ON THE CYCLIC SUBGROUP SEPARABILITY OF FREE PRODUCTS OF TWO GROUPS WITH AMALGAMATED SUBGROUP

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Abstract. Let G be a free product of two groups with amalgamated subgroup, π be either the set of all prime numbers or the one-element set $\{p\}$ for some prime number p. Denote by Σ the family of all cyclic subgroups of group G, which are separable in the class of all finite π-groups.

Obviously, cyclic subgroups of the free factors, which aren't separable in these factors by the family of all normal subgroups of finite π-index of group G, the subgroups conjugated with them and all subgroups, which aren't π'-isolated, don't belong to Σ. Some sufficient conditions are obtained for Σ to coincide with the family of all other π'-isolated cyclic subgroups of group G.

It is proved, in particular, that the residual p-finiteness of a free product with cyclic amalgamation implies the p-separability of all p'-isolated cyclic subgroups if the free factors are free or finitely generated residually p-finite nilpotent groups.

1. Introduction. Main results

Let $\mathcal{K}$ be a class of groups. We recall (see [1]), that a subgroup F of a group G is said to be separable by the groups of class $\mathcal{K}$ if, to any element $g \in G \setminus F$, there exists a homomorphism ψ of group G onto a group of $\mathcal{K}$ such that $g\psi \notin F\psi$. Group G is called residually $\mathcal{K}$ if it's trivial subgroup is separable by the groups of class $\mathcal{K}$. If class $\mathcal{K}$ coincides with the class of all finite groups, then we shall say about residual finiteness and about finite separability of subgroups. A group, all cyclic subgroups of which are finitely separable, is called π_c-group.

If Ψ is a family of normal subgroups of group G, then we shall say also that subgroup F of group G is separable by the subgroups of Ψ if $\bigcap_{N\in\Psi} FN = F$. Thus, the separability of subgroup F in class $\mathcal{K}$ is equivalent to the separability of F by the family of all normal subgroups of group G, the factor-groups by which belong to $\mathcal{K}$.

It is obvious that the separability of all cyclic subgroups of group G in class $\mathcal{K}$ implies the "residually $\mathcal{K}$" property of G. The converse, in general, isn't true, and so the problem arises to describe all cyclic subgroups of a residually $\mathcal{K}$ group, which

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are separable by the groups of $\mathcal{K}$. The case considered in the present paper is that $\mathcal{K}$ coincides with the class of all finite groups or all finite p-groups and group G is a free product of two groups with amalgamated subgroup.

Let A and B be some groups, H be a subgroup of A, K be a subgroup of B and let φ: $H \to K$ be an isomorphism. Let $G = (A * B; H = K, \varphi)$ be the free product of groups A and B with subgroups H and K amalgamated according to isomorphism φ. Obviously, an arbitrary cyclic subgroup of group G conjugated with a subgroup of one of the free factors A and B, that isn't separable in class $\mathcal{K}$, will not be separable by the groups of $\mathcal{K}$ itself. Thus, our task is to determine which of the remaining cyclic subgroups are separable in $\mathcal{K}$ and, in particular, to find the conditions for them all being separable by the groups of $\mathcal{K}$.

More precisely, the problem may be formulated as follows. Let Δ_A and Δ_B be the families of all cyclic subgroups of groups A and B, respectively, which aren't finitely separable in these groups. It is necessary to find the conditions guaranteeing the truth of the following statement:

($*$) *An arbitrary cyclic subgroup of group G, which isn't conjugate with any subgroup of the family $\Delta_A \cup \Delta_B$, is finitely separable.*

We note that in a series of papers (see, e. g., [2]), dealing with generalized free products of two π_c-groups, the special case of this task was considered, when the family $\Delta_A \cup \Delta_B$ was empty.

If, as above, H and K are subgroups of groups A and B, respectively, and φ: $H \to K$ is an isomorphism, then, following G. Baumslag [3], we shall call subgroups $R \leqslant A$ and $S \leqslant B$ (H, K, φ)-compatible if $(R \cap H)\varphi = S \cap K$. Let denote by Ω the family of all pairs of normal (H, K, φ)-compatible subgroups of finite index of groups A and B and by Ω_A and Ω_B it's projections onto groups A and B.

It is easy to see that, if N is an arbitrary normal subgroup of finite index of group G, then the pair $(A \cap N, B \cap N)$ belongs to family Ω. It follows from this remark that a finitely separable cyclic subgroup of group G contained in one of groups A and B is separable by the subgroups of families Ω_A or Ω_B, respectively.

Let now Λ_A and Λ_B denote the families of all cyclic subgroups of groups A and B, which aren't separable by the subgroups of families Ω_A or Ω_B. Then the condition just stated can be formulated in the form of the following

Proposition 1.1. *If a cyclic subgroup of group G is finitely separable, then it conjugates with no subgroup of the family $\Lambda_A \cup \Lambda_B$.*

We note now that inclusions $\Delta_A \subseteq \Lambda_A$ and $\Delta_B \subseteq \Lambda_B$ take place, since the finite separability of a subgroup of a given group means precisely the separability by the family of all normal subgroups of finite index. Thus, statement ($*$) is equivalent to simultaneous realizability of the next two ones:
a) $\Delta_A = \Lambda_A$ and $\Delta_B = \Lambda_B$, and
b) an arbitrary cyclic subgroup of group G, that isn't conjugate with any subgroup of $\Lambda_A \cup \Lambda_B$, is finitely separable.

The following statement, the first of the main results of the paper, gives a sufficient condition for the second claim to be true.

Theorem 1.2. *Let family Ω_A be an H-filtration and family Ω_B be a K-filtration. Then an arbitrary cyclic subgroup of group G, that conjugates with no subgroup of $\Lambda_A \cup \Lambda_B$, is finitely separable.*

We recall (see [3]) that a family Ψ of normal subgroups of a group X is said to be a Y-filtration, where Y is a subgroup of X, if $\bigcap_{N\in\Psi} N = 1$ and Y is separable by the subgroups of Ψ. Proposition 2 of paper [3] asserts that, if family Ω_A is an H-filtration and family Ω_B is a K-filtration, then G is a residually finite group. Thus, theorem 1.2 may be considered as a generalization of this statement.

Having slightly increased our restrictions, we may obtain the maximal property $(*)$ for the family of finitely separable cyclic subgroups of group G.

Theorem 1.3. *Let groups A and B be residually finite, subgroups H and K be finitely separable in the free factors and, to any two normal subgroups of finite index $M \leqslant A$ and $N \leqslant B$ there exists a pair of subgroups $(R, S) \in \Omega$ such that $R \leqslant M$ and $S \leqslant N$. Then group G satisfies condition $(*)$. In particular, if A and B are π_c-groups, then G is also a π_c-group.*

Indeed, any subgroup F of group A or group B, which is finitely separable in A or B, turns out separable by the subgroups of Ω_A or Ω_B, respectively, in this case. Therefore, in particular, $\Delta_A = \Lambda_A$ and $\Delta_B = \Lambda_B$. Besides, groups A and B being residually finite, Ω_A is an H-filtration and Ω_B is a K-filtration. The desired claim follows now from theorem 1.2.

We note that theorems 1.2 and 1.3 are a generalization of the results obtained by G. Kim [2, theorem 1.1 and proposition 1.2] for generalized free products of π_c-groups.

Let turn now to description of cyclic subgroups of group $G = (A * B; H = K, \varphi)$, which are separable in the class of finite p-groups (or, briefly, p-separable).

We remind, first of all, that a subgroup Y of a group X is called p'-isolated if, for any element $g \in Y$ and for any prime number q, which doesn't equal p, $g^q \in Y$ implies $g \in Y$. It is easy to see that every p-separable subgroup must be p'-isolated, and so the original task takes the following form.

Let Δ_A^p and Δ_B^p be the families of all p'-isolated cyclic subgroups of groups A and B, respectively, which aren't p-separable in these groups. It is necessary to find the conditions guaranteeing the truth of the following statement:

$(**)$ *An arbitrary p'-isolated cyclic subgroup of group G, which isn't conjugate with any subgroup of $\Delta_A^p \cup \Delta_B^p$, is p-separable.*

We remark that the p-separability of all p'-isolated cyclic subgroups of some group doesn't necessarily imply the residual p-finiteness of this group.

Let Ω^p denotes the family of all ordered pairs $(A \cap N, B \cap N)$, where N is an arbitrary normal subgroup of group G of finite p-index. Let also Ω_A^p and Ω_B^p denote the families of the first and the second components of elements of Ω^p. The next proposition is obtained by E. D. Loginova in the paper [4].

Proposition 1.4. *A pair of subgroups (R, S) belongs to family Ω^p if, and only if there exist sequences of subgroups $R = R_0 \leqslant \ldots \leqslant R_m = A$, $S = S_0 \leqslant \ldots \leqslant S_n = B$ such that:*

1) R_i, S_j are normal subgroups of groups A and B, respectively $(0 \leqslant i \leqslant m, 0 \leqslant j \leqslant n)$;

2) $|R_{i+1}/R_i| = |S_{j+1}/S_j| = p$ $(0 \leqslant i \leqslant m-1, 0 \leqslant j \leqslant n-1)$;

3) isomorphism φ maps the set $\{R_i \cap H\}$ onto the set $\{S_j \cap K\}$.

Following to [4] we shall call subgroups R and S satisfying the conditions of proposition 1.4 (H, K, φ, p)-compatible.

Let us denote by Λ_A^p and Λ_B^p the families of all p'-isolated cyclic subgroups of groups A and B, which aren't separable by the subgroups of Ω_A^p and Ω_B^p, respectively. Then, as above, the inclusions $\Delta_A^p \subseteq \Lambda_A^p$, $\Delta_B^p \subseteq \Lambda_B^p$ take place, and the following proposition is true.

Proposition 1.5. *If a p'-isolated cyclic subgroup of group G is p-separable, then it conjugates with no subgroup of the family $\Lambda_A^p \cup \Lambda_B^p$.*

In the same paper [4] the analog of the mentioned above sufficient condition by Baumslag is obtained: if family Ω_A^p is an H-filtration and family Ω_B^p is a K-filtration, then group G is residually p-finite. It turns out that the statements, similar to theorems 1.2 and 1.3, also take place.

Theorem 1.6. *Let family Ω_A^p be an H-filtration and family Ω_B^p be a K-filtration. Then an arbitrary p'-isolated cyclic subgroup of group G, which conjugates with no subgroup of $\Lambda_A^p \cup \Lambda_B^p$, is p-separable.*

Theorem 1.7. *Let groups A and B be residually p-finite, subgroups H and K be p-separable in the free factors and, to any two normal subgroups of finite p-index $M \leqslant A$ and $N \leqslant B$, there exists a pair of subgroups $(R, S) \in \Omega^p$ such that $R \leqslant M$ and $S \leqslant N$. Then group G satisfies condition $(**)$.*

The last theorem is deduced from theorem 1.6 in exactly the same way as theorem 1.3 from theorem 1.2.

2. Some applications

Let A be a free group with the set of free generators $\{a, b\}$, B be a free group with the set of free generators $\{c, d\}$, and let H be the subgroup of group A generated by the elements a and $a_1 = b^{-1}ab$, K be the subgroup of group B generated by the elements c and $c_1 = d^{-1}c^2d$. It is obvious that the indicated generators of subgroups H and K generate these subgroups freely, and so the map, which associates a with c and a_1 with c_1, defines an isomorphism φ of subgroup H onto subgroup K.

Thus, the group $G = \langle a, b, c, d; a = c, b^{-1}ab = d^{-1}c^2d\rangle$ is a free product of groups A and B with subgroups H and K amalgamated according to isomorphism φ.

Theorem 2.1. *An arbitrary cyclic subgroup of the group $G = \langle a, b, c, d; a = c, b^{-1}ab = d^{-1}c^2d\rangle$, that isn't conjugate with any subgroup of $\Lambda_A \cup \Lambda_B$, is finitely separable. At the same time families Λ_A and Λ_B aren't empty, and family Ω_B isn't a K-filtration.*

The given statement demonstrates that the sufficient condition stated in theorem 1.2 isn't necessary. Besides, all finitely generated subgroups of an arbitrary free group being finitely separable, the first two conditions of theorem 1.3 are fulfilled here. But G isn't a π_c-group. Thus, the third condition of this theorem isn't true and hence doesn't follow, in general, from the first two ones.

It isn't difficult to verify that the (H, K, φ)-compatibility of normal subgroups of finite p-index implies their (H, K, φ, p)-compatibility in a free product of two groups with cyclic amalgamation. It is easy to see also that, to any element g of a residually p-finite group and to any p-number x, there exists a normal subgroup of finite p-index, which intersects with the cyclic subgroup generated by g at the subgroup $\langle g^x\rangle$.

Thus, if groups A and B are residually p-finite, and subgroups H and K are cyclic, then families Ω_A^p and Ω_B^p coincide with the families of all normal subgroups of groups A and B of finite p-index. This remark results in the next statement following directly from theorem 1.7.

Theorem 2.2. *Let A and B be residually p-finite groups, H and K be cyclic subgroups, which are p-separable in the free factors. Then group G is residually p-finite and satisfies condition $(**)$.*

The same reasons are used in the proof of one more result.

Theorem 2.3. *Let H and K are infinite cyclic subgroups, and their centralizers in groups A and B, respectively, don't contain elements of finite order. If group G is residually p-finite, then it satisfies condition $(**)$.*

Let us formulate now two statements following directly from theorems 2.2 and 2.3, respectively.

Corollary 2.4. *Let A and B be finitely generated residually p-finite nilpotent groups (i. e. their torsion parts are p-groups), $\langle h\rangle \leqslant A$ and $\langle k\rangle \leqslant B$ be maximal infinite cyclic subgroups and $H = \langle h^m\rangle$, $K = \langle k^n\rangle$. If m and n are p-numbers, then all p'-isolated cyclic subgroups of group G are p-separable.*

Corollary 2.5. *Let A and B be free groups, $\langle h\rangle \leqslant A$ and $\langle k\rangle \leqslant B$ be maximal cyclic subgroups and $H = \langle h^m\rangle$, $K = \langle k^n\rangle$. If $m = 1$ or $n = 1$ or m and n are p-numbers, then all p'-isolated cyclic subgroups of group G are p-separable.*

It is proved in [5] and [6] that the conditions of corollaries 2.4 and 2.5 are necessary and sufficient for the residual p-finiteness of group G. So the only remark which is needed for the proof is that all p'-isolated cyclic subgroups of free and finitely generated nilpotent groups are p-separable (see [5] and [4], respectively).

We note that, as it followes from corollary 2.5, neither the p-separability of the amalgamated subgroups, nor even their p'-isolation isn't the necessary condition for the p-separability of all p'-isolated cyclic subgroups of group G.

The other applications of theorems 1.3 and 1.7 can be founded in the author's papers [7] and [8].

3. The proof of theorems 1.2 and 1.6

To any pair of subgroups $(R, S) \in \Omega$, the map $\varphi_{R,S} : HR/R \to KS/S$, which associates an element hR, $h \in H$, with the element $(h\varphi)S$, is correctly defined and serves as an isomorphism of subgroups. Therefore we may construct the group $G_{R,S} = (A/R * B/S; HR/R = KS/S, \varphi_{R,S})$. The natural homomorphisms of group A onto A/R and of group B onto B/S are extendable to a homomorphism $\pi_{R,S}$ of group G onto group $G_{R,S}$.

It is well known that generalized free product of two finite groups is residually finite and moreover a π_c-group. So, to any pair of subgroups $(R, S) \in \Omega$, $G_{R,S}$ is a π_c-group.

Generalized free product of two finite p-groups isn't, in general, residually p-finite. The corresponding criteria was founded by G. Higman in [9]. It follows directly from this criteria and proposition 1.4 that, if $(R, S) \in \Omega$, then the group $G_{R,S}$ is residually p-finite if, and only if $(R, S) \in \Omega^p$. We'll show also that all p'-isolated cyclic subgroups of group $G_{R,S}$ are p-separable in this case.

Proposition 3.1. *Let A and B be finite groups. If G is a residually p-finite group, then all it's p'-isolated cyclic subgroups are p-separable.*

Proof. For group G is residually p-finite, there exists it's homomorphism onto a finite p-group, the kernel of which intersects trivially with the free factors and, because of known theorem by H. Neumann [10], is a free group. As it was noted above, all p'-isolated cyclic subgroups of free group are p-separable, and so the desired claim results from the following statement.

Proposition 3.2. *Let a group X be an extension of a group Y by a finite p-group and let all p'-isolated cyclic subgroups of group Y be p-separable. Then all p'-isolated cyclic subgroups of group X are p-separable too.*

Proof. Let F be a p'-isolated cyclic subgroup of group X, $g \in X \setminus F$. It is sufficient for proving to point out a normal subgroup N of finite p-index such that $g \notin FN$.

If $g \notin FY$, then subgroup Y is desired. So g will be considered to be an element of FY.

We write g in the form $g = fy$, where $f \in F$, $y \in Y$. Since $g \notin F$, $y \notin F \cap Y$.

Obviously, $F \cap Y$ is a p'-isolated cyclic subgroup of group Y. Hence it is p-separable in Y and there exists a normal subgroup M of group Y of finite p-index such that $y \notin (F \cap Y)M$. To every element $y \in Y$, the subgroup $y^{-1}My$ is included in Y, is normal and has finite p-index in this group. Owing to finiteness of the index $[X : Y]$, the number of different subgroups of such form is also finite. Thus, their intersection N, say, is a subgroup of finite p-index of group Y, normal in X.

If $g \in FN$, then $g = f'u$ for some elements $f' \in F$, $u \in N$ and $f^{-1}f' = yu^{-1} \in F \cap Y$. But $y = (f^{-1}f')u \in (F \cap Y)N \subseteq (F \cap Y)M$ in this case, what contradicts the choice of subgroup M. Thus, $g \notin FN$, and subgroup N is required.

Proposition 3.3. *Let X be a residually p-finite group, $g \in X$ be an element of infinite order. The subgroup $\langle g \rangle$ isn't p'-isolated if, and only if there exist an element $h \in X$ and a prime number q, which doesn't equal p, such that $g = h^q$.*

Proof. The sufficiency of this condition is obvious, we'll show it's necessity.

Let $f \in X \setminus \langle g \rangle$ be such an element that $f^q \in \langle g \rangle$ for some prime number q, which isn't equal to p, and $f^q = g^k$.

The residual p-finiteness of group X results that the centralizer $C(g)$ of element g in group X is a p-separable subgroup and therefore a p'-isolated one. Hence $f \in C(g)$.

If we suppose that $k = qk'$, then $1 = f^q g^{-qk'} = (fg^{-k'})^q$, and, owing to the residual p-finiteness of X, $f = g^{k'}$. We obtain a contradiction with the choice of element f. Thus, $(k, q) = 1$ and $ku + qv = 1$ for some integer numbers u and v. From this it follows that $g = g^{ku+qv} = f^{qu}g^{qv} = (f^u g^v)^q$, as claimed.

We shall carry out **the proof of theorems 1.2 and 1.6** simultaneously and say about separability and compatibility of subgroups without specifying of the concrete class of groups.

Let h and g be arbitrary elements of group G such that $g \neq 1$, the cyclic subgroup $\langle g \rangle$ conjugates with no subgroup of $\Lambda_A \cup \Lambda_B$ (respectively, is p'-isolated and conjugates with no subgroup of $\Lambda_A^p \cup \Lambda_B^p$), and $h \notin \langle g \rangle$. Let also $h = h_1 h_2 \ldots h_m$, $g = g_1 g_2 \ldots g_n$ be reduced forms of elements h and g. Applying an appropriate inner automorphism of group G we may consider element g as cyclically reduced.

To find a homomorphism θ of group G onto a finite group mapping h to an element, which doesn't belong to $\langle g\theta\rangle$, it is sufficient to point out a pair of subgroups $(R,S)\in\Omega$ satisfying the property $h\pi_{R,S}\notin\langle g\pi_{R,S}\rangle$. Since $G_{R,S}$ is a π_c-group, the homomorphism $\pi_{R,S}$ can be extended to the desired homomorphism θ.

We may use the same idea for constructing a homomorphism of group G onto a finite p-group, but with minor restriction.

Indeed, if subgroups R and S are compatible, then all cyclic subgroups of the free factors of the group $G_{R,S}$ (which are finite p-groups) are p'-isolated. Therefore, if $n=1$, i. e. $g\in A\cup B$, and if we succeed to point out such a pair of subgroups $(R,S)\in\Omega^p$ that $h\pi_{R,S}\notin\langle g\pi_{R,S}\rangle$, then the existence of the required homomorphism follows from proposition 3.1.

But if $n\geqslant 2$, then the mere presence of a pair of compatible subgroups R and S satisfying the property $h\pi_{R,S}\notin\langle g\pi_{R,S}\rangle$ may turn out insufficient, because the subgroup $\langle g\pi_{R,S}\rangle$ need not be p'-isolated in the group $G_{R,S}$ (the corresponding example is given at the end of the proof). To make use of proposition 3.1 in this case we shall find such a pair of subgroups $(R,S)\in\Omega^p$ that the image of h under the action of homomorphism $\pi_{R,S}$ doesn't belong to some p'-isolated cyclic subgroup including the subgroup $\langle g\pi_{R,S}\rangle$.

Let, at first, $n=1$, and let $g\in A$ for definiteness.

By the condition the subgroup $\langle g\rangle$ is separable by the subgroups of family Ω_A (respectively, of family Ω^p_A). Therefore, if $h\in A$, there exists a pair of compatible subgroups R and S such that $h\notin\langle g\rangle R$, and hence $h\pi_{R,S}\notin\langle g\pi_{R,S}\rangle$.

Let $h\notin A$. Then $h\in B\setminus K$ if $m=1$ or every syllable h_i of it's reduced form belongs to one of the free factors but isn't contained in the amalgamated subgroup if $m>1$. So, to every i $(1\leqslant i\leqslant m)$, we can point out a pair of compatible subgroups R and S such that $h_i\notin HR_i$ if $h_i\in A$ and $h_i\notin KS_i$ if $h_i\in B$. Let $R=\bigcap R_i$, $S=\bigcap S_i$.

It is easy to see that subgroups R and S are compatible, $l(h\pi_{R,S})=l(h)$ (here $l(\cdot)$ denotes syllable length), and, if $m=1$, then $h\pi_{R,S}\in B\pi_{R,S}\setminus K\pi_{R,S}$. Thus, $h\pi_{R,S}\notin\langle g\pi_{R,S}\rangle$ in this case too.

Let now $n\geqslant 2$. We find, as above, a pair of compatible subgroups R and S such that $l(h\pi_{R,S})=l(h)$ and $l(g\pi_{R,S})=l(g)$. Obviously, the form of the element $g\pi_{R,S}$ is cyclically reduced as before.

We shall finish the proof of theorem 1.2 at first.

It is not difficult to show that, for any two elements $u,v\in G_{R,S}$, if one of these elements is cyclically reduced and $v\in\langle u\rangle$, then the other element is also cyclically reduced and $l(u)|l(v)$. Hence, if n doesn't divide m, then $h\pi_{R,S}\notin\langle g\pi_{R,S}\rangle$.

Let $m=nk$ for some positive k. Since $h\notin\langle g\rangle$, then $h\neq g^{\pm k}$ and, because of the residual finiteness of group G, there exists it's normal subgroup L of finite index not containing the elements $h^{-1}g^k$ and $h^{-1}g^{-k}$. Putting $R'=R\cap L$, $S'=S\cap L$ we have $l(h\pi_{R',S'})=l(h)$, $l(g\pi_{R',S'})=l(g)$, and $h\pi_{R',S'}\neq(g\pi_{R',S'})^{\pm k}$, whence follows that $h\pi_{R',S'}\notin\langle g\pi_{R',S'}\rangle$.

Thereby, theorem 1.2 is proved, and we turn to the proof of theorem 1.6.

Obviously, $l(g\pi_{R',S'})=l(g)>1$ for any pair of compatible subgroups R' and S', which are included in R and S, respectively. Applying proposition 3.3 it isn't difficult to see that the subgroup $\langle g\pi_{R',S'}\rangle$ is contained in some p'-isolated cyclic subgroup $F_{R',S'}$, it's index in this subgroup being mutually distinct with p. We shall prove that subgroups R' and S' can be chosen in a such way that the element $h\pi_{R',S'}$ doesn't belong to $F_{R',S'}$.

Let write the number n in the form $n = p^l n'$, where $(n', p) = 1$, and consider the two cases.

Case 1. n doesn't divide mn'.

Suppose that $h\pi_{R,S} \in F_{R,S}$. It is clear that the index of the subgroup $\langle g\pi_{R,S}\rangle$ in group $F_{R,S}$ divides n', and so $(h\pi_{R,S})^{n'} \in \langle g\pi_{R,S}\rangle$. But this contradicts the supposition that n doesn't divide mn'. Thus, $h\pi_{R,S} \notin F_{R,S}$.

Case 2. $mn' = nk$ for some positive k.

Since the subgroup $\langle g\rangle$ is p'-isolated in G and $h \notin \langle g\rangle$, then $h^{n'} \neq g^{\pm k}$. The residual p-finiteness of group G results that there exists a normal subgroup L of group G of finite p-index such that $h^{-n'}g^k, h^{-n'}g^{-k} \notin L$. Let $R' = R \cap L$, $S' = S \cap L$.

Then $(h\pi_{R',S'})^{n'} \neq (g\pi_{R',S'})^{\pm k}$, and so $(h\pi_{R',S'})^{n'} \notin \langle g\pi_{R',S'}\rangle$. It follows, as above, that $h\pi_{R',S'} \notin F_{R',S'}$, and the proof is finished.

Let us make a remark now in connection with the given proof. Let F be a cyclic subgroup of group G generated by a cyclically reduced element g of a syllable length greater than 1. It is interesting that, even if subgroup F is p-separable, it may be impossible to find such a pair of subgroups $(R, S) \in \Omega^p$ that the element $g\pi_{R,S}$ has a reduced form of a non-unit length, as before, and at the same time the subgroup $\langle g\pi_{R,S}\rangle$ is p'-isolated in $G_{R,S}$.

Let $G = \langle a, b; a^p = b^p\rangle$ and $g = (ab)^q a^p$, where p, q are different prime numbers. It is easy to see that the subgroup $\langle g\rangle$ is p'-isolated in G and hence is p-separable in G by virtue of theorem 2.2.

From the other hand, for every pair of subgroups $(R, S) \in \Omega^p$, where $R \neq A$ and $S \neq B$, the group $G_{R,S}$ has the presentation $\langle a, b; a^{p^n} = b^{p^n} = 1, a^p = b^p\rangle$ for a convenient natural n. Let $h = aba^{px_n}$, where x_n is a solution of the congruence $qx \equiv 1$ $(mod\ p^n)$. Then, obviously, $h\pi_{R,S} \notin \langle g\pi_{R,S}\rangle$ while $(h\pi_{R,S})^q \in \langle g\pi_{R,S}\rangle$, and, thus, the subgroup $\langle g\pi_{R,S}\rangle$ isn't p'-isolated in $G_{R,S}$.

4. The proof of theorems 2.1 and 2.3

The proof of theorem 2.1. We put $t = bd^{-1}$ and then use the obvious Tietze transformations to convert the presentation $G = \langle a, b, c, d; a = c, b^{-1}ab = d^{-1}c^2d\rangle$ of group G to the presentation $G = \langle a, b, t; t^{-1}at = a^2\rangle$, which means that group G is the ordinary free product of the group $C = \langle a, t; t^{-1}at = a^2\rangle$ and an infinite cyclic subgroup with generator b.

Owing to the residual finiteness of group G and theorem 1.3 a cyclic subgroup of group G isn't finitely separable if, and only if it conjugates with a subgroup of Δ_C. It is well known that family Δ_C consists of those subgroups of group C, which conjugate with the subgroups generated by the elements of form a^k.

We shall prove now the two auxiliary statements.

Proposition 4.1. *If a normal subgroup M of finite index of group A (of group B) belongs to family Ω_A (respectively, to family Ω_B), then the order of element a (respectively, of element c) modulo subgroup M is an odd number.*

Proof. Let a normal subgroup M of finite index of group A and a normal subgroup N of finite index of group B are (H, K, φ)-compatible. We put $H \cap M = U$ and $K \cap N = V$, so that $U\varphi = V$.

It is obvious that the orders of elements a and a_1 modulo subgroup M must coincide, and, the factor-group H/U being embeddable naturally to the factor-group A/M, the orders of these elements modulo subgroup U coincide too.

Considering the images according to isomorphism φ we get coincidence of the orders of elements c and c_1 of group K modulo subgroup V, and so coincidence of the orders of these elements modulo subgroup N. It follows that elements c and c^2 have the same order, and therefore the order of element c modulo subgroup N is an odd number.

Since the element cN corresponds to the element aM under the isomorphism of the subgroup HM/M of A/M onto the subgroup KN/N of B/N, which is induced by isomorphism φ, the order of element a modulo subgroup M is an odd number too.

Proposition 4.2. *A cyclic subgroup of group A (of group B) belongs to family Λ_A (respectively, to family Λ_B) if, and only if it conjugates with a subgroup generated by an element a^{2k} (respectively, c^{2k}) for some $k \neq 0$.*

Proof. The cyclic subgroup F of group A generated by the element a^{2k} doesn't contain the element a^k. Let M be an arbitrary subgroup of family Ω_A. Owing to proposition 4.1 the order m of element a modulo subgroup M is an odd number, and so the congruence $2l \equiv 1\ (mod\ m)$ is solvable for some integer number l. Therefore $a \equiv a^{2l}\ (mod\ M)$, $a^k \equiv (a^{2k})^l\ (mod\ M)$, whence $a^k \in FM$. Thus, subgroup F isn't separable by family Ω_A.

Conversely, the elements a and a^2 conjugated in G, an arbitrary cyclic subgroup F, which is contained in A and conjugates with no subgroup generated by an element a^{2k}, is finitely separable in G, and, in accordance with proposition 1.1, is separable by family Ω_A.

The argument for group B is analogous.

Since the elements a and a^2 are conjugated in group G, the statement of theorem follows directly from propositions 4.1 and 4.2.

The proof of theorem 2.3. If families Ω_A^p and Ω_B^p are an H- and a K-filtration, respectively, the desired claim results from theorem 1.6. So this condition will be considered to be false.

Let, for definiteness, family Ω_A^p be not an H-filtration. Owing to the residual p-finiteness of group G, this means that subgroup H isn't separable by the subgroups of family Ω_A^p, and hence there exists an element $f \in A \setminus H$ moving to H under the action of any homomorphism of group G onto a finite p-group (we will denote the family of all such homomorphisms by Ψ). Let us remark that family Ω_B^p must be a K-filtration then: otherwise there exists an element g of the set $B \setminus K$ with the analogous property, and the commutator $[f, g]$ turns out a non-trivial element of group G, mapped to unit under any homomorphism $\psi \in \Psi$.

Let further h and k be a generators of subgroups H and K, respectively, and $h\varphi = k$. First of all we'll show that, to any natural p-number n, there exists such an element $f_n \in A \setminus H$ that $f_n\psi \in H^n\psi$ under every homomorphism $\psi \in \Psi$.

Let $f \in A \setminus H$ be an element moving to H under the action of any homomorphism of Ψ. The residual p-finiteness of group G results that the centralizer $C(H)$ of subgroup H of group A is a p-separable subgroup, and so $f \in C(H)$.

Obviously, if element f has an infinite order modulo subgroup H (i. e. $f^n \notin H$ for any natural n), it is sufficient to put $f_n = f^n$. Therefore the order of f modulo H is considered to be finite and equal to q. We'll show that q isn't a p-number.

Let $f^q = h^m$. Since, by the condition, subgroup $C(H)$ doesn't contain elements of finite order, $(m, q) = 1$. From the order hand, there exists a homomorphism

$\psi \in \Psi$, mapping h to a non-identity element. By virtue of the choice of element f one can find such a number x that $f\psi = h^x\psi$. Then $qx \equiv m$ $(mod\ |h\psi|)$, and, the order of the element $h\psi$ being a non-unit p-number, the property $p|q$ would imply $p|m$ and $(m, q) \neq 1$.

Thus, q isn't a p-number, and we can put $f_n = f^n$ again.

Let now $b \in B \setminus K$ be an arbitrary element. Suppose that $b^{-1}Kb \cap K \neq 1$, and $b^{-1}k^nb \in b^{-1}Kb \cap K$ for some $n > 0$. Since subgroup K is separable by the subgroups of family Ω_B^p, it is p'-isolated in group B, hence n may be considered as a p-number. Putting $g = [b^{-1}f_nb, f_n]$, where element f_n is defined above, we get $g \neq 1$ and at the same time $g\psi = 1$ for any homomorphism $\psi \in \Psi$. This contradicts the residual p-finiteness of group G.

Thus, $b^{-1}Kb \cap K = 1$ for every element $b \in B \setminus K$. It follows, in particular, that an arbitrary non-unit element of subgroup K doesn't commutate with any element of group G having a reduced form of a syllable length greater than 1.

Let now a p'-isolated cyclic subgroup $\langle u\rangle$ of group G be not p-separable in G, and let $v \in G$ be such an element that $v \notin \langle u\rangle$, but $v\psi \in \langle u\psi\rangle$ for every homomorphism $\psi \in \Psi$. Then, as it was noted above, $[u, v] = 1$.

Suppose, at first, that element u belongs to some subgroup C conjugated with A or with B. Then v comes to be an element of the same subgroup C: it follows from general considerations (see, e. g., [11, theorem 4.5]) if u isn't contained in a subgroup conjugated with K, and from proved before otherwise. Hence, the subgroup $\langle u\rangle$ conjugates with a subgroup of family $\Lambda_A^p \cup \Lambda_B^p$, which coincides with $\Delta_A^p \cup \Delta_B^p$.

The case, when element v belongs to a subgroup conjugated with A or with B, is considered similarly.

Let, at last, neither u, nor v be contained in such a subgroup. Then $u = g^{-1}k^mgw^s$, $v = g^{-1}k^ngw^t$, where g, $w \in G$ and $[g^{-1}k^mg, w] = [g^{-1}k^ng, w] = 1$ [ibid.]. We'll show that it is impossible.

It follows from $[g^{-1}k^mg, w] = [g^{-1}k^ng, w] = 1$ that $[k^m, gwg^{-1}] = [k^n, gwg^{-1}] = 1$, and so either $m = n = 0$, or $gwg^{-1} \in A \cup B$. The second case just gives a contradiction, since elements $u = g^{-1}k^m(gwg^{-1})^sg$ and $v = g^{-1}k^n(gwg^{-1})^tg$ turn out in the subgroup conjugated with A or with B by element g.

Thus, $u = w^s$ and $v = w^t$, and s is a p-number. As it was noted above, the residual p-finiteness of group G gives the existence of it's normal subgroup N, say, of finite p-index, which intersects with the cyclic subgroup generated by element w at the subgroup $\langle w^s\rangle$. Since $v \notin \langle u\rangle$, $vN \neq 1$ in the group G/N, i. e. $vN \notin \langle uN\rangle$. We get a contradiction with the choice of element v.

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