in Γ_H . \square

Theorem 48. [70] *Let G be a finitely generated subgroup of $CDR(\mathbb{Z}^n, X)$, where X is arbitrary. Then there exists a finite alphabet Y and an embedding $\phi : G \rightarrow H$, where H is a finitely generated subgroup of $CDR(\mathbb{Z}^n, Y)$ with a regular length function, such that*

- (a) $|g|_G = |\phi(g)|_H$ for every $g \in G$,
- (b) if A is a maximal abelian subgroup of G then $\phi(A)$ is a maximal abelian subgroup of H ,
- (c) if A and B are maximal abelian subgroups of G which are non-conjugate in G then $\phi(A)$ and $\phi(B)$ are non-conjugate in H .

Proof. We use the induction on n . If $n = 1$ then the result follows from Lemma 51. Finally, the induction step follows from Theorem 47. \square

As a simple corollary of the above theorem we get the following result.

Theorem 49. [70] *Every finitely generated \mathbb{Z}^n -free group G has a length-preserving embedding into a finitely generated complete \mathbb{Z}^n -free group H .*

10.5 Description of \mathbb{Z}^n -free groups

Given two $\mathbb{Z}[t]$ -free groups G_1, G_2 and maximal abelian subgroups $A \leq G_1, B \leq G_2$ such that

- (a) A and B are cyclically reduced with respect to the corresponding embeddings of G_1 and G_2 into infinite words,
- (b) there exists an isomorphism $\phi : A \rightarrow B$ such that $|\phi(a)| = |a|$ for any $a \in A$.

Then we call the amalgamated free product

$$\langle G_1, G_2 \mid A \stackrel{\phi}{=} B \rangle$$

the *length-preserving amalgam* of G_1 and G_2 .

Given a $\mathbb{Z}[t]$ -free group H and non-conjugate maximal abelian subgroups $A, B \leq H$ such that

- (a) A and B are cyclically reduced with respect to the embedding of H into infinite words,
- (b) there exists an isomorphism $\phi : A \rightarrow B$ such that $|\phi(a)| = |a|$ and a is not conjugate to $\phi(a)^{-1}$ in H for any $a \in A$.

Then we call the HNN extension

$$\langle H, t \mid t^{-1}At = B \rangle$$

the *length-preserving separated HNN extension* of H .

As a corollary of Theorem 43 and Theorem 44 we get the description of complete \mathbb{Z}^n -free groups in the following form.

Theorem 50. *A finitely generated group G is complete \mathbb{Z}^n -free if and only if it can be obtained from free groups by finitely many length-preserving separated HNN extensions and centralizer extensions.*

Using the results of previous Subsection 10.4 we can prove a theorem similar to Theorem 50 for the class of finitely generated $\mathbb{Z}[t]$ -free groups (not necessarily complete). For that we need the following theorems.

Theorem 51. *Let G_1 and G_2 be finitely generated \mathbb{Z}^n -free groups. Then the length-preserving amalgam*

$$G = \langle G_1, G_2 \mid A \stackrel{\phi}{=} B \rangle$$

of G_1 and G_2 is a finitely generated $\mathbb{Z}^{n'}$ -free group and the length function on G extends the ones on G_1 and G_2 .

Proof. Both G_1 and G_2 are finitely generated $\mathbb{Z}[t]$ -free. Hence, the free product $P = G_1 * G_2$ is also finitely generated $\mathbb{Z}[t]$ -free (see Example 16 and [24, Proposition 5.1.1]) and canonical embeddings of G_1 and G_2 into P preserve length. Note that A and B are non-conjugate maximal abelian subgroups of P .

Next, by Theorem 48, there exists a finitely generated complete $\mathbb{Z}[t]$ -free group H such that P embeds into H and the embedding preserves length. Moreover, A and B stay non-conjugate maximal abelian in H .

By Theorem 44, the HNN extension $K = \langle H, t \mid t^{-1}At = B \rangle$ is a finitely generated complete $\mathbb{Z}[t]$ -free group. Observe that K contains a subgroup $K_0 = \langle t^{-1}\widehat{G_1}t, \widehat{G_2} \rangle$, where $\widehat{G_i}$ denotes the copy of G_i through the embedding $G_i \hookrightarrow P \hookrightarrow K$. Finally, it is easy to see that K_0 is isomorphic to G . \square

Theorem 52. *Let H be a finitely generated \mathbb{Z}^n -free group. Then the length-preserving separated HNN extension*

$$G = \langle H, t \mid t^{-1}At = B \rangle$$

of H is a finitely generated $\mathbb{Z}^{n'}$ -free group and the length function on G extends the one on H .

Proof. The proof is very similar to the one of Theorem 51.

By Theorem 48, there exists a finitely generated complete $\mathbb{Z}[t]$ -free group P and an embedding $\psi : H \rightarrow P$ such that ψ preserves length and $\psi(A), \psi(B)$ are non-conjugate maximal abelian in P . Observe that since $\psi(A)$ and $\psi(B)$ are isomorphic to the original subgroups A and B , and $\phi : A \rightarrow B$ is an isomorphism, there exists a natural isomorphism from $\psi(A)$ to $\psi(B)$ which we also denote by ϕ . Since ψ preserves length, we have $|\phi(a)| = |a|$ for any $a \in \psi(A)$. Next, if there exists $g \in P$ such that $g^{-1}ag = \phi(a)$ for some $a \in \psi(A)$ then from the CSA property of P we get $g^{-1}\psi(A)g = \psi(B)$, that is, $\psi(A)$ and $\psi(B)$ are conjugate - a contradiction. It follows that both conditions (a) and (b) hold for $\psi(A)$ and $\psi(B)$.

By Theorem 44, the HNN extension $K = \langle P, t \mid t^{-1}\psi(A)t = \psi(B) \rangle$ is a finitely generated complete $\mathbb{Z}[t]$ -free group. Observe that K contains a subgroup $K_0 = \langle \psi(H), t \rangle$ which is isomorphic to G . \square

Theorem 53. *Let H be a finitely generated \mathbb{Z}^n -free group and let A be a maximal abelian subgroup of H . Then the centralizer extension*

$$G = \langle H, t \mid t^{-1}At = A \rangle$$

is a finitely generated $\mathbb{Z}^{n'}$ -free group and the length function on G extends the one on H .

Proof. By Theorem 48, there exists a finitely generated complete $\mathbb{Z}[t]$ -free group P and an embedding $\psi : H \rightarrow P$ such that ψ preserves length and $\psi(A)$ is maximal abelian in P .

By Theorem 44, the HNN extension $K = \langle P, t \mid t^{-1}\psi(A)t = \psi(A) \rangle$ is a finitely generated complete $\mathbb{Z}[t]$ -free group. Observe that K contains a subgroup $K_0 = \langle \psi(H), t \rangle$ which is isomorphic to G . \square

Theorem 54. *A finitely generated group G is \mathbb{Z}^n -free if and only if it can be obtained from free groups by a finite sequence of length-preserving amalgams, length-preserving separated HNN extensions, and centralizer extensions.*

Proof. Since every finitely generated $\mathbb{Z}[t]$ -free group embeds into a finitely generated complete $\mathbb{Z}[t]$ -free group, from Bass-Serre Theory we get the required. \square

11 Elimination process over finitely presented Λ -free groups

In this section we will describe the Elimination process over finitely presented Λ -free groups which we will use in Section 12 to prove Theorems 57, 58, 43. From now on we assume that $G = \langle X \mid R \rangle$ is a finitely presented group which acts freely and regularly on a Λ -tree, where Λ is a discretely ordered abelian group, or, equivalently, G can be represented by Λ -words over some alphabet Z and the length function on G induced from $CDR(\Lambda, Z)$ is regular. Let us fix the embedding $\xi : G \hookrightarrow CDR(\Lambda, Z)$ for the rest of this section. For all the details please refer to [68].

11.1 The notion of a generalized equation

Definition 1. *A combinatorial generalized equation Ω (which is convenient to visualize as shown on the picture below) consists of the following objects.*

1. *A finite set of bases $\mathcal{M} = BS(\Omega)$. The set of bases \mathcal{M} consists of $2n$ elements $\mathcal{M} = \{\mu_1, \dots, \mu_{2n}\}$. The set \mathcal{M} comes equipped with two functions: a function $\varepsilon : \mathcal{M} \rightarrow \{1, -1\}$ and an involution $\Delta : \mathcal{M} \rightarrow \mathcal{M}$ (that is, Δ is a bijection such that Δ^2 is an identity on \mathcal{M}). Bases μ and $\bar{\mu} = \Delta(\mu)$ are called dual bases. We denote bases by letters μ, λ , etc.*
2. *A set of boundaries $BD = BD(\Omega) = \{1, 2, \dots, \rho + 1\}$, that is, integer points of the interval $I = [1, \rho + 1]$. We use letters i, j , etc. for boundaries.*

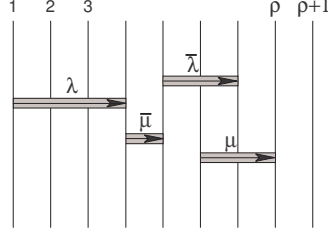


Figure 12: A typical generalized equation.

3. Two functions $\alpha : BS \rightarrow BD$ and $\beta : BS \rightarrow BD$. We call $\alpha(\mu)$ and $\beta(\mu)$ the initial and terminal boundaries of the base μ (or endpoints of μ). These functions satisfy the following conditions: for every base $\mu \in BS$: $\alpha(\mu) < \beta(\mu)$ if $\varepsilon(\mu) = 1$ and $\alpha(\mu) > \beta(\mu)$ if $\varepsilon(\mu) = -1$.
4. A set of boundary connections (p, λ, q) , where p is a boundary on λ (that is a number between $\alpha(\lambda)$ and $\beta(\lambda)$) and q on $\bar{\lambda}$. In this case we say that p and q are λ -tied. If (p, λ, q) is a boundary connection then $(q, \bar{\lambda}, p)$ is also a boundary connection. (The meaning of boundary connections will be explained in the transformation (ET5)).

With a combinatorial generalized equation Ω one can canonically associate a system of equations in variables $h = (h_1, \dots, h_p)$ (variables h_i are also called *items*). This system is called a *generalized equation*, and (slightly abusing the terminology) we denote it by the same symbol Ω , or $\Omega(h)$ specifying the variables it depends on. The generalized equation Ω consists of the following two types of equations.

1. Each pair of dual bases $(\lambda, \bar{\lambda})$ provides an equation

$$[h_{\alpha(\lambda)} h_{\alpha(\lambda)+1} \dots h_{\beta(\lambda)-1}]^{\varepsilon(\lambda)} = [h_{\alpha(\bar{\lambda})} h_{\alpha(\bar{\lambda})+1} \dots h_{\beta(\bar{\lambda})-1}]^{\varepsilon(\bar{\lambda})}.$$

These equations are called *basic equations*.

2. Every boundary connection (p, λ, q) gives rise to a *boundary equation*

$$[h_{\alpha(\lambda)} h_{\alpha(\lambda)+1} \dots h_{p-1}] = [h_{\alpha(\bar{\lambda})} h_{\alpha(\bar{\lambda})+1} \dots h_{q-1}],$$

if $\varepsilon(\lambda) = \varepsilon(\bar{\lambda})$ and

$$[h_{\alpha(\lambda)} h_{\alpha(\lambda)+1} \dots h_{p-1}] = [h_q h_{q+1} \dots h_{\alpha(\bar{\lambda})-1}]^{-1},$$

if $\varepsilon(\lambda) = -\varepsilon(\bar{\lambda})$.

Remark 8. We assume that every generalized equation comes from a combinatorial one.

Given a generalized equation $\Omega(h)$ one can define the *group of $\Omega(h)$*

$$G_\Omega = \langle h \mid \Omega(h) \rangle.$$

Definition 2. Let $\Omega(h) = \{L_1(h) = R_1(h), \dots, L_s(h) = R_s(h)\}$ be a generalized equation in variables $h = (h_1, \dots, h_\rho)$. A set $U = (u_1, \dots, u_\rho) \subseteq R(\Lambda, Z)$ of nonempty Λ -words is called a *solution of Ω* if:

1. all words $L_i(U), R_i(U)$ are reduced,
2. $L_i(U) = R_i(U)$, $i \in [1, s]$.

Observe that a solution U of $\Omega(h)$ defines a homomorphism $\xi_U : G_\Omega \rightarrow R(\Lambda, Z)$ induced by the mapping $h_i \rightarrow u_i$, $i \in [1, \rho]$ since after this substitution all the equations of $\Omega(h)$ turn into identities in $R(\Lambda, Z)$.

If we specify a particular solution U of a generalized equation Ω then we use a pair (Ω, U) .

Definition 3. A *cancelation table* $C(U)$ of a solution $U = (u_1, \dots, u_\rho)$ is defined as follows

$$C(U) = \{h_i^\epsilon h_j^\sigma \mid \text{there is cancelation in the product } u_i^\epsilon * u_j^\sigma, \text{ where } \epsilon, \sigma = \pm 1\}.$$

Definition 4. A solution U^+ of a generalized equation Ω is called *consistent* with a solution U if $C(U^+) \subseteq C(U)$.

11.2 From a finitely presented group to a generalized equation

Recall that $G = \langle X \mid R \rangle$ is finitely presented and let $X = \{x_1, \dots, x_n\}$ and $R = \{r_1(X), \dots, r_m(X)\}$. Adding, if necessary, auxiliary generators, we can assume that every relator involves at most three generators.

Since ξ is a homomorphism it follows that after the substitution $x_i \rightarrow \xi(x_i)$, $i \in [1, n]$ all products $r_i(\xi(X))$, $i \in [1, m]$ cancel out. Hence, we have finitely many *cancelation diagrams* over $CDR(\Lambda, Z)$, which give rise to a generalized equation Ω corresponding to the embedding $\xi : G \hookrightarrow CDR(\Lambda, Z)$.

The precise definition and all the details concerning cancelation diagrams over $CDR(\Lambda, Z)$ can be found in [58]. Briefly, a cancelation diagram for $r_i(\xi(X))$ can be viewed as a finite directed tree T_i in which every positive edge e has a label λ_e so that every occurrence x^δ , $\delta \in \{-1, 1\}$ of $x \in X$ in r_i corresponds to a reduced path $e_1^{\epsilon_1} \dots e_k^{\epsilon_k}$, where $\epsilon_i \in \{-1, 1\}$, in T_i and $\xi(x^\delta) = \lambda_{e_1}^{\epsilon_1} \circ \dots \circ \lambda_{e_k}^{\epsilon_k}$. In other words, each λ_e is a piece of some generator of G viewed as a Λ -word. Moreover, we assume that $|\lambda_e|$ is known (since we know the homomorphism ξ).

Now we would like to construct a generalized equation Ω_i corresponding to T_i . Denote by $X(T_i)$ all generators of G which appear in r_i . Next, consider a segment J in Λ of length

$$\sum_{x \in X(T_i)} |\xi(x)|$$

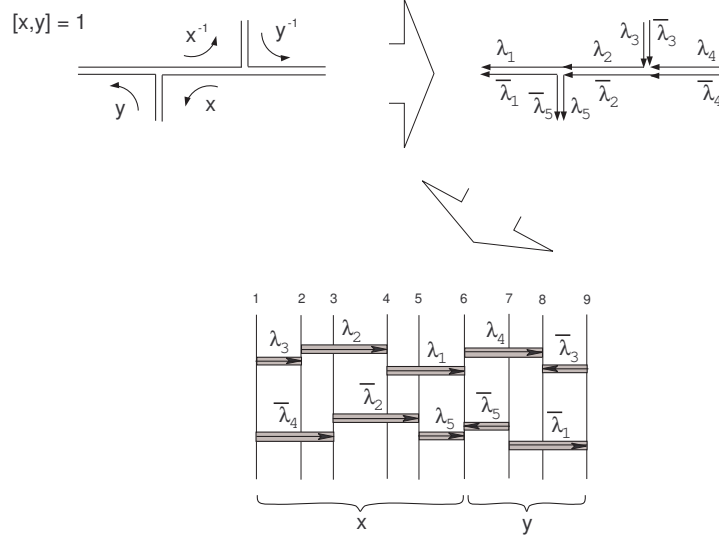


Figure 13: From the cancelation diagram for the relation $[x, y] = 1$ to the generalized equation.

which is naturally divided by the lengths of $\xi(x)$, $x \in X(T_i)$ into subsegments with respect to any given order on $X(T_i)$. Since every $\xi(x)$, $x \in X(T_i)$ splits into at least one reduced product $\lambda_{e_1}^{\epsilon_1} \circ \dots \circ \lambda_{e_k}^{\epsilon_k}$, every such splitting gives a subdivision of the corresponding subsegment of J . Hence, we subdivide J using all product representations of all $\xi(x)$, $x \in X(T_i)$. As a result we obtain a subdivision of J into ρ_i items whose endpoints become boundaries of Ω_i . Observe that each λ_e appears exactly twice in the products representing some $\xi(x)$, $x \in X(T_i)$ and each such entry covers several adjacent items of J . This pair of entries defines a pair of dual bases $(\lambda_e, \overline{\lambda_e})$. Hence,

$$\mathcal{M}_i = BS(\Omega_i) = \{\lambda_e, \overline{\lambda_e} \mid e \in E(T_i)\}.$$

$\epsilon(\lambda_e)$ depends on the sign of λ_e in the corresponding product representing a variable from $X(T_i)$ (similarly for $\overline{\lambda_e}$).

In the same way one can construct T_i and the corresponding Ω_i for each r_i , $i \in [1, m]$. Combining all combinatorial generalized equations Ω_i , $i \in [1, m]$ we obtain the equation Ω with items h_1, \dots, h_ρ and bases $\mathcal{M} = \cup_i \mathcal{M}_i$. By definition

$$G_\Omega = \langle h_1, \dots, h_\rho \mid \Omega(h_1, \dots, h_\rho) \rangle.$$

At the same time, since each item can be obtained in the form

$$(\lambda_{i_1}^{\epsilon_1} \circ \dots \circ \lambda_{i_k}^{\epsilon_k}) * (\lambda_{j_1}^{\delta_1} \circ \dots \circ \lambda_{j_l}^{\delta_l})^{-1},$$

it follows that G_Ω can be generated by \mathcal{M} with the relators obtained by rewriting $\Omega(h_1, \dots, h_\rho)$ in terms of \mathcal{M} .

It is possible to transform the presentation $\langle h_1, \dots, h_\rho \mid \Omega \rangle$ into $\langle X \mid R \rangle$ using Tietze transformations as follows. From the cancelation diagrams constructed for each relator in R it follows that $x_i = w_i(h_1, \dots, h_\rho) = w_i(\bar{h})$, $i \in [1, n]$. Hence

$$\langle h_1, \dots, h_\rho \mid \Omega \rangle \simeq \langle h_1, \dots, h_\rho, X \mid \Omega \cup \{x_i = w_i(\bar{h}), i \in [1, n]\} \rangle.$$

Next, from the cancelation diagrams it follows that R is a set of consequences of $\Omega \cup \{x_i = w_i(\bar{h}), i \in [1, n]\}$, hence,

$$\langle h_1, \dots, h_\rho \mid \Omega \rangle \simeq \langle h_1, \dots, h_\rho, X \mid \Omega \cup \{x_i = w_i(\bar{h}), i \in [1, n]\} \cup R \rangle.$$

Finally, since the length function on G is regular, for each h_i there exists a word $u_i(X)$ such that $h_i = u_i(\xi(X))$ and all the equations in $\Omega \cup \{x_i = w_i(\bar{h}), i \in [1, n]\}$ follow from R after we substitute h_i by $u_i(X)$ for each i . It follows that

$$\begin{aligned} & \langle h_1, \dots, h_\rho, X \mid \Omega \cup \{x_i = w_i(\bar{h}), i \in [1, n]\} \cup R \rangle \\ & \simeq \langle h_1, \dots, h_\rho, X \mid \Omega \cup \{x_i = w_i(\bar{h}), i \in [1, n]\} \cup R \cup \{h_j = u_j(X), j \in [1, \rho]\} \rangle \\ & \simeq \langle X \mid R \rangle. \end{aligned}$$

It follows that $G \simeq G_\Omega$.

Let \tilde{G} be a finitely presented group with a free length function in Λ (not necessary regular). It can be embedded isometrically in the group \hat{G} with a free regular length function in Λ by [27]. That group can be embedded in $R(\Lambda', X)$. When we make a generalized equation Ω for \tilde{G} , we have to add only finite number of elements from \hat{G} . Let G be a subgroup generated in \hat{G} by \tilde{G} and these elements. Then G is the quotient of G_Ω containing \tilde{G} as a subgroup.

11.3 Elementary transformations

In this subsection we describe *elementary transformations* of generalized equations. Let (Ω, U) be a generalized equation together with a solution U . An elementary transformation (ET) associates to a generalized equation (Ω, U) a generalized equation (Ω_1, U_1) and an epimorphism $\pi : G_\Omega \rightarrow G_{\Omega_1}$ such that for the solution U_1 the following diagram commutes

$$\begin{array}{ccc} G_\Omega & \xrightarrow{\pi} & G_{\Omega_1} \\ \xi_U \downarrow & \swarrow \xi_{U_1} & \\ R(\Lambda, Z) & & \end{array}$$

One can view (ET) as a mapping $ET : (\Omega, U) \rightarrow (\Omega_1, U_1)$.

(ET1) (*Cutting a base (see Fig. 14)*). Let λ be a base in Ω and p an internal boundary of λ (that is, $p \neq \alpha(\lambda), \beta(\lambda)$) with a boundary connection (p, λ, q) . Then we cut the base λ at p into two new bases λ_1 and λ_2 , and cut $\bar{\lambda}$ at q into the bases $\bar{\lambda}_1$ and $\bar{\lambda}_2$.

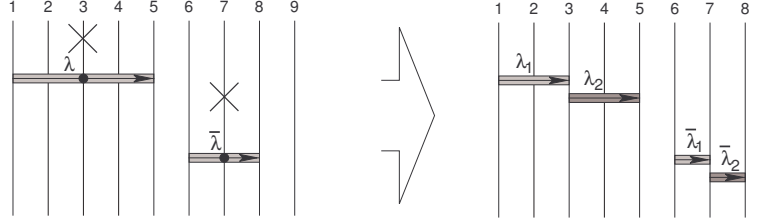


Figure 14: Elementary transformation (ET1).

(ET2) (*Transferring a base* (see Fig. 15)). If a base λ of Ω contains a base μ (that is, $\alpha(\lambda) \leq \alpha(\mu) < \beta(\mu) \leq \beta(\lambda)$) and all boundaries on μ are λ -tied by boundary some connections then we transfer μ from its location on the base λ to the corresponding location on the base $\bar{\lambda}$.

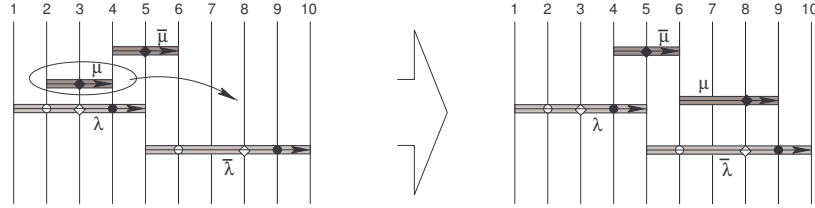


Figure 15: Elementary transformation (ET2).

(ET3) (*Removal of a pair of matched bases* (see Fig. 16)). If the bases λ and $\bar{\lambda}$ are *matched* (that is, $\alpha(\lambda) = \alpha(\bar{\lambda}), \beta(\lambda) = \beta(\bar{\lambda})$) then we remove $\lambda, \bar{\lambda}$ from Ω .

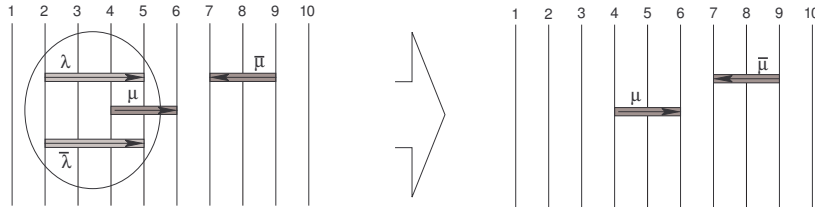


Figure 16: Elementary transformation (ET3).

Remark 9. Observe, that Ω and Ω_1 , where $\Omega_1 = ETi(\Omega)$ for $i \in \{1, 2, 3\}$ have the same set of variables h and the bijection $h_i \rightarrow h_i$, $i \in [1, \rho]$

induces an isomorphism $G_\Omega \rightarrow G_{\Omega_1}$. Moreover, U is a solution of Ω if and only if U is a solution of Ω_1 .

- (ET4) (*Removal of a lone base (see Fig. 17)*). Suppose, a base λ in Ω does not intersect any other base, that is, the items $h_{\alpha(\lambda)}, \dots, h_{\beta(\lambda)-1}$ are contained only inside of the base λ .

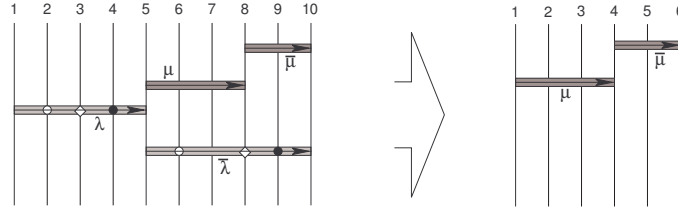


Figure 17: Elementary transformation (ET4).

Suppose also that all boundaries in λ are λ -tied, that is, for every i ($\alpha(\lambda) < i \leq \beta(\lambda)-1$) there exists a boundary $b(i)$ such that $(i, \lambda, b(i))$ is a boundary connection in Ω . Then we remove the pair of bases λ and $\bar{\lambda}$ together with all the boundaries $\alpha(\lambda)+1, \dots, \beta(\lambda)-1$ (and rename the rest $\beta(\lambda)-\alpha(\lambda)-1$ of the boundaries correspondingly).

We define the isomorphism $\pi : G_\Omega \rightarrow G_{\Omega_1}$ as follows:

$$\pi(h_j) = h_j \text{ if } j < \alpha(\lambda) \text{ or } j \geq \beta(\lambda)$$

$$\pi(h_i) = \begin{cases} h_{b(i)} \cdots h_{b(i)-1}, & \text{if } \varepsilon(\lambda) = \varepsilon(\bar{\lambda}), \\ h_{b(i)} \cdots h_{b(i-1)-1}, & \text{if } \varepsilon(\lambda) = -\varepsilon(\bar{\lambda}) \end{cases}$$

for $\alpha+1 \leq i \leq \beta(\lambda)-1$.

- (ET5) (*Introduction of a boundary (see Fig. 18)*). Suppose a point p in a base λ is not λ -tied. The transformation (ET5) λ -ties it. To this end, denote by u_λ the element of $CDR(\Lambda, Z)$ corresponding to λ and let u'_λ be the beginning of this word ending at p . Then we perform one of the following two transformations according to where the end of u'_λ on $\bar{\lambda}$ is situated:

- (a) If the end of u'_λ on $\bar{\lambda}$ is situated on the boundary q then we introduce the boundary connection (p, λ, q) . In this case the corresponding isomorphism $\pi : G_\Omega \rightarrow G_{\Omega_1}$ is induced by the bijection $h_i \rightarrow h_i$, $i \in [1, \rho]$. (If we began with the group \tilde{G} with non-regular length function, this is the only place where $\pi : G_\Omega \rightarrow G_{\Omega_1}$ may be a proper epimorphism, but its restriction on \tilde{G} is still an isomorphism.)
- (b) If the end of u'_λ on $\bar{\lambda}$ is situated between q and $q+1$ then we introduce a new boundary q' between q and $q+1$ (and rename all the boundaries), and also introduce a new boundary connection (p, λ, q') .

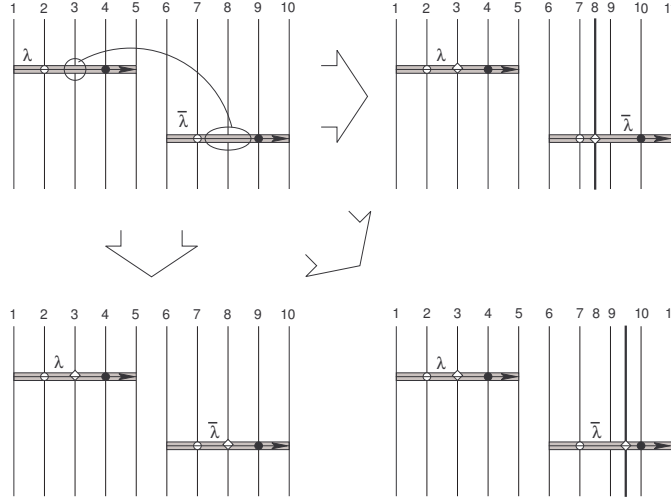


Figure 18: Elementary transformation (ET5).

In this case the corresponding isomorphism $\pi : G_\Omega \rightarrow G_{\Omega_1}$ is induced by the map $\pi(h) = h$, if $h \neq h_q$, and $\pi(h_q) = h_{q'}h_{q'+1}$.

11.4 Derived transformations and auxiliary transformations

In this section we define complexity of a generalized equation and describe several useful “derived” transformations of generalized equations. Some of them can be realized as finite sequences of elementary transformations, others result in equivalent generalized equations but cannot be realized by finite sequences of elementary moves.

A boundary is *open* if it is an internal boundary of some base, otherwise it is *closed*. A section $\sigma = [i, \dots, i+k]$ is said to be *closed* if the boundaries i and $i+k$ are closed and all the boundaries between them are open.

Sometimes it will be convenient to subdivide all sections of Ω into *active* (denoted $A\Sigma_\Omega$) and *non-active* sections. For an item h denote by $\gamma(h)$ the number of bases containing h . An item h is called *free* if it meets no base, that is, if $\gamma(h) = 0$. Free variables are transported to the very end of the interval behind all items in Ω and they become non-active.

(D1) (*Closing a section*). Let σ be a section of Ω . The transformation (D1) makes the section σ closed. Namely, (D1) cuts all bases in Ω through the end-points of σ .

(D2) (*Transporting a closed section*). Let σ be a closed section of a generalized equation Ω . We cut σ out of the interval $[1, \rho_\Omega]$ together with all the bases

on σ and put σ at the end of the interval or between any two consecutive closed sections of Ω . After that we correspondingly re-enumerate all the items and boundaries of the latter equation to bring it to the proper form. Clearly, the original equation Ω and the new one Ω' have the same solution sets and their coordinate groups are isomorphic

(D3) (*Moving free variables to the right*). Suppose that Ω contains a free variable h_q in an active section. Here we close the section $[q, q+1]$ using (D1), transport it to the very end of the interval behind all items in Ω using (D2). In the resulting generalized equation Ω' the transported section becomes a non-active section.

(D4) (*Deleting a complete base*). A base μ of Ω is called *complete* if there exists a closed section σ in Ω such that $\sigma = [\alpha(\mu), \beta(\mu)]$.

Suppose μ is an active complete base of Ω and σ is a closed section such that $\sigma = [\alpha(\mu), \beta(\mu)]$. In this case using (ET5), we transfer all bases from μ to $\bar{\mu}$, then using (ET4) we remove the lone base μ together with the section σ .

(D5) (*Linear elimination*).

We first explain how to find *the kernel of the generalized equation*. We will give a definition of eliminable base for an equation Ω that does not have any boundary connections. An active base $\mu \in A\Sigma_\Omega$ is called *eliminable* if at least one of the following holds

- (a) μ contains an item h_i with $\gamma(h_i) = 1$,
- (b) at least one of the boundaries $\alpha(\mu), \beta(\mu)$ is different from $1, \rho+1$ and does not touch any other base (except for μ).

The process of finding the kernel works as follows. We cut the bases of Ω along all the boundary connections thus obtaining the equation without boundary connections, then consequently remove eliminable bases until no eliminable base is left in the equation. The resulting generalized equation is called the *kernel* of Ω and we denote it by $Ker(\Omega)$. One can show that it does not depend on a particular removal process. We say that an item h_i belongs to the kernel ($h_i \in Ker(\Omega)$), if h_i belongs to at least one base in the kernel. Notice that the kernel can be empty.

Lemma 55. [68] *If Ω is a generalized equation, then*

$$G_\Omega \simeq G_{Ker(\Omega)} * F(K)$$

where $F(K)$ is a free group on K . The set K can be empty.

Suppose that in Ω there is h_i in an active section with $\gamma(h_i) = 1$ and such that $|h_i|$ is comparable with the length of the active section. In this case we say that Ω is *linear in h_i* .

If Ω is linear in h_i in an active section such that both boundaries i and $i + 1$ are closed then we remove the closed section $[i, i + 1]$ together with the lone base using (ET4).

If there is no such h_i but Ω is linear in some h_i in an active section such that one of the boundaries $i, i + 1$ is open, say $i + 1$, and the other is closed, then we perform (ET5) and μ -tie $i + 1$ through the only base μ it intersects. Next, using (ET1) we cut μ in $i + 1$ and then we delete the closed section $[i, i + 1]$ by (ET4).

Suppose there is no h_i as above but Ω is linear in some h_i in an active section such that both boundaries i and $i + 1$ are open. In addition, assume that there is a closed section σ containing exactly two (not matched) bases μ_1 and μ_2 , such that $\sigma = \sigma(\mu_1) = \sigma(\mu_2)$ and in the generalized equation $\tilde{\Omega}$ (see the derived transformation (D3)) all the bases obtained from μ_1, μ_2 by (ET1) in constructing $\tilde{\Omega}$ from Ω , do not belong to the kernel of $\tilde{\Omega}$. Here, using (ET5), we μ_1 -tie all the boundaries inside of μ_1 , then using (ET2) we transfer μ_2 onto $\bar{\mu}_1$, and remove μ_1 together with the closed section σ using (ET4).

Suppose now that Ω satisfies the first assumption of the previous paragraph and does not satisfy the second one. In this event we close the section $[i, i + 1]$ using (D1) and remove it using (ET4).

Lemma 56. [68] *Suppose that the process of linear elimination continues infinitely and there is a corresponding sequence of generalized equations*

$$\Omega \rightarrow \Omega_1 \rightarrow \cdots \rightarrow \Omega_k \rightarrow \cdots$$

Then

- (a) ([64, Lemma 15]) *The number of different generalized equations that appear in the process is finite. Therefore some generalized equation appears in this process infinitely many times.*
- (b) ([64, Lemma 15]) *If $\Omega_j = \Omega_k$, $j < k$ then $\pi(j, k)$ is an isomorphism, invariant with respect to the kernel, namely $\pi(j, k)(h_i) = h_i$ for any variable h_i that belongs to some base in $\text{Ker}(\Omega)$.*
- (c) ([68, Lemma 7]) *The interval for the equation Ω_j can be divided into two disjoint parts, each being the union of closed sections, such that one part is a generalized equation $\text{Ker}(\Omega)$ and the other part is non-empty and corresponds to a generalized equation Ω' , such that $G_{\Omega'} = F(K)$ is a free group on variables K and $G_{\Omega} = G_{\text{Ker}(\Omega)} * F(K)$.*

(D6) (*Tietze cleaning*). Suppose that in Ω is linear in some h_i in an active section such that $|h_i|$ is comparable with the length of the active section. This transformation consists of four transformations performed consecutively

- (a) linear elimination: if the process of linear elimination goes infinitely we replace the equation by its kernel,

- (b) deleting all pairs of matched bases,
- (c) deleting all complete bases,
- (d) moving all free variables to the right.

(D7) (*Entire transformation*). We need a few definitions. A base μ of the equation Ω is called a *leading* base if $\alpha(\mu) = 1$. A leading base is said to be *maximal* (or a *carrier base*) if $\beta(\lambda) \leq \beta(\mu)$ for any other leading base λ . Let μ be a carrier base of Ω . Any active base $\lambda \neq \mu$ with $\beta(\lambda) \leq \beta(\mu)$ is called a *transfer* base (with respect to μ).

Suppose now that Ω is a generalized equation with $\gamma(h_i) \geq 2$ for each h_i in the active part of Ω and such that $|h_i|$ is comparable with the length of the active part. *Entire transformation* is a sequence of elementary transformations which are performed as follows. We fix a carrier base μ of Ω . We transfer all transfer bases from μ onto $\bar{\mu}$. Now, there exists some $i < \beta(\mu)$ such that h_1, \dots, h_i belong to only one base μ , while h_{i+1} belongs to at least two bases. Applying (ET1) we cut μ along the boundary $i + 1$. Finally, applying (ET4) we delete the section $[1, i + 1]$.

11.5 Complexity of a generalized equation and Delzant-Potyagailo complexity $c(G)$ of a group G

Denote by ρ_A the number of variables h_i in all active sections of Ω , by $n_A = n_A(\Omega)$ the number of bases in all active sections of Ω , by ν' the number of open boundaries in the active sections, and by σ' the number of closed boundaries in the active sections.

For a closed section $\sigma \in \Sigma_\Omega$ denote by $n(\sigma), \rho(\sigma)$ the number of bases and, respectively, variables in σ .

$$\rho_A = \rho_A(\Omega) = \sum_{\sigma \in A\Sigma_\Omega} \rho(\sigma),$$

$$n_A = n_A(\Omega) = \sum_{\sigma \in A\Sigma_\Omega} n(\sigma).$$

The *complexity* of the generalized equation Ω is the number

$$\tau = \tau(\Omega) = \sum_{\sigma \in A\Sigma_\Omega} \max\{0, n(\sigma) - 2\}.$$

Notice that the entire transformation (D7) as well as the cleaning process (D4) do not increase complexity of equations.

Below we recall Delzant-Potyagailo's result (see [33]). A family \mathcal{C} of subgroups of a torsion-free group G is called *elementary* if

- (a) \mathcal{C} is closed under taking subgroups and conjugation,
- (b) every $C \in \mathcal{C}$ is contained in a maximal subgroup $\bar{C} \in \mathcal{C}$,

- (c) every $C \in \mathcal{C}$ is small (does not contain F_2 as a subgroup),
- (d) all maximal subgroups from \mathcal{C} are malnormal.

G admits a *hierarchy* over \mathcal{C} if the process of decomposing G into an amalgamated product or an HNN-extension over a subgroup from \mathcal{C} , then decomposing factors of G into amalgamated products and/or HNN-extensions over a subgroup from \mathcal{C} etc. eventually stops.

Proposition 8. ([33]) *If G is a finitely presented group without 2-torsion and \mathcal{C} is a family of elementary subgroups of G then G admits a hierarchy over \mathcal{C} .*

Corollary 15. [68] *If G is a finitely presented Λ -free group then G admits a hierarchy over the family of all abelian subgroups.*

There is a notion of complexity of a group G defined in [33] and denoted by $c(G)$. We will only use the following statement that follows from there.

Proposition 9. ([33]) *If G is a non-trivial free product of finitely presented groups G_1 and G_2 . Then $c(G_i) < c(G)$, $i = 1, 2$. Let G be a finitely presented freely indecomposable Λ -free group (therefore CSA). Let Γ be an abelian decomposition of G as a fundamental group of a graph of groups with at least two vertices with non-cyclic vertex groups, maximal abelian subgroups being elliptic, and with each edge group being maximal abelian at least in one of its vertex groups. Then for each vertex group G_v , $c(G_v) < c(G)$.*

11.6 Rewriting process for Ω

In this section we describe a rewriting process (elimination process) for a generalized equation Ω and its solution corresponding to G . Performing the elimination process we eventually detect a decomposition of G as a free product or (if it is freely indecomposable) as the fundamental group of a graph of groups with vertex groups of three types: QH vertex groups, abelian vertex groups (corresponding to periodic structures, see below), non-QH, non-abelian vertex groups (we will call them *weakly rigid* meaning that we do not split them in this particular decomposition). We also can detect splitting of G as an HNN-extension with stable letter infinitely longer than generators of the abelian associated subgroups. After obtaining such a decomposition we continue the elimination process with the generalized equation corresponding to free factors of G or to weakly rigid subgroups of G (we will show that this generalized equation can be naturally obtained from the generalized equation Ω .) The Delzant-Potyagailo complexity of factors in a free decomposition and complexity of weakly rigid subgroups is smaller than the complexity of G . In the case of an HNN extension we will show that the complexity τ of the generalized equation corresponding to a weakly rigid subgroup is smaller than the complexity of Ω .

We assume that Ω is in standard form, namely, that transformations (ET3), (D3) and (D4) have been applied to Ω and that on each step we apply them to the generalized equation before applying any other transformation.

Let Ω be a generalized equation. We construct a path $T(\Omega)$ (with associated structures), as a directed path oriented from the root v_0 , starting at v_0 and proceeding by induction on the distance n from the root.

We start with a general description of the path $T(\Omega)$. For each vertex v in $T(\Omega)$ there exists a unique generalized equation Ω_v associated with v . The initial equation Ω is associated with the root v_0 , $\Omega_{v_0} = \Omega$. In addition there is a homogeneous system of linear equations Σ_v with integer coefficients on the lengths of variables of Ω_v . We take Σ_{v_0} to be empty. For each edge $v \rightarrow v'$ (here v and v' are the origin and the terminus of the edge) there exists an epimorphism $\pi(v, v') : G_{\Omega_v} \rightarrow G_{\Omega_{v'}}$ associated with $v \rightarrow v'$. If Ω was constructed for the group G with regular free length function, then $\pi(v, v')$ is an isomorphism. If Ω was constructed for the group \tilde{G} with free but not regular length function, then $\pi(v, v')$ is a monomorphism on \tilde{G} .

If

$$v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_s \rightarrow u$$

is a subpath of $T(\Omega)$, then by $\pi(v, u)$ we denote composition of corresponding isomorphisms

$$\pi(v, u) = \pi(v, v_1) \circ \cdots \circ \pi(v_s, u).$$

If $v \rightarrow v'$ is an edge then there exists a finite sequence of elementary or derived transformations from Ω_v to $\Omega_{v'}$ and the isomorphism $\pi(v, v')$ is a composition of the isomorphisms corresponding to these transformations. We also assume that active (and non-active) sections in $\Omega_{v'}$ are naturally inherited from Ω_v , if not said otherwise. Recall that initially all sections are active.

Suppose the path $T(\Omega)$ is constructed by induction up to level n and suppose v is a vertex at distance n from the root v_0 . We describe now how to extend the path from v . The construction of the outgoing edge at v depends on which case described below takes place at vertex v . We say that two elements are comparable (or their lengths are comparable) if they have the same height. There are three possible cases.

- **Linear case:** there exists h_i in the active part such that $|h_i|$ is comparable with the length of the active part and $\gamma(h_i) = 1$.
- **Quadratic and almost quadratic case:** $\gamma(h_i) = 2$ for all h_i in the active part such that $|h_i|$ is comparable with the length of the active part.
- **General JSJ case:** $\gamma(h_i) \geq 2$ for all h_i in the active part such that $|h_i|$ is comparable with the length of the active part, and there exists such h_i that $\gamma(h_i) > 2$.

11.6.1 Linear case

We apply Tietze cleaning (D6) at the vertex v_n if it is possible. We re-write the system of linear equations Σ_{v_n} in new variables and obtain a new system $\Sigma_{v_{n+1}}$.

If $\Omega_{v_{n+1}}$ splits into two parts, $\Omega_{v_{n+1}}^{(1)} = \text{Ker}(\Omega_{v_n})$ and $\Omega_{v_{n+1}}^{(2)}$ that corresponds to a free group $F(K)$, then when we put the free group section $\Omega_{v_{n+1}}^{(2)}$ into a non-active part we decrease both complexities τ and Delzant-Potyagailo's complexity c . It may happen that the kernel is empty, then the process terminates.

If it is impossible to apply Tietze cleaning (that is $\gamma(h_i) \geq 2$ for any h_i in the active part of Ω_v comparable to the length of the active part), we apply the entire transformation.

Termination condition: Ω_v does not contain active sections. In this case the vertex v is called a *leaf* or an *end vertex*.

11.6.2 Quadratic and almost quadratic case

Suppose that $\gamma_i = 2$ for each h_i in the active part comparable with the length of the active part of Ω_v . First of all, we fill in all the h'_i s in the active part such that $\gamma_i = 1$ by new (infinitely short) bases μ such that $\overline{\mu}$ covers a new variable that we add to the non-active part.

We apply the entire transformation (D7), then apply Tietze cleaning (D6), if possible, then again apply entire transformation, etc. In this process we, maybe, will remove some pairs of matching bases decreasing the complexity τ . Eventually we either end up with empty active part or the process will continue infinitely, and the number of bases in the active part will be constant.

Lemma 57. [68] *If a closed section σ has quadratic-coefficient bases, and the entire transformation goes infinitely, then after a finite number of steps there will be quadratic bases belonging to σ that have length infinitely larger than all participating quadratic coefficient bases on σ .*

If σ does not have quadratic-coefficient bases then $G_{R(\Omega_v)}$ splits as a free product with one factor being a closed surface group or a free group. We move σ into a non-active part and thus decrease the complexity τ .

We repeat the described transformation until there is no quadratic base on the active part that has length comparable with the length of the remaining active part. Then we consider the remaining generalized equation in the active part. We remove from the active part doubles of all quadratic coefficient bases that belong to non-active part (doing this we may create new boundaries). We will remember the relations corresponding to these pairs of bases. In this case the remaining generalized equation has smaller complexity τ . Relations corresponding to the quadratic sections that we made non-active show that G_Ω is an HNN-extension of the subgroup generated by the variables in the active part and (maybe) a free group. Removing these double bases we have to add equations to $\Sigma_{v_{i+1}}$ that guarantee that the associated cyclic subgroups are generated by elements of the same length.

11.6.3 General JSJ-case

Generalized equation Ω_v satisfies the condition $\gamma_i \geq 2$ for each h_i in the active part such that $|h_i|$ is comparable with the length of the active part, and $\gamma_i > 2$

for at least one such h_i . First of all, we fill in all the h'_i s in the active part such that $\gamma_i = 1$ by new (infinitely short) bases with doubles corresponding to free variables in the non-active part. We apply the transformation (D1) to close the quadratic part and put it in front of the interval.

(a) QH-subgroup case. Suppose that the entire transformation of the quadratic part (D7) goes infinitely. Then the quadratic part of Ω_v (or the initial section from the beginning of the quadratic part until the first base on the quadratic part that does not participate in the entire transformation) corresponds to a QH-vertex or to the representation of G_Ω as an HNN-extension, and there is a quadratic base (on this section) that is infinitely longer than all the quadratic coefficient bases (on this section). We work with the quadratic part the same way as in the quadratic case until there is no quadratic base satisfying the condition above. Then we make the quadratic section non-active, and consider the remaining generalized equation where we remove doubles of all the quadratic coefficient bases. We certainly have to remember that the bases that we removed express some variables in the quadratic part (that became non-active) in the variables in the active part. We have to add an equation to $\Sigma_{v_{i+1}}$ that guarantees that the associated cyclic subgroups are generated by elements of the same length. In this case the subgroup of G_Ω that is isomorphic to the coordinate group of the new generalized equation in active part is a vertex group in an abelian splitting of G_Ω and has smaller Delzant–Potyagailo complexity.

(b) QH-shortening

Lemma 58. [68] *Suppose that the quadratic part of Ω does not correspond to the HNN-splitting of G_Ω (there are only quadratic coefficient bases), or, we cannot apply the entire transformation to the quadratic part infinitely. In this case either G_Ω is a non-trivial free product or, applying the automorphism of G_Ω , one can replace the words corresponding to the quadratic bases in the quadratic part by their automorphic images such that in the new solution H^+ of Ω the length of the quadratic part is bounded by some function $f_1(\Omega)$ times the length of the non-quadratic part. Solution H^+ can be chosen consistent with H .*

If there is a matching pair, we replace G_Ω by the group obtained by removing a cyclic free factor corresponding to a matching pair. We also replace Ω by the generalized equation obtained by removing the matching pair. The Delzant–Potyagailo complexity decreases.

(c) Abelian splitting: short shift.

Proposition 10. [68] *Suppose Ω_v satisfies the following condition: the carrier base μ of the equation Ω_v intersects with its dual $\bar{\mu}$ (form an overlapping pair) and is at least twice longer than $|\alpha(\bar{\mu}) - \alpha(\mu)|$. Then G_{Ω_v} either splits as a fundamental group of a graph of groups that has a free abelian vertex group or splits as an HNN-extension with abelian associated subgroups.*

The proof is given in [68], it uses the technique of so-called periodic structures introduced by Razborov in his Ph.D thesis and almost repeats the proof given in [64] to show that the coordinate group of a generalized equation splits in this case as a fundamental group of a graph of groups that has a free abelian vertex group or splits as an HNN-extension with abelian associated subgroups. In the HNN-extension case, the base group is the coordinate group of the generalized equation obtained from the original by removing the corresponding stable letter. This reduces the complexity τ of the generalized equation.

(d) Abelian splitting: long shift. If Ω does not satisfy the conditions of (a)–(c), we perform QH-shortening, then apply the entire transformation and then, if possible, the transformation (D6).

Lemma 59. [68] *Let*

$$v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_r \rightarrow \cdots$$

be an infinite path in $T(\Omega)$. Then there exists a natural number N such that all the generalized equations in vertices v_n , $n \geq N$ satisfy the general JSJ-case (d).

Proof. Indeed, the Tietze cleaning either replaces the group by its proper free factor or decreases the complexity. Every time when the case (a) holds we replace G by some vertex group in a non-trivial abelian splitting of G . This can be done only finitely many times [33]. Every time when case (c) takes place, we decrease the complexity τ . \square

Proposition 11. [68] *The general JSJ case (d) cannot be repeated infinitely many times.*

12 Structure theorems for Λ -free groups

12.1 Finitely generated \mathbb{R} -free groups (Rips' Theorem)

R. Lyndon in [78] conjectured that any group acting freely on an \mathbb{R} -tree can be embedded into a free product of finitely many copies of \mathbb{R} . Counterexamples were initially given in [2] and [103].

Later, J. Morgan and P. Shalen in [87] showed that the fundamental groups of closed surfaces (except non-orientable of genus 1, 2 and 3) are \mathbb{R} -free. Since such groups are not free products of subgroups of \mathbb{R} this gives a wide class of counterexamples to Lyndon's Conjecture.

In 1991 I. Rips came with an idea of a proof of the Morgan and Shalen conjecture about finitely generated \mathbb{R} -free groups. This result can be formulated as follows.

Theorem 55. [40],[14] (*Rips' Theorem*) *Let G be a finitely generated group acting freely and without inversions on an \mathbb{R} -tree. Then G can be written as a free product $G = G_1 * \cdots * G_n$ for some integer $n \geq 1$, where each G_i is either a finitely generated free abelian group, or the fundamental group of a closed surface.*

Proof. To prove the theorem we are going to use the techniques of Section 11. In this case $\Lambda = \mathbb{R}$. Suppose G is a finitely presented group with free Lyndon length function L in \mathbb{R} . By Theorem 60, G can be embedded into a finitely presented group with a free regular length function in \mathbb{R} , so we assume from the beginning that G has a free regular length function in \mathbb{R} . By Corollary 5 there exists an embedding $\psi : G \rightarrow CDR(\mathbb{Z} \oplus \mathbb{R}, X)$ such that $|\psi(g)| = (0, L(g))$ for any $g \in G$.

We construct the generalized equation Ω for G and apply the elimination process Ω . Linear case always splits off free factors of G . Quadratic case, almost quadratic case, general JSJ case 11.6.3 (a) will produce closed surface groups factors. Indeed, in these cases, by Lemma 57, the height of some quadratic bases is higher than the height of quadratic coefficient bases. Since all the bases have length $(0, r)$, $r \in \mathbb{R}$, there are no quadratic coefficient bases in these cases. General JSJ case (c) for Λ produces abelian vertex groups corresponding to periodic structures and HNN-extensions with stable letter infinitely longer than the generators of associated abelian subgroups. Since $\Lambda = \mathbb{R}$, we do not have such HNN-extensions. If the edge group were non-trivial, then applying automorphisms of G we could shorten generators of the abelian vertex group. Namely, there exists a number N depending only on Ω such that the carrier base μ of the current equation Ω_v intersects with its double and in no longer than $N|\alpha(\bar{\mu}) - \alpha(\mu)|$. This situation is similar to the general JSJ case (d), because the length of μ is bounded in terms of $|\alpha(\bar{\mu}) - \alpha(\mu)|$. One can similarly prove that it cannot be repeated infinitely many times. Therefore, the edge group of the abelian vertex group is trivial.

We have shown that G is a free product of free abelian groups and closed surface groups (notice that a free group is also a free product of free abelian (cyclic) groups). Therefore any subgroup of G has the same structure. Since there is no proper infinite chain of quotients of G that are also free products of free abelian groups and closed surface groups, we can prove the theorem for a finitely generated group \tilde{G} by adding relations of G one-by-one and considering the chain of finitely presented quotients. \square

Rips' Theorem does not hold for infinitely generated groups. Counterexamples were given in [39] and [130]. In particular, Dunwoody showed that both groups

$$G_1 = \langle a_1, b_1, a_2, b_2, \dots \mid b_1 = [a_2, b_2], b_2 = [a_3, b_3], \dots \rangle$$

and

$$G_2 = \langle a_1, b_1, a_2, b_2, \dots \mid b_1 = a_2^2 b_2^2, b_2 = a_3^2 b_3^2, \dots \rangle$$

are \mathbb{R} -free but cannot be decomposed as free products of surface groups and subgroups of \mathbb{R} .

Recently, Berestovskii and Plaut gave a new method to construct \mathbb{R} -free groups [11]. In particular, they provide new examples of \mathbb{R} -free groups that are not free products of free abelian and surface groups. In fact, some of these groups are locally free but not free (so, obviously, not subgroups of free products of free abelian and surface groups).

12.2 Finitely generated \mathbb{R}^n -free groups

In 2004 Guirardel proved the following result that reveals the structure of finitely generated \mathbb{R}^n -free groups, which is reminiscent of the Bass' structural theorem for \mathbb{Z}^n -free groups. This is not by chance, since every \mathbb{Z}^n -free group is also \mathbb{R}^n -free, and ordered abelian groups \mathbb{Z}^n and \mathbb{R}^n have a similar convex subgroup structure. However, it is worth to point out that the original Bass argument for $\Lambda = \mathbb{Z} \oplus \Lambda_0$ does not work in the case of $\Lambda = \mathbb{R} \oplus \Lambda_0$.

Theorem 56. [49] *Let G be a finitely generated, freely indecomposable \mathbb{R}^n -free group. Then G can be represented as the fundamental group of a finite graph of groups, where edge groups are cyclic and each vertex group is a finitely generated \mathbb{R}^{n-1} -free.*

In fact, there is a more detailed version of this result, Theorem 7.2 in [49], which is rather technical, but gives more for applications. Observe also that neither Theorem 56 nor the more detailed version of it, does not "characterize" finitely generated \mathbb{R}^n -free groups, i.e. the converse of the theorem does not hold. Nevertheless, the result is very powerful and gives several important corollaries.

Corollary 16. [49] *Every finitely generated \mathbb{R}^n -free group is finitely presented.*

This comes from Theorem 56 and elementary properties of free constructions by induction on n .

Theorem 56 and the Combination Theorem for relatively hyperbolic groups proved by F. Dahmani in [30] imply the following.

Corollary 17. *Every finitely generated \mathbb{R}^n -free group is hyperbolic relative to its non-cyclic abelian subgroups.*

A lot is known about groups which are hyperbolic relative to its maximal abelian subgroups (*toral relatively hyperbolic groups*), so all of this applies to \mathbb{R}^n -free groups. We do not mention any of these results here, because we discuss their much more general versions in the next section in the context of Λ -free groups for arbitrary Λ .

12.3 Finitely presented Λ -free groups

In this section we discuss recent results obtained on finitely presented Λ -free groups for an arbitrary abelian ordered group Λ . Notice that finitely generated \mathbb{R}^n -free (or \mathbb{Z}^n -free) groups are finitely presented, so all the results below apply to arbitrary finitely generated \mathbb{R}^n -free (or \mathbb{Z}^n -free) groups.

The elimination process (the Λ -Machine) developed in Section 11 allows one to prove the following theorems.

Theorem 57 (The Main Structure Theorem [68]). *Any finitely presented group G with a regular free length function in an ordered abelian group Λ can be represented as a union of a finite series of groups*

$$G_1 < G_2 < \cdots < G_n = G,$$

where

1. G_1 is a free group,
2. G_{i+1} is obtained from G_i by finitely many HNN-extensions in which associated subgroups are maximal abelian, finitely generated, and length isomorphic as subgroups of Λ .

Theorem 58. [68] *Any finitely presented Λ -free group is \mathbb{R}^n -free.*

In his book [24] Chiswell (see also [108]) asked the following principal question (Question 1, page 250): If G is a finitely generated Λ -free group, is G Λ_0 -free for some finitely generated abelian ordered group Λ_0 ? The following result answers this question in the affirmative in the strongest form. It comes from the proof of Theorem 58 (not the statement of the theorem itself).

Theorem 59. *Let G be a finitely presented group with a free Lyndon length function $l : G \rightarrow \Lambda$. Then the subgroup Λ_0 generated by $l(G)$ in Λ is finitely generated.*

Theorem 60. [68] *Any finitely presented group \tilde{G} with a free length function in an ordered abelian group Λ can be isometrically embedded into a finitely presented group G that has a free regular length function in Λ .*

The following result automatically follows from Theorem 57 and Theorem 60 by simple application of Bass-Serre Theory.

Theorem 61. *Any finitely presented Λ -free group G can be obtained from free groups by a finite sequence of amalgamated free products and HNN extensions along maximal abelian subgroups, which are free abelian groups of finite rank.*

This theorem would have another proof provided Delzant-Potyagailo's proof of hierarchical accessibility [33] were correct (it has a gap in the case of HNN extension). Indeed, a finitely presented group acting freely on a Λ -tree has a stable action on an \mathbb{R} -tree. The proof is the same as the proof of Fact 5.1 in [49]. Therefore G splits over a finitely generated abelian group [14]. Then we could apply the result about hierarchical accessibility if it was available. Starting with this hierarchy one could obtain the one with edge groups maximal abelian.

The following result concerns with abelian subgroups of Λ -free groups. For $\Lambda = \mathbb{Z}^n$ it follows from the main structural result for \mathbb{Z}^n -free groups and [66], for $\Lambda = \mathbb{R}^n$ it was proved in [49]. The statement 1) below answers to Question 2 (page 250) from [24] in the affirmative for finitely presented Λ -free groups.

Theorem 62. *Let G be a finitely presented Λ -free group. Then:*

- 1) *every abelian subgroup of G is a free abelian group of finite rank, which is uniformly bounded from above by the rank of the abelianization of G .*
- 2) *G has only finitely many conjugacy classes of maximal non-cyclic abelian subgroups,*

- 3) G has a finite classifying space and the cohomological dimension of G is at most $\max\{2, r\}$ where r is the maximal rank of an abelian subgroup of G .

Proof. It comes from Theorem 57 by the standard properties of free product with amalgamation and HNN-extensions. Another way to prove the theorem is to notice that finitely presented Λ -free groups are \mathbb{R}^n -free (Theorem 58) and then apply the corresponding results for \mathbb{R}^n -free groups from [49]. \square

Theorem 63. *Every finitely presented Λ -free group is hyperbolic relative to its non-cyclic abelian subgroups.*

Proof. It follows from Theorem 58 and Corollary 17 on \mathbb{R}^n -free groups, or directly from the structural Theorem 57 and the Combination Theorem for relatively hyperbolic groups [30]. \square

The following results answers affirmatively in the strongest form to the Problem (GO3) from the Magnus list of open problems [10] in the case of finitely presented groups.

Corollary 18. *Every finitely presented Λ -free group is biautomatic.*

Proof. It follows from Theorem 63 and Rebbechi's result [107]. \square

Theorem 64. *Every finitely presented Λ -free group G has a quasi-convex hierarchy.*

Proof. By Theorem 61, G can be obtained by a finite sequence \mathcal{S} of amalgamated free products and HNN extensions along maximal abelian subgroups starting from free groups. Hence, each group H in this sequence is either an amalgamated free product of $H_1 *_{A=B} H_2$, or an HNN extension $K *_{A^t=B}$, where the groups H_1, H_2, K are constructed on previous steps. To prove the corollary we assume that all groups involved in the construction of H have quasi-convex hierarchies and we have to show that A is quasi-convex in H both when $H = H_1 *_{A=B} H_2$ and $H = K *_{A^t=B}$.

Note that each group in the list $\{H_1, H_2, K\}$ is Λ -free, hence, biautomatic by Corollary 18, and from [15] it follows that all their maximal abelian subgroups are regular and quasi-convex in the corresponding groups.

Consider these cases.

Case I. $H = \langle K, t \mid t^{-1}At = B \rangle$, where A and B are maximal abelian subgroups of K .

Suppose A and B are not conjugate in K . Then by [42, Lemma 2] both are maximal abelian in H and therefore quasi-convex there.

If $A^g = B$ for some $g \in K$, then we have

$$H \simeq \langle K, s \mid s^{-1}As = A \rangle,$$

where s represents $tg^{-1} \in H$ and A embeds into a maximal free abelian subgroup $C = \langle A, s \rangle$ as a direct factor (see [66, Lemma 3]). Hence, A is quasi-convex in C and, hence, in H .

Case II. $H = H_1 *_{A=B} H_2$, where A and B are maximal abelian subgroups respectively of H_1 and H_2 .

From [79, Theorem 4.5] it follows that both A and B are maximal abelian in H and therefore quasi-convex there. □

Theorems 64 and 63 imply the following result (the argument is straightforward but rather technical and we omit it, see also Lemma 17.10 in [129], though there is no proof there neither).

Theorem 65. *Every finitely presented Λ -free group G is locally undistorted, that is, every finitely generated subgroup of G is quasi-isometrically embedded into G .*

Since a finitely generated \mathbb{R}^n -free group G is hyperbolic relative to its non-cyclic abelian subgroups and G admits a quasi-convex hierarchy then recent results of D. Wise [129] imply the following.

Corollary 19. *Every finitely presented Λ -free group G is virtually special, that is, some subgroup of finite index in G embeds into a right-angled Artin group.*

In his book [24] Chiswell posted Question 3 (page 250): Is every Λ -free group orderable, or at least right-orderable? The following result answers in the affirmative to Question 3 (page 250) from [24] in the case of finitely presented groups.

Theorem 66. *Every finitely presented Λ -free group is right orderable.*

Proof. I. Chiswell proved that every finitely generated \mathbb{R}^n -free group is right orderable [26], so the result now follows from Theorem 58. □

The following addresses Chiswell's question whether Λ -free groups are orderable or not.

Theorem 67. *Every finitely presented Λ -free group is virtually orderable, that is, it contains an orderable subgroup of finite index.*

Proof. Indeed, in [38], G. Duchamp and D. Krob show that right-angled Artin groups are residually torsion free nilpotent. Hence, right-angled Artin groups are residually p -groups for any p , so, they are orderable (see [109]). Now the result follows from Corollary 19. □

Note that one cannot remove “virtually” in the formulation of Theorem 67 since S. Rourke recently showed that there are finitely presented Λ -free (even \mathbb{Z}^n -free) groups which are not orderable (see [113]).

Since right-angled Artin groups are linear (see [55, 54, 32] and the class of linear groups is closed under finite extension we get the following

Theorem 68. *Every finitely presented Λ -free group is linear.*

Since every linear group is residually finite we get the following.

Corollary 20. *Every finitely presented Λ -free group is residually finite.*

It is known that linear groups are equationally Noetherian (see [7] for discussion on equationally Noetherian groups), therefore the following result holds.

Corollary 21. *Every finitely presented Λ -free group is equationally Noetherian.*

Hint of the proof of Theorem 57. We perform the elimination process for the generalized equation $\Omega = \Omega_{v_0}$ corresponding to G . If the process goes infinitely we obtain one of the following:

- free splitting of G with at least on non-trivial free factor 11.6.1 and, maybe, some surface group factor 11.6.2,
- a decomposition of G as the fundamental group of a graph of groups with QH-vertex groups corresponding to quadratic sections 11.6.3 (a),
- decomposition of G as the fundamental group of a graph of groups with abelian vertex groups corresponding to periodic structures (see Proposition 10 in 11.6.3 (c) and HNN-extensions with stable letters infinitely longer than the generators of the abelian associated subgroups 11.6.3 (c).

Then we continue the elimination process with the generalized equation Ω_{v_1} where the active part corresponds to weakly rigid subgroups. In the case of an HNN-extension, Ω_{v_1} is obtained from Ω_{v_0} by removing a pair of bases from the generalized equation corresponding to the periodic structure (and this equation has the same complexity as Ω_{v_0} . This means that the complexity $\tau(\Omega_{v_1})$ is smaller than $\tau(\Omega_{v_0})$. In the other cases the Delzant–Potyagailo complexity of weakly rigid subgroups is smaller by Proposition 9 (which follows from the correct part of the paper 9. Therefore this procedure stops. At the end we obtain some generalized equation $\Omega_{v_{fin}}$ with all non-active sections. Continuing the elimination process till the end, we obtain, in addition, a complete system of linear equations with integer coefficients $\Sigma_{complete}$ on the lengths of items h_i 's that is automatically satisfied and guarantees that the associated maximal abelian subgroups are length-isomorphic.

This completes the proof of Theorem 57.

Remark 10. *If we begin with the group \tilde{G} with free but not necessary regular length function in Λ then in the Elimination process we work with the generalized equation Ω and add a finite number of elements from \hat{G} (see the end of Section 11.2). Thus we got an embedding of \tilde{G} in a group that can be represented as a union of a finite series of groups*

$$G_1 < G_2 < \cdots < G_n = G,$$

where

1. G_1 is a free group,

2. G_{i+1} is obtained from G_i by finitely many HNN-extensions in which associated subgroups are maximal abelian, finitely generated, and length isomorphic as subgroups of Λ .

Remark 11. As a result of the Elimination process, the equation $\Omega_{v_{fin}}$ (we will denote it Ω_{fin}) is defined on the multi-interval I , that is, a union of closed sections which have a natural hierarchy: a section σ_1 is smaller than a section σ_2 if the largest base on σ_2 is infinitely larger than the largest base on σ_1 .

The lengths of bases satisfy the system of linear equations $\Sigma_{complete}$.

Hint of the Proof of Theorem 58. Suppose G , as above, is a finitely presented Λ -free group. By Theorem 60 we may assume that the action of G is regular. Let Ω be a generalized equation for G corresponding to the union of closed sections I , $G = \langle \mathcal{M} \mid \Omega(\mathcal{M}) \rangle$. Consider the Cayley graph $X = \text{Cay}(G, \mathcal{M})$ of G with respect to the generators \mathcal{M} . Assign to edges of $\text{Cay}(G, \mathcal{M})$ their lengths in Λ (and infinite words that represent the generators) and consider edges as closed intervals in Λ of the corresponding length. For each relation between bases of Ω , $\lambda_{i_1} \cdots \lambda_{i_k} = \lambda_{j_1} \cdots \lambda_{j_m}$ (without cancelation) there is a loop in X labeled by this relation. Then the path labeled by $\lambda_{i_1} \cdots \lambda_{i_k}$ has the same length in Λ as the path labeled by $\lambda_{j_1} \cdots \lambda_{j_m}$. If x is a point on the path $\lambda_{i_1} \cdots \lambda_{i_k}$ at the distance $d \in \Lambda$ from the beginning of the path, and x' is a point on the path $\lambda_{j_1} \cdots \lambda_{j_m}$ at the distance d from the beginning, then we say that x and x' are in the same leaf. In other words, after we substitute generators in \mathcal{M} by their infinite word representations, we “fold” loops into segments. We consider the equivalence relation between points on the edges of X generated by all such pairs $x \sim x'$. Equivalence classes of this relation are called *leaves*. We also glue an arc isometric to a unit interval between each x and x' . Let \mathcal{F} be the foliation (the set of leaves). One can define a foliated complex $\Sigma = \Sigma(X, \mathcal{F})$ associated to X as a pair (X, \mathcal{F}) . The paths in Σ can travel vertically (along the leaves) and horizontally (along the intervals in Λ). The length of a path γ in $\Sigma(X, \mathcal{F})$ (denoted $\|\gamma\|$) is the sum of the lengths of horizontal intervals in γ . Therefore Σ is a graph with points on horizontal intervals being vertices. We now identify all points of Σ that belong to the same leaf (identify vertices in the leaf and points on the vertical edges of this leaf in Σ). We denote by T the obtained graph. A reduced path in T is a path without backtracking. We call removing of a backtracking in T a reduction of a path.

Lemma 60. [68] *The image of a loop in Σ can be reduced to a point in T .*

Indeed, a loop in Σ is a composition of vertical and horizontal subpaths. If a vertical path of length one connects two points of Σ , then they are also connected in Σ by a horizontal path that is mapped to a path of the form rr^{-1} in T . Then a loop in Σ can be transformed into a loop where all paths are horizontal. Therefore any simple loop in Σ is mapped into a path with backtracking in T , and if we reduce this loop in T we get a point. This implies that there is a unique reduced path between any two points of T . A distance between two points in T is the length of the reduced path between them. We

extend naturally the left action of G on X to the action on T . The following lemma holds.

Lemma 61. [68]

- (1) T is a Λ -tree.
- (2) The left action of G on T is free.

Similarly, we can construct a Λ -tree where G acts freely beginning not with the original generalized equation Ω but with a generalized equation obtained from Ω by the application of the Elimination process. In this case we may have closed sections with some items h_i with $\gamma(h_i) = 1$. Then to construct the Cayley graph of G in the new generators we cover items h_i with $\gamma(h_i) = 1$ by bases without doubles (these bases were previously removed at some stages of the Elimination process as matching pairs).

We begin by considering the union of the closed sections of the minimal height in the hierarchy introduced in Remark 11. Denote the union of these sections by σ . The group H of the generalized equation corresponding to their union is a free product of free groups, free abelian groups and closed surface groups because for these groups we stop the Elimination process. Notice that for those sections for which $\gamma(h_i) = 1$ for some maximal height items, these items h_i are also products of some bases because initially every item is a product of bases. We can assume the following:

1. For all closed sections of σ such that $\gamma(h_i) = 2$ for all items of the maximal height, the number of bases of maximal height cannot be decreased using entire transformation or similar transformation applied from the right of the section (*right entire transformation*).
2. For all closed sections of σ where we apply linear elimination, the number of bases of maximal height cannot be decreased using the transformations as above as well as transformation (E2) (transfer) preceded by creation of necessary boundary connections as in (E5) (denote it $(E2_5)$), (E3) and first two types of linear elimination (D5) (denote it $(D5_{1,2})$).

Indeed, if we can decrease the number of bases of maximal height using the transformations described above then we just do this and continue. Since these transformations do not change the total number of bases we can also assume that they do not decrease the number of bases of second maximal height etc. We now re-define the lengths of bases belonging to σ in \mathbb{R}^k . We are going to show that all components of the length of every base can be made zeros except for the components which appear to be maximal in the lengths of bases from σ .

Let Ω_σ be the generalized equation corresponding to the sections from σ . Let H be the group of the equation $H = \langle \mathcal{M}_\sigma \mid \Omega_\sigma(M) \rangle$.

Denote by Λ_1 the minimal convex subgroup of Λ containing lengths of all bases in σ , and by Λ' a maximal convex subgroup of Λ_1 not containing lengths of maximal height bases in σ (it exists by Zorn's lemma). Then the quotient

Λ_1/Λ' is a subgroup of \mathbb{R} . Denote by $\hat{\ell}$ the length function in \mathbb{R} on this quotient induced from ℓ . We consider elements of Λ' as infinitesimals. Denote by \hat{T} the \mathbb{R} -tree constructed from T by identifying points at zero distance (see [24], Theorem 2.4.7). Then H acts on \hat{T} . Denote by $\|\gamma\|_{\mathbb{R}}$ the induced length of the path γ in \hat{T} , and by $d_{\mathbb{R}}(\bar{x}, \bar{y})$ the induced distance.

However, the action of H on \hat{T} is not free. The action is minimal, that is, there is no non-empty proper invariant subtree. Notice that the canonical projection $f : T \rightarrow \hat{T}$ preserves alignment, and the pre-image of the convex set is convex. The pre-image of a point in \hat{T} is an infinitesimal subtree of T .

Lemma 62. [68] *The action of H on \hat{T} is superstable: for every non-degenerate arc $J \subset \hat{T}$ with non-trivial fixator, and for every non degenerate subarc $S \subset J$, one has $\text{Stab}(S) = \text{Stab}(J)$.*

The proof is the same as the proof of Fact 5.1 in [49].

Proposition 12. [68] *One can define the lengths of bases in σ in \mathbb{R}^k .*

We will show how to prove the proposition only for the case of a closed section σ_1 on the lowest level corresponding to a closed surface group.

Lemma 63. [68] *Let σ_1 be a closed section on the lowest level corresponding to a closed surface group and the lengths of the bases satisfy some system of linear equations Σ . Then one can define the lengths of bases in σ_1 in \mathbb{R}^k .*

Proof. Denote by $\mathcal{M}_{\sigma+}$ the set of bases (on all the steps of the process of entire transformation applied to the lowest level) with non-zero oldest component and by $\mathcal{M}_{\sigma 0}$ the rest of the bases (infinitesimals). Denote by $\hat{\ell}$ the projection of the length function ℓ to Λ_1/Λ' as before. Then $\lambda \in \mathcal{M}_{\sigma+}$ if and only if $\hat{\ell}(\lambda) > 0$. We apply the entire transformation to σ_1 . If we obtain an overlapping pair or an infinitesimal section, where the process goes infinitely, we declare it non-active and move to the right. This either decreases, or does not change the number of bases in the active part. Therefore, we can assume that the process goes infinitely and the number of bases in $\mathcal{M}_{\sigma+}$ never decreases. Therefore bases from $\mathcal{M}_{\sigma 0}$ are only used as transfer bases.

We will show that the stabilizer of a pre-image of a point, an infinitesimal subtree T_0 of T is generated by some elements in $\mathcal{M}_{\sigma 0}$. An element h from H belongs to such a stabilizer if $\hat{\ell}(h) = 0$. If some product of bases not only from $\mathcal{M}_{\sigma 0}$ has infinitesimal length (denote this product by g), then by Lemma 61, the identity and g belong to leaves at the infinitesimal distance δ in Σ . Therefore using elementary operations we can obtain a base of length δ . Denote by Λ'' the minimal convex subgroup of Λ containing all elements of the same height as bases from $\mathcal{M}_{\sigma 0}$.

This implies the following lemma which we need to finish the proof of Lemma 63.

Lemma 64. [68] *Let $\bar{\sigma}_1$ be the projection of the quadratic section σ_1 to $\Lambda_1/(\Lambda'' \cap \Lambda_1)$. Suppose the process of entire transformation for σ_1 goes infinitely and the*

number of bases in $\mathcal{M}_{\sigma+}$ never decreases. Then the process of entire transformation for $\bar{\sigma}_1$ goes infinitely too and the number of bases in $\mathcal{M}_{\sigma+}$ never decreases.

If $\ell(g) \in \Lambda'$ then g is a product of bases in \mathcal{M}_{σ_0} and $\ell(g) \in \Lambda''$.

This lemma implies that there is no element in H infinitely larger than all bases in \mathcal{M}_{σ_0} , but infinitely smaller than all bases in $\mathcal{M}_{\sigma+}$.

Therefore, we can make all components of the lengths of bases in \mathcal{M} zeros except for those which are maximal components of some bases in \mathcal{M} . Since \mathcal{M} is a finite set, the number of such components is finite, and the length is defined in \mathbb{R}^k for some k not larger than the number of pairs of bases. \square

Similarly we can prove the Proposition for sections corresponding to the linear case and the case of an abelian vertex group. Using induction on the number of levels obtained in the Elimination process, we prove the statement of Theorem 58.

The points of an \mathbb{R}^n -tree, where G acts freely are the leaves in the foliation corresponding to the new length of bases in \mathbb{R}^n . The new lengths of bases are exactly their Lyndon lengths. \square

Hint of the Proof of Theorem 60. Notice that in the case when \tilde{G} is a finitely presented group with a free length function in Λ (not necessary regular) it can be embedded in the group with a free regular length function in Λ . That group can be embedded in $R(\Lambda', X)$. When we make a generalized equation for \tilde{G} , we have to add only a finite number of elements from $R(\Lambda', X)$. We run the elimination process for this generalized equation as we did in the proof of Theorem 57 and obtain a group G as in Remark 10, where \tilde{G} is embedded, and then redefine the length of elements of G in \mathbb{R}^n as above. Therefore G acts freely and regularly on a \mathbb{R}^n -tree. The theorem is proved.

Moreover, one shows by induction that the length function in \mathbb{R}^n and in Λ defined on G is regular. This proves Theorem 60.

12.4 Algorithmic problems for finitely presented Λ -free groups

The structural results of the previous section give solution to many algorithmic problems on finitely presented Λ -free groups.

Theorem 69. [68] *Let G be a finitely presented Λ -free group. Then the following algorithmic problems are decidable in G :*

- the Word Problem;
- the Conjugacy Problems;
- the Diophantine Problem (decidability of arbitrary equations in G).

Proof. By Theorem 63 a finitely presented Λ -free group G is hyperbolic relative to its non-cyclic abelian subgroups. Decidability of the Conjugacy Problem for

such groups is known - it was done by Bumagin [16] and Osin [99]. Decidability of the Diophantine Problems in such groups was proved by Dahmani [30]. \square

Theorem 63 combined with results of Dahmani and Groves [31] immediately implies the following two corollaries.

Corollary 22. *Let G be a finitely presented Λ -free group. Then:*

- *G has a non-trivial abelian splitting and one can find such a splitting effectively,*
- *G has a non-trivial abelian JSJ-decomposition and one can find such a decomposition effectively.*

Corollary 23. *The Isomorphism Problem is decidable in the class of finitely presented groups that act freely on some Λ -tree.*

Theorem 70. *The Subgroup Membership Problem is decidable in every finitely presented Λ -free group.*

Proof. By Theorem 65 every finitely generated subgroup of a finitely presented Λ -tree group G is quasi-isometrically embedded into G . Obviously, the Membership Problem for every fixed quasi-isometrically embedded subgroup in a finitely generated group with decidable Word Problem is decidable. \square

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