# Context-Free Groups and Their Structure Trees 

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#### Abstract

Let $\Gamma$ be a connected, locally finite graph of finite tree width and $G$ be a group acting on it with finitely many orbits and finite node stabilizers. We provide an elementary and direct construction of a tree $T$ on which $G$ acts with finitely many orbits and finite vertex stabilizers. Moreover, the tree is defined directly in terms of the structure tree of optimally nested cuts of $\Gamma$. Once the tree is constructed, Bass-Serre theory yields that $G$ is virtually free. This approach simplifies the existing proofs for the fundamental result of Muller and Schupp that characterizes context-free groups as f.g. virtually free groups. Our construction avoids the explicit use of Stallings' structure theorem and it is self-contained.

We also give a simplified proof for an important consequence of the structure tree theory by Dicks and Dunwoody which has been stated by Thomassen and Woess. It says that a f.g. group is accessible if and only if its Cayley graph is accessible.


Keywords. Combinatorial group theory, context-free group, structure tree, finite treewidth, accessible graph.
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## 1 Introduction

A seminal paper of Muller and Schupp [23] showed that a group $G$ is context-free if and only if it is a finitely generated virtually free group. A group $G$ is contextfree if there is some finite set $\Sigma$ and a surjective homomorphism $\varphi: \Sigma^{*} \rightarrow G$ such that the associated group language $L_{G}=\varphi^{-1}(1)$ is context-free in the sense of formal language theory. A group $G$ is virtually free if it has a free subgroup of finite index. Finitely generated (f.g.) virtually free groups were the basic examples for context-free groups because the standard algorithm to solve their word problem runs on a deterministic pushdown automaton; and these automata recognize a proper subfamily of context-free languages. The deep insight by Muller and Schupp is that the converse holds: If $G$ is context-free, then $G$ is a finitely generated virtually free group. Over the past decades a wide
range of other characterizations of context-free (or f.g. virtually free) groups have been found showing the importance of this class.

The various equivalent characterizations include: (1) fundamental groups of finite graphs of finite groups [18], (2) f.g. groups having a Cayley graph which can be $k$-triangulated [23] (3) f.g. groups having a Cayley graph of finite treewidth 21, (4) universal groups of finite pregroups 24], (5) groups having a finite presentation by some geodesic string rewriting system [15], and (6) f.g. groups having a Cayley graph with decidable monadic second-order theory [21]. For some other related results see the recent surveys [3] or 6].

The result of Muller and Schupp was stated in 23 as a conjecture and proved only under the assumption that finitely presented groups are accessible. The accessibility of finitely presented groups was proved later by Dunwoody [13]. (There are examples of finitely generated groups which not accessible by [14].) Accessibility means that the process of splitting the group with Stallings' structure theorem [28] 1 eventually terminates. In subsequent proofs the result in [13] could be replaced by showing explicit upper bounds on how often splittings according to Stallings' structure theorem can be performed, see e.g. 26.

However, the reference to 28 remained. Indeed, almost all proofs in the literature showing that a context-free group is virtually free use the structure theorem by Stallings. Recently, in [3] another proof was given by Antolin which instead of Stallings' structure theorem and a separate result for accessibility uses a more general result due to Dunwoody [11].

The starting point for our contribution has been as follows: Circumvent the deep theorems of Dunwoody and Stallings by starting with a f.g. group $G$ having a Cayley graph of finite treewidth. Construct from these data a tree on which $G$ acts with finite node stabilizers and with finitely many orbits. Apply Bass-Serre theory [27] to see that $G$ can be realized as a fundamental group of a finite graph of groups with finite vertex groups. It is known by [18] that these groups are f.g. and virtually free.

To follow this roadmap became possible due to a recent paper by Krön 19 which presents a simplified version of Dunwoody's cut construction [12]. We realized that Krön's proof of Stallings' structure theorem can be modified such that it yields the tree we were looking for. We could not use Krön's result as a black box because in his paper he deals with cuts of globally minimal weight, only. Thus all cuts have the same weight whereas we need to consider cuts of different weight in order to get a non-refinable decomposition as fundamental group of a graph of groups.

Our approach leads to the following result: Let $\Gamma$ be a connected, locally finite graph of finite treewidth, and let $G$ be a group acting on $\Gamma$ such that $G \backslash \Gamma$ is finite and each node stabilizer $G_{v}$ is finite. Then $G$ is finitely generated and virtually free.

This is the essence of Corollary 5.10. To the best of our knowledge this result has not been formulated elsewhere. On the other hand, it is also clear that Corollary 5.10 can be derived rather easily from existing results in the literature. So, the main contribution of the present paper is the new construction of optimally nested cuts (optimal cuts for short) and a direct self-contained

[^0]combinatorial proof of Theorem 5.9, which implies Corollary 5.10 by Bass-Serre theory.

In Theorem 7.4 we also give a new elegant self-contained proof for another important result in this area by Thomassen and Woess which is a consequence of [8, Thm. II 2.20]: Let $\Gamma$ be a locally finite, connected, accessible graph, and let a f.g. group $G$ act on $\Gamma$ such that $G \backslash \Gamma$ is finite and each node stabilizer $G_{v}$ is finite. Then the group $G$ is accessible.

The outline of the paper is as follows: Section 2 fixes some notation.
In Section 3 we follow [19] introducing the necessary modifications. The focus in this section is on accessible graphs c.f. Definition 3.5. We work with bi-infinite simple paths rather than with ends. This avoids some technical definitions and is more intuitive when drawing pictures as in Figure 3 or Figure 4 The key point in Section 3 is Proposition 3.7, which is valid for optimally nested cuts of different weights. It generalizes the corresponding results in [12] and [19] on globally minimal cuts. This leads to Proposition 3.12 saying that the set of optimally nested cuts forms a tree set in the sense of [11. This means that they can be viewed as the edge set of the so-called structure tree.

In Section 4 we want to obtain some more information about the vertex stabilizers of the the action on the structure tree. In order to do so we define blocks as in 30. The central result is Proposition 4.8. It says that blocks have at most one end, which finally leads to Theorem 5.9 and Corollary 7.5 ,

Section 5 recalls the notion of finite treewidth. The results of Section 3 and Section 4 yield the desired proof of Theorem 5.9,

Section 6.1 shows how to derive the result of Muller and Schupp [23] using our approach. This section does not contain any new material, but we tried to have a concise presentation. In particular, we omit the technical notion of a $k$-triangulation of a graph by showing directly that the Cayley graph of a context-free group has finite treewidth. This can be done with the very same ideas which are present in [23. Then, we can apply Corollary 5.10 to show that a context-free group is virtually free.

## 2 Preliminaries

### 2.1 Preliminaries on graphs

A directed graph $\Gamma$ is given by the following data: A set of vertices $V=V(\Gamma)$, a set of edges $E=E(\Gamma)$ together with two mappings $s: E \rightarrow V$ and $t: E \rightarrow V$. The vertex $s(e)$ is the source of $e$ and $t(e)$ is the target of $e$. A vertex $u$ and an edge $e$ are incident, if $u \in\{s(e), t(e)\}$. The degree of $u$ is the number of incident edges, and $\Gamma$ is called locally finite if the degree of all vertices is finite.

An undirected graph $\Gamma$ is a directed graph such that the set of edges $E$ is equipped with a fixed point free involution $e \mapsto \bar{e}$. (i.e., a map such that $e=\overline{\bar{e}}$ and $e \neq \bar{e}$ for all $e \in E)$. Furthermore we demand $s(e)=t(\bar{e})$. An undirected edge is the set $\{e, \bar{e}\}$. By abuse of language we denote an undirected edge simply by $e$, too.

If we speak about a graph, then we always mean an undirected graph, otherwise we say specifically directed graph. Most of the time we only consider (undirected) graphs without loops and multi-edges. In this case we identify $E$
with two-element sets of incident vertices $\{u, v\}$ and write $e=u v$ if either $s(e)=u$ and $t(e)=v$ or $s(\bar{e})=u$ and $t(\bar{e})=v$.

For $S \subseteq V(\Gamma)$ and $v \in V(\Gamma)$ define as usual in graph theory $\Gamma(S)$ (resp. $\Gamma-S)$ to be the subgraph of $\Gamma$ which is induced by the vertex set $S$ (resp. $\underline{V}(\Gamma) \backslash S)$ and $\Gamma-v=\Gamma-\{v\}$. We also write $\bar{S}$ for the complement of $S$, i.e., $\bar{S}=V(\Gamma) \backslash S$. Likewise for $e \in E(\Gamma)$ we let $\Gamma-e=(V(\Gamma), E(\Gamma) \backslash\{e\})$.

A path is a subgraph $\left(\left\{v_{0}, \ldots, v_{n}\right\},\left\{e_{1}, \ldots, e_{n}\right\}\right)$ such that $s\left(e_{i}\right)=v_{i-1}$ and $t\left(e_{i}\right)=v_{i}$ for all $1 \leq i \leq n$. It is simple if the vertices are pairwise disjoint. It is closed if $v_{0}=v_{n}$. A cycle is a closed path with $n \geq 3$ such that $v_{1}, \ldots, v_{n}$ is a simple path.

The distance $d(u, v)$ between $u$ and $v$ is defined as the length (i.e., the number of edges) of the shortest path connecting $u$ and $v$. We let $d(u, v)=\infty$ if there is no such path. A path $v_{0}, \ldots, v_{n}$ is called geodesic if $n=d\left(v_{0}, v_{n}\right)$. An infinite path is defined as geodesic if all its finite subpaths are geodesic. For $A, B \subseteq V(\Gamma)$ the distance is defined as $d(A, B)=\min \{d(u, v) \mid u \in A, v \in B\}$.

A graph $\Gamma$ is called connected if $d(u, v)<\infty$ for all vertices $u$ and $v$. A tree is a connected graph without any cycle. If $T=(V, E)$ is a tree, we may fix a root $r \in V$. This gives an orientation $E^{+} \subseteq E$ by directing all edges "away from the root". In this way a rooted tree becomes a directed graph $\left(V, E^{+}\right)$which refers to the tree $T=\left(V, E^{+} \cup E^{-}\right)$, where $E^{-}=E \backslash E^{+}$.

In the following, when we write $\Gamma$ we always mean a locally finite and connected graph, whereas the capital letter $T$ refers to a tree, which does not need to be locally finite, in general.

### 2.2 Preliminaries on groups

The paper is mainly concerned with finitely generated groups. Let $G$ be a group with 1 as neutral element. The Cayley graph $\Gamma$ of $G$ depends on $G$ and on a generating set $X \subseteq G$. It is defined by $V(\Gamma)=G$ and $E(\Gamma)=\left\{(g, g a) \mid g \in G\right.$ and $\left.a \in X \cup X^{-1}\right\}$, with the obvious incidence functions $s(g, g a)=g, t(g, g a)=g a$, and involution $\overline{(g, g a)}=(g a, a)$. For an edge $(g, g a)$ we call $a$ the label of $(g, g a)$ and extend this definition also to paths. Thus, the label of a path is a sequence (or word) in the free monoid $X^{*}$. The Cayley graph is without loops and without multi-edges. It is connected because $X$ generates $G$. The Cayley graph $\Gamma$ is locally finite if and only if $X$ is finite. Sometimes we suppress $X$ if there is a standard choice for the generating set. For example, if $G=F(X)$ is the free group over $X$, then the Cayley graph of $G$ refers to $X$ and it is a tree. By the infinite grid we mean the Cayley graph of $\mathbb{Z} \times \mathbb{Z}$ with generators $(1,0)$ and $(0,1)$.

A group $G$ acts on a graph $\Gamma=(V, E)$ if there is an action of $G$ on $V$, denoted by $v \mapsto g \cdot v$, and an action on $E$, denoted by $e \mapsto g \cdot e$, such that $s(g \cdot e)=g \cdot s(e), t(g \cdot e)=g \cdot t(e)$, and $g \cdot \bar{e}=\overline{g \cdot e}$ for all $g \in G$ and $e \in E$. If $G$ acts on $\Gamma$, then we can define its quotient graph $G \backslash \Gamma$. Its vertices (resp. edges) are the orbits $G \cdot u$ for $u \in V$ (resp. $G \cdot e$ for $e \in E$ ). We say that $G$ acts with finitely many orbits if $G \backslash \Gamma$ is finite.

Let $\mathcal{G}$ denote some class of groups. A group $G$ is called virtually $\mathcal{G}$ if it has a subgroup of finite index which is in $\mathcal{G}$. Virtually finite groups are finite. The focus in this paper is on virtually free groups.

## 3 Cuts and structure trees

The constructions in this section follow the paper by Krön [19] which gives a simplified approach to Dunwoody's constructions of cuts 12. The main difference between this section and the paper of Krön lies in the definition of minimal cuts.

### 3.1 Cuts and optimally nested cuts

Let $\Gamma=(V(\Gamma), E(\Gamma))$ be a connected and locally finite graph. For a subset $C \subseteq V(\Gamma)$ we define the edge- and vertex-boundaries of $C$ as follows:

$$
\begin{array}{ll}
\text { Edge-boundary: } & \delta C=\{u v \in E(\Gamma) \mid u \in C, v \in \bar{C}\} . \\
\text { Vertex-boundary: } & \beta C=\{u \in V(\Gamma) \mid \exists v \in V(\Gamma) \text { with } u v \in \delta C\} .
\end{array}
$$

Definition 3.1 $A$ cut is a subset $C \subseteq V(\Gamma)$ such that the following conditions hold.

1. $C$ and $\bar{C}$ are non-empty and connected.
2. $\delta C$ is finite.

The weight of a cut is defined by $|\delta C|$. If $|\delta C| \leq k$, then $C$ a called a $k$-cut.
We are mainly interested in cuts where both parts $C$ and $\bar{C}$ are infinite. However it might be that there are no such cuts. Consider the infinite grid $\mathbb{Z} \times \mathbb{Z}$, i.e., the graph with vertex set $\mathbb{Z} \times \mathbb{Z}$ where $(i, j)$ is adjacent to the four vertices $(i, j \pm 1)$ and $(i \pm 1, j)$. It is connected and locally finite, but there are no cuts of finite weight splitting the grid into two infinite parts.

The following well-known observation is crucial. It can be found e.g. in 30 in a slightly different formulation:

Lemma 3.2 Let $S \subseteq V(\Gamma)$ be finite and $k \geq 1$. There are only finitely many $k$-cuts $C$ with $\beta C \cap S \neq \emptyset$.

Proof. It is enough to prove the result for $S=\{u, v\}$ where $e=u v \in E(\Gamma)$ is some fixed edge. Since $\Gamma$ is locally finite, it is enough to show that the set of $k$-cuts $C$ with $e \in \delta C$ is finite. This is now trivial for $k=1$ because there is at most one cut with $\{e\}=\delta C$. If the graph $\Gamma-e$ becomes disconnected, i.e., $e$ is a so-called bridge, then all cuts with $e \in \delta C$ have weight $k=1$. Thus, we may assume that the graph $\Gamma-e$ is still connected; and we may fix a path from $u$ to $v$ in $\Gamma-e$. Every $k$-cut $C$ with $e \in \delta C$ becomes a $k-1$-cut $C$ in the graph $\Gamma-e$. Such a cut must use one edge of the path from $u$ to $v$ in $\Gamma-e$ because otherwise we had either both $u, v \in C$ or both $u, v \in \bar{C}$. By induction, there are only finitely many $k-1$-cuts using edges from a fixed path. Thus, we are done.

We are interested in bi-infinite simple paths which can be split into two infinite pieces by some cut of finite weight. For a bi-infinite simple path $\alpha$ denote:

$$
\begin{aligned}
\mathcal{C}(\alpha) & =\{C \subseteq V(\Gamma) \mid C \text { is a cut and }|\alpha \cap C|=\infty=|\alpha \cap \bar{C}|\} \\
\mathcal{C}_{\min }(\alpha) & =\{C \in \mathcal{C}(\alpha)| | \delta C \mid \text { is minimal in } \mathcal{C}(\alpha)\} .
\end{aligned}
$$

Thus, $\mathcal{C}(\alpha) \neq \emptyset$ if and only if there is a cut of finite weight such that the graph $\alpha-\delta C$ has exactly two infinite components each of these two being a one-sided infinite subpath of $\alpha$. We define the set of minimal cuts $\mathcal{C}_{\text {min }}$ by

$$
\mathcal{C}_{\min }=\bigcup\left\{\mathcal{C}_{\min }(\alpha) \mid \alpha \text { is a bi-infinite simple path }\right\} .
$$

In the infinite grid $\mathbb{Z} \times \mathbb{Z}$ we have $\mathcal{C}_{\text {min }}=\emptyset$. Note that the set of minimal cuts may contain cuts of very different weight. Actually we might have $C, D \in \mathcal{C}(\alpha) \cap \mathcal{C}_{\text {min }}$ with $C \in \mathcal{C}_{\text {min }}(\alpha)$, but $D \notin \mathcal{C}_{\text {min }}(\alpha)$. In such a case, there must be another biinfinite simple path $\beta$ with $D \in \mathcal{C}(\alpha) \cap \mathcal{C}_{\text {min }}(\beta)$ and $|\delta C|<|\delta D|$. Here is an example: Let $\Gamma$ be the subgraph of the infinite grid $\mathbb{Z} \times \mathbb{Z}$ which is induced by the pairs $(i, j)$ satisfying $j \in\{0,1\}$ or $i=0$ and $j \geq 0$. Let $\alpha$ be the bi-infinite simple path with $i=0$ or $j=1$ and $i \geq 0$ and let $\beta$ be the bi-infinite simple path defined by $j=0$. Then there are such cuts with $|\delta C|=1$ and $|\delta D|=2$, as depicted in Figure 1 .


Figure 1: The subgraph of the grid $\mathbb{Z} \times \mathbb{Z}$ induced by the pairs $(i, j)$ satisfying $j \in\{0,1\}$ or $i=0$ and $j \geq 0$. Here we have $D \in \mathcal{C}(\alpha) \cap \mathcal{C}_{\text {min }}$ but $D \notin \mathcal{C}_{\text {min }}(\alpha)$.

Definition 3.3 Two cuts $C$ and $D$ are called nested if one of the four inclusions $C \subseteq D, C \subseteq \bar{D}, \bar{C} \subseteq D$, or $\bar{C} \subseteq \bar{D}$ holds.

The set $\{C \cap D, C \cap \bar{D}, \bar{C} \cap D, \bar{C} \cap \bar{D}\}$ is called the set of corners of $C$ and $D$, see Figure 2. Two corners $E, E^{\prime}$ of $C$ and $D$ are called opposite if either $\left\{E, E^{\prime}\right\}=\{C \cap D, \bar{C} \cap \bar{D}\}$ or $\left\{E, E^{\prime}\right\}=\{\bar{C} \cap D, C \cap \bar{D}\}$. Two different corners are called adjacent if they are not opposite. Note that two cuts $C, D$ are nested if and only if one of the four corners of $C$ and $D$ is empty.

We define for every cut $C$ and $k \geq 1$ a cardinality $m_{k}(C)$ as follows:

$$
m_{k}(C)=\mid\{D \mid C \text { and } D \text { are not nested and } D \text { is a } k \text {-cut. }\} \mid .
$$

Lemma 3.4 Let $k \in \mathbb{N}$ and $C$ be a cut, then $m_{k}(C)$ is finite.
Proof. Let $S$ be a finite connected subgraph of $\Gamma$ containing all vertices of $\beta C$. The number of $k$-cuts $D$ with $\beta D \cap S \neq \emptyset$ is finite by Lemma 3.2. For all other


Figure 2: The corners of $C$ and $D$. Nested cuts have one empty corner.
cuts we may assume (by symmetry) that $\beta C \subseteq D$. Now assume that both, $C \cap \bar{D} \neq \emptyset$ and $\bar{C} \cap \bar{D} \neq \emptyset$. Then we can connect a vertex $c \in C$ with some vertex $\bar{c} \in \bar{C}$ inside the connected set $\bar{D}$. This must involve a vertex from $\beta C$, but $\beta C \subseteq D$. Hence, either $C \subseteq \bar{D}$ or $\bar{C} \subseteq \bar{D}$.

We are mainly interested in graphs $\Gamma$ where the weight over all cuts in $\mathcal{C}_{\text {min }}$ can be bounded by some constant. This leads to the notion of accessible graph due to (30]:

Definition 3.5 A graph is called accessible if there exists a constant $k \in \mathbb{N}$ such that for every bi-infinite simple path $\alpha$ either $\mathcal{C}(\alpha)$ is empty or $\mathcal{C}(\alpha)$ contains some $k$-cut

For the rest of this section we assume that $\Gamma$ is accessible. Thus, there is some constant $k$ such that for all bi-infinite simple paths $\alpha$ with $\mathcal{C}(\alpha) \neq \emptyset$ there exists some cut $C \in \mathcal{C}(\alpha)$ with $|\delta C| \leq k$.

Fixing this number $k$ let us define, by Lemma 3.4 for each cut $C$ a natural number as follows:

$$
m(C)=\mid\{D \mid C \text { and } D \text { are not nested and } D \text { is a } k \text {-cut }\} \mid
$$

We use the following notation, where $\alpha$ denotes a bi-infinite simple path:

$$
\begin{aligned}
m_{\alpha} & =\min \left\{m(C) \mid C \in \mathcal{C}_{\min }(\alpha)\right\} \\
\mathcal{C}_{\mathrm{opt}}(\alpha) & =\left\{C \in \mathcal{C}_{\min }(\alpha) \mid m(C)=m_{\alpha}\right\} \\
\mathcal{C}_{\text {opt }} & =\bigcup\left\{\mathcal{C}_{\mathrm{opt}}(\alpha) \mid \alpha \text { is a bi-infinite simple path }\right\}
\end{aligned}
$$

Definition 3.6 $A$ cut $C \in \mathcal{C}_{\text {opt }}$ is called an optimally nested cut. For simplicity, an optimally nested cut is also called optimal cut.

In some sense we can forget all other cuts, we just focus on optimal cuts. This viewpoint is possible because every "cuttable" bi-infinite simple path is "cut" into two infinite parts at least by one optimal cut. The next proposition is the main result in this section.

Proposition 3.7 Let $C, D \in \mathcal{C}_{\mathrm{opt}}$. Then $C$ and $D$ are nested.
Proof. We choose bi-infinite simple paths $\alpha$ and $\beta$ such that $C \in \mathcal{C}_{\text {opt }}(\alpha)$ and $D \in \mathcal{C}_{\text {opt }}(\beta)$. If possible, we let $\alpha=\beta$. In any case, we may assume that $m_{\alpha} \geq m_{\beta}$. The proof is by contradiction. Hence, we assume that $C$ and $D$ are not nested.

We distinguish two cases: First, let $D \in \mathcal{C}_{\text {min }}(\alpha)$. Since $m(D)=m_{\beta} \leq m_{\alpha}$, this implies $D \in \mathcal{C}_{\text {opt }}(\alpha)$ and therefore $\alpha=\beta$. In particular, there are opposite corners $E$ and $E^{\prime}$ such that $|\alpha \cap E|=\left|\alpha \cap E^{\prime}\right|=\infty$.

In the other case we have $D \notin \mathcal{C}_{\text {min }}(\alpha)$ and therefore $\alpha \neq \beta$. We claim that there must be one corner $K$ of $C$ and $D$ such that $|\alpha \cap K|<\infty$ and $|\beta \cap K|<\infty$. Indeed, if there is no such corner $K$, then infinite parts of $\alpha$ and $\beta$ are in opposite corners. In particular, $\alpha$ and $\beta$ are split by both by $C$ as well as by $D$ in two infinite pieces. This implies $|\delta C|=|\delta D|$, and hence $D \in \mathcal{C}_{\text {min }}(\alpha)$. Thus such a corner $K$ exists and we define $E$ and $E^{\prime}$ to be the adjacent corners of $K$. Without loss of generality, $E$ splits $\alpha$ into two infinite pieces and $E^{\prime}$ splits $\beta$ into two infinite pieces.


Figure 3: For all four corners $K$ we have $\max \{|K \cap \alpha|,|K \cap \beta|\}=\infty$.


Figure 4: For one corner $K$ we have $\max \{|K \cap \alpha|,|K \cap \beta|\}<\infty$.

Thus, in both cases, $E$ and $E^{\prime}$ are defined such that $|\alpha \cap E|=\left|\beta \cap E^{\prime}\right|=\infty$. By interchanging, if necessary, $C$ with $\bar{C}$ and $D$ with $\bar{D}$, we may assume that $E=C \cap D$ and $E^{\prime}=\bar{C} \cap \bar{D}$, too.

Thus, in all cases we are in the following situation:
$C$ and $D$ are not nested, $C \in \mathcal{C}_{\text {opt }}(\alpha), D \in \mathcal{C}_{\text {opt }}(\beta), E=C \cap D, E^{\prime}=\bar{C} \cap \bar{D}$, and $|\alpha \cap E|=\left|\beta \cap E^{\prime}\right|=\infty$. Possibly $\alpha=\beta$, but it is not yet clear that $E$ and $E^{\prime}$ are cuts.

The graph $\Gamma(E)$ contains an infinite connected component $F \subseteq E$ such that $|\alpha \cap F|=\infty$. Let us show that $\bar{F}$ is non-empty and connected. The set $\bar{F}$ is non-empty and infinite because $E^{\prime} \subseteq \bar{F}$. Now fix a vertex $v \in E^{\prime}$ and let $u \in \bar{F}$. There is a path $\gamma$ from $u$ to $v$ in $\Gamma$ and on this path there is a first vertex $w$ with $w \in \bar{C} \cup \bar{D}$. If the initial path from $u$ to $w$ was using a point of $F$, then it would be a path in $E$, and $u$ would be in the connected component $F$, which was excluded. Hence, we can connect $u$ to $w$ in $\Gamma-F$. Now, by symmetry $w \in \bar{C}$. But then $w, v \in \bar{C} \subseteq \Gamma-F$ and $\bar{C}$ is connected. Hence, $F$ is a cut.

In a symmetric way we find a cut $F^{\prime} \subseteq E^{\prime}$ such that $\left|\beta \cap F^{\prime}\right|=\infty$. Let us show that $F=E \in \mathcal{C}_{\text {min }}(\alpha)$ and $F^{\prime}=E^{\prime} \in \mathcal{C}_{\text {min }}(\beta)$.

We can write $|\delta E|=a+b+c+d$, where

$$
\begin{aligned}
a & =|\{x y \mid x \in F \wedge y \in \bar{C} \cap D\}| \\
b & =\left|\left\{x y \mid x \in F \wedge y \in E^{\prime}\right\}\right| \\
c & =|\{x y \mid x \in F \wedge y \in C \cap \bar{D}\}| \\
d & =|\{x y \mid x \in E \backslash F \wedge y \notin E\}|
\end{aligned}
$$

Likewise, we have $\left|\delta E^{\prime}\right|=a^{\prime}+b^{\prime}+c^{\prime}+d^{\prime}$, where

$$
\begin{aligned}
a^{\prime} & =\left|\left\{x y \mid x \in F^{\prime} \wedge y \in \bar{C} \cap D\right\}\right| \\
b^{\prime} & =\left|\left\{x y \mid x \in F^{\prime} \wedge y \in E\right\}\right| \\
c^{\prime} & =\left|\left\{x y \mid x \in F^{\prime} \wedge y \in C \cap \bar{D}\right\}\right| \\
d^{\prime} & =\left|\left\{x y \mid x \in E^{\prime} \backslash F^{\prime} \wedge y \notin E^{\prime}\right\}\right|
\end{aligned}
$$

With the minimality of $|\delta C|$ and $|\delta D|$ we derive the following:

$$
\begin{aligned}
a+b+c^{\prime} & \leq|\delta C| \leq|\delta F|=a+b+c \\
a^{\prime}+b^{\prime}+c & \leq|\delta D| \leq\left|\delta F^{\prime}\right|=a^{\prime}+b^{\prime}+c^{\prime}
\end{aligned}
$$

We conclude $|\delta C|=|\delta F|$ and $|\delta D|=\left|\delta F^{\prime}\right|$. This implies $F \in \mathcal{C}_{\text {min }}(\alpha)$ and $F^{\prime} \in \mathcal{C}_{\text {min }}(\beta)$.

We still have to show $E=F$ and $E^{\prime}=F^{\prime}$. For this it is enough to show that $d=d^{\prime}=0$. Assume by contradiction that $d+d^{\prime} \geq 1$. Say, $d \geq 1$. Then we have $|\delta C|+|\delta D|>a+b+c+a^{\prime}+b^{\prime}+c^{\prime}$. This contradicts the assertion $|\delta C|=|\delta F|$ and $|\delta D|=\left|\delta F^{\prime}\right|$. This yields $F=E \in \mathcal{C}_{\min }(\alpha)$ and $F^{\prime}=E^{\prime} \in \mathcal{C}_{\min }(\beta)$. Since $C$ and $D$ are optimal cuts, we conclude $m(E) \geq m(C)$ and $m\left(E^{\prime}\right) \geq m(D)$.

The crucial step in the proof is the following assertion:

$$
\begin{equation*}
m(E)+m\left(E^{\prime}\right)<m(C)+m(D) . \tag{1}
\end{equation*}
$$

Once we have established Equation 1 we get an obvious contradiction to $m(E) \geq$ $m(C)$ and $m\left(E^{\prime}\right) \geq m(D)$.

To see Equation we show two claims:

1. If a cut $F$ is nested with $C$ or nested with $D$, then $F$ is nested with $E$ or nested with $E^{\prime}$ :
By symmetry let $F$ be nested with $C$. If $F \subseteq C$ (resp. $\bar{F} \subseteq C$ ), then $F \subseteq \overline{E^{\prime}}$ (resp. $\bar{F} \subseteq \overline{E^{\prime}}$ ). If $C \subseteq F$ (resp. $C \subseteq \bar{F}$ ), then $E \subseteq F$ (resp. $E \subseteq \bar{F})$.
2. If a cut $F$ is nested with $C$ and nested with $D$, then $F$ is nested with $E$ and nested with $E^{\prime}$ :
By symmetry in $F, \bar{F}$ we may assume $C \subseteq F$ or $\bar{C} \subseteq F$. Using now the symmetry in $E, E^{\prime}$ we may assume that $C \subseteq F$. Hence we have $E \subseteq F$; and it remains to show that $E^{\prime}$ and $F$ are nested. For $D \subseteq \bar{F}$, we had $C \cap D=\emptyset$. For $\bar{D} \subseteq \bar{F}$, we had $C \cap \bar{D}=\emptyset$. Both is impossible because $C$ and $D$ are not nested. For $D \subseteq F$ we obtain $\overline{E^{\prime}}=C \cup D \subseteq F$ what implies that $E^{\prime}$ and $F$ are nested. Finally let $\bar{D} \subseteq F$, then $E^{\prime} \subseteq F$. Again $E^{\prime}$ and $F$ are nested.

Putting claims 1 and 2 together yields: $m(E)+m\left(E^{\prime}\right) \leq m(C)+m(D)$. Now, $C$ is nested with both corners $E$ and $E^{\prime}$. Hence, $C$ is not counted on the left-hand side of the inequality. However, $C$ is counted on the right-hand side because $C$ is not nested with $D$. That means the inequality in Equation 1 is strict. Hence, we have shown the result of the proposition.

Analog results to Proposition 3.7 are Theorem 1.1 in [12] or Theorem 3.3 of [19]. In contrast to these results, Proposition 3.7 allows that $\mathcal{C}_{\text {opt }}$ may contain cuts of different weights. We have to deal with cuts of different weights because we wish to get a "complete" decomposition of virtually free groups like ( $\mathbb{Z} \times$ $\mathbb{Z} / 2 \mathbb{Z}) * \mathbb{Z} / 2 \mathbb{Z}$. Like in the graph in Figure $\mathbb{1}$ in the Cayley graph of this group cuts with weight 1 and 2 are necessary to split all bi-infinite paths into two infinite pieces, see Figure 5


Figure 5: The Cayley graph of the group $(\mathbb{Z} \times \mathbb{Z} / 2 \mathbb{Z}) * \mathbb{Z} / 2 \mathbb{Z}$ moving very fast towards your eyes.

### 3.2 The structure tree

The notion of structure tree is due to Dunwoody [11]. Recall that $\Gamma$ is assumed to be accessible, hence $\mathcal{C}_{\text {opt }}$ is defined and there is some $k \in \mathbb{N}$ such that every
cut in $\mathcal{C}_{\text {opt }}$ is a $k$-cut.

Lemma 3.8 Let $C, D \in \mathcal{C}_{\mathrm{opt}}$. Then the set $\left\{E \in \mathcal{C}_{\mathrm{opt}} \mid C \subseteq E \subseteq D\right\}$ is finite.
Proof. Choose two vertices $u \in C$ and $v \in \bar{D}$, and a path $\gamma$ in $\Gamma$ connecting them. Every cut $E$ with $C \subseteq E \subseteq D$ must separate $u$ and $v$ and thus contain a vertex of of $\gamma$. With Lemma 3.2 and the accessibility of $\Gamma$ it follows that there are only finitely many such cuts.

The set $\mathcal{C}_{\text {opt }}$ is partially ordered by $\subseteq$. By Lemma 3.8, the partial order is induced by its so-called Hasse diagram. In the Hasse diagram there is an arc from $\bar{C} \in \mathcal{C}_{\text {opt }}$ to $D \in \mathcal{C}_{\text {opt }}$ if and only if $\bar{C} \varsubsetneqq D$ and there is no $E \in \mathcal{C}_{\text {opt }}$ between them. In dense orderings, like $(\mathbb{Q}, \leq)$, the Hasse diagram is empty, whereas in discrete orderings, like $\left(\mathcal{C}_{\text {opt }}, \subseteq\right)$, the partial order is the reflexive and transitive closure of the arc relation in the Hasse diagram.

If there is an arc from $\bar{C}$ to $D$, then there is also an arc from $\bar{D}$ to $C$. In such a situation we put $C$ and $D$ in one class:

Definition 3.9 For $C, D \in \mathcal{C}_{\text {opt }}$ we define the relation $C \sim D$ by the following condition:

Either $C=D$ or both $\bar{C} \varsubsetneqq D$ and $\forall E \in \mathcal{C}_{\mathrm{opt}}: \bar{C} \varsubsetneqq E \subseteq D \Longrightarrow D=E$.
The intuition behind this definition is as follows: Consider $(C, \bar{C})$ for $C \in \mathcal{C}_{\text {opt }}$ as an edge set of some graph. Call edges $(C, \bar{C})$ and $(D, \bar{D})$ to be adjacent if $C \sim D$. This makes sense due the following property.

Lemma 3.10 The relation $\sim$ is an equivalence relation.
Proof. Reflexivity and symmetry are immediate. Transitivity requires to check all inclusions how the cuts can be nested. A proof can be found e.g. in [11. In order to keep the paper self-contained we repeat the proof for transitivity. Let $C \sim D \neq C$ and $D \sim E \neq D$. This implies $\emptyset \neq \bar{D} \subseteq C \cap E$. We have to show that $C \sim E$. The cuts $C$ and $E$ are nested due to Proposition 3.7 Hence we have one of the following four inclusions:

- $C \subseteq E$ : This implies $\bar{D} \varsubsetneqq C \subseteq E$. Hence, $C=E$ because $D \sim E$.
- $E \subseteq C$ : This implies $\bar{D} \varsubsetneqq E \subseteq C$, Hence, $C=E$ because $D \sim C$.
- $E \subseteq \bar{C}$ : This contradicts $C \cap E \neq \emptyset$.
- $\bar{C} \subseteq E$ : Since $C \cap E \neq \emptyset$, we see $\bar{C} \varsubsetneqq E$. Now, let $\bar{C} \varsubsetneqq F \subseteq E$ for some $F \in \mathcal{C}_{\text {opt }}$. Since $F$ and D are nested, we obtain one of the following inclusions:
- $D \subseteq F$ : This implies $D \subseteq E$, in contradiction to $\bar{D} \varsubsetneqq E$.
- $F \varsubsetneqq D$ : This implies $\bar{C} \varsubsetneqq F \varsubsetneqq D$, in contradiction to $C \sim D$.
- $F \subseteq \bar{D}$ : This implies $\bar{C} \varsubsetneqq F \subseteq \bar{D}$, in contradiction to $\bar{C} \varsubsetneqq D$.
$-\bar{D} \varsubsetneqq F$ : This implies $\bar{D} \varsubsetneqq F \subseteq E$. Hence, $F=E$ because $D \sim E$.

Definition 3.11 Let $T\left(\mathcal{C}_{\text {opt }}\right)$ denote the following graph:

$$
\begin{aligned}
& V\left(T\left(\mathcal{C}_{\mathrm{opt}}\right)\right)=\left\{[C] \mid C \in \mathcal{C}_{\mathrm{opt}}\right\} \\
& E\left(T\left(\mathcal{C}_{\mathrm{opt}}\right)\right)=\left\{(C, \bar{C}) \mid C \in \mathcal{C}_{\mathrm{opt}}\right\}
\end{aligned}
$$

The incidence maps are defined by $s((C, \bar{C}))=[C]$ and $t((C, \bar{C}))=[\bar{C}]$.
The directed edges are in canonical bijection with the pairs $([C],[\bar{C}])$. Indeed, let $C \sim D$ and $\bar{C} \sim \bar{D}$. It follows $C=D$ because otherwise $C \varsubsetneqq \bar{D} \varsubsetneqq C$. Thus, $T\left(\mathcal{C}_{\text {opt }}\right)$ is an undirected graph without self-loops and multi-edges.

The graph $T\left(\mathcal{C}_{\text {opt }}\right)$ is locally finite if and only if the equivalence classes $[C]$ are finite. Hence it is not locally finite, in general: For each $C \in \mathcal{C}_{\text {opt }}$ there might be infinitely many $D \in \mathcal{C}_{\text {opt }}$ with $D \sim C$. For example, consider the Cayley graph of the group $(\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z} / 2 \mathbb{Z}$. There is one $\mathbb{Z} \times \mathbb{Z}$-plane through the origin and for every $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ there is one edge leaving this plane. Removing this edge defines a unique 1-cut $C_{i, j}$ with $\mathbb{Z} \times \mathbb{Z} \subseteq C_{i, j}$. It is in $\mathcal{C}_{\text {opt }}$ because all other minimal cuts are nested with $C_{i, j}$. We have $\overline{C_{i, j}} \subseteq C_{0,0}$ for $(i, j) \neq(0,0)$, but there is no $E \in \mathcal{C}_{\text {opt }}$ with $\overline{C_{i, j}} \varsubsetneqq E \varsubsetneqq C_{0,0}$.

Proposition 3.12 The graph $T\left(\mathcal{C}_{\mathrm{opt}}\right)$ is a tree.
Proof. Let $\gamma$ be a simple path in $T\left(\mathcal{C}_{\mathrm{opt}}\right)$ of length at least two. Then $\gamma$ corresponds to a sequence of cuts

$$
C_{0}, \bar{C}_{0} \sim C_{1}, \ldots, \bar{C}_{n-2} \sim C_{n-1}, \bar{C}_{n-1}=C_{n}
$$

with $\left[C_{i-1}\right] \neq\left[C_{i+1}\right]$, so in particular $\bar{C}_{i-1} \neq C_{i}$ for $1 \leq i \leq n-1$ (otherwise we would have $\left.C_{i-1}=\bar{C}_{i} \sim C_{i+1}\right)$. So we get a sequence

$$
C_{0} \varsubsetneqq C_{1} \varsubsetneqq C_{2} \varsubsetneqq \cdots \varsubsetneqq C_{n-1}
$$

Therefore we have $C_{0} \neq \bar{C}_{n-1}$ and $\bar{C}_{0} \nsubseteq \bar{C}_{n-1}$. So $C_{0} \nsim \bar{C}_{n-1}=C_{n}$ and the original path is not a cycle. So $T\left(\mathcal{C}_{\mathrm{opt}}\right)$ has no cycles.

It remains to show that $T\left(\mathcal{C}_{\text {opt }}\right)$ is connected. Let $[C],[D] \in V\left(T\left(\mathcal{C}_{\text {opt }}\right)\right)$. Since $C$ and $D$ are nested and $(C, \bar{C}),(D, \bar{D}) \in E\left(T\left(\mathcal{C}_{\text {opt }}\right)\right)$, we may assume $C \subseteq D$. By Lemma 3.8, there are only finitely many cuts $E \in \mathcal{C}_{\text {opt }}$, with $C \subseteq E \subseteq D$. Now, let $C_{0}, C_{1}, \ldots, C_{n}$ be a not refinable sequence of cuts in $\mathcal{C}_{\text {opt }}$ such that

$$
C=C_{0} \varsubsetneqq C_{1} \varsubsetneqq C_{2} \varsubsetneqq \cdots \varsubsetneqq C_{n-1} \varsubsetneqq C_{n}=D
$$

Then we obtain a path from $C$ to $D$ :

$$
C=C_{0}, \bar{C}_{0} \sim C_{1}, \bar{C}_{1} \sim C_{2}, \ldots, \bar{C}_{n-1} \sim C_{n}=D
$$

Hence, $T\left(\mathcal{C}_{\text {opt }}\right)$ is connected and therefore a tree.
Remark 3.13 According to Dunwoody [11] a tree set is a set of pairwise nested cuts, which is closed under complementation and such that for each $C, D \in \mathcal{C}$ the set $\{E \in \mathcal{C} \mid C \subseteq E \subseteq D\}$ is finite. Thus, using this terminology, Proposition 3.7 and Lemma 3.8 show that $\mathcal{C}_{\text {opt }}$ is a tree set. Once this is established Proposition 3.12 becomes a general fact due to Dunwoody [11, Thm. 2.1].

## 4 Actions on $\Gamma$ and its structure tree $T\left(\mathcal{C}_{\text {opt }}\right)$

In this section, $\Gamma$ denotes a connected, locally finite, and accessible graph such that the group of automorphisms $\operatorname{Aut}(\Gamma)$ acts with finitely many orbits on $\Gamma$. The action on $\Gamma$ induces an action of $\operatorname{Aut}(\Gamma)$ on $\mathcal{C}_{\text {opt }}$ and on the structure tree $T\left(\mathcal{C}_{\text {opt }}\right)$. For example, if $\Gamma$ is the Cayley graph of a group $G$ with respect to some finite generating set $\Sigma \subseteq G$, then $\Gamma$ is connected, locally finite, and there is only one orbit: $|\operatorname{Aut}(\Gamma) \backslash \Gamma|=1$.

Lemma 4.1 Let $|\operatorname{Aut}(\Gamma) \backslash \Gamma|$ be finite and $k \in \mathbb{N}$. Then the canonical action of $\operatorname{Aut}(\Gamma)$ on the set of $k$-cuts has finitely many orbits, only. In particular $\operatorname{Aut}(\Gamma)$ acts on $\mathcal{C}_{\mathrm{opt}}$ and on the tree $T\left(\mathcal{C}_{\mathrm{opt}}\right)$ with finitely many orbits.

Proof. Let $\operatorname{Aut}(\Gamma) \backslash V(\Gamma)$ be represented by some finite vertex set $U \subseteq V(\Gamma)$. With Lemma 3.2 it follows that there are only finitely many $k$-cuts $C$ such that $U \cap \beta C \neq \emptyset$. Since every cut is in the same orbit as some cut $C$ with $U \cap \beta C \neq \emptyset$, the group $\operatorname{Aut}(\Gamma)$ acts on the set of $k$-cuts with finitely many orbits.

Since $\Gamma$ is accessible, there is a $k$ such that for all cuts $C \in \mathcal{C}_{\text {opt }}$ holds $|\delta C| \leq k$. For the last statement observe that $\left\{(C, \bar{C}) \mid C \in \mathcal{C}_{\text {opt }}\right\}$ is the edge set of $T\left(\mathcal{C}_{\mathrm{opt}}\right)$. Thus, the action of $\operatorname{Aut}(\Gamma)$ on $T\left(\mathcal{C}_{\mathrm{opt}}\right)$ has only finitely many orbits, too.

For $S \subseteq V(\Gamma)$ and $k \geq 1$ we let $N^{k} S=\{v \in V(\Gamma) \mid d(v, S) \leq k\}$ denote the $k$-th neighborhood of $S$. Now, for a cut $C$ we can choose $k$ large enough such that $N^{k} C \cap \bar{C}$ is connected because $\bar{C}$ is connected. (Indeed, all points in $\beta C \cap \bar{C}$ can be connected in $\bar{C}$, hence for some $k$ large enough these points can be connected in $N^{k} C \cap \bar{C}$. This $k$ suffices to make $N^{k} C \cap \bar{C}$ connected.) By Lemma 4.1, there are only finitely many orbits of optimal cuts. Thus we can choose some $\kappa \in \mathbb{N}$ which works for all $C \in \mathcal{C}_{\text {opt }}$ and fix it for the rest of this section.

Now, we want to deduce some more information about the structure of the vertex stabilizers $G_{[C]}=\{g \in G \mid g C \sim C\}$ of vertices of the tree $T\left(\mathcal{C}_{\text {opt }}\right)$. Therefore, we assign to each vertex of $T\left(\mathcal{C}_{\text {opt }}\right)$ a so-called block. The definition has been taken from 30. In Lemma 4.7 we show that the blocks are somehow "small". They are defined as follows.

Definition 4.2 Let $\operatorname{Aut}(\Gamma) \backslash \Gamma$ be finite and $\mathcal{C}_{\text {opt }}$ be the set of optimal cuts. Let $\kappa \geq 1$ be defined as above such that $N^{\kappa} C \cap \bar{C}$ is connected for all $C \in \mathcal{C}_{\mathrm{opt}}$. The block assigned to $[C] \in V\left(T\left(\mathcal{C}_{\mathrm{opt}}\right)\right)$ is defined by:

$$
B[C]=\bigcap_{D \sim C} N^{\kappa} D
$$

Lemma 4.3 We have

$$
B[C]=\bigcap_{D \sim C} D \cup \bigcup_{D \sim C} N^{\kappa} D \cap \bar{D}
$$

Proof. The inclusion from left to right is trivial. It is therefore enough to show that we have $N^{\kappa} C \cap \bar{C} \subseteq B[C]$. Clearly, $N^{\kappa} C \cap \bar{C} \subseteq N^{\kappa} C$. Thus is enough to consider $D \sim C, D \neq C$ and to show that $N^{\kappa} C \cap \bar{C} \subseteq N^{\kappa} D$. This follows from:

$$
N^{\kappa} C \cap \bar{C} \subseteq \bar{C} \varsubsetneqq D \subseteq N^{\kappa} D
$$

Example 4.4 Figure 6 shows a part of the Cayley graph of the free product $\mathbb{Z} / 2 \mathbb{Z} * \mathbb{Z} / 3 \mathbb{Z}=\left\langle a, b \mid a^{2}=1=b^{3}\right\rangle$. The minimal cuts cut the edges with label $a$, i.e., they cut through cosets of $\mathbb{Z} / 2 \mathbb{Z}$. The optimal cuts are exactly the minimal cuts. The three cuts depicted with dashed lines belong to the same equivalence class and the bold vertices form the respective block. Here, we can choose $\kappa=1$ for the definition of the blocks.


Figure 6: Block of six vertices in the Cayley graph of $\mathbb{Z} / 2 \mathbb{Z} * \mathbb{Z} / 3 \mathbb{Z}$

Lemma 4.5 The following assertions hold.

1. For all $C \in \mathcal{C}_{\text {opt }}$ the block $B[C]$ is connected .
2. There is a number $\ell \in \mathbb{N}$ such that for all $C \in \mathcal{C}_{\text {opt }}$ and all $S \subseteq B[C]$ we have: Whenever two vertices $u, v \in B[C]-N^{\ell} S$ can be connected by some path in $\Gamma-N^{\ell} S$, then they can be connected by some path in $B[C]-S$.

Proof. Note that 1, is a special case of 2, by choosing $S=\emptyset$. Let $\ell=$ $\max \left\{d(u, v) \mid D \in \mathcal{C}_{\text {opt }}, u, v \in \bar{D} \cap N^{\kappa} D\right\}$. Thus $\ell$ is a uniform bound on the diameters for the sets $N^{\kappa} D \cap \bar{D}$ for $D \in \mathcal{C}_{\text {opt }}$. It exists because there are only finitely many orbits of optimal cuts.

Now, let $u, v \in B[C]-N^{\ell} S$ be two vertices which are connected by some path $\gamma$ in $\Gamma-N^{\ell} S$. We are going to transform the path $\gamma$ into some path $\gamma^{\prime}$ where all vertices are in $B[C]-S$. If $\gamma$ is entirely in $B[C]$ we are done. Hence we may assume that there exist a first vertex $v_{m}$ of $\gamma$ which does not
lie in $B[C]$. Thus for some $D \sim C$ we have $v_{m} \notin N^{\kappa} D$. Since $\kappa \geq 1$, we have $v_{m-1} \in N^{\kappa} D \cap \bar{D}$. For some $n>m$ we find a vertex $v_{n}$ which is the first vertex after $v_{m}$ lying in $N^{\kappa} D$ again. As $v_{n}$ is the first one, we have $v_{n} \in N^{\kappa} D \cap \bar{D}$, too. Since $N^{\kappa} D \cap \bar{D}$ is connected, we can choose a path from $v_{m-1}$ to $v_{n}$ inside $N^{\kappa} D \cap \bar{D}$. This is a path inside $B[C]$ by Lemma 4.3 Note that this path does not use $v_{m}$ anymore. Moreover, the new segment cannot meet any point in $S$ because otherwise $v_{n} \in N^{\ell} S$. The path from $v_{n} \in B[C]-N^{\ell} S$ to $v$ is shorter than $\gamma$. Hence, by induction, $v_{n}$ is connected to $v$ in $B[C]-S$; and we can transform $\gamma$ as desired.

Lemma 4.6 Let $C \in \mathcal{C}_{\text {opt }}$ and $g \in \operatorname{Aut}(\Gamma)$ be such that $g(C) \sim C$. Then we have $g(B[C])=B[C]$.

Proof. Let $v \in B[C]=\bigcap\left\{N^{\kappa} D \mid D \sim C\right\}$, then $g v \in \bigcap\left\{N^{\kappa} g D \mid g D \sim g C\right\}$ for all $g \in \operatorname{Aut}(\Gamma)$. Now, if $D \sim C$ and $g C \sim C$ for some $g \in G$, then $g D \sim g C \sim C$. Hence, $g v \in B[C]$.

Lemma 4.7 Let $\Gamma$ be a connected, locally finite, and accessible graph such that a group $G$ acts on $\Gamma$ with finitely many orbits. Let $C \in \mathcal{C}_{\text {opt }}$. Then the stabilizer $G_{[C]}=\{g \in G \mid g C \sim C\}$ of the vertex $[C]=\{D \mid C \sim D\} \in V\left(T\left(\mathcal{C}_{\mathrm{opt}}\right)\right)$ acts with finitely many orbits on the block $B[C]$.

Proof. Since $G$ acts with finitely many orbits on $\Gamma$, it acts with finitely many orbits on the set $\mathcal{C}_{\text {opt }}$. For $D \sim g D \sim C$ we have $g \in G_{[C]}$ by Lemma 4.6, Hence, $G_{[C]}$ acts with finitely many orbits on $[C]$. This implies that $G_{[C]}$ acts with finitely many orbits on the union $\bigcup\{\beta D \mid D \sim C\}$.

We are going to show that there is some $m \in \mathbb{N}$ such that for every $v \in B[C]$ there is a cut $D \in[C]$ with $d(v, \beta D) \leq m$. This implies the lemma since $\Gamma$ is locally finite.

Let $v \in B[C]$. If $v \in N^{\kappa} D \cap \bar{D}$ for some $D \sim C$, then we have $d(v, \beta D) \leq \kappa$ (recall that $\kappa$ is a fixed constant). Thus it remains to consider the case $v \in D$ for all $D \sim C$.

Let $U$ be a finite subset of $B[C]$ such that $B[C] \subseteq G \cdot U$. There is a constant $m \geq \kappa$ such that $d(u, \beta C) \leq m$ for $u \in U$. We conclude that for the node $v \in B[C]$ there is some $g \in G$ and $E=g C$ such that $d(v, \beta E) \leq m$. Thus, we actually may assume $v \in \beta E$ and show that this implies $v \in \bigcup_{D \sim C} \beta D$.

Because $C$ and $E$ are nested, we can assume (after replacing $E$ with $\bar{E}$ if necessary) that $C \subseteq \bar{E}$ or $\bar{E} \varsubsetneqq C$. If $C \subseteq \bar{E}$ (thus $E \subseteq \bar{C}$ ), then $\beta E \subseteq \beta \bar{C} \cup \bar{C}$. But $v \in C$, hence $v \in \beta \bar{C}=\beta C$. On the other hand, if $\bar{E} \varsubsetneqq C$, then there is a $D \sim C$ such that $\bar{E} \subseteq \bar{D} \varsubsetneqq C$. It follows that $v \in D \cap \beta \bar{E} \subseteq D \cap(\beta \bar{D} \cup \bar{D}) \subseteq \beta D$.

A graph $\Gamma$ is said to have more than one end if there is a finite set $S \subseteq V(\Gamma)$ such that $\Gamma-S$ has at least two infinite connected components. Otherwise, it has at most one end. Since we only consider connected and locally finite graphs, it follows that $\Gamma$ has more than one end if and only if there exists a bi-infinite simple path $\alpha$ such that $\mathcal{C}(\alpha) \neq \emptyset$.

The key property of blocks is that blocks cannot have more than one end:
Proposition 4.8 For $C \in \mathcal{C}_{\text {opt }}$ the block $B[C]$ has at most one end.

Proof. Assume by contradiction that $B[C]$ has more than one end. By Lemma 4.5 $B[C]$ is connected, hence there is a bi-infinite simple path $\alpha$ and a finite subset $S \subseteq B[C]$ such that two different connected components of $B[C]-S$ contain infinitely many elements of $\alpha$. However, for all $D \sim C$ we have $\alpha \subseteq B[C] \subseteq N^{\kappa} D$ and $N^{\kappa} D \cap \bar{D}$ is finite. Hence for all $D \sim C$ almost all nodes of $\alpha$ are in $D$ and $|\alpha \cap \bar{D}|<\infty$.

By Lemma 4.5, there are two different connected components of $\Gamma-N^{\ell} S$ containing each infinitely many elements of $\alpha$. Thus, the set $\mathcal{C}(\alpha)$ is not empty, hence there is an optimal cut $E \in \mathcal{C}_{\text {opt }}(\alpha)$. This means $|\alpha \cap E|=\infty=|\alpha \cap \bar{E}|$. The cuts $C$ and $E$ are nested. We cannot have $E \subseteq \bar{C}$ or $\bar{E} \subseteq \bar{C}$ because $|\alpha \cap \bar{C}|<\infty$. Hence, by symmetry $E \varsubsetneqq C$. By Lemma 3.8, there is some $D \in[C]$ such that $E \subseteq \bar{D} \varsubsetneqq C$. But we have just seen that almost all nodes of $\alpha$ belong to $D$. Thus, $|\alpha \cap E|<\infty$. This is a contradiction.

### 4.1 Actions on accessible graphs

In this section $\mathcal{G}$ denotes a class of groups which is closed under taking normal subgroups of finite index. In our application $\mathcal{G}$ will be the class of all finite groups. But actually many other classes of groups are closed under taking finite-index normal subgroups as e.g. the class of f.g. virtually free groups or e.g. the class of finitely presented groups.

Proposition 4.9 Let $\Gamma$ is a connected, locally finite, and accessible graph such that $\operatorname{Aut}(\Gamma) \backslash \Gamma$ is finite and let $G$ be a group acting on $\Gamma$ such that all vertex stabilizers $G_{v}=\{g \in G \mid g v=v\}$ belong to the class $\mathcal{G}$. Then we have:

1. The group $G$ acts with virtually $\mathcal{G}$ edge stabilizers on the tree $T\left(\mathcal{C}_{\mathrm{opt}}\right)$.
2. If $B[C]$ is finite for all $C \in \mathcal{C}_{\mathrm{opt}}$, then $G$ acts with virtually $\mathcal{G}$ vertex stabilizers on the tree $T\left(\mathcal{C}_{\mathrm{opt}}\right)$.

Proof. First, let $\emptyset \neq U \subseteq V(\Gamma)$ be any finite set. The action of $G$ induces a homomorphism from the stabilizer $G_{U}=\{g \in G \mid g U \subseteq U\}$ to the finite group of permutations on $U$. Its kernel is $\bigcap_{u \in U} G_{u}$.

Now fix one vertex $v \in U$. Then for every $k \in \mathbb{N}$ an element $g \in G_{v}$ defines a permutation on the set of vertices $\{u \in V(\Gamma) \mid d(v, u) \leq k\}$. Choose $k$ large enough such that $U \subseteq\{u \in V(\Gamma) \mid d(v, u) \leq k\}$ and let $N$ be the kernel of the $\operatorname{map} G_{v} \longrightarrow \operatorname{Sym}(\{u \in V(\Gamma) \mid d(v, u) \leq k\})$. Then $N$ has finite index in $G_{v}$ because $\Gamma$ is locally finite. The action of $G$ on $V(\Gamma)$ is with $\mathcal{G}$-stabilizers and $\mathcal{G}$ closed under forming finite index normal subgroups, so $N$ is $\mathcal{G}$. Furthermore, $N \leq \bigcap \bigcap_{u \in U} G_{u} \leq G_{U}$ with finite index and so $G_{U}$ is virtually $\mathcal{G}$.

An element in an edge stabilizer $G_{\{C, \bar{C}\}}$ maps $\beta C$ to itself. Since $\beta C$ is finite, $G_{\{C, \bar{C}\}}$ is virtually $\mathcal{G}$.

Now, let $g \in G_{[C]}$, then we have $g(B[C])=B[C]$ by Lemma4.6. If $B[C]$ is finite, $G_{[C]}$ is virtually $\mathcal{G}$.

## 5 Finite treewidth

Tree decompositions were introduced by Robertson and Seymour in connection with their famous result on graph minors, [25]. For some basic properties of tree decompositions see 10 .

Definition 5.1 Let $\Gamma=(V(\Gamma), E(\Gamma))$ be a graph and $\mathcal{P}(\Gamma)$ the family of subsets of $V(\Gamma)$. A tree $T=(V(T), E(T))$ together with a mapping

$$
V(T) \rightarrow \mathcal{P}(\Gamma), t \mapsto X_{t}
$$

is called tree decomposition of $\Gamma$ if the following conditions are fulfilled:
(T1) For every node $v \in V(\Gamma)$ there is some $t \in V(T)$ such that $v \in X_{t}$, i.e., $V(\Gamma)=\bigcup_{t \in V(T)} X_{t}$.
(T2) For every edge $u v \in E(\Gamma)$ there is some $t \in V(T)$ such that $u, v \in X_{t}$.
(T3) If $v \in X_{t} \cap X_{s}$, then we have $v \in X_{r}$ for all vertices $r$ of the tree which are on the unique geodesic path from s to $t$, i.e., the set $\left\{t \in V(T) \mid v \in X_{t}\right\}$ forms a subtree of $T$.
Let $k \in \mathbb{N}$. A graph $\Gamma$ is said to have treewidth $k$ if there exists a tree decomposition such that $\left|X_{t}\right| \leq k+1$ for all $t \in V(T)$. We say that $\Gamma$ has finite treewidth if it has treewidth $k$ for some $k \in \mathbb{N}$. The sets $X_{t}$ are called buckets or bags.

Lemma 5.2 If $\Gamma$ has finite treewidth, then all subgraphs of $\Gamma$ have finite treewidth, too.
Proof. Trivial.
Lemma 5.3 If $\Gamma$ is locally finite of finite treewidth $k$, then there is a tree decomposition $T=(V(T), E(T))$ satisfying the following conditions.

1. Each vertex $v \in \Gamma$ occurs in finitely many bags, only.
2. We have $1 \leq|X| \leq k$ for all bags $X$. In particular, bags are not empty.
3. If two bags $X$ and $Y$ are connected by some edge in the tree $E(T)$, then $X \cap Y \neq \emptyset$.
4. The tree $T$ is locally finite.

Proof. We start with a tree decomposition $T=(V(T), E(T))$ such that $\left|X_{t}\right| \leq$ $k$ for all $t \in V(T)$ and show that we can transform it into one meeting the desired conditions. For every edge $u v \in E(\Gamma)$ choose and fix some vertex $t=t_{u v} \in V(T)$ with $u, v \in X_{t}$. Now, for each vertex $u$ let $T_{u}$ be the finite subtree spanned by the $t_{u v}$ for $v \in V(\Gamma)$. It is finite because $\Gamma$ is locally finite. Remove $u$ from all bags which do not belong to $T_{u}$. This yields still a tree decomposition.

Next, let $x \in X$ and $y \in Y$ where $X$ and $Y$ are two bags, and let $x=$ $x_{0}, \ldots, x_{n}=y$ be some path in $\Gamma$ connecting $x$ and $y$. Let $Z$ be on the geodesic in the tree $T$ from bag $X$ to bag $Y$. An induction on $n$ shows that $Z \cap\left\{x_{0}, \ldots, x_{n}\right\} \neq \emptyset$. Removing all empty bags we therefore have still a tree decomposition.

Now, if in addition, $x \in X, y \in Y$ and $X$ and $Y$ are neighbors in the tree $T$, then we can define $i=\max \left\{i \mid x_{i} \in X\right\}$. We have $i \geq 0$ and if $i=n$, then $y \in X \cap Y$. Thus, we may assume $i<n$. Looking at the location where $x_{i}, x_{i+1}$ are in the same bag, we see that $x_{i} \in Y$.

Now, we can put things together to derive that $T$ is locally finite: For each bag $X$ each of the neighbors contains at least one element of $X$. But every $x$ is contained in at most finitely many bags. Hence, the result follows.

Lemma 5.4 Let $\Gamma$ be a graph of finite treewidth and uniformly bounded degree. Then there exists some $k \in \mathbb{N}$ such that: For every one-sided infinite simple path $\gamma$, every $v_{0} \in V(\Gamma)$, and every $n \in \mathbb{N}$ there is a $k$-cut $D$ with $d\left(v_{0}, \bar{D}\right) \geq n$, $v_{0} \in D$, and $|\bar{D} \cap \gamma|=\infty$.

Remark 5.5 It follows from the following proofs that in the case of $\Gamma$ being a locally finite Cayley graph also the converse of the lemma holds. Thus, when restricting to Cayley graphs of f.g. groups, the statement of Lemma 5.4 gives a characterization of Cayley graphs of context-free groups by its own. A very similar result is due to Woess [31]. Is states that a group is context-free if and only if the ends of its Cayley graph have uniformly bounded diameter.

Proof. Let $d$ be the maximal degree of $\Gamma$ and let $m=\max \left\{\left|X_{t}\right| \mid t \in V(T)\right\}$ be the maximal size of a bag in the tree decomposition $(T, \mathcal{X})$. We let $k=d m$.

Let $t_{0} \in V(T)$ such that $v_{0} \in X_{t_{0}}$. Consider vertices $u, v \in V(\Gamma)-X_{t_{0}}$ which are in bags of two different connected components of $T-t_{0}$. Then every path from $u$ to $v$ has a vertex in $X_{t_{0}}$, so $u$ and $v$ are not in the same connected component of $\Gamma-X_{t_{0}}$. Now let $C_{t_{0}, \gamma}$ be the connected component of $\Gamma-X_{t_{0}}$ which contains infinitely many vertices of $\gamma$. Then the set $C_{t_{0}, \gamma}$ is contained in the union of the bags of one connected component of $T-t_{0}$. Let $t_{1}$ be the neighbor of $t_{0}$ in this connected component, which is uniquely defined because $T$ is a tree.

Repeating this procedure yields a simple path $t_{0}, t_{1}, t_{2}, \ldots$ in $T$ and a sequence of connected sets $C_{t_{0}, \gamma}, C_{t_{1}, \gamma}, C_{t_{2}, \gamma}, \ldots$ such that $\left|\gamma \cap C_{t_{i}, \gamma}\right|=\infty$ for all $i \in \mathbb{N}$. By Lemma 5.3 we may assume that every node $v \in V(\Gamma)$ is contained in only finitely many bags. Hence, we can choose $\ell$ large enough such that $X_{t_{\ell}}$ does not contain any $v \in V(\Gamma)$ with $d\left(v_{0}, v\right) \leq n$.

Now, let $D$ be the connected component of $\overline{C_{t_{\ell}, \gamma}}$ which contains $v_{0}$. Then $\bar{D}$ is connected because every vertex in another connected component of $\overline{C_{t_{\ell}, \gamma}}$ is connected with $C_{t_{\ell}, \gamma}$ inside of $\bar{D}$.

Since every edge of $\delta D$ has one node in $X_{t_{\ell}}$, we have $|\delta D| \leq d m=k$. Thus, $D$ is a $k$-cut with $v_{0} \in D$ and $|\bar{D} \cap \gamma|=\infty$. Furthermore, since every path from $v_{0}$ to a vertex $v \in \bar{D}$ uses a vertex of $X_{\ell}$, we have $d\left(v_{0}, \bar{D}\right) \geq n$.

Proposition 5.6 Let $\Gamma$ be a graph of finite treewidth and uniformly bounded degree. Then $\Gamma$ is accessible.

Proof. Let $\alpha$ be a bi-infinite simple path such that $\mathcal{C}(\alpha) \neq \emptyset$ and let $C \in \mathcal{C}(\alpha)$. We fix a vertex $v_{0} \in \beta C$ and we let $n=\max \left\{d\left(v_{0}, w\right) \mid w \in \beta C\right\}$. Let $k \in \mathbb{N}$ be according to Lemma 5.4. It follows that there is a $k$-cut $D$ with $|\alpha \cap \bar{D}|=\infty$, $v_{0} \in D$, and $d\left(v_{0}, \bar{D}\right) \geq n$. Because of the choice of $n$, we also have $\beta C \subseteq D$ what means that either $C \subseteq D$ or $\bar{C} \subseteq D$. In either case $D$ splits $\alpha$ in two infinite pieces.

Lemma 5.7 Let $\Gamma$ be a connected, locally finite, and infinite graph such that Aut $(\Gamma) \backslash \Gamma$ is finite. Then there is a bi-infinite geodesic.

Proof. There are arbitrarily long geodesics, hence geodesics of every length. For each geodesics $\gamma$ with an odd number of vertices let $m(\gamma)$ be the vertex in
the middle. Because $\operatorname{Aut}(\Gamma) \backslash \Gamma$ is finite, there exists a single vertex $v_{0}$ such that infinitely many geodesics $\gamma$ satisfy $m(\gamma)=v_{0}$. These geodesics form the vertices of a tree as follows: The root is $v_{0}$ (viewed as a geodesic of length $0)$. The parent of a geodesic $\left(v_{-k}, v_{-k+1}, \ldots, v_{0}, \ldots, v_{k-1}, v_{k}\right)$ is defined as $\left(v_{-k+1}, \ldots, v_{0}, \ldots, v_{k-1}\right)$. Since $\Gamma$ is locally finite, we obtain an infinite tree where each node has finite degree. By Königs Lemma there is an infinite path, which defines a bi-infinite geodesic through $v_{0}$.

Note that we cannot remove any of the requirements in Lemma 5.7. In particular, we cannot remove that $\operatorname{Aut}(\Gamma) \backslash \Gamma$ is finite. For example consider the graph $\Gamma$ with $V(\Gamma)=\mathbb{Z}$ and $E(\Gamma)=\{(n, n \pm 1),(n,-n) \mid n \in \mathbb{Z}\}$. This graph is connected, locally finite, and infinite. It has a bi-infinite simple path, but there is no bi-infinite geodesic.

Lemma 5.8 Let $\Gamma$ be connected, locally finite, and infinite such that $\operatorname{Aut}(\Gamma) \backslash \Gamma$ is finite and let $\Gamma$ have finite treewidth. Then $\Gamma$ has more than one end.

Proof. The graph $\Gamma$ has uniformly bounded degree because it is locally finite and $\operatorname{Aut}(\Gamma) \backslash \Gamma$ is finite. By Lemma [5.4 there is some $k$ such that for every $n \in \mathbb{N}, v_{0} \in V(\Gamma)$ and every one-sided infinite simple path $\alpha$ there is a $k$-cut $C$ with $v_{0} \in C, d\left(v_{0}, \bar{C}\right) \geq n$, and $|\bar{C} \cap \alpha|=\infty$.

By Lemma 4.1 there are only finitely many orbits of $k$-cuts under the action of $\operatorname{Aut}(\Gamma)$. Therefore, there is some $m \in \mathbb{N}$ such that $\max \{d(u, v) \mid u, v \in \beta C\} \leq m$ for all $k$-cuts $C$.

Assume that $\Gamma(G)$ has only one end. Now, by Lemma 5.7, there is a biinfinite geodesic $\alpha=\ldots, v_{-2}, v_{-1}, v_{0}, v_{1}, v_{2} \ldots$ Let $C$ be a $k$-cut with $d\left(v_{0}, \bar{C}\right)>$ $m$ such that $v_{0} \in C$ and $|\alpha \cap \bar{C}|=\infty$. Then $|\alpha \cap C|<\infty$, for otherwise $\mathcal{C}(\alpha) \neq \emptyset$.

Hence, there are $i, j>m$ with $v_{-i}, v_{j} \in \beta C \cap \bar{C}$. But this implies $d\left(v_{-i}, v_{j}\right)=$ $d\left(v_{-i}, v_{0}\right)+d\left(v_{0}, v_{j}\right)>2 m$ in contradiction to $d(u, v) \leq m$ for all $u, v \in \beta C$.

Now we have all the tools to state and prove our main theorem.
Theorem 5.9 Let $\mathcal{G}$ be a class of groups which is closed under taking finiteindex normal subgroups. Let $\Gamma$ be a connected, locally finite graph of finite treewidth. Let a group $G$ act on $\Gamma$ such that $G \backslash \Gamma$ is finite and each node stabilizer $G_{v}$ is in $\mathcal{G}$.

Then $G$ acts on the tree $T\left(\mathcal{C}_{\mathrm{opt}}\right)$ such that all vertex and edge stabilizers are virtually $\mathcal{G}$ and $G \backslash T\left(\mathcal{C}_{\mathrm{opt}}\right)$ is finite.

Proof. The blocks $B[C]$ have finite treewidth by Lemma 5.2. By Lemma 4.7 , $G_{[C]}$ acts with finitely many orbits on $B[C]$. Hence, we can apply Lemma 5.8 what implies that the blocks are finite or have more than one end. The latter case is excluded by Proposition 4.8, which states that they have at most one end. That means that the blocks are finite. The theorem then follows with Lemma 4.1 and Proposition 4.9

Corollary 5.10 Let a group $G$ act on a connected, locally finite graph $\Gamma$ of finite treewidth such that $G \backslash \Gamma$ is finite and each node stabilizer $G_{v}$ is finite. Then $G$ is the fundamental group of a finite graph of finite groups.

Proof. By Theorem 5.9, $G$ acts on a tree $T$ with finite vertex stabilizers such that $G \backslash T$ is finite. Bass-Serre theory ( $[27]$ ) yields the result.

Note that if we know that $G$ is finitely generated, then the condition $|G \backslash \Gamma|<$ $\infty$ in Theorem 5.9 and Corollary 5.10 is no real restriction since in this case we always can construct a subgraph of $\Gamma$ on which $G$ acts with finitely many orbits. To do that we proceed as follows: Let $\Sigma$ be a finite generating set of $G$ and let $v_{0} \in V(\Gamma)$ be some arbitrary vertex. For all $a \in \Sigma$ we fix paths $\gamma_{a}$ from $v_{0}$ to $a v_{0}$. Let $\Delta$ be the subgraph of $\Gamma$ induced by the vertex set $G \cdot \bigcup_{a \in \Sigma} \gamma_{a}$. This graph is connected, locally finite and it has finite treewidth by Lemma 5.2,

Another interesting observation about the tree $T\left(\mathcal{C}_{\text {opt }}\right)$ is that together with the blocks $B[C]$ it forms a tree decomposition of $\Gamma$ of finite width.

## 6 Context-free groups

A formal language is a subset $L$ of the free monoid $\Sigma^{*}$ over some alphabet $\Sigma$. Here, an alphabet simply means any finite set. We say that a class $\mathcal{K}$ of formal languages is closed under inverse homomorphism if $L \in \mathcal{K}$ implies $\psi^{-1}(L) \in \mathcal{K}$ for all homomorphisms $\psi: \Sigma^{*} \rightarrow \Sigma^{*}$. Almost all classes investigated in formal language theory or complexity theory are closed under inverse homomorphism, see e.g. 17. For example, all classes in the Chomsky hierarchy have this property. Other examples are the classes of deterministic context-free languages, the class of languages where the membership problem can be solved in polynomial time, and the class of recursive languages.

Let $\mathcal{K}$ be a class of languages which is closed under inverse homomorphisms. We say that the word problem of a group $G$ belongs to the class $\mathcal{K}$ if there is homomorphism $\pi: \Sigma^{*} \rightarrow G$ onto $G$ such that $\pi^{-1}(1) \in \mathcal{K}$. This is a property of $G$ and does not depend on the presentation $\pi: \Sigma^{*} \rightarrow G$ : Indeed, let $\pi^{\prime}: \Sigma^{* *} \rightarrow$ $G$ be another presentation of $G$. Since $\Sigma^{* *}$ is free, we find a homomorphism $\psi: \Sigma^{\prime *} \rightarrow \Sigma^{*}$ such that $\pi^{\prime}=\pi \circ \psi$. Hence, $\pi^{\prime-1}(1)=\psi^{-1}\left(\pi^{-1}(1)\right) \in \mathcal{K}$. For simplicity, we say that a group $G$ is context-free if the word problem of $G$ is context-free. By well-known and classical results of Anisimov it is known that all context-free groups are finitely presented [2, Thm. 2] (see also Section 6.0.3); and the word problem of a group $G$ is regular if and only if $G$ is finite 1, Thm. 1]. The proofs of these facts are actually very easy by using the standard "pumping properties" of context-free (resp. regular) languages.

### 6.0.1 Solving the word problem using deterministic pushdown automata

Let $G$ be a finitely generated virtually free group and $F(X)$ be a free subgroup of finite index. Choose a set $R$ with $1 \in R \subseteq G$ such that the canonical projection $G \rightarrow F(X) \backslash G$ induces a bijection between $R$ and the finite quotient $F(X) \backslash G$. We use the disjoint union $\Sigma=X^{ \pm} \cup R$ as a finite generating alphabet, where $X^{ \pm}=X \cup X^{-1}$. For all letters $a, b \in \Sigma$ we can define rewrite rules as follows:
$a b \rightarrow x_{a b} r \quad$ if $x_{a b}$ is a word over $X^{ \pm}$and $r \in R$ such that $a b=x_{a b} r \in G$.
This system can be used by a deterministic pushdown automaton transforming an input word $w \in \Sigma^{*}$ into its normal form $w=x r$ with $x \in\left(X^{ \pm}\right)^{*}$ and
$r \in R$ : First, we choose $k \in \mathbb{N}$ such that $k \geq\left|x_{a b} r\right|$ for all rules $a b \rightarrow x_{a b} r$. The pushdown stack contains freely reduced words over $X^{ \pm}$, the set of states are the words $y r \in F(X) \cdot R$ of length at most $k$. We start with an empty stack in state $1 \in R$ and with the input word $w$. We perform the following instructions:

- If the input is empty and the state is a letter $r \in R$, then stop.
- If the state is a letter $s \in R$, but the input is not empty, then read the next input letter $b$ and change the state to $x_{s b} r$ according to the rule $s b \rightarrow x_{a b} r$.
- If the state is a word $y s \in F(X) \cdot R$ with $1 \neq y \in F(X)$ and the stack content is a freely reduced word $z$ over $X^{ \pm}$, then replace (within less than $k$ steps) $z$ by the freely reduced word corresponding the group element $z y \in F(X)$, and after that switch to the state $s \in R$.

The description how the pushdown automaton works is just standard way how to compute normal forms in linear time. Indeed, if we start with an input word $w$, then we stop in a configuration where $x$ is a freely reduced word on the stack and we are in some state $r \in R$. It is clear that $w=x r \in G$. Hence, in order to solve the word problem we only have to check whether $x=1$ and $r=1$.

### 6.0.2 Finitely generated virtually free groups are context-free

The statement itself follows from the precedent subsection and standard facts how to transform a pushdown automaton into a context-free grammar, see any textbook on formal languages like [17. Let us recall however that, a priori, the class of context-free groups could be larger than the class of deterministic context-free groups.

It is well-known that there are context-free languages which are not deterministic context-free. Indeed, consider the group $\mathbb{Z} \times \mathbb{Z}$ with generators $a=(1,0), b=(0,1)$, and $c=(-1,-1)$. A standard exercise shows that set of the words $w \in\{a, b, c\}^{*}$ which are equal to $(0,0)$ is not context-free, but its complement is context-free. It cannot be deterministic context-free because deterministic context-free languages are closed under complementation, 17. Thus, $\mathbb{Z} \times \mathbb{Z}$ is co-context-free in the sense of [16]. The class of co-context-free groups is very interesting in its own, for example it includes the Higman-Thompson group [22].

### 6.0.3 Context-free groups are finitely presented

Anisimov [2 used the so-called uvwxy-Theorem in order to show that contextfree groups are finitely presented. We obtain however a more concise finite presentation by using the production rules of a context-free grammar as defining relations. To be more precise, let $\pi: \Sigma^{*} \rightarrow G$ a surjective homomorphism such that $L_{G}=\left\{w \in \Sigma^{*} \mid \pi(w)=1\right\}$ is context-free. Let $(V, \Sigma, P, S)$ be a contextfree grammar which generates $L_{G}$ according to the notation of [17]: This means $V \cap \Sigma=\emptyset$ and all production rules of $P$ have the form $A \rightarrow \alpha$ where $A \in V$ is a variable and $\alpha \in(V \cup \Sigma)^{*}$ is a word. We may assume that every variable $A \in V$ appears in some derivation

$$
S \underset{P}{\stackrel{*}{\longrightarrow}} \gamma A \delta \underset{P}{\stackrel{*}{\longrightarrow}} w \in \Sigma^{*} .
$$

(If there is no such derivation, we may remove $A$ from the grammar.) Now, the canonical homomorphisms $\Sigma^{*} \rightarrow F(\Sigma) \rightarrow F(V \cup \Sigma) \rightarrow F(V \cup \Sigma) / P$ yield an isomorphism:

$$
G=\Sigma^{*} /\left\{u=1 \mid u \in L_{G}\right\} \rightarrow F(V \cup \Sigma) / P
$$

This fact has a straightforward verification. It has been generalized to other languages and grammar types leading to the notion of Hotz-isomorphism. We refer to 9 for details and some open problems in this area.

### 6.0.4 Quasi-isometric sections

This section yields a direct construction of a context-free grammar (in Chomsky normal form) associated to a f.g. virtually free group. Thus, we do not rely on any formal definition for a push-down automaton or the result that the accepted language of push-down automaton is always context-free. This is standard fact in formal language theory, but its proof is non-trivial. So we prefer to circumvent these constructions. We shall use the fact that virtually free groups have a presentation with a quasi-isometric section as defined below. In 5] Bridson and Gilman introduced quasi-isometric sections as broomlike combings and proved that the groups with quasi-isometric sections are exactly the virtually free groups.

Throughout this section we assume that $G$ is finitely generated and $\pi: \Sigma^{*} \rightarrow$ $G$ refers to a a monoid presentation. This means $\Sigma$ is a finite alphabet and $\pi$ is a surjective homomorphism. By abuse of language, we simply write $g a$ for $g \pi(a)$. The set of words $\Sigma^{*}$ forms a tree. The empty word $\varepsilon$ is the root and a word $u$ has the children $u a$ for letters $a \in \Sigma$. The geodesic distance $d(u, v)$ in the tree $\Sigma^{*}$ yields a natural metric on $\Sigma^{*}$. That means, we have $d(u, v)=d$ if and only if $d=\left|u^{\prime}\right|+\left|v^{\prime}\right|$ where $u=p u^{\prime}$ and $v=p v^{\prime}$ and $p$ is the longest common prefix of $u$ and $v$. We are interested in sections of $\pi$ which define quasiisometric embeddings of the Cayleygraph of $G$ (w.r.t. $\pi$ ) into the tree $\Sigma$. We abbreviate this as a quasi-isometric section and use the following definition. A quasi-isometric section of $G$ is a mapping $\sigma: G \rightarrow \Sigma^{*}$ such that
(1) we have $\sigma(1)=\varepsilon$,
(2) we have $\pi(\sigma(g))=g$ for all $g \in G$,
(3) there is some $1 \leq k \in \mathbb{N}$ such that $d(\sigma(g), \sigma(g a)) \leq k$ for all $g \in G$ and $a \in \Sigma$.

Note that $\sigma(G)$ yields a set of normal forms with $\varepsilon \in \sigma(G)$. The important property is however that vertices $g, h$ of distance $d$ in the Cayley graph of $G$ (w.r.t. $\pi$ ) have representing words of distance at most $k d$ in the tree $\Sigma^{*}$.

The existence of a quasi-isometric section depends only on the group $G$ and not on its presentation $\pi: \Sigma^{*} \rightarrow G$ : Indeed, let $\sigma: G \rightarrow \Sigma^{*}$ be a quasi-isometric section of $G$ and $\pi^{\prime}: \Sigma^{* *} \rightarrow G$ be another monoid presentation. Then we find a homomorphism $\tau: \Sigma^{*} \rightarrow \Sigma^{* *}$ such that $\pi(w)=\pi^{\prime}(\tau(w))$ for all words $w \in \Sigma^{*}$. Now, the set of normal forms $\sigma(G)$ is mapped onto the set of normal forms $\tau(\sigma(G))$ satisfying (1) and (2). Moreover, consider $u=p u^{\prime}$ and $v=p v^{\prime}$ with $\left|u^{\prime}\right|+\left|v^{\prime}\right| \leq k$. Then there is some constant $\ell$ (depending only on $\tau$ ) such that
$\left|\tau\left(u^{\prime}\right)\right|+\left|\tau\left(v^{\prime}\right)\right| \leq k \ell$. This shows (3) for $\tau \circ \sigma: G \rightarrow \Sigma^{\prime *}$. Thus, we can say that $G$ has a quasi-isometric section.

It follows from Section 6.0.1 that f.g. virtually groups have quasi-isometric sections.

Now, let $G$ have a quasi-isometric section $\sigma: G \rightarrow \Sigma^{*}$ for some monoid presentation $\pi: \Sigma^{*} \rightarrow G$. We let $k \geq 1$ such that $d(\sigma(g), \sigma(g a)) \leq k$ for all $(g, a) \in G \times \Sigma$. We are going to define a context-free grammar for the language $L_{G}=\left\{w \in \Sigma^{*} \mid \pi(w)=1\right\}$. The grammar will be in Chomsky normal form. First we choose a symbol $S$ (which is outside of $G \cup \Sigma^{*}$ ) as axiom, then we let

$$
V=\{S\} \cup\{g \in G| | \sigma(g) \mid \leq k\}
$$

Thus, the set of variables consists of the axiom $S$ and a finite subset of $G$. We have the following set $P$ of rules:

1. $S \rightarrow \varepsilon$ is the so-called $\varepsilon$-rule in order to produce the empty word.
2. $S \rightarrow a$ for all $a \in \Sigma$ such that $\pi(a)=1$.
3. $S \rightarrow B C$ for all $B, C \in V \cap G$ such that $1=B C$ in $G$.
4. $A \rightarrow B C$ for all $A, B, C \in V \cap G$ such that $A=B C$ in $G$.
5. $A \rightarrow a$ for all $A \in V \cap G$ and all $a \in \Sigma$ such that $A=\pi(a)$ in $G$.

It is clear that whenever $S \xrightarrow[P]{*} w \in \Sigma^{*}$, then we have $\pi(w)=1$. Now we show the converse. For words $u, v \in \Sigma^{*}$ we denote by $\left[u^{-1} v\right.$ ] the group element $\pi(u)^{-1} \pi(v)$ in $G$. Thus, $\left[u^{-1} v\right]$ is a short hand for the expression $\pi(u)^{-1} \pi(v)$.

Now, let $w=a_{1} \cdots a_{n}$ with $a_{i} \in \Sigma$ and $\pi(w)=1$. We have to show that there is a derivation $S \xrightarrow[P]{*} w$. The first two types of the rules in $P$ show that this is true if $n \leq 1$. Hence we may assume $n \geq 2$. Let us define words $u_{i}=a_{1} \cdots a_{i} \in \Sigma^{*}$ for $0 \leq i \leq n$. Then we have $\pi\left(u_{0}\right)=\pi\left(u_{n}\right)=1$. Note that $\left[u_{i-1}^{-1} u_{i}\right]=\pi\left(a_{i}\right)$ for all $1 \leq i \leq n$. In particular, $\left[u_{i-1}^{-1} u_{i}\right] \in V \cap G$ for all $1 \leq i \leq n$ because $\left|\sigma\left(\pi\left(a_{i}\right)\right)\right| \leq k$ for $1 \leq i \leq n$ by the choice of $k$. We have rules $\left[u_{i-1}^{-1} u_{i}\right] \rightarrow a_{i}$ for all $1 \leq i \leq n$ and it remains to show that there is some derivation

$$
S \xlongequal[P]{*}\left[u_{0}^{-1} u_{1}\right] \cdots\left[u_{n-1}^{-1} u_{n}\right] .
$$

Now let $u_{0}, \ldots u_{n}$ be any sequence of words words $u_{i} \in \Sigma^{*}$ such that $n \geq 2$, $\pi\left(u_{0}\right)=\pi\left(u_{n}\right)=1$ and $\left[u_{i-1}^{-1} u_{i}\right] \in V$ for $1 \leq i \leq n$. We are going to show that this already implies that there is a derivation $S \xrightarrow[P]{*}\left[u_{0}^{-1} u_{1}\right] \cdots\left[u_{n-1}^{-1} u_{n}\right]$. For $n=2$ we have a rule

$$
S \rightarrow\left[u_{0}^{-1} u_{1}\right]\left[u_{1}^{-1} u_{2}\right] .
$$

Hence, we may assume $n \geq 3$ and we use induction. As $n \geq 2$ we may choose and fix some index $m$ with $0<m<n$ such that $\left|\sigma\left(\pi\left(u_{m}\right)\right)\right|$ is at least as large as any other $\left|\sigma\left(\pi\left(u_{i}\right)\right)\right|$ for $0 \leq i \leq n$. It follows

$$
\left|\sigma\left(\left[u_{m-1}^{-1} u_{m+1}\right]\right)\right| \leq \max \left\{\left|\sigma\left(\left[u_{m-1}^{-1} u_{m}\right]\right)\right|,\left|\sigma\left(\left[u_{m}^{-1} u_{m+1}\right]\right)\right|\right\} \leq k
$$

The set $P$ includes a rule

$$
\left[u_{m-1}^{-1} u_{m+1}\right] \rightarrow\left[u_{m-1}^{-1} u_{m}\right]\left[u_{m}^{-1} u_{m+1}\right] .
$$

Now we are done since by induction

$$
S \xlongequal[P]{*}\left[u_{0}^{-1} u_{1}\right] \cdots\left[u_{m-1}^{-1} u_{m+1}\right] \cdots\left[u_{n-1}^{-1} u_{n}\right] .
$$

### 6.0.5 Cayley graphs of context-free groups have finite treewidth

Muller and Schupp have shown that a Cayley graph of a context-free group has a $k$-triangulation [23]. The definition of a $k$-triangulation is technical. We skip it here because the proof in [23] can also be used to show directly that a Cayley graph of a context-free group has finite treewidth. This suffices for our purposes.

Proposition 6.1 Let $\Gamma$ be a Cayley graph of a context-free group $G$ with respect to a finite generating set $X$. Then $\Gamma$ has finite treewidth.

Proof. If $G$ is finite, then the assertion is trivial. Hence, let $G$ be infinite. We may assume that $1 \notin X \subseteq G$.

The vertex set of $\Gamma$ is the group $G$, by $B_{n}$ we denote the ball of radius $n$ around the origin $1 \in G$. Hence $B_{n}=\{g \in G \mid d(1, g) \leq n\}$. We are heading for a tree decomposition where certain finite subsets of $G$ become nodes in the tree. For $n \in \mathbb{N}$ we define sets $V_{n}$ of level $n$ such that $V_{0}=\{\Gamma-1\}$ and $V_{n}=\left\{C \mid C\right.$ is a connected component of $\left.\Gamma-B_{n}\right\}$ for $n \geq 1$. This defines a tree $T$ with root $B_{1}$ as follows:

$$
\begin{aligned}
& V(T)=\left\{\beta C \mid C \in V_{n}, n \in \mathbb{N}\right\} \\
& E(T)=\left\{\{\beta C, \beta D\} \mid D \subseteq C \in V_{n}, D \in V_{n+1}, n \in \mathbb{N}\right\}
\end{aligned}
$$

The nodes are subsets of $G$, hence we can identify nodes $t \in T$ with their bags $X_{t} \subseteq G$. If $\{g, h\}$ is an edge in the Cayley graph $\Gamma$, then there are essentially two cases; either $d(1, g)=n$ and $d(1, h)=n+1$ or $d(1, g)=d(1, h)=n+1$ for some $n$. In both cases the elements $g, h$ are in some bag $\beta C$ for some $C \in V_{n}$ and $n \in \mathbb{N}$.

It remains to show that $|\beta C|$ is bounded by some constant for all $C \in$ $V_{n}, n \in \mathbb{N}$. It is here where the context-freeness comes into the play. We denote $\Sigma=X \cup X^{-1}$. This is a set of monoid generators of $G$. We let $L_{G}=$ $\left\{w \in \Sigma^{*} \mid w=1 \in G\right\}$ its associated group language. By hypothesis, $L_{G}$ is generated by some context-free grammar $(V, \Sigma, P, S)$, and we may assume that it is in Chomsky normal form. This means all rules are either of the form $A \rightarrow B C$ with $A, B, C \in V$ or of the form $A \rightarrow a$ with $A \in V$ and $a \in \Sigma^{*}$ such that $|a| \leq 1$. We write $A \xlongequal[P]{\stackrel{*}{P}} \alpha$, if we can derive $\alpha \in(V \cup \Sigma)^{*}$ with production rules from $P$. We define a constant $k \in \mathbb{N}, k \geq 1$ such that

$$
k \geq \max _{A \in V} \min \left\{|w| \mid A \xlongequal[P]{*} w \in \Sigma^{*}\right\} .
$$

Consider $C \in V_{n}$ and $n \in \mathbb{N}$. Let $g, h \in \beta C$. We are going to show that $d(g, h) \leq 3 k$. For $n=0$ we have $\beta C=B_{1}$. Hence, we may assume $n \geq 1$.

Let $\alpha$ be a geodesic path from 1 to $g$ with label $u \in \Sigma^{*}, \gamma$ a geodesic path from $h$ to 1 with label $w \in \Sigma^{*}$, and $\beta$ some path from $g$ to $h$ with label $v \in \Sigma^{*}$ which is entirely contained in $C$. Such a path exists since $C$ is connected. The composition of these paths forms a closed path $\alpha \beta \gamma$ with label uvw. We have
$u v w \in L_{G}$ and there is a derivation $S \xlongequal{*} u v w$. We may assume that $|v| \geq 2$ because otherwise there is nothing to do.

Since the grammar is in Chomsky normal form we can find a rule $A \rightarrow B C$ and derivations as follows:

$$
S \xlongequal[P]{*} u^{\prime} A w^{\prime} \underset{P}{\Longrightarrow} u^{\prime} B C w^{\prime} \stackrel{P}{\neq} u^{\prime} v^{\prime} v^{\prime \prime} w^{\prime}=u v w
$$

such that $B \underset{P}{*} v^{\prime}, C \xrightarrow[P]{*} v^{\prime \prime}$, and $\left|u^{\prime}\right| \leq|u|<\left|u^{\prime} v^{\prime}\right|<|u v| \leq\left|u^{\prime} v^{\prime} v^{\prime \prime}\right|$.
This yields three nodes $x \in \alpha, y \in \beta$, and $z \in \gamma$ such that $d(x, y), d(y, z)$, $d(x, z) \leq k$. (These three nodes correspond exactly to a triangle with endpoints $x, y, z$ in the $k$-triangulation of the closed path $\alpha \beta \gamma$ in [23.)


Figure 7: The distance between $g$ and $h$ is bounded by $3 k$.
Now we have:

$$
d(x, g)=d(1, g)-d(1, x) \leq d(1, y)-d(1, x) \leq d(x, y)
$$

The first equality holds because $\alpha$ is geodesic and $x$ lies on $\alpha$; the second one because $d(1, g) \leq n+1 \leq d(1, y)$. Likewise we obtain $d(z, h) \leq d(z, y)$. Thus, it follows

$$
\begin{aligned}
d(g, h) & \leq d(g, x)+d(x, z)+d(z, h) \\
& \leq d(y, x)+d(x, z)+d(z, y) \leq 3 k
\end{aligned}
$$

This implies that the size of the bags is uniformly bounded by some constant since $\Gamma$ has uniformly bounded degree.

### 6.1 The result of Muller and Schupp revisited

To date various equivalent characterizations of context-free groups are known. The following theorem mentions only those characterizations which we met in this paper for proving the fundamental result of Muller and Schupp that contextfree groups are virtually free.

Theorem 6.2 Let $G$ be a finitely generated group and $\Gamma$ be its Cayley graph with respect to some finite set of generators. The following assertions are equivalent.

1. $G$ is virtually free.
2. $G$ is deterministic context-free.
3. $\Gamma$ has a quasi-isometric section.
4. $G$ is context-free.
5. $\Gamma$ has finite treewidth.
6. The group $G$ is the fundamental group of a finite graph of finite groups.

Proof. A review on the implications $1>2,1] 3,2 \Longrightarrow 4$, and $3 \Longrightarrow 4 \Longrightarrow$ 5 has been given in this section. The implication $5 \Longrightarrow 6$ is a direct consequence of Corollary 5.10. The last implication $6 \Longrightarrow 1$ follows from 18 .

## 7 Accessibility of groups

In this section we assume all groups to be finitely generated. As another application of the construction in Section 3 and Section 4 we give a proof of a theorem of Thomassen and Woess [30, Thm. 1.1]. It is an important corollary of [8, Thm. II 2.20] where Dicks and Dunwoody develop their the structure tree theory. This result allows us to consider all groups which act on a locally finite, connected, accessible graph with finite stabilizers and finitely many orbits, and not only those which act on graphs of finite treewidth. The result in 30 gave birth to the notion of accessibility for graphs.

We need some standard facts of Bass-Serre theory. The following lemma is well-known, see e.g. [7]. For convenience of the reader, we give a proof.

Lemma 7.1 Let $G$ be a f.g. fundamental group of a finite graph of groups with finite edge groups. Then every vertex group is finitely generated.

Proof. We give a sketch only. Let $V$ be the set of vertices, $Y$ be the set of edges of the finite graph, and $Z$ be the union over all edge groups. For each vertex $v \in V$ let $X_{v}$ be some generating set of the vertex group $G_{v}$. Then there is a finite generating set $X$ inside $\bigcup\left\{X_{v} \mid v \in V\right\} \cup Y \cup Z$ such that $Y \cup Z \subseteq X$. Now consider any $x \in X_{v}$, it is enough to show that $x$ can be expressed as a product over $X \cap X_{v}$. To see this, write $x$ as shortest word in $X$. Assume this word contained a factor $y z y^{-1}$ with $y \in Y$ and where $z$ belongs to edge group of $y$ sitting in $G_{t(y)}$, then we could perform a "Britton reduction" replacing $y z y^{-1}$ by some $z^{\prime}$ in the edge group of $\bar{y}$ sitting in $G_{s(y)}$. This would lead to a shorter word, since $Y \cup Z \subseteq X$. Hence, this is impossible; and the word representing $x$ is "Britton reduced". This implies that the word uses letters from $X \cap X_{v}$, only.

Definition 7.2 1. A group is called more than one ended (resp. at most one ended) if its Cayley graph has more than one end (resp. at most one end). (This definition does not depend on the choice of the finite generating set for the Cayley graph.)
2. A group $G$ is called accessible if it acts on a tree with finitely many orbits, finite edge stabilizers, and vertex stabilizers with at most one end.

If a group $G$ is accessible, then Bass-Serre theory yields an upper bound on the number how often $G$ can be split properly as an HNN-extension or amalgamated product over finite subgroups. This observation is also another definition of accessibility used frequently in literature. The link to accessibility of the corresponding Cayley graphs is due to the next proposition.

Proposition 7.3 Let $G$ be a f.g. group which acts on a tree with finitely many orbits, finite edge stabilizers and no vertex stabilizer having more than one end. Then the Cayley graph $\Gamma$ of $G$ is accessible.

Proof. Again, we give only a sketch. Bass-Serre theory tells us that $G$ is the fundamental group of a finite graph of groups with finite edge groups. By Lemma 7.1, every vertex group $G_{v}$ is finitely generated. We only consider the case where $G=A *_{H} B$ is an amalgamated product of two f.g. groups $A$ and $B$ over a common finite subgroup. The case of HNN-extensions follows analogously and is left to the reader.

We assume that $A$ and $B$ have accessible Cayley graphs and show that this implies that $G=A *_{H} B$ has an accessible Cayley graph. Then Proposition 7.3 follows by induction.

Let $A$ be generated by $X_{A}$ and $B$ be generated by $X_{B}$, where $X_{A}$ and $X_{B}$ are finite. As a generating set for $G$ we use $X=H \cup H X_{A} H \cup H X_{B} H$ and we may assume that $\Gamma$ is the Cayley graph of $G$ w.r.t. $X$. We may regard the Cayley graphs of $A$ and $B$ as subgraphs of $\Gamma$ and refer to them as $A$ or $B$. Now, consider any bi-infinite simple path $\alpha$ in $\Gamma$ such that there is a cut $C$ (of finite weight) with $|C \cap \alpha|=|\bar{C} \cap \alpha|=\infty$. We can assume that $|\delta C|$ is minimal among all such cuts. In order to show that $\Gamma$ is accessible we need a uniform bound on $|\delta C|$. The path $\alpha$ gives us a bi-infinite sequence of labels in $X$. We may assume the origin $1 \in G$ is a vertex of $\alpha$. If all the labels belong to $H \cup H X_{A} H$, then the path is entirely in $A$. So by hypothesis there is an upper bound on $|\delta C|$. Thus we may assume that there is at least one label in $A \backslash H$ and one label in $B \backslash H$ and that 1 is sitting between two such labels of minimal distance. Without restriction the label on the right of 1 belongs to $A \backslash H$ and on the left it belongs to $B$. Let $1=x_{0}, x_{1}, x_{2}, \ldots$ be the one-sided infinite sequence of vertices of $\alpha$ going to the right of 1 and $\ldots, y_{2}, y_{1}, y_{0}=1$ the corresponding one on the left. For every $x \in G$ the set $H x H$ is finite. Hence, switching to infinite subsequences of $x_{0}, x_{1}, x_{2}, \ldots$ and $\ldots, y_{2}, y_{1}, y_{0}$ we may assume that no $x_{i}^{-1} x_{j}$ or $y_{j} y_{i}^{-1}$ belongs to $H$ for $i<j$. Grouping consecutive labels from $A \backslash H$ (resp. $B \backslash H$ ) into blocks we obtain sequences

$$
h_{j}, \ldots, h_{2}, h_{1}, g_{1}, g_{2} \ldots g_{i}
$$

such that $g_{1} \in A \backslash H, h_{1} \in B \backslash H$, and the $g$ - and the $h$-vertices alternate between $A \backslash H$ and $B \backslash H$. It might happen that $i$ or $j$ remains bounded, but there are sequences with $1 \leq i, j$. The final step is to observe that every path connecting $g_{1} \cdots g_{i}$ to $\left(h_{j} \cdots h_{1}\right)^{-1}$ must use a vertex from $H$. This is due to the normal form theorem for amalgamated products.

Now, we can state the main result of this section. We use the notation of Section 4

Theorem 7.4 Let $\Gamma$ be a locally finite, connected, accessible graph. Let $G$ act on $\Gamma$ such that $G \backslash \Gamma$ is finite and each node stabilizer $G_{v}$ is finite. Then $G$ acts on the tree $T\left(\mathcal{C}_{\mathrm{opt}}\right)$ with finitely many orbits, finite edge stabilizers, and no vertex stabilizer has more than one end.

Proof. By Lemma 4.1 and Proposition 4.9, we know that $G$ acts with finitely many orbits and finite edge stabilizers on $T\left(\mathcal{C}_{\text {opt }}\right)$. Now consider a vertex stabilizer $G_{[C]}$ for some $C \in \mathcal{C}_{\text {opt }}$. By Lemma 7.1] we have that $G_{[C]}$ is finitely generated. Thus, its Cayley graph is locally finite and the number of ends is defined.

The block $B[C]$ has at most one end by Proposition 4.8. So it suffices to show that if the Cayley graph of $G_{[C]}$ has more than one end, then $B[C]$ has more than one end, too.

Because of Lemma4.7, there is a finite set of representatives $U \subseteq B[C]$ such that $G_{[C]} \cdot U=B[C]$. More precisely, we can identify $B[C]$ with $G_{[C]} \times U$. Let $Z=\left\{g \in G_{[C]} \mid \exists u, v \in U:(u, g v) \in E(\Gamma)\right\}$. Then we have $|Z|<\infty$ since $U$ is finite, $\Gamma$ is locally finite, and all vertex stabilizers are finite. Thus, we can define $m=\max \{d(1, a) \mid a \in Z\}<\infty$ (here, $d$ denotes the distance in the Cayley graph of $\left.G_{[C]}\right)$.

Assume that $G_{[C]}$ has more than one end. Then the Cayley graph of $G_{[C]}$ has a cut of finite weight $D \subseteq G_{[C]}$ with $|D|=|\bar{D}|=\infty$. We claim that there are only finitely many pairs $g, h$ such that $g \in D, h \in G_{[C]}-D$ and $g^{-1} h \in Z$.

Indeed, since $g^{-1} h \in Z$, there is is a path of length at most $m$ from $g$ to $h$ in the Cayley graph of $G_{[C]}$. Since $g \in D$ and $h \in G_{[C]}-D$, this path uses an edge of $\delta D$. Since $\delta D$ is finite and the Cayley graph is locally finite, there are only finitely many such paths of length at most $m$. Hence, there are only finitely many such $g$ and $h$.

Now consider $E=\{g u \mid g \in D, u \in U\} \subseteq B[C]$. Every edge of the boundary $\delta E$ inside $B[C]$ has endpoints $g u$ and $h v$ with $g \in D, h \in G_{[C]} \backslash D$ and $u, v \in U$. By the above claim there are only finitely many choices for $g$ and $h$. Together with the finiteness of $U$ this implies that $E$ has finite boundary inside $B[C]$.

Thus, $\beta E \subseteq B[C]$ is a finite set of vertices. Since $|D|=|\bar{D}|=\infty$ and $B[C]=G_{[C]} \times U$, we see that $|E|=\left|B_{C} \backslash E\right|=\infty$, too. Since $B[C]$ is connected, $B[C]-\beta E$ has more than one infinite connected component. This in turn implies that $B[C]$ has more than one end.

Corollary $7.5([8],[30])$ A finitely generated group is accessible if and only if its Cayley graph is accessible.

Proof. If the Cayley graph of $G$ is accessible, then Theorem 7.4 shows that the group $G$ is accessible. The converse is stated in Proposition 7.3 ,

## 8 Conclusion

The paper gives direct and simplified proofs for two fundamental results: 1. The Theorem of Muller and Schupp [23] (context-free groups are exactly the f.g. virtually free groups, Theorem 6.2) and 2. the accessibility result Corollary 7.5 by Dicks and Dunwoody resp. Thomassen and Woess. This became possible due
to the paper of Krön 19. The intuition behind our construction is that having a Cayley graph of finite tree width should unravel a simplicial tree on which the group acts with finitely many orbits and finite node stabilizers. This intuition is worked out into a mathematical fact here. In particular, if we start with a context-free group $G$, the construction yields that $G$ is a fundamental group of a finite graph of finite groups by standard Bass-Serre theory. By [18, fundamental groups of finite graphs of finite groups are f.g. virtually free. Together this yields the proof for the difficult direction in the characterization of context-free groups by Muller and Schupp.

A future research program is to investigate whether our constructions can be performed effectively. The problem is to find the minimal cuts, i.e., to decide whether a given cut is minimal with respect to some bi-infinite simple path. If this could be done in elementary time for graphs of finite tree width, it would lead to an elementary time algorithm for the isomorphism problem of contextfree groups by first constructing the graphs of groups and then using Krstic's algorithm $([20)$ to check whether the fundamental groups are isomorphic.

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[^0]:    ${ }^{1}$ The structure theorem was first proved for finitely presented torsion-free groups by Stallings [29] and for finitely generated torsion-free groups by Bergman [4].

