HOW MANY TUPLES OF GROUP ELEMENTS HAVE A GIVEN PROPERTY?

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with an appendix by Dmitrii V. Trushin

Generalising Solomon's theorem, C. Gordon and F. Rodriguez-Villegas have proven recently that, in any group, the number of solutions to a system of coefficient-free equations is divisible by the order of this group whenever the rank of the matrix composed of the exponent sums of *i*th unknown in *j*th equation is less than the number of unknowns. We generalise this result in two directions: first, we consider equations with coefficients, and secondly, we consider not only systems of equations but also any first-order formulae in the group language (with constants). Our theorem implies some amusing facts; for example, the number of group elements whose squares lie in a given subgroup is divisible by the order this subgroup.

0. Introduction

Solomon theorem [Solo69]. In any group, the number of solutions to a system of coefficient-free equations is divisible by the order of this group if the number of equations is less than the number of unknowns.

This topic was developed in different directions (see, e.g. [Stru95], [AmV11], [Isaa70], and references therein), but the simplest and most natural generalisation of this theorem was obtained quite recently.

Gordon–Rodriguez-Villegas theorem [GRV12]. In any group, the number of solutions to a system of coefficientfree equations is divisible by the order of this group if the rank of the matrix composed of exponent sums of *i*th unknown in *j*th equation is less than the number of unknowns.

For example, the Solomon theorem does not apply to the system $x^2y^3[x, y]y^{-1} = 1 = (yx)^2$, but nevertheless, the number of its solutions in a group is divisible by the order of the group according to the Gordon–Rodriguez-Villegas theorem, because the rank of the corresponding matrix $\binom{2}{2}{2}$ is one while there are two unknowns.

What if the rank of the matrix is much less than the number of unknowns? Does this imply that the number of solutions must be divisible by a higher power of the order of the group? The answer is *no*. And even a weaker conjecture is false. In Section 2, we give an example.

We generalise the Gordon-Rodriguez-Villegas theorem in other directions. We study equations with coefficients, and not only systems of equations but also any first-order formulae. The main theorem implies a lot of non-obvious facts, e.g. that mentioned in the abstract. Section 1 contains the main theorem. In Section 2, we give several examples. In Section 3, we prove the main theorem. In particular, our argument proves the Gordon-Rodriguez-Villegas theorem and is (in this case) somewhat simpler than the original one (from our point of view), although our proof is based on the same ideas. In Section 4, we give a direct proof of the amusing fact from the abstract. We have not succeeded in finding this fact in literature,^{*)} though it might be easily obtained from a result of P. Hall (see Section 1) generalising the well-known Frobenius theorem [Frob03] (see also [Hall59]) which says that the number of solutions to the equation $x^n = g$ is divisible by GCD(n, |C(g)|). The Frobenius theorem was generalised in various directions (see, e.g. [Hall36], [Kula38], [Sehg62], [BrTh88], [AsTa01], and references therein).

The appendix written by D. V. Trushin contains a proof of a pure logical proposition which makes it possible to simplify somewhat the statement of a corollary of the main theorem at the expense of preliminarily transformation of the logic formula.

Our notation is mainly standard. Note only that if $k \in \mathbb{Z}$ and x and y are elements of a group, then x^y , x^{ky} , and x^{-y} denote $y^{-1}xy$, $y^{-1}x^ky$ and $y^{-1}x^{-1}y$, respectively. A commutator [x, y] is $x^{-1}y^{-1}xy$. If X is a subset of a group, then $|X|, \langle X \rangle$, and C(X) denote the cardinality of X, the subgroup generated by X, and the centraliser of X. The letter \mathbb{Z} denotes the set of integers.

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^{*)} In 2017, we learned that this fact was proven in [Iwa82].

1. Main theorem

Consider the group language L over a group G. This language has two functional symbols: \cdot and $^{-1}$, and also, for each element of G, there is a constant symbol g. We do not assume that the group is finite (although it is in the majority of interesting cases); the assertions about divisibility should be understood in the sense of cardinal arithmetics: any infinite cardinal is divisible by any smaller or equal nonzero cardinal (and zero is divisible by any cardinal).

Consider an arbitrary first-order formula φ in the language L. Each atomic subformula can be written in the form

$$u = 1$$
,

where the words $u \in G * F$ and F is the free group generated by all (free and bound) variables of φ . Thus, the words u (possibly, different for different subformulae) can contain free and bound variables and elements of G (called *coefficients* of φ).

Now, we define the digraph $\Gamma(\varphi)$ of the formula φ as follows. The vertices of $\Gamma(\varphi)$ are all bound variables of φ . Each atomic subformula containing bound variables has the form

$$v_1(y_1)w_1(x_1\ldots,x_n)\ldots v_r(y_r)w_r(x_1\ldots,x_n)=h,$$

where y_i are bound variables of the formula φ (not necessarily different), $x_1 \dots, x_n$ are all (different) free variables of φ , the words $v_i(y_i)$ belong to the free product $G * \langle y_i \rangle_{\infty}$ of G and the infinite cyclic group generated by the letter y_i , the words $w_i(x_1 \dots, x_n)$ belong to the free product $G * F(x_1 \dots, x_n)$ of G and the free group with basis $x_1 \dots, x_n$, and $h \in G$. Let us connect the vertices y_i and y_{i+1} (subscripts modulo r) by a directed edge labelled by an integer tuple $(\alpha_1, \dots, \alpha_n)$, where α_j is the exponent sum of x_j in w_i ; loops labelled by zero tuples are excluded. We apply this construction to each atomic subformula containing bound variables.

For instance, if the formula $\varphi(x_1, x_2)$ has the form^{*)}

$$\forall y \exists z \left(\left([ygy, x_1gx_2]x_1z^{-1}azx_2^{-3}x_1^3hx_2^7 = 1 \right) \land \neg \left(z^{-1}bz(x_2x_1)^2 = 1 \right) \lor \left((x_1^2x_2^2)^5 = 1 \right) \right), \tag{1}$$

where $g, h, a, b \in G$ are some fixed elements (not necessarily different), then the graph $\Gamma(\varphi)$ looks as follows:



Fig. 1

Now, in the graph $\Gamma(\varphi)$, we choose cycles c_1, c_2, \ldots generating the first homology group (e.g. we can take generators of the fundamental groups of all components), and compose the matrix $A(\varphi)$ of φ as follows: for each generating cycle c_i we write a row which is the sum of labels of the edges of this cycle (the edge labels are summed with signs plus or minus depending on the orientation), and then we add rows consisting of exponent sums of atomic subformulae containing no bound variables.

This matrix depends on the choice of generating cycles but the integer linear hull of its rows is determined uniquely by the formula φ . For the example above, the matrix $A(\varphi)$ is

$$A(\varphi) = \begin{pmatrix} -1 & -1\\ 5 & 5\\ 2 & 2\\ 10 & 10 \end{pmatrix} \quad \text{(for an obvious choice of three generating cycles).}$$
(2)

A bound variable t is called *isolating* if it occurs in atomic subformulae only in subwords of the form $t^{-1}g_i t$, where $g_i \in G$. The corresponding coefficients g_i are called *isolated*. More precisely, an element g of G is called *isolated* if it occurs in φ only in subwords o the form $t_i^{-1}gt_i$, where all t_i are isolating variables. In the example under consideration, z is the only isolating variable and a and b are isolated coefficients.

^{*)} We do not assume that the formula is always in the prenex normal form (i.e. all quantifiers are moved outside), as in this example.

Main theorem. If the rank of the matrix $A(\varphi)$ of a formula φ is less than the number of free variables of this formula, then the number of tuples of group elements satisfying formula φ is divisible by the order of the centraliser of the set of all non-isolated coefficients of φ . In particular, this number is divisible by the order of the group if all non-isolated coefficients equal 1.

In the example above, the rank of the matrix is one and there are two free variables. Therefore, the theorem asserts that the cardinality of the set

$$\left\{ (x_1, x_2) \in G^2 ; \forall y \exists z \ \left(\left([ygy, x_1gx_2]x_1z^{-1}azx_2^{-3}x_1^3hx_2^7 = 1 \right) \land \neg \left(z^{-1}bz(x_2x_1)^2 = 1 \right) \lor \left((x_1^2x_2^2)^5 = 1 \right) \right) \right\}$$

is divisible by |C(g,h)| (even if g = a, we should consider the coefficient g as non-isolated). In the following section, we give more instructive examples.

Corollary 1. The number of solutions to a system of equations in a group is divisible by the order of the centraliser of the set of coefficients if the rank of the matrix of this system is less than the number of unknowns.

This corollary transforms into the Gordon–Rodriguez-Villegas theorem in the case where all coefficients equal 1.

Gordon-Rodriguez-Villegas conjugation theorem ([GRV12], Corollary 3.5^{*}).

Let $\{w_j(x_1, \ldots, x_n)\} \subset F(x_1, \ldots, x_n)$ be a set of words (elements of a free group) such that the rank of the matrix composed of exponent sums of x_i in w_j is less than n. Then, for any group G and any elements $h_j \in G$, the number of tuples satisfying the formula

$$\bigwedge_{j} \left(\exists q_j \ w_j(x_1, \dots, x_n) = q_j^{-1} h_j q_j \right),$$

is divisible by the order of G.

This statement (generalizing a conjugation theorem from [Solo69]) is obviously stronger than the Gordon–Rodriguez-Villegas theorem from the introduction. Our theorem gives even stronger fact: the pointwise conjugation is replaced by a common conjugation.

Corollary 2. Under the condition of the Gordon–Rodriguez-Villegas conjugation theorem, the number of tuples satisfying the formula

$$\exists q \left(\bigwedge_{j} \left(w_{j}(x_{1},\ldots,x_{n}) = q^{-1}h_{j}q \right) \right),$$

is divisible by the order of G.

(If $w_j(x_1, \ldots, x_n) \in G * F(x_1, \ldots, x_n)$, then the number of satisfying tuples is divisible by the order of the centraliser of the set of all coefficients of the words w_j .)

Proof. The graph contains the only vertex corresponding to an isolated variable q; coefficients h_j are isolated; and the matrix of the formula coincides with the matrix from the Gordon–Rodriguez-Villegas conjugation theorem. So, Corollary 2 follows from the main theorem.

The next proposition generalises the Gordon–Rodriguez-Villegas conjugation theorem in another direction.

Invariant set theorem. Suppose that U_j, V_j are subsets of a finite group G that are conjugation invariant (i.e. they are unions of some conjugation classes) and $\{w_j(x_1, \ldots, x_n)\} \subset F(x_1, \ldots, x_n) * G$ is an arbitrary set of words such that the rank o the matrix composed of exponent sums of x_i in w_j is less than n. Then the number of tuples of elements of G satisfying the formula

$$\bigwedge_j \left(w_j U_j \subseteq V_j \right)$$

is divisible by the order of the centraliser of the set of all coefficients of all words w_j . In particular this number is a multiple of |G| if $\{w_j(x_1, \ldots, x_n)\} \subset F(x_1, \ldots, x_n)$.

Proof. Consider one inclusion $wU \subseteq V$. Let us decompose U and V into unions of conjugation classes:

$$U = a_1^G \cup a_2^G \cup \dots, \quad V = b_1^G \cup b_2^G \cup \dots$$

The inclusion $wU \subseteq V$ is equivalent to the following first-order formula:

$$\forall y_1 \forall y_2 \dots \exists z_1 \exists z_2 \dots \bigwedge_k \bigvee_l w y_k^{-1} a_k y_k = z_l^{-1} b_l z_l.$$

^{*)} translated into a language convenient for our purposes.

Thus, the conjunction of inclusions is transformed into a first-order formula where all bound variables are isolating and the coefficients a_i and b_i are isolated. The matrix of this formula differ from the exponent-sum matrix only in repeated rows and the assertion follows from the main theorem.

This theorem transforms into the Gordon–Rodriguez-Villegas conjugation theorem in the case where $U_j = \{1\}$ and $w_j \in F(x_1, \ldots, x_n)$. Note that the assertion of the invariant set theorem remains true if we replace the conjunction of inclusions by the conjunction of arbitrary (maybe different) set-theoretic relations "logically expressible" via inclusions. For instance, \subseteq may be replaced by \subset , \supseteq , \neg , =, \neq , \notin , "intersects",... The conjunction itself may be replaced by any quantifier-free first-order formula For example, if $A = a^G$ and $B = b^G$ are some conjugation classes of a group G, then the number of pairs $(x, y) \in G^2$ such that

$$x^2 y^3 [x, y] y^{-1} A B = A$$
 or $(yx)^2 A$ does not intersects B ,

is divisible by |G|. This follows from the main theorem. (The second term of the disjunction is equivalent to the formula $\forall z \forall t \ (yx)^2 z^{-1} az \neq t^{-1} bt$.)

The following statement is an analogue of the Solomon theorem (and transforms into it in the case when all coefficients equal 1, formula is quantifier-free and is a conjunction of equalities).

Corollary 3. The number of tuples of group elements satisfying formula φ is divisible by the order of the centraliser of the set of all non-isolated coefficients of φ (in particular, this number is divisible by the order of the group provided all non-isolated coefficients equal 1) if, in the formula φ ,

(the number of proper occurrences of bound variables)+
+(the number of components of
$$\Gamma(\varphi)$$
)+ (*)

+(the number of atomic subformulae containing no bound variables) < (the number of all variables).

An occurrence of a variable y is a maximal subword in the left-hand side of an equation (the left-hand side is considered as a cyclic word) containing the variable y and no other variables; An occurrence is called *proper* if it does not coincide with the entire left-hand side of the equation. In the example above, there are two occurrences of y and two occurrences of z, that is four occurrences of bound variables; all these occurrences are proper. The number of occurrences of the special variable q is always equal to the number of inhomogeneous equations (because we consider left-hand sides of equations as cyclic words).

Proof. The rank of the first homology group of a graph equals to the number of edges minus the number of vertices plus the number of connected components. The number of vertices equals to the number of bound variables, the number of edges equals to the number of occurrences of bound variables; improper occurrences give loops with zero labels. Therefore, the rank of the matrix $A(\varphi)$ is an most the left-hand side of (*) minus the number of bound variables. It remains to apply the theorem.

It is interesting that actually the graph $\Gamma(\varphi)$ can always be assumed to be connected, in the sense that the graph of a formula written economically, i.e. with minimal possible number of bound variables (among formulae equivalent to the given one) is always connected. For example, any formula with two bound variables that do not occur together in atomic subformulae is equivalent to a formula with one bound variable. Dima Trushin proves this fact in the appendix.

Corollary 4. The number of group elements whose k-th powers belong to a given subgroup is divisible by the order of this subgroup.^{*)}

Proof. Let H be a subgroup of a group G. We are interested in elements x such that $x^k \in H$. First, suppose that the subgroup H is the centraliser of a subgroup A. Then the inclusion $x \in H$ is equivalent to the system of equations $\{[x^k, a] = 1 ; a \in A\}$ satisfying the main theorem (there are no bound variables and the matrix is zero). Therefore, the number of solutions is divisible by the order of the centraliser of the set of coefficients, i.e. by |H|, as required.

Now let H be an arbitrary subgroup. Let us use the following trick. We embed the group G in a larger group \widehat{G} in such a way that H becomes the centraliser of a subgroup A of \widehat{G} . In addition, we must guarantee that all solutions of our system of equations over \widehat{G} belong to G. For this sake, we make G to be the centraliser of another subgroup $B \subset \widehat{G}$ and consider the system of equations

$$\left(\bigwedge_{a \in A} \left([x^k, a] = 1 \right) \right) \land \left(\bigwedge_{b \in B} \left([x, b] = 1 \right) \right).$$

This will prove Corollary 4 in the general case.

^{*)} In 2017, we learned that this fact was proven in [Iwa82].

The rôle of \widehat{G} can be played by the amalgamated free product $\widehat{G} = (B \times G) \underset{H}{*} (H \times A)$, where A and B are arbitrary nontrivial centreless groups. Clearly, C(A) = H and C(B) = G, i.e. the solutions to our system of equations are precisely the elements of G whose kth powers lie in H. According to the main theorem, the number of solutions is divisible by $|C(A) \cap C(B)| = |H \cap G| = |H|$, as required.

This proof of Corollary 4 uses only a very particular case of the main theorem, where the formula φ is a system of equations with one unknown. This case of the main theorem follows immediately from an old result of P. Hall (generalising the Frobenius theorem).

Hall theorem ([Hall36], Theorem II). In any group, the number of solutions to a system of equations with one unknown is divisible by $\text{GCD}(|C|, n_1, n_2, \ldots)$, where C is the centraliser of the set of coefficients and n_i is the exponent sum of the unknown in *i*-th equation.

A trick (another) allowing to transform an arbitrary subgroup into a centraliser can also be found in [Hall36]. In Section 4, we give a direct proof of Corollary 4 illustrating a part of the proof of the main theorem.

A more general fact can be obtained similarly.

Corollary 5. Suppose that H is a subgroup of a group G and W is a subgroup (or a subset) of a finitely generated group F with infinite abelianisation F/F'. Then the number of homomorphisms $f: F \to G$ such that $f(W) \subseteq H$ is divisible by |H|.

Proof. Consider a presentation $F = \langle X | R \rangle$ of the group F. The number of homomorphisms $f: F \to G$ such that $f(W) \subseteq H$ coincides with the number of solutions to the system of equations

in the group $\widehat{G} = (B \times G) \underset{H}{\star} (H \times A)$, where A and B are arbitrary nontrivial centreless groups. The rank of the matrix of this system coincides with the rank of the matrix of the system $\{r = 1 ; r \in R\}$ (as the other equations are commutators) and is less than the number of unknowns X, because the abelianisation of $F = \langle X | R \rangle$ is infinite. According to Corollary 1, the number of solutions is divisible by $|C(A) \cap C(B)| = |H \cap G| = |H|$, as required.

Note that Corollary 5 coincides with Corollary 4 in the case where $F = \mathbb{Z}$ and turns into the Gordon–Rodriguez-Villegas theorem when H = G.

2. Examples

We start with a curious application of the Solomon theorem.

Example 1. We say that two elements of a group belong to the same *tribe* if their squares are equal. Clearly, the total size of all tribes is the order of the group. It is less obvious that

the sum of 2018th powers of tribe sizes is a multiple of the order of the group.

To prove this fact, it suffices to consider the system of equations $x_1^2 = \ldots = x_{2018}^2$. Clearly, the number of solutions is the sum of 2018th powers of tribe sizes. The number of equations is less than that of unknowns. So, the statement is a corollary of the Solomon theorem. The assertion remains valid if 2018 is replaced by an arbitrary positive integer; the squares (in the definition of tribes) can also be replaced by any (equal) positive integer powers.

Example 2. The number of pairs of group elements whose product of squares is a cube is divisible by the order of the group. This follows from Corollary 3, because the formula $\exists z \ x^2y^2 = z^3$ has one bound variable, it occurs once, there are no equations without bound variables, the graph is connected, and there are two free variables: 1 + 0 + 1 < 2 + 1. This fact can also be derived from the Gordon–Rodriguez-Villegas conjugation theorem. Indeed, this theorem implies that the order of the group divides the number of pairs of group elements whose product of squares is conjugate to any given element.

By the same reason, the order of any group divides, e.g. the following numbers:

- the number of pairs of noncommuting elements whose product of squares is the cube of a noncentral element;
- the number of pairs of noncommuting elements whose product of squares is a cube if and only if the cube of their product lies in the centre;
- the number of pairs of elements such that either the product of their squares is a cube or their commutator is not a square;

Example 3. The order of a group divides the number of pairs of elements of this group whose product of squares is the cube of a commutator $(x_1^2 x_2^2 = [z, t]^3)$ and square of product is the commutator of cubes of the same elements

^{- ...}

 $((x_1x_2)^2 = [z^3, t^3])$. It is difficult to derive this fact from the Gordon–Rodriguez-Villegas conjugation theorem, but it follows immediately from our main theorem. Corollary 2 gives a stronger statement: the order of a group divides the number of pairs of elements whose square of product and product of squares are simultaneously conjugate to any given pair of elements.

The analogy between the Gordon–Rodriguez-Villegas theorem and well-known properties of solutions to systems of linear equations (over finite fields) could suggest an idea that, if the rank of the matrix is much less than the number of unknowns, then the number of solutions must be divisible by a higher power of the order of the group. A more realistic question is the following.

Is it true that the number of homomorphisms from a finitely generated group H into a group G is divisible by $|G|^m$ if H admits an epimorphism onto a free group of rank m?

The point is that the number of solutions to a system of coefficient-free equations

$$\{u(x_1,\ldots,x_n)=v(x_1,\ldots,x_n)=\ldots=1\}$$

equals to the number of homomorphisms from the group $H = \langle x_1, \ldots, x_n | u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n) = \ldots = 1 \rangle$ to the group G. The matrix of the system has rank at most r if and only if the group H admits an epimorphism onto the free abelian group of rank n - r. The existence of an epimorphism onto the absolutely free group of the same rank is a much stronger property, but nevertheless, the conjecture under consideration is false for m > 1, as the following example shows.

Example 4. The group $\langle x, y, z | z = z^x z^y \rangle$ has an epimorphism onto the free group of rank two (sending z to 1), but the number of solutions to the equation $z = z^x z^y$ in the symmetric group S_3 is not divisible by $|S_3|^2 = 36$. Indeed, with z = 1, there are 36 solutions (x and y can be arbitrary). With z = (123), there are $3 \cdot 3 = 9$ solutions (x and y are arbitrary transpositions). With z = (321), there are also 9 solutions. If z is a transposition, then there are no solutions (by parity). Thus, the total amount of solutions is $36 + 2 \cdot 9$.

3. Proof of the main theorem

Lemma 1. Under the conditions of the theorem, the first column of the matrix $A(\varphi)$ vanishes after a suitable invertible change of free variables. In particular, the exponent sum of x_1 (in new variables) in each cycle of the graph $\Gamma(\varphi)$ is zero.

Proof. The rank of the matrix $A(\varphi)$ is less than the number of its columns; therefore, some integer (invertible) elementary transformations of columns produces a matrix with zero first column. Elementary transformations of columns are induced by obvious changes of variables, e.g. the change $x_i \to x_i x_j^k$ adds *i*th column multiplied by *k* to *j*th column. Lemma 1 is proven.

For example, to annihilate the first column of matrix (2) from Section 1, it suffices to subtract the second column from the first one, i.e. the change of variables $x_2 \to x_2 x_1^{-1}$ transform formula (1) into the formula

$$\forall y \exists z \ \left(\left([ygy, x_1gx_2x_1^{-1}]x_1z(x_2x_1^{-1})^{-3}x_1^3h(x_2x_1^{-1})^7 = 1 \right) \land \neg \left(z^8x_2^2 = 1 \right) \lor \left(x_1^2(x_2x_1^{-1})^2 \right)^5 = 1 \right) \right). \tag{3}$$

The graph of this formula is shown in Figure 2



Fig. 2

and the matrix is

$$\begin{pmatrix} 0 & -1 \\ 0 & 5 \\ 0 & 2 \\ 0 & 10 \end{pmatrix}.$$

In what follows, we assume that the first column of $A(\varphi)$ is zero.

For each variable t (free or bound), we introduce new symbols $t^{(i)}$, where $i \in \mathbb{Z}$, (their meaning will be $t^{(i)} = tx_1^{i} = x_1^{-i}tx_1^{i}$ if t is not isolating and $t^{(i)} = tx_1^{i}$, if t is isolating but we act formally at this step). Similarly, for each coefficient g, we introduce new symbols $g^{(i)}$, where $i \in \mathbb{Z}$. Now, we transform formula φ as follows:

- 1) in all atomic subformulae, replace each symbol t by $t^{(0)}$, where t is a variable different from x_1 or a coefficient;
- 2) replace each subword of the form $(t^{(i)})^k x_l^l$, where t is a non-isolating variable or a coefficient, by $x_1^l(t^{(i+l)})^k$; replace each subword of the form $(g^{t^{(i)}})x_l^l$, where t is an isolating variable and $g \in G$, by $x_1^l(g^{t^{(i+l)}})$;
- 3) repeat step 2) while it is possible.

This shift of symbols x_1 produces a formula without x_1 (the exponent sum of x_1 vanishes in each cycle of the graph and in each atomic subformula without bound variables and, hence, the exponent sum of x_1 vanishes in all atomic subformulae). For example, formula (3) after these transformations will have the following atomic subformulae:

$$\begin{split} [y^{(0)}g^{(0)}y^{(0)}, g^{(-1)}x_2^{(-1)}]a^{z^{(-1)}} \left(x_2^{(-4)}x_2^{(-3)}x_2^{(-2)}\right)^{-1}h^{(-7)}x_2^{(-7)}x_2^{(-6)}x_2^{(-5)}x_2^{(-4)}x_2^{(-3)}x_2^{(-2)}x_2^{(-1)} = 1, \\ b^{z^{(0)}}(x_2^{(0)})^2 &= 1, \\ (x_2^{(-2)}x_2^{(-1)})^5 &= 1. \end{split}$$

Now, we proceed with transformation of the formula.

4) In each homogeneous equation α , we replace all symbols $t^{(i)}$ by $t^{(i+j_{\alpha})}$, where the integers j_{α} are chosen such that, for each bound variable t, the symbols $t^{(i)}$ will occur in the entire formula with at most one value of the superscript (i); this is possible, because the exponent sum of x_1 vanishes in each cycle of the graph.

In formula (4), it suffices to decrease the superscripts by one in the second equation. In the general case, we can act as follows. In each connected component K of Γ , we choose a vertex (variable) y_K . In each homogeneous equation α containing bound variables, we choose one of such variables y_{α} and connect each vertex y_{α} by a path p_{α} with the vertex y_K such that $y_{\alpha} \in K$. The sum s_{α} of the first coordinates of labels of edges of the path p_{α} does not depend on the choice of the path by the condition. Put $j_{\alpha} = -i_{\alpha} - s_{\alpha}$, where i_{α} is the unique number such that $y_{\alpha}^{(i_{\alpha})}$ occurs in the equation α . The sum of the first coordinates of labels of edges vanishes in each cycle, hence, $s_{\alpha} = s_{\beta}$ if $y_{\alpha} = y_{\beta}$, the value j_{α} does not depend on the choice of variables y_{α} in equation α , and the changes $t^{(i)} \to t^{(i+j_{\alpha})}$ in each equation α produce a formula such that each bound variable t occurs in the entire formula only with one superscript (i), where i is the sum of the first coordinates of labels of edges of any path from y_K to t.

- 5) Now, we replace each quantifier $\forall y$ and $\exists y$ by $\forall y^{(p)}$ and $\exists y^{(p)}$, where $p \in \mathbb{Z}$ is the unique number such that $y^{(p)}$ occurs in atomic subformulae.
- 6) Finally, we add equalities defining new symbols to the obtained formula $\hat{\varphi}$, i.e. we replace $\hat{\varphi}$ by the infinite formula

$$\varphi' = \widehat{\varphi} \wedge \underbrace{\left(\bigwedge_{g} \left(g^{(0)} = g\right)\right) \wedge \left(\bigwedge_{t,i} \left(t^{(i)} = x_1^{-1} t^{(i-1)} x_1\right)\right)}_{\delta}, \qquad (**)$$

where t ranges over all free variables and all coefficients of the initial formula, i ranges over all integers, and g ranges over all coefficients. The symbols $g^{(i)}$, where $g \in G$, are considered as free variables of the formula φ' .

The numbers of tuples satisfying the obtained formula φ' and the initial formula φ are equal. Indeed, the formula $\varphi' = \widehat{\varphi} \wedge \delta$ admits as many satisfying tuples as the formula $\overline{\varphi} = \widehat{\varphi} \big|_{t^{(i)} = t^{x_1^i}}$ (i.e. the formula $\widehat{\varphi}$ with each symbol $t^{(i)}$, where t is a free variable of the initial formula or a coefficient, is substituted by the expression $x_1^{-i}tx_1^i$). The formula $\overline{\varphi}$ is equivalent to φ (i.e. this formulae are satisfied by the same tuples). Indeed, $\overline{\varphi}$ differs from φ in two aspects:

- a) in quantifiers of $\overline{\varphi}$, symbols $y^{(p)}$ occurs instead of y;
- b) in atomic subformulae of $\overline{\varphi}$, each bound variable y is replaced by the expressions $(y^{(p)})x_1^{-p}$ or $(y^{(p)})x_1^{-p}$ (depending on whether y is isolating), where p is the same for all occurrences of y.

Formulae different only in these aspects are obviously equivalent: e.g. $(\forall y \, \alpha(y, z, \ldots)) \equiv (\forall t \, \alpha(t^{z^2}, z, \ldots))$, because for any $g \in G$, if y ranges over the whole group, then y^g ranges over the whole group.

Thus, it suffices to show that the number of tuples satisfying φ' (such tuples are called *solutions* henceforth), is divisible by |C|, where the letter C denotes the centraliser of the set of coefficients of φ (or of φ' , equivalently).

Consider a solution $X = (\tilde{x}_1, \tilde{x}_i^{(j)}, \tilde{g}^{(j)} \ i = 2, ..., n, \ j \in \mathbb{Z})$. The tuple $(\tilde{x}_i^{(j)}, \tilde{g}^{(j)})$, i.e. everything but \tilde{x}_1 , is called the *tail* of the solution X. Let B_X denote the centraliser of the tail of X. Note that $B_X \subseteq C$ (because of the equations $g^{(0)} = g$ in formula (**)).

We say that two solutions are *similar* if their tails are conjugate by an element of C. Clearly, this is an equivalence relation. The theorem is a corollary of the following proposition.

Proposition. Each class of similar solutions contains exactly |C| solutions.

Let us find the number of solutions similar to X. The number of all possible tails of such solutions is $|C|/|B_X|$, because, on the set of tails of solutions similar to X, the group C acts by conjugation (since a tuple conjugate to the tail of a solution Y by an element $c \in C$ is also the tail of a solution, e.g., of Y^c) and B_X is the stabiliser of the tail of X.

The number of solutions with the same tail as that of X equals to $|B_X|$, since, if a tuple with the same tail and with the first coordinate \tilde{x}'_1 is also a solution, then the quotient $\tilde{x}'_1(\tilde{x}_1)^{-1}$ must commute with the tail because of the equations δ in formula (**), i.e. $\tilde{x}'_1 \in B_X \tilde{x}_1$. On the other hand, any element $\tilde{x}'_1 \in B_X \tilde{x}_1$ gives a solution with the same tail (as that of X), because the variable x_1 occurs in φ' only in subformula δ .

If X' is a solution similar to X, then the number of solutions with the same tail as that of X' equals to $|B_{X'}| = |B_X|$ (if tails are conjugate, then their centralisers are conjugate and have the same order).

Thus, we obtain that the number of solutions similar to X equals $(|C|/|B_X|) \cdot |B_X| = |C|$, that proves the proposition and the theorem.

4. Roots of subgroups

In Section 1, the following assertion was derived from the main theorem.

Corollary 4. The number of group elements whose k-th powers belong to a given subgroup is divisible by the order of this subgroup.

Here, we give a direct proof that exemplify the concluding part of the proof of the main theorem. For simplicity, we assume that k = 2. Let H be a subgroup of a group G. We are interested in elements $x \in G$ such that $x^2 \in H$; such elements are called *solutions* henceforth. The assertion is implied by the following lemma.

Lemma. Each double coset HxH contains either 0 or |H| solutions.

Proof. Let x be a solution; its *tail* is the coset Hx.

The group H acts (on the right) on the set of tails of solutions from the double coset HxH by the right multiplication:

$$Hy \circ h \stackrel{\text{def}}{=} Hyh \ (= the \ tail \ of \ the \ solution \ y^h).$$

The stabiliser of the tail Hx is $B_x \stackrel{\text{def}}{=} H \cap H^x$:

$$Hx = Hxh \iff h \in H^x.$$

Therefore, all possible solutions lying in HxH have precisely $|H|/|B_x|$ different tails.

How many solutions have the same tail as that of x?

$$Hx = Hy \Longrightarrow yx^{-1} \in H,$$

but if y is also a solution, then

$$(Hx)x = Hx^2 = H = Hy^2 = (Hx)y$$
, i.e. $yx^{-1} \in H^x$.

Thus, each solution y with the same tail as that of x lies in $B_x x$. On the other hand, each element from this coset is a solution:

$$(bx)^2 = bxbx = bb^{x^{-1}}x^2 \in H$$
, because $b, b^{x^{-1}}$, and x^2 lie in H .

So, the number of solutions with the same tail as that of x is $|B_x|$.

Since $|B_x| = |B_y|$ if x and y lie in the same double coset HxH (because B_x and B_y are conjugate in this case), HxH contains $|B_x| \cdot (|H|/|B_x|) = |H|$ solutions, that completes the proof.

Recently, I. M. Isaacs [Isaa12] obtained a character-theoretic proof of this corollary. In 2017, we learned that this fact was proven in [Iwa82].

APPENDIX. ON THE MINIMISATION OF THE NUMBER OF BOUND VARIABLES IN FIRST-ORDER FORMULAE

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Let φ be a first-order formula (in some language) with bound variables y_1, \ldots, y_k and free variables x_1, \ldots, x_m . Consider the following graph $\Delta(\varphi)$ with vertices y_1, \ldots, y_k : vertices y_i and y_j are connected by an edge if there exists an atomic subformula in φ containing the both variables.

The following assertion shows that the formula φ with disconnected graph $\Delta(\varphi)$ is equivalent to a formula with fewer bound variables.

Proposition. Any formula φ is equivalent to a formula φ' with connected graph $\Delta(\varphi')$ such that

 $|\Delta(\varphi')| \leq \max(|\Delta_1|, \dots, |\Delta_s|),$ where $\Delta_1, \dots, \Delta_s$ are connected components of $\Delta(\varphi)$.

Proof. Let Φ_i be the set of formulae ψ such that

1) all variables of ψ (free and bound) lie in the set $\{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_k\}$;

2) all bound variables of ψ lie in the set $\{y_1, \ldots, y_k\}$;

3) if a variable y_i occurs in ψ (as a bound or free variable), then $y_i \in \Delta_i$.

Note that the classes Φ_i are closed under logical operators and quantifications on y_i .

Let Λ be the closure of the union $\bigcup_i \Phi_i$ with respect to logical operators $(\vee, \wedge, \text{ and } \neg)$. Clearly, each formula from Λ can be written in the form

$$\psi = \bigvee_{i=1}^{l} \bigwedge_{j=1}^{s} r_{ij}, \quad \text{where } r_{ij} \in \Phi_j, \tag{D}$$

and the in form

$$\psi = \bigwedge_{i=1}^{l} \bigvee_{j=1}^{s} r_{ij}, \quad \text{where } r_{ij} \in \Phi_j, \tag{C}$$

because the conjunction and disjunction are mutually distributive, the classes Φ_i are closed with respect to logical operations, $\neg(A \lor B) = (\neg A) \land (\neg B)$, and $\neg(A \land B) = (\neg A) \lor (\neg B)$.

Lemma. The class Λ is closed with respect to quantifications on the variables y_i .

Proof. Note that

$$\forall y(\psi_1(y) \land \psi_2(y)) = (\forall \psi_1(y)) \land (\forall y \psi_2(y)) \quad \text{and} \quad \forall y(\psi_1(y) \lor \psi_2) = (\forall y \psi_1(y)) \lor \psi_2,$$

where, in the second equality, y is not a free variable of the formula ψ_2 . For the existence quantifier we have similar equalities

$$\exists y(\psi_1(y) \lor \psi_2(y)) = (\exists \psi_1(y)) \lor (\exists y\psi_2(y)) \quad \text{and} \quad \exists y(\psi_1(y) \land \psi_2) = (\exists y\psi_1(y)) \land \psi_2,$$

where, in the second equality, y is not a free variable of the formula ψ_2 .

Suppose that a formula $\psi \in \Lambda$ is written in the form (D):

$$\psi = (r_{11} \wedge \ldots \wedge r_{1s}) \vee \ldots \vee (r_{l1} \wedge \ldots \wedge r_{ls})$$

and a variable y_j belongs to a component Δ_t . Then

$$\exists y_j (r_{11} \wedge \ldots \wedge r_{1s}) \vee \ldots \vee (r_{l1} \wedge \ldots \wedge r_{ls}) = (\exists y_j (r_{11} \wedge \ldots \wedge r_{1s})) \vee \ldots \vee (\exists y_j (r_{l1} \wedge \ldots \wedge r_{ls})) = \\ = (r_{11} \wedge \ldots \wedge (\exists y_j r_{1t}) \wedge \ldots \wedge r_{1s}) \vee \ldots \vee (r_{l1} \wedge \ldots \wedge (\exists y_j) r_{lt} \wedge \ldots \wedge r_{ls}),$$

where the first equality is valid for any formulae and the second equality is valid, because, in *i*th term of the disjunction, only r_{it} may depend on y_j . A similar transformation can be made for the universal quantifier using the form (C) of the formula $\psi \in \Lambda$. Lemma is proven.

Let us proceed with the proof of proposition. By condition, each atomic subformula of φ belongs to a class Φ_i , i.e., in particular, this subformula lies in Λ . Since Λ is closed with respect to logic operators and quantifications on y_j (by Lemma), we obtain that $\varphi \in \Lambda$, i.e.

$$\varphi = \bigvee_{i=1}^{l} \bigwedge_{j=1}^{s} r_{ij}.$$

Let us assume that the maximum of $|\Delta_i|$ is attained at Δ_1 . Then, in each formula r_{ij} , where $j \neq 1$, we change the names of bound variables from Δ_j for the names of variables from Δ_1 . Since Δ_1 is the largest component, we can assign different names from Δ_1 to different variables (in each particular formula r_{ij}). This renaming produces a formula φ' equivalent to φ and all bound variables of φ' belong to Δ_1 .

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