Compatibility JSJ decomposition of graphs of free abelian groups

Benjamin Beeker

June 22, 2018

Abstract

A group G is a $vGBS$ group if it admits a decomposition as a finite graph of groups with all edge and vertex groups finitely generated and free abelian. We describe the compatibility JSJ decomposition over abelian groups. We prove that in general this decomposition is not algorithmically computable.

1 Introduction

The theory of JSJ splittings starts with the work of Jaco-Shalen and Johansson on orientable irreducible closed 3-manifolds giving a canonical family of 2-dimensional tori. Kropholler first introduced the notion into group theory giving a JSJ decomposition for some Poincaré duality groups [10]. Then Sela gave a construction for torsion-free hyperbolic groups [13]. This notion has been more generally developed by Rips and Sela [12], Dunwoody and Sageev [4], Fujiwara and Papasoglu [6] for various classes of groups.

However, in general, given a group G and a class of subgroups, there is not a unique JSJ splitting of G over these subgroups. Guirardel and Levitt define in [8] the JSJ deformation spacse which generalize the previous notions. They also introduce in [9] a new object called the compatibility JSJ splitting of G over these subgroups. Unlike the usual JSJ splittings, the compatibility JSJ splitting is unique, and so is invariant under is automorphisms. However, it is often harder to construct it, the purpose of this article is to give some positive and negative results about the constructibility of this object.

In this paper we focus on the construction of the compatibility JSJ tree over abelian groups for the following class of groups. Let G be a group acting on a simplicial tree T with free abelian vertex (and edge) stabilizers. We call such a group a Generalized Baumslag-Solitar group of variable rank, or $vGBS$ group, and such a tree T a $vGBS$ (G-)tree. We call GBS_n the collection of groups which admits an action on a simplicial tree with vertex stabilizers $\mathbb{Z}^n$ with n a fixed integer. This two classes of groups generalize the one of GBS_1 groups (or

GBS groups) introduced by Forester in [5] as examples of groups for which we do not have a canonical (usual) JSJ tree.

Recall that an element of G is *elliptic* if it fixes a vertex of T, and *hyperbolic* otherwise. Given a hyperbolic element $g \in G$, it acts by translation on a line $\mathcal{A}_g$ of T called the *axis* of g or its *characteristic space*. The characteristic space of an elliptic element is the set of its fixed points. A subgroup of G is *elliptic*, if it is included in the stabilizer of a vertex. A subgroup of G is *universally elliptic* if it is elliptic in every G-tree. With no restriction on the G-trees, almost any tree can be universally elliptic, we thus consider G-trees whose edge groups are included in a set $\mathcal{A}$ of subgroups of G. We then talk about G-trees over $\mathcal{A}$, and a subgroup is $\mathcal{A}$-universally elliptic if it is elliptic in every G-tree over $\mathcal{A}$.

Given two G-trees T and T', the tree T *refines* T' if T' may be obtained from T by equivariantly collapsing edges. The tree T *dominates* T' if every elliptic group of T is elliptic in T'. In particular, if T refines T' then it also dominates it. We say that T and T' are *compatible* if there exists a G-tree T'' which refines both T and T'. A G-tree is $\mathcal{A}$-*universally compatible* if it is compatible with every G-tree (over $\mathcal{A}$).

A G-tree over $\mathcal{A}$ is a *JSJ tree* over $\mathcal{A}$ if it is $\mathcal{A}$-universally elliptic and dominates every universally elliptic G-tree over $\mathcal{A}$. A G-tree over $\mathcal{A}$ is a *compatibility JSJ tree* over $\mathcal{A}$ if it is $\mathcal{A}$-universally compatible and dominates every universally compatible G-tree over $\mathcal{A}$.

The present paper splits into two parts. We first describe the compatibility JSJ tree over $\mathbb{Z}^n$ groups of the GBS_n groups. In the second part, we describe the compatibility JSJ tree over abelian groups of the *vGBS* groups.

Let G be a GBS_n group. We propose to describe the compatibility JSJ tree over $\mathbb{Z}^n$ groups in the following sense. Starting from a JSJ tree T of G over $\mathbb{Z}^n$ groups (except for some degenerated cases, any GBS_n tree is a JSJ tree [1]), we explicit a set of edges and a set of vertices such that the compatibility JSJ tree is obtained by expanding (in a precise way) these vertices and collapsing these edges in T.

The compatibility relation is very restrictive. For example, two G-trees with an isomorphic quotient graph of groups may be not compatible. Take the *GBS* group $\mathbb{Z} *_{2\mathbb{Z}} \mathbb{Z} *_{4\mathbb{Z}} \mathbb{Z}$. Its JSJ deformation space contains infinitely many reduced trees and any reduced JSJ tree of this deformation space have same isomorphic quotient graphs of groups. However, each of these reduced JSJ tree is compatible with exactly two others reduced JSJ trees.

Here are the two main examples of edges to collapse. The labelled graphs represent *GBS* groups, and must be understood in the following manner: all vertices and edges carry the infinite cyclic group $\mathbb{Z}$, the number n at the end of each edge indicates that the injection of the edge group into the group of its endpoint is $k \mapsto nk$.

In Figure 1, the two graphs of groups have isomorphic fundamental group, and are related by a slide along the edge denoted by $\mathfrak{e}$. In this case, the edge $\mathfrak{e}$

is called *slippery* (see Section 2 for the complete definition) and is collapsed in the compatibility JSJ. In fact, after collapsing the edge $\mathfrak{e}$, we exactly obtain a compatibility JSJ over group $\mathbb{Z}$.
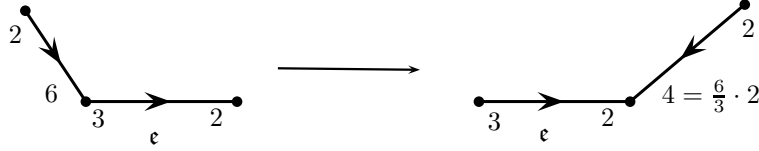
Figure 1: The two graphs of groups are related by a slide.

In Figure 2, the two graphs have isomorphic fundamental groups, and are related by a deformation called *induction along* $\mathfrak{e}$ (see definition in Section 2). In this case, the edge $\mathfrak{e}$ is called *strictly ascending*. Collapsing $\mathfrak{e}$, we obtain a compatibility JSJ.
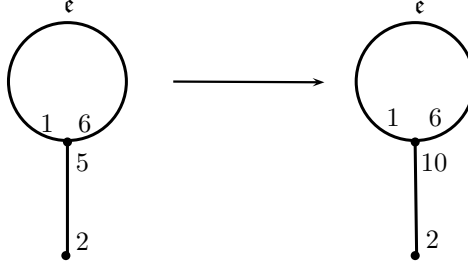
Figure 2: The two graphs of groups are related by an induction.

Given a usual abelian JSJ tree of a GBS_n group, we obtain a universally compatible tree by collapsing four types of edges. The definitions of these edges is technical and described in Section 2. .

Proposition 1.1. *Let G be a GBS_n group. Let T be a reduced abelian JSJ tree of G. Let $\mathcal{E}$ be the set of edges containing the vanishing edges of T, the potentially strictly ascending edges of T, the non-ascending slippery edges of T, the toric 2-slippery edges of T.*

1. *The tree T' obtained from T by collapsing the edges of $\mathcal{E}$ is compatible with every G-tree over $\mathbb{Z}^n$ groups.*
2. *Let T'' be a collapse of T in which an edge of $\mathcal{E}$ is not collapsed. Then T'' is not compatible with every G-tree over $\mathbb{Z}^n$ groups.*

3

However the tree T' is not always the compatibility JSJ tree: some vertices could have to be expanded. Roughly speaking, some vertices could act as dead end: no edge "arriving" at this vertex by a slide may continue to slide further. These vertices must be expanded in the compatibility JSJ tree in a precise way called a *blow up* (see Section 2). The vertex $\mathfrak{v}$ of Figure 3 is an example. The top edge may slide along $\mathfrak{e}$ or along $\mathfrak{f}$, but after performing one of these slides, no new slide is allowed (except the converse one). The compatibility JSJ is obtained by collapsing $\mathfrak{e}$ and $\mathfrak{f}$ (which are slippery) in the graph of groups on the right. The complete description of *dead ends* is given in Section 2.
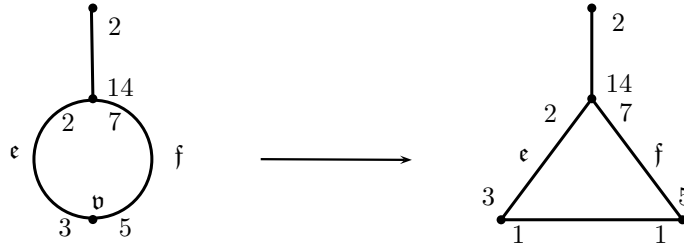

Figure 3: The edges $\mathfrak{e}$ and $\mathfrak{f}$ are slippery and $\mathfrak{v}$ is a dead end.

Theorem 1.2. *Let G be a GBS_n group. Let T be a reduced abelian JSJ tree of G. Call T_{comp} the G-tree obtained from T by blowing up up the dead ends and collapsing the vanishing edges of T, the p.s.a. edges of T, the non-ascending slippery edges of T, the toric 2-slippery edges of T.*

Then T_{comp} is a compatibility JSJ tree over $\mathbb{Z}^n$ groups.

This construction is algorithmic whenever no vertex group is conjugated to one of its proper subgroup. In the general case, the decidability of the construction is unknown.

The GBS_n groups also admit splittings over $\mathbb{Z}^{n+1}$ groups. If for the usual JSJ tree, this does not have any incidence - the abelian JSJ tree and the JSJ tree over $\mathbb{Z}^n$ are isomorphic -, it has one for the compatibility JSJ tree. For example, applying Theorem 1.2, we may easily see that the compatibility JSJ tree over $\mathbb{Z}^n$ groups of $BS(2,2) = \langle a, t | ta^2 t^{-1} = a^2 \rangle$ is the one associated to the HNN extension, but this tree is not compatible with the tree associated to the amalgamated product

$$BS(2,2) = \langle a, s | sa^2 s^{-1} = a^2 \rangle *_{a^2 = b, s = t^2} \langle b, t | bt = tb \rangle.$$

The incompatibility comes from the fact that the element t is hyperbolic in the first splitting but stabilizes an edge in the second. Such an element is said to be *potentially bi-elliptic*.

In the case of *vGBS* groups, to obtain universally compatible tree over abelian groups, it suffices to collapse in any JSJ tree the four types of edges described previously, and every edge in the axis of a potentially bi-elliptic element. Again, the obtained tree is not always the abelian compatibility JSJ tree, even if we blow up the dead ends. Some other edges have to be expanded:

In some very specific cases (see description in Section 4), the axes of potentially bi-elliptic elements must be separated from the rest of the tree, and then the axes must be collapsed (see Figure 4).
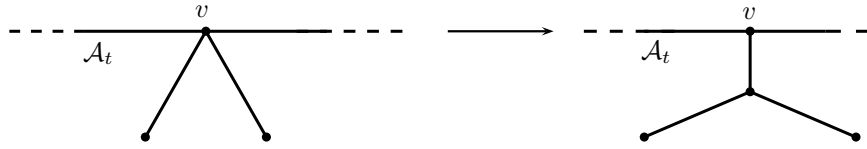

Figure 4: The axis $\mathcal{A}_t$ of a potentially bi-elliptic element t must be "taken away" from the rest of the tree.

These separations are called *expansions (of inert edges)*. We obtain the following theorem.

Theorem 1.3. *Let G be a vGBS group. Let T be a reduced abelian JSJ tree of G. Call $\mathcal{E}$ the set of non-ascending slippery, potentially strictly ascending and toric 2-slippery edges.*

We define T_{ab} as the G-tree obtained by expanding inert edges, blowing up dead ends, then collapsing every edge of $\mathcal{E}$ and the axes of potentially bi-elliptic elements.

Then T_{ab} is an abelian compatibility JSJ tree of G.

This gives a description of the abelian compatibility JSJ tree. However this construction is not and cannot be algorithmic.

Theorem 1.4. *There is no algorithm that constructs the abelian compatibility JSJ tree of vGBS groups.*

This inconstructibility follows from the fact that in *vGBS* trees, we may not dectect whether an edge is slippery or not. We describe this problem in the last section.

2 Preliminaries

Let G be a finitely generated group. A *G-tree* is a simplicial tree T equipped with an action of G that we suppose without inversion (the stabilizer of an edge

is included in the stabilizer of its endpoints) and minimal. Given an (oriented) edge e, the opposite edge is denoted by $\bar{e}$. We denote by G_e the stabilizer of e. We have $G_e = G_{\bar{e}}$. The stabilizer of a vertex v is denoted by G_v. The orbit of a vertex v and an edge e is denoted by Gothic font $\mathfrak{v}$ and $\mathfrak{e}$. Similarly the opposite orbit of $\mathfrak{e}$ is denoted by $\bar{\mathfrak{e}}$. The *valency* of a vertex v (or an orbit of vertices $\mathfrak{v}$) is the number of orbits of edges with initial vertex in $\mathfrak{v}$.

An edge is *reduced* if its endpoints are in the same orbit under the action of G or its stabilizer is strictly included in both stabilizers of its endpoints. An orbit of edges is *reduced* if one (or equivalently every) representative is reduced. A tree is reduced if all its edges are reduced. The *deformation space* $\mathcal{D}_T$ of T is the set of G-trees having the same set of elliptic subgroups as T. The *reduced deformation space* of T is the subset of reduced trees of $\mathcal{D}_T$.

To *collapse* an orbit of edges $\mathfrak{e}$ consists in collapsing every connected component of edges in the orbit $\mathfrak{e}$ to a point. This operation produces a new G-tree. The converse of collapsing is called *expanding*. Given a G-tree T and an orbit of edges $\mathfrak{e}$, the tree obtained by collapsing $\mathfrak{e}$ is in the deformation space of T if and only if $\mathfrak{e}$ is not reduced. We may thus construct a reduced G-tree in the deformation space of T by collapsing one by one orbits of non-reduced edges until the tree is reduced (we assume here that number of orbits of edges is finite). Note that if we collapse at the same time all non reduced edges, the obtained tree may be in a different deformation space.

Whitehead moves for G-trees   In [3], Clay and Forester describe three deformation moves on G-trees such that any two reduced G-trees of a given deformation space are related by a finite sequence of these three moves, and every intermediate tree in this sequence is reduced.

The first deformation move is the *slide* of an edge e along an edge f: if two edges e and f are such that

- e is not in the orbit of f or $\bar{f}$,
- the terminal vertex of e is equal to the initial vertex v of f,
- the stabilizer of e is included in the stabilizer of f,

then e may slide along f. Call w the terminal vertex of f. The new G-tree is obtained by changing the terminal vertex of every vertex $g \cdot e$ in the orbit of e from $g \cdot v$ to $g \cdot w$ (see Figure 5 for the changes on the associated graph of groups). The case were $\mathfrak{v} = \mathfrak{w}$ is not excluded. This move does not change the stabilizers of vertices and edges. This move is fully determined by the data of e and f. Note that the converse move is also a slide.

The slide of e along f will be denoted by e/f.

The second deformation move is the *induction*. Let e be an edge with initial and terminal vertices v and w and A a subgroup of G_w such that
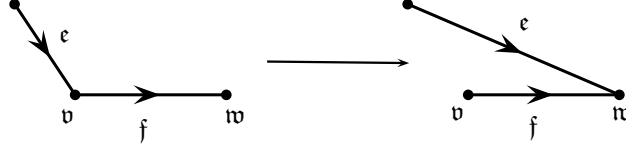
Figure 5: Slide of 𝔢 along 𝔣

- v and w are in the same orbit,
- the stabilizers G_v and G_e are equal,
- the group A contains G_e.

Then we may perform an induction on e with group A. We first add an edge f with terminal vertex w, a new initial vertex v' which is also the new terminal vertex of e and with $G_f = G_{v'} = A$. Every edge of another orbit with initial vertex w keeps w as initial vertex. We then collapse e (see Figure 6). This description is given around the edge e but must be made equivariantly.

This move does not change the underlying graph, but the edge e and the vertex v have been replaced by an edge f and a vertex v' with a distinct stabilizer A. It is fully determined by the data of e and the group A.

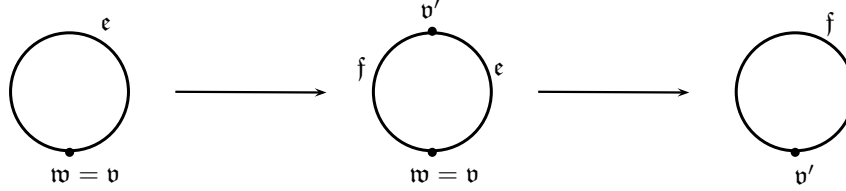The induction on the edge e with group A will be denoted $i_A(e)$ or e/e if we do not precise the group.

Figure 6: Induction on 𝔢

If we have an inclusion of groups $G_e \subset A \subset B \subset G_w$, we may first perform an induction on A and then an induction on B. The composition of these two inductions is equal to the induction on B: $i_A(e) \cdot i_B(f) = i_B(e)$ (where f is the edge appearing in the induction on e).

The induction with $A = G_e$ has no effect on the G-tree. We call this move a *trivial induction*. At the opposite we may take $A = G_w$. In this case performing the induction is the same as sliding along *e*every edge f (not in the orbit of e or $\bar{e}$) with terminal vertex v.

The converse move of an induction with any group is just an induction with group G_w and slides along $\bar{e}$. Indeed from the first remark, we have $i_A(e) \cdot i_{G_w}(f) = i_{G_w}(e)$. And by the second remark, the move $i_{G_w}(e)$ is the same as a

finite sequence of slides.

By extension, performing the converse of an induction with group A consists in performing an induction with group A and then sliding every edge around w along $\bar{e}$. This move is possible whenever, for every edge g with initial vertex w not in the orbit of $\mathfrak{e}$, the group A contains G_g.

The third move is the $\mathcal{A}^{\pm 1}$-*move*. We first describe the $\mathcal{A}^{-1}$-move. Suppose that e is an edge with initial and terminal vertices v and w, and that f is an edge with terminal vertex w and initial vertex not in $\mathfrak{w}$, such that

- v and w are in the same orbit,
- the stabilizers G_v and G_e are equal,
- for every edge g with initial vertex w not in the orbit of $\mathfrak{e}$, the stabilizer of f contains G_g.

We may then perform an $\mathcal{A}^{-1}$-move on e by first performing the converse of an induction on e with group G_f and then collapsing f which is now non reduced (see Figure 7 for the changes on the underlying graph of groups). We will say that v (and $\mathfrak{v}$) is a *vanishing vertex* and f (and $\mathfrak{f}$) a *vanishing edge*.

For some technical reasons, the $\mathcal{A}^{-1}$-move is not exactly the same as the one described in [3]: we allow here the vertex w to be of valence more than 3.

However the definition of an $\mathcal{A}$-move remains the same: Let e be an edge with initial and terminal vertices v and w and $a \in G$ such that

- $w = a \cdot v$ (so v and w belong to the same orbit of edges),
- we have $G_e \subsetneq a G_e a^{-1}$.

Call $\mathcal{F}$ the set of the edges with initial vertex w not in the orbit of e or $\bar{e}$. Performing a $\mathcal{A}$-move consists in performing the two following deformations. First expand w in an edge f with initial and terminal vertex w' and w'', with $G_f = G_{w''} = G_e$ and $G_{w'} = G_w$ and such that the initial vertex of e and the new terminal vertex of $a \cdot e$ are w'' and the initial vertices of the edges of $\mathcal{F}$ are w'. Then perform an induction on e with group $a G_e a^{-1}$.

The $\mathcal{A}^{-1}$-move is determined by the data of the orbits of e and f, however the $\mathcal{A}$-move is determined by the data of two edges e and e' in the same orbit such that the terminal vertex of e is the initial edge of e', and $G_e \subsetneq G_{e'}$.

A $\mathcal{A}^{\pm 1}$-move on e is denoted $\mathcal{A}^{\pm 1}(e)$ or e/e.

A move is *admissible* if we may perform it, if the obtained tree it is reduced, and if it is not a trivial induction. A sequence of moves is *admissible* if after performing the first n moves of the sequence the $n + 1$th is admissible.

If S is an admissible sequence of moves on the G-tree T, the tree obtained by applying S is denoted by $S \cdot T$.
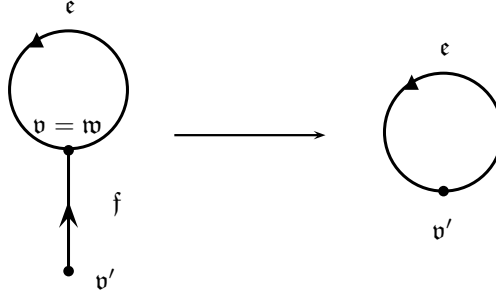
Figure 7: $\mathcal{A}^{-1}$-move

The number of orbits of vertices and edges in the G-trees of a reduced deformation space is not fixed. However, the only way to make a vertex or an edge disappear is to perform an $\mathcal{A}^{-1}$-move. Thus if two G-trees T and T' are related by a single move, we may identify the orbits of vertices and edges of T with the ones of T', except for the one that vanish. Moreover if the move is not an induction or an $\mathcal{A}^{\pm 1}$-move, the edge and vertex stabilizers does not change, we may then identify not only the orbits of edges but each edge. To be more precise, let T be a G-tree, e an edge of T and $\mathbf{m}$ an admissible move on T such that

- the move $\mathbf{m}$ is not an induction on the orbit $\mathfrak{e}$,
- the move $\mathbf{m}$ is not an $\mathcal{A}^{\pm 1}$-move on the orbit $\mathfrak{e}$,
- if $\mathbf{m}$ is a $\mathcal{A}^{-1}$-move, its vanishing orbit of edges is not $\mathfrak{e}$.

We may then identify e with an edge of $\mathbf{m} \cdot T$ and this identification is equivariant.

By extension, an orbit of vertices or edges is *vanishing* if we may perform an admissible sequence of moves ending by a $\mathcal{A}^{-1}$-move in which it vanishes. A sequence of moves *preserves* an orbit $\mathfrak{e}$ if no move of the sequence is an $\mathcal{A}^{-1}$-move in which $\mathfrak{e}$ vanishes.

As two reduced G-trees in the same reduced deformation space are related by a finite admissible sequence of deformation moves, given two reduced G-trees T and T' in the same deformation space, we may identify any non-vanishing orbit of vertices or edges of T to one of T'.

Edges properties Let T be a reduced G-tree. An orbit of edges $\mathfrak{e}$ of T is *slippery* if there exists an admissible sequence of moves on T, preserving $\mathfrak{e}$ and ending by a slide along $\mathfrak{e}$. The orbit $\mathfrak{e}$ is 2-*slippery* if there exist an admissible sequence S of moves on T, preserving $\mathfrak{e}$ and two distinct orbits of edges $\mathfrak{f}$ and $\mathfrak{f}'$ of $S \cdot T$ such that the slides $\mathfrak{f}/\mathfrak{e}$ and $\mathfrak{f}'/\mathfrak{e}$ are simultaneously admissible in $S \cdot T$. Note that $\mathfrak{f}' = \bar{\mathfrak{f}}$ is allowed.

An edge e of T is *ascending* if its endpoints are in the same orbit, and if its

9

stabilizer is equal to the stabilizer of its initial vertex. It is *strictly ascending* if moreover its stabilizer is strictly contained in the stabilizer of its terminal vertex. An edge is *toric* if its stabilizer is equal to the stabilizer of both its endpoints. The opposite edge of a (strictly) ascending edge is *(strictly) descending.*

An edge e is *pre-ascending* in T if its initial and terminal vertices v and v' are in the same orbit and if there exists $t \in G$ such that $t \cdot v' = v$ and $tG_et^{-1} \subsetneq G_e$. The opposite edge of a pre-ascending edge is said to be *pre-descending.* The edge e is *potentially strictly ascending* (or *p.s.a.*) if $\mathfrak{e}$ is pre-ascending after a finite admissible sequence of moves preserving $\mathfrak{e}$. Note that a strictly ascending edge is also pre-ascending. If an edge e is such that e or $\bar{e}$ is p.s.a. then e is said to be p.s.a.d..

An orbit of edges is slippery, 2-slippery, pre-ascending, pre-descending, (potentially) strictly ascending (or descending) or toric if one of its representative is.

The inductions and $\mathcal{A}^{-1}$-moves may only be performed on strictly ascending (or descending) edges. An $\mathcal{A}$-move may only be performed on the pre-ascending edges which are not ascending. Note that an $\mathcal{A}$-move changes a pre-ascending edge into a strictly ascending edge.

Figure 8 represent a Generalized Baumslag-Solitar group and is constructed in the following way: all vertices and edges carry the infinite cyclic group $\mathbb{Z}$, the number n at the end of each edge indicate that the injection of the edge group into the group of its endpoint is $k \mapsto nk$.

In such a representation, pre-ascending edges and admissible slides may be seen as divisibility relation. For example in Figure 8, the edge $\mathfrak{e}$ is pre-descending, the edges $\mathfrak{f}$ and $\mathfrak{g}$ are slippery since $\mathfrak{h}$ may first slide along $\mathfrak{g}$ then along $\mathfrak{f}$, and $\mathfrak{h}$ is p.s.a. since it is pre-ascending after sliding along $\mathfrak{g}$ and $\mathfrak{f}$.
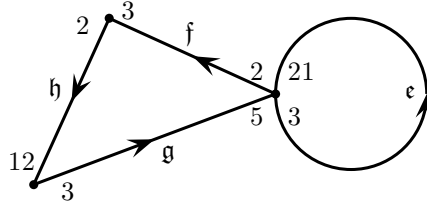

Figure 8

Expansion Let T be a G-tree, and v a vertex of T. Take $\mathcal{F}$ a set of edges with pairwise distinct orbits and initial vertex v, and H a subgroup of G_v containing G_f for all $f \in \mathcal{F}$. Then the *expansion* of v of group H and set $\mathcal{F}$ is the tree $\tilde{T}$ obtained from T by expanding v in an edge e of initial and terminal vertex $\tilde{v}$ and $\tilde{v}'$, such that $G_e = G_{\tilde{v}} = H$, $G_{\tilde{v}'} = G_v$, the edges of $\mathcal{F}$ have initial vertex $\tilde{v}$, and every edge of initial vertex v in T and not in the orbit of any edge of $\mathcal{F}$

has initial vertex $\tilde{v}'$ (see Figure 9 for the changes in the graph of groups). This construction produces a non-reduced tree.
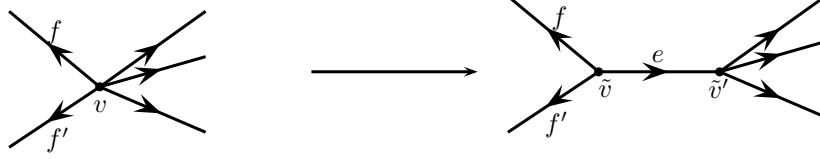

Figure 9: Expansion of v of set $\{f, f'\}$.

Dead ends   Let T be a reduced G-tree. Let v be a non-vanishing vertex and f an edge with initial vertex v. The vertex v is a *dead end* with *wall* f (or $\mathfrak{f}$) if

- $G_f \subsetneq G_v$,
- the edge f is slippery or p.s.a.d.,
- for every edge h with initial vertex v not in the orbit of f, we have the equality $\langle G_h, G_f \rangle = G_v$,
- there exists an edge g not in the orbit of f, with initial vertex v, with $G_g \subsetneq G_v$, such that for all edges h not in the orbit of f with initial vertex v there exists $a \in G_v$ such that $G_{a \cdot h} \subset G_g$,
  (we may notice that automatically $\mathfrak{g}$ and $\mathfrak{f}$ do not slide one along the other)

and if for every G-tree T' in the reduced deformation space of T and any representative v' of $\mathfrak{v}$ in T', there exists an edge with initial edge v' which has the listed properties.

The orbit $\mathfrak{v}$ of v is a *dead end* if v is a dead end.

If there exists two edges f and g with distinct orbits such that v is a dead end with wall f and with wall g in T, then there are exactly two orbits of edges with initial vertex $\mathfrak{v}$. However given another G-tree T' in the reduced deformation space of T, the orbits of the walls of v in T and T' may differ.

For example, Figure 10 represents the two reduced graphs of groups of the deformation space of a GBS decomposition. We may reach the right graph from the left one by a slide of $\mathfrak{h}_2$ along $\bar{\mathfrak{h}}_1$.

The vertex $\mathfrak{v}$ is a dead end. In the graph on the left, $\mathfrak{h}_1$ is the wall $\mathfrak{f}$ and $\bar{\mathfrak{h}}_1$ is the edge orbit $\mathfrak{g}$ as in the fourth point. In the one on the right, $\bar{\mathfrak{h}}_1$ is the wall $\mathfrak{f}$ and $\bar{\mathfrak{h}}_1$ plays the role of $\mathfrak{g}$. The edge orbit $\mathfrak{g}$ may be non-unique since another edge may have the same stabilizer as the one of g.

To *blow up* this dead end vertex v in T consists in expanding v with group G_v and set $\{f\}$ (note that in this case the expansion does not depends of the
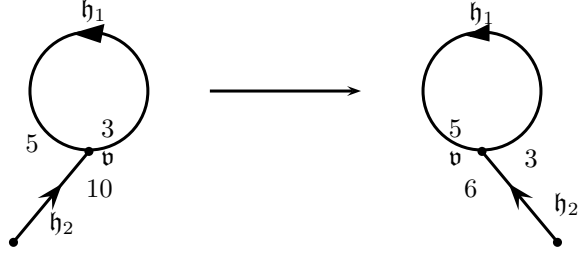
Figure 10: Example of dead end vertex.

choice of $f \in \mathfrak{f}$). Figure 11 represents the blow up of $\mathfrak{v}$ in the left graph of Figure 10. Performing a blow-up does not change the deformation space. If the dead end $\mathfrak{v}$ has valence 2, we may choose equivalently one of the two edges to be the wall. However the blow up does not depend on this choice.
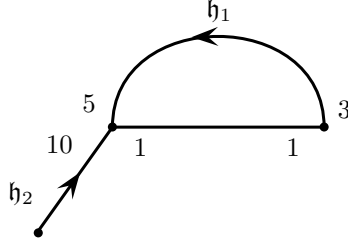

Figure 11: Blow up of a dead end vertex

3 Compatibility JSJ tree of GBS_n groups over $\mathbb{Z}^n$ subgroups

3.1 Construction

Given a group G and any deformation space $\mathcal{D}$ of G, we may canonically construct a G-tree associated to $\mathcal{D}$, as follows.

Definition 3.1. *Let T be a reduced G-tree in the deformation space $\mathcal{D}$. Define $T_{\mathcal{D}}$ as the G-tree obtained from T by blowing up the dead ends and collapsing the following edges:*

- *the vanishing edges of T*
- *the p.s.a. edges of T,*
- *the non-ascending slippery edges of T,*

12

- *the toric 2-slippery edges of T.*

The non-vanishing edge and vertice orbits of a reduced G-tree T may be identified with another orbit in any G-tree T' in the same reduced deformation space. Being a dead end vertex, being p.s.a., being slippery, and being 2-slippery does not depend on the chosen tree in the deformation space. But a toric edge may be changed in a non-ascending edge and vice-versa.

Note that as an edge is either non-ascending, strictly ascending (or descending) or toric, all 2-slippery edges are collapsed in $T_\mathcal{D}$.

Proposition 3.2. *The tree $T_\mathcal{D}$ does not depend on the choice of the reduced tree in $\mathcal{D}$ taken for the construction.*

Proof. By Clay-Forester [3, Corollaire 1.2], two reduced G-trees of $\mathcal{D}$ are related by a finite sequence of deformation moves. We just have to show that we obtain the same tree $T_\mathcal{D}$ starting from two trees T and T' which differ by a single move.

For each move we proceed in three steps. We first prove that the set of orbits we have to collapse is the same before or after performing the move. We then prove that the move commutes with the collapses. Finally we prove that the move commutes with the blow-ups.

For the first step as p.s.a. edges are collapsed, we have to prove that if a non-ascending slippery edge is turned into a toric edge, then it is 2-slippery.

1. Assume that T and T' differ by a slide of an edge f along another edge e.

   Call v the initial vertex of f. The only edge which may possibly become toric is f. If f is non-ascending slippery, then as f is reduced we have $G_f \subsetneq G_v$, in T, thus also in T'. Then f is not toric in T'. Thus no non-ascending slippery edge may be turned into a toric edge. Thus the edges in T and T' that must be collapsed are the same.

   For the second step, it suffices to to check that either e is collapsed in $T_\mathcal{D}$ or $T = T'$.

   As f slides along e, the edge e is slippery. If e is not collapsed in $T_\mathcal{D}$, then e must be toric and not 2-slippery. Then in the associated graph of group $\mathfrak{e}$ is a loop and $\mathfrak{f}$ is the only edge adjacent to $\mathfrak{e}$ (see Figure 12).This case is exactly the rigidity case describe in [11, Theorem 1, case 3], that is, the slide leaves T unchanged.

   It remains to show that any blow-up of a dead end vertex commutes with the slide. The only non-trivial cases is when an endpoint of e is a dead end. We may assume that the initial vertex v of e is a dead end. As f slides along e, neither e nor $\bar{f}$ may be a wall. Thus if we blow up v, the edges e and $\bar{f}$ will be on the same side of the new edge. Hence this is equivalent to blow up a vertex then slide f along e or slide f along e and then blow up a vertex.

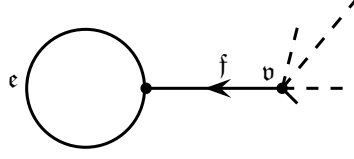2. Assume that T and T' differ by an induction on an orbit of edges $\mathfrak{e}$.

Figure 12: The only orbits of edges adjacent to $\mathfrak{v}$ are $\mathfrak{e}$, $\bar{\mathfrak{e}}$ and $\mathfrak{f}$.

Call $\mathfrak{v}$ the orbit of the endpoints of $\mathfrak{e}$.

Here $\mathfrak{e}$ (or $\bar{\mathfrak{e}}$) is strictly ascending in both trees. Assume that an edge f was non-ascending slippery and becomes toric. Then the endpoints of f are in $\mathfrak{v}$. And as f is toric in T', the orbit of edges $\mathfrak{e}$ and $\bar{\mathfrak{e}}$ may slide along f in T'. Then f is 2-slippery. Thus the edges in T and T' that must be collapsed are the same.

As $\mathfrak{e}$ is strictly ascending in both trees, then it is collapsed in both T and T', and the trees obtained are the same.. Moreover as $\mathfrak{e}$ is ascending the vertex orbit $\mathfrak{v}$ is not a dead end, thus blowing up dead end vertices and performing the induction commute.

3. Assume that T and T' differ by an $\mathcal{A}^{-1}$-move of an orbit of edges $\mathfrak{e}$ with collapse of an orbit $\mathfrak{f}$.

When we perform the $\mathcal{A}^{\pm 1}$-move on the orbit of edges $\mathfrak{e}$, if an orbit of edges $\mathfrak{g}$ becomes toric (or stops beeing toric), then $\mathfrak{e}$, $\bar{\mathfrak{e}}$, $\mathfrak{g}$ ans $\bar{\mathfrak{g}}$ have same terminal vertex orbit, thus $\mathfrak{e}$ and $\bar{\mathfrak{e}}$ may slide along $\mathfrak{g}$ thus $\mathfrak{g}$ is two slippery.

In T, the orbit $\mathfrak{e}$ is strictly ascending and $\mathfrak{f}$ is vanishing, thus both $\mathfrak{e}$ and $\mathfrak{f}$ are collapsed in $T_{\mathcal{D}}$. In T', the orbit $\mathfrak{f}$ is already collapsed and $\mathfrak{e}$ is pre-ascending, thus collapsed in $T_{\mathcal{D}}$. Thus the collapses commute with the move.

It remains to show that the move commutes with the blow up of dead ends. Before performing the $\mathcal{A}^{-1}$-move the vertex of $\mathfrak{e}$ is not a dead end, since $\mathfrak{e}$ is ascending. However the terminal orbit of vertices $\mathfrak{v}$ of $\mathfrak{f}$ (which is also the vertex of $\mathfrak{e}$ after performing the $\mathcal{A}^{-1}$-move) may be a dead end. This is the only non-trivial case of commutativity. Assume $\mathfrak{v}$ is a dead end. Call $\mathfrak{g}$ a wall in T. If $\mathfrak{g} \neq \mathfrak{f}$, then $\mathfrak{g}$ remains the wall in T' and it is easy to see that the $\mathcal{A}^{-1}$-move and the blow up commute.

Two distinct orbits may play the role of the wall if and only if the orbit of vertices $\mathfrak{v}$ is of valence 2. Assume that $\mathfrak{f}$ is the unique wall of $\mathfrak{v}$, then $\mathfrak{v}$ is of valence at least 3. Let $\mathfrak{g}$ be as in the definition of the dead end and $\mathfrak{h}$ another orbit of edges with initial vertex $\mathfrak{v}$ in T. Take v a representative of $\mathfrak{v}$ and f, and g representatives of $\mathfrak{f}$, and $\mathfrak{g}$ with initial vertex v. By assumption G_g is maximal for inclusion and there exists h in the orbit of $\mathfrak{h}$ with initial vertex v such that $G_h \subset G_g \not\subset G_f$. But in T', as $\mathfrak{e}$ is pre-ascending, thus there exists two edges e' and e in the orbit of $\mathfrak{e}$ such that the initial vertex of e and the terminal vertex of e' are v and $G_{e'} \subset G_e$ and we still have $G_h \subset G_g \not\subset G_e(= G_f)$ with G_g and G_e maximal for the inclusion among

14

groups of edges adjacent to v. There is no wall to v, this is a contradiction we the fact that v is a dead end.

We obtain the same tree $T_\mathcal{D}$ starting from two reduced tree related by a move. Thus the tree $T_\mathcal{D}$ only depends from the deformation space $\mathcal{D}$. $\qquad\square$

We denote by T_{comp} the G-tree associated to Guirardel-Levitt's JSJ deformation space of G.

Lemma 3.3. *Let T be a JSJ tree. Let $\tilde{T}$ be a reduced Guirardel-Levitt's JSJ tree refined by T. Let $\tilde{e}$ be an edge of $\tilde{T}$ with initial vertex $\tilde{v}$ and let H be a subgroup of $G_{\tilde{v}}$ containing $G_{\tilde{e}}$ such that for every edge $\tilde{f} \notin \tilde{\mathfrak{e}}$ with initial vertex $\tilde{v}$ we have $H \subset \langle G_{\tilde{e}}, G_{\tilde{f}} \rangle$. Denote by $\tilde{T}_{\tilde{e}}$ the tree obtained from $\tilde{T}$ by performing an expansion of v with group H and set $\{\tilde{e}\}$. Then T is compatible with $\tilde{T}_{\tilde{e}}$.*

Proof. Let e be the lift of $\tilde{e}$ in T and v a lift of $\tilde{v}$ such that $G_v = G_{\tilde{v}}$. Such a v exists since T and $\tilde{T}$ are both JSJ trees. Call $[w,v]$ the path between the initial vertex w of e and v. Call v_1 the vertex of $[w,v]$ the closest of w such that $H \subset G_{v_1}$.

If $w = v_1$, then the G-tree obtained by the expansion of w with group H and set $\{e\}$ obviously refines $\tilde{T}_{\tilde{e}}$.

Otherwise, denote by e_1 the last edge of $[w, v_1]$. We have $G_e \subset G_{e_1} \subsetneq H$. Moreover as for every edge $\tilde{f} \notin \tilde{\mathfrak{e}}$ of initial vertex $\tilde{v}$ in $\tilde{T}$, we have $H \subset \langle G_{\tilde{f}}, G_{\tilde{e}} \rangle$, we obtain $G_{\tilde{f}} \not\subset G_{e_1}$. Hence for f the lift of $\tilde{f}$ in T, the edges e and f are in distinct components of $T \setminus v_1$. Let T_e be the tree obtained by performing an expansion of v_1 with group H and set $\{e_1\}$. Then T_e refines $\tilde{T}_{\tilde{e}}$. $\qquad\square$

Corollary 3.4. *Let T be a JSJ tree. The G-tree T_{comp} is compatible with T. Moreover there exists a common refining tree which is a JSJ tree.*

Proof. For f an edge of T_{comp} call T_f the tree obtained by collapsing every edge of T_{comp} except f.

By the point 1. of [9, Proposition 3.22], we have to show that T is compatible with T_f for every edge f of T_{comp}.

Call $\tilde{T}$ a reduced JSJ tree refined by T. By construction, $\tilde{T}$ refines every tree T_f with f an edge of T_{comp} not obtained by a blow-up. Thus T also refines all these trees.

If f comes from a blow-up, then we may apply Lemma 3.3, taking T the JSJ tree, $\tilde{T}$ the reduced JSJ tree, v the dead end from f comes from, e the wall of v and $H = G_v$. We obtain that T is compatible with T_f.

If follows that T is compatible with T_{comp}. $\qquad\square$

3.2 Universal compatibility

Proposition 3.5. *Let G be a group that admits a JSJ tree over a set of subgroup $\mathcal{A}$ and such that the edge groups of every minimal G-tree over $\mathcal{A}$ are elliptic in the JSJ deformation space of G. Then T_{comp} is universally compatible.*

Proof. From Lemma 5.3 of [8], every G-tree is refined by a JSJ tree. Thus by Corollary 3.4, the tree T_{comp} is universally compatible. $\qquad\square$

A GBS_n group G is *generic* if the trivial G-tree is not abelian JSJ G-tree. The description of the generic GBS_n groups is given in [1]. In particular, if a GBS_n group is not isomorphic to a semi-direct product $\mathbb{Z}^n \rtimes \mathbb{Z}$ then it is generic.

Lemma 3.6. *Let G be a generic GBS_n group. If T is a minimal G-tree with an edge group $A \simeq \mathbb{Z}^r$ with $r \leq n$, then $r = n$ and A is universally elliptic over abelian G-trees.*

Proof. Since G is generic, any GBS_n tree is a JSJ G-tree ([1, Theorem 1.2]). Then all edge stabilizers of a given JSJ tree are universally elliptic and commensurable. Let E be one of these stabilizers. Its commensurator is G. In every other G-tree T over groups $\mathbb{Z}^r$ with $r \leq n$, the group E is elliptic, thus included in the stabilizer of some vertex v. As its commensurator is G, it is then virtually contained in every stabilizer of vertex in the orbit of v. If we assume T is minimal, it implies that E is virtually included in every edge stabilizer. Hence every edge stabilizer is a $\mathbb{Z}^n$ which contains with finite index a universally elliptic group. Thus every edge stabilizer is universally elliptic. $\qquad\square$

Corollary 3.7. *Let G be a GBS_n group. The tree T_{comp} associated to the JSJ deformation space of G is compatible with every G-tree over the subgroups of $\subset \mathbb{Z}^n$.*

Proof. If G is generic, this is a direct consequence of Proposition 3.5 and Lemma 3.6.

If G is not generic, then the trivial G-tree is a JSJ decomposition. Hence T_{comp} is trivial and universally compatible. $\qquad\square$

3.3 Maximality

In this section, we prove that in the case of $vGBS$ groups (and not only GBS_n groups), the tree T_{comp} dominates every other universally compatible trees.

For this we need some technical lemmas, that we divide in three categories. The lemmas of the first category give conditions on trees to be compatible. the lemmas of the second category give existence of deformation sequences in the deformation space and the lemmas of the third category give some refinement conditions.

Compatibility lemmas For S a set of elements of G and T a G-tree, call $\mathcal{E}_S(T)$ the convex hull of all characteristic spaces of elements of S in T. Note that if S is reduced to one element then $\mathcal{E}_S(T)$ is just the characteristic space of this element.

Lemma 3.8. *Let T and T' be two G-trees and let a, b, c, and d be four elements of G.*

- *If the intersection $\mathcal{E}_{a,b}(T) \cap \mathcal{E}_c(T)$ is empty, and the intersection $\mathcal{E}_{a,b}(T') \cap \mathcal{E}_c(T')$ contains an edge, then T and T' are not compatible.*
- *If $\mathcal{E}_{\{a,b\}}(T)$ and $\mathcal{E}_{\{c,d\}}(T)$ are disjoint in T and $\mathcal{E}_{\{a,c\}}(T')$ and $\mathcal{E}_{\{b,d\}}(T')$ are disjoint in T', then T and T' are not compatible.*

Proof. First notice that if $\tilde{T}$ is a refinement of T, then $\mathcal{E}_S(\tilde{T})$ surjects onto $\mathcal{E}_S(T)$ via the natural map.

Thus for the first point, the two assumptions are stable by refinement. Hence a common refinement should have both properties. This is impossible.

For the second point, assume there exists a common refinement $\tilde{T}$. As $\mathcal{E}_{\{a,b\}}(T)$ and $\mathcal{E}_{\{c,d\}}(T)$ are disjoint in T then $\mathcal{E}_{\{a,b\}}(\tilde{T})$ and $\mathcal{E}_{\{c,d\}}(\tilde{T})$ are also disjoint in $\tilde{T}$. Call B the bridge in $\tilde{T}$ between $\mathcal{E}_{\{a,b\}}(\tilde{T})$ and $\mathcal{E}_{\{c,d\}}(\tilde{T})$. Then B is contained in $\mathcal{E}_{\{a,c\}}(\tilde{T})$ and $\mathcal{E}_{\{b,d\}}(\tilde{T})$ in $\tilde{T}$. Thus $\mathcal{E}_{\{a,c\}}(\tilde{T}) \cap \mathcal{E}_{\{b,d\}}(\tilde{T})$ is not empty in $\tilde{T}$, hence $\mathcal{E}_{\{a,c\}}(T') \cap \mathcal{E}_{\{b,d\}}(T')$ is not empty in T'. $\qquad\square$

Lemma 3.9. *Let G be a vGBS group and T be an abelian JSJ G-tree. Let e be a reduced edge with initial vertex v in T such that $G_e \subsetneq G_v$. Then there exists $g \in G_v \setminus G_e$ that centralizes G_e.*

Proof. If G_v is abelian, the lemma is trivial. Otherwise by [1], the group G_v is a semi-direct product $\mathbb{Z}^n \rtimes_\varphi \langle t \rangle$, with φ a non-trivial automorphism of $\mathbb{Z}^n$, and $G_e \subsetneq \mathbb{Z}^n$, thus any element in $\mathbb{Z}^n \setminus G_e$ centralizes G_e. $\qquad\square$

Corollary 3.10. *Let T be a G-tree. Let $\mathfrak{e}$ be an orbit of reduced edges of T. Assume $\mathfrak{e}$ has one of the following properties:*

1. *$\mathfrak{e}$ is pre-ascending in T and G is not an ascending HNN-extension,*
2. *$\mathfrak{e}$ is not ascending and there exists an orbit of reduced edges $\mathfrak{f}$ which slides along two consecutive edges of $\mathfrak{e} \cup \bar{\mathfrak{e}}$ in T,*
3. *T is a vGBS-tree, the orbit $\mathfrak{e}$ is not ascending and there exists an orbit of reduced edges $\mathfrak{f}$ which slides along $\mathfrak{e}$ in T.*
4. *$\mathfrak{e}$ is toric of endpoint $\mathfrak{v}$ and there exist two distinct orbits of reduced edges $\mathfrak{f}$ and $\mathfrak{g}$ with terminal vertex $\mathfrak{v}$, distinct from the orbits $\mathfrak{e}$ or $\bar{\mathfrak{e}}$.*

Let T' be a collapse of T such that e is not collapsed in T'. Then T' is not universally compatible.

Here we do not need the slides to be admissible.

Proof.

1. If $\mathfrak{e}$ is pre-ascending in T, then $\mathfrak{e}$ is pre-ascending in T'. By [7, Proposition 7.1] the tree T' is not universally compatible.

2. Let e, and e' be edges of $\mathfrak{e}\cup\bar{\mathfrak{e}}$ and f an edge of $\mathfrak{f}$ such that f may consecutively slide along e and $\bar{e}'$ in T. Let v and v' be the initial and terminal vertices of e, call v'' the terminal vertex of e' and w the initial vertex of f (note that the terminal vertex of f is v and the initial vertex of e' is v').

   As e is reduced and non ascending, we may find three elements $b \in G_v \setminus G_e$, $c \in G_{v'} \setminus (G_e \cup G_{e'}) \neq \emptyset$ and $d \in G_{v''} \setminus G_{e'}$. If f is not ascending let a be in $G_w \setminus G_f$. If f is ascending, let t be such that $t \cdot w = v$, an call $a = t^{-1}ct$. The sets $\mathcal{E}_{\{a,b\}}(T)$ and $\mathcal{E}_{\{c,d\}}(T)$ are separated by an edge in $\mathfrak{e}$. As $\mathfrak{e}$ is not collapsed in T', the sets $\mathcal{E}_{\{a,b\}}(T')$ and $\mathcal{E}_{\{c,d\}}(T')$ are disjoint in T'. But making slide f along e and $\bar{e}'$, we now have a new tree T'' in which $\mathcal{E}_{\{b,c\}}(T'')\cap\mathcal{E}_{\{a,d\}}(T'') = \emptyset$ (see Figure 13). By Lemma 3.8, the tree T' is not universally compatible.
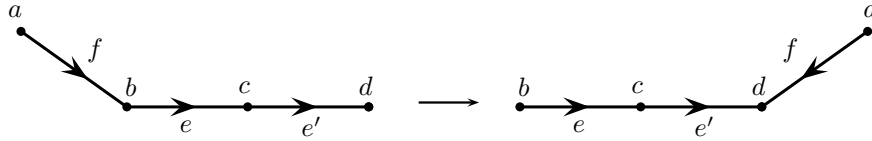

Figure 13

3. Assume that T is a $vGBS$-tree. Let f be an edge which slides along a reduced non ascending edge e. Call v the terminal vertex of e. The set $G_v \setminus G_e$ is not empty since e is reduced and non-ascending. Applying Lemma 3.9, there exists λ in this set such that $G_e = G_{\lambda \cdot e}$, thus f slides consecutively on e and $\lambda \cdot \bar{e}$. Thus the point 3 is implied by the point 2.

4. As $\mathfrak{e}$ is toric then $\mathfrak{f}$ slides along two consecutive edges in $\mathfrak{e}$. If $\mathfrak{f} \neq \bar{\mathfrak{g}}$, collapsing $\mathfrak{g}(\neq \mathfrak{e}, \bar{\mathfrak{e}})$ in T, the edge e is no more ascending (and still reduced) in the new tree T'', we may apply the point 2. Note that T'' may perhaps not refines T', however it refines the tree obtained from T' by collapsing $\mathfrak{g}$, this is enough to obtain the result.

   If $\mathfrak{g}$ is in the orbit of $\bar{\mathfrak{f}}$. Let f be an edge of $\mathfrak{f}$. Call v its terminal vertex and let g be an edge of $\mathfrak{g} = \bar{\mathfrak{f}}$ with terminal vertex v. Let e be an edge of $\mathfrak{e}$ with initial vertex v (on which f may slide). Let v' be the terminal vertex of e and w be the initial vertex of g. Call $t \in G$ an element which sends v onto v' and $s \in G$ which sends v onto w.

   As e is toric $G_e = G_{t \cdot e}$, and f slides consecutively along e and $t \cdot e$. Define the four elements $a = s^{-1}ts$, $b = sts^{-1}$, $c = t^{-1}sts^{-1}t^1$ and $d = t^{-2}sts^{-1}t^2$ (see Figure 14). Then the sets $\mathcal{E}_{\{a,b\}}(T')$ and $\mathcal{E}_{\{c,d\}}(T')$ are disjoint in T' but after making slide f along e, and $t \cdot e$, we have $\mathcal{E}_{\{b,c\}} \cap \mathcal{E}_{\{a,d\}} = \emptyset$. Thus by Lemma 3.8, the tree T' cannot be universally compatible. $\square$
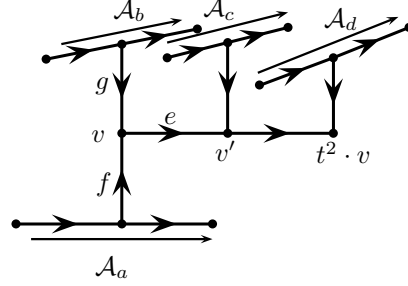
18

Figure 14

Corollary 3.11. *Let T be a G-tree. Let e be an edge of T with initial and terminal vertices v and v'. Call T_e the tree obtained from T by collapsing all orbits of edges except $\mathfrak{e}$.*

If there are two reduced non-ascending edges, f with initial vertex v and g with initial vertex v', such that $\mathfrak{f} \neq \bar{\mathfrak{g}}$, $G_f \subset G_g$, and $G_e \not\subset G_g$, then T_e is not universally compatible.

Here the edge e is not necessarily reduced.

Proof. As f and g are reduced and non ascending, we may take a an element which fixes the terminal vertex of f but not f and d which fixes the terminal vertex of g but not g. Let r be an element of $G_e \setminus (G_g \cap G_e)$ and define $f' = r \cdot f$ and $g' = r \cdot g$. Define $b = rar^{-1}$ and $c = rdr^{-1}$. We are as in Figure 15. Here $\mathcal{E}_{\{a,b\}}(T_e)$ and $\mathcal{E}_{\{c,d\}}(T_e)$ are disjoint. But $\bar{f}$ may slides along e and g. By equivariance, at the same time $\bar{f}'$ slides along e and g'.

After performing these slides, we obtain a tree T' in which $\mathcal{E}_{\{a,d\}}(T')$ and $\mathcal{E}_{\{b,c\}}(T')$ are disjoint. By Lemma 3.8, the tree T_e is not universally compatible. $\square$
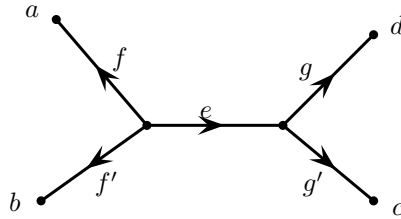

Figure 15

19

Lemma 3.12. *Let T be a G-tree. Let e be an edge of T whose endpoints are in distinct orbits, call v the initial vertex of e and v' its terminal vertex. Call T_e the tree obtained from T by collapsing all orbits of edges except $\mathfrak{e}$. If there exists two reduced edges f and f' in the same orbit $\mathfrak{f} \neq \mathfrak{e}$, such that the initial vertex of f is v, the terminal vertex of f' is v', and $G_f \subsetneq G_{f'}$, then T_e is not universally compatible.*

Proof. As f and f' are in the same orbit we may take $t \in G$ such that $t \cdot f = f'$. Let us define $e' = t \cdot e$ and $f'' = t \cdot f'$.

First assume $G_v = G_e \subset G_{f'}$. As $G_f \subsetneq G_{f'}$, we have $G_{f'} \subsetneq G_{f''}$ and $G_e \subsetneq G_{e'}$. Thus collapsing f in T, the edge e becomes pre-ascending, hence it stays pre-ascending in T_e. By Corollary 3.10, the tree T_e is not universally compatible.

Assume now that $G_v \neq G_e$ or $G_e \not\subset G_{f'}$. We have $G_{f'} \subsetneq G_{f''}$. Take a in $G_{f''} \setminus G_{f'}$, and call $f''' = a \cdot f'$. As $G_v \neq G_e$ or $G_e \not\subset G_{f'}$, as f' is reduced and $\mathfrak{v} \neq \mathfrak{v}'$, we have $G_{f'} \subsetneq G_{v'}$.

We may now take an element b which fixes the terminal vertex of f''' but not f''' and an element c which fixes the terminal vertex of f but not f.

If $G_v \neq G_e$, take d in $G_v \setminus G_e$ (automatically $d \notin G_f$ and $d \notin G_{f'}$). If $G_v = G_e$ then $G_e \not\subset G_{f'}$. Take $h \in G_e \setminus (G_{f'} \cap G_e)$. Define $\tilde{f} = h \cdot f$ and take d which fixes the terminal vertex of $\tilde{f}$ but not $\tilde{f}$.

Then $\mathcal{E}_{\{a,b\}}(T)$ and $\mathcal{E}_{\{c,d\}}(T)$ are separated at least by e. Thus $\mathcal{E}_{\{a,b\}}(T_e) \cap \mathcal{E}_{\{c,d\}}(T_e)$ is empty.

Now call T' the tree obtained by collapsing e in T and call w the initial vertex of f in T'. Perform the expansion of the vertex w with group G'_f and set $\{f, \bar{f}'\}$. Call g the edge obtained in this expansion, and w_1 the terminal edge of g.

We may note that:

- the element a still stabilizes f'' but not f',
- the element b stabilizes $a \cdot w_1$ but not $a \cdot g$,
- the element c stabilizes $t^{-1} \cdot w_1$ but not $t^{-1} \cdot g$,
- if $G_v \neq G_e$, the element d stabilizes w_1 but not g,
- if $G_v = G_e$, the edge g is between f and the characteristic space of d.

Then making the edge $a \cdot g$ slide along $\bar{f}'''$ and f', we obtain a tree in which $\mathcal{E}_{\{a,d\}} \cap \mathcal{E}_{\{b,c\}}$ is empty. By Lemma 3.8, the edge T_e is not universally compatible (see the two slides in Figure 16). $\square$

Deformation lemmas

Lemma 3.13. *If f is slippery or p.s.a.d., up to exchanging the role of f and $\bar{f}$, there exists an admissible sequence of moves*
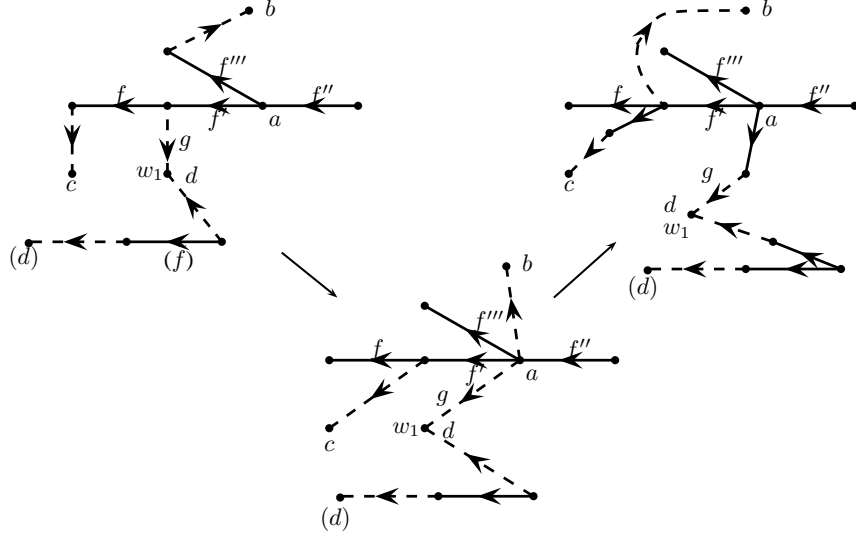
$$S = g_1/h_1 \ldots g_p/h_p \cdot f/i_1 \ldots f/i_q$$

Figure 16: The dashed edges are in the orbit $\mathfrak{g}$ and the plain ones in the orbit $\mathfrak{f}$.

which preserves f with $\mathfrak{h}_j, \mathfrak{i}_j, \mathfrak{g}_j \neq \mathfrak{f}, \bar{\mathfrak{f}}$ for all j such that, after performing the sequence, the edge F is pre-ascending or there exists an admissible slide along $\bar{f}$.

Proof. As f is slippery or p.s.a.d., there exists an admissible sequence of moves $S' = \lambda_1/\mu_1 \cdot \cdots \cdot \lambda_r/\mu_r)$ such that after performing the sequence either f is pre-ascending (or pre-descending) or there exists an admissible slide along f or $\bar{f}$. We may chose S' of minimal length.

Suppose there exists i such that $\mu_i \in \mathfrak{f}, \bar{\mathfrak{f}}$, then after the $i-1$ first moves either f or $\bar{f}$ is pre-descending (if $\lambda_i = f$ or $\bar{f}$) or there exists an edge g which slides along f (take $g = \lambda_i$ if $\lambda \neq f, \bar{f}$). As we chose S' of minimal length, this is impossible, thus $\mu_i \notin \mathfrak{f}, \bar{\mathfrak{f}}$.

Up to replacing f by $\bar{f}$, we may assume that, in $S' \cdot T$, the edge f is pre-ascending or that the admissible slide is along $\bar{f}$.

We will first show that S' may be changed into the product of two sequences of move S_1 and S_2 such that no edge of $\mathfrak{f}$ or $\bar{\mathfrak{f}}$ occurs in S_1, and S_2 is a product of slides of f and $\bar{f}$. Then it will remain to remove the slides of $\bar{f}$ in S_2.

If S is not already of this form there exists an i such that $\lambda_i = f$ or $\bar{f}$ and $\lambda_{i+1} \neq \mathfrak{f}, \bar{\mathfrak{f}}$. We only consider the case $\lambda_i = f$, the following formulas remain true replacing f by $\bar{f}$. We replace the sequence $f/\mu_i \cdot \lambda_{i+1}/\mu_{i+1}$ by a sequence of 2, 3 or 4 terms in which f appears in the last terms.

21

If λ_{i+1} and μ_{i+1} are not in the same orbit (it is not an induction or a $\mathcal{A}^{\pm 1}$-move), there are tree possibilities:

- either λ_{i+1} is not in the orbit of μ_i or $\bar{\mu}_i$, then the two slides commute,
- or λ_{i+1} and μ_i are in the same orbit (we may assume $\lambda_{i+1} = \mu_i$), then $G_f \subset G_{\mu_{i+1}}$ and we have the equality $f/\mu_i \cdot \lambda_{i+1}/\mu_{i+1} = \lambda_{i+1}/\mu_{i+1} \cdot f/\mu_i \cdot f/\bar{\mu}_{i+1}$,
- or λ_{i+1} and $\bar{\mu}_i$ are in the same orbit (we may assume $\lambda_{i+1} = \bar{\mu}_i$), then $G_f \subset G_{\mu_{i+1}}$ and we have the equality $f/\mu_i \cdot \lambda_{i+1}/\mu_{i+1} = \lambda_{i+1}/\mu_{i+1} \cdot f/\mu_{i+1} \cdot f/\mu_i$.

If λ_{i+1} and μ_{i+1} are in the same orbit then the move is an induction or an $\mathcal{A}^{\pm 1}$-move.

If λ_{i+1}/μ_{i+1} is an induction then the two moves f/μ_i and λ_{i+1}/μ_{i+1} commute.

If λ_{i+1}/μ_{i+1} is an $\mathcal{A}^{\pm 1}$-move we have three cases to consider:

- If λ_{i+1} is not in the orbit of μ_i or $\bar{\mu}_i$ and μ_i does not vanish in the move λ_{i+1}/μ_{i+1} then the two moves commute.
- If λ_{i+1} is not in the orbit of μ_i or $\bar{\mu}_i$ but μ_i vanishes in the move λ_{i+1}/μ_{i+1} then $f/\mu_i \cdot \lambda_{i+1}/\mu_{i+1} = \lambda_{i+1}/\mu_{i+1}$.
- If λ_{i+1} is in the orbit of μ_i or $\bar{\mu}_i$.
  If λ_{i+1}/μ_{i+1} is an $\mathcal{A}^{-1}$-move, then the two moves commute. Else it is an $\mathcal{A}$-move, then calling h the new edge we have $f/\mu_i \cdot \lambda_{i+1}/\mu_{i+1} = \lambda_{i+1}/\mu_{i+1} \cdot f/h \cdot f/\mu_i \cdot f/\bar{h}$.

Using these equalities, we may put all slides of f and $\bar{f}$ at the end of the sequence.

Let us prove that the process of transformation finishes.

First note that the number of moves not involving f stays constant. Call n this number. Let $\mathcal{S}$ be the set of move sequences with exactly n terms not involving f or $\bar{f}$. Let $\varphi : \mathcal{S} \to \mathbb{N}^n$ be defined as follows: for S a sequence $\varphi(S) = (k_1, \ldots, k_n)$ where k_i counts the number of terms involving f or $\bar{f}$ appearing before the ith term not involving f or $\bar{f}$ in S. Note that $\varphi(S) = (0, \ldots 0)$ if and only if $S = S_1 S_2$ with S_1 not involving f or $\bar{f}$ and S_2 composed of slides of f or $\bar{f}$.

Then φ decreases strictly for the lexicographic order on $\mathbb{N}^n$ when we apply one of the transformation described previously. Thus the process finishes.

Now, it remains to prove that we may exchange S_2 by a sequence in which only slides of f occurs. As a slide of f and a slide of $\bar{f}$ commute, we may assume that S_2 is a product of S_3 and S_4, with S_3 a sequence of slides of f and S_4 a sequence of slides of $\bar{f}$.

After performing the sequence S', either the edge f is pre-ascending or there exists an edge g which slides along $\bar{f}$.

First assume that after performing the sequence of moves S', the slide $g/\bar{f}$ is admissible, and that f is not pre-ascending. If a sequence $\bar{f}/h \cdot g/\bar{f}$ is admissible in a tree T, this implies that $G_g \subset G_f$ and that the terminal vertices of f and

g are the same in T. Thus $g/\bar{f}$ is also admissible in T (the admissibility comes from the fact that f is pre-ascending: g is still reduced after the slide). Thus sequence $S_1 \cdot S_3 \cdot g/\bar{f}$ is admissible in the initial tree, and $S = S_1 \cdot S_3$ is as required.

Now assume that after performing the sequence of move S, the edge f is pre-ascending.

We now use the following simple fact.

Fact. *Let T be a G-tree. Let f and $g \notin \mathfrak{f}, \bar{\mathfrak{f}}$ be two edges of T. If the move $\bar{f}/g$ is admissible in T and f is pre-ascending in $\bar{f}/g \cdot T$, then there exists $g' \in \mathfrak{g}$ such that $f/\bar{g}'$ is admissible in T and f is pre-ascending in $f/\bar{g}' \cdot T$*

Proof of the fact. Call v and v' the initial and terminal vertices of g. As f is pre-ascending in $\bar{f}/g \cdot T$, there exists $f' \in \mathfrak{f}$ with terminal vertex v', such that $G_{f'} \subset G_f (\subset G_g)$. As $g \notin \mathfrak{f}, \bar{\mathfrak{f}}$, the slide $f'/\bar{g}$ is admissible in T and f is pre-ascending in $f'/\bar{g} \cdot T$. Let a be an element such that $f = a \cdot f'$. Then the edge $g' = a \cdot g$ is as required (the moves $f'/\bar{g}$ and $f/\bar{g}'$ are equal). $\qquad\square$

Call T the tree obtained by performing S_1 and S_3 on the initial tree. Assume S_4 is the product $\prod_{i=1}^{p} \bar{f}/k_i$. We prove the lemma by recurrence. If $p = 0$ then the sequence $S_1 \cdot S_3$ is as required.

If $p > 0$, applying Fact 3.3, there exists $k_p' \in \mathfrak{k}_p$ such that we may change S_4 by $(\prod_{i=1}^{p-1} \bar{f}/k_i) \cdot f/\bar{k}_p'$. Moreover the move $f/\bar{k}_p'$ commutes with every move $(\bar{f}/k_i)$. Thus S_4 may be changed by $f/\bar{k}_p \cdot (\prod_{i=1}^{p-1} \bar{f}/k_i)$. We may conclude by applying the recurrence hypothesis. $\qquad\square$

Lemma 3.14. *Let T be a G-tree. Let f be a vanishing non-slippery edge of T.*

There exists a strictly ascending orbit of edges $\mathfrak{e}$ of T whose endpoint $\mathfrak{w}$ is an endpoint of $\mathfrak{f}$ and of valence 3, and there exists an admissible sequence S of inductions on $\mathfrak{e}$ and slides of $\mathfrak{f}$ along $\mathfrak{e}$ or $\bar{\mathfrak{e}}$ finishing by a $\mathcal{A}^{-1}$-move on $\mathfrak{e}$ in which $\mathfrak{f}$ and $\mathfrak{w}$ vanish.

Proof. As f vanishes, there exists an admissible sequence S such that there exists an admissible $\mathcal{A}^{-1}$-move on $T' = S \cdot T$ in which $\mathfrak{f}$ vanishes. Call w the vanishing vertex of this move. As f is non-slippery, there are exactly three orbits $\mathfrak{f}, \mathfrak{e}, \bar{\mathfrak{e}}$ adjacent to w (by definition every other edge should slide along $\bar{f}$), and the sequence S does not involve slides along $\mathfrak{f}$ or $\bar{\mathfrak{f}}$.

We prove the lemma by recurrence on the length n of the sequence S. If $n = 0$ then $v = w$, and the lemma is trivial.

Assume $n \geq 1$, let $\mathfrak{g}/\mathfrak{g}'$ the first move of S, and S' the sequence of the $n-1$ last moves of S. Call $T'' = \mathfrak{g}/\mathfrak{g}' \cdot T$. By recurrence hypothesis, we may replace S' by a sequence S'' of induction on $\mathfrak{e}$ and slides along $\mathfrak{e}$ and $\bar{\mathfrak{e}}$. As f is not slippery we have $\mathfrak{g}' \neq \mathfrak{f}, \bar{\mathfrak{f}}$. If $\mathfrak{g}$ and $\mathfrak{g}'$ differ from $\mathfrak{f}, \mathfrak{e}$ and $\bar{\mathfrak{e}}$, then we may remove the slide from S, without changing anything around $\mathfrak{v}$, thus the sequence S''

23

is admissible in T and the lemma is proven. If $\mathfrak{g} = \mathfrak{e}$ (or $\bar{\mathfrak{e}}$), as the only edges adjacent to $\mathfrak{e}$ are in $\mathfrak{f}$ and $\bar{\mathfrak{e}}$, and that $\mathfrak{f}$ is non-slippery, then $\mathfrak{g}/\mathfrak{g}'$ is an induction on e, thus the sequence $\mathfrak{g}/\mathfrak{g}' \cdot S''$ is a sequence of inductions on e and slides of $\mathfrak{f}$ along $\mathfrak{e}$ and $\bar{\mathfrak{e}}$. Finally, if $\mathfrak{g} = \mathfrak{f}$, as $\mathfrak{e}$, $\bar{\mathfrak{e}}$ and $\mathfrak{f}$ are the only three orbits adjacent to $\mathfrak{w}$, we have $\mathfrak{g}' = \mathfrak{e}$ or $\bar{\mathfrak{e}}$. Thus $\mathfrak{g}/\mathfrak{g}' \cdot S''$ is as required. That finishes the proof on the lemma. $\square$

Lemma 3.15. *Let T be a reduced G-tree, e an edge of T and $\mathbf{m}$ an admissible move. If e is toric in T but not in $\mathbf{m} \cdot T$, then there exists a strictly ascending edge f adjacent to e in T and e is p.s.a.d..*

Proof. Note that a slide of a toric edge in a reduced tree must be along an ascending edge. As e is not toric in $\mathbf{m} \cdot T$, if $\mathbf{m}$ is a slide, it is a slide of e along a strictly ascending edge f, and e is strictly ascending in $\mathbf{m} \cdot T$. In this case the lemma is proved.

As e is toric in T, the move $\mathbf{m}$ is not an induction or an $\mathcal{A}^{\pm 1}$-move on e. The move $\mathbf{m}$ may not be an $\mathcal{A}$-move on any edge f, since an $\mathcal{A}$-move leaves unchanged the set of toric edges.

If $\mathbf{m}$ is an induction or an $\mathcal{A}^{-1}$-move on an orbit of edges $\mathfrak{f}$, then $\mathfrak{f}$ is strictly ascending in T and adjacent to e. Thus sliding e along f in T, it become strictly ascending. $\square$

Lemma 3.16. *Let T be a reduced G-tree, and e be a 2-slippery toric edge. Then there exists an admissible sequence S of moves which does not involve $\mathfrak{e}$ and two edges f and g of distinct orbits in $S \cdot T$ such that the slides f/e and g/e are both admissible in $S \cdot T$.*

Proof. Note that if e is toric in a tree T', then there exists two admissible slides on e if and only if the initial vertex of e is of valence at least four.

By definition of being 2-slippery there exists an admissible sequence S of moves and two edges f and g of distinct orbits in $S \cdot T$ such that f/e and g/e are both admissible in $S \cdot T$. But a move involving e may occur in S. We may assume that for every initial subsequence $S_1 \neq S$ of S, the initial vertex of e in $S_1 \cdot T$ is of valence 3. Hence $\mathfrak{e}$ has no adjacent strictly ascending edges, thus by Lemma 3.15, the edge e is toric in every intermediate tree.

Recall that no induction and no $\mathcal{A}^{\pm 1}$-move on a toric edge are admissible, and note that if a toric edge slides along an edge f then f is strictly ascending. Hence, no induction, no $\mathcal{A}^{\pm 1}$-moves on e and no slide of e appears in S.

If a slide on e occurs in S, as the initial vertex of e is of valence 3, by [11, Theorem 1, case 3] this slide may be remove from S.

Thus S may be taken as required. $\square$

Refinement lemmas

Lemma 3.17. *Let T be a G-tree, let $\mathfrak{e}$ be an orbit of edges of T. Let T_e be the tree obtained from T by collapsing every orbit of edges except $\mathfrak{e}$. Let S be a sequence of moves on T, in which no slide along $\mathfrak{e}$ or $\bar{\mathfrak{e}}$, no induction and no $\mathcal{A}^{\pm 1}$-move on $\mathfrak{e}$ occurs, and such that $\mathfrak{e}$ does not vanish (all moves involving $\mathfrak{e}$ except slides of $\mathfrak{e}$ and $\bar{\mathfrak{e}}$). Then $S \cdot T$ refines T_e.*

Proof. In [3], Clay and Forester give a description of the Whitehead moves in terms of expansions and collapses. If a tree $\tilde{T}$ refines T_e and if we perform an expansion on $\tilde{T}$, clearly the resulting tree also refines T_e. If we perform a collapse of an orbit distinct from $\mathfrak{e}$ in $\tilde{T}$ it will still refines T_e (since T_e is obtained from $\tilde{T}$ by collapsing all orbits of edges except $\mathfrak{e}$ in T).

A move which is not a slide along $\mathfrak{e}$ or $\bar{\mathfrak{e}}$, not an induction or an $\mathcal{A}^{\pm 1}$-move on $\mathfrak{e}$ or in which $\mathfrak{e}$ vanishes, may be seen as expansions and collapses with no collapse of $\mathfrak{e}$ (see [3]). Thus $S \cdot T$ still refines T_e. $\qquad\square$

Lemma 3.18. *Let T' be a G-tree. Let f be an edge of T' with initial vertex w and H a group such that $G_f \subset H \subset G_w$. Let T be a refinement of T' obtained by performing an expansion on w with group H and set $\{f\}$. Call e the new edge and v its initial vertex. Let T_e be the tree obtained from T by collapsing all orbits of edges except $\mathfrak{e}$.*

Let $\mathbf{m}$ be a move of T' which is not a slide of $\bar{f}$, a slide along f, an induction of f, an $\mathcal{A}^{\pm 1}$-move of f or in which f vanishes. Then we may perform $\mathbf{m}$ in T, and $\mathbf{m} \cdot T$ refines both T_e and $\mathbf{m} \cdot T'$.

Moreover if $\mathbf{m}$ is not a slide along $\bar{f}$, then the vertex v in $\mathbf{m} \cdot T$ is still of valence two.

Proof. Call v' the terminal vertex of e in T. Let W be the subtree of T which collapses onto w in T'. As $G_e = G_v$, for each vertex v'' in the orbit $\mathfrak{v}$ in T there is only one edge in $\mathfrak{e}$ with initial vertex v''. This implies that W is of diameter 2 and that $W \cap \mathfrak{v}' = \{v'\}$ (and every other vertex of W is in $\mathfrak{v}$).

Given g and h two edges with a common initial vertex in T', from the previous remark, if g and h are not in the orbit $\mathfrak{f}$ (but may be in $\bar{\mathfrak{f}}$), their lifts in T have a common initial vertex. Thus if the move $\mathbf{m}$ is an induction or an $\mathcal{A}^{\pm 1}$-move on an orbit of edges $\mathfrak{g} \neq \mathfrak{f}$, the edge $\mathfrak{g}$ is still pre-ascending in T, and $\mathbf{m}$ may be performed in T. If $\mathbf{m}$ is a slide, then the two concerned orbits still have a common vertex in T and the slide may be performed.

From Lemma 3.17, the tree $\mathbf{m} \cdot T$ refines T_e.

As the edges of initial vertex v in T are either e or in $\mathfrak{f}$, if $\mathbf{m}$ is not a slide along $\bar{f}$, the vertex stays of valence 2 in $\mathbf{m} \cdot T$. $\qquad\square$

Corollary 3.19. *Let T', T and T_e be as in Lemma 3.18. Let S be a sequence of moves on T' in which no move is a slide of $\bar{f}$, a slide along f or $\bar{f}$, an induction of f, an $\mathcal{A}^{\pm 1}$-move of f or in which f vanishes, then S may be performed in T and $S \cdot T$ refines T_e and $S \cdot T'$.*

Proof. This is just a recurrence on the length of S, noticing that as in S no slide along $\bar{f}$ occurs, for all initial subsequence S_1 of S, the valence of v in $S_1 \cdot T$ remains 2, and we may apply Lemma 3.18. $\qquad\square$

Maximality

Lemma 3.20. *Let T_e be a universally compatible G-tree with exactly one orbit of edges $\mathfrak{e}$ (we choose for $\mathfrak{e}$ an arbitrary orientation). There exists a JSJ tree T which refines T_e, such that every edge collapsed in T_e is reduced.*

Proof. Let T' be a reduced JSJ G-tree. As T_e is universally compatible, there exists a common refinement tree between T_e and T'. Call T their least common refinement (for the existence of such a tree see [9]). As T' is a JSJ tree, it dominates T_e (which is universally elliptic), and thus T is also a JSJ tree (see [9]).

Two cases may appear. Either $\mathfrak{e}$ is reduced in T, then $T' = T$, and T is reduced, or $\mathfrak{e}$ is not reduced, and then T has one orbit of (unoriented) edges more than T'.

It remains to prove that we may assume that the only non reduced edges orbit of T is $\mathfrak{e}$.

If there exists another non reduced orbit $\mathfrak{f} \neq \mathfrak{e}, \bar{\mathfrak{e}}$ in T, then collapsing $\mathfrak{f}$ we obtain a JSJ tree which still refines T_e. We may thus collapse one by one non-reduced edge which are not in the orbit of $\mathfrak{e}$ or $\bar{\mathfrak{e}}$ until every edge except maybe $\mathfrak{e}$ is reduced. $\qquad\square$

Proposition 3.21. *Let G be a vGBS group. Assume that G is not an ascending HNN-extension. Let T_e be a universally compatible G-tree with exactly one orbit of edges $\mathfrak{e}$. If T_e is refined by a reduced JSJ tree, then T_{comp} dominates T_e*

Proof. Let T be a reduced JSJ G-tree that refines T_e. Let $\mathfrak{e}$ be the orbit of edge of T' not collapsed in T_e.

If $\mathfrak{e}$ is not collapsed in T_{comp}, then T_{comp} refines T_e thus dominates it. If $\mathfrak{e}$ is collapsed in T_{comp}, by construction of T_{comp} the orbit $\mathfrak{e}$ is non-ascending slippery, p.s.a.d., toric 2-slippery, or vanishing.

1. If $\mathfrak{e}$ non-ascending and slippery or p.s.a.d. in T.

   By Lemma 3.13 there exists an admissible sequence of moves S without slide along $\mathfrak{e}$ or $\bar{\mathfrak{e}}$ and no induction or $\mathcal{A}^{\pm 1}$-move of $\mathfrak{e}$, at the end of which either $\mathfrak{e}$ is pre-ascending or pre-descending, or there exists an admissible slide along $\mathfrak{e}$ or $\bar{\mathfrak{e}}$.

   By Lemma 3.17, the tree $S \cdot T$ refines T_e. By Lemma 3.15, we may assume that e is not toric. Then by Corollary 3.10 cases 1. and 3, the tree T_e is not universally compatible. This is a contradiction.

2. If $\mathfrak{e}$ is 2-slippery and toric in T.

   By Lemmas 3.16 and 3.17, we may assume that there exists two edges f and g in T of distinct orbits with terminal vertices v which may (directly) slide along e . Then by Corollary 3.10 case 3, the tree T_e is not universally compatible. This is a contradiction.

3. If $\mathfrak{e}$ is vanishing non-slippery in T.

   Take e an edge in the orbit $\mathfrak{e}$ and call v the terminal vertex of e. By Lemma 3.14, there exists a strictly ascending edge f with initial vertex v such that v has exactly 3 orbits of adjacent edges $\mathfrak{e}$, $\mathfrak{f}$ and $\bar{\mathfrak{f}}$. Moreover up to perform inductions on $\mathfrak{f}$ and slides of e along $\mathfrak{f}$, we may assume that there exists an admissible $\mathcal{A}^{-1}$-move on $\mathfrak{f}$ in T in which e vanishes. It implies that $G_f \subset G_e$, and f slides along e (this slide is not admissible, since f is ascending). By Corollary 3.10 case 3. the tree T_e is not universally compatible. This is a contradiction.

4. If $\mathfrak{e}$ is vanishing, slippery and toric in T.

   As $\mathfrak{e}$ is vanishing there exists a tree in the deformation space of T in which $\mathfrak{e}$ is not ascending. Then by Lemma 3.15, the orbit $\mathfrak{e}$ is p.s.a.d., and the case is already done in Case 1. $\qquad\square$

Proposition 3.22. *Let G be a vGBS group. Let T_e be a universally compatible tree with one orbit of edges $\mathfrak{e}$ (whose orientation is fixed arbitrarily). If there is a JSJ tree T which refines T_e, in which the unique non-reduced orbit of edges is the one not collapsed in T_e, then T_{comp} dominates T_e.*

Proof. The orbit of edges of T not collapsed in T_e is also called $\mathfrak{e}$. Let T' be the tree obtained by collapsing $\mathfrak{e}$ in T. Then T' is a reduced JSJ tree.

If $\mathfrak{e}$ is obtained by blowing up a dead end vertex in T', then $\mathfrak{e}$ is not collapsed in T_{comp}, thus T_{comp} refines (hence dominates) T_e . We may assume that $\mathfrak{e}$ is not obtained by blowing up a dead end vertex in T'.

Fix e a representative of $\mathfrak{e}$, and let v and v' be the initial and terminal vertices of e. Since e is not reduced, the orbits of v and v' are distinct. We assume $G_v = G_e$. Then there exists f not in the orbit $\mathfrak{e}$ with initial vertex v (otherwise the tree is not minimal).

Lemma 3.23. *If none of the edges with initial vertex v except those in $\mathfrak{e}$ is collapsed in T_{comp} then T_{comp} dominates T_e.*

Proof of Lemma 3.23. Given a hyperbolic h in T_e, its axis contains an edge in the orbit $\mathfrak{e}$ (we may assume it is e), thus in T its axis contains v. As $G_e = G_v$ (and e is not reduced), the edge e is the only edge of $\mathfrak{e}$ with initial or terminal vertex v and thus the axis of h also contains an edge which is not collapsed in T_{comp}. Then h is also hyperbolic in T_{comp}. Then all hyperbolic elements of T_e are hyperbolic in T_{comp}, as G is finitely generated this implies T_{comp} dominates T_e. $\qquad\square$

27

We may thus assume there always exists an edge of T distinct from e with initial vertex v which is collapsed in T_{comp}.

1. We first assume that $G_{v'} \neq G_e$.

   This hypothesis will not be used in the cases 1.2.1.2, 1.2.1.4 and 1.2.2.

1.1. Assume that v has at least three orbits of adjacent edges $\mathfrak{e}$, $\mathfrak{f}$, and $\mathfrak{g}$.

1.1.1. Assume that $\mathfrak{f}$ (or $\mathfrak{g}$) is strictly ascending/descending in T. Up to taking $\bar{\mathfrak{f}}$, we may assume that $\mathfrak{f}$ is ascending. Then $\bar{e}$ may slide along $\mathfrak{f}$ and becomes a reduced edge. We obtain a reduced JSJ tree which refines T_e. This case is done in Proposition 3.21.

1.1.2. Assume that $\mathfrak{f}$ and $\mathfrak{g}$ are not ascending/descending in T.
   Let f and g be representatives of $\mathfrak{f}$ and $\mathfrak{g}$ with initial vertices v.
   We may take a, b, c and d as described in Figure 17: call w and w' the terminal vertices of f and g. Take a in $G_w \setminus G_f$, b in $G_{w'} \setminus G_g$, and c in $G_{v'} \setminus G_e$ that centralizes G_f (it exists by Lemma 3.9). Call $e' = c \cdot e$, $g' = c \cdot g$ and $w'' = c \cdot w'$. Take d in $G_{w''} \setminus G_{g'}$. The sets $\mathcal{E}_{\{a,b\}}(T)$ and $\mathcal{E}_{\{c,d\}}(T)$ are separated by the edge e. In T_e every edge is collapsed except the orbit $\mathfrak{e}$, thus $\mathcal{E}_{\{a,b\}}(T_e) \cap \mathcal{E}_{\{c,d\}}(T_e) = \emptyset$.
   As c centralizes G_f, the group G_f is contained in $G_{e'}$, the edge $\bar{f}$ may slide along e and $\bar{e}'$. We obtain a new tree T'' in which $\mathcal{E}_{\{b,c\}}(T'') \cap \mathcal{E}_{\{a,d\}}(T'') = \emptyset$. By Lemma 3.8, the tree T_e is not universally compatible.
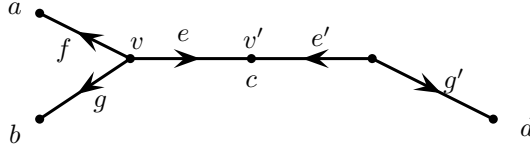

Figure 17

1.1.3. Assume that $\mathfrak{f}$ is toric in T.
   We may assume $\mathfrak{g} = \bar{\mathfrak{f}}$. We proceed as in Figure 18: by Lemma 3.9, we may take a in $G_{v'} \setminus G_e$ that centralizes G_f, and $e' = a \cdot e$ (note that $G_e = G_{e'}$ thus $\bar{f}$ may slide consecutively along e and $\bar{e}'$). Call v'' the initial vertex of e', call $f' = a \cdot f$ and $g' = a \cdot g$ the edges in the orbit of f and g with initial vertex v'', and $e'' = a \cdot e'$ such that when $\bar{f}$ slides along e and $\bar{e}'$, then f' slide along e' and $\bar{e}''$ (note that if G_e is of index 2 in $G_{v'}$, we have $e'' = e$). Call w the terminal vertex of f, w' the terminal vertex of f' and w'' the terminal vertex of g'. Let t be such that $t \cdot v = w$, and t' such that $t' \cdot v' = w'$. Define $b = tat^{-1}$, $c = t'^{-1}at'$ and $d = t'at'^{-1}$.
   Then in T, the sets $\mathcal{E}_{\{a,b\}}(T)$ and $\mathcal{E}_{\{c,d\}}(T)$ are separated by the edge e. In T_e every edge is collapsed except the orbit $\mathfrak{e}$, thus $\mathcal{E}_{\{a,b\}}(T_e) \cap \mathcal{E}_{\{c,d\}}(T_e) = \emptyset$. As a centralizes G_f, we have $G_f \subset G_{e'}$. Making slide $\bar{f}$ along e and $\bar{e}'$, simultaneously $\bar{f}'$ slides along e' and $\bar{e}''$ and g' slides along $t^{-1} \cdot e'$ and $t^{-1} \cdot \bar{e}$.

28

We obtain $\mathcal{E}_{\{b,c\}} \cap \mathcal{E}_{\{a,d\}} = \emptyset$. By Lemma 3.8, the tree T_e is not universally compatible.
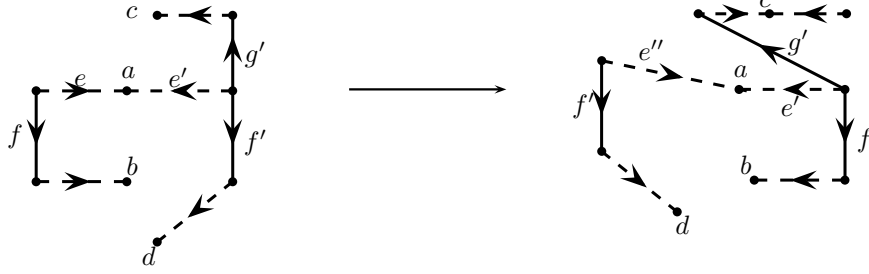

Figure 18: The plain lines are in the orbit $\mathfrak{f}$, the dashed ones in the orbit $\mathfrak{e}$

1.2. Assume that v has only two orbits of adjacent edges $\mathfrak{e}$ and $\mathfrak{f}$.

As $\mathfrak{f}$ is the only orbit with initial orbit of vertices $\mathfrak{v}$ distinct from $\mathfrak{e}$, we may assume that $\mathfrak{f}$ is collapsed in T_{comp} (see Lemma 3.23). Fix f a representative of $\mathfrak{f}$ with initial vertex v.

As every edge of T except the ones in the orbit $\mathfrak{e}$ is reduced and the terminal vertex of $\mathfrak{f}$ is not adjacent to $\mathfrak{v}$, we have $G_f \subsetneq G_e$.

As f is reduced in T, the edge f is not toric in T', thus it is slippery, p.s.a.d., or vanishing.

1.2.1. Assume that f is slippery or p.s.a.d..

From Lemma 3.13, there exists a sequence of moves S admissible in T' such that either f is pre-ascending or pre-descending in $S \cdot T'$, or there exists an admissible slide g/f or $g/\bar{f}$ in $S \cdot T'$. We thus have four cases to treat here. Moreover this sequence may be split in two subsequence S_1 and S_2, such that no move involving f occurs in S_1, and S_2 is composed either of slides of f or of slides of $\bar{f}$. By Corollary 3.19, the sequence S_1 may be performed in T and the tree $S_1 \cdot T$ still refines T_e. Up to replacing T by $S_1 \cdot T$, we may thus assume that $S = S_2$.

1.2.1.1. Assume that S is composed of slides of f and there exists an admissible slide $g/\bar{f}$ in $S \cdot T'$, then by Corollary 3.19 and Lemma 3.18, we may apply the sequence $S \cdot g/\bar{f}$ to T, and obtain a tree $(S \cdot g/\bar{f}) \cdot T$ which refines T_e. Moreover v is now of valence three, we may thus apply Case 1.1.

1.2.1.2. Assume that S is composed of slides of f and f is pre-ascending in $S \cdot T'$. By Corollary 3.19, we may perform S in T, the tree $S \cdot T$ refines T_e, and the vertex v in $S \cdot T$ has valence 2. Thus, up to replacing T by $S \cdot T$ we may assume that the sequence S is empty. Now f is pre-ascending in T'. As in T, the terminal vertex of f is not in $\mathfrak{v}$ it must be in $\mathfrak{v}'$, that is there exists an edge f' in the orbit of f, with terminal vertex v' and $G_{f'} \subsetneq G_f$. We may apply

29

Lemma 3.12 (interchanging f and f'). Thus the tree T_e is not universally compatible, which is a contradiction.

1.2.1.3. Assume here that there exists a sequence $S \cdot g/\bar{f} = \bar{f}/i_1 \ldots \bar{f}/i_q \cdot g/f$ admissible in T'.

In T we may perform the sequence $\bar{f}/e \cdot S \cdot g/\bar{f}$. Up to collapsing all orbits of edges different from $\mathfrak{e}$, $\mathfrak{f}$ and $\mathfrak{g}$, we may assume that S is composed of slides of $\bar{f}$ along edge in $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ (that is every i_j is in $\mathfrak{g} \cup \bar{\mathfrak{g}}$).

Lemma 3.24. *Let T be a G-tree (not necessarily a vGBS tree). Let f and g be two edges of T' verifying the following properties:*

- *for w' the terminal vertex of f, there exists an edge $f' \neq f$ in $\mathfrak{f}$ with terminal vertex w', such that $G_f = G_{f'}$,*
- *there exists an admissible sequence S of slides in T', composed of slides of $\bar{f}$ along edges in $\mathfrak{g}$ or $\bar{\mathfrak{g}}$, ending with a slide of g along f.*

Let w be the initial edge of f and let H be a subgroup such $G_f \subsetneq H \subsetneq G_w$, call T the tree obtained from T' by performing an expansion of w with group H and set $\{f\}$ and T_H the tree obtained by collapses every orbit of edges of T except the one created in the expansion. Then T_H is not universally compatible.

This lemma proves that in Case 1.2.1.3, the tree T_e is not universally compatible. Indeed taking $H = G_e$ we obtain $T_e = T_H$.

Proof of Lemma 3.24. We may assume that for every slide $\bar{f}/g'$ in S we have $G_f \subsetneq G_{g'}$, otherwise $\bar{g}'$ may slide along f, we may take $g = \bar{g}'$ and truncate the sequence S.

Call e the edge of T' obtained in the expansion of w and call v and v' the initial and terminal vertices of e (the vertex v is the initial vertex of f in T). These two cases appear, depending on whether g is strictly ascending (or descending) or not (note that g cannot be toric, otherwise g may never slide along f).

First assume that g is not ascending. Let c such that $f' = c \cdot f$. Let a, b and d be three elements such that a fixes the initial vertex of g but not g (this is possible since g is not ascending), the element b fixes v' but not e and d fixes $c \cdot v'$ but not $e' = c \cdot e$.

Then $\mathcal{E}_{\{a,b\}}(T_H) \cap \mathcal{E}_{\{c,d\}}(T_H) = \emptyset$ (since $\mathcal{E}_{\{a,b\}}(T)$ and $\mathcal{E}_{\{c,d\}}(T)$ are separated by e). Now perform the sequence of deformations $f/e \cdot S \cdot g/f'$ in T. Let g' be an edge on which $\bar{f}$ slides in S. When g slides along $\bar{f}$ and f', g' slides along an edge in $\mathfrak{f} \cup \bar{\mathfrak{f}}$. However since $G_f \subsetneq G_{g'}$, neither g' nor $\bar{g}'$ slides along f when g slides along f. This ensure us that f and f' are between g' and g after performing the slides. Thus the sets $\mathcal{E}_{\{b,c\}}$ and $\mathcal{E}_{\{a,d\}} = \emptyset$ are separated by f' in the resulting tree. Hence T_H is not universally compatible.

Figure 19

If g is strictly ascending, as the sequence S is admissible in T, this implies that f is a loop in T' (otherwise g becomes non-reduced after sliding along f). Thus in T the terminal vertex of f is in $\mathfrak{v}$.

Take f'' in the orbit of f such that its terminal vertex is the initial vertex of g. We may take a' which fixes the initial vertex of f'' but not f''. We may then proceed exactly as in the case where g is not ascending, replacing a by a'. Note that a' does not fix g since f'' is between g and the characteristic space of a'. $\qquad\square$

1.2.1.4. The last possibility is when S is composed of slides of $\bar{f}$ and after performing the sequence $f/e \cdot S$ in T , the edge f is pre-descending.
Collapsing all edges along which f slides in S, we obtain a tree $\tilde{T}$ in which e is not collapsed, and T_e is tree obtained from $\tilde{T}$ by collapsing all orbits of edges except $\mathfrak{e}$, the edge f has initial vertex v, there exists an edge f' in the orbit $\mathfrak{f}$ with terminal vertex v' and $G_f \subset G_{f'}$. By Lemma 3.12, the tree T is not universally compatible.

1.2.2. Assume that f is vanishing non-slippery.
By Lemma 3.14, the edge f is not a loop and there exists in T' a strictly ascending edge g adjacent to f and we may perform an admissible sequence S of inductions on g and slides of f (or $\bar{f}$ depending on whether the vertices of g are in the orbit of the initial or terminal vertex of f) along $\mathfrak{g}$ or $\bar{\mathfrak{g}}$ followed by a $\mathcal{A}^{-1}$-move in which f vanishes.
If the endpoints of g are in the same orbit as the terminal vertex of f then S is also admissible in T and by Lemma 3.17 the resulting tree still refines T_e but there is now three distinct orbits $\mathfrak{e}$, $\mathfrak{g}$, and $\bar{\mathfrak{g}}$ adjacent to v, thus the case is already treated in Case 1.1.
If g is adjacent to (the image of) v in T', then g and $\bar{\mathfrak{g}}$ are adjacent to v' in T. Call $\bar{g}'$ the edge in the orbit of $\bar{\mathfrak{g}}$ adjacent to v', we have $G_{g'} \subsetneq G_g = G_{v'}$. We prove that T_e is not universally compatible by induction on the length of S.
If the sequence S is empty, we have $G_{g'} \subset G_f \subset G_e \subset G_g = G_{v'}$, thus g' may slide consecutively along $\bar{e}$ and f. As f is reduced, g' may then slide along an edge $\bar{f}'$ in the orbit $\bar{\mathfrak{f}}$ different from $\bar{f}$, and then along an edge e' in the orbit $\mathfrak{e}$ (see Figure 20). Collapsing f, the obtained tree $\tilde{T}$ still refines T_e, but then e is reduced and non ascending in $\tilde{T}$ and g' slides consecutively along two distinct edges in the orbits $\mathfrak{e} \cup \bar{\mathfrak{e}}$. By Corollary 3.10 case 2. the tree T_e is not universally compatible.
If the sequence S is not empty, let $\mathbf{m}$ be the first move of S. Then $\mathbf{m}$ is either an induction on $\mathfrak{g}$ or a slide of f on $\mathfrak{g}$ or $\mathfrak{g}'$.
If $\mathbf{m}$ is an induction, then by Lemma 3.18, we may perform the induction in T, and the obtained tree still refines T_e, thus we may reduce the length of S by one. If $\mathbf{m}$ is a slide of f along an edge $\tilde{g}$ in the orbit of $\mathfrak{g}$ or $\bar{\mathfrak{g}}$, then if $G_e \not\subset G_{\tilde{g}}$, then by Lemma 3.12, the tree T_e is not universally compatible. Else $G_e \subset G_{\tilde{g}}$ and $\mathbf{m} \cdot T'$ is refined by $e/\tilde{g} \cdot T$ which also refines T_e. Thus the sequence S may be shortened.

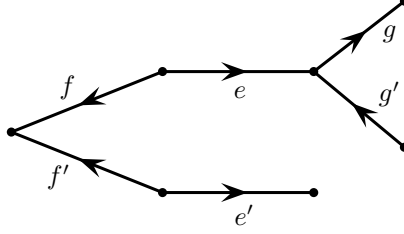2. Assume now that $G_{v'} = G_e$.

31

Figure 20

Let g be an edge with initial vertex v', $g \neq \bar{e}$ (then g is not in the orbit $\bar{\mathfrak{e}}$).

2.1. Assume that there exist at least three orbits of edges g, i and $\bar{e}$ adjacent to v' and at least three orbits of edges f, h and e adjacent to v.

If one of the edges f, g, h or i is strictly ascending, with the same argument of 1.1.1 the edge e (or $\bar{e}$) may slide along it and we are now in Case 1.2.

2.1.1. First assume none of the edges f, g, h or i is toric. We will find a contradiction using Lemma 3.8.
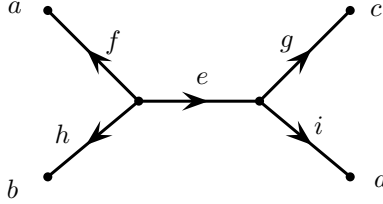

Figure 21

We take a, b, c and d stabilizing respectively the terminal vertices of f, h, g and i but not stabilizing the edges (see Figure 21). In T the spaces $\mathcal{E}_{\{a,b\}}(T)$ and $\mathcal{E}_{\{c,d\}}(T)$ are separated by the edge e which is not collapsed in T_e, thus $\mathcal{E}_{\{a,b\}}(T_e) \cap \mathcal{E}_{\{c,d\}}(T_e) = \emptyset$. But making $\bar{f}$ and $\bar{g}$ slide respectively along e and $\bar{e}$, we obtain $\mathcal{E}_{\{b,c\}} \cap \mathcal{E}_{\{a,d\}} = \emptyset$. Thus T_e is not universally compatible.

2.1.2. Assume that g (or i) is toric and $\mathfrak{h}$ and $\mathfrak{f}$ are not toric. Then $\bar{h}$ may slide along e, then along g and finally along another edge $\bar{e}'$ in $\bar{\mathfrak{e}}$. Call w the terminal vertex of h and w' the terminal vertex of f. Then we may take a in $G_w \setminus G_h$, b in $G_{w'} \setminus G_f$, c which send the initial vertex of g onto its terminal vertex and $d = cbc^{-1}$ (see Figure 22).

In T the spaces $\mathcal{E}_{\{a,b\}}$ and $\mathcal{E}_{\{c,d\}}$ are separated by the edge e which is not collapsed in T, thus $\mathcal{E}_{\{a,b\}} \cap \mathcal{E}_{\{c,d\}} = \emptyset$ in T_e. But making $\bar{h}$ sliding consecutively along e, g and $\bar{e}'$, the spaces $\mathcal{E}_{\{b,c\}}$ and $\mathcal{E}_{\{a,d\}}$ are separated by e'. Thus T_e is not universally compatible.

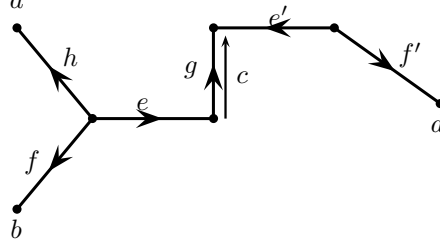2.1.3. Up to symmetries the only case that remains is when f and g are both toric.

32

Figure 22

Define e' as in Case 2.1.2. The edge $\bar{f}$ may slide along e and g and $\bar{e'}$. Collapsing g, we are reduced to the case where $G_e \subsetneq G_{v'}$. The difference here is that $G_{v'}$ is not abelian, however the argument of Case 1.1.3 only use the fact that $\bar{f}$ slides consecutively along two edges in the orbits $\mathfrak{e}$ or $\bar{\mathfrak{e}}$. This is also the case here, we thus have a contradiction with the universal compatibility of T.

2.2. Assume that there are exactly two orbits of edges $\mathfrak{e}$ and $\mathfrak{f}$ adjacent to $\mathfrak{v}$. (Up to symmetries, this is the last remaining case).

As f is collapsed in T_{comp} but is not toric in T', it is slippery, p.s.a.d. or vanishing non slippery.

If f is vanishing non-slippery, the case is already treated in 1.2.2 (which does not use the fact that $G_e \subsetneq G_{v'}$). If f is slippery or p.s.a.d., by Lemma 3.13, there exists a sequence of moves S admissible in T' such that in $S \cdot T'$, either f is pre-ascending or pre-descending, or there exists an admissible slide g/f or $g/\bar{f}$. Moreover this sequence may be split in two subsequences S_1 and S_2, such that no move involving f occurs in S_1, and such that

- either S_2 is composed of slides of f, and there exists a slide along $\bar{f}$ in $S \cdot T'$,
- or S_2 is composed of slides of $\bar{f}$, and there exists a slide along f in $S \cdot T'$,
- or S_2 is composed of slides of f, and f is strictly ascending in $S \cdot T'$,
- or S_2 is composed of slides of $\bar{f}$, and $\bar{f}$ is strictly ascending in $S \cdot T'$.

The last two cases are done in 1.2.1. It remains the first and second cases.

As before, we may assume that there is no strictly ascending edge with initial vertex v'.

We now have to use the fact the vertex w obtained by collapsing e is not a dead end of wall f.

Recall that in T', the non-vanishing vertex w is a dead of wall f if:

- the vertex group of G_f is strictly include in G_w,
- for every edge h with initial vertex w and not in $\mathfrak{f}$ we have $\langle G_f, G_h \rangle = G_w$,
- for every edge h with initial vertex w, we have $G_h \neq G_w$,

33

- there exists an edge g with initial vertex w and not in $\mathfrak{f}$ such that for every edge h with initial vertex w and not in $\mathfrak{f}$ we have $G_h \subset G_g$,

and if for every tree T'' in the deformation space there exists an edge f' (not necessarily in the orbit $\mathfrak{f}$) with initial vertex w having these properties.

Note that we may assume that G_f is strictly include in G_w (because f is reduced in T).

We distinguish the cases depending on whether the fact that w is not a dead end may be seen in T or not.

2.2.1. Assume that one of the listed properties does not hold in T'.

There are three possibilities:

2.2.1.1. There exists an edge h of initial vertex w with $\langle G_f, G_h \rangle \subsetneq G_w$.

Then $G_h \subsetneq G_w$, thus in T the edge is not ascending. Call $\tilde{v}$ the terminal vertex of f and $\tilde{v}'$ the terminal vertex of h. Define $\tilde{T}$ the tree obtained from T' by performing an expansion on the vertex w with group $G_{\tilde{w}'} = \langle G_f, G_h \rangle$ and set $\{f, h\}$.

Call $\tilde{e}$ the edge of the expansion. We prove that $\tilde{T}$ is not compatible with T_e, contradiction the universal compatibility of T_e.

Take $a \in G_{\tilde{v}} \setminus G_f$, $b \in G_{\tilde{v}'} \setminus G_h$ and c in $G_w \setminus \langle G_f, G_h \rangle$. In T_e, the sets $\mathcal{E}_{a,b}(T_e)$ and $\mathcal{E}_c(T_e)$ both contain e, but in $\tilde{T}$, the sets $\mathcal{E}_{a,b}(\tilde{T})$ and $\mathcal{E}_c(\tilde{T})$ are separated be $\tilde{e}$. By Lemma 3.8, the trees T_e and $\tilde{T}$ are not compatible. We have a contradiction.

2.2.1.2. There exists g of initial vertex w with $G_g = G_w$.

By Lemma 3.15, as no edge of endpoint w is strictly ascending, the edge g is toric and remains toric with initial vertex w in any tree of the reduced deformation space.

We use now the fact that there exists a sequence $S = S_1 \cdot S_2$ such that S_2 which is composed either of slide of $\bar{f}$, that finishes by a slide of an edge g' along f, or of slide of f, that finishes by a slide of an edge g' along $\bar{f}$.

In both cases we may assume that $g' \neq g$. Indeed, after preforming S in T', the edge g is still toric. Thus if g slides along f then f is ascending. Let S_1 be the minimal initial subsequence of S such that f is ascending in $S_1 \cdot T'$. Either f is strictly ascending, or it is toric, and then by Lemma 3.15, f may be change in a strictly ascending edge in one slide. Thus up to change S, we may assume that f is pre-ascending, and the case is already done.

If S_2 is composed of slides of f, then by Corollary 3.19 and Lemma 3.18, the tree $(S \cdot g'/f) \cdot T$ refines T_e, but now there v is of valence 3 (with orbits $\mathfrak{f}$, $\bar{\mathfrak{g}}'$ and $\mathfrak{e}$) and v' is of valence at least 3 (with orbits $\mathfrak{g}$, $\bar{\mathfrak{g}}$ and $\bar{\mathfrak{e}}$). We are reduced to Case 2.1.

If S_2 is composed of slides of $\bar{f}$, as $g' \neq g$, first call $T_1 = S_1 \cdot T$. Then collapsing in T_1 the orbit $\mathfrak{g}$ and every edge not in $\mathfrak{f}$, $\mathfrak{e}$ and $\mathfrak{g}'$ in T, we obtain a tree $\tilde{T}$ in which there still exists an admissible sequence of slides of $\bar{f}$ along edges in $\mathfrak{g}'$ and $\bar{\mathfrak{g}}'$, followed by a slide g'/f, but now $G_e \subsetneq G_{v'}$. We may now apply Lemma 3.24 in which $T = \tilde{T}$ and $T_e = T_H$ with $H = G_e$.

2.2.1.3. There are two edges g and g' such that $G_{g'} \not\subset G_g$ and $G_g \not\subset G_{g'}$, and there is no edge h such that $G_{g'} \subset G_h$ and $G_g \subset G_h$. If there exists an edge

h with initial vertex v' such that $G_f \subset G_h$, or $G_f \subset G_h$, by Lemma 3.11, the tree T_e is not universally elliptic.

Thus the case where S_2 is composed of slide of $\bar{f}$ is not possible. It remains the case where S_2 is composed of slide of f, and then there exists a slide $h/\bar{f}$. By Corollary 3.19 and Lemma 3.18 we may perform $(S \cdot h/\bar{f})$ in T, and the obtained tree still refines T_e. The conditions on f, g and g' implies, that G_f, G_g and $G_{g'}$ are maximal for the inclusion between groups of edge with an endpoint w. Applying [7, Proposition 4.10], there are at least 3 edges with initial vertex w and distinct maximal edge groups in $(S \cdot h/\bar{f}) \cdot T'$. As $G_h \subset G_f$, the vertex w is of valence at least 4. As in $(S \cdot h/\bar{f}) \cdot T$ the vertex v is of valence 3 (including the one added by e), this implies that v' is of valences at least 3, we are thus reduced to Case 2.1.

2.2.2. Assume that there exists another tree T'' in which w vanishes or in which the listed properties does not hold.

Let S' be an admissible sequence such that at the end $\mathfrak{w}$ vanishes, or the listed properties does not hold in $T'' = S' \cdot T'$.

2.2.2.1. Assume that no moves involving $\mathfrak{f}$ or $\bar{\mathfrak{f}}$ (except slides of f), occurs in S'. Then by Corollary 3.19, we may apply S' to T and the obtained tree is still refines T_e.

If one of the listed properties does not hold in T'', we are reduced to Case 2.2.1.

If w vanishes, it is now of valence 3 with a strictly ascending vertex g. If g is still strictly ascending in $S' \cdot T$ (that is the endpoints of g are in $\mathfrak{v}'$, then e may slide along g. We obtained a new tree which still refines T_e in which $G_e \subsetneq G_{v'}$. We may thus apply Case 1.2.

If g is no more ascending in $S \cdot T$, then its endpoints are in the orbit $\mathfrak{v}$ and $\mathfrak{v}'$, and by Lemma 3.12 the tree T_e is not universally elliptic.

2.2.2.2. If a move involving $\mathfrak{f}$ or $\bar{\mathfrak{f}}$ different from a slide of f occurs in S', call S'_1 the maximal initial subsequence of S with no moves involving f and let $\mathbf{m}$ be the next move. Then $S'_1 \cdot T$ refines T_e, and the vertex v is still of valence 2. If $\mathbf{m}$ is a slide on $\bar{f}$, in $(S'_1 \cdot \mathbf{m}) \cdot T$ the vertex v is now of valence 3. If v' is of valence greater than 3 then we may apply Case 2.1, if v' is of valence 2 in $(S'_1 \cdot \mathbf{m}) \cdot T$, then we may exchange the role of v and v' and continue to preform the sequence S (which is strictly shorter).

If $\mathbf{m}$ is an induction or an $\mathcal{A}^{\pm 1}$-move on f, then f is pre-ascending in $S'_1 \cdot T'$, and the terminal vertex of f is in the orbit v'. By Lemma 3.12 the tree T_e is not universally compatible.

If $\mathbf{m}$ is an $\mathcal{A}^{-1}$-move in which f vanishes, if the move occurs at w, then w is the vanishing vertex and the case is already treated in 2.2.2.1. If the move occurs at the terminal vertex of f, then $(S'_1 \cdot \mathbf{m}) \cdot T$ still refines T_e and now the valence of v is 3, as previously, if the valence of v' is greater than 3, we may apply Case 2.1, and if v' is of valence 2 in $(S'_1 \cdot \mathbf{m}) \cdot T$, then we may exchange the role of v and v' and continue to preform the sequence S' (which is strictly shorter).

If $\mathbf{m}$ is a slide of $\bar{f}$ along an edge g. If $G_g \neq G_w$ then Corollary 3.11, the tree

T_e is not universally elliptic.

If $G_g = G_w$ then we are in Case 2.2.1.2. □

Theorem 3.25. *Let G be a GBS_n group which is not an ascending HNN-extension, then T_{comp} is the compatibility JSJ tree for decomposition over $\mathbb{Z}^n$ groups.*

Proof. By Proposition 3.7, the tree T_{comp} is universally compatible. By Lemma 3.20, Propositions 3.21 and 3.22, the tree T_{comp} dominates every universally compatible tree with one edge. By [9, Proposition 3.22], this imply that T_{comp} dominates every universally compatible tree. □

4   The Case of $vGBS$ groups

In the previous section, we showed that T_{comp} is the compatibility JSJ tree of a GBS_n group over $\mathbb{Z}^n$ groups. However some GBS_n groups may split over $\mathbb{Z}^{n+1}$. Thus the abelian compatibility JSJ tree may be different from T_{comp}. This is actually the case even for very simple GBS_1 groups such as $BS(2,2) = \langle a, t | ta^2 t^{-1} = a^2 \rangle$.

Indeed, the reduced (abelian and cyclic) JSJ deformation space of $BS(2,2)$ is reduced to one tree with one orbit of (non-oriented) non-ascending edges. Thus T_{comp} is also this JSJ tree.

However $BS(2,2)$ may split over $\mathbb{Z}^2$ is the following way: define

$$A = \langle a, s | sa^2 s^{-1} = a^2 \rangle \simeq BS(2,2)$$

and

$$B = \langle b, t | bt = tb \rangle \simeq \mathbb{Z}^2.$$

Define $G = A *_{a^2 = b, s = t^2} B$, then

$$G = \langle a, t | ta^2 t^{-1} = a^2 \rangle \simeq BS(2,2).$$

The element t^2 is hyperbolic in the JSJ decomposition, but belongs to the edge group of the decomposition $A *_{a^2 = b, s = t^2} B$. Thus the tree associated to the decomposition $A *_{a^2 = b, s = t^2} B$ is not compatible with the JSJ tree (which is T_{comp}). This implies that T_{comp} is not the abelian compatible JSJ tree. In fact in this case the compatibility JSJ tree is the trivial G-tree.

4.1   Definitions and construction tools

An element $t \in G$ is *bi-elliptic* in a G-tree T if it stabilizes an edge of the tree, *potentially bi-elliptic* if there exists a reduced G-tree over abelian groups in which it is bi-elliptic. An edge is *bearing* if it is contained in the axis of some hyperbolic element which is potentially bi-elliptic.

Let T be a reduced G-tree. Let e be an edge of T with initial vertex v. Let w be the vertex obtained by collapsing the ascending edges adjacent to v. Assume that G_v is abelian.

Inert edges Let e be an edge of T verifying the following four properties in every tree of the deformation space of G :

- the initial vertex v of e is of valence at least 2,
- the edge e is bearing,
- the edge e is not vanishing, not slippery and not p.s.a.d.,
- the centralizer of G_e in G_w is G_v.

The edge e is *inert* if furthermore it has one of the following properties:

- the group G_v/G_e is not isomorphic to $\mathbb{Z}/p^k\mathbb{Z}$ for any prime number p, and for all $f \notin \mathfrak{e}$ with initial vertex v, we have $\langle G_f, G_e \rangle = G_v$,
- there exit a prime number p and an integer k such that $G_v/G_e \simeq \mathbb{Z}/p^k\mathbb{Z}$, and for all $f \notin \mathfrak{e}$ of initial vertex v, we have the implication if $G_e \subset G_f$ then f is toric,

and if e has these properties in every tree of the reduced deformation space (note that as e is not p.s.a.d., we may follow e in every tree of the reduced deformation space).

The edge e is said to be *inert of type 1* in the first case and *inert of type 2* in the second.

An orbit of edge $\mathfrak{e}$ is *inert* if one of its representative is inert.

The reason why we have to distinguish inert edges of type 1 and 2 comes from the following fact:

Fact. *Let G_e be a subgroup of $G_v \simeq \mathbb{Z}^n$. The subgroups of G_v containing G_e are pairwise comparable for inclusion if and only if $G_v/G_e \simeq \mathbb{Z}/p^k\mathbb{Z}$ for some prime number p and integer k.*

Lemma 4.1. *If e is inert then G_v is maximal abelian and if there exists an edge $f \notin \mathfrak{e}$ of initial vertex v such that $G_e \subset G_f$ then f is toric.*

Proof. Let us first prove the implication $G_e \subset G_f \Rightarrow f$ is toric.

If e is of type 2, this point is in the definition. If e is of type 1, call v' the terminal vertex of f. By definition $\langle G_f, G_e \rangle = G_v$ thus the edge f is ascending and e may slide along f, thus after sliding as e is still inert we must have $\langle G_e, G_f \rangle = G_{v'}$, as $G_e \subset G_f$ we have $G_f = G_{v'}$ and f is toric.

A consequence is that an ascending edge adjacent to v is toric. Call $\mathcal{V}$ the subtree fixed by G_v. Then every vertex of $\mathcal{V}$ is in $\mathfrak{v}$ and every edge of $\mathcal{V}$ is toric, thus $\mathcal{V}$ collapses onto w. Any element that commute to G_v must either fix a

37

point of $\mathcal{V}$ if it is elliptic, or have its axis contained in $\mathcal{V}$ if it is hyperbolic. In both cases it belongs to G_w. $\qquad\square$

4.2 Compatibility JSJ tree over abelian groups

Expansion of inert edges If an edge e is inert of type 1, to *expand* e consists in performing an expansion on v with group G_v and set $\{e\}$ (see definition in Section 2).

If an edge e is inert of type 2, we define H_e as the largest subgroup such that for every tree in the reduced deformation space of T, and every edge f with same initial vertex as e, we have $H_e \subset \langle G_e, G_f \rangle$.

The edge e is also bearing, and $G_v/G_e \simeq \mathbb{Z}/p^k\mathbb{Z}$. Let a be an element of G_v whose projection into G_v/G_e is a generator. Let i be the minimal integer such that there exist q with $p \nmid q$ and a potentially bi-elliptic element whose axis contains both e and $a^{p^i q} \cdot e$. Note that as $a^{p^k} \cdot e = e$, we have $i \leq k$. Define $F_e = \langle G_e, a^{p^i} \rangle$

If $F_e \subset H_e$, to *expand* e consists in performing an expansion on v with group F_e and set $\{e\}$. The expansion of e is not defined when $F_e \not\subset H_e$

Construction of the abelian compatibility JSJ tree

Definition 4.2. *Let G be a vGBS group. Let T be a reduced abelian JSJ tree of G. Call $\mathcal{E}$ the set of non-ascending slippery, strictly ascending and toric 2-slippery edges.*

We define T_{ab} as the G-tree obtained by expanding inert edges, blowing up dead ends, then collapsing simultaneously every edge of $\mathcal{E}$ and every bearing edges.

Note that the wall of a dead end cannot be inert. There may be several blow up and expansion around a vertex, but the blowing up of dead ends, and the expansion of inert edges are made on distinct edges.

Lemma 4.3. *The tree T_{ab} does not depend on the choice of T in the reduced JSJ deformation space of G.*

Proof. We show that the tree obtained before collapsing bearing edges does not depend on T.

Using Proposition 3.2, we just have to prove that the expansion of an inert edge commutes with the Whitehead moves and collapses.

Let T be a reduced JSJ tree and **m** be an admissible move on T.

- Assume that **m** is a slide e/f. Let g be an inert edge. If $g \neq f, \bar{f}, \bar{e}$, then obviously the slide commutes with the expansion. As f and $\bar{f}$ are slippery,

38

they are not inert. If $\bar{e}$ is inert, by Lemma 4.1, the edge f is toric. Then either f is 2-slippery and must be collapsed, and thus it is the same to expand $\bar{e}$ and collapse f than sliding e, expand $\bar{e}$ and collapse f, or f is not 2-slippery, thus the slide has no effect on the tree [11, Theorem 1, case 3].

- If **m** is an induction, or an $\mathcal{A}^{-1}$-move on an edge e of initial vertex v, as e is strictly ascending, the group G_v is not maximal abelian, thus no edge adjacent to v is inert. $\qquad\square$

Call $\tilde{T}_{ab}$ the tree obtained from T_{ab} by collapsing edges coming from an expansion or a blow-up.

Note that $\tilde{T}_{ab}$ is refined by every JSJ tree.

Lemma 4.4. *The tree T_{ab} is universally elliptic.*

Proof. As every edge of $\tilde{T}_{ab}$ comes from an edge of a reduced JSJ tree, the tree $\tilde{T}_{ab}$ is universally elliptic.

We have to show that the edges obtained by a blow up or an expansion are universally elliptic. Let T_J be an abelian JSJ tree, and let e be an edge of T_J of initial vertex v.

If the vertex v is a dead end with wall e or if e is an inert edge of type 1, let $f \notin \mathfrak{c}$ be an edge of T_J with initial vertex v. Then $G_v = \langle G_e, G_f \rangle$, from [1, Theorem 7.3] this implies that G_v is abelian. The group G_v is abelian and generated by universally elliptic elements thus universally elliptic. Thus the edge comming from a blow up of v or an expansion of e is universally elliptic

If e is inert of type 2, call f the edge coming from the expansion. Then G_e is universally elliptic and of finite index in G_f, thus G_f is also universally elliptic. $\qquad\square$

4.3 Universal compatibility

Lemma 4.5. *The tree T_{ab} is compatible with every abelian JSJ tree T.*

Proof. For f an edge of T_{ab} call T_f the tree obtained by collapsing every edge of T_{ab} except f.

By the point 1. of [9, Proposition 3.22], we have to show that T is compatible with T_f for every edge f of T_{ab}.

Call $\tilde{T}$ a reduced JSJ tree refined by T. By construction, $\tilde{T}$ refines every tree T_f with f an edge of T_{ab} not obtained by a blow up or an expansion, thus refines $\tilde{T}_{ab}$. Hence T also refines $\tilde{T}_{ab}$.

If f comes from a blow up or an expansion, then we may apply Lemma 3.3. We obtain that T is compatible with T_f.

If follows that T is compatible with T_{ab}. $\qquad\square$

39

Lemma 4.6. *Let G be a vGBS group. Any G-tree over abelian groups is elliptic with respect to T_{ab} and $\tilde{T}_{ab}$*

Proof. Let E be an abelian subgroup of G and assume that G splits non-trivially over E as an amalgam or an HNN extension. By definition every element of E is potentially bi-elliptic. As every bearing edges are collapsed, every element of E is elliptic in T_{ab}, but as E is finitely generated, this implies that E is elliptic. $\qquad\square$

Lemma 4.7. *Let T_1 and T_2 be two reduced universally elliptic G-trees. If every JSJ tree refines T_2 and if T_1 dominates T_2 then T_1 refines T_2.*

Proof. By [8, Lemma 5.3], there exists a JSJ tree T_{JSJ} which refines T_1. As any JSJ refines T_2, the tree T_{JSJ} also refines T_2. We prove that every edge of T_{JSJ} collapsed in T_1 is also collapsed in T_2.

We may collapse one by one the non-reduced edges of T_{JSJ} which must be collapsed in T_1. The remaining tree is still a JSJ tree, hence still refines T_2. Let e be a reduced edge that is collapsed in T_1. As e is reduced, there are more elliptic elements after collapsing e. Yet T_1 dominates T_2, thus elliptic of T_1 are elliptic of T_2. Necessarily, the edge e is collapsed in T_2. $\qquad\square$

Lemma 4.8. *Let T_1 and T_2 be G-trees. If every JSJ tree refines T_2 and if T_1 is elliptic with respect to T_2, then T_1 and T_2 are compatible.*

Proof. Note that as T_2 is refined by every JSJ tree it is reduced.

As T_1 is elliptic with respect to T_2, by [8, Lemma 3.2] there exists T' a refining tree of T_1 which dominates T_2. As T_2 is universally elliptic, the tree T'' obtained from T' by collapsing the non universally elliptic edges still dominates T_2. We may also assume that T'' is reduced. The trees T'' and T_2 are universally elliptic and reduced. By Lemma 4.7 the tree T'' refines T_2. Thus T' is a common refinement of T_1 and T_2. $\qquad\square$

Corollary 4.9. *Let G be a vGBS group. Then $\tilde{T}_{ab}$ is universally compatible.*

Proof. That is an immediate consequence of Lemmas 4.6 and 4.8. $\qquad\square$

Lemma 4.10. *Let T_J be a reduced JSJ tree. Let e be an inert edge. Call T'_J the tree obtained by expending e in T_J. Call f the new edge of T'_J adjacent to e. Call v and v' the initial and terminal vertices of f, and T_v and $T_{v'}$ the connected components of v and v' in $T'_J \setminus f$. Then*

- *every t whose characteristic space is contained in $T_{v'}$ and which belongs to the centralizer of G_e belongs to $G_{v'}$,*
- *for every edge $e' \neq f$ adjacent to v' in $T_{v'}$ and every element t whose characteristic space is contained in T_v, the group $G_{e'}$ is not contained in the centralizer of t.*

Moreover, in both point, we may replace G_e and $G_{e'}$ by G_f.

Proof. Call w the vertex obtained by collapsing in T_J every toric edge adjacent to the initial vertex of e. Let t be an element whose characteristic space is contained in $T_{v'}$. Assume that it commutes with every element of G_e. If t is hyperbolic then G_e is contained in every edge group of it axis. But applying Lemma 4.1, we easily see that every edge of the axis is toric, and that the whole axis of t in T_J collapses onto w. Thus by the fourth point of the definition of an inert edge, the element t belongs to $G_{v'}$.

If t is elliptic, then again applying Lemma 4.1, then path between v' and the characteristic space of t contains only toric edges. Thus t belongs to G_w, thus to $G_{v'}$.

As $G_e \subset G_f$ the result is also true for G_f.

Let us treat now the second point of the lemma. As e is inert, it is neither slippery nor pre-ascending. Thus $G_{e'}$ is not include in G_e. As every edge of T_v adjacent to v has edge group G_e, if $G_{e'}$ centralizes an element t with characteristic space intersecting T_v, then t belongs to G_v. As $G_v \subset G_{v'}$, the characteristic space of t is not contained in T_v.

As G_f is not include in G_e, the result is also true replacing $G_{e'}$ by G_f. $\square$

Let e be an edge of T_{ab} coming from the expansion of an inert edge or the blow up of a dead end. Call T_e the tree obtained by collapsing every edge in T_{ab} except the ones in $\mathfrak{e}$.

Lemma 4.11. *Let T be an abelian G-tree which dominates T_e, then T is compatible with T_e.*

Proof. The method consists in recreating the edge e in T.

Let T' be the G-tree obtained from T by collapsing every non-universally elliptic edge. By Lemma 4.4 l'arbre T_e is universally elliptic, thus the tree T' still dominates T_e.

Now, T' is universally elliptic, thus refined by a JSJ tree. By Lemma 4.5, the tree T_e is compatible with any JSJ tree, therefore T' and T_e are compatible. Call T'_e the least common refinement. The tree T'_e is also universally elliptic. As T' dominates T_e, the trees T' and T'_e belong to the same deformation space.

If $T'_e = T'$ then T refines T_e thus is compatible with it. Otherwise, T'_e has one orbit of edges $\mathfrak{e}$ more than T' and this orbit is not reduced. Call v and v' the endpoints of e in T'_e such that $G_{v'} = G_e \subset G_v$.

Fix $\tilde{T}_e$ a JSJ tree refining T'_e. We may assume that if an edge of $\tilde{T}_e$ is not reduced, then it is not collapsed in T'_e. Call also e the unique lift of e in $\tilde{T}_e$. As $G_e = G_{v'}$, the vertex v' as a unique preimage in $\tilde{T}_e$. Indeed the terminal vertex $\tilde{v}$ of e in $\tilde{T}_e$ has group $G_{v'}$ and collapses onto v', if an edge adjacent to $\tilde{v}$ collapse

onto v', as $G_{\tilde{v}} = G_{v'}$, this edge may not be reduced. This is in contradiction with the construction of $\tilde{T}_e$.

Let $\widehat{T}_e$ be a reduced JSJ tree refined by $\tilde{T}_e$. As e in T_{ab} comes from an expansion or a blow up, call $\hat{v}$ the vertex on which e collapses in $\widehat{T}_e$. Let $\mathcal{G}$ be the set of edges of $\widehat{T}_e$ adjacent to $\hat{v}$.

Let $f \neq e, \bar{e}$ be an edge of T'_e with initial vertex v', call $\tilde{f}$ its lift in $\tilde{T}_e$. Then $\tilde{f}$ has initial vertex $\tilde{v}$. If $\tilde{f}$ is reduced then it is the lift of an edge of $\mathcal{G}$. If it is not reduced, as $\tilde{T}_e$ is minimal, it is in the path of between $\tilde{v}$ and the lift of some edge g of $\mathcal{G}$. But $G_g \subset G_{\tilde{v}'}$, thus $G_g \subset G_{\tilde{f}}$. Thus in both cases there exists an edge $g \in \mathcal{G}$ such that $G_g \subset G_f$.

Let w' be the vertex on which e collapses in T'. We have $G_w = G_v$. Call T_w the subtree of T which collapses on w'. As $G_e \subset G_w$ is universally elliptic, there exists a vertex w of T_w fixed by G_e. We claim that w is uniquely determined.

Indeed, the tree T' is obtained from T by collapsing non-universally elliptic edges. Thus T_w has no universally elliptic edges. But the centralizer of G_e in $G_{w'}$ is $G_{\hat{v}}$ (see Lemma 4.10) which is universally elliptic. Thus if an edge of T_w is fixed by G_e, its stabilizer is contained in $G_{\hat{v}}$, and thus universally elliptic. Which is absurd.

Call $\mathcal{F}$ the set of edges of T'_e with initial vertex v'. Take $f \in \mathcal{F}$ with $f \neq \bar{e}$ an edge of T'_e of initial vertex v', as f is not collapsed in T', it has a unique lift in T (that we also call f), let us show that f has initial vertex w in T.

There exists $g \in \mathcal{G}$ such that $G_g \subset G_f$, now by Lemma 4.10, the element which commutes with G_f and fixes a point in T_w belongs to $G_{\hat{v}}$. But again, as $G_{\hat{v}}$ is universally elliptic, no edge of T_w has its stabilizer contained in $G_{\hat{w}}$, thus the initial edge of f must be w. We may thus perform and expansion on w with group G_e, and set $\mathcal{F}$.

The obtained tree refines T'_e, thus T_e. $\qquad\square$

Proposition 4.12. *The tree T_{ab} is universally compatible.*

Proof. Let $\tilde{T}$ be an abelian G-tree. By Lemma 4.6, the tree $\tilde{T}$ is elliptic with respect to T_{ab}, applying [8, Lemma 3.2], there exists T a refining tree of $\tilde{T}$ which dominates T_{ab}. Let e be an edge of T_{ab}, and T_e the tree obtained by collapsing every edge orbit except $\mathfrak{e}$.

If e is not collapsed in $\tilde{T}_{ab}$, applying Corollary 4.9, the tree T is compatible with T_e. If e is collapsed in $\tilde{T}_{ab}$, applying Lemma 4.11, the tree T is compatible with T_e. From [9, Proposition 3.14], the T is compatible with $\tilde{T}$, and consequently $\tilde{T}$ is compatible with T_{ab}. $\qquad\square$

4.4 Maximality

Proposition 4.13. *Let G be a vGBS group. The tree T_{ab} dominates every universally compatible G-tree.*

Proof. Let T' be a universally compatible G-tree with a unique orbit of edges $\mathfrak{e}$. Let $\tilde{T}_J$ be a JSJ tree which refines T'. We may assume that $\tilde{T}_J$ has a most one orbit of non-reduced edges and if so this orbit is $\mathfrak{e}$.

If $\tilde{T}_J$ is reduced, the only case not treated in Proposition 3.21, is when $\mathfrak{e}$ is bearing. But if $\mathfrak{e}$ is bearing, this exactly means that there exists an hyperbolic element of T' which belongs to an edge group of another abelian G-tree. This immediately contradict the universal compatibility of T'.

If $\tilde{T}_J$ is not reduced, recall the notation of Proposition 3.22, fix e a representative of $\mathfrak{e}$, call v the initial vertex of e and v' its terminal vertex. We assume that $G_v = G_e$. Call w the vertex obtained by collapsing every toric edge adjacent to v'. The vertex v is of valence at least 2, and $f \notin \mathfrak{e}$ is an edge of initial vertex v. The cases not treated in Proposition 3.22 are the ones where f is an inert edge and e is not the expansion on f. To be more specific, after excluding the subcases already treated, the remaining cases are

the valence of v is 2, the edge f is bearing but not pre-ascending, not pre-descending, not slippery, not vanishing (and not toric) and

1. either $G_e = G_v \subsetneq G_{v'}$ and there is no prime p such that $G_{v'}/G_f \simeq \mathbb{Z}/p^k\mathbb{Z}$,

2. or the centralizer of G_e in G_w strictly contains G_v.

3. or $G_e = G_v \subsetneq G_{v'}$, the edge f is inert of type 2, and $G_e \subsetneq F_f$,

4. or $G_e = G_v \subsetneq G_{v'}$ and f is inert of type 2, and $F_f \subsetneq G_e$,

5. or one of the property of inert edges does not hold for f in a reduced JSJ tree T'_J.

Remember that if f is inert of type 2, as $G_{v'}/G_f \simeq \mathbb{Z}/p^k\mathbb{Z}$ with p a prime number, we must have $G_e \subset H_f$ or $H_f \subset G_e$. The definition of F_f is given in Section 4.2.

In the two first cases there exists an abelian subgroup E of G_w containing G_f and not comparable with G_e. Let a be an element in G_e but not in E and b an element in E but not in G_e.

Let $\widehat{T}$ be the tree obtained from T_J by performing an expansion on v' with group E, and set $\{f\}$.

Let c be an element fixing the terminal edge of f but not f (such an element exists since f is not ascending). Then a fixes the edge e in $\mathcal{E}_{\{c,bcb^{-1}\}}(T_e)$, but no vertex of $\mathcal{E}_{\{c,bcb^{-1}\}}(\widehat{T})$, by Lemma 3.8, these T_e and $\widehat{T}$ are not compatible. Thus T_e is not universally compatible.

In the third case, by construction of F_f, there exist an element $a \in G_{v'} \subsetneq F_f$ and a potentially bi-elliptic element whose axis contains f and $a \cdot f$, as e is between f and $a \cdot f$ this implies that e is bearing, thus T_e cannot be universally compatible.

In the fourth case, the tree T_{ab} obviously dominates G_e.

In the last case, call T_J the reduced JSJ obtained by collapsing e in $\tilde{T}_J$. There exists a sequence S of admissible moves such that $S \cdot T_J = T'_J$. We prove that we may apply a sequence S' at $\tilde{T}_J$ such that $S \cdot \tilde{T}_J$ still refines T'. As f is not slippery, and not pre-ascending, the only way f may appear in S is in a slide f/g or $\bar{f}/g$. From Lemma 3.18, if no slide of $\bar{f}$ occurs, then S is admissible in $\tilde{T}_J$ and $S \cdot \tilde{T}_J$ refines T'. If a slide of $\bar{f}$ along an edge g occurs, then g is toric. Thus in $\tilde{T}_J$ it suffices to replace f/g by e/g. $\qquad\square$

5   Algorithmic construction

5.1   Inconstructibility of the abelian compatibility JSJ of $vGBS$ groups

In [2], Bogopolski, Martino and Ventura show that there exist four automorphisms $(\varphi_i)_{1 \le i \le 4}$ of $\mathbb{Z}^4$ such that the semi-direct product $G = \mathbb{Z}^4 \rtimes_{(\varphi_i)} F_4$, where F_4 is the free group in four generators, has an undecidable conjugacy problem. From this example, we give a family of $vGBS$ groups for which it is impossible to describe algorithmically the abelian compatibility JSJ tree.

Proposition 5.1. *Let G the group described in the previous paragraph and T be the G-tree associated to the presentation with one orbit of vertices (whose stabilizer are all equal to $\mathbb{Z}^4$) and four orbits of edges. Let x_1 and x_2 be two primitive elements of $\mathbb{Z}^4$. We build $G(x_1, x_2)$ the fundamental group of the following graph of groups $\Gamma(x_1, x_2)$: we add to Γ two loops ℓ_1 and ℓ_2 whose edge groups an both $\mathbb{Z}$. The injections of G_{ℓ_i} are defined by $n \mapsto x_i^n$.*

The group $G(x_1, x_2)$ is then isomorphic to $\langle G, t_1, t_2 | t_i x_i t_i^{-1} = x_i \rangle$. Then the compatibility JSJ tree of $G(x_1, x_2)$ is trivial if and only if x_1 et x_2 are conjugated.

Proof. Call $T(x_1, x_2)$ the Bass-Serre tree associated to $\Gamma(x_1, x_2)$. The G-tree $T(x_1, x_2)$ is an abelian JSJ tree of $G(x_1, x_2)$. The four toric edges are collapsed in T_{ab} since they are 2-slippery, and the vertex is not a dead end. As $\langle x_1, x_2 \rangle$ is not of finite index in the vertex group, no edge is inert. As x_1 and x_2 are primitive and the φ are in $Gl_n(\mathbb{Z})$, the edges l_i are not p.s.a.. Thus l_1 and l_2 are collapsed if and only if they are slippery. But the only edge which may slide along l_1 is l_2. Thus l_1 is collapsed if and only if x_2 is conjugate to a power of x_1 in G, but again as the elements x_1 and x_2 are primitive elements, the edge l_1 is collapsed if and only if x_1 and x_2 are conjugated in G. $\qquad\square$

Corollary 5.2. *There is no algorithm that permit to decide whether $G(x_1, x_2)$ has a trivial compatibility JSJ tree.*

Proof. The conjugacy problem between hyperbolic elements in $vGBS$ groups is decidable. From [2] we may deduce that the conjugacy problem between elliptic

elements of G is not solvable. Proposition 5.1 allows to conclude. $\square$

References

[1] B. Beeker. Abelian JSJ decomposition of graphs of free abelian groups. *arxiv.org/abs/1012.1311*, 2010.

[2] O. Bogopolski, A. Martino, and E. Ventura. Orbit decidability and the conjugacy problem for some extensions of groups. *Trans. Am. Math. Soc.*, 362:2003–2036, 2010.

[3] M. Clay and M. Forester. Whitehead moves for G-trees. *Bull. Lond. Math. Soc.*, 41(2):205–212, 2009.

[4] M. J. Dunwoody and M. E. Sageev. JSJ-splittings for finitely presented groups over slender groups. *Invent. Math.*, 135(1):25–44, 1999.

[5] M. Forester. On uniqueness of jsj decompositions of finitely generated groups. *Comment. Math. Helv.*, 78:740–751, 2003.

[6] K. Fujiwara and P. Papasoglu. JSJ-decompositions of finitely presented groups and complexes of groups. *Geom. Funct. Anal.*, 16(1):70–125, 2006.

[7] V. Guirardel and G. Levitt. Deformation spaces of trees. *Groups Geom. Dyn.*, 1(2):135–181, 2007.

[8] V. Guirardel and G. Levitt. JSJ decompositions: definitions, existence, uniqueness. I: The JSJ deformation space. *arxiv.org/abs/0911.3173*, 2009.

[9] V. Guirardel and G. Levitt. JSJ decompositions: definitions, existence, uniqueness. II. Compatibility and acylindricity. *arxiv.org/abs/1002.4564*, 2010.

[10] P. H. Kropholler. An analogue of the torus decomposition theorem for certain poincaré duality groups. *Proc. London Math. Soc.*, 60(3):503–529, 1990.

[11] G. Levitt. Characterizing rigid simplicial actions on trees. In *Geometric methods in group theory*, volume 372 of *Contemp. Math.*, pages 27–33. Amer. Math. Soc., Providence, RI, 2005.

[12] E. Rips and Z. Sela. Cyclic splittings of finitely presented groups and the canonical JSJ decomposition. *Ann. of Math. (2)*, 146(1):53–109, 1997.

[13] Z. Sela. Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie groups. II. *Geom. Funct. Anal.*, 7(3):561–593, 1997.

Benjamin Beeker
MAPMO,
Université d'Orléans, UFR Sciences
Bâtiment de mathématiques - Rue de Chartres
B.P. 6759
45067 Orléans Cedex 2
France
benjamin.beeker@univ-orleans.fr