# The stabilizers in a Drinfeld modular group of the vertices of its Bruhat-Tits tree: an elementary approach 

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#### Abstract

Let $K$ be an algebraic function field of one variable with constant field $k$ and let $\mathcal{C}$ be the Dedekind domain consisting of all those elements of $K$ which are integral outside a fixed place $\infty$ of $K$. When $k$ is finite the group $G L_{2}(\mathcal{C})$ plays a central role in the theory of Drinfeld modular curves analagous to that played by $S L_{2}(\mathbb{Z})$ in the classical theory of modular forms. When $k$ is finite (resp. infinite) we refer to a group $G L_{2}(\mathcal{C})$ as an arithmetic (resp. nonarithmetic) Drinfeld modular group. Associated with $G L_{2}(\mathcal{C})$ is its BruhatTits tree, $\mathcal{T}$. The structure of the group is derived from that of the quotient graph $G L_{2}(\mathcal{C}) \backslash \mathcal{T}$. Using an elementary approach which refers explicitly to matrices we determine the structure of all the vertex stabilizers of $\mathcal{T}$. This extends results of Serre, Takahashi and the authors. We also determine all possible valencies of the vertices of $G L_{2}(\mathcal{C}) \backslash \mathcal{T}$ for the important special case where $\infty$ has degree 1 .


Key words: Drinfeld modular group; Bruhat-Tits tree; vertex stabilizer; inseparable function field; isolated vertex; amalgamated product;

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## Introduction

Let $K$ be an algebraic function field of one variable with constant field $k$. As usual we assume that $k$ is algebraically closed in $K$. Let $\infty$ be a fixed place of $K$ of degree $\delta$ and let $\mathcal{C}$ be the set of all those elements of $K$ which are integral outside $\infty$. Then $\mathcal{C}$ is a Dedekind domain whose unit group, $\mathcal{C}^{*}$, is $k^{*}$. (For the simplest example let $K=k(t)$, the rational function field over $k$. Then, when $\infty$ corresponds to the usual "point at infinity" of $k(t), \mathcal{C}$ is the polynomial ring $k[t]$.) Here our focus of attention is the group $G=G L_{2}(\mathcal{C})$. When $k$ is finite (the arithmetic case) this group plays a central role [1], [2] in the theory of Drinfeld modular curves, analogous to that of the classical modular group $S L_{2}(\mathbb{Z})$ in the theory of modular forms. When $k$ is finite we refer to $G L_{2}(\mathcal{C})$ as an (arithmetic) Drinfeld modular group. Otherwise we call $G L_{2}(\mathcal{C})$ a non-arithmetic Drinfeld modular group.

Let $K_{\infty}$ be the completion of $K$ with respect to $\infty$. The group $G L_{2}\left(K_{\infty}\right)$ acts [9, Chapter II, Section 1.1] on its associated Bruhat-Tits building which in this case is a tree, $\mathcal{T}$. Classical Bass-Serre theory [9, Theorem 10, p.39] shows how to obtain a presentation for $G L_{2}(\mathcal{C})$ from the structure of the quotient graph $G L_{2}(\mathcal{C}) \backslash \mathcal{T}$. Serre's approach $[9,2.1, \mathrm{p} .96]$ to the structure of $\mathcal{T}$ is based on the theory of vector bundles. Here, as in [3], we adopt a more elementary approach which explicitly refers to matrices. One of our aims in so doing is to make our results more accessible to group theorists less familiar with algebraic geometry. Moreover the approach based on vector bundles does not provide more detailed versions of our principal results (which describe the group-theoretic structure of the vertex stabilizers in $G L_{2}(\mathcal{C})$ ). Serre [9, Theorem 9, p.106] has determined the "shape" of $G L_{2}(\mathcal{C}) \backslash \mathcal{T}$. He has shown that this quotient is the union of a central subgraph $X$, of bounded width, together with a number of (pairwise disjoint) infinite half-lines (or rays). ("Bounded width" refers to the geodesic length in $G L_{2}(\mathcal{C}) \backslash \mathcal{T}$.) In the arithmetic case $X$ is finite and there are only finitely many rays. The structure of the rays (by which we mean the structure of the stabilizers of vertices and edges of $\mathcal{T}$ which project onto those of the rays) is well understood [9, p.118]. However much less is known about $X$, except for a number of special cases. For example the structure of $X$ is known [4], when $g=0$ and $K \cong k(t)$, and [9, 2.4.3, p.114], [6], when $g=0, K \neq k(t)$ and $\delta=2$. In addition Takahashi [12] has determined $X$ precisely when $g=1$ and $\delta=1$. Among the few known [9, p.97] general properties of $X$ is that it contains at least one vertex whose stabilizer is isomorphic to $G L_{2}(k)$.

A number of important properties of $G L_{2}(\mathcal{C})$ derive from $X$. For example, $G L_{2}(\mathcal{C})$ has a free quotient whose rank is that of $\pi_{1}(X)$, the fundamental group of $X$. (See [9, p.43].) The theory of Drinfeld modular curves [2] provides a formula [5] for the (finite) rank of $\pi_{1}(X)$, for the case where $k$ is finite.

The principal aim in this paper is to determine the structure of the stabilizers of all the vertices of $X$ thus extending the above results of Serre, Takahashi and the
authors. Our elementary approach shows that this structure depends entirely on the nature of the eigenvalues of the relevant matrices. Our main result is the following. Suppose that there exists a quadratic polynomial over $k$ which has no roots in $k$. (Then $k$ is said to be not quadratically closed.)

Theorem A. Let $G_{v}$ be the stabilizer in $G$ of $v \in \operatorname{vert}(\mathcal{T})$.
(a) Suppose that the eigenvalues of every matrix in $G_{v}$ lie in $k$ and that at least one such matrix has distinct eigenvalues. Then

$$
G_{v} / N \cong k^{*} \times k^{*},
$$

where $N \cong V^{+}$, the additive group of $V$, a finite-dimensional vector space over $k$.
(b) Suppose that every matrix in $G_{v}$ has repeated eigenvalues in $k$. Then

$$
G_{v}=Z \times N
$$

where $N$ is as above and $Z \cong k^{*}$.
(c) Suppose that $G_{v}$ contains a matrix whose eigenvalues are distinct and do not lie in $k$. Then $G_{v}$ is isomorphic to one of the following.
(i) $L^{*}$, where $L$ is a separable quadratic extension of $k$,
(ii) $G L_{2}(k)$,
(iii) $Q^{*}$, the units of a quaternion division algebra $Q$ over $k$.
(d) Suppose that every matrix of $G_{v}$ has repeated eigenvalues and that one such matrix has eigenvalues not in $k$. Then $\operatorname{char}(k)=2$ and

$$
G_{v} \cong F^{*}
$$

where $F$ is a totally inseparable finite extension of $k$ of degree $2^{m}$, where $m \geq 1$. Moreover $2^{m-1}$ divides the degree of every place of $K$.

Non-arithmetic groups are more complicated than their arithmetic counterparts. For example, stabilizers of types (c)(iii) and (d) only arise when $k$ is infinite. The situation when $k$ is quadratically closed turns out to be much simpler.

Our results show that although edge stabilizers are of the same type they are subject to some restrictions. For the remainder of the paper we focus on the valency (or degree) of the projection of a vertex of $\mathcal{T}$ in $G L_{2}(\mathcal{C}) \backslash \mathcal{T}$. For one important special case we determine all the possibilities.

Theorem B. Suppose that $\delta=1$. Let $\widetilde{v}$ denote the projection of a vertex $v$ of $\mathcal{T}$ in $G L_{2}(\mathcal{C}) \backslash \mathcal{T}$. Then

$$
\operatorname{val}(\widetilde{v})=1,2,3 \text { or } \operatorname{card}(k)+1
$$

Of particular interest here are the isolated vertices in the quotient graph for the following reason. If a vertex $v$ and edge $e$ project onto such a vertex and its only incident edge in $G L_{2}(\mathcal{C}) \backslash \mathcal{T}$, then from standard Bass-Serre theory [9, Theorem 13, p.55] it follows that

$$
G L_{2}(\mathcal{C}) \cong H *{ }_{L} K
$$

where $H=G_{v}$ and $L=G_{e}$, the stabilizers of $v$ and $e$ in $\mathcal{T}$, with $H \neq L$ and $K \neq L$. For isolated vertices the situation is again less complicated for arithmetic groups. For example we prove here that, in contrast to the non-arithmetic case, isolated vertices exist in the quotient graph only when $\delta=1$.

In the case where $k$ is finite, $G L_{2}(\mathcal{C}) \backslash \mathcal{T}$ is a particularly interesting object from a number theoretic point of view, because it encodes a lot of information concerning the Drinfeld modular curve associated with $G L_{2}(\mathcal{C})$.

In [7] we work out a precise connection between the elliptic points of this Drinfeld modular curve (that is, points on the Drinfeld upper half-plane with nontrivial stabilizer under the action of $G$ ) and vertices of $G \backslash \mathcal{T}$ with certain stabilizers. In the case $\delta=1$ we also get a precise relation between the elliptic points and the isolated vertices of $G \backslash \mathcal{T}$. This in turn then yields information on the possibilities for $P G L_{2}(\mathcal{C})$ to decompose as a free product.

In contrast to the mainly group theoretic approach in the present paper, the arguments used in [7] are largely number theoretic, involving properties of ideal class groups in function fields and $L$-functions.

## 1. Preliminaries

For convenience we list at this point the notation which will be used throughout:

| $k$ | a field; |
| :--- | :--- |
| $\mathbb{F}_{q}$ | the finite field of order $q ;$ |
| $K$ | an algebraic function field of one variable with constant field $k ;$ |
| $g=g(K)$ | the genus of $K ;$ |
| $\infty$ | a chosen place of $K ;$ |
| $\delta$ | the degree of the place $\infty ;$ |
| $\nu$ | the discrete valuation of $K$ defined by $\infty ;$ |
| $\pi$ | a local parameter at $\infty$ in $K ;$ |
| $\mathcal{O}$ | the valuation ring of $\infty$ in $K ;$ |
| $\mathfrak{m}=(\pi)$ | the maximal ideal of $\mathcal{O} ;$ |
| $k_{\infty}$ | the residue field, $\mathcal{O} / \mathfrak{m} ;$ |
| $K_{\infty}$ | the completion of $K$ with respect to $\infty ;$ |
| $\mathcal{O}_{\infty}$ | the completion of $\mathcal{O}$ with respect to $\infty ;$ |
| $\mathcal{T}$ | the Bruhat-Tits tree of $G L_{2}\left(K_{\infty}\right) ;$ |
| $\mathcal{C}$ | the ring of all elements of $K \operatorname{that}$ are integral outside $\infty ;$ |
| $G$ | the group $G L_{2}(\mathcal{C}) ;$ |
| $G_{w}$ | the stabilizer in $G$ of $w \in \operatorname{vert}(\mathcal{T}) \cup \operatorname{edge}(\mathcal{T}) ;$ |
| $\widetilde{w}$ | the image in $G \backslash \mathcal{T}$ of $w \in \operatorname{vert}(\mathcal{T}) \cup \operatorname{edge}(\mathcal{T}) ;$ |
| $Z$ | the set of scalar matrices in $G ;$ |
| $Z_{\infty}$ | the set of scalar matrices in $G L_{2}\left(K_{\infty}\right)$. |

Our basic reference for the theory of algebraic function fields is Stichtenoth's book [11]. We recall that $k_{\infty}$ is a finite extension of $k$ of degree $\delta$. Associated with the group $G L_{2}\left(K_{\infty}\right)$ is its Bruhat-Tits building which in this case is a tree, $\mathcal{T}$. See [9, Chapter II, Section 1]. It is known [9, Corollary, p.75] that $G$ acts on $\mathcal{T}$ without inversion. Classical Bass-Serre theory [9, Theorem 13, p.55] shows how the structure of $G$ can be derived from that of the quotient graph $G \backslash \mathcal{T}$. The "shape" of this graph is described in the following.

Theorem 1.1.(Serre) To each element $\sigma$ of $\mathrm{Cl}(\mathcal{C})$, the ideal class group of $\mathcal{C}$, there corresponds a ray (i.e. an infinite half-line without backtracking), $R(\sigma)$, with terminal vertex $v_{\sigma}$, together with a subgraph $X$ such that

$$
G \backslash \mathcal{T}=\left(\bigcup_{\sigma \in \mathrm{Cl}(\mathcal{C})} R(\sigma)\right) \cup X
$$

where
(i) $X$ is bounded (with respect to geodesic length in $G \backslash \mathcal{T}$ ),
(ii) $R(\sigma) \cap R(\tau)=\emptyset(\sigma \neq \tau)$,
(iii) $\operatorname{vert}(X) \cap \operatorname{vert}(R(\sigma))=\left\{v_{\sigma}\right\}$,
(iv) $\operatorname{edge}(X) \cap \operatorname{edge}(R(\sigma))=\emptyset$.

Serre's approach [9, Theorem 9, p.106] uses the theory of vector bundles. For a more elementary approach which refers specifically to matrices see [3, Theorem 4.7]. A presentation for $G$ can be derived from a lift

$$
j: \mathcal{T}_{0} \longrightarrow \mathcal{T}
$$

where $\mathcal{T}_{0}$ is a maximal tree of $G \backslash \mathcal{T}$. Each ray $R(\sigma)$ is then realised as a subgraph of $\mathcal{T}$.
The structure of the stabilizers of the vertices of $\mathcal{T}$ which project onto those of each $R(\sigma)$ is well understood. See, for example, [3, Theorem 4.2]. However much less is known about $X$. One of the few known [9, p.97] properties of $X$ is that it contains at least one vertex whose stabilizer is isomorphic to $G L_{2}(k)$. The precise structure of $X$ is only known for a number of special cases, for example, when $g=0$ [4], [6] and when $g=1$ and $\delta=1$ [12].

A number of important properties of $G$ are derived from $X$. For example, it is known $[9, \mathrm{p} .43]$ that $G$ has a free quotient whose rank is that of, $\pi_{1}(X)$, the fundamental group of $X$. The theory of Drinfeld modular curves [2] provides a formula [5] for the (finite) rank of $\pi_{1}(X)$, for the case where $k$ is finite. When $k$ is infinite this rank is only known in a number of particular instances $[4],[6],[12]$.

We require a detailed model for $\mathcal{T}$. For convenience we make use of that used by Takahashi [12]. The vertices of $\mathcal{T}$ are the left cosets of $Z_{\infty} G L_{2}\left(\mathcal{O}_{\infty}\right)$ in $G L_{2}\left(K_{\infty}\right)$, where $Z_{\infty} G L_{2}\left(\mathcal{O}_{\infty}\right)$ is the subgroup generated by $G L_{2}\left(\mathcal{O}_{\infty}\right)$ and $Z_{\infty}$. Recall that

$$
\mathcal{O}=\{z \in K: \nu(z) \geq 0\}
$$

$\nu(\pi)=1$ and that $\mathcal{O} / \mathfrak{m} \cong k_{\infty}$. In addition $\nu(c) \leq 0$, for all $c \in \mathcal{C}$ and $\mathcal{O} \cap \mathcal{C}=k$. The edges of $\mathcal{T}$ are defined in the following way. The vertices $g_{1} Z_{\infty} G L_{2}\left(\mathcal{O}_{\infty}\right)$ and $g_{2} Z_{\infty} G L_{2}\left(\mathcal{O}_{\infty}\right)$ are adjacent if and only if

$$
g_{2}^{-1} g_{1} \equiv\left[\begin{array}{ll}
\pi & z \\
0 & 1
\end{array}\right] \text { or }\left[\begin{array}{ll}
\pi^{-1} & 0 \\
0 & 1
\end{array}\right]\left(\bmod Z_{\infty} G L_{2}\left(\mathcal{O}_{\infty}\right)\right)
$$

for some $z \in k_{\infty}$. Then $\mathcal{T}$ is a tree on which $G$ acts (by left multiplication) without inversion. It is clear that $\mathcal{T}$ is regular in the sense that the edges of $\mathcal{T}$ incident with each $v \in \operatorname{vert}(\mathcal{T})$ are in one-one correspondence with the elements of $\mathbb{P}_{1}\left(k_{\infty}\right)$. It is also clear that if $e \in \operatorname{edge}(\mathcal{T})$ has vertex extremities $u, v$ then $G_{e}=G_{u} \cap G_{v}$.
The model used by Serre [9, Chapter II, 1.1] is different but equivalent. A lattice $L$ is a $\mathcal{O}_{\infty}$-submodule of $K_{\infty}^{2}$ of rank 2 and the class containing $L$ is the set $\{z L: z \in$ $\left.K_{\infty}^{*}\right\}$. Then $G$ acts (naturally) on the set of lattice classes which for this model are the vertices of $\mathcal{T}$. Let $g Z_{\infty} G L_{2}\left(\mathcal{O}_{\infty}\right)$ be any left coset and let

$$
g=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]
$$

If $\Lambda_{g}$ denotes the lattice class containing the lattice generated by $(a, c)$ and $(b, d)$ then the correspondence

$$
g Z_{\infty} G L_{2}\left(\mathcal{O}_{\infty}\right) \longleftrightarrow \Lambda_{g}
$$

demonstates that the models for $\mathcal{T}$ are equivalent.
Since $\mathcal{O}_{\infty}$ is a PID we may represent every coset of $Z_{\infty} G L_{2}\left(\mathcal{O}_{\infty}\right)$ by an element of the form

$$
\left[\begin{array}{ll}
\pi^{n} & z \\
0 & 1
\end{array}\right]
$$

for some $n \in \mathbb{Z}$ and $z \in K_{\infty}$. We denote this vertex of $\mathcal{T}$ by $v(n, z)$. It is clear that

$$
v(n, z)=v\left(m, z^{\prime}\right) \Longleftrightarrow n=m \text { and } \nu\left(z-z^{\prime}\right) \geq n
$$

We may assume therefore that $z \in K$. When $v=v(n, z)$ we put

$$
G_{v}=G(n, z) .
$$

Definition. Let $H$ be any subgroup of $G$. Vertices $u, v$ of $\mathcal{T}$ are said to be $H$ equivalent, written

$$
u \equiv v(\bmod H)
$$

if and only if $u=h(v)$, for some $h \in H$.
If, as above, $u, v$ are $H$-equivalent it is clear that $G_{v}=h G_{u} h^{-1}$ and that $\bar{u}=\widetilde{v}$. A similar definition can be made for the elements of $\operatorname{edge}(\mathcal{T})$.

We record the following well-known result.
Lemma 1.2. With the above notation, let $v=v(n, z)$ and

$$
M=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \in G
$$

Then $M \in G(n, z)$ if and only if

$$
\text { (i) } \nu(c) \geq-n \text {, }
$$

(ii) $\nu(a-z c) \geq 0, \nu(d+z c) \geq 0$,
(iii) $\nu\left(b+z(a-d)-z^{2} c\right) \geq n$.

Lemma 1.2 shows that there exists a constant $\kappa=\kappa(v)$ such that $\nu(x) \geq \kappa$, for every entry of the matrices in $G_{v}$. Hence the entries of the matrices in $G_{v}$ lie in a
bounded subset of $K_{\infty}$ with respect to the metric on $K_{\infty}$ defined by $\nu$. (Serre [9, Proposition 2, p.76] proves an alternative version of this result.) For each $n \in \mathbb{Z}$ let

$$
\mathcal{C}(n)=\{c \in \mathcal{C}: \nu(c) \geq-n\}
$$

By the Riemann-Roch Theorem [11, I.5.15, p.28] $\mathcal{C}(n)$ is a finite-dimensional vector space over $k$. It follows that, when $k$ is finite, each $G_{v}$ is finite. The following notation is useful for our purposes.

Notation. We put

$$
M(n, z, \alpha, \beta, c)=\left[\begin{array}{cc}
\alpha+c z & b \\
c & \beta-c z
\end{array}\right]
$$

where
(i) $\nu(c) \geq-n$,
(ii) $c, c z \in \mathcal{C}$,
(iii) $b=(\beta-\alpha) z-c z^{2} \in \mathcal{C}$,
(iv) $\alpha, \beta \in k^{*}$.

By Lemma 1.2 it is clear that $M(n, z, \alpha, \beta, c) \in G(n, z)$ and that $\operatorname{det}(M(z, n, \alpha, \beta, c))=$ $\alpha \beta$. Let

$$
V=\{c \in \mathcal{C}: M(n, z, 1,1, c) \in G(n, z)\}
$$

From the above, if $c \in V$, then $c \in \mathcal{C}(n), c z, c z^{2} \in \mathcal{C}$ and $V$ is a finite-dimensional $k$-vector space. We note also that $M(n, z, \alpha, \alpha, 0)=\operatorname{diag}(\alpha, \alpha) \in G(n, z)$ and that

$$
M(n, z, \alpha, \alpha, c) \in G(n, z) \Longleftrightarrow c \in \mathcal{C}(n), c z, c z^{2} \in \mathcal{C}
$$

Lemma 1.3. Let $v \in \operatorname{vert}(\mathcal{T})$ and let $M \in G_{v}$. Then the characteristic polynomial of $M$ has coefficients in $k$.

Proof. Let

$$
M=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]
$$

It suffices to prove that $\operatorname{tr}(M)=a+d \in k$, since $\operatorname{det}(M) \in k^{*}$. By Lemma 1.2 $\nu(a+d) \geq 0$ and so $a+d \in \mathcal{C} \cap \mathcal{O}=k$.

Our study of stabilizers depends crucially on the next result.
Lemma 1.4. Let $w \in \operatorname{vert}(\mathcal{T}) \cup \operatorname{edge}(\mathcal{T})$. Suppose that the matrices $M_{1}, M_{2} \in G_{w}$. Then, if

$$
\operatorname{det}\left(\alpha_{1} M_{1}+\alpha_{2} M_{2}\right) \in k^{*}
$$

where $\alpha_{1}, \alpha_{2} \in k$, then

$$
\alpha_{1} M_{1}+\alpha_{2} M_{2} \in G_{w} .
$$

Proof. We may assume that $w \in \operatorname{vert}(\mathcal{T})$. Then $M_{i}$ "fixes" some $\operatorname{coset} g Z_{\infty} G L_{2}\left(\mathcal{O}_{\infty}\right)$, say. Hence $g^{-1} M_{i} g \in Z_{\infty} G L_{2}\left(\mathcal{O}_{\infty}\right)$ and so $g^{-1} M_{i} g \in G L_{2}\left(\mathcal{O}_{\infty}\right)$, since $\operatorname{det}\left(M_{i}\right) \in k^{*}$ $(i=1,2)$. It follows that

$$
g^{-1}\left(\alpha_{1} M_{1}+\alpha_{2} M_{2}\right) g \in M_{2}\left(\mathcal{O}_{\infty}\right)
$$

We note that $Z=\left\{\alpha I_{2}: \alpha \in k^{*}\right\}$ and consequently that

$$
Z \leq G_{w}
$$

for all $w \in \operatorname{vert}(\mathcal{T}) \cup \operatorname{edge}(\mathcal{T})$.
Let the characteristic polynomial $\chi_{M}(X)$ of $M \in G L_{2}\left(K_{\infty}\right)$ have coefficients in $k$. It is clear that, for all $\alpha, \beta \in k$, with $\beta \neq 0$,

$$
\operatorname{det}\left(\alpha I_{2}+\beta M\right)=\beta^{2} \chi_{M}\left(-\alpha \beta^{-1}\right) \in k
$$

Notation. We put

$$
I(M)=\left\{M^{\prime}=\alpha I_{2}+\beta M: \alpha, \beta \in k, \operatorname{det}\left(M^{\prime}\right) \in k^{*}\right\} .
$$

It is clear by Lemmas 1.3 and 1.4 that, if $M \in G_{v}$ then

$$
Z \leq I(M) \leq G_{v}
$$

This will be the starting point for our study of $G_{v}$. We record an immediate consequence of Lemma 1.4.

Lemma 1.5. Let $M \in G$ and $w \in \operatorname{vert}(\mathcal{T}) \cup \operatorname{edge}(\mathcal{T})$. Then

$$
I(M) \cap G_{w}=Z \text { or } I(M)
$$

Lemma 1.6. Suppose that $M \in G_{v} \backslash Z$. Then there are three possibilities.
(i) If $M$ has distinct eigenvalues in $k$, then

$$
I(M) \cong k^{*} \times k^{*} .
$$

(ii) If $M$ has repeated eigenvalues in $k$, then

$$
I(M) \cong k^{*} \times k^{+}
$$

where $k^{+}$is the additive group of $k$.
(iii) If $M$ has an eigenvalue $\lambda \notin k$, then

$$
I(M) \cong k(\lambda)^{*}
$$

Proof. For part (i) $M$ is conjugate to $D_{0}=\operatorname{diag}\left(\lambda_{1}, \lambda_{2}\right)$, where $\lambda_{1}, \lambda_{2} \in k^{*}$, with $\lambda_{1} \neq \lambda_{2}$. Then

$$
I(M) \cong I\left(D_{0}\right)=\left\{\operatorname{diag}(\lambda, \mu): \lambda, \mu \in k^{*}\right\} .
$$

Part (ii) is very similar. Here

$$
I(M) \cong Z \times B
$$

where

$$
B=\left\{\left[\begin{array}{ll}
1 & x \\
0 & 1
\end{array}\right]: x \in k\right\}
$$

For part (iii) we note that $\operatorname{det}\left(\alpha I_{2}+\beta M\right)=0$ only when $\alpha=\beta=0$. It is clear then that in this case $I(M)$ is the multiplicative group of a quadratic extension of $k$.

We record separately a special case of Lemma 1.6.
Lemma 1.7. Suppose that $k=\mathbb{F}_{q}$. Then with the above notation

$$
|I(M)|= \begin{cases}\left(q^{2}-1\right) & , \text { the eigenvalues of } M \text { are not in } \mathbb{F}_{q} \\ (q-1)^{2} & , \text { the eigenvalues of } M \text { are distict and in } \mathbb{F}_{q} \\ q(q-1) & , \text { otherwise }\end{cases}
$$

Note that $I(M) \cong \mathbb{F}_{q^{2}}^{*}$, when $|I(M)|=q^{2}-1$.
We conclude this section by stating a well-known result which follows immediately from the definition of $\mathcal{T}$. Let $\Re \subset K$ denote a complete set of representatives for $\mathcal{O} / \mathfrak{m}\left(\cong k_{\infty}\right)$.

Lemma 1.8. The vertices in $\mathcal{T}$ adjacent to $v(n, z)$ are

$$
v(n-1, z) \text { and } v\left(n+1, z+u \pi^{n}\right)
$$

where $u \in \mathfrak{R}$.

## 2. Vertex stabilizers

This section is devoted to our principal aim, namely the determination of the structure of each $G_{v}$. We require the following.

Definition. The field $k$ is called quadratically closed if and only if every quadratic polynomial over $k$ has a root in $k$.

Every algebraically closed field is, of course, quadratically closed. Examples of quadratically closed fields which are not algebraically closed include the separable closure of an imperfect field of odd characteristic, and

$$
\bigcup_{n \geq 0} \mathbb{F}_{q(n)}
$$

where $q(n)=q^{2^{n}}$. Examples of fields which are not quadratically closed include all subfields of $\mathbb{R}$, all finite fields, and all fields with a discrete valuation, in particular local fields, algebraic number fields, and algebraic function fields over any constant field.

Theorem 2.1. Suppose that all the matrices in $G_{v}$ have eigenvalues in $k$ with at least one with distinct eigenvalues. Then there are two possibilities.
(a) There exists a normal subgroup $N$ of $G_{v}$, such that
(i) $N \cong V^{+}$, the additive group of a finite-dimensional $k$-vector space $V$,
(ii) $G_{v} / N \cong k^{*} \times k^{*}$.

In this case there is a homomorphism $\theta: k^{*} \times k^{*} \rightarrow \operatorname{Aut}\left(V^{+}\right)$given by

$$
\theta((\alpha, \beta)): v \mapsto\left(\alpha \beta^{-1}\right) v \quad(v \in V)
$$

(b) Only when $k$ is quadratically closed,

$$
G_{v} \cong G L_{2}(k)
$$

Proof. Let $M_{0} \in G_{v}$ have distinct eigenvalues in $\lambda, \mu \in k^{*}$. Then we may replace $G_{v}$ with a conjugate $G_{v}^{\prime}$ (over $G L_{2}(K)$ ) containing

$$
I\left(D_{0}\right)=D
$$

where $D_{0}=\operatorname{diag}(\lambda, \mu)$ and $D=\left\{\operatorname{diag}(\sigma, \tau): \sigma, \tau \in k^{*}\right\}$. Now $G_{v}^{\prime}$ satisfies the same hypotheses as $G_{v}$. In addition Lemmas $1.3,1.4$ and 1.6 apply to $G_{v}^{\prime}$. Let

$$
M=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \in G_{v}^{\prime}
$$

By multiplying $M$ and $\operatorname{diag}(\lambda, 1)$ and considering traces we conclude by Lemma 1.3 that

$$
\lambda a+d \in k
$$

for all $\lambda \in k^{*}$. It follows that $a, d \in k$ and hence that $b c \in k$. (Note that the hypotheses ensure that $k \neq \mathbb{F}_{2}$.) There are two possibilities.
(a) Suppose that $b c=0$, for all $M \in G_{v}^{\prime}$. It follows then that either $c=0$, for all $M \in G_{v}^{\prime}$, or $b=0$, for all $M \in G_{v}^{\prime}$. We will assume the former.
We define a subset $V$ of $K$ by

$$
x \in V \Longleftrightarrow\left[\begin{array}{ll}
1 & x \\
0 & 1
\end{array}\right] \in G_{v}^{\prime}
$$

Let

$$
B=\left\{\left[\begin{array}{ll}
1 & x \\
0 & 1
\end{array}\right]: x \in V\right\}
$$

It is clear that

$$
G_{v}^{\prime}\left\{\left[\begin{array}{cc}
\alpha & x \\
0 & \beta
\end{array}\right]: \alpha, \beta \in k^{*}, x \in V\right\} .
$$

By Lemma $1.4 V$ is a vector space over $k$. Moreover $B \cong V^{+}$. Under the conjugacy between $G_{v}$ and $G_{v}^{\prime} B$ is conjugate to a subgroup $B^{\prime}$, say, of $G_{v}$. We now apply Lemma 1.2 to the entries of $B^{\prime}$. It follows that there exists $t \in K^{*}$ and a real constant $\kappa$ such that, for all $x \in V$,
(i) $t x \in \mathcal{C}$,
(ii) $\nu(t x) \geq \kappa$.

Then $V$ is a finite-dimensional $k$-space by the Riemann-Roch theorem.
(b) Suppose that there exists

$$
M_{0}=\left[\begin{array}{ll}
a_{0} & b_{0} \\
c_{0} & d_{o}
\end{array}\right] \in G_{v}^{\prime}
$$

for which $b_{0}, c_{0} \in k^{*}$. Then, for any $M \in G_{v}^{\prime}$, as above, since $M M_{0} \in G_{v}^{\prime}$ it follows from the above that $b c_{0}, c b_{0} \in k$. We deduce that

$$
G_{v}^{\prime}=\left\{\left[\begin{array}{cc}
\alpha & \beta b_{0} \\
\gamma c_{0} & \delta
\end{array}\right]: \alpha, \beta, \gamma, \delta \in k, \alpha \delta-\beta \gamma \neq 0\right\} .
$$

By multiplying $M_{0}$ with a suitable element of $D$ we may assume that $b_{0} c_{0}=1$. It is clear then that

$$
G_{v}^{\prime} \cong G L_{2}(k)
$$

in which case $k$ is quadratically closed.

The arithmetic version of Theorem 2.1 is as follows.

Corollary 2.2. Suppose that $k=\mathbb{F}_{q}$. Then, with the hypotheses of Theorem 2.1,

$$
G_{v} / N \cong \mathbb{F}_{q}^{*} \times \mathbb{F}_{q}^{*}
$$

where

$$
N \cong V^{+}
$$

the additive group of a finite-dimensional $\mathbb{F}_{q}$-vector space $V$ (and hence elementary p-abelian). In addition

$$
\left|G_{v}\right|=(q-1)^{2} q^{n}
$$

where $n=\operatorname{dim}_{q}(V) \geq 0$.
Theorem 2.1 applies in particular [3, Theorem 4.2] to the vertices of $\mathcal{T}$ which map onto those of the rays $R(\sigma)$ (in Serre's Theorem). If $v_{1}, v_{2}$ map onto adjacent vertices of $R(\sigma)$ in $G \backslash \mathcal{T}$, then $G_{v_{i}} \cong V_{i}^{+} \rtimes\left(k^{*} \times k^{*}\right)(i=1,2)$. Moreover (relabelling if necessary) it is known that $G_{v_{1}} \leq G_{v_{2}}$ and that $\operatorname{dim}\left(V_{2}\right)-\operatorname{dim}\left(V_{1}\right)=\delta$. Consequently $\operatorname{dim}(V)$ in Theorem 2.1 can be arbitrarily large.

We record a known "minimal size" example of a stabilizer as described by Theorem 2.1. Suppose that $g=\delta=1$. Then $\mathcal{C}$ is the coordinate ring of an elliptic curve. In which case there exist $x, y \in \mathcal{C}$, where $\nu(x)=-2$ and $\nu(y)=-3$ which satisfy a Weierstrass equation, $F(x, y)=0$, for which

$$
\mathcal{C}=k[x, y] .
$$

Takahashi [12] has completely determined $G \backslash \mathcal{T}$ in this case. In particular [12, Theorem 5] he has shown that

$$
G\left(2, \pi^{-1}+\pi \lambda\right) \cong k^{*} \times k^{*}
$$

whenever there exist $\mu_{1}, \mu_{2} \in k$, with $\mu_{1} \neq \mu_{2}$, such that

$$
F\left(\lambda, \mu_{1}\right)=F\left(\lambda, \mu_{2}\right)=0 .
$$

Theorem 2.3. Suppose that every matrix in $G_{v}$ has repeated eigenvalues in $k$. Then

$$
G_{v}=Z \times B,
$$

where $B \cong V^{+}$, the additive group of a finite-dimensional vector space $V$ over $k$.
Proof. We recall that $Z \leq G_{v}$. We may assume that $G_{v} \neq Z$. Then $G_{v}$ contains a non-central matrix. As in the proof of Theorem 2.1 we replace $G_{v}$ with a conjugate $G_{v}^{\prime}$ which contains $Z$ and a matrix

$$
\left[\begin{array}{ll}
\alpha & x \\
0 & \alpha
\end{array}\right]
$$

where $\alpha \in k^{*}$ and $x \neq 0$. Now as above $G_{v}^{\prime}$ satisfies the same hypotheses as $G_{v}$ and Lemmas 1.3, 1.4 and 1.6 apply to it. By Lemma 1.4 it follows that, for all $\lambda \in k^{*}$,

$$
U=\left[\begin{array}{ll}
1 & \lambda x \\
0 & 1
\end{array}\right] \in G_{v}^{\prime}
$$

Let

$$
M=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \in G_{v}^{\prime}
$$

Then there are two possibilities.
Suppose that $\operatorname{char}(k) \neq 2$. Since the eigenvalues are repeated, we have $(a+d)^{2}=$ $4(a d-b c)$, that is $(a-d)^{2}=-4 b c$. But $U M \in G_{v}^{\prime}$ and so

$$
(a-d+\lambda c x)^{2}=-4 c(b+\lambda d x)
$$

It follows that

$$
2 c(a-d)+c^{2} \lambda x=-4 c d,
$$

and hence that $c=0$.
Suppose that $\operatorname{char}(k)=2$. The hypotheses ensure that in this case $a=d$, for all $M$. Again by considering $U M$ it follows that $c=0$.

In all cases therefore $c=0$ and hence $a=d$. Defining $V$ and $B$ as in the proof of Theorem 2.1 we deduce that

$$
G_{v}^{\prime}=\left\{\left[\begin{array}{cc}
\alpha & b \\
0 & \alpha
\end{array}\right]: \alpha \in k^{*}, b \in V\right\}
$$

Again $V$ is a $k$ - vector space which is finite-dimensional by the Riemann-Roch theorem.

The arithmetic version of Theorem 2.3 is as follows.
Corollary 2.4. Suppose that $k=\mathbb{F}_{q}$. Then, with the hypotheses of Theorem 2.3,
(i) $G_{v} \cong \mathbb{F}_{q}^{*} \times V^{+}$, where $V^{+}$is the additive group of a vector space over $\mathbb{F}_{q}$,
(ii) $\left|G_{v}\right|=(q-1) q^{n}$, where $n=\operatorname{dim}(V) \geq 0$.

The images in $G \backslash \mathcal{T}$ of the vertices to which Theorem 2.3 and Corollary 2.4 apply lie in $X$ (in Serre's theorem), except when $k=\mathbb{F}_{2}$.

We record 'minimal size" examples of stabilizers as described in Theorem 2.3. Suppose that $g>0$ and that $\delta=1$. Then by the Weierstrass gap theorem [11, I.6.7, p.32] 1 is a gap number for $\nu$, i.e. there is no $x \in \mathcal{C}$ for which $\nu(x)=-1$. It follows readily from Lemma 1.2 that
(i)

$$
G\left(1, \pi^{-1}\right) \cong k^{*}
$$

(ii)

$$
G\left(0, \pi^{-1}\right) \cong k^{*} \times k^{+}
$$

The remaining cases are more complicated. Let $\overline{K_{\infty}}$ denote the algebraic closure of $K_{\infty}$. The group $G L_{2}\left(\overline{K_{\infty}}\right)$ acts as a group of linear fractional transformations on $\mathbb{P}^{1}\left(\overline{K_{\infty}}\right)=\overline{K_{\infty}} \cup\{\infty\}$ in the usual way. For each subgroup $H$ of $G L_{2}\left(\overline{K_{\infty}}\right)$ and each $z \in \mathbb{P}^{1}\left(\overline{K_{\infty}}\right)$ let $H_{z}$ denote the stabilizer in $H$ of $z$. We record without proof the following.

Lemma 2.5. Let $g \in G L_{2}\left(\overline{K_{\infty}}\right)$. Then $\left[\begin{array}{l}\alpha \\ \beta\end{array}\right]$ is an eigenvector of $g$ if and only if

$$
g(\gamma)=\gamma
$$

(as a linear fractional transformation) where $\gamma=\alpha / \beta . \quad(\gamma=\infty$, when $\beta=0$ ).
Theorem 2.6. Suppose that $G_{v}$ contains a matrix $M_{0}$ with eigenvalues $\alpha_{0}, \beta_{0}$, where $\alpha_{0}, \beta_{0} \notin k$ and $\alpha_{0} \neq \beta_{0}$. Let $L=k\left(\alpha_{0}\right)\left(=k\left(\beta_{0}\right)\right)$ and let $\sigma$ denote the (Galois) $k$-automorphism of (the quadratic extension) L. Then there exist $x, y$, for which either $x=y=0$ or $x y=\kappa \in k^{*}$, such that

$$
G_{v} \cong\left\{\left[\begin{array}{cc}
\lambda & x \mu \\
y \mu^{\sigma} & \lambda^{\sigma}
\end{array}\right]: \lambda, \mu \in L, \lambda \lambda^{\sigma} \neq \kappa \mu \mu^{\sigma}\right\}
$$

There are then two possibilities.
(i)

$$
G_{v} \cong L^{*}
$$

(ii) There exists a quaternion algebra $Q$ over $k$ such that

$$
G_{v} \cong Q^{*}
$$

Proof. The nontrivial $k$-automorphism of $L$ extends naturally to a $K$-automorphism of the (quadratic) extension $L K / K$. Note that $\beta_{0}=\alpha_{0}^{\sigma}$. Let $\left[\begin{array}{l}\alpha \\ \beta\end{array}\right]$ and $\left[\begin{array}{l}\alpha^{\sigma} \\ \beta^{\sigma}\end{array}\right]$ be corresponding eigenvectors of $M_{0}$. Then there exists $g \in G L_{2}\left(\overline{K_{\infty}}\right)$ such that

$$
\left(I\left(M_{0}\right)\right)^{g}=\bar{Z}=\left\{\operatorname{diag}\left(\lambda, \lambda^{\sigma}\right): \lambda \in L^{*}\right\} \leq\left(G_{v}\right)^{g}
$$

We may assume that $g$ maps $\left[\begin{array}{l}\alpha \\ \beta\end{array}\right]$ and $\left[\begin{array}{c}\alpha^{\sigma} \\ \beta^{\sigma}\end{array}\right]$ onto $\left[\begin{array}{l}1 \\ 0\end{array}\right]$ and $\left[\begin{array}{l}0 \\ 1\end{array}\right]$, respectively. Let $X=\left(G_{v}\right)^{g}$. Although $\left(G_{v}\right)^{g}$ is not necessarily contained in $G$, Lemmas 1.3, 1.4 and 1.6 apply to the matrices in $X$. Then by Lemma 2.5 and the above

$$
X_{0}=X_{\infty}=\bar{Z} .
$$

We first of all dispose of the simplest possibility, namely $\bar{Z}=X$, i.e. $G_{v}=I\left(M_{0}\right) \cong$ $L^{*}$. (See Lemma 1.6(iii).) We may suppose then that from now on there exists

$$
M=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \in X \backslash \bar{Z} .
$$

By considering the trace of the product $D M$, where $D=\operatorname{diag}\left(\lambda, \lambda^{\sigma}\right)$, with $\lambda \in L^{*}$. It follows that

$$
\lambda a+\lambda^{\sigma} d \in k, \text { for all } \lambda \in L
$$

Now $\lambda(a+d) \in L$. From the case where $\lambda \neq \lambda^{\sigma}$ we deduce that $a, d \in L$.
Also $\lambda a+\lambda^{\sigma} a^{\sigma} \in k$ and so

$$
\lambda^{\sigma}\left(d-a^{\sigma}\right) \in k, \text { for all } \lambda \in L .
$$

We conclude that $d=a^{\sigma}$ and hence that $b c \in k$.
We now show that $b c=0$ if and only if $b=c=0$. Suppose then that $c=0$. By an obvious modified version of Lemma 1.4 it follows that

$$
B=\left[\begin{array}{ll}
1 & b \\
0 & 1
\end{array}\right] \in X .
$$

Hence $B \in X_{\infty}$ and so $B \in X_{0}$, from the above. We deduce that $b=0$. Similarly $c=0$ when $b=0$.

We may now suppose that $b c \in k^{*}$. Again by a modified version of Lemma 1.4 and the above

$$
N=\left[\begin{array}{ll}
0 & b \\
c & 0
\end{array}\right] \in X .
$$

Let

$$
N_{1}=\left[\begin{array}{ll}
0 & b_{1} \\
c_{1} & 0
\end{array}\right] \in X
$$

Now $N N_{1} \in X$. It follows from the above that $b c_{1}, b_{1} c \in L$ with $\left(b c_{1}\right)^{\sigma}=b_{1} c$ and $b c, b_{1} c_{1} \in k^{*}$. We deduce that

$$
b_{1}=b \beta \text { and } c_{1}=c \beta^{\sigma} \text {, }
$$

for some $\beta \in L$, i.e. $N_{1}=\operatorname{diag}\left(\beta, \beta^{\sigma}\right) N$.
Let $D_{0}=\operatorname{diag}\left(\alpha_{0}, \beta_{0}\right)$. By Lemma 1.4 and the above it is clear that $X$ consists of all those matrices of nonzero determinant in the (4-dimensional) $k$-vector space $Q$ spanned by $I_{2}, D_{0}, N$ and $D_{0} N$. (Let $b=x$ and $c=y$.) Equivalently $X$ is the set of units in the $k$-algebra $Q$ generated by $I_{2}, D_{0}$ and $N$. It is clear that $Q$ is central. To prove that $Q$ is quaternion it suffices therefore to prove that, if $J$ is a nonzero two-sided ideal in $Q$, then $J=Q$.

Let

$$
M=\lambda I_{2}+\mu D_{0}+\nu N+\rho N_{0} \in J
$$

where $N_{0}=D_{0} N$ and $\lambda, \mu, \nu, \rho \in k$, with $(\lambda, \mu, \nu, \rho) \neq(0,0,0,0)$. Then

$$
C=M D_{0}-D_{0} M=\nu\left(N D_{0}-D_{0} N\right)+\rho\left(N_{0} D_{0}-D_{0} N_{0}\right)=(\beta-\alpha)\left[\begin{array}{ll}
0 & u \\
v & 0
\end{array}\right] \in J,
$$

where $u=x(\nu+\rho \alpha)$ and $v=-y(\nu+\rho \beta)$. Then $C^{2}$ is a scalar matrix. If $C^{2}$ is nonzero it follows that $J=Q$. If on the other hand it is zero then $\nu=\rho=0$. Thus

$$
\lambda I_{2}+\mu D_{0} \in J
$$

where $(\lambda, \mu) \neq(0,0)$. In this case $M$ is invertible and again $J=Q$.
The proof of Theorem 2.6 is based on a representation of $G_{v}$ which derives from a quadratic extension of $k$ generated by the eigenvalues of a particular element of $G_{v}$. In general any quadratic extension of $k$ arises in this way. For example the stabilizer of $G(0,0)$ is $G L_{2}(k)$ (by Lemma 1.2 ) and every quadratic extension of $k$ is generated by the eigenvalues of some matrix in $G L_{2}(k)$.
By Theorem 2.6 there are three possibilities for $G_{v}$.
Corollary 2.7. With the notation of Theorem 2.6,

$$
G_{v} \cong L^{*} \Longleftrightarrow x=y=0 .
$$

Such vertices exist, for example, when $g=\delta=1$. See [12, Theorem 5].
Notation. Let $N_{L / k}: L \longrightarrow k$ denote the norm map, that is $N_{L / k}(a)=a \sigma(a)$.
Corollary 2.8. With the notation of Theorem 2.6, suppose that $\kappa \in N_{L / k}\left(L^{*}\right)$. Then

$$
G_{v} \cong G L_{2}(k)
$$

Proof. With the notation of Theorem 2.6, replacing $N$ with $Y N$, for some $Y \in \bar{Z}$, we may assume that $x y=\kappa=1$. It is easily verified that
$G_{v} \cong\left\{\left[\begin{array}{cc}\lambda & x \mu \\ y \mu^{\sigma} & \lambda^{\sigma}\end{array}\right]: \lambda, \mu \in L, \lambda \lambda^{\sigma} \neq \mu \mu^{\sigma}\right\} \cong\left\{\left[\begin{array}{cc}\lambda & \mu \\ \mu^{\sigma} & \lambda^{\sigma}\end{array}\right]: \lambda, \mu \in L, \lambda \lambda^{\sigma} \neq \mu \mu^{\sigma}\right\}$.

The latter group is the set of matrices of nonzero determinant (in $k$ ) in the $k$-vector space, $V$, say, spanned by $\left\{I_{2}, D_{0}, N_{1}, D_{0} N_{1}\right\}$, where

$$
N_{1}=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] .
$$

Now let

$$
M=\left[\begin{array}{ll}
\alpha_{0} & \alpha_{0}^{\sigma} \\
1 & 1
\end{array}\right]
$$

It is readily verified that $M D_{0} M^{-1}, M N_{1} M^{-1} \in M_{2}(k)$. Since the $k$-dimension of $V$ is 4 it follows that

$$
M V M^{-1}=M_{2}(k)
$$

The result follows.
Corollary 2.8 arises in all cases since (to repeat from the above) the stabilizer of $G(0,0)$ is always $G L_{2}(k)$. The remaining case can be stated without proof.

Corollary 2.9. With the notation of Theorem 2.6, suppose that $\kappa \notin N_{L / k}(L)$. Then $G_{v}$ is isomorphic to the set of units in the quaternion algebra, $Q$, over $k$, with $k$-basis $\left\{I_{2}, D_{0}, N, D_{0} N\right\}$.

The existence of vertex stabilizers as described in Corollary 2.9 imposes restrictions on $K$.

Lemma 2.10. A necessary condition for the existence of a non-split quaternion algebra $Q$ for which $Q^{*}$ is a vertex stabilizer is that all places of $K$ have even degree.

Proof. We start from $Q^{*} \subseteq G L_{2}(\mathcal{C}) \subseteq G L_{2}(K)$, fix a $k$-basis of $Q$ and consider its norm form. After tensoring up to $K$ the norm form will have a nontrivial zero; either because the $k$-basis becomes linearly dependent over $K$, or if not because the above inclusion implies that $Q \otimes_{k} K \cong M_{2}(K)$. Thus for every place $v$ of $K$ the norm form of $Q$ has a nontrivial zero in the completion $K_{v}$ and hence also in its valuation ring $\mathcal{O}_{v}$. Obviously we can achieve that the coordinates of such a zero are not all in the maximal ideal. So after reduction the norm form of $Q$ will have a nontrivial zero in the residue field $k_{v}$. In other words, $k_{v}$ is a splitting field of $Q$. Hence the degree of $k_{v}$ over $k$ must be even by [8, Lemma in Section 13.4].

Lemma 2.10 explains the absence of stabilizers of this type from Takahashi's list [12, Theorem 5] (since there $\delta=1$ ). We are also able to prove the following interesting existence theorem.

Theorem 2.11. The set of units of every non-split quaternion algebra $Q$ over
$k$ occurs as a vertex stabilizer of $G L_{2}(\mathcal{C})$ for a nonrational genus zero function field $K$ and a place $\infty$ of degree 2.

Proof. If the characteristic of $k$ is different from 2, there exists a $k$-basis $\{1, i, j, i j\}$ of $Q$ with $i^{2}=-\rho, j^{2}=-\sigma$, and $i j=-j i$. Since $Q$ is non-split, $-\rho$ is not a square in $k$, and for all $x, y \in k$ the element $y+x i+j$ has norm $y^{2}+\rho x^{2}+\sigma \neq 0$. We define

$$
\mathcal{C}:=k[X, Y] \text { with } Y^{2}+\rho X^{2}+\sigma=0 .
$$

Then $K=k(X, Y)$ is a non-rational function field of genus 0 (compare [6, Theorem 1.1 case (i)]). By [6, Lemma 2.5] and [6, Remark 2.7] there exists a vertex of $\mathcal{T}$ whose stabilizer in $G L_{2}(\mathcal{C})$ is isomorphic to $Q$.

If the characteristic is 2 , there exists a $k$-basis $\{1, i, j, i j\}$ of $Q$ with $i^{2}+i=\rho$, $j^{2}=\sigma$, and $i j=j(i+1)$. For all $x, y \in k$ the element $y+x i+j$ has norm $y^{2}+x y+\rho x^{2}+\sigma \neq 0$. Moreover, $\rho$ is not of the form $\alpha^{2}+\alpha$ for any $\alpha \in k$. Then the claim follows from [6, Theorem 1.1 (iv)], [6, Lemma 2.6] and [6, Remark 2.7].

In particular [6, Example 3.3] for this case, when $k=\mathbb{R}$, there is a vertex whose stabilizer is Hamilton's quaternions. We record separately the arithmetic version of Theorem 2.6.

Corollary 2.12. Suppose that $k=\mathbb{F}_{q}$. With the hypotheses of Theorem 2.6,

$$
G_{v} \cong \mathbb{F}_{q^{2}}^{*} \text { or } G L_{2}\left(\mathbb{F}_{q}\right)
$$

Proof. In this case $N_{L / k}$ is always surjective. (Alternatively we could argue that there are no non-split quaternion algebras over a finite field, or that a function field with finite constant field always has places of odd degree.)

There remains one further possibility.
Theorem 2.13. Suppose that (a) every matrix in $G_{v}$ has repeated eigenvalues and (b) the eigenvalues of at least one matrix in $G_{v}$ do not lie in $k$. Then $k$ is infinite, $\operatorname{char}(k)=2$ and

$$
G_{v} \cong F^{*}
$$

where
(i) $F$ is a finite, totally inseparable extension of $k$,
(ii) $u^{2} \in k$, for all $u \in F$,
(iii) $|F: k|=2^{m}$, for some $m>0$,
(iv) $2^{m-1}$ divides the degree of every place of $K$, in particular $2^{m-1}$ divides $\delta$ and $2 g-2$.

Proof. The fact that $k$ is infinite of characteristic 2 follows from Lemma 1.3. Let

$$
M=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \in G_{v}
$$

Then the hypotheses imply that
(i) $a=d$,
(ii) either $b=c=0$ or $b c \neq 0$,
(iii) $b=\gamma c$, where $\gamma$ is a nonzero constant determined by $G_{v}$.

As usual we denote the set of all squares of elements of $K$ by $K^{2}$. Suppose that $\gamma \in K^{2}$. Then

$$
\operatorname{det}(M)=a^{2}+\gamma c^{2} \in K^{2} \cap k
$$

Since $k$ is algebraically closed in $K$ it follows that the eigenvalues of $M$ lie in $k$. Hence

$$
\gamma \notin K^{2} .
$$

Every matrix in $G_{v}$ has trace 0 and hence is a multiple of its inverse. From this one easily sees that $G_{v}$ is abelian. Consequently, if $M \in G_{v}$ is a matrix with eigenvalues outside $k$, then $G_{v}$ must be contained in the commutator of $K(M)$ in $G L_{2}(K)$, that is, in $K(M)$.

Let

$$
M_{i}=\left[\begin{array}{ll}
a_{i} & b_{i} \\
c_{i} & d_{i}
\end{array}\right] \quad(i=1,2)
$$

Then $\operatorname{det}\left(M_{1}+M_{2}\right)=\left(a_{1}+a_{2}\right)^{2}+\gamma\left(c_{1}+c_{2}\right)^{2} \in k$. It is clear that $\operatorname{det}\left(M_{1}+M_{2}\right)=0$ if and only if $M_{1}=M_{2}$. Now $M_{1}+M_{2} \in G_{v}$, when $\operatorname{det}\left(M_{1}+M_{2}\right) \neq 0$, by Lemma 1.4. It is clear that $G_{v}$ is the multiplicative group of an extension $F$ of $k$ which satisfies property (ii). We use $[a, c]$ as a shorthand for

$$
\left[\begin{array}{ll}
a & \gamma c \\
c & a
\end{array}\right]
$$

Then from the above
(i) If $\left[a_{i}, c_{i}\right] \in G_{v}, i=1,2$, then $\left[a_{1}+a_{2}, c_{1}+c_{2}\right] \in G_{v}$.
(ii) If $[a, c],\left[a_{0}, c\right] \in G_{v}$, then $a+a_{0} \in k$.

Let

$$
V=\left\{c \in \mathcal{C}:[*, c] \in G_{v}\right\} .
$$

It is clear from the above, Lemma 1.2 and the Riemann-Roch theorem that $V$ is a finite-dimensional vector space over $k$. It follows that $F$ is a finite extension of $k$.

Property (ii) implies that $F / k$ is totally inseparable, whence (i). Consequently $F$ satisfies property (iii).

Choose $M_{0} \in F \backslash k$. Then, for all $M \in F$, there exist $\lambda, \mu \in K$ such that

$$
M=\lambda I_{2}+\mu M_{0}
$$

For part (iv) let $L=K F$. Then $L / K$ is a constant field extension (with constant field $F$ ) which by the above is also quadratic. Some of the unusual properties of such extensions (where the constant field extension is inseparable) can be found in [10]. Alternatively see [13, Section 8.6] for the same material in English.

Combining [10, Korollar 9] (or [13, Corollary 8.6.15]) with [10, Lemma 4] (or [13, Theorem 8.6.8]) it follows that $2^{m-1}(=|F: k| /|L: K|)$ divides the degree of every place of $K$.

We note that, in contrast to Theorem 2.13, the (quadratic) extension in Corollary 2.7 (and Corollary 2.12) is separable.

It is known that stabilizers of the type described in Theorem 2.13 do exist. For example, when $g=0$, $\operatorname{char}(k)=2, K$ has no places of degree 1 and $\delta=2$, there are $G_{v} \cong F^{*}$, where $|F: k|=4$. See [6, Lemma 2.5] (with $\operatorname{char}(k)=2$ and $\tau=0$ ). See also [6, Remark 2.7].

The case $m=1$ can also occur. Let for example $k=\mathbb{F}_{2}(t)$ and $\mathcal{C}=k[X, Y]$ with $Y^{2}+X Y=X^{3}+t$. Then by the results of Takahashi [12] the part of the quotient graph corresponding to $X=0$ is a vertex with stabilizer $k(\sqrt{t})$.

The existence of stabilizers of the type $F^{*}$, where $F$ is an inseparable extension of $k$ and $|F: k|>4$ remains an open question. However the existence of such stabilizers is equivalent to the existence of a constant field extension of the type described in the proof of Theorem 2.13 for the following reason. If $L / K$ is a quadratic extension with constant field $F$, then the stabilizer of the vertex defined by the lattice class of $\mathcal{O}_{\infty} \oplus \mathcal{O}_{\infty} \alpha$ with $\alpha \in F \backslash k$ is $F^{*}$. It does not seem to be known whether there exist function fields $K / k$ in characteristic $p$ such that a suitable extension $L / K$ of degree $p$ has a constant field $F$ of degree $p^{m}$ over $k$ with $m>2$.

## Terminology.

(i) When $G_{v}$ is determined by Theorem 2.1(a) we call it rational non-abelian.
(ii) When $G_{v}$ is determined by Theorem 2.3 we call it rational abelian.
(iii) When $G_{v} \cong L^{*}$, as in Corollary 2.7, we call it of type $C M$ (for "complex multiplication").
(iv) When $G_{v} \cong G L_{2}(k)$, as in Corollary 2.8 and Theorem 2.1(b), we call it split quaternionic.
(v) When $G_{v} \cong Q^{*}$, as in Corollary 2.9, we call it non-split quaternionic.
(vi) When $G_{v}$ is determined by Theorem 2.13 we call it inseparable.

Note that if $u, v$ are $G$-equivalent they are of the same type. We conclude this section by separately recording the results for the case where $k$ is quadratically closed. From all the previous results of this section (including the proof of Theorem 2.1) the situation here is considerably simplified.

Theorem 2.14. Suppose that $k$ is quadratically closed. Then either $G_{v}$ is rational or split quaternionic.

Theorems 2.1, 2.3, 2.6 and 2.13 apply to every edge stabilizer $G_{e}$ by virtue of Lemma 1.3. Our results show however that the structure of $G_{e}$ is restricted in some interesting ways.

Proposition 2.15. Suppose that $\delta$ is odd. Let the matrix $M$ lie in $G_{e}$, where $e \in \operatorname{edge}(\mathcal{T})$. Then the eigenvalues of $M$ lie in $k$.

Proof. Let the characteristic polynomial of $M$ be

$$
t^{2}+\rho t+\tau
$$

where $\rho=-\operatorname{tr}(M) \in k$ and $\tau=\operatorname{det}(M) \in k^{*}$. (See Lemma 1.3.) Assume to the contrary that this polynomial is irreducible over $k$. Now there exists $g \in G L_{2}(K)$ such that

$$
g M g^{-1}=\left[\begin{array}{cc}
0 & -1 \\
\tau & -\rho
\end{array}\right] .
$$

Let $C=<M>$. Then $C^{\prime}=g C g^{-1} \leq G L_{2}(k)$ and so

$$
C^{\prime} \leq G_{g(v)} \cap G_{g\left(v^{\prime}\right)}
$$

where $v, v^{\prime}$ are the endpoints of $e($ in $\mathcal{T})$. Suppose now that

$$
\left[\begin{array}{cc}
0 & -1 \\
\tau & -\rho
\end{array}\right] \in G(n, z)
$$

for some $n, z$. We now apply Lemma 1.2. By parts (i) and (ii) it follows that $\nu(z), n \geq 0$. Suppose now that $n>0$. Then by Lemma 1.2 (iii) it follows that $z \equiv \lambda(\bmod \mathfrak{m})$, for some $\lambda \in k$, since $\delta$ is odd. (Recall that the degree of $\mathcal{O} / \mathfrak{m}$ over
$k$ is $\delta$.) This contradicts the irreducibility of the above polynomial. Hence $n=0$ and so $v(n, z)=v(0,0)$. It follows that $g(v)=g\left(v^{\prime}\right)=v(0,0)$ and hence that $v=v^{\prime}$. The result follows.

When $\delta$ is odd therefore the structure of $G_{e}$ is given by Theorems 2.1 and 2.3. For the arithmetic case Corollaries 2.2, 2.4 and 2.12 ensure that Proposition 2.15 reduces to the following.

Corollary 2.16. Suppose that $k=\mathbb{F}_{q}$ and that $\delta$ is odd. Then $\left|G_{e}\right|$ is not divisible by $q^{2}-1$.

Proposition 2.15 has another interesting consequence. In the above terminology any $G_{v}$ containing a matrix with eigenvalues not in $k$ must be of types CM, quaternionic or inseparable. However when $\delta$ is odd $G_{v}$ cannot be non-split quaternionic by Lemma 2.10.

Corollary 2.17. Suppose that $\delta$ is odd and that $G_{v}$ is $C M$ or inseparable. Then $G_{e}$ is trivial (i.e. $\cong k^{*}$ ), for every $e \in \operatorname{edge}(\mathcal{T})$ incident with $v$.

Proof. Theorem 2.13 ensures that in both cases $G_{v} \cong F^{*}$, where $F$ is a quadratic extension of $k$. If $G_{e}$ is not trivial then $G_{e}=G_{v}$ by Lemma 1.4, which contradicts Proposition 2.15.

Nagao's Theorem [9, Corollary, p.87] provides an example (for the case $\delta=1$ ) of a split quaternionic $G_{v}$ and an edge $e$ incident with $v$ for which $G_{e}$ is non-trivial. The restriction on $\delta$ in Proposition 2.15 is necessary.

Proposition 2.18. Suppose that $k=\mathbb{F}_{q}$ and that $\delta$ is even. Let $v \in \operatorname{vert}(\mathcal{T})$ be any vertex for which

$$
G_{v} \cong G L_{2}\left(\mathbb{F}_{q}\right)
$$

(e.g. $v=v(0,0)$ ) Then, for at least one edge e incident with $v,\left|G_{e}\right|$ is divisible by $q^{2}-1$.

Proof. Let $e$ be such an edge. Now $G_{v}$ acts on the $q^{\delta}+1$ edges incident with $v$ and the order of the orbit for this action containing $e$ is $\left|G_{v}: G_{e}\right|$. (See Lemma 3.1.) If $q^{2}-1$ does not divide $\left|G_{e}\right|$ then

$$
\left|G_{e}\right|=q^{s}(q-1)^{t},
$$

where $s \geq 0$ and $t=1,2$ by Corollaries $2.2,2.4$ and 2.6. Then the order of the orbit containing this $e$ is divisible by $q+1$. This cannot apply to all these orbits since then $q^{\delta}+1$ would be divisible by $q+1$, contradicting the fact that $\delta$ is even. The result follows.

## 3. The quotient graph: non-isolated vertices

Throughout this paper the terminology "degree" has been used in the context of algebraic function fields. To avoid confusion in the context of graph theory we will use "valency" instead of "degree" for number of edges attached to a vertex.

Notation. We denote the valency of a vertex $v$ in a graph $\mathcal{G}$ by $\operatorname{val}(v)$.
In this section we are primarily concerned with the values of $\operatorname{val}(\widetilde{v})$. When $\delta=1$ the situation turns out to be surprisingly straightforward. In view of Lemma 1.4 is is clear that all the results of the Theorems 2.1, 2.3, 2.6 and 2.13 apply to all $G_{e}$, where $e \in \operatorname{edge}(\mathcal{T})$.

Now, for each $v \in \operatorname{vert}(\mathcal{T})$, the group $G_{v}$ acts on, $\operatorname{star}(v)$, the set of edges of $\mathcal{T}$ which are incident with $v$. Let $\operatorname{Orbs}(v)$ denote the set of equivalence classes of this action and, for each $e \in \operatorname{star}(v)$, let orb $(e)$ be the equivalence class containing $e$. We recall that $\mathcal{T}$ is a regular graph of valency $\operatorname{card}\left(k_{\infty}\right)+1$. The fact that $G$ acts on $\mathcal{T}$ without inversion leads to the following well-known result.

## Lemma 3.1.

(i) For each $v \in \operatorname{vert}(\mathcal{T})$,

$$
\operatorname{card}(\operatorname{Orbs}(v))=\operatorname{val}(\widetilde{v})
$$

(ii) For each $e \in \operatorname{star}(v)$, there is a one-one correspondence

$$
\operatorname{orb}(e) \longleftrightarrow G_{v} / G_{e}
$$

To repeat from the above the valency of every vertex of $G \backslash \mathcal{T}$ which lies on a ray $R(\sigma)$ (as in Theorem 1.1) is 2 .
We recall that $\mathfrak{R}$ denotes a complete set of representatives for $\mathcal{O} / \mathfrak{m}\left(\cong k_{\infty}\right)$.
Lemma 3.2. Let

$$
g=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \in G(n, z)
$$

and let $u \in \mathfrak{R}$. Then $g$ maps $v\left(n+1, z+u \pi^{n}\right)$ onto $v(n-1, z)$ if and only if

$$
(c z+d)+u c \pi^{n} \in \mathfrak{m}
$$

Proof. Note that $c z+d, c \pi^{n} \in \mathcal{O}$ by Lemma 1.2. The result follows from Lemma 3.1.

Our next result is essential for dealing with rational stabilizers.
Lemma 3.3. Suppose that $G(n, z)$ is rational. Then there exist $n^{\prime}, z^{\prime}$ such that
(i) $v\left(n^{\prime}, z^{\prime}\right) \equiv v(n, z)(\bmod G)$,
(ii) $G\left(n^{\prime}, z^{\prime}\right)$ consists of all matrices of the form $M\left(n^{\prime}, z^{\prime}, \alpha, \beta, c\right)$.

Proof. Let

$$
g=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \in G(n, z) .
$$

From the proofs of Theorems 2.1 and 2.3 it is clear that every element of $G(n, z)$ is of the form $M(n, z,-,-,-)$ unless, for all $g \in G(n, z)$, either (i) $b=0$ or (ii) $c=0$.

Let $v\left(n^{\prime}, z^{\prime}\right)=g_{0} v(n, z)$, where, for any nonzero $c^{\prime} \in \mathcal{C}$,

$$
\text { (i) } g_{0}=\left[\begin{array}{ll}
1 & c^{\prime} \\
0 & 1
\end{array}\right] \text { and (ii) } g_{0}=\left[\begin{array}{cc}
1 & 0 \\
c^{\prime} & 1
\end{array}\right] \text {. }
$$

Then $G\left(n^{\prime}, z^{\prime}\right)=g_{0} G(n, z) g_{0}^{-1}$ has the required properties.
For now we will focus on rational stabilizers. Our next result shows that isolated vertices in $G \backslash \mathcal{T}$ never arise from such vertices.

Theorem 3.4. Suppose that $G_{v}$ is rational. Then

$$
\operatorname{val}(\widetilde{v}) \geq 2
$$

Proof. Let $v=v(n, z)$. By Lemma 3.3 we may assume that every element of $G(n, z)$ is of the form

$$
M(n, z, \alpha, \beta, c) .
$$

Then

$$
G(n, z) \leq G(n+1, z)
$$

by Lemma 1.2. On the other hand for each

$$
g=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \in G(n, z)
$$

$c z+d \in k^{*}$. It follows from Lemma 3.2 that there does not exist $g^{\prime} \in G(n, z)$ such that $g^{\prime} v(n-1, z)=v(n+1, z)$. The result follows from Lemma 3.1.

We will require the following special case.
Lemma 3.5. For any $z$, either $G(0, z) \cong G L_{2}(k)$ or

$$
G(0, z)=\left\{\left[\begin{array}{cc}
\alpha & \beta \\
0 & \alpha
\end{array}\right]: \alpha \in k^{*}, \beta \in k\right\} .
$$

Proof. If $\nu(z) \geq 0$, then $G(0, z)=G(0,0)=G L_{2}(k)$. We may suppose then that $\nu(z)<0$.

Let

$$
g=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \in G(0, z) .
$$

Now $c \in k$ by Lemma 1.2. Suppose that $c \neq 0$. By Lemma $1.2 \nu(a-c z) \geq 0$. It follows that $G(0, z)=G\left(0, a^{\prime}\right)$, for some $a^{\prime} \in \mathcal{C}$. Then $g^{\prime} v\left(0, a^{\prime}\right)=v(0,0)$, where

$$
g^{\prime}=\left[\begin{array}{cc}
1 & -a^{\prime} \\
0 & 1
\end{array}\right]
$$

We may assume then that $c=0$. Hence by Lemma $1.2 a, d \in k^{*}$ and $\nu(b+z(a-d)) \geq$ 0 . If $a \neq d$ then by a previous argument $G(0, z) \cong G L_{2}(k)$. Otherwise $a=d$ and $b \in k$.

The latter possibility occurs, for example, when $\nu(z)$ is a gap number for $\nu$ [11, I.6.7, p.32]. From now on nearly all our results will apply to the case where $\delta=1$, i.e. when $\mathcal{O} / \mathfrak{m} \cong k$. Here a stronger version of Theorem 3.4 holds.

Theorem 3.6. Suppose that $\delta=1$. If $G_{v}$ is rational non-abelian, then

$$
\operatorname{val}(\widetilde{v})=2 \text { or } 3 .
$$

Proof. Let $v=v(n, z)$. By Lemma 3.3 we may suppose that every element of $G(n, z)$ is of the form $M(n, z,-,-,-)$. If

$$
g=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \in G(n, z)
$$

where $n<0$, then $c=0$ by Lemma 1.2. It follows form this and Lemma 3.5 that $n \geq 1$. As in Theorem 3.4

$$
G(n, z) \leq G(n+1, z)
$$

Let $\rho, \epsilon \in k^{*}$. Choose $g_{0}=M\left(n, z, \alpha_{0}, \beta_{0},-\right) \in G(n, z)$ such that $\alpha_{0} \rho=\beta_{0} \epsilon$. Then it is easily verified that

$$
g_{0} v\left(\pi^{n+1}, z+\rho \pi^{n}\right)=v\left(\pi^{n+1}, z+\epsilon \pi^{n}\right)
$$

i.e.

$$
v\left(\pi^{n+1}, z+\rho \pi^{n}\right) \cong v\left(\pi^{n+1}, z+\epsilon \pi^{n}\right)(\bmod G(n, z)) .
$$

However by Lemma 3.2

$$
v(n-1, z) \not \approx v(n+1, z)(\bmod G(n, z)) .
$$

The result follows from Lemma 1.8.

We repeat that the valency of the projection of a vertex of $\mathcal{T}$ which projects onto those of one of the rays (in Serre's Theorem) is 2. Takahashi [12] however has shown that the valency 3 can occur for rational non-abelian $G_{v}$ (when $g=\delta=1$ ).

Theorem 3.7. Suppose that $\delta=1$. If $G_{v}$ is rational abelian, then

$$
\operatorname{val}(\widetilde{v})=2 \quad \text { or } \quad \operatorname{val}(\widetilde{v})=\operatorname{card}(k)+1 .
$$

Proof. If $G_{v}$ is trivial (i.e. $G_{v} \cong k^{*}$ ), then by Lemma 3.1 each $G_{v}$-orbit on $\operatorname{star}(v)$ contains only one element. We may suppose then by Lemma 3.3 that $G_{v}$ consists of elements of the form $M(n, z, \alpha, \beta, c)$, where for some $c \neq 0$. By an argument used in the proof of Theorem 3.6, together with Lemma 3.5, we may further assume that $n \geq 1$.

As before $G(n, z) \leq G(n+1, z)$ and

$$
v(n-1, z) \not \equiv v(n+1, z)(\bmod G(n, z)),
$$

by Lemma 3.2. Suppose that there exists $c_{0} \in \mathcal{C}(n) \backslash \mathcal{C}(n-1)$ for which $M\left(n, z, \alpha, \beta, c_{0}\right) \in$ $G(n, z)$, where $\alpha \neq \beta$. Then using this element in conjunction with Lemma 1.4 it follows from Lemma 3.2 that

$$
v(n-1, z) \equiv v\left(n+1, z+\lambda \pi^{n}\right)
$$

for all $\lambda \in k$. In this case

$$
\operatorname{val}(\widetilde{v})=2
$$

We are left with the case where $c \in \mathcal{C}(n-1)$, for all $M(n, z, \alpha, \beta, c) \in G(n, z)$. Here it is easily verified that $G(n, z) \leq G(n-1, z)$ and that

$$
G(n, z) \leq G\left(n+1, z+\lambda \pi^{n}\right)
$$

for all $\lambda \in k$. By Lemmas 3.1 and 1.8 it follows that

$$
\operatorname{val}(\widetilde{v})=\operatorname{card}(k)+1
$$

Examples of vertices with trivial stabilizers are given in the previous section. In addition Takahashi [12] (for the case $g=\delta=1$ ) has shown the existence of rational abelian stabilizers $G_{v}\left(\cong k^{*} \times k^{+}\right)$for which $\operatorname{deg}(\widetilde{v})=2$.

## 4. The quotient graph: isolated vertices

Definition. A vertex $\widetilde{v} \in G \backslash \mathcal{T}$ is isolated if and only if

$$
\operatorname{val}(\widetilde{v})=1
$$

By Lemma 3.1 for such a vertex we have

$$
G_{v} / G_{e} \longleftrightarrow \operatorname{card}\left(k_{\infty}\right)+1,
$$

for all $e \in \operatorname{star}(v)$. Isolated vertices always exist when $\delta=1$. For example the image of $v(0,0)$ is always isolated in this case [9, Exercise 6, p.99]. (Its stabilizer is $G L_{2}(k)$.) On the other hand $G \backslash \mathcal{T}$ need not have any isolated vertices. If $g=0$ and $K$ has a place of degree 1 (so that $K \cong k(t)$ ) then the results of [4] show that this holds for all $\delta>1$. We will prove that when $\delta=1$ isolated vertices arise from non-rational stabilizers.

Theorem 4.1. Suppose that $\delta=1$ and that, for some $v \in \operatorname{vert}(\mathcal{T}), G_{v}$ contains a matrix $M$ with no eigenvalues in $k$. Then
(i) $\operatorname{val}(\widetilde{v})=1$;
(ii)

$$
I(M) \cap G_{e}=Z,
$$

for all $e \in \operatorname{star}(v)$
Proof. Let $v=v(n, z)$ as above and let $e$ be the edge joining $v(n, z)$ and $v(n-1, z)$. Now $G_{v}$ contains all matrices of the form

$$
\alpha I_{2}+\beta M,
$$

where $\alpha, \beta \in k$, with $(\alpha, \beta) \neq(0,0)$, by Lemma 1.4. Let

$$
M=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]
$$

Let $u \in \mathfrak{R}$. Since $\delta=1$ it is then clear that there exist $\alpha^{\prime} \in k, \beta^{\prime} \in k^{*}$ such that

$$
\alpha^{\prime}+\beta^{\prime}(d+c z)+u \beta^{\prime} \pi^{n} c \in \mathfrak{m} .
$$

Then by Lemmas 1.8, 3.1 and 3.2 all edges in $\operatorname{star}(v)$ are $G(n, z)$-equivalent and so $\widetilde{v}$ is isolated.

For part (ii) suppose that $I(M) \cap G_{e} \neq Z$. Then $I(M) \leq G_{v_{0}}$, where $v_{0}=v(n-1, z)$, by Lemma 1.5, which contradicts the proof of part (i). Hence $I(M) \cap G_{e}=Z$. Let $e^{\prime} \in \operatorname{star}(v)$. From the proof of part (i) there exists $g \in G_{v}$ such that $e^{\prime}=g(e)$. Part (ii) follows.

Theorem 4.1 applies to all quaternionic and inseparable $G_{v}$, where $k$ is not quadratically closed. The restriction on $\delta$ in Theorem 4.1 is necessary. For the case where $g=0, K$ has no degree 1 places and $\delta=2$ it is known [6, Corollary 2.10] that there exists $e \in \operatorname{edge}(\mathcal{T})$ for which $G_{e}=I(M)(\neq Z)$. It is also known from this case [6, Lemma 2.8] that isolated vertices can occur when $\delta \neq 1$. We do however prove the following partial converse.

Theorem 4.2. Suppose that $G_{v}=I(M)$, where $M$ has no eigenvalues in $k$. Then

$$
\operatorname{val}(\widetilde{v})=1 \Longleftrightarrow \delta=1
$$

Proof. Now $\widetilde{v}$ is isolated when $\delta=1$ by Theorem 4.1. Assume then that $\widetilde{v}$ is isolated. Let $v=v(n, z)$ and

$$
M=\left[\begin{array}{ll}
e & f \\
g & h
\end{array}\right]
$$

By a previous argument we may assume that $n \geq 1$. Let $u \in \mathfrak{R}$. As in the proof of Theorem 2.9 there exist $\alpha \in k, \beta \in k^{*}$ such that

$$
\alpha+\beta(h+g z)+\beta u \pi^{n} g \in \mathfrak{m} .
$$

We recall from Lemma 1.2 that $\nu(g) \geq-n$. Suppose that $\nu(g)>-n$. Then $G_{v} \leq G(n-1, z)$ by Lemmas 1.2 and 1.4 which implies that $\widetilde{v}$ cannot be isolated. Hence $\pi^{n} g \in \mathcal{O} \backslash \mathfrak{m}$. It follows that there exist $\gamma \in k$ and constants $r, s \in \mathcal{O}$ such that

$$
u \equiv \gamma r+s(\bmod \mathfrak{m})
$$

Hence $\delta$, the degree of $\mathcal{O} / \mathfrak{m}$ over $k$, is 1 .
For the case where $g=0, K$ has no degree 1 places and $\delta=2$ it is known there exists $v \in \operatorname{vert}(\mathcal{T})$ for which (i) $\widetilde{v}$ is isolated and (ii) $G_{v} \neq I(M)$. See [6, Lemmas 2.5, 2.8].

Corollary 4.3. Suppose that $\delta=1$ and let $\left\{\widetilde{v_{\lambda}}: \lambda \in \Lambda\right\}$ be the set of all $\widetilde{v_{\lambda}} \in \operatorname{vert}(G \backslash \mathcal{T})$ such that

$$
G_{v_{\lambda}}=I\left(M_{\lambda}\right)
$$

for some $M_{\lambda}$ with no eigenvalues in $k$. Let $L_{\lambda}$ be the quadratic extension of $k$ generated by the eigenvalues of $M_{\lambda}$. Then there exists an epimorphism

$$
\theta: G \rightarrow \underset{\lambda \in \Lambda}{*}\left(L_{\lambda}^{*} / k^{*}\right) .
$$

Proof. By Theorem 4.1

$$
\operatorname{val}\left(\widetilde{v_{\lambda}}\right)=1
$$

Let $e_{\lambda} \in \operatorname{edge}(\mathcal{T})$ be a lift of its adjacent edge to one adjacent to $v_{\lambda}(\in \operatorname{vert}(\mathcal{T}))$. Then $G_{e_{\lambda}}=Z$ by Theorem 4.1.

From the fundamental theorem of the theory of groups acting on trees [9, Theorem 13, p.55] $G$ is isomorphic to the fundamental group of the graph of groups given by $G \backslash \mathcal{T}$. (See [9, p.42].) Hence, for each $\lambda \in \Lambda$,

$$
G / Z \cong\left(G_{v_{\lambda}} / G_{e_{\lambda}}\right) * H
$$

where $H$ is non-trivial. The result follows.
Even when $\delta=1$ it can happen that $\Lambda=\emptyset$. For the simplest case consider $g=0, \delta=1$ (so that $\mathcal{C} \cong k[t]$ ). By Nagao's theorem, [9, Corollary, p.87] if $G_{v}$ is non-rational then, $G_{v}$ is split quaternionic.

However Corollary 4.3 does occur. Suppose, once again, that $g=\delta=1$. Then $\mathcal{C}$ is the coordinate ring of an elliptic curve. In which case there exist $X, Y \in \mathcal{C}$, where $\nu(X)=-2$ and $\nu(Y)=-3$ which satisfy a Weierstrass equation, $F(X, Y)=0$, for which

$$
\mathcal{C}=k[X, Y] .
$$

Takahashi [12, Theorem 5] has shown that, if for some $\lambda \in k$ there does not exist any $\mu \in k$ for which $F(\lambda, \mu)=0$, then

$$
G\left(2, \pi^{-1}+\pi \lambda\right) \cong k(\omega)^{*}
$$

where $k(\omega)$ is the quadratic extension of $k$ given by any $\omega$ for which $F(\lambda, \omega)=0$. An explicit example can be found in [12, p.87].

## 5. Special constant fields

We list separately the results for the arithmetic case.
Theorem 5.1. Suppose that $k=\mathbb{F}_{q}$. Let $v \in \operatorname{vert}(\mathcal{T})$. Then $\operatorname{val}(\widetilde{v})=1$ if and only if
(i) $\delta=1$,
(ii) $\mathbb{F}_{q^{2}}^{*} \hookrightarrow G_{v}$.

Proof. The conditions are sufficient by Theorem 4.1 and Corollary 2.12. Suppose now that $\widetilde{v}$ is isolated. Then $q^{\delta}+1$ divides $\left|G_{v}\right|$ by Lemma 3.1. (We recall that $\mathcal{O} / \mathfrak{m} \cong k_{\infty}$ and that $\left|k_{\infty}: k\right|=\delta$.) From the list of all possible $\left|G_{v}\right|$ listed in Corollaries 2.2, 2.4 and 2.12 it follows that $\delta=1$. If the second condition is not satisfied then by the same results $G_{v}$ is of rational type. Then $\widetilde{v}$ cannot be isolated by Theorem 3.4.

The significant difference here from the non-arithmetic case is that when $k$ is finite isolated vertices arise only when $\delta=1$. We repeat that, from [6, Lemma 2.8], it is possible, when $k$ is infinite, for $G \backslash \mathcal{T}$ to have an isolated vertex when $\delta>1$. From the results of the two previous sections we can state.

Corollary 5.2. Suppose that $k=\mathbb{F}_{q}$ and that $\delta=1$. Let $v \in \operatorname{vert}(\mathcal{T})$. Then

$$
\operatorname{val}(\widetilde{v})=1,2,3 \text { or } q+1
$$

The possible patterns of orbit sizes are, respectively, $(q+1),(1, q),(1,1, q-1)$ and $(1, \cdots, 1)$. Takahashi [12] has shown (for the case $g=\delta=1$ ) that all possibilities can occur. We now record the arithmetic version of Corollary 4.3. Recall that here $X$ is finite.

Corollary 5.3. Suppose that $k=\mathbb{F}_{q}$ and that $\delta=1$. Let $\left\{\widetilde{v}_{i}: 1 \leq i \leq n\right\}$ be the set of all $\widetilde{v}_{i} \in \operatorname{vert}(G \backslash \mathcal{T})$ such that

$$
G_{v_{i}} \cong \mathbb{F}_{q^{2}}^{*} .
$$

Then there exists an epimorphism

$$
\theta: G \rightarrow \mathbb{Z} /(q+1) \mathbb{Z} * \cdots * \mathbb{Z} /(q+1) \mathbb{Z}
$$

to $n$ factors.
Takahashi [12, p.87] provides an example of Corollary 5.3 , for the case $k=\mathbb{F}_{3}$, where $n=1$. Here then

$$
\theta: G / Z \rightarrow \mathbb{Z} / 4 \mathbb{Z}
$$

In [7] we will, among other things, derive more information about $n$. This requires, however, quite a lengthy detour and the use of arguments from algebraic number theory.

In conclusion we deal with the one case yet to be considered.
Theorem 5.4. Suppose that $k$ is quadratically closed and that $\delta=1$. If $G_{v}$ is split quaternionic, (i.e. $\cong G L_{2}(k)$ ), then

$$
\operatorname{val}(\widetilde{v})=1
$$

Proof. From the proof of Theorem 2.1, part (b), by conjugating $G_{v}^{\prime}$ by

$$
\left[\begin{array}{ll}
0 & b_{0} \\
c_{0} & 0
\end{array}\right]
$$

(where $b_{0} c_{0}=1$ ) it follows that $G_{v}$ is conjugate in $G L_{2}(K)$ to $G L_{2}(k)$. Then

$$
G L_{2}(k)=G(n, z),
$$

for some $n, z$.

As before we may assume that $n \geq 0$ since $G L_{2}(k)$ is not metabelian. It follows from Lemma 1.2 that $\nu(z) \geq 0$. Suppose that $n \geq 1$. Then by Lemma 1.2

$$
b+z(a-d)-z^{2} c \in \mathfrak{m}
$$

for all

$$
\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \in G L_{2}(k) .
$$

Hence $n=0$ and so

$$
G(n, z)=G(0,0)
$$

The result follows from [9, Exercise 6, p.99].

## 6. Concluding remarks

Let $\mathcal{T}_{m}$ be a maximal subtree of $G \backslash \mathcal{T}$ and let
$\omega=\operatorname{card}\left(\left\{e \in \operatorname{edge}(G \backslash \mathcal{T}): e \notin \operatorname{edge}\left(\mathcal{T}_{m}\right)\right\}\right)=\operatorname{card}\left(\left\{e \in \operatorname{edge}(X): e \notin \operatorname{edge}\left(\mathcal{T}_{m}\right)\right\}\right)$.
Now we put

$$
G_{V}=\left\langle G_{v}: v \in \operatorname{vert}(\mathcal{T})\right\rangle
$$

From standard Bass-Serre theory [9, p.43] it is known that

$$
G / G_{V} \cong F_{\omega},
$$

the free group of rank $\omega$. This raises the following question.

$$
\text { When is } G \backslash \mathcal{T} \text { a tree? }
$$

The theory of Drinfeld modular curves provides a complete answer [5] when $k$ is finite. When $k$ is infinite very little is known. If $g=0$ and $K$ has a place of degree one (so that $K \cong k(t)$ ) a complete answer is known [4]. It is also known [6] that $G \backslash \mathcal{T}$ is a tree when $g=0, K$ has no places of degree one (i.e. $K \not \approx k(t)$ ) and $\delta=2$. Takahashi [12] has shown that the quotient graph is a tree for the elliptic case $g=\delta=1$.

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