# SYNDETIC SUBMEASURES AND PARTITIONS OF $G$-SPACES AND GROUPS

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ABSTRACT. We prove that for every $k \in \mathbb{N}$ each countable infinite group $G$ admits a partition $G = A \cup B$ into two sets which are $k$-*meager* in the sense that for every $k$-element subset $K \subset G$ the sets $KA$ and $KB$ are not thick. The proof is based on the fact that $G$ possesses a *syndetic submeasure*, i.e., a left-invariant submeasure $\mu : \mathcal{P}(G) \to [0, 1]$ such that for each $\varepsilon > \frac{1}{|G|}$ and subset $A \subset G$ with $\mu(A) < 1$ there is a set $B \subset G \setminus A$ such that $\mu(B) < \varepsilon$ and $FB = G$ for some finite subset $F \subset G$.

In this paper we continue the studies [7]–[13] of combinatorial properties of partitions of $G$-spaces and groups.

By a $G$-*space* we understand a non-empty set $X$ endowed with a left action of a group $G$. The image of a point $x \in X$ under the action of an element $g \in G$ is denoted by $gx$. For two subsets $F \subset G$ and $A \subset X$ we put $FA = \{fa : f \in F,\ a \in A\} \subset X$.

## 1. Prethick sets in partitions of $G$-spaces

A subset $A$ of a $G$-space $X$ is called

- *large* if $FA = X$ for some finite subset $F \subset G$;
- *thick* if for each finite subset $F \subset G$ there is a point $x \in X$ with $Fx \subset A$;
- *prethick* if $KA$ is thick for some finite set $K \subset G$.

Now we insert number parameters in these definitions. Let $k, m \in \mathbb{N}$. A subset $A$ of a $G$-space $X$ is called

- $m$-*large* if $FA = X$ for some subset $F \subset G$ of cardinality $|F| \le m$;
- $m$-*thick* if for each finite subset $F \subset G$ of cardinality $|F| \le m$ there is a point $x \in X$ with $Fx \subset A$;
- $(k, m)$-*prethick* if $KA$ is $m$-thick for some set $K \subset G$ of cardinality $|K| \le k$;
- $k$-*prethick* if $KA$ is thick for some set $K \subset G$ of cardinality $|K| \le k$;
- $k$-*meager* if $A$ is not $k$-prethick (i.e., $KA$ is not thick for any subset $K \subset G$ of cardinality $|K| \le k$).

In the dynamical terminology [6, 4.38], large subsets are called syndetic and prethick subsets are called piecewise syndetic. We note also that these notions can be defined in much more general context of balleans [11], [13].

The following proposition is well-known [6, 4.41], [9, 1.3], [11, 11.2].

**Proposition 1.1.** *For any finite partition $X = A_1 \cup \cdots \cup A_n$ of a $G$-space $X$ one of the cells $A_i$ is prethick and hence $k$-prethick for some $k \in \mathbb{N}$.*

For finite groups the number $k$ in this proposition can be bounded from above by $n(\ln(\frac{|G|}{n})+1)$. We consider each group $G$ as a $G$-space endowed the natural left action of $G$.

**Proposition 1.2.** *Let $G$ be a finite group and $n, k \in \mathbb{N}$ be numbers such that $k \ge n \cdot \big(\ln(\frac{|G|}{n}) + 1\big)$. For any $n$-partition $G = A_1 \cup \cdots \cup A_n$ of $G$ one of the cells $A_i$ is $k$-large and hence $k$-prethick.*

*Proof.* One of the cells $A_i$ of the partition has cardinality $|A_i| \ge \frac{|G|}{n}$. Then by [15] or [2, 3.2], there is a subset $B \subset G$ of cardinality $|B| \le \frac{|G|}{|A_i|}(\ln|A_i| + 1) \le n(\ln(\frac{|G|}{n}) + 1) \le k$ such that $G = BA_i$. It follows that the set $A_i$ is $k$-large and hence $k$-prethick. □

For $G$-spaces we have the following quantitative version of Proposition 1.1.

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**Proposition 1.3.** *Let $m, n \in \mathbb{N}$. For any $n$-partition $X = A_1 \cup \dots \cup A_n$ of a $G$-space $X$ one of the cells $A_i$ is $(m^{n-1}, m)$-prethick in $X$.*

*Proof.* For $n = 1$ the proposition is trivial. Assume that it has been proved for some $n$ and take any partition $X = A_0 \cup \dots \cup A_n$ of $X$ into $(n+1)$ pieces. If the cell $A_0$ is $(1, m)$-prethick, then we are done. If not, then there is a set $F \subset G$ of cardinality $|F| \le m$ such that $Fx \not\subset A_0$ for all $x \in X$. This implies that $x \in F^{-1}(A_1 \cup \dots \cup A_n)$ and then by the inductive assumption, there is an index $1 \le i \le n$ such that the set $F^{-1}A_i$ is $(m^{n-1}, m)$-prethick. The latter means that there is a subset $E \subset G$ of cardinality $|E| \le m^{n-1}$ such that $EF^{-1}A_i$ is $m$-thick. Since $|EF^{-1}| \le |E| \cdot |F| \le m^{n-1}m = m^n$, the set $A_i$ is $(m^n, m)$-prethick. □

Looking at Proposition 1.3 it is natural to ask what happens for $m = \omega$. Is there any hope to find for every $n \in \mathbb{N}$ a finite number $k_n$ such that for each $n$-partition $X = A_1 \cup \dots \cup A_n$ some cell $A_i$ of the partition is $k_n$-prethick? In fact, $G$-spaces with this property do exist.

**Example 1.4.** *Let $X$ be an infinite set endowed with the natural action of the group $G = S_X$ of all bijections of $X$. Then each subset $A \subset X$ of cardinality $|A| = |X|$ is 2-large, which implies that for each finite partition $X = A_1 \cup \dots \cup A_n$ one of the cells $A_i$ has cardinality $|A_i| = |X|$ and hence is 2-large and 2-prethick.*

The action of the normal subgroup $FS_X \subset S_X$ consisting of all bijections $f : X \to X$ with finite support $\mathrm{supp}(f) = \{x \in X : f(x) \ne x\}$ has a similar property.

**Example 1.5.** *Let $X$ be an infinite set endowed with the natural action of the group $G = FS_X$ of all finitely supported bijections of $X$. Then each infinite subset $A \subset X$ is thick, which implies that for each finite partition $X = A_1 \cup \dots \cup A_n$ one of the cells $A_i$ is infinite and hence is thick and 1-prethick.*

However the $G$-spaces described in Examples 1.4 and 1.5 are rather pathological. In the next section we shall show that each $G$-space admitting a syndetic submeasure for every $k \in \mathbb{N}$ can be covered by two $k$-meager (and hence not $k$-prethick) subsets. In Section 6 using syndetic submeasures we shall prove that each countable infinite group admits a partition into two $k$-meager subsets for every $k \in \mathbb{N}$.

## 2. Syndetic submeasures on $G$-spaces

A function $\mu : \mathcal{P}(X) \to [0, 1]$ defined on the family of all subsets of a $G$-space $X$ is called

- *$G$-invariant* if $\mu(gA) = \mu(A)$ for each $g \in G$ and a subset $A \subset X$;
- *monotone* if $\mu(A) \le \mu(B)$ for any subsets $A \subset B \subset G$;
- *subadditive* if $\mu(A \cup B) \le \mu(A) + \mu(B)$ for any sets $A, B \subset X$;
- *additive* if $\mu(A \cup B) = \mu(A) + \mu(B)$ for any disjoint sets $A, B \subset X$;
- a *submeasure* if $\mu$ is monotone, subadditive, and $\mu(\emptyset) = 0$, $\mu(X) = 1$;
- a *measure* if $\mu$ is an additive submeasure;
- a *syndetic submeasure* if $\mu$ is a $G$-invariant submeasure such that for each subset $A \subset X$ with $\mu(A) < 1$ and each $\varepsilon > \frac{1}{|X|}$ there is a large subset $L \subset X \setminus A$ of submeasure $\mu(L) < \varepsilon$.

In this definition we assume that $\frac{1}{|X|} = 0$ if the $G$-space $X$ is infinite.

**Proposition 2.1.** *A finite $G$-space $X$ possesses a syndetic submeasure if and only if $X$ is transitive.*

*Proof.* If $X$ is transitive, then the counting measure $\mu : \mathcal{P}(X) \to [0, 1]$, $\mu : A \mapsto |A|/|X|$, is syndetic.

Now assume conversely that a finite $G$-space $X$ admits a syndetic submeasure $\mu : \mathcal{P}(X) \to [0, 1]$. If $X$ is a singleton, then $X$ is transitive. So, we assume that $X$ contains more than one point. Since the empty set $A = \emptyset$ has submeasure $\mu(A) = 0 < 1$, for the number $\varepsilon = \frac{1}{|X|-1} > \frac{1}{|X|}$ there is a large subset $L \subset X \setminus A = X$ of submeasure $\mu(L) < \varepsilon$. It follows that $L$, being large in $X$, has non-empty intersection with each orbit $Gx$, $x \in X$. Replacing $L$ by a smaller subset we can assume that $L$ meets each orbit in exactly one point. For every point $x \in L$ we can find a finite subset $F_x \subset G$ of cardinality $|Gx| - 1$ such that $F_x x = Gx \setminus \{x\}$. Then the set $F = \{1_G\} \cup \bigcup_{x \in L} F_x$ has cardinality $|F| = 1 + \sum_{x \in L}(|G_x| - 1) = 1 - |L| + \sum_{x \in L} |G_x| = 1 - |L| + |X|$ and $FL = X$. By the subadditivity and the $G$-invariance of the submeasure $\mu$, we get

$$1 = \mu(FL) \le |F| \cdot \mu(L) < |F| \cdot \varepsilon = \frac{|F|}{|X| - 1} = \frac{1 - |L| + X}{|X| - 1},$$

which implies $|L| = 1$. This means that $X$ has exactly one orbit and hence is transitive. □

For $G$-spaces admitting a syndetic submeasure we have the following result completing Propositions 1.1–1.3.

**Theorem 2.2.** *Let $G$ be a countable group and $X$ be an infinite $G$-space possessing a syndetic submeasure $\mu : \mathcal{P}(X) \to [0,1]$. Then for every $k \in \mathbb{N}$ there is a partition $X = A \cup B$ of $X$ into two $k$-meager subsets.*

*Proof.* Fix any $k \in \mathbb{N}$ and choose an enumeration $(K_n)_{n=1}^\infty$ of all $k$-element subsets of $G$.

Using the definition of a syndetic submeasure, we can inductively construct two sequences $(A_n)_{n=1}^\infty$ and $(B_n)_{n=1}^\infty$ of large subsets of $X$ satisfying the following conditions for every $n \in \mathbb{N}$:

(1) $A_n \subset X \setminus \bigcup_{i<n} K_n^{-1} K_i B_i$;
(2) $\mu(A_n) < \frac{1}{k^2 2^n}$;
(3) $B_n \subset X \setminus \bigcup_{i \le n} K_n^{-1} K_i A_i$;
(4) $\mu(B_n) < \frac{1}{k^2 2^n}$.

At each step the choice of the set $A_n$ is possible as

$$\mu\Big(\bigcup_{i<n} K_n^{-1} K_i B_i\Big) \le \sum_{i<n} \sum_{x \in K_n^{-1} K_i} \mu(xB_i) = \sum_{i<n} |K_n^{-1} K_i| \cdot \mu(B_i) \le \sum_{i<n} k^2 \frac{1}{k^2 2^i} < 1$$

by the subadditivity of $\mu$. By the same reason, the set $B_n$ can be chosen.

After completing the inductive construction, we get the disjoint sets $A = \bigcup_{n=1}^\infty K_n A_n$ and $B = \bigcup_{n=1}^\infty K_n B_n$.

It remains to check that the sets $A$ and $X \setminus A$ are $k$-meager. Given any $k$-element subset $K \subset G$ we need to prove that the sets $KA$ and $K(X \setminus A)$ are not thick. Find $n \in \mathbb{N}$ such that $K_n = K^{-1}$.

Since the set $K_n B_n$ is disjoint with $A$, the large set $B_n$ is disjoint with $K_n^{-1} A = KA$, which implies that $X \setminus KA$ is large and $KA$ is not thick.

Next, we show that the set $K(X \setminus A) = K_n^{-1}(X \setminus A)$ is not thick. We claim that $A_n \subset X \setminus K_n^{-1}(X \setminus A)$. Assuming the converse, we can find a point $a \in A_n \cap K_n^{-1}(X \setminus A)$. Then $K_n a$ intersects $X \setminus A$, which is not possible as $K_n a \subset K_n A_n \subset A$. So, the set $X \setminus K(X \setminus A) \supset A_n$ is large, which implies that $K(X \setminus A)$ is not thick. □

## 3. Toposyndetic submeasures on $G$-spaces

In light of Theorem 2.2 it is important to detect $G$-spaces possessing a syndetic submeasure. We shall find such spaces among $G$-spaces possessing a toposyndetic submeasure. To define such submeasures, we need to recall some information from Measure Theory.

Let $\mu : \mathcal{P}(X) \to [0,1]$ be a submeasure on a set $X$. A subset $A \subset X$ is called *$\mu$-measurable* if $\mu(B) = \mu(B \cap A) + \mu(B \setminus A)$ for each subset $B \subset X$. By (the proof of) [4, 2.1.3], the family $\mathcal{A}_\mu$ of all $\mu$-measurable subsets of $X$ is an algebra (called the *measure algebra of* $\mu$) and the restriction $\mu|\mathcal{A}_\mu$ is additive in the sense that $\mu(A \cup B) = \mu(A) + \mu(B)$ for any disjoint $\mu$-measurable sets $A, B \in \mathcal{A}_\mu$.

A $G$-invariant submeasure $\mu : \mathcal{P}(X) \to [0,1]$ on a $G$-space $X$ will be called *toposyndetic* if $\mathcal{A}_\mu \cap \tau$ is a base of some $G$-bounded $G$-invariant regular topology $\tau$ on $X$. The *$G$-boundedness* of the topology $\tau$ means that each non-empty open set $U \in \tau$ is large in $X$. The $G$-boundedness of $\tau$ implies the density of all orbits $Gx$, $x \in X$, in the topology $\tau$.

**Theorem 3.1.** *If a $G$-space $X$ admits a toposyndetic submeasure, then each non-empty $G$-invariant subspace $Y \subset X$ possesses a syndetic submeasure.*

*Proof.* Let $\mu : \mathcal{P}(X) \to [0,1]$ be a toposyndetic submeasure on $X$ and $\tau$ be a $G$-bounded $G$-invariant Tychonoff topology on $X$ such that $\mathcal{A}_\mu \cap \tau$ is a base of the topology $\tau$.

Fix any non-empty $G$-invariant subspace $Y \subset X$. The $G$-boundedness of the topology $\tau$ implies that $Y$ is dense in the topological space $(X, \tau)$. If the regular topological space $(X, \tau)$ has an isolated point $x$, then by the $G$-boundedness of the topology $\tau$ for the open set $U = \{x\}$ there is a finite set $F \subset G$ with $X = FU \subset Gx$, which means that $X$ is a finite transitive space. By the density of $Y$ in $X$, $Y = X$ and by Proposition 2.1, $Y$ possesses a syndetic submeasure.

So, we assume that the topological space $(X, \tau)$ has no isolated points. The $G$-invariant submeasure $\mu$ induces a $G$-invariant submeasure $\lambda : \mathcal{P}(Y) \to [0,1]$ defined by $\lambda(A) = \mu(\bar{A})$ for every subset $A \subset Y$, where $\bar{A}$ is the closure of $A$ in the topological space $(X, \tau)$. To see that the submeasure $\lambda$ is syndetic, fix any $\varepsilon < \frac{1}{|Y|} = 0$ and any subset $A \subset Y$ with $\lambda(A) < 1$. Then $\mu(\bar{A}) = \lambda(A) < 1$, which implies that $X \setminus \bar{A}$ is an open non-empty subset of $X$. Since $\mathcal{A}_\mu \cap \tau$ is a base of the topology $\tau$, there is a non-empty $\mu$-measurable open set $U \subset X \setminus \bar{A} \subset X \setminus A$. Since the topological space $(X, \tau)$ has no isolated points, we can fix pairwise disjoint non-empty open sets $U_1, \dots, U_n \subset U$ for some integer number $n > 1/\varepsilon$. Since $\mathcal{A}_\mu \cap \tau$ is a base of the topology $\tau$, we can additionally assume that these open sets $U_1, \dots, U_n$ are $\mu$-measurable, which implies that $\sum_{i=1}^n \mu(U_i) \le 1$ and hence $\mu(U_i) \le \frac{1}{n} < \varepsilon$ for some $i \le n$. By the regularity of the topological space $(X, \tau)$, the open set $U_i$ contains the closure $\bar{V}$ of some non-empty open set $V \subset X$. The $G$-boundedness of $X$ guarantees that $V$ is large in $X$ and hence $V \cap Y$ is large in $Y$. Also $\lambda(V \cap Y) = \mu(\overline{V \cap Y}) \le \mu(\bar{V}) \subset \mu(U_i) < \varepsilon$. This means that the submeasure $\lambda$ on $Y$ is syndetic. □

Many examples of $G$-spaces having a toposyndetic submeasure occur among subspaces of minimal compact measure $G$-spaces. By a *compact* (*measure*) *$G$-space* we understand a $G$-space $X$ endowed with a compact Hausdorff $G$-invariant topology $\tau_X$ (and a $G$-invariant probability Borel $\sigma$-additive measure $\lambda_X : \mathcal{B}(X) \to [0,1]$ defined on the $\sigma$-algebra $\mathcal{B}(X)$ of Borel subsets of $X$). A compact $G$-space $X$ is called *minimal* if each orbit $Gx$, $x \in X$, is dense in $X$.

**Theorem 3.2.** *If $(X, \tau_X, \lambda_X)$ is a minimal compact measure $G$-space, then each non-empty $G$-invariant subspace $Y$ of $X$ possesses a (topo)syndetic submeasure.*

*Proof.* By the minimality of $X$, the $G$-invariant subspace $Y$ is dense in $X$. Let $\tau = \{U \cap Y : U \in \tau_X\}$ be the induced topology on $Y$. The $G$-invariant measure $\lambda_X : \mathcal{B}(X) \to [0,1]$ induces a $G$-invariant submeasure $\mu : \mathcal{P}(Y) \to [0,1]$ defined by the formula $\mu(A) = \lambda_X(\bar{A})$ for $A \subset Y$, where $\bar{A}$ denotes the closure of $A$ in the compact space $(X, \tau_X)$. To prove that the submeasure $\mu$ is toposyndetic, it remains to prove that the topology $\tau$ is $G$-bounded and $\mathcal{A}_\tau \cap \tau$ is a base of the topology $\tau$.

Consider the algebra $\mathcal{A}_X = \{A \subset X : \lambda_X(\partial A) = 0\}$ consisting of subsets $A \subset X$ whose boundary $\partial A$ in $X$ have measure $\lambda_X(\partial A) = 0$, and let $\mathcal{A}_Y = \{A \cap Y : A \in \mathcal{A}_X\}$. It can be shown that each set $A \subset \mathcal{A}_Y$ is $\mu$-measurable and $\mathcal{A}_Y \cap \tau \subset \mathcal{A}_\mu \cap \tau$ is a base of the topology $\tau$. The $G$-boundedness of the topology $\tau$ on $Y$ is proved in the following lemma. Therefore, $\mu$ is a toposyndetic submeasure on $X$. By the proof of Theorem 3.1, the submeasure $\mu$ is syndetic. □

**Lemma 3.3.** *For each minimal compact $G$-space $X$, the induced topology on each $G$-invariant subspace $Y \subset X$ is $G$-bounded.*

*Proof.* To show that the induced topology on $Y$ is $G$-bounded, fix any non-empty open subset $U \subset Y$. Find an open set $\widetilde{U} \subset X$ such that $\widetilde{U} \cap Y = U$. By the regularity of the compact Hausdorff space $X$, there is a non-empty open subset $V \subset X$ with $\bar{V} \subset \tilde{U}$.

By a classical Birkhoff theorem in Topological Dynamics (see e.g. Theorem 19.26 [6]), the minimal compact $G$-space $X$ contains a uniformly recurrent point $y \in X$. The uniform recurrence of $y$ means that for each open neighborhood $O_y \subset X$ of $y$ the set $\{g \in G : gy \in O_y\}$ is large in $G$. By the density of the orbit $Gy$ there is $s \in G$ with $sy \in V$. Then $s^{-1}V$ is a neighborhood of $y$ and by the uniform recurrence of $y$, the set $L = \{g \in G : gy \in s^{-1}V\}$ is large in $G$. Consequently, we can find a finite subset $F \subset G$ such that $G = FL$. Then $Gy = FLy \subset Fs^{-1}V$, which implies that the open set $Fs^{-1}V$ is dense in $X$. Consequently, $X = Fs^{-1}\bar{V} \subset Fs^{-1}\tilde{U}$ and $Y = Fs^{-1}(Y \cap \tilde{U}) = Fs^{-1}U$, witnessing that the topology of $Y$ is $G$-bounded. □

## 4. Groups possessing a toposyndetic submeasure

In this section we shall detect groups possessing a toposyndetic submeasure. Each group $G$ will be considered as a $G$-space endowed with the natural left action of the group $G$. A group $G$ is called *amenable* if it admits a $G$-invariant additive measure $\mu : \mathcal{P}(G) \to [0,1]$.

We shall say that a $G$-space $X$ has a *free orbit* if for some $x \in X$ the map $\alpha_x : G \to X$, $\alpha_x : g \mapsto gx$, is injective.

**Theorem 4.1.** *A group $G$ admits a toposyndetic submeasure if one of the following conditions holds:*

1. *there is a minimal compact measure $G$-space $X$ with a free orbit;*
2. *$G$ is a subgroup of a compact topological group;*
3. *$G$ is countable;*
4. *$G$ is amenable.*

*Proof.* 1. Assume that $(X, \tau_X, \lambda_X)$ is a minimal compact measure $G$-space with a free orbit. In this case there is a point $x \in X$ for which the map $\alpha_x : G \to Gx \subset X$, $\alpha_x : g \mapsto gx$, is injective. This map allows us to define a Tychonoff $G$-invariant topology

$$\tau = \{\alpha_x^{-1}(U) : U \in \tau_X\}$$

on the group $G$. By Lemma 3.3, the topology $\tau$ is $G$-bounded.

Since the orbit $Gx$ is dense in $X$ (which follows from the minimality of $X$), the formula

$$\mu(A) = \lambda(\overline{Ax}) \quad \text{for} \quad A \subset G$$

determines a $G$-invariant submeasure on $G$. Observe that $\mathcal{B} = \{U \in \tau_X : \lambda(\bar{U}) = \lambda(U)\}$ is a base of the topology $\tau_X$ on $X$ and $\mathcal{A} = \{\alpha_x^{-1}(U) : U \in \mathcal{B}\}$ is a base of the topology $\tau$ on $G$. It can be verified that each set $A \in \mathcal{A}$ is $\mu$-measurable, which implies that $\mathcal{A}_\mu \cap \tau \supset \mathcal{A}$ is a base of the topology $\tau$. This means that the submeasure $\mu$ is toposyndetic.

2. The second statement follows immediately from the first statement and the well-known fact that each compact topological group carries an invariant probability Borel measure (namely, the Haar measure).

3. The third statement follows from the first one an a recent deep result of B.Weiss [16] stating that for each countable group $G$ there is a compact minimal measure $G$-space with a free orbit.

4. The fourth statement follows from the first statement and the well-known fact [5, §449] stating for any amenable group $G$, each compact $G$-space $X$ possesses a $G$-invariant probability Borel measure. □

**Problem 4.2.** *Is the class of groups admitting a toposyndetic submeasure hereditary with respect to taking subgroups?*

**Problem 4.3.** *Has every group a toposyndetic submeasure?*

**Problem 4.4.** *Has the group $S_X$ of all bijections of an infinite set $X$ a toposyndetic submeasure?*

## 5. Groups possessing a syndetic submeasure

In this section we shall detect groups possessing a syndetic submeasure. By Theorem 3.1 the class of such groups contains all groups possessing a toposyndetic submeasure, in particular, all countable groups.

**Theorem 5.1.** *A group $G$ possesses a syndetic submeasure if one of the following conditions is satisfied:*

(1) *there is an infinite transitive $G$-space possessing a syndetic submeasure;*
(2) *there is an infinite minimal compact measure $G$-space;*
(3) *$G$ admits a homomorphism onto an infinite group possessing a (topo)syndetic submeasure;*
(4) *$G$ admits a homomorphism onto a countable infinite group;*
(5) *$G$ contains an amenable infinite normal subgroup.*

*Proof.* 1. Assume that $X$ is an infinite transitive $G$-space possessing a syndetic submeasure $\lambda : \mathcal{P}(X) \to [0,1]$. Fix any point $x \in X$ and consider the map $\alpha_x : G \to X$, $\alpha_x : g \mapsto gx$, which is surjective (by the transitivity of the $G$-space $X$). One can check that the syndetic submeasure $\lambda$ on $X$ induces a syndetic submeasure $\mu : \mathcal{P}(G) \to [0,1]$ defined by $\mu(A) = \lambda(\alpha_x(A)) = \lambda_X(Ax)$ for $A \subset G$.

2. Let $(X, \tau_X, \mu_X)$ be an infinite minimal compact measure $G$-space. By the minimality, the orbit $Gx$ of any point $x \in X$ is dense in $(X, \tau_X)$. Then the formula $\mu(A) = \mu_X(\overline{Ax})$, $A \subset X$, determines a $G$-invariant submeasure $\mu : \mathcal{P}(G) \to [0,1]$ on the group $G$. We claim that the submeasure $\mu$ is syndetic. Given any $\varepsilon > \frac{1}{|G|}$ and a set $A \subset G$ with $\mu(A) < 1$, we should find a large set $L \subset G \setminus A$ with $\mu(L) < \varepsilon$. Since $\mu_X(\overline{Ax}) = \mu(A) < 1$, the closed subset $\overline{Ax}$ is not equal to $X$. By the minimality, the infinite compact $G$-space $(X, \tau_X)$ has no isolated points, which allows us to find an open non-empty set $U \subset X \setminus \overline{Ax}$ such that $\mu_X(\overline{U}) < \varepsilon$. By Lemma 3.3, the topology $\tau_X$ is $G$-bounded, which implies that the set $U \subset X$ is large in $X$ and hence $V = \alpha_x^{-1}(U) \subset X \setminus A$ is large in $G$ and has submeasure $\mu(V) \le \mu_X(\bar{U}) < \varepsilon$.

3. The third statement follows from the first statement and Theorem 3.1.

4. The fourth statement follows from the third statement and Theorem 4.1(3).

5. Suppose that the group $G$ contains a normal infinite amenable subgroup $H$. Denote by $P_\omega(H)$ the set of finitely supported probability measures on $H$. Each measure $\mu \in P_\omega(H)$ can be written as a convex

combination $\mu = \sum_{i=1}^n \alpha_i \delta_{x_i}$ of Dirac measures concentrated at points $x_i$ of $H$. This allows us to identify $P_\omega(H)$ with a convex subset of the Banach space $\ell_1(H)$ endowed with the norm $\|f\| = \sum_{x\in H}|f(x)|$.

We claim that the function

$$\sigma_H : \mathcal{P}(G) \to [0,1], \ \ \sigma_H : A \mapsto \inf_{\mu\in P_\omega(H)} \sup_{y\in G} \mu(Ay),$$

is a syndetic left-invariant submeasure on $G$.

First we prove that $\sigma_H$ is left-invariant. Given any $x \in G$ and $A \subset G$ it suffices to check that $\sigma_H(xA) \le \sigma_H(A)+\varepsilon$ for every $\varepsilon > 0$. The definition of $\sigma_H$ guarantees that $\sigma_H$ is right-invariant. Consequently, $\sigma_H(xA) = \sigma_H(xAx^{-1})$. By the definition of $\sigma_H(A)$, there is a finitely supported probability measure $\mu \in P_\omega(H)$ such that $\sup_{y\in G}\mu(Ay) < \sigma_H(A) + \varepsilon$. Write $\mu$ as a convex combination $\mu = \sum_{i=1}^n \alpha_i\delta_{a_i}$ of Dirac measures concentrated at points $a_1,\dots,a_n \in H$. Since $H$ is a normal subgroup of $G$, the probability measure $\mu' = \sum_{i=1}^n \alpha_i \delta_{xa_ix^{-1}}$ belongs to $P_\omega(H)$. Taking into account that for every $y \in G$

$$\mu'(xAx^{-1}y) = \mu'(xAx^{-1}yxx^{-1}) = \mu(Ax^{-1}yx),$$

we conclude that

$$\sigma_H(xAx^{-1}) \le \sup_{y\in G} \mu'(xAx^{-1}y) \le \sup_{y\in G}\mu(Ax^{-1}yx) < \sigma_H(A) + \varepsilon.$$

So, $\sigma_H$ is left-invariant.

Next, we prove that $\sigma_H$ is subadditive. Given two subsets $A, B \subset G$, it suffices to check that $\sigma_H(A\cup B) \le \sigma_H(A) + \sigma_H(B) + 3\varepsilon$ for every $\varepsilon > 0$. By the definition of the numbers $\sigma_H(A)$ and $\sigma_H(B)$, there are finitely supported probability measures $\mu_A, \mu_B \in P_\omega(H)$ such that $\sup_{y\in G}\mu_A(Ay) < \sigma_H(A)+\varepsilon$ and $\sup_{y\in G}\mu_B(By) < \sigma_H(By) + \varepsilon$. By Emerson's characterization of amenability [3, 1.7], for the probability measures $\mu_A$ and $\mu_B$ there are probability measures $\mu'_A, \mu'_B \in P_\omega(H)$ such that

$$\sup_{C\subset H} |\mu_A * \mu'_A(C) - \mu_B * \mu'_B(C)| \le \|\mu_A * \mu'_A - \mu_B * \mu'_B\| < \varepsilon.$$

Write the measures $\mu_A$, $\mu_B$, $\mu'_A$ and $\mu'_B$ as convex combinations of Dirac measures:

$$\mu_A = \sum_i \alpha_i\delta_{x_i},\ \mu'_A = \sum_j \alpha'_j\delta_{x'_j},\ \mu_B = \sum_i \beta_i\delta_{y_i},\ \ \mu'_B = \sum_j \beta'_j\delta_{y'_i}.$$

Then $\mu_A * \mu'_A = \sum_{i,j}\alpha_i\alpha'_j\delta_{x_ix'_j}$ and $\mu_B * \mu'_B = \sum_{i,j}\beta_i\beta'_j\delta_{y_iy'_j}$. For every $y \in G$ we get

$$\mu_A * \mu'_A(Ay) = \sum_{i,j}\alpha_i\alpha'_j\delta_{x_ix'_j}(Ay) = \sum_j \alpha'_j \sum_i \alpha_i\delta_{x_i}(Ay(x'_j)^{-1}) =$$
$$= \sum_j \alpha'_j\,\mu_A(Ay(x_j)'^{-1}) \le \sum_j \alpha'_j \sup_{z\in G}\mu_A(Az) = \sup_{z\in G}\mu_A(Az) < \sigma_H(A) + \varepsilon.$$

By analogy we can prove that $\mu_B * \mu'_B(By) \le \sigma_H(B) + \varepsilon$. Now consider the measure $\nu = \mu_A * \mu'_A$ and observe that for every $y \in B$ we get

$$\nu(By) = \mu_A * \mu'_A(By) \le \mu_B * \mu'_B(By) + \|\mu_A * \mu'_A - \mu_B * \mu'_B\| < \sigma_H(B) + \varepsilon + \varepsilon.$$

Then

$$\sigma_H(A\cup B) \le \sup_{y\in G}\nu((A\cup B)y) \le \sup_{y\in G}\nu(Ay) + \sup_{y\in G}\nu(By) < \sigma_H(A) + \varepsilon + \sigma_H(B) + 2\varepsilon = \sigma_H(A) + \sigma_H(B) + 3\varepsilon,$$

which proves the subadditivity of $\sigma_H$.

Finally we prove that the left-invariant submeasure $\sigma_H$ on $G$ is syndetic. Fix a subset $A \subset G$ of submeasure $\sigma_H(A) < 1$ and take an arbitrary $\varepsilon > 0$. Since $\sigma_H(A) < 1$, there is a finitely supported measure $\mu \in P_\omega(H)$ such that $\sup_{y\in G}\mu(Ay) < 1$. Write $\mu$ as the convex combination $\mu = \sum_{i=1}^n \alpha_i\delta_{x_i}$ of Dirac measures. We can assume that each coefficient $\alpha_i$ is positive. Then the finite set $F = \{x_1,\dots,x_n\}$ coincides with the support $\mathrm{supp}(\mu)$ of the measure $\mu$.

It follows that for every $y \in G$ we get $\mu(Ay) < 1$ and hence $F = \mathrm{supp}(\mu) \not\subset Ay$. This ensures that the set $Fy^{-1}$ meets the complement $X \setminus A$ and hence $y^{-1} \in F^{-1}(G\setminus A)$. So, $G = F^{-1}(G\setminus A)$ and the set $X\setminus A$ is large in $G$. Now take any finite subset $E \subset H$ of cardinality $|E| > 1/\varepsilon$. Using Zorn's Lemma, choose a maximal subset $B \subset G\setminus A$ which is $E$-separated in the sense that $Ex \cap Ey = \emptyset$ for any distinct points $x, y \in B$.

The maximality of the set $B$ guarantees that for each $x \in G \setminus A$ the set $Ex$ meets $EB$, which implies that $G \setminus A \subset E^{-1}EB$ and $G = F^{-1}(G \setminus A) = F^{-1}E^{-1}EB$. This means that the set $B$ is large in $G$. We claim that $|E^{-1} \cap By| \le 1$ for each $y \in G$. Assume conversely that $E^{-1} \cap By$ contains two distinct points $by$ and $b'y$ with $b, b' \in B$. Then $b'b^{-1} = b'y(by)^{-1} \in E^{-1}E$ and hence $Eb' \cap Eb \neq \emptyset$, which is not possible as $B$ is $E$-separated. Now consider the uniformly distributed probability measure $\nu = \frac{1}{|E|}\sum_{x \in E^{-1}} \delta_x \in P_\omega(H)$ and observe that $\sigma_H(B) \le \sup_{y \in G} \nu(By) \le \frac{|E^{-1} \cap By|}{|E|} \le \frac{1}{|E|} < \varepsilon$, which means that the submeasure $\sigma_H$ is syndetic. □

**Remark 5.2.** For an infinite amenable group $G$ and the subgroup $H = G$ the syndetic submeasure $\sigma_H$ (used in the proof of Theorem 5.1(5)) coincides with the right Solecki submeasure $\sigma^R$ introduced in [14] and studied in [1].

Theorem 5.1(5) implies:

**Corollary 5.3.** *The group $S_X$ of bijections of any set $X$ possesses a syndetic submeasure.*

*Proof.* If $X$ is finite, then the finite group $S_X$ has a syndetic submeasure according to proposition 2.1. So, we assume that the set $X$ is infinite. Observe that the subgroup $FS_X$ of finitely supported permutations of $X$ is locally finite and hence amenable. By Theorem 5.1(5) the group $S_X$ admits a syndetic submeasure as it contains the infinite amenable normal subgroup $FS_X$. □

**Problem 5.4.** *Has every group a syndetic submeasure?*

**Problem 5.5.** *Has the quotient group $S_\omega/FS_\omega$ a syndetic submeasure?*

## 6. Partitions of groups into $k$-meager pieces

Now we return to the problem of partitioning groups into $k$-meager pieces, which was posed and partly resolved in [12]. Combining Theorems 2.2 and 5.1(5), we get:

**Theorem 6.1.** *Each countable infinite group $G$ for every $k \in \mathbb{N}$ admits a partition into two $k$-meager subsets.*

This theorem admits a self-generalization.

**Corollary 6.2.** *If a group $G$ has a countable infinite quotient group, then for every $k \in \mathbb{N}$ the group $G$ admits a partition into two $k$-meager subsets.*

*Proof.* Let $h : G \to H$ be a homomorphism of $G$ onto a countable infinite group $H$. By Theorem6.1, for every $k \in \mathbb{N}$ the countable group $H$ admits a partition $H = A \cup B$ into two $k$-meager subsets. Then $G = h^{-1}(A) \cup h^{-1}(B)$ is a partition of the group $G$ into two $k$-meager subsets. □

Corollary 6.2 gives a partial answer to the following (still open) problem posed in and partially answered in [12].

**Problem 6.3.** *Is it true that each infinite group $G$ for every $k \in \mathbb{N}$ admits a partition into two $k$-meager sets?*

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## References


[1] T. Banakh, *The Solecki submeasures and densities on groups*, preprint (http://arxiv.org/abs/1211.0717).

[2] B. Bollobás, S. Janson, O. Riordan, *On covering by translates of a set*, Random Structures Algorithms **38**:1-2 (2011), 33–67.

[3] W. Emerson, *Characterizations of amenable groups*, Trans. Amer. Math. Soc. **241** (1978), 183–194.

[4] H. Federer, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.

[5] D. Fremlin, *Measure Theory*, V.4, Torres Fremlin, Colchester, 2006.

[6] N. Hindman, D. Strauss, *Algebra in the Stone-Čech compactification: Theory and Applications*, Walter de Grueter, Berlin, New York, 2011.

[7] I. Protasov, *Partition of groups into large subsets*, Math. Notes, **73** (2003), 271–281.

[8] I. Protasov, *Small systems of generators of groups*, Math. Notes, **76** (2004), 420–426.

[9] I. Protasov, *Selective survey on Subset Combinatorics of Groups*, Ukr. Math. Bull.,**7**(2010), 220-257.

[10] I. Protasov, *Partition of groups into thin subsets*, Algebra Disc. Math., **11**(2011), 88–92.

[11] I. Protasov, T. Banakh, *Ball Structures and Colorings of Groups and Graphs*, Math. Stud. Monogr. Ser., Vol. 11, VNTL Publishers, Lviv, 2003.

[12] I. Protasov, S. Slobodyanuk, *Prethick subsets in partitions of groups*, Algebra Discr. Math. **14**:2 (2012), 267–275.

[13] I. Protasov, M. Zarichnyi, *General Asymptology*, Math. Stud. Monogr. Ser., Vol. 12, VNTL Publishers, Lviv, 2007

[14] S. Solecki, *Size of subsets of groups and Haar null sets*, Geom. Funct. Anal. **15** (2005), 246–273.

[15] G. Weinstein, *Minimal complementary sets*, Trans. Amer. Math. Soc., **212** (1975), 131–137.

[16] B. Weiss, *Minimal models for free actions*, in: Dynamical Systems and Group Actions (L. Bowen, R. Grigorchuk, Ya. Vorobets eds.), Contemp. Math. **567**, Amer. Math. Soc. Providence, RI, (2012), 249–264.



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