LATTICES OF REGULAR CLOSED SUBSETS OF CLOSURE SPACES

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Abstract. For a closure space (P,φ) with $\varphi(\varnothing)=\varnothing$, the closures of open subsets of P, called the *regular closed* subsets, form an ortholattice $\mathrm{Reg}(P,\varphi)$, extending the poset $\mathrm{Clop}(P,\varphi)$ of all clopen subsets. If (P,φ) is a finite convex geometry, then $\mathrm{Reg}(P,\varphi)$ is pseudocomplemented. The Dedekind-MacNeille completion of the poset of regions of any central hyperplane arrangement can be obtained in this way, hence it is pseudocomplemented. The lattice $\mathrm{Reg}(P,\varphi)$ carries a particularly interesting structure for special types of convex geometries, that we call *closure spaces of semilattice type*. For finite such closure spaces,

— $\mathrm{Reg}(P,\varphi)$ satisfies an infinite collection of stronger and stronger quasi-identities, weaker than both meet- and join-semidistributivity. Nevertheless it may fail semidistributivity.

— If $\mathrm{Reg}(P,\varphi)$ is semidistributive, then it is a bounded homomorphic image of a free lattice.

— $\mathrm{Clop}(P,\varphi)$ is a lattice iff every regular closed set is clopen.

The extended permutohedron $\mathsf{R}(G)$ on a graph G, and the extended permutohedron $\mathrm{Reg}\,S$ on a join-semilattice S, are both defined as lattices of regular closed sets of suitable closure spaces. While the lattice of regular closed sets is, in the semilattice context, always the Dedekind Mac-Neille completion of the poset of clopen sets, this does not always hold in the graph context, although it always does so for finite block graphs and for cycles. Furthermore, both $\mathsf{R}(G)$ and $\mathrm{Reg}\,S$ are bounded homomorphic images of free lattices.

Contents

Date: September 11, 2018.

2010 *Mathematics Subject Classification.* 06A15, 05C40, 05C63, 05C05, 06A12, 06B25, 20F55.

Key words and phrases. Lattice; pseudocomplemented; semidistributive; bounded; join-irreducible; join-dependency; permutohedron; orthocomplementation; closed; open; clopen; regular closed; graph; block graph; clique; bipartite.

1. Introduction

The lattice of permutations $\mathsf{P}(n)$, also known as the permutohedron, even if well known and studied in combinatorics, is a relatively young object of study from a pure lattice-theoretical perspective. Its elements, the permutations of n elements, are endowed with the weak Bruhat order; this order turns out to be a lattice.

There are many possible generalization of this order, arising from the theory of Coxeter groups (Björner [4]), from graph and order theory (Pouzet *et al.* [33], Santocanale and Wehrung [36]; see also Section 14), from language theory (Flath [13], Bennett and Birkhoff [3]), from geometry (Edelman [11], Björner, Edelman, and Ziegler [5], Reading [34]).

While trying to understand those generalizations in a unified framework, we observed that the most noticeable property of permutohedra—at least from a lattice-theoretical perspective—is that they arise as lattices of clopen (that is, closed and open) subsets for a closure operator. We started thus investigating this kind of construction.

While closed subsets of a closure space naturally form a lattice when ordered under subset inclusion, the same need not be true for clopen subsets. Yet, we can tune our attention to a larger kind of subsets, the closures of open subsets, called here regular closed subsets; they always form, under subset inclusion, a lattice. Thus, for a closure space (P, φ), we denote by $\mathrm{Reg}(P, \varphi)$ the lattice of regular closed subsets of P; $\mathrm{Reg}(P, \varphi)$ is then an orthocomplemented lattice, which contains a copy of $\mathrm{Clop}(P, \varphi)$, the poset of all clopen subsets of P. There are many important classes of closure spaces (P, φ) for which $\mathrm{Reg}(P, \varphi)$ is the Dedekind-MacNeille completion of $\mathrm{Clop}(P, \varphi)$. One of them is the closure space giving rise to relatively convex subsets of real affine spaces (cf. Corollary 5.4). As a particular case, we describe the Dedekind-MacNeille completion L of the poset of regions of any central hyperplane arrangement as the lattice of all regular closed subsets of a convex geometry of the type above (Theorem 6.2). This implies, in particular, that the lattice L is always pseudocomplemented (Corollary 6.4).

After developing some basic properties of $\mathrm{Reg}(P, \varphi)$, we restrict our focus to a class of closure spaces (P, φ) that arise in the concrete examples we have in mind—we call them *closure spaces of semilattice type*. For such closure spaces, P is a poset, and every minimal covering $\boldsymbol{x}$ of $p \in P$, with respect to the closure operator φ, joins to p (i.e., $p = \bigvee \boldsymbol{x}$). A closure space of semilattice type turns out to be an atomistic convex geometry. For finite such closure spaces, we can prove the following facts:

— $\mathrm{Reg}(P, \varphi)$ satisfies an infinite collection of stronger and stronger quasi-identities, weaker than semidistributivity (cf. Theorem 10.4 and the discussion following). Nevertheless it may fail semidistributivity (cf. Example 10.1).
— If $\mathrm{Reg}(P, \varphi)$ is semidistributive, then it is a bounded homomorphic image of a free lattice (cf. Theorem 11.3).
— $\mathrm{Clop}(P, \varphi)$ is a lattice iff $\mathrm{Clop}(P, \varphi) = \mathrm{Reg}(P, \varphi)$ (cf. Theorem 17.3).

While it is reasonable to conjecture that $\mathrm{Reg}(P, \varphi)$ is the Dedekind-MacNeille completion of $\mathrm{Clop}(P, \varphi)$—and this is actually the case for many examples—we disprove this conjecture in the general case, with various finite counterexamples (cf. Example 17.4 and Corollary 18.2). Yet we prove that, in the finite case, the inclusion map of $\mathrm{Clop}(P, \varphi)$ into $\mathrm{Reg}(P, \varphi)$ preserves all existing meets and joins (cf. Theorem 17.2).

We focus then on two concrete examples of closure spaces of semilattice type. In the first case, P is the collection $\boldsymbol{\delta}_G$ of all nonempty connected subsets of a *graph* G, endowed with set inclusion, while in the second case, P is an arbitrary *join-semilattice*, endowed with its natural ordering. In case $P = \boldsymbol{\delta}_G$, we define the closure operator in such a way that, if G is a Dynkin diagram of type A_n, then we obtain $\mathrm{Reg}(P, \varphi) = \mathrm{Clop}(P, \varphi)$ isomorphic to the permutohedron $\mathsf{P}(n + 1)$ (symmetric group on $n + 1$ letters, with the weak Bruhat ordering). In case P is a join-semilattice, the closure operator associates to a subset $\boldsymbol{x}$ of P the join-subsemilattice of P generated by $\boldsymbol{x}$, and then we write $\mathrm{Reg}\, P$ instead of $\mathrm{Reg}(P, \varphi)$.

In the finite case and for both classes above, we prove that $\mathrm{Reg}(P, \varphi)$ is a bounded homomorphic image of a free lattice (cf. Theorems 12.2 and 14.9). For the closure space defined above in $P = \boldsymbol{\delta}_G$,

— We characterize those graphs G for which $\mathrm{Clop}(P, \varphi)$ is a lattice; these turn out to be the block graphs without any 4-clique (cf. Theorem 15.1).
— We give a nontrivial description of the completely join-irreducible elements of $\mathrm{Reg}(P, \varphi)$, in terms of so-called *pseudo-ultrafilters* on nonempty connected subsets of G (cf. Theorem 16.8). It follows that if G has no diamond-contractible induced subgraph, then every completely join-irreducible regular closed set is clopen (cf. Theorem 16.10).
— It follows that if G is finite and either a block graph or a cycle, then $\mathrm{Reg}(P, \varphi)$ is the Dedekind-MacNeille completion of $\mathrm{Clop}(P, \varphi)$ (cf. Corollary 16.11).
— We find a finite graph G for which $\mathrm{Reg}(P, \varphi)$ is not the Dedekind-MacNeille completion of $\mathrm{Clop}(P, \varphi)$ (cf. Corollary 18.2).
– If G is a complete graph on seven vertices, we find a regular open subset of $\boldsymbol{\delta}_G$ which is not a union of clopen subsets (cf. Theorem 19.1).

For the closure space defined above on a join-semilattice S,

— We give a precise description of the minimal neighborhoods of elements of S (cf. Theorem 9.1) and the completely join-irreducible elements of $\mathrm{Reg}\, S$ (cf. Theorem 13.2), in terms of differences of ideals of S. It follows that these sets are all clopen.
— We prove that every open subset of S is a union of clopen subsets of S, thus that $\mathrm{Reg}\, S$ is the Dedekind-MacNeille completion of $\mathrm{Clop}\, S$ (cf. Corollary 9.2).

— It follows that $\operatorname{Reg} S = \operatorname{Clop} S$ iff $\operatorname{Clop} S$ is a lattice, iff $\operatorname{Clop} S$ is a complete sublattice of $\operatorname{Reg} S$ (cf. Corollary 9.3).

We illustrate our paper with many examples and counterexamples.

2. Basic concepts

We refer the reader to Grätzer [17] for basic facts and notation about lattice theory.

We shall denote by 0 (resp., 1) the least (resp., largest) element of a partially ordered set (from now on *poset*) $(P, \leq)$, if they exist. For subsets $\boldsymbol{a}$ and $\boldsymbol{x}$ in a poset P, we shall set

$$\begin{aligned}\boldsymbol{a} \downarrow \boldsymbol{x} &= \{p \in \boldsymbol{a} \mid (\exists x \in \boldsymbol{x})(p \leq x)\}\,,\\ \boldsymbol{a} \downdownarrows \boldsymbol{x} &= \{p \in \boldsymbol{a} \mid (\exists x \in \boldsymbol{x})(p < x)\}\,,\\ \boldsymbol{a} \uparrow \boldsymbol{x} &= \{p \in \boldsymbol{a} \mid (\exists x \in \boldsymbol{x})(p \geq x)\}\,.\end{aligned}$$

We shall say that $\boldsymbol{x}$ is a *lower subset* of P if $\boldsymbol{x} = P \downarrow \boldsymbol{x}$. For $x \in P$, we shall write $\boldsymbol{a} \downarrow x$ ($\boldsymbol{a} \downdownarrows x$, $\boldsymbol{a} \uparrow x$, respectively) instead of $\boldsymbol{a} \downarrow \{x\}$ ($\boldsymbol{a} \downdownarrows \{x\}$, $\boldsymbol{a} \uparrow \{x\}$, respectively). For posets P and Q, a map $f \colon P \to Q$ is *isotone* (resp., *antitone*) if $x \leq y$ implies that $f(x) \leq f(y)$ (resp., $f(y) \leq f(x)$), for all $x, y \in P$.

A *lower cover* of an element $p \in P$ is an element $x \in P$ such that $x < p$ and there is no y such that $x < y < p$; then we write $x \prec p$. If p has a unique lower cover, then we shall denote this element by p_*. *Upper covers*, and the notation p^*, are defined dually. A nonzero element p in a join-semilattice L is *join-irreducible* if $p = x \vee y$ implies that $p \in \{x, y\}$, for all $x, y \in L$. We say that p is *completely join-irreducible* if it has a unique lower cover p_* and every element $y < p$ is such that $y \leq p_*$. *Meet-irreducible* and *completely meet-irreducible* elements are defined dually. We denote by $\operatorname{Ji} L$ (resp., $\operatorname{Mi} L$) the set of all join-irreducible (resp., meet-irreducible) elements of L.

Every completely join-irreducible element is join-irreducible and, in a finite lattice, the two concepts are equivalent. A lattice L is *spatial* if every element of L is a (possibly infinite) join of completely join-irreducible elements of L. Equivalently, a lattice L is spatial if, for all $a, b \in L$, $a \nleq b$ implies that there exists a completely join-irreducible element p of L such that $p \leq a$ and $p \nleq b$. For a completely join-irreducible element p and a completely meet-irreducible element u of L, let $p \nearrow u$ hold if $p \leq u^*$ and $p \nleq u$. Symmetrically, let $u \searrow p$ hold if $p_* \leq u$ and $p \nleq u$. The *join-dependency relation* D is defined on completely join-irreducible elements by

$$p \mathbin{D} q \quad \text{if} \quad \bigl(p \neq q \text{ and } (\exists x)(p \leq q \vee x \text{ and } p \nleq q_* \vee x\bigr)\,.$$

It is well-known (cf. Freese, Ježek, and Nation [16, Lemma 11.10]) that the join-dependency relation D on a finite lattice L can be conveniently expressed in terms of the arrow relations $\nearrow$ and $\searrow$ between $\operatorname{Ji} L$ and $\operatorname{Mi} L$, as stated in the next Lemma.

Lemma 2.1. *Let p, q be distinct join-irreducible elements in a finite lattice L. Then $p \mathbin{D} q$ iff there exists $u \in \operatorname{Mi} L$ such that $p \nearrow u \searrow q$.*

A lattice L is *join-semidistributive* if $x \vee z = y \vee z$ implies that $x \vee z = (x \wedge y) \vee z$, for all $x, y, z \in L$. *Meet-semidistributivity* is defined dually. A lattice is *semidistributive* if it is both join- and meet-semidistributive.

A lattice L is a *bounded homomorphic image of a free lattice* if there are a free lattice F and a surjective lattice homomorphism $f \colon F \twoheadrightarrow L$ such that $f^{-1}\{x\}$ has both a least and a largest element, for each $x \in L$. These lattices, introduced by McKenzie in [32], play a key role in the theory of lattice varieties; often called "bounded", they are not to be confused with lattices with both a least and a largest element. A finite lattice is bounded (in the sense of McKenzie) iff the join-dependency relations on L and its dual lattice are both cycle-free (cf. Freese, Ježek, and Nation [16, Corollary 2.39]). Every bounded lattice is semidistributive (cf. Freese, Ježek, and Nation [16, Theorem 2.20]), but the converse fails, even for finite lattices (cf. Freese, Ježek, and Nation [16, Figure 5.5]).

An *orthocomplementation* on a poset P with least and largest element is a map $x \mapsto x^\perp$ of P to itself such that

(O1) $x \leq y$ implies that $y^\perp \leq x^\perp$,
(O2) $x^{\perp\perp} = x$,
(O3) $x \wedge x^\perp = 0$ (in view of (O1) and (O2), this is equivalent to $x \vee x^\perp = 1$),

for all $x, y \in P$. Elements $x, y \in P$ are *orthogonal* if $x \leq y^\perp$, equivalently $y \leq x^\perp$.

An *orthoposet* is a poset endowed with an orthocomplementation. Of course, any orthocomplementation of P is a dual automorphism of $(P, \leq)$. In particular, if P is a lattice, then *de Morgan's rules*

$$(x \vee y)^\perp = x^\perp \wedge y^\perp\,, \quad (x \wedge y)^\perp = x^\perp \vee y^\perp$$

hold for all $x, y \in P$. An *ortholattice* is a lattice endowed with an orthocomplementation.

The *parallel sum* $L = A \parallel B$ of lattices A and B is defined by adding a top and a bottom element to the disjoint union $A \cup B$.

A *graph* is a structure $(G, \sim)$, where $\sim$ is an irreflexive and symmetric binary relation on the set G. We shall often identify a subset $X \subseteq G$ with the corresponding induced subgraph $\bigl(X, \sim \cap (X \times X)\bigr)$. Let $x, y \in G$; a *path* from x to y in $(G, \sim)$ is a finite sequence $x = z_0, z_1, \dots, z_n = y$ such that $z_i \sim z_{i+1}$ for each $i < n$. If the z_i are distinct and $z_i \sim z_j$ implies $i - j = \pm 1$, then we say that the path is *induced*. A subset X of $(G, \sim)$ is *connected* if, for each $x, y \in X$, there exists a path from x to y in X. A connected subset X of G is *biconnected* if it is connected and $X \setminus \{x\}$ is connected for each $x \in X$. We shall denote by $\mathcal{K}_n$ the complete graph (or *clique*) on n vertices, for any positive integer n.

We say that G is a *block graph* if each biconnected subset of G is a clique (*we do not assume that block graphs are connected*). Equivalently, none of the cycles $\mathcal{C}_n$, for $n \geq 4$, nor the diamond $\mathcal{D}$ (cf. Figure 2.1) embeds into G as an induced subgraph.

Also, block graphs are characterized as those graphs where there is at most one induced path between any two vertices, respectively the graphs where any intersection of connected subsets is connected. For references, see Bandelt and Mulder [2], Howorka [21], Kay and Chartrand [27], and the wonderful online database `http://www.graphclasses.org/`. Block graphs have been sometimes (for example in Howorka [21]) called *Husimi trees*.

We shall denote by $\operatorname{Pow} X$ the powerset of a set X. For every positive integer n, $[n]$ will denote the set $\{1, 2, \dots, n\}$.

3. Regular closed subsets with respect to a closure operator

A *closure operator* on a set P is usually defined as an extensive, idempotent, isotone map $\varphi\colon \operatorname{Pow} P \to \operatorname{Pow} P$; that is, $\boldsymbol{x} \subseteq \varphi(\boldsymbol{x})$, $\varphi(\varphi(\boldsymbol{x})) = \varphi(\boldsymbol{x})$, and $\varphi(\boldsymbol{x}) \subseteq \varphi(\boldsymbol{y})$ if $\boldsymbol{x} \subseteq \boldsymbol{y}$, for all $\boldsymbol{x}, \boldsymbol{y} \subseteq P$. Throughout this paper we shall require that a

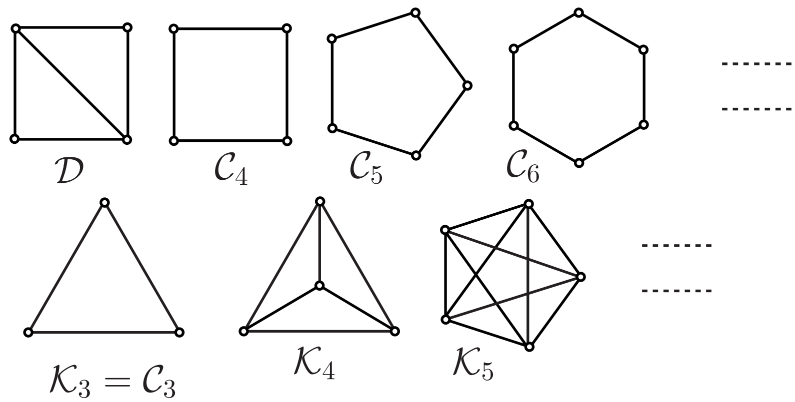

FIGURE 2.1. Cycles, diamond, and cliques

closure operator φ satisfies the additional condition $\varphi(\varnothing) = \varnothing$. A *closure space* is a pair (P, φ), where φ is a closure operator on P.

We say that the closure space (P, φ) is *atomistic* if $\varphi(\{p\}) = \{p\}$ for each $p \in P$. The associated *kernel* (or *interior*) *operator* is defined by $\breve{\varphi}(\boldsymbol{x}) = P \setminus \varphi(P \setminus \boldsymbol{x})$ for each $\boldsymbol{x} \subseteq P$. We shall often call $\varphi(\boldsymbol{x})$ the *closure* of $\boldsymbol{x}$ and $\breve{\varphi}(\boldsymbol{x})$ the *interior* of $\boldsymbol{x}$. Then both φ and $\breve{\varphi}$ are idempotent operators, with $\breve{\varphi} \leq \mathrm{id} \leq \varphi$. It is very easy to find examples with $\varphi \neq \varphi\breve{\varphi}\varphi$. However,

Lemma 3.1. *The operators $\varphi\breve{\varphi}$ and $\breve{\varphi}\varphi$ are both idempotent. Thus, $\breve{\varphi}\varphi$ is a closure operator on the collection of open sets, and $\varphi\breve{\varphi}$ is a kernel operator on the collection of closed sets.*

Proof. Let $\boldsymbol{x} \subseteq P$. From $\breve{\varphi} \leq \mathrm{id}$ it follows that $\varphi\breve{\varphi}\varphi\breve{\varphi}(\boldsymbol{x}) \subseteq \varphi\varphi\breve{\varphi}(\boldsymbol{x}) = \varphi\breve{\varphi}(\boldsymbol{x})$. From $\mathrm{id} \leq \varphi$ it follows that $\varphi\breve{\varphi}\varphi\breve{\varphi}(\boldsymbol{x}) \supseteq \varphi\breve{\varphi}\breve{\varphi}(\boldsymbol{x}) = \varphi\breve{\varphi}(\boldsymbol{x})$.

If $\boldsymbol{x}$ is open, then $\boldsymbol{x} = \breve{\varphi}(\boldsymbol{x}) \subseteq \varphi\breve{\varphi}(\boldsymbol{x})$. As $\breve{\varphi}\varphi$ is isotone, it follows that $\breve{\varphi}\varphi$ is a closure operator on the collection of all open sets. Dually, $\varphi\breve{\varphi}$ is a kernel operator on the collection of all closed sets. $\square$

Definition 3.2. For a closure space (P, φ), a subset $\boldsymbol{x}$ of P is

— *closed* if $\boldsymbol{x} = \varphi(\boldsymbol{x})$,
— *open* if $\boldsymbol{x} = \breve{\varphi}(\boldsymbol{x})$,
— *regular closed* if $\boldsymbol{x} = \varphi\breve{\varphi}(\boldsymbol{x})$,
— *regular open* if $\boldsymbol{x} = \breve{\varphi}\varphi(\boldsymbol{x})$,
— *clopen* if it is simultaneously closed and open.

We denote by $\mathrm{Clop}(P, \varphi)$ ($\mathrm{Reg}(P, \varphi)$, $\mathrm{Reg}_{\mathrm{op}}(P, \varphi)$, respectively) the set of all clopen (regular closed, regular open, respectively) subsets of P, ordered by set inclusion. Due to the condition $\varphi(\varnothing) = \varnothing$, the sets $\varnothing$ and P are both clopen.

Of course, a set $\boldsymbol{x}$ is open (closed, regular closed, regular open, clopen, respectively) iff its complement $\boldsymbol{x}^{\mathsf{c}} = P \setminus \boldsymbol{x}$ is closed (open, regular open, regular closed, clopen, respectively). A straightforward application of Lemma 3.1 yields the following.

Lemma 3.3.

(i) *A subset* $\boldsymbol{x}$ *of* P *is regular closed iff* $\boldsymbol{x} = \varphi(\boldsymbol{u})$ *for some open set* $\boldsymbol{u}$*.*

(ii) *The poset* $\operatorname{Reg}(P, \varphi)$ *is a complete lattice, with meet and join given by*

$$\bigvee(\boldsymbol{a}_i \mid i \in I) = \varphi\big(\bigcup(\boldsymbol{a}_i \mid i \in I)\big)\,,$$
$$\bigwedge(\boldsymbol{a}_i \mid i \in I) = \varphi\breve{\varphi}\big(\bigcap(\boldsymbol{a}_i \mid i \in I)\big)\,,$$

for any family $(\boldsymbol{a}_i \mid i \in I)$ *of regular closed sets.*

Remark 3.4. The previous Lemma is an immediate consequence of the fact that $\varphi\breve{\varphi}$ is a kernel operator on closed sets. For a direct proof, we need to argue that $\varphi\big(\bigcup(\boldsymbol{a}_i \mid i \in I)\big)$ is regular closed. To this goal, observe that this set is equal to $\varphi\big(\bigcup(\breve{\varphi}(\boldsymbol{a}_i) \mid i \in I)\big)$ and, more generally,

$$\bigvee(\boldsymbol{a}_i \mid i \in I) = \varphi\big(\bigcup(\boldsymbol{a}_j \mid j \in J) \cup \bigcup(\breve{\varphi}(\boldsymbol{a}_j) \mid j \notin J)\big)\,,$$

whenever J is a subset of I.

The complement of a regular closed set may not be closed. Nevertheless, we shall now see that there is an obvious "complementation-like" map from the regular closed sets to the regular closed sets.

Definition 3.5. We define the *orthogonal* of $\boldsymbol{x}$ as $\boldsymbol{x}^{\perp} = \varphi(\boldsymbol{x}^{\mathsf{c}})$, for any $\boldsymbol{x} \subseteq P$.

Lemma 3.6.

(i) $\boldsymbol{x}^{\perp}$ *is regular closed, for any* $\boldsymbol{x} \subseteq P$*.*

(ii) *The assignment* ${}^{\perp}\colon \boldsymbol{x} \mapsto \boldsymbol{x}^{\perp}$ *defines an orthocomplementation of* $\operatorname{Reg}(P, \varphi)$*.*

Proof. (i). This follows right away from Lemma 3.3(i).

(ii). It is obvious that the map ${}^{\perp}$ is antitone. Now, using Lemma 3.1, we obtain

$$\boldsymbol{x}^{\perp\perp} = \varphi((\boldsymbol{x}^{\perp})^{\mathsf{c}}) = \varphi(\varphi(\boldsymbol{x}^{\mathsf{c}})^{\mathsf{c}}) = \varphi(\breve{\varphi}(\boldsymbol{x})) = \boldsymbol{x}\,, \quad \text{for each } \boldsymbol{x} \in \operatorname{Reg}(P, \varphi)\,.$$

Therefore, ${}^{\perp}$ defines a dual automorphism of the lattice $\operatorname{Reg}(P, \varphi)$. As $\boldsymbol{x}^{\perp}$ contains $\boldsymbol{x}^{\mathsf{c}}$, $P = \boldsymbol{x} \cup \boldsymbol{x}^{\perp}$ for each $\boldsymbol{x} \subseteq P$, hence $P = \boldsymbol{x} \vee \boldsymbol{x}^{\perp}$ in case $\boldsymbol{x} \in \operatorname{Reg}(P, \varphi)$. □

In particular, $\operatorname{Reg}(P, \varphi)$ is self-dual. As $\boldsymbol{x} \mapsto \boldsymbol{x}^{\mathsf{c}}$ defines a dual isomorphism from $\operatorname{Reg}(P, \varphi)$ to $\operatorname{Reg}_{\mathrm{op}}(P, \varphi)$, we obtain the following.

Corollary 3.7. *Let* (P, φ) *be a closure space. Then the lattices* $\operatorname{Reg}(P, \varphi)$ *and* $\operatorname{Reg}_{\mathrm{op}}(P, \varphi)$ *are both self-dual. Moreover, the maps* $\breve{\varphi}\colon \operatorname{Reg}(P, \varphi) \to \operatorname{Reg}_{\mathrm{op}}(P, \varphi)$ *and* $\varphi\colon \operatorname{Reg}_{\mathrm{op}}(P, \varphi) \to \operatorname{Reg}(P, \varphi)$ *are mutually inverse isomorphisms.*

As the following result shows, there is nothing special about orthoposets of the form $\operatorname{Clop}(P, \varphi)$, or complete ortholattices of the form $\operatorname{Reg}(P, \varphi)$.

Proposition 3.8. *Let* $(L, 0, 1, \leq, {}^{\perp})$ *be an orthoposet. Then there exists a closure space* (Ω, φ) *such that* $L \cong \operatorname{Clop}(\Omega, \varphi)$*, and such that, in addition,* $\operatorname{Reg}(\Omega, \varphi)$ *is the Dedekind-MacNeille completion of* $\operatorname{Clop}(\Omega, \varphi)$*.*

Outline of proof. We invoke a construction due to Mayet [30], or, equivalently, Katrnoška [26]. As everything needed here is already proved in those papers, we just give an outline of the proof, leaving the details as an exercise.

For such an orthoposet L, we say that a subset X of L is *anti-orthogonal* if its elements are pairwise non-orthogonal. We denote then by Ω the set of all maximal anti-orthogonal subsets of L. We set $\mathrm{Z}(p) = \{X \in \Omega \mid p \in X\}$, for each $p \in L$, and we call the sets $\mathrm{Z}(p)$ *elementary clopen*. We define $\varphi(\boldsymbol{x})$ as the intersection of all elementary clopen sets containing $\boldsymbol{x}$, for each $\boldsymbol{x} \subseteq \Omega$. The pair (Ω, φ) is a closure space. It turns out that the clopen sets, with respect to that closure space, are exactly the elementary clopen sets. A key property, to be verified in the course of the proof above, is that $\mathrm{Z}(p^{\perp}) = \Omega \setminus \mathrm{Z}(p)$ for every $p \in L$. Hence, the assignment $p \mapsto \mathrm{Z}(p)$ defines an isomorphism from $(L, 0, 1, \leq, ^{\perp})$ onto $(\mathrm{Clop}(\Omega, \varphi), \varnothing, \Omega, \subseteq, \complement)$, and clopen is the same as elementary clopen.

Every closed set is, by definition, an intersection of clopen sets. Hence, by Lemma 4.1, $\mathrm{Reg}(\Omega, \varphi)$ is then the Dedekind-MacNeille completion of $\mathrm{Clop}(\Omega, \varphi)$. □

While Proposition 3.8 implies that every finite orthocomplemented lattice has the form $\mathrm{Reg}(P, \varphi)$, for some finite closure space (P, φ), we shall now establish a restriction on $\mathrm{Reg}(P, \varphi)$ in case (P, φ) is a *convex geometry*, that is (cf. Edelman and Jamison [12]), $\varphi(\boldsymbol{x} \cup \{p\}) = \varphi(\boldsymbol{x} \cup \{q\})$ implies that $p = q$, for all $\boldsymbol{x} \subseteq P$ and all $p, q \in P \setminus \varphi(\boldsymbol{x})$.

Recall that a lattice L with zero is *pseudocomplemented* if for each $x \in L$, there exists a largest $y \in L$, called the *pseudocomplement* of x, such that $x \wedge y = 0$. It is mentioned in Chameni-Nembua and Monjardet [8] (and credited there to a personal communication by Le Conte de Poly-Barbut) that every permutohedron is pseudocomplemented; see also Markowsky [29].

While not every orthocomplemented lattice is pseudocomplemented (the easiest counterexample is M_4, see Figure 3.1), we shall now see that the lattice of regular closed subsets of a finite convex geometry is always pseudocomplemented. Our generalization is formally similar to Hahmann [18, Lemma 4.17], although the existence of a precise connection between Hahmann's work and the present paper remains, for the moment, mostly hypothetical.

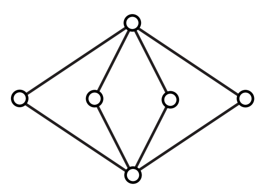

FIGURE 3.1. The orthocomplemented, non pseudocomplemented lattice M_4

We set $\partial \boldsymbol{a} = \{x \in \boldsymbol{a} \mid x \notin \varphi(\boldsymbol{a} \setminus \{x\})\}$, for every subset $\boldsymbol{a}$ in a closure space (P, φ). Observe that $p \in \varphi(\boldsymbol{x})$ implies $p \in \boldsymbol{x}$, for any $p \in \partial P$ and any $\boldsymbol{x} \subseteq P$. It is well-known that $P = \varphi(\partial P)$ for any finite convex geometry (P, φ) (cf. Edelman and Jamison [12, Theorem 2.1]), and an easy exercise to find finite examples, with $P = \varphi(\partial P)$, which are not convex geometries.

Proposition 3.9. *The lattice* $\mathrm{Reg}(P, \varphi)$ *is pseudocomplemented, for any closure space* (P, φ) *such that* $P = \varphi(\partial P)$*. In particular,* $\mathrm{Reg}(P, \varphi)$ *is pseudocomplemented in case* (P, φ) *is a finite convex geometry.*

Proof. Let $\boldsymbol{b} \in \mathrm{Reg}(P, \varphi)$ and let $(\boldsymbol{a}_i \mid i \in I)$ be a family of elements of $\mathrm{Reg}(P, \varphi)$ with join $\boldsymbol{a}$ such that $\boldsymbol{a}_i \wedge \boldsymbol{b} = \varnothing$. We must prove that $\boldsymbol{a} \wedge \boldsymbol{b} = \varnothing$. Suppose otherwise and set $\boldsymbol{d} = \breve{\varphi}(\boldsymbol{a} \cap \boldsymbol{b})$. From $\boldsymbol{a} \wedge \boldsymbol{b} = \varphi(\boldsymbol{d})$ it follows that $\boldsymbol{d} \neq \varnothing$. If $\partial P \cap \boldsymbol{d} = \varnothing$, then $\partial P \subseteq \boldsymbol{d}^{\mathsf{c}}$, thus, as $P = \varphi(\partial P)$ and $\boldsymbol{d}^{\mathsf{c}}$ is closed, $\boldsymbol{d} = \varnothing$, a contradiction.

Pick $p \in \partial P \cap \boldsymbol{d}$. As $p \in \boldsymbol{d} \subseteq \boldsymbol{a} = \varphi\big(\bigcup_{i\in I} \boldsymbol{a}_i\big)$ and $p \in \partial P$, we get $p \in \boldsymbol{a}_i$ for some $i \in I$. Furthermore, $p \in \boldsymbol{d} \subseteq \boldsymbol{b}$, so $p \in \boldsymbol{a}_i \cap \boldsymbol{b}$. On the other hand, from $\boldsymbol{a}_i \wedge \boldsymbol{b} = \varnothing$ it follows that $\breve{\varphi}(\boldsymbol{a}_i \cap \boldsymbol{b}) = \varnothing$, thus $\varphi(P \setminus (\boldsymbol{a}_i \cap \boldsymbol{b})) = P$, and thus (as $p \in \partial P$) we get $p \in P \setminus (\boldsymbol{a}_i \cap \boldsymbol{b})$, a contradiction. □

As we shall see in Example 14.10, the result of the second part of Proposition 3.9 cannot be extended to the infinite case. Observe that $\mathrm{Reg}(P, \varphi)$ being pseudo-complemented is, in the finite case, an immediate consequence of $\mathrm{Reg}(P, \varphi)$ being meet-semidistributive (which is here, by self-duality, equivalent to being semidistributive). Example 10.1 shows that the lattice $\mathrm{Reg}(P, \varphi)$ need not be semidistributive, even in case (P, φ) is a convex geometry.

4. Regular closed as Dedekind-MacNeille completion of clopen

For any closure space (P, φ), the lattice $\mathrm{Reg}(P, \varphi)$ contains the poset $\mathrm{Clop}(P, \varphi)$. It turns out that in many cases, the inclusion is a Dedekind-MacNeille completion. The following result will be always used for $K = \mathrm{Clop}(P, \varphi)$, except in Sections 5 and 6.

Lemma 4.1. *The following statements hold, for any closure space (P, φ) and any subset K of $\mathrm{Reg}(P, \varphi)$.*

(i) $\mathrm{Reg}(P, \varphi)$ *is the Dedekind-MacNeille completion of K iff every regular closed set is a join of members of K. This occurs, in particular, if every regular open set is a union of members of K.*
(ii) *If $\mathrm{Reg}(P, \varphi)$ is the Dedekind-MacNeille completion of K, then every completely join-irreducible element in $\mathrm{Reg}(P, \varphi)$ belongs to K. If $\mathrm{Reg}(P, \varphi)$ is spatial, then the converse holds.*

Proof. It is well-known that a complete lattice L is the Dedekind-MacNeille completion of a subset K iff every element of L is simultaneously a join of elements of K and a meet of elements of K (cf. Davey and Priestley [10, Theorem 7.41]). Item (i) follows easily.

If $\mathrm{Reg}(P, \varphi)$ is the Dedekind-MacNeille completion of K, then every element of $\mathrm{Reg}(P, \varphi)$ is a join of elements of K, thus every completely join-irreducible element of $\mathrm{Reg}(P, \varphi)$ belongs to K. Conversely, if $\mathrm{Reg}(P, \varphi)$ is spatial and every completely join-irreducible element in $\mathrm{Reg}(P, \varphi)$ belongs to K, then every element of $\mathrm{Reg}(P, \varphi)$ is a join of members of K, thus, using the orthocomplementation, also a meet of clopen subsets. By (i), the conclusion of (ii) follows. □

Definition 4.2. A subset K of a poset L is *tight* in L if the inclusion map preserves all existing (not necessarily finite) joins and meets. Namely,

$$a = \bigvee X \text{ in } K \Longrightarrow a = \bigvee X \text{ in } L\,, \quad \text{for all } a \in K \text{ and all } X \subseteq K\,. \tag{4.1}$$

$$a = \bigwedge X \text{ in } K \Longrightarrow a = \bigwedge X \text{ in } L\,, \quad \text{for all } a \in K \text{ and all } X \subseteq K\,. \tag{4.2}$$

It is well-known (and quite easy to verify) that if the lattice L is the Dedekind-MacNeille completion of the poset K, then K is tight in L. We shall observe—see Theorem 17.2—that $\mathrm{Clop}(P, \varphi)$ is often tight in $\mathrm{Reg}(P, \varphi)$; in particular this holds if P is a finite set and φ has *semilattice type*, as defined later in 7.3. Yet, even under those additional assumptions, there are many examples where $\mathrm{Reg}(P, \varphi)$ is not the Dedekind-MacNeille completion of $\mathrm{Clop}(P,\varphi)$ (even in the finite case), see Examples 7.5 and 17.4.

In our next section, we shall discuss a well-known class of convex geometries for which $\mathrm{Reg}(P,\varphi)$ is always the Dedekind-MacNeille completion of $\mathrm{Clop}(P,\varphi)$.

5. Convex subsets in affine spaces

Denote by $\mathrm{conv}(X)$ the convex hull of a subset X in any left affine space Δ over a linearly ordered division ring $\mathbb{K}$. For a subset E of Δ, the *convex hull operator relatively to* E is the map $\mathrm{conv}_E \colon \mathrm{Pow}\,E \to \mathrm{Pow}\,E$ defined by

$$\mathrm{conv}_E(X) = \mathrm{conv}(X) \cap E\,, \quad \text{for any } X \subseteq E\,.$$

The map conv_E is a closure operator on E. It is well-known that (E, conv_E) is an atomistic convex geometry (cf. Edelman and Jamison [12, Example I]). The fixpoints of conv_E are the *relatively convex subsets* of E. The poset $\mathrm{Clop}(E, \mathrm{conv}_E)$ consists of the *relatively bi-convex* subsets of E, that is, those $X \subseteq E$ such that both X and $E \setminus X$ are relatively convex; equivalently, $\mathrm{conv}_E(X) \cap \mathrm{conv}_E(E \setminus X) = \varnothing$. A subset X of E is *strongly bi-convex* (relatively to E) if $\mathrm{conv}(X) \cap \mathrm{conv}(E \setminus X) = \varnothing$. We denote by $\mathrm{Clop}^*(E, \mathrm{conv}_E)$ the set of all strongly bi-convex subsets of E. This set is contained in $\mathrm{Clop}(E, \mathrm{conv}_E)$, and the containment may be proper.

An *extended affine functional on* Δ is a map of the form $\ell \colon \Delta \to {}^*\mathbb{K}$, where ${}^*\mathbb{K}$ is a ultrapower of $\mathbb{K}$ and

$$\ell((1-\lambda)x + \lambda y) = (1-\lambda)\ell(x) + \lambda\ell(y)\,, \quad \text{for all } x, y \in \Delta \text{ and all } \lambda \in \mathbb{K}\,.$$

If ${}^*\mathbb{K} = \mathbb{K}$, then we say that ℓ is an *affine functional* on Δ.

Lemma 5.1. *Let $X \subseteq E \subseteq \Delta$, with E finite, and let $p \in E \setminus \mathrm{conv}(X)$. Then there exists an affine functional $\ell \colon \Delta \to \mathbb{K}$ such that*

(i) $E \cap \ell^{-1}\{0\} = \{p\}$;
(ii) $\ell(x) < 0$ *for each* $x \in X$.

Proof. We may assume without loss of generality that X is a maximal subset of E with the property that $p \notin \mathrm{conv}(X)$. Since X is finite, there exists an affine functional ℓ such that $\ell(p) = 0$ and $\ell(x) < 0$ for each $x \in X$.

Suppose that $\ell(y) = 0$ for some $y \in E \setminus \{p\}$. Necessarily, $y \in E \setminus X$, thus, by the maximality assumption on X, we get $p \in \mathrm{conv}(X \cup \{y\})$, so $p = (1-\lambda)x + \lambda y$ for some $x \in \mathrm{conv}(X)$ and some $\lambda \in \mathbb{K}$ with $0 \leq \lambda \leq 1$. From $p \neq y$ it follows that $\lambda < 1$. Since $\ell(p) = \ell(y) = 0$, it follows that $\ell(x) = 0$, a contradiction. □

Lemma 5.2. *Let $X \subseteq E \subseteq \Delta$ and let $p \in E \setminus \mathrm{conv}(X)$. Then there are a ultrapower ${}^*\mathbb{K}$ of $\mathbb{K}$ and an extended affine functional $\ell \colon \Delta \to {}^*\mathbb{K}$ such that*

(i) $E \cap \ell^{-1}\{0\} = \{p\}$;
(ii) $\ell(x) < 0$ *for each* $x \in X$.

Proof. Denote by I the set of all finite subsets of E and let $\mathcal{U}$ be a ultrafilter on I such that $\{I \uparrow F \mid F \in I\} \subseteq \mathcal{U}$. We denote by ${}^*\mathbb{K}$ the ultrapower of $\mathbb{K}$ by $\mathcal{U}$. It follows from Lemma 5.1 that for each $F \in I$, there exists an affine functional $\ell_F \colon \Delta \to \mathbb{K}$ such that $F \cap \ell_F^{-1}\{0\} = \{p\}$ and $\ell_F(x) < 0$ for each $x \in X \cap F$. For each $v \in \Delta$, denote by $\ell(v)$ the equivalence class modulo $\mathcal{U}$ of the family $(\ell_F(v) \mid F \in I)$. Then ℓ is as required. □

Say that an extended affine functional $\ell\colon \Delta \to {}^*\mathbb{K}$ is *special*, with respect to a subset E of Δ, if $\ell^{-1}\{0\} \cap E$ is a singleton. A *special relative half-space* of E is a subset of E of the form $\ell^{-1}[<0] \cap E$ (where we set $\ell^{-1}[<0] = \{x \in E \mid \ell(x) < 0\}$), for some special affine functional ℓ. It is obvious that every special relative half-space is a strongly bi-convex, proper subset of E. The converse statement for E finite is an easy exercise. For E infinite, the converse may fail (take $\Delta = \mathbb{R}$, $E = \{1/n \mid 0 < n < \omega\} \cup \{0\}$, and $X = \{0\}$).

Corollary 5.3. *Let $E \subseteq \Delta$. Then every relatively convex subset of E is the intersection of all special relative half-spaces of E containing it.*

Proof. Any relatively convex subset X of E is trivially contained in the intersection $\widetilde{X}$ of all special relative half-spaces of E containing X. Let $p \in \widetilde{X} \setminus X$. Since X is relatively convex, $p \notin \operatorname{conv}(X)$. By Lemma 5.2, there are a ultrapower ${}^*\mathbb{K}$ of $\mathbb{K}$ and an extended affine functional $\ell\colon \Delta \to {}^*\mathbb{K}$ such that $E \cap \ell^{-1}\{0\} = \{p\}$ and $X \subseteq \ell^{-1}[<0]$. The set $\widetilde{X}$ is, by definition, contained in $\ell^{-1}[<0]$, whence $\ell(p) < 0$, a contradiction. Therefore, $\widetilde{X} = X$. □

Since every (special) relative half-space of Δ is strongly bi-convex, a simple application of Lemma 4.1 yields the following.

Corollary 5.4. *Let $E \subseteq \Delta$. Then $\operatorname{Reg}(E, \operatorname{conv}_E)$ is the Dedekind-MacNeille completion of $\operatorname{Clop}^*(E, \operatorname{conv}_E)$ (thus also of $\operatorname{Clop}(E, \operatorname{conv}_E)$).*

In addition, we point in the following result a few noticeable features of the completely join-irreducible elements of $\operatorname{Reg}(E, \operatorname{conv}_E)$.

Theorem 5.5. *Let $E \subseteq \Delta$. For every completely join-irreducible element P of $\operatorname{Reg}(E, \operatorname{conv}_E)$, there are $p \in P$, a ultrapower ${}^*\mathbb{K}$ of $\mathbb{K}$, and a special extended affine functional $\ell\colon \Delta \to {}^*\mathbb{K}$ such that the following statements hold:*

(i) $\ell(p) = 0$;
(ii) $P = \ell^{-1}[\geq 0] \cap E$;
(iii) $P_* = P \setminus \{p\}$;
(iv) *both P and P_* are strongly bi-convex.*

In particular, the element p above is unique. Furthermore, if E is finite, then ℓ can be taken an affine functional (i.e., ${}^*\mathbb{K} = \mathbb{K}$).

Proof. It follows from Corollary 5.4 that P is strongly bi-convex. Since $E \setminus P$ is a completely meet-irreducible element of $\operatorname{Reg}(E, \operatorname{conv}_E)$ and since, by Corollary 5.3, $E \setminus P$ is an intersection (thus also a meet) of special relative half-spaces of E, $E \setminus P$ is itself a special relative half-space of E, so $E \setminus P = \ell^{-1}[<0] \cap E$ for some special extended affine functional $\ell\colon \Delta \to {}^*\mathbb{K}$. Denote by p the unique element of $\ell^{-1}\{0\} \cap E$.

Setting $Q = \ell^{-1}[>0] \cap E = P \setminus \{p\}$, we get $E \setminus Q = \ell^{-1}[\leq 0] \cap E$, so Q is strongly bi-convex as well.

Let $X \in \operatorname{Reg}(E, \operatorname{conv}_E)$ be properly contained in P. If X is not contained in Q, then $p \in X$, thus $P = Q \cup X$, and thus, *a fortiori*, $P = Q \vee X$ in $\operatorname{Reg}(E, \operatorname{conv}_E)$. Since P is join-irreducible and since $Q \neq P$, it follows that $X = P$, a contradiction. Therefore, $X \subseteq Q$, thus completing the proof of (iii).

If E is finite, then Lemma 5.1 can be used in place of Lemma 5.2 in the proof of Corollary 5.3, so "extended affine functional" can be replaced by "affine functional" in the argument above. □

6. Posets of regions of central hyperplane arrangements

In this section we shall fix a positive integer d, together with a *central hyperplane arrangement* in $\mathbb{R}^d$, that is, a finite set $\mathcal{H}$ of hyperplanes of $\mathbb{R}^d$ through the origin. The open set $\mathbb{R}^d \setminus \bigcup \mathcal{H}$ has finitely many connected components, of course all of them open, called the *regions*. We shall denote by $\mathcal{R}$ the set of all regions. Set

$$\operatorname{sep}(X, Y) = \{H \in \mathcal{H} \mid H \text{ separates } X \text{ and } Y\}\,, \quad \text{for all } X, Y \in \mathcal{R}\,.$$

After Edelman [11], we fix a distinguished "base region" B and define a partial ordering $\leq_B$ on the set $\mathcal{R}$ of all regions, by

$$X \leq_B Y \quad \text{if} \quad \operatorname{sep}(B, X) \subseteq \operatorname{sep}(B, Y)\,, \quad \text{for all } X, Y \in \mathcal{R}\,.$$

The poset $\operatorname{Pos}(\mathcal{H}, B) = (\mathcal{R}, \leq_B)$ has a natural orthocomplementation, given by $X \mapsto -X = \{-x \mid x \in X\}$. This poset is not always a lattice, see Björner, Edelman, and Ziegler [5, Example 3.3].

Denote by $(x, y) \mapsto \langle x \mid y \rangle$ the standard inner product on $\mathbb{R}^d$ and pick, for each $H \in \mathcal{H}$, a vector $z_H \in \mathbb{R}^d$, on the same side of H as B, such that

$$H = \{x \in \mathbb{R}^d \mid \langle z_H \mid x \rangle = 0\}\,.$$

Fix $b \in B$. Observing that $\langle z_H \mid b \rangle > 0$ for each $H \in \mathcal{H}$, we may scale z_H and thus assume that

$$\langle z_H \mid b \rangle = 1 \quad \text{for each} \quad H \in \mathcal{H}\,. \tag{6.1}$$

The set $\Delta = \{x \in \mathbb{R}^d \mid \langle x \mid b \rangle = 1\}$ is an affine hyperplane of $\mathbb{R}^d$, containing $E = \{z_H \mid H \in \mathcal{H}\}$. For each $R \in \mathcal{R}$ and each $z \in E$, the sign of $\langle z \mid x \rangle$, for $x \in R$, is constant. Accordingly, we shall write $\langle z \mid R \rangle > 0$ instead of $\langle z \mid x \rangle > 0$ for some (every) $x \in R$; and similarly for $\langle z \mid R \rangle < 0$. The following result is contained in Björner, Edelman, and Ziegler [5, Remark 5.3]. Due to (6.1), z_H can be expressed in the form $\sum_{i \in I} \lambda_i z_{H_i}$, with all $\lambda_i \geq 0$, iff it belongs to the convex hull of $\{z_{H_i} \mid i \in I\}$ (i.e., one can take $\sum_{i \in I} \lambda_i = 1$); hence our formulation involves convex sets instead of convex cones.

Lemma 6.1. *The assignment $\varepsilon\colon R \mapsto \{z \in E \mid \langle z \mid R \rangle < 0\}$ defines an order-isomorphism from $\operatorname{Pos}(\mathcal{H}, B)$ onto the set $\operatorname{Clop}^*(E, \operatorname{conv}_E)$ of all strongly bi-convex subsets of E.*

Proof. For any $r \in R$, the set $\varepsilon(R) = \{z \in E \mid \langle z \mid r \rangle < 0\}$ has complement $E \setminus \varepsilon(R) = \{z \in E \mid \langle z \mid r \rangle \geq 0\}$, hence it is strongly bi-convex in E. Furthermore, for $R, S \in \mathcal{R}$, $\varepsilon(R) \subseteq \varepsilon(S)$ iff $\langle z_H \mid R \rangle < 0$ implies that $\langle z_H \mid S \rangle < 0$ for any $H \in \mathcal{H}$, iff $\operatorname{sep}(B, R) \subseteq \operatorname{sep}(B, S)$, iff $R \leq_B S$; whence ε is an order-embedding.

Finally, we must prove that every strongly bi-convex subset U of E belongs to the image of ε. Since $\varnothing = \varepsilon(B)$ and $E = \varepsilon(-B)$, we may assume that $U \neq \varnothing$ and $U \neq E$. Since U and $E \setminus U$ have disjoint convex hulls in Δ, there exists an affine functional ℓ on Δ such that

$$U = \{z \in E \mid \ell(z) < 0\} \quad \text{and} \quad E \setminus U = \{z \in E \mid \ell(z) > 0\}\,.$$

Let c be any normal vector to the unique hyperplane of $\mathbb{R}^d$ containing $\{0\} \cup \ell^{-1}\{0\}$, on the same side of that hyperplane as $E \setminus U$. Then

$$(\forall z \in U)\bigl(\langle z \mid c \rangle < 0\bigr) \quad \text{and} \quad (\forall z \in E \setminus U)\bigl(\langle z \mid c \rangle > 0\bigr)\,.$$

In particular, $c \notin \bigcup \mathcal{H}$. Furthermore, if R denotes the unique region such that $c \in R$, we get

$$\varepsilon(R) = \{z \in E \mid \langle z \mid R \rangle < 0\} = \{z \in E \mid \langle z \mid c \rangle < 0\} = U\,. \qquad \square$$

According to Lemma 6.1, we shall identify $\operatorname{Pos}(\mathcal{H}, B)$ with the collection of all strongly bi-convex subsets of E.

Theorem 6.2. *The lattice* $\operatorname{Reg}(E, \operatorname{conv}_E)$ *is the Dedekind-MacNeille completion of* $\operatorname{Pos}(\mathcal{H}, B)$ *(via the embedding ε).*

Proof. By Theorem 5.5, every completely join-irreducible element P of the lattice $\operatorname{Reg}(E, \operatorname{conv}_E)$ is strongly bi-convex. By Lemma 6.1, P belongs to the image of ε. The conclusion follows then from Lemma 4.1. $\square$

The following corollary is a slight strengthening of Björner, Edelman, and Ziegler [5, Theorem 5.5], obtained by changing "bi-convex" to "regular closed".

Corollary 6.3. *The poset of regions* $\operatorname{Pos}(\mathcal{H}, B)$ *is a lattice iff every regular closed subset of E is strongly bi-convex, that is, it has the form $\varepsilon(R)$ for some region R.*

Proof. By Theorem 6.2, the lattice $\operatorname{Reg}(E, \operatorname{conv}_E)$ is generated by the image of ε. $\square$

Corollary 6.4. *The Dedekind-MacNeille completion of* $\operatorname{Pos}(\mathcal{H}, B)$ *is a pseudocomplemented lattice.*

Proof. By Theorem 6.2, $\operatorname{Reg}(E, \operatorname{conv}_E)$ is the Dedekind-MacNeille completion of $\operatorname{Pos}(\mathcal{H}, B)$. Now $(E, \operatorname{conv}_E)$ is a convex geometry, so the conclusion follows from Proposition 3.9. $\square$

Remark 6.5. There are many important cases where $\operatorname{Pos}(\mathcal{H}, B)$ is a lattice, see Björner, Edelman, and Ziegler [5]. Further lattice-theoretical properties of $\operatorname{Pos}(\mathcal{H}, B)$ are investigated in Reading [34]. In particular, even if $\operatorname{Pos}(\mathcal{H}, B)$ is a lattice, it may not be semidistributive (cf. Reading [34, Figure 3]); and even if it is semidistributive, it may not be bounded (cf. Reading [34, Figure 5]).

Remark 6.6. We established in Lemma 6.1 that the poset of all regions of any central hyperplane arrangement with base region is isomorphic to the poset of all strongly bi-convex subsets of some finite set E. Conversely, the collection of all strongly bi-convex subsets of any finite subset E in any finite-dimensional real affine space Δ arises in this fashion. Indeed, embed Δ as a hyperplane, avoiding the origin, into some $\mathbb{R}^d$, and pick $b \in \mathbb{R}^d$ such that $\langle b \mid x \rangle = 1$ for all $x \in \Delta$. The set $\mathcal{H}$ of orthogonals of all the elements of E is a central hyperplane arrangement of $\mathbb{R}^d$, and the set $\{z_H \mid H \in \mathcal{H}\}$ associated to $\mathcal{H}$ and b as above is exactly E. The corresponding base region B is the one containing b, that is,

$$B = \{x \in \mathbb{R}^d \mid \langle z \mid x \rangle > 0 \text{ for each } z \in E\}\,.$$

By the discussion above, $\operatorname{Pos}(\mathcal{H}, B) \cong \operatorname{Clop}^*(E, \operatorname{conv}_E)$.

7. Closure operators of poset and semilattice type

We study in this section closure spaces (P,φ) where P is a poset and the closure operator φ is related to the order of P; such a relation will make it possible to derive properties of $\operatorname{Reg}(P,\varphi)$ from the order. Closure spaces of this kind originate from concrete examples generalizing permutohedra, investigated in further sections.

Let (P,φ) be a closure space. A *covering* of an element $p\in P$ is a subset $\boldsymbol{x}$ of P such that $p\in\varphi(\boldsymbol{x})$. If $\boldsymbol{x}$ is minimal, with respect to set inclusion, for the property of being a covering, then we shall say that $\boldsymbol{x}$ is a *minimal covering* of p. We shall denote by $\mathcal{M}_\varphi(p)$, or $\mathcal{M}(p)$ if φ is understood, the set of all minimal coverings of p. Due to the condition $\varphi(\varnothing)=\varnothing$, every minimal covering of an element of P is nonempty. We say that a covering $\boldsymbol{x}$ of p is *nontrivial* if $p\notin\boldsymbol{x}$.

We say that (P,φ) is *algebraic* if $\varphi(\boldsymbol{x})$ is the union of the $\varphi(\boldsymbol{y})$, for all finite subsets $\boldsymbol{y}$ of $\boldsymbol{x}$, for any $\boldsymbol{x}\subseteq P$. A great deal of the relevance of algebraic closure spaces for our purposes is contained in the following straightforward lemma.

Lemma 7.1. *Let p be an element in an algebraic closure space (P,φ). Then every minimal covering of p is a finite set, and every covering of p contains a minimal covering of p.*

The following trivial observation is quite convenient for the understanding of open sets and the interior operator $\check{\varphi}$.

Lemma 7.2. *Let (P,φ) be an algebraic closure space, let $p\in P$, and let $\boldsymbol{a}\subseteq P$. Then $p\in\check{\varphi}(\boldsymbol{a})$ iff every minimal covering of p meets $\boldsymbol{a}$.*

Proof. Assume first that $p\in\check{\varphi}(\boldsymbol{a})$ and let $\boldsymbol{x}$ be a minimal covering of p. If $\boldsymbol{x}\subseteq\boldsymbol{a}^{\mathsf{c}}$ then $p\in\varphi(\boldsymbol{a}^{\mathsf{c}})$, a contradiction. Conversely, suppose that $p\notin\check{\varphi}(\boldsymbol{a})$, that is, $p\in\varphi(\boldsymbol{a}^{\mathsf{c}})$. By Lemma 7.1, there exists $\boldsymbol{x}\in\mathcal{M}(p)$ contained in $\boldsymbol{a}^{\mathsf{c}}$. $\square$

Definition 7.3. We say that an algebraic closure space (P,φ), with P a poset, has

— *poset type*, if $\boldsymbol{x}\subseteq P\downarrow p$ whenever $\boldsymbol{x}\in\mathcal{M}_\varphi(p)$,

— *semilattice type*, if $p=\bigvee\boldsymbol{x}$ whenever $\boldsymbol{x}\in\mathcal{M}_\varphi(p)$.

We say that an algebraic closure space (P,φ), with P just a set, has poset (resp., semilattice) type if there exists a poset structure on P such that, with respect to that structure, (P,φ) has poset (resp., semilattice) type.

Remark 7.4. We do not require, in the statement of Definition 7.3, that P be a join-semilattice; and indeed, in many important examples, this will not be the case.

Example 7.5. Let P be a poset and set $\varphi(\boldsymbol{x})=P\uparrow\boldsymbol{x}=\{p\in P\mid(\exists x\in\boldsymbol{x})(x\leq p)\}$, for each $\boldsymbol{x}\subseteq P$. Then (P,φ) is an algebraic closure space, and the elements of $\mathcal{M}(p)$ are exactly the singletons $\{q\}$ with $q\leq p$, for each $p\in P$. In particular, (P,φ) has poset type, and it has semilattice type iff the ordering of P is trivial.

The lattice $\operatorname{Reg}(P,\varphi)$ turns out to be complete and Boolean. It plays a fundamental role in the theory of set-theoretical forcing, where it is usually called the completion of P (or the Boolean algebra of all regular open subsets of P), see for example Jech [22]. Any complete Boolean algebra can be described in this form, so $\operatorname{Reg}(P,\varphi)$ may not be spatial. Further, if P has a largest element, then $\operatorname{Clop}(P,\varphi)=\{\varnothing,P\}$, so even in the finite case, $\operatorname{Reg}(P,\varphi)$ may not be the Dedekind-MacNeille completion of $\operatorname{Clop}(P,\varphi)$.

Another example of a closure space of poset type, but not of semilattice type, is the following.

Example 7.6. Consider a four-element set $P = \{p_0, p_1, q_0, q_1\}$ and define the closure operator φ on P by setting $\varphi(\boldsymbol{x}) = \boldsymbol{x}$ unless $\{q_0, q_1\} \subseteq \boldsymbol{x}$, in which case $\varphi(\boldsymbol{x}) = P$. Then the partial ordering $\trianglelefteq$ on P with the only nontrivial coverings $q_j \trianglelefteq p_i$, for $i, j < 2$, witnesses (P, φ) being a closure space of poset type.

As $\{q_0, q_1\}$ is a minimal covering, with respect to φ, of both p_0 and p_1, any ordering on P witnessing (P, φ) being of semilattice type would thus satisfy that $p_0 = q_0 \vee q_1$ and $p_1 = q_0 \vee q_1$ (with respect to that ordering), contradicting $p_0 \neq p_1$.

An important feature of closure spaces of poset type, reminiscent of the absence of D-cycles in lower bounded homomorphic images of free lattices, is the following.

Lemma 7.7. *Let (P, φ) be a closure space of poset type, let $p, q \in P$, and let $\boldsymbol{a} \subseteq P$. If $p \in \varphi(\boldsymbol{a} \cup \{q\}) \setminus \varphi(\boldsymbol{a})$, then $p \geq q$.*

Proof. By Lemma 7.1, there exists $\boldsymbol{x} \in \mathcal{M}(p)$ such that $\boldsymbol{x} \subseteq \boldsymbol{a} \cup \{q\}$. From $p \notin \varphi(\boldsymbol{a})$ it follows that $q \in \boldsymbol{x}$, while, as (P, φ) has poset type, $\boldsymbol{x} \subseteq P \downarrow p$. $\square$

It follows easily from the previous Lemma that *every closure space of poset type is a convex geometry.* For semilattice type, we get the following additional property.

Lemma 7.8. *Every closure space of semilattice type is atomistic.*

Proof. If $y \in \varphi(\{x\})$, then $\{x\} \in \mathcal{M}(y)$ (because $\varphi(\varnothing) = \varnothing$), so $y = \bigvee\{x\} = x$. $\square$

Example 7.9. Let $\boldsymbol{e}$ be a transitive binary relation on some set. It is obvious that the transitive closure on subsets of $\boldsymbol{e}$ gives rise to an algebraic closure operator τ on $\boldsymbol{e}$, the latter being viewed as a set of pairs. We study in our paper [36] the lattice of all regular closed subsets of $\boldsymbol{e}$. In particular, we prove the following statement: *If $\boldsymbol{e}$ is finite and antisymmetric, then the lattice* $\operatorname{Reg}(\boldsymbol{e}, \tau)$ *is a bounded homomorphic image of a free lattice.*

If $\boldsymbol{e}$ is antisymmetric (*not necessarily reflexive*), then we can define a partial ordering $\sqsubseteq$ between elements of $\boldsymbol{e}$ as follows:

$$(x, y) \sqsubseteq (x', y') \quad \text{if} \quad \bigl(x' = x \text{ or } (x', x) \in \boldsymbol{e}\bigr) \text{ and } \bigl(y = y' \text{ or } (y, y') \in \boldsymbol{e}\bigr). \tag{7.1}$$

We argue next that, with respect to this ordering, $(\boldsymbol{e}, \tau)$ is a closure space of semilattice type. For $(x, y) \in \boldsymbol{e}$ and $\boldsymbol{z} \in \mathcal{M}((x, y))$, the pair (x, y) belongs to the (transitive) closure of $\boldsymbol{z}$, hence there exists a subdivision $x = z_0 < z_1 < \cdots < z_n = y$ such that each $(z_i, z_{i+1}) \in \boldsymbol{z}$. As (x, y) does not belong to the closure of any proper subset of $\boldsymbol{z}$, it follows that $\boldsymbol{z} = \{(z_i, z_{i+1}) \mid i < n\}$; whence (x, y) is the join of $\boldsymbol{z}$ with respect to the ordering $\sqsubseteq$.

In case $\boldsymbol{e}$ is the *strict ordering* associated to a partial ordering $(E, \leq)$, the poset $\operatorname{Clop}(\boldsymbol{e}, \tau)$ is the "permutohedron-like" poset denoted, in Pouzet *et al.* [33], by $\mathbf{N}(E)$. In particular, it is proved there that $\mathbf{N}(E)$ *is a lattice iff E contains no copy of the two-atom Boolean lattice* B_2. The latter fact is extended in our paper [36] to all transitive relations. In particular, this holds for the full relation $\boldsymbol{e} = E \times E$ on any set E. The corresponding lattice $\operatorname{Reg}(\boldsymbol{e}, \tau) = \operatorname{Clop}(\boldsymbol{e}, \tau)$ is called the *bipartition lattice* of E. This structure originates in Foata and Zeilberger [14] and Han [19]. Its poset structure is investigated further in Hetyei and Krattenthaler [20]. However, it can be verified that the closure space $(E \times E, \tau)$ does not have poset type if $\operatorname{card} E \geq 3$.

Example 7.10. Let $(S, \vee)$ be a join-semilattice. We set, for any $\boldsymbol{x} \subseteq S$, $\mathrm{cl}(\boldsymbol{x}) = \boldsymbol{x}^{\vee}$, the set of joins of all nonempty finite subsets of $\boldsymbol{x}$. The closure lattice of (S, cl) is the lattice of all (possibly empty) join-subsemilattices of S. We shall call (S, cl) the *closure space canonically associated to the join-semilattice* S. For any $p \in S$, a nonempty subset $\boldsymbol{x} \subseteq S$ belongs to $\mathcal{M}(p)$ iff p is the join of a nonempty finite subset of $\boldsymbol{x}$ but of no proper subset of $\boldsymbol{x}$; thus, $\boldsymbol{x}$ is finite and $p = \bigvee \boldsymbol{x}$.

Therefore, the closure space (S, cl) thus constructed has semilattice type. The ortholattice $\mathrm{Reg}\, S = \mathrm{Reg}(S, \mathrm{cl})$ and the orthoposet $\mathrm{Clop}\, S = \mathrm{Clop}(S, \mathrm{cl})$ will be studied in some detail in the subsequent sections, in particular Sections 9 and 11.

Another large class of examples, obtained from *graphs*, will be studied in more detail in subsequent sections, in particular Sections 14 and 15.

8. Minimal neighborhoods in closure spaces

Minimal neighborhoods are a simple, but rather effective, technical tool for dealing with lattices of closed, or regular closed, subsets of an algebraic closure space.

Definition 8.1. Let (P, φ) be a closure space and let $p \in P$. A *neighborhood* of p is a subset $\boldsymbol{u}$ of P such that $p \in \check{\varphi}(\boldsymbol{u})$.

Since $\check{\varphi}(\boldsymbol{u})$ is also a neighborhood of p, it follows that every minimal neighborhood of p is open. The following result gives a simple sufficient condition, in terms of minimal neighborhoods, for $\mathrm{Reg}(P, \varphi)$ being the Dedekind-MacNeille completion of $\mathrm{Clop}(P, \varphi)$.

Proposition 8.2. *The following statements hold, for any algebraic closure space (P, φ).*

(i) *Every open subset of P is a union of minimal neighborhoods.*
(ii) *Every minimal neighborhood in P is clopen iff every open subset of P is a union of clopen sets, and in that case,* $\mathrm{Reg}(P, \varphi)$ *is the Dedekind-MacNeille completion of* $\mathrm{Clop}(P, \varphi)$.

Proof. (i). Let $\boldsymbol{u}$ be an open subset of P. Since (P, φ) is an algebraic closure space, every downward directed intersection of open sets is open, so it follows from Zorn's Lemma that every element of $\boldsymbol{u}$ is contained in some minimal neighborhood of p.

(ii) follows easily from (i) together with Lemma 4.1. □

Minimal neighborhoods can be easily recognized by the following test.

Proposition 8.3. *The following are equivalent, for any algebraic closure space (P, φ), any open subset $\boldsymbol{u}$ of P, and any $p \in \boldsymbol{u}$:*

(i) *$\boldsymbol{u}$ is a minimal neighborhood of p.*
(ii) *For each $x \in \boldsymbol{u}$, there exists $\boldsymbol{x} \in \mathcal{M}(p)$ such that $\boldsymbol{x} \cap \boldsymbol{u} = \{x\}$.*

Proof. (i)⇒(ii). For each $x \in \boldsymbol{u}$, the interior $\check{\varphi}(\boldsymbol{u} \setminus \{x\})$ is a proper open subset of $\boldsymbol{u}$, thus, by the minimality assumption on $\boldsymbol{u}$, it does not contain p as an element. Since (P, φ) is an algebraic closure space, there is $\boldsymbol{x} \in \mathcal{M}(p)$ such that $\boldsymbol{x} \cap (\boldsymbol{u} \setminus \{x\}) = \varnothing$. Since $\boldsymbol{u}$ is a neighborhood of p, $\boldsymbol{x} \cap \boldsymbol{u} \neq \varnothing$, so $\boldsymbol{x} \cap \boldsymbol{u} = \{x\}$.

(ii)⇒(i). Let $\boldsymbol{v}$ be a neighborhood of p properly contained in $\boldsymbol{u}$, and pick $x \in \boldsymbol{u} \setminus \boldsymbol{v}$. By assumption, there is $\boldsymbol{x} \in \mathcal{M}(p)$ such that $\boldsymbol{x} \cap \boldsymbol{u} = \{x\}$. Since $\boldsymbol{v}$ is a neighborhood of p, $\boldsymbol{x} \cap \boldsymbol{v} \neq \varnothing$, thus, as $\boldsymbol{v} \subseteq \boldsymbol{u}$, we get $\boldsymbol{x} \cap \boldsymbol{v} = \{x\}$, and thus $x \in \boldsymbol{v}$, a contradiction. □

9. Minimal neighborhoods in semilattices

The present section will be devoted to the study of minimal neighborhoods in a join-semilattice S, endowed with its canonical closure operator introduced in Example 7.10. It will turn out that those minimal neighborhoods enjoy an especially simple structure.

The following crucial result gives a simple description of minimal neighborhoods (not just minimal regular open neighborhoods) of elements in a join-semilattice.

Theorem 9.1. *Let p be an element in a join-semilattice S. Then the minimal neighborhoods of p are exactly the subsets of the form $(S \downarrow p) \setminus \boldsymbol{a}$, for maximal proper ideals $\boldsymbol{a}$ of $S \downarrow p$. In particular, every minimal neighborhood of p is clopen.*

Note. We allow the empty set $\varnothing$ as an ideal of S.

Proof. Let $\boldsymbol{a}$ be an ideal of $S \downarrow p$. It is straightforward to verify that the subset $\boldsymbol{u} = (S \downarrow p) \setminus \boldsymbol{a}$ is clopen. Now, assuming that $\boldsymbol{a}$ is a maximal proper ideal of $S \downarrow p$, we shall prove that $\boldsymbol{u}$ is a minimal neighborhood of p. For each $x \in \boldsymbol{u}$, it follows from the maximality assumption on $\boldsymbol{a}$ that p belongs to the ideal of S generated by $\boldsymbol{a} \cup \{x\}$, thus either $p = x$ or there exists $a \in \boldsymbol{a}$ such that $p = a \vee x$. Therefore, the set $\boldsymbol{x}$, defined as $\{p\}$ in the first case and as $\{a, x\}$ in the second case, is a minimal covering of p that meets $\boldsymbol{u}$ in $\{x\}$. By Proposition 8.3, $\boldsymbol{u}$ is a minimal neighborhood of p.

Conversely, any minimal neighborhood $\boldsymbol{u}$ of p is open. Since $\boldsymbol{u} \downarrow p$ is a lower subset of $\boldsymbol{u}$, it is open as well, hence $\boldsymbol{u} = \boldsymbol{u} \downarrow p$. Since $\boldsymbol{u}$ is an open subset of the ideal $S \downarrow p$, the subset $\boldsymbol{a} = (S \downarrow p) \setminus \boldsymbol{u}$ is closed, that is, $\boldsymbol{a}$ is a subsemilattice of $S \downarrow p$.

Claim. *For every $x \in \boldsymbol{u} \setminus \{p\}$, there exists $a \in \boldsymbol{a}$ such that $p = x \vee a$.*

Proof of Claim. Since $\boldsymbol{u}$ is a minimal neighborhood of p, it follows from Proposition 8.3 that there exists $\boldsymbol{x} \in \mathcal{M}(p)$ such that $\boldsymbol{x} \cap \boldsymbol{u} = \{x\}$. From $x \neq p$ it follows that $\boldsymbol{x} \neq \{x\}$, so $\boldsymbol{x} \setminus \{x\}$ is a nonempty subset of $\boldsymbol{a}$, and so the element $a = \bigvee(\boldsymbol{x} \setminus \{x\})$ is well-defined and belongs to $\boldsymbol{a}$. Therefore,

$$p = \bigvee \boldsymbol{x} = x \vee \bigvee(\boldsymbol{x} \setminus \{x\}) = x \vee a\,,$$

as desired. □ Claim.

Now let $x < y$ with $y \in \boldsymbol{a}$, and suppose, by way of contradiction, that $x \notin \boldsymbol{a}$, that is, $x \in \boldsymbol{u}$. By the Claim above, there exists $a \in \boldsymbol{a}$ such that $p = x \vee a$, thus $p \leq y \vee a$, and thus, as $\{y, a\} \subseteq \boldsymbol{a} \subseteq S \downarrow p$, we get $p = y \vee a \in \boldsymbol{a}$, a contradiction. Therefore, $x \in \boldsymbol{a}$, thus completing the proof that $\boldsymbol{a}$ is an ideal of $S \downarrow p$.

By definition, $p \notin \boldsymbol{a}$. For each $x \in (S \downarrow p) \setminus \boldsymbol{a}$ (i.e., $x \in \boldsymbol{u}$), there exists, by the Claim, $a \in \boldsymbol{a}$ such that $p = x \vee a$. This proves that there is no proper ideal of $S \downarrow p$ containing $\boldsymbol{a} \cup \{x\}$, so $\boldsymbol{a}$ is a maximal proper ideal of $S \downarrow p$. □

Obviously, every set-theoretical difference of ideals of S is clopen. By combining Lemma 4.1, Proposition 8.2, and Theorem 9.1, we obtain the following results (recall that the open subsets of S are exactly the complements in S of the join-subsemilattices of S).

Corollary 9.2. *The following statements hold, for any join-semilattice S.*

(i) *Every open subset of S is a set-theoretical union of differences of ideals of S; thus it is a set-theoretical union of clopen subsets of S.*

(ii) $\operatorname{Reg} S$ *is generated, as a complete ortholattice, by the set of all ideals of* S.
(iii) $\operatorname{Reg} S$ *is the Dedekind-MacNeille completion of* $\operatorname{Clop} S$.
(iv) *Every completely join-irreducible element of* $\operatorname{Reg} S$ *is clopen.*
(v) $\operatorname{Clop} S$ *is tight in* $\operatorname{Reg} S$.

Corollary 9.3. *The following are equivalent, for any join-semilattice* S:
(i) $\operatorname{Clop} S$ *is a lattice.*
(ii) $\operatorname{Clop} S$ *is a complete sublattice of* $\operatorname{Reg} S$.
(iii) $\operatorname{Clop} S = \operatorname{Reg} S$.
(iv) *The join-closure of any open subset of* S *is open.*

Proof. It is trivial that (ii) implies (i). Suppose, conversely, that (i) holds and let $\{\boldsymbol{a}_i \mid i \in I\}$ be a collection of clopen subsets of S, with join $\boldsymbol{a}$ in $\operatorname{Reg} S$. We must prove that $\boldsymbol{a}$ is clopen. Since $\operatorname{Clop} S$ is a lattice, for each finite $J \subseteq I$, the set $\{\boldsymbol{a}_i \mid i \in J\}$ has a join, that we shall denote by $\boldsymbol{a}_{(J)}$, in $\operatorname{Clop} S$. Since $\operatorname{Clop} S$ is tight in $\operatorname{Reg} S$, $\boldsymbol{a}_{(J)}$ is also the join of $\{\boldsymbol{a}_i \mid i \in J\}$ in $\operatorname{Reg} S$. Since $\boldsymbol{a}$ is the directed join of the clopen sets $\boldsymbol{a}_{(J)}$, for $J \subseteq I$ finite, it is clopen as well, thus completing the proof that $\operatorname{Clop} S$ is a complete sublattice of $\operatorname{Reg} S$.

It is obvious that (iii) and (iv) are equivalent, and that they imply (i). Hence it remains to prove that (ii) implies (iv). By Corollary 9.2, every regular closed subset of S is a join of clopen subsets, hence $\operatorname{Clop} S$ generates $\operatorname{Reg} S$ as a complete sublattice. Since $\operatorname{Clop} S$ is a complete sublattice of $\operatorname{Reg} S$, (iii) follows. □

Example 9.4. Denote by S_m the join-semilattice of all nonempty subsets of $[m]$, for any positive integer m. It is an easy exercise to verify that $\operatorname{Clop} \mathsf{S}_2 = \operatorname{Reg} \mathsf{S}_2$ is isomorphic to the permutohedron on three letters $\mathsf{P}(3)$, which is the six-element "benzene lattice".

On the other hand, the lattice $\operatorname{Reg} \mathsf{S}_3 = \operatorname{Clop} \mathsf{S}_3$ has apparently not been met until now.

Denote by a, b, c the generators of the join-semilattice S_3 (see the left hand side of Figure 9.1). The lattice $\operatorname{Clop} \mathsf{S}_3$ is represented on the right hand side of Figure 9.1, by using the following labeling convention:

$$\{a\} \mapsto a\,, \quad \{a, a \vee b\} \mapsto a^2 b\,, \quad \{a, b, a \vee b\} \mapsto a^2 b^2\,,$$

(the "variables" a, b, c being thought of as pairwise commuting, so for example $a^2 b = b a^2$) and similarly for the pairs $\{b, c\}$ and $\{a, c\}$, and further, $\overline{\varnothing} = \mathsf{S}_3$, $\overline{a}^2 \overline{b} = \mathsf{S}_3 \setminus (a^2 b)$, and so on.

Going to higher dimensions, it turns out that $\operatorname{Clop} \mathsf{S}_4$ is not a lattice. In order to see this, observe that (denoting by a, b, c, d the generators of S_4) the subsets $\boldsymbol{x} = \{a, a \vee b\}$ and $\boldsymbol{y} = \{c, c \vee d\}$ are clopen, and their join $\boldsymbol{x} \vee \boldsymbol{y}$ in $\operatorname{Reg} \mathsf{S}_4$ is the regular closed set $\boldsymbol{z} = \{a, c, a \vee b, c \vee d, a \vee c, a \vee b \vee c, a \vee c \vee d, a \vee b \vee c \vee d\}$, which is not clopen (for $a \vee b \vee c \vee d = (a \vee d) \vee (b \vee c)$ with neither $a \vee d$ nor $b \vee c$ in $\boldsymbol{z}$). Brute force computation shows that $\operatorname{card} \operatorname{Reg} \mathsf{S}_4 = 162$ while $\operatorname{card} \operatorname{Clop} \mathsf{S}_4 = 150$. Every join-irreducible element of $\operatorname{Reg} \mathsf{S}_4$ belongs to $\operatorname{Clop} \mathsf{S}_4$ (cf. Corollary 9.2).

Example 9.5. Unlike the situation with graphs (cf. Theorem 15.1), the property of $\operatorname{Clop} S$ being a lattice is not preserved by subsemilattices, so it cannot be expressed by the exclusion of a list of "forbidden subsemilattices". For example, consider the subsemilattice P of S_3 represented on the left hand side of Figure 9.2.

Then the sets $\{a_i\}$ and $\{a_0, a_1, 1, b_j\}$ are clopen in P, with $\{a_i\} \subset \{a_0, a_1, 1, b_j\}$, for all $i, j < 2$. However, there is no $\boldsymbol{c} \in \operatorname{Clop} P$ such that $\{a_i\} \subseteq \boldsymbol{c} \subseteq \{a_0, a_1, 1, b_j\}$

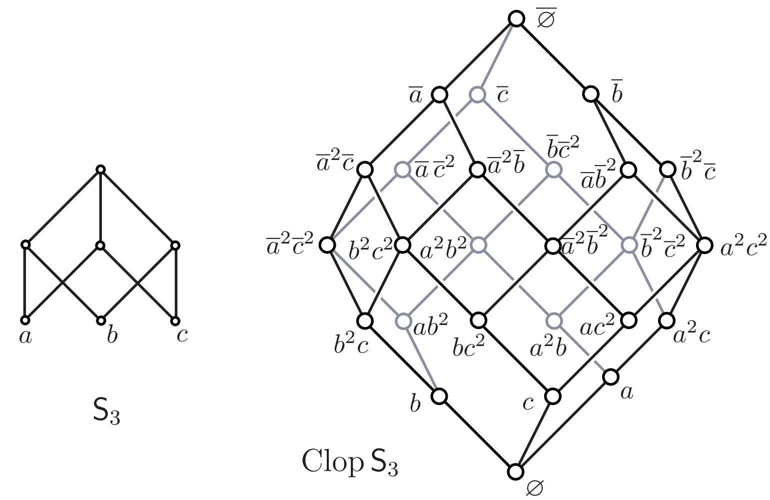

Figure 9.1. The permutohedron on the join-semilattice S_3

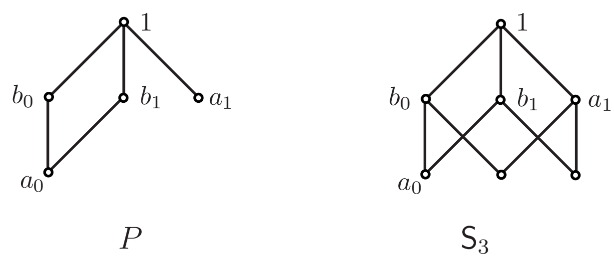

Figure 9.2. A subsemilattice of S_3

for all $i, j < 2$. Hence $\operatorname{Clop} P$ is not a lattice. On the other hand, P is a subsemilattice of S_3 and $\operatorname{Clop} \mathsf{S}_3$ is a lattice (cf. Example 9.4).

10. A collection of quasi-identities for closure spaces of poset type

A natural strengthening of pseudocomplementedness, holding in particular for all permutohedra, and even in all finite Coxeter lattices (see Le Conte de Poly-Barbut [28]), is meet-semidistributivity. Although we shall verify shortly (Example 10.1) that semidistributivity may not hold in $\operatorname{Reg}(P, \varphi)$ even for (P, φ) of semilattice type, we shall prove later that once it holds, then it implies, in the finite case, a much stronger property, namely being bounded (cf. Theorem 11.3). Recall that the implication "semidistributive $\Rightarrow$ bounded" does not hold for arbitrary finite ortholattices, see Example 11.4, also Reading [34, Figure 5].

Example 10.1. *A finite closure space (P, φ) of semilattice type such that the lattice $\operatorname{Reg}(P, \varphi)$ is not semidistributive.*

Proof. Denoting by a, b, c the atoms of the five-element modular nondistributive lattice M_3 (see the left hand side of Figure 10.1), we endow $P = \mathsf{M}_3^- = \{a, b, c, 1\}$ with the restriction of the ordering of M_3. For any subset $\boldsymbol{x}$ of P, we set $\varphi(\boldsymbol{x}) = \boldsymbol{x}$, unless $\boldsymbol{x} = \{a, b, c\}$, in which case we set $\varphi(\boldsymbol{x}) = P$. The only nontrivial covering of (P, φ) is $1 \in \varphi(\{a, b, c\})$, and indeed $1 = a \vee b \vee c$ in P, hence (P, φ) has semilattice type.

The lattice $\operatorname{Reg}(P, \varphi) = \operatorname{Clop}(P, \varphi)$ is represented on the right hand side of Figure 10.1. Its elements are labeled as $\{a\} \mapsto a$, $\{1, a, b\} \mapsto 1ab$, and so on.

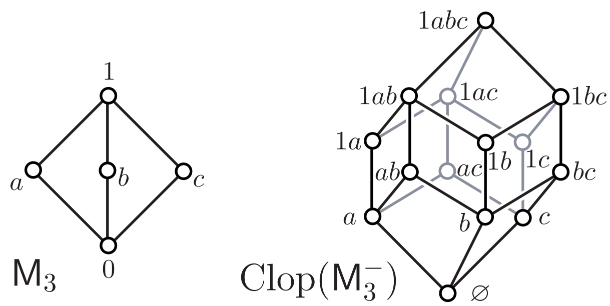

FIGURE 10.1. The lattice $\operatorname{Clop}(\mathsf{M}_3^-, \varphi)$

It is not semidistributive, as $ab \wedge 1b = bc \wedge 1b = b$ while $(ab \vee bc) \wedge 1b = 1b > b$. $\square$

We shall now introduce certain weakenings of semidistributivity, which are always satisfied by lattices of regular closed subsets of well-founded closure spaces of poset type (a poset P is *well-founded* if every nonempty subset of P has a minimal element). We begin with an easy lemma.

Lemma 10.2. *Let (P, φ) be a closure space. The following statements hold, for any $p \in P$ and any $\boldsymbol{a} \subseteq P$.*

(i) *If (P, φ) has semilattice type, then $\varphi(\boldsymbol{a} \downarrow p) = \varphi(\boldsymbol{a}) \downarrow p$.*
(ii) *If (P, φ) has poset type, then $\check{\varphi}(\boldsymbol{a} \downarrow p) = \check{\varphi}(\boldsymbol{a}) \downarrow p$.*
(iii) *If (P, φ) has semilattice type and if $\boldsymbol{a}$ is closed (resp., open, regular closed, regular open, clopen, respectively), then so is $\boldsymbol{a} \downarrow p$.*

Proof. (i). Let $q \in \varphi(\boldsymbol{a} \downarrow p)$. By Lemma 7.1, there exists $\boldsymbol{x} \in \mathcal{M}(q)$ such that $\boldsymbol{x} \subseteq \boldsymbol{a} \downarrow p$, so $q = \bigvee \boldsymbol{x} \leq p$, and so $q \in \varphi(\boldsymbol{a}) \downarrow p$. Conversely, let $q \in \varphi(\boldsymbol{a}) \downarrow p$. By Lemma 7.1, there exists $\boldsymbol{x} \in \mathcal{M}(q)$ such that $\boldsymbol{x} \subseteq \boldsymbol{a}$; as moreover $\bigvee \boldsymbol{x} = q \leq p$, we get $\boldsymbol{x} \subseteq \boldsymbol{a} \downarrow p$, whence $q \in \varphi(\boldsymbol{a} \downarrow p)$.

(ii). The containment $\check{\varphi}(\boldsymbol{a} \downarrow p) \subseteq \check{\varphi}(\boldsymbol{a}) \downarrow p$ is trivial: $\check{\varphi}(\boldsymbol{a} \downarrow p) \subseteq \check{\varphi}(\boldsymbol{a})$ since $\check{\varphi}$ is isotone, while $\check{\varphi}(\boldsymbol{a} \downarrow p) \subseteq \boldsymbol{a} \downarrow p \subseteq P \downarrow p$. Conversely, let $q \in \check{\varphi}(\boldsymbol{a}) \downarrow p$ and let $\boldsymbol{x} \in \mathcal{M}(q)$. Since (P, φ) has poset type, $\boldsymbol{x} \subseteq P \downarrow q \subseteq P \downarrow p$. From $q \in \check{\varphi}(\boldsymbol{a})$ it follows (cf. Lemma 7.2) that $\boldsymbol{x} \cap \boldsymbol{a} \neq \varnothing$; hence $\boldsymbol{x} \cap (\boldsymbol{a} \downarrow p) \neq \varnothing$. By using again Lemma 7.2, we obtain that $q \in \check{\varphi}(\boldsymbol{a} \downarrow p)$.

(iii) follows trivially from the combination of (i) and (ii). $\square$

Lemma 10.3. *Let $\boldsymbol{a}$ and $\boldsymbol{c}$ be subsets in a closure space (P, φ) of poset type. Then every minimal element x of $\varphi(\boldsymbol{a} \cup \boldsymbol{c}) \setminus \varphi(\boldsymbol{c})$ belongs to $\boldsymbol{a}$.*

Proof. Pick $\boldsymbol{u} \in \mathcal{M}(x)$ such that $\boldsymbol{u} \subseteq \boldsymbol{a} \cup \boldsymbol{c}$ and suppose that $\boldsymbol{u}$ is nontrivial. Since (P, φ) has poset type, this implies that every $u \in \boldsymbol{u}$ is smaller than x, thus, if $u \in \boldsymbol{a}$ (so *a fortiori* $u \in \varphi(\boldsymbol{a} \cup \boldsymbol{c})$), it follows from the minimality assumption on x that $u \in \varphi(\boldsymbol{c})$. Hence $\boldsymbol{u} \cap \boldsymbol{a} \subseteq \varphi(\boldsymbol{c})$, but $\boldsymbol{u} \subseteq \boldsymbol{a} \cup \boldsymbol{c} \subseteq \boldsymbol{a} \cup \varphi(\boldsymbol{c})$, thus $\boldsymbol{u} \subseteq \varphi(\boldsymbol{c})$, and thus, as $\boldsymbol{u} \in \mathcal{M}(x)$ and $\varphi(\boldsymbol{c})$ is closed, we get $x \in \varphi(\boldsymbol{c})$, a contradiction. Therefore, $\boldsymbol{u}$ is the trivial covering $\{x\}$, so that we get $x \in \boldsymbol{a}$ from $x \notin \boldsymbol{c}$. $\square$

Although semidistributivity may fail in finite lattices of regular closed sets (cf. Example 10.1), we shall now prove that a certain weak form of semidistributivity always holds in those lattices, whenever (P, φ) has poset type. This will be sufficient to yield, in Corollary 10.6, a characterization of semidistributivity by the exclusion of a single lattice.

Theorem 10.4. *Let (P, φ) be a well-founded closure space of poset type, let $o \in I$, let $(\boldsymbol{a}_i \mid i \in I)$ be a nonempty family of regular closed subsets of P, and let $\boldsymbol{c}, \boldsymbol{d} \subseteq P$ be regular closed subsets such that (the joins and meets being evaluated in* $\operatorname{Reg}(P, \varphi)$*)*

(i) $\boldsymbol{a}_i \vee \boldsymbol{c} = \boldsymbol{d}$ *for each* $i \in I$*;*
(ii) $\boldsymbol{a}_o \wedge \boldsymbol{c} = \bigwedge_{i \in I} \boldsymbol{a}_i$.

Then $\boldsymbol{a}_i \subseteq \boldsymbol{c}$ for each $i \in I$.

Proof. Suppose, by way of contradiction, that $\bigcup_{i \in I} \boldsymbol{a}_i \nsubseteq \boldsymbol{c}$. Then the set $\boldsymbol{d} \setminus \boldsymbol{c}$ is nonempty, thus, since P is well-founded, $\boldsymbol{d} \setminus \boldsymbol{c}$ has a minimal element e. Observing that $\varphi\big(\breve{\varphi}(\boldsymbol{a}_i) \cup \boldsymbol{c}\big) = \boldsymbol{d}$ for each $i \in I$, it follows from Lemma 10.3 that

$$e \in \breve{\varphi}(\boldsymbol{a}_i) \text{ for each } i \in I\,. \tag{10.1}$$

On the other hand, from $e \notin \boldsymbol{c}$ together with Assumption (ii) it follows that $e \notin \bigwedge_{i \in I} \boldsymbol{a}_i$, so, *a fortiori*, $e \notin \breve{\varphi}\big(\bigcap_{i \in I} \boldsymbol{a}_i\big)$; consequently, there exists $\boldsymbol{u} \in \mathcal{M}(e)$ such that

$$\boldsymbol{u} \cap \bigcap_{i \in I} \boldsymbol{a}_i = \varnothing\,. \tag{10.2}$$

From (10.1) and (10.2) it follows that $e \notin \boldsymbol{u}$. Since (P, φ) has poset type, it follows that $u < e$ for each $u \in \boldsymbol{u}$.

Let $u \in \boldsymbol{u}$. From (10.2) together with Assumption (ii) it follows that $u \notin \boldsymbol{a}_o \wedge \boldsymbol{c}$, hence $u \notin \breve{\varphi}(\boldsymbol{a}_o \cap \boldsymbol{c})$, and hence there exists $\boldsymbol{z}_u \in \mathcal{M}(u)$ such that

$$\boldsymbol{z}_u \cap \boldsymbol{a}_o \cap \boldsymbol{c} = \varnothing\,. \tag{10.3}$$

Since (P, φ) has poset type, $\boldsymbol{z}_u \subseteq P \downarrow u$ for each $u \in \boldsymbol{u}$. Set $\boldsymbol{z} = \bigcup_{u \in \boldsymbol{u}} \boldsymbol{z}_u$. Since $u \in \varphi(\boldsymbol{z}_u)$ for each $u \in \boldsymbol{u}$ and as $e \in \varphi(\boldsymbol{u})$, we get $e \in \varphi(\boldsymbol{z})$. Since $e \in \breve{\varphi}(\boldsymbol{a}_o)$ (cf. (10.1)), it follows that $\boldsymbol{z} \cap \boldsymbol{a}_o \neq \varnothing$. Pick $z \in \boldsymbol{z} \cap \boldsymbol{a}_o$. There exists $u \in \boldsymbol{u}$ such that $z \in \boldsymbol{z}_u$. From $z \leq u$ and $u < e$ it follows that $z < e$. Since $z \in \boldsymbol{a}_o \subseteq \boldsymbol{d}$ and by the minimality statement on e, we get $z \in \boldsymbol{c}$, so $z \in \boldsymbol{z}_u \cap \boldsymbol{a}_o \cap \boldsymbol{c}$, which contradicts (10.3). $\square$

In particular, whenever (P, φ) is a closure system of poset type with P well-founded and m is a positive integer, the lattice $\operatorname{Reg}(P, \varphi)$ satisfies the following quasi-identity, weaker than join-semidistributivity:

$$\Big(a_0 \vee c = a_1 \vee c = \cdots = a_m \vee c \text{ and } a_0 \wedge c = \bigwedge_{0 \leq i \leq m} a_i\Big) \Longrightarrow a_0 \leq c\,. \tag{RSD_m}$$

Proposition 10.5. *The following statements hold, for every positive integer m.*

(i) *Meet-semidistributivity and join-semidistributivity both imply* (RSD_m).
(ii) (RSD_{m+1}) *implies* (RSD_m).

Proof. (i). Let $a_0, \dots, a_m, c$ be elements in a lattice L, satisfying the premise of (RSD_m). If L is meet-semidistributive, then, as $a_0 \wedge c = \bigwedge_{0\leq i\leq m} a_i$ and by Jónsson and Kiefer [24, Theorem 2.1] (see also Freese, Ježek, and Nation [16, Theorem 1.21]),

$$a_0 \wedge c = \bigwedge_{0\leq i\leq m} (a_0 \vee a_i) \wedge \bigwedge_{0\leq i\leq m} (c \vee a_i) = a_0\,,$$

so $a_0 \leq c$, as desired. If L is join-semidistributive, then

$$a_0 \vee c = \Big(\bigwedge_{0\leq i\leq m} a_i\Big) \vee c = (a_0 \wedge c) \vee c = c\,,$$

so $a_0 \leq c$ again.

(ii) is trivial. □

A computer search, using the software `Mace4` (see McCune [31]), yields that every 24-element (or less) lattice satisfying (RSD_1) also satisfies (RSD_2), nevertheless that there exists a 25-element lattice K satisfying both (RSD_1) and its dual, but not (RSD_2). It follows that the 52-element ortholattice $K \parallel K^{\mathrm{op}}$ (cf. Section 2) satisfies (RSD_1) but not (RSD_2). We do not know whether (RSD_{m+1}) is properly stronger than the conjunction of (RSD_m) and its dual, for each positive integer m, although this seems highly plausible.

The quasi-identity (RSD_1) does not hold in any of the lattices M_3, L_3, and L_4 represented in Figure 10.2. (We are following the notation of Jipsen and Rose [23] for those lattices.) Since $\mathrm{Reg}(P,\varphi)$ is self-dual, none of those lattices neither their duals can be embedded into $\mathrm{Reg}(P,\varphi)$, whenever P is well-founded of poset type.

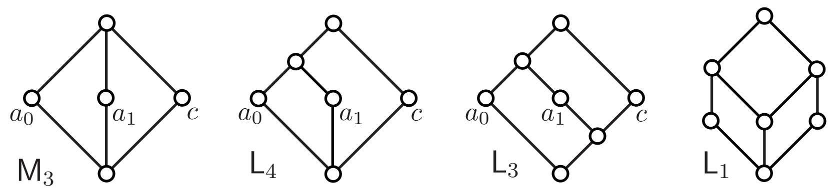

FIGURE 10.2. Non-semidistributive lattices, with failures of (RSD_1) marked whenever possible

As, in the finite case, semidistributivity is characterized by the exclusion, as sublattices, of M_3, L_3, L_4, together with the lattice L_1 of Figure 10.2, and the dual lattices of L_1 and L_4 (cf. Davey, Poguntke, and Rival [9] or Freese, Ježek, and Nation [16, Theorem 5.56]), it follows from the self-duality of $\mathrm{Reg}(P,\varphi)$ together with Theorem 10.4 that the semidistributivity of $\mathrm{Reg}(P,\varphi)$ takes the following very simple form.

Corollary 10.6. *Let (P,φ) be a finite closure system of poset type. Then $\mathrm{Reg}(P,\varphi)$ is semidistributive iff it contains no copy of L_1.*

The following example shows that the assumption, in Theorem 10.4, of (P,φ) being of poset type cannot be relaxed to (P,φ) being a convex geometry.

Example 10.7. *A finite atomistic convex geometry (P,φ) such that $\mathrm{Reg}(P,\varphi)$ does not satisfy (RSD_1).*

Proof. Consider a six-element set $P = \{a, b, c, d, e, u\}$, and let a subset $\boldsymbol{x}$ of P be *closed* if

$$\begin{aligned}
\{c, d, u\} \subseteq \boldsymbol{x} &\Rightarrow \{a, b, e\} \subseteq \boldsymbol{x}\,,\\
\{a, b, u\} \subseteq \boldsymbol{x} &\Rightarrow e \in \boldsymbol{x}\,,\\
\{c, d, e\} \subseteq \boldsymbol{x} &\Rightarrow \{a, b\} \subseteq \boldsymbol{x}\,.
\end{aligned}$$

Denote by $\varphi(\boldsymbol{x})$ the least closed set containing $\boldsymbol{x}$, for each $\boldsymbol{x} \subseteq P$. It is obvious that (P, φ) is an atomistic closure space. Brute force calculation also shows that (P, φ) is a convex geometry. There are 51 closed sets and 40 regular closed sets, the latter all clopen. The following subsets

$$\begin{aligned}
\boldsymbol{a}_0 &= \{a, d, e\}\,,\\
\boldsymbol{a}_1 &= \{b, d, e\}\,,\\
\boldsymbol{c} &= \{c, d\}
\end{aligned}$$

are all clopen. Moreover, $\boldsymbol{a}_0 \cap \boldsymbol{a}_1 = \{e, d\}$ and $e \notin \check{\varphi}(\boldsymbol{a}_0 \cap \boldsymbol{a}_1)$ (because $\{a, b, u\}$ is a minimal covering of e disjoint from $\boldsymbol{a}_0 \cap \boldsymbol{a}_1$), so $\boldsymbol{a}_0 \wedge \boldsymbol{a}_1 = \{d\}$. Furthermore, $\boldsymbol{a}_0 \vee \boldsymbol{c} = \boldsymbol{a}_1 \vee \boldsymbol{c} = \{a, b, c, d, e\}$ and $(\boldsymbol{a}_0 \vee \boldsymbol{a}_1) \cap \boldsymbol{c} = \{a, b, d, e\} \cap \boldsymbol{c} = \{d\}$. Therefore, $\{\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{c}\}$ *generates a sublattice of* $\operatorname{Reg}(P, \varphi)$ *isomorphic to* L_4, with labeling as given by Figure 10.2. In particular, $\operatorname{Reg}(P, \varphi)$ *does not satisfy the quasi-identity* (RSD_1). $\square$

Nevertheless, the elements $\{c, u\}$, $\{d, u\}$, and $\{e, u\}$ are the atoms of a copy of L_1 in $\operatorname{Reg}(P, \varphi)$. Hence the construction of Example 10.7 is not sufficient to settle whether Corollary 10.6 can be extended to the case of convex geometries. Observe, also, that although there are lattice embeddings from both L_1 and L_4 into $\operatorname{Reg}(P, \varphi)$, there is no 0-lattice embedding from either L_1 or L_4 into $\operatorname{Reg}(P, \varphi)$: this follows from Proposition 3.9 (indeed, neither L_1 nor L_4 is pseudocomplemented).

Figure 10 illustrates the closure lattice of the closure space (P, φ) of Example 10.7, with the copy of L_1 in gray and the copy of L_4 in black.

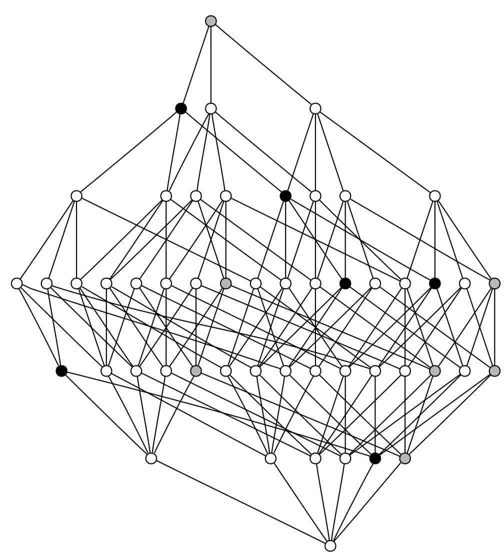

FIGURE 10.3. The closure lattice of Example 10.7

11. From semidistributivity to boundedness for semilattice type

In Lemmas 11.1–11.2 let (P, φ) be a closure space of semilattice type.

We begin with a useful structural property of the completely join-irreducible elements of $\operatorname{Reg}(P, \varphi)$. We refer to Section 3 for the notation $\partial\boldsymbol{a}$.

Lemma 11.1. *Every completely join-irreducible element $\boldsymbol{a}$ of the lattice $\operatorname{Reg}(P, \varphi)$ has a largest element p, and $\boldsymbol{a} \setminus \boldsymbol{a}_* = \breve{\varphi}(\boldsymbol{a}) \setminus \breve{\varphi}(\boldsymbol{a}_*) = \{p\}$. Furthermore, for every $x \in \partial\boldsymbol{a}$, there exists $\boldsymbol{x} \in \mathcal{M}(p)$ such that $\boldsymbol{x} \cap \boldsymbol{a} = \{x\}$.*

Proof. From the trivial observation that $\boldsymbol{a}$ is the union of all $\boldsymbol{a} \downarrow x$, for $x \in \boldsymbol{a}$, and by Lemma 10.2, it follows that $\boldsymbol{a} = \bigvee(\boldsymbol{a} \downarrow x \mid x \in \boldsymbol{a})$ in $\operatorname{Reg}(P, \varphi)$. Since $\boldsymbol{a}$ is completely join-irreducible, there exists $p \in \boldsymbol{a}$ such that $\boldsymbol{a} = \boldsymbol{a} \downarrow p$. Of course, p is necessarily the largest element of $\boldsymbol{a}$.

Claim 1. *Let $\boldsymbol{x} \in \operatorname{Reg}(P, \varphi)$ be contained in $\boldsymbol{a}$. Then $p \in \boldsymbol{x}$ implies that $\boldsymbol{x} = \boldsymbol{a}$.*

Proof of Claim. From $p \in \boldsymbol{x}$ and $\boldsymbol{x} \subseteq \boldsymbol{a}$ it follows that $\boldsymbol{a} = \boldsymbol{x} \cup (\boldsymbol{a} \setminus \{p\})$, hence

$$\boldsymbol{a} = \boldsymbol{x} \cup \bigcup(\boldsymbol{a} \downarrow x \mid x \in \boldsymbol{a} \setminus \{p\}),$$

and hence, by Lemma 10.2,

$$\boldsymbol{a} = \boldsymbol{x} \vee \bigvee(\boldsymbol{a} \downarrow x \mid x \in \boldsymbol{a} \setminus \{p\}) \text{ in } \operatorname{Reg}(P, \varphi).$$

Since $\boldsymbol{a}$ is completely join-irreducible and $p \in \boldsymbol{a} \setminus (\boldsymbol{a} \downarrow x)$ for each $x \in \boldsymbol{a} \setminus \{p\}$, the desired conclusion follows. □ Claim 1.

Claim 2. *The set $\boldsymbol{a} \setminus \{p\}$ is regular closed.*

Proof of Claim. Evaluate the join $\boldsymbol{x} = \bigvee(\boldsymbol{a} \downarrow x \mid x \in \boldsymbol{a} \setminus \{p\})$ in $\operatorname{Reg}(P, \varphi)$. Since each of the joinands $\boldsymbol{a} \downarrow x$ is smaller than $\boldsymbol{a}$ and the latter is completely join-irreducible, we obtain that $\boldsymbol{x} \subsetneq \boldsymbol{a}$, thus $p \notin \boldsymbol{x}$ (cf. Claim 1), so $\boldsymbol{x} \subseteq \boldsymbol{a} \setminus \{p\}$. The converse containment being trivial, $\boldsymbol{x} = \boldsymbol{a} \setminus \{p\}$. □ Claim 2.

From Claims 1 and 2 it follows that $\boldsymbol{a}_* = \boldsymbol{a} \setminus \{p\}$.

Let $q \in \breve{\varphi}(\boldsymbol{a}) \setminus \breve{\varphi}(\boldsymbol{a}_*)$. Since $q \notin \breve{\varphi}(\boldsymbol{a}_*)$, there exists $\boldsymbol{x} \in \mathcal{M}(q)$ such that $\boldsymbol{x} \cap \boldsymbol{a}_* = \varnothing$. From $q \in \breve{\varphi}(\boldsymbol{a})$ it follows that $\boldsymbol{x} \cap \boldsymbol{a} \neq \varnothing$, thus $p \in \boldsymbol{x}$ (as $\boldsymbol{a} = \boldsymbol{a}_* \cup \{p\}$) and $p \leq q$ (as $q = \bigvee \boldsymbol{x}$); whence, as also $p = \max \boldsymbol{a}$, we get $p = q$. Therefore, $\breve{\varphi}(\boldsymbol{a}) \setminus \breve{\varphi}(\boldsymbol{a}_*) \subseteq \{p\}$. Since $\breve{\varphi}(\boldsymbol{a}) \setminus \breve{\varphi}(\boldsymbol{a}_*) \neq \varnothing$ (because $\boldsymbol{a}_*$ is properly contained in $\boldsymbol{a}$ and both sets are regular closed), it follows that $\breve{\varphi}(\boldsymbol{a}) \setminus \breve{\varphi}(\boldsymbol{a}_*) = \{p\}$.

Finally, let $x \in \partial\boldsymbol{a}$. Since $\boldsymbol{a} \setminus \{x\}$ is closed and does not contain x as an element, $\breve{\varphi}(\boldsymbol{a} \setminus \{x\})$ is regular open and properly contained in $\breve{\varphi}(\boldsymbol{a})$, thus it is contained in $\breve{\varphi}(\boldsymbol{a})_* = \breve{\varphi}(\boldsymbol{a}_*) = \breve{\varphi}(\boldsymbol{a}) \setminus \{p\}$. Hence $p \notin \breve{\varphi}(\boldsymbol{a} \setminus \{x\})$, which means that there exists $\boldsymbol{x} \in \mathcal{M}(p)$ such that $\boldsymbol{x} \cap (\boldsymbol{a} \setminus \{x\}) = \varnothing$. From $p \in \breve{\varphi}(\boldsymbol{a})$ it follows that $\boldsymbol{x} \cap \boldsymbol{a} \neq \varnothing$, so $\boldsymbol{x} \cap \boldsymbol{a} = \{x\}$. □

As we shall see from Example 17.4, not every join-irreducible element of $\operatorname{Reg}(P, \varphi)$ needs to be clopen.

Lemma 11.2. *The following statements hold, for any completely join-irreducible elements $\boldsymbol{a}$ and $\boldsymbol{b}$ of $\operatorname{Reg}(P, \varphi)$.*

(i) *If $\boldsymbol{a}^\perp \searrow \boldsymbol{b}$, then $\max \boldsymbol{a} \geq \max \boldsymbol{b}$; if, moreover, $\max \boldsymbol{a} = \max \boldsymbol{b}$, then $[\boldsymbol{a}^\perp, (\boldsymbol{a}_*)^\perp]$ is down-perspective to $[\boldsymbol{b}_*, \boldsymbol{b}]$.*

(ii) *If* $\boldsymbol{a} \nearrow \boldsymbol{b}^{\perp}$, *then* $\max \boldsymbol{a} \geq \max \boldsymbol{b}$*; if, moreover,* $\max \boldsymbol{a} = \max \boldsymbol{b}$*, then* $[\boldsymbol{a}_*, \boldsymbol{a}]$ *is up-perspective to* $[\boldsymbol{b}^{\perp}, (\boldsymbol{b}_*)^{\perp}]$.

Let us observe that the first parts of items (i) and (ii) in the Lemma immediately imply the following property (as well as its dual): if $[\boldsymbol{a}^{\perp}, (\boldsymbol{a}_*)^{\perp}]$ is down-perspective to $[\boldsymbol{b}_*, \boldsymbol{b}]$, then $\max \boldsymbol{a} = \max \boldsymbol{b}$. The situation can be visualized on Figure 11.1. (Following the convention used in Freese, Ježek, and Nation [16], prime intervals are highlighted by crossing them with a perpendicular dash.)

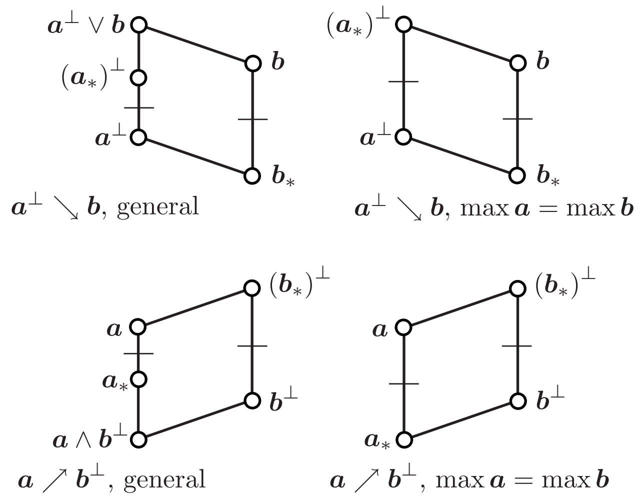

FIGURE 11.1. Illustrating $\boldsymbol{a}^{\perp} \searrow \boldsymbol{b}$ and $\boldsymbol{a} \nearrow \boldsymbol{b}^{\perp}$

Proof. Since (ii) is dual of (i) (*via* the dual automorphism $\boldsymbol{x} \mapsto \boldsymbol{x}^{\perp}$), it suffices to prove (i).

Set $p = \max \boldsymbol{a}$ and $q = \max \boldsymbol{b}$. Let us state the next observation as a Claim.

Claim. *The relation* $\boldsymbol{a}^{\perp} \searrow \boldsymbol{b}$ *holds iff* $\breve{\varphi}(\boldsymbol{a}) \cap \boldsymbol{b} = \{q\}$.

Proof of Claim. Since $\boldsymbol{a}^{\perp} = \big(\breve{\varphi}(\boldsymbol{a})\big)^{\mathsf{c}}$, the relation $\boldsymbol{a}^{\perp} \searrow \boldsymbol{b}$ means that $\breve{\varphi}(\boldsymbol{a}) \cap \boldsymbol{b} \neq \varnothing$ and $\breve{\varphi}(\boldsymbol{a}) \cap \boldsymbol{b}_* = \varnothing$. Recalling that $\boldsymbol{b} = \boldsymbol{b}_* \cup \{q\}$ (cf. Lemma 11.1), the Claim follows immediately. □ Claim.

Using our Claim, we see that if $\boldsymbol{a}^{\perp} \searrow \boldsymbol{b}$, then, as $p = \max \boldsymbol{a}$, we get $q \in \boldsymbol{a}$, thus $q \leq p$.

Suppose next that $\boldsymbol{a}^{\perp} \searrow \boldsymbol{b}$ and $p = q$. Since $\breve{\varphi}(\boldsymbol{a}) \setminus \breve{\varphi}(\boldsymbol{a}_*) = \{p\}$ (cf. Lemma 11.1) and $\breve{\varphi}(\boldsymbol{a}) \cap \boldsymbol{b} = \{p\}$ (by our Claim together with $p = q$), it follows that $\breve{\varphi}(\boldsymbol{a}_*) \cap \boldsymbol{b} = \varnothing$, that is, $\boldsymbol{b} \subseteq (\boldsymbol{a}_*)^{\perp} = (\boldsymbol{a}^{\perp})^*$. Since $\boldsymbol{b} \nleq \boldsymbol{a}^{\perp}$ follows from $\boldsymbol{a}^{\perp} \searrow \boldsymbol{b}$, we get $\boldsymbol{b} \nearrow \boldsymbol{a}^{\perp}$, showing that the interval $[\boldsymbol{a}^{\perp}, (\boldsymbol{a}_*)^{\perp}]$ is down-perspective to $[\boldsymbol{b}_*, \boldsymbol{b}]$. □

Theorem 11.3. *Let* (P, φ) *be a finite closure space of semilattice type. Then* $\operatorname{Reg}(P, \varphi)$ *is semidistributive iff it is a bounded homomorphic image of a free lattice.*

Proof. It is well-known that every bounded homomorphic image of a free lattice is semidistributive, see Freese, Ježek, and Nation [16, Theorem 2.20].

Conversely, suppose that P is finite and that $\operatorname{Reg}(P,\varphi)$ is semidistributive. Since $\operatorname{Reg}(P,\varphi)$ is self-dual (*via* the natural orthocomplementation), it suffices to prove that it is a lower bounded homomorphic image of a free lattice. This amounts, in turn, to proving that $\operatorname{Reg}(P,\varphi)$ has no cycle of join-irreducible elements with respect to the join-dependency relation D (cf. Freese, Ježek, and Nation [16, Corollary 2.39]). In order to prove this, it is sufficient to prove that $\boldsymbol{a}\ D\ \boldsymbol{b}$ implies that $\max\boldsymbol{a} > \max\boldsymbol{b}$, for all join-irreducible elements $\boldsymbol{a}$ and $\boldsymbol{b}$ of $\operatorname{Reg}(P,\varphi)$. By Lemma 2.1, there exists a meet-irreducible $\boldsymbol{u}\in\operatorname{Reg}(P,\varphi)$ such that $\boldsymbol{a}\nearrow\boldsymbol{u}\searrow\boldsymbol{b}$. The element $\boldsymbol{c}=\boldsymbol{u}^{\perp}$ is join-irreducible, and, by Lemma 11.2, $\max\boldsymbol{a}\geq\max\boldsymbol{c}\geq\max\boldsymbol{b}$. Suppose that $\max\boldsymbol{a}=\max\boldsymbol{c}=\max\boldsymbol{b}$. By Lemma 11.2, $(\boldsymbol{c}_*)^{\perp}=\boldsymbol{a}\vee\boldsymbol{c}^{\perp}=\boldsymbol{b}\vee\boldsymbol{c}^{\perp}$, thus, as $\operatorname{Reg}(P,\varphi)$ is join-semidistributive, $(\boldsymbol{c}_*)^{\perp}=(\boldsymbol{a}\wedge\boldsymbol{b})\vee\boldsymbol{c}^{\perp}$. On the other hand, $\boldsymbol{a}\ D\ \boldsymbol{b}$ thus $\boldsymbol{a}\neq\boldsymbol{b}$, and thus $\boldsymbol{a}\wedge\boldsymbol{b}$ lies either below $\boldsymbol{a}_*$ or below $\boldsymbol{b}_*$, hence below $\boldsymbol{c}^{\perp}$. It follows that $(\boldsymbol{c}_*)^{\perp}=\boldsymbol{c}^{\perp}$, a contradiction. Therefore, $\max\boldsymbol{a}>\max\boldsymbol{b}$. □

Example 11.4. The lattice K represented on the left hand side of Figure 11.2, taken from Jónsson and Nation [25] (see also Freese, Ježek, and Nation [16, page 111]), is semidistributive but not bounded. It follows that the parallel sum $L=K\parallel K$ (cf. Section 2), represented on the right hand side of Figure 11.2, is also semidistributive and not bounded. Now observe that K has an involutive dual automorphism α. Sending each x in one copy of K to $\alpha(x)$ in the other copy of K (and exchanging 0 and 1) defines an orthocomplementation of L.

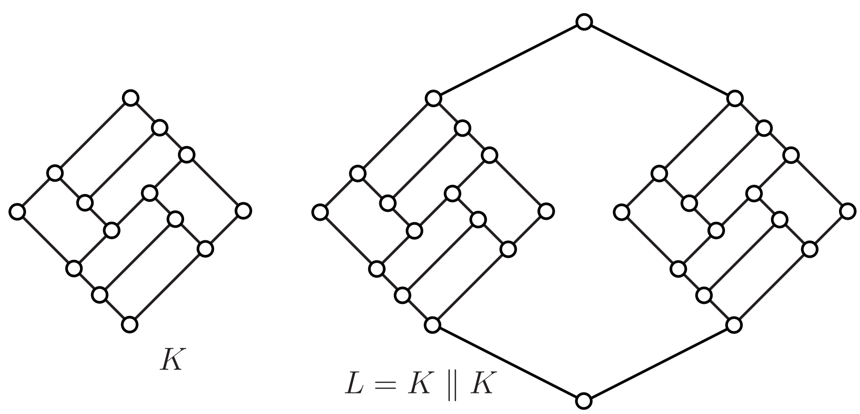

FIGURE 11.2. The lattices K and $L=K\parallel K$

This shows that *a finite, semidistributive ortholattice need not be bounded.*

Although $\operatorname{Reg}(P,\varphi)$ may not be semidistributive, even for a finite closure space (P,φ) of semilattice type (cf. Example 10.1), there are important cases where semidistributivity holds, such as the case of the closure space associated to an antisymmetric, transitive binary relation (cf. Santocanale and Wehrung [36]). Further such situations will be investigated in Section 12 and 14.

12. Boundedness of lattices of regular closed subsets of semilattices

Although $\mathrm{Reg}(P, \varphi)$ may not be semidistributive, even for a finite closure space (P, φ) of semilattice type, the situation changes for the closure space associated to a finite semilattice. We first state an easy lemma.

Lemma 12.1. *Let S be a join-semilattice. Then every completely join-irreducible member $\boldsymbol{a}$ of $\mathrm{Reg}\,S$ is a minimal neighborhood of some element of S.*

Proof. Since $\mathrm{int}(\boldsymbol{a}) = \bigcup_{i\in I} \boldsymbol{a}_i$ for minimal neighborhoods $\boldsymbol{a}_i$ (cf. Proposition 8.2) and every minimal neighborhood is clopen (cf. Theorem 9.1), $\boldsymbol{a} = \mathrm{cl}\,\mathrm{int}(\boldsymbol{a}) = \bigvee_{i\in I} \boldsymbol{a}_i$ is a join of minimal neighborhoods in $\mathrm{Reg}\,S$. Since $\boldsymbol{a}$ is completely join-irreducible, it follows that $\boldsymbol{a}$ is itself a minimal neighborhood. □

On the other hand, there are easy examples of finite join-semilattices containing join-reducible minimal neighborhoods. The structure of completely join-irreducible elements of $\mathrm{Reg}\,S$ will be further investigated in Section 13.

A further illustration of Theorem 11.3 is provided by the following result.

Theorem 12.2. *For any* finite *join-semilattice S, the lattice $\mathrm{Reg}\,S$ is a bounded homomorphic image of a free lattice.*

Proof. As in the proof of Theorem 11.3, it is sufficient to prove that $\boldsymbol{a}\,D\,\boldsymbol{b}$ implies that $\max\boldsymbol{a} > \max\boldsymbol{b}$, for all join-irreducible elements $\boldsymbol{a}$ and $\boldsymbol{b}$ of $\mathrm{Reg}\,S$. By Lemma 2.1, there exists a meet-irreducible $\boldsymbol{u} \in \mathrm{Reg}\,S$ such that $\boldsymbol{a} \nearrow \boldsymbol{u} \searrow \boldsymbol{b}$. The element $\boldsymbol{c} = \boldsymbol{u}^{\perp} (= \boldsymbol{u}^{\mathsf{c}})$ is join-irreducible and it follows from Lemma 11.2 that $\max\boldsymbol{a} \geq \max\boldsymbol{c} \geq \max\boldsymbol{b}$. Suppose that $\max\boldsymbol{a} = \max\boldsymbol{c} = \max\boldsymbol{b}$ and denote that element by p. It follows from Lemma 12.1 that $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are minimal neighborhoods of p. Thus, by Theorem 9.1, the sets $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are all clopen and their respective complements $\tilde{\boldsymbol{a}}$, $\tilde{\boldsymbol{b}}$, $\tilde{\boldsymbol{c}}$ in $S\downarrow p$ are maximal proper ideals of $S\downarrow p$. From $\boldsymbol{a} \nearrow \boldsymbol{c}^{\mathsf{c}} \searrow \boldsymbol{b}$ and Lemma 11.1 it follows that $\boldsymbol{a}\cap\boldsymbol{c} = \boldsymbol{c}\cap\boldsymbol{b} = \{p\}$, that is, $\tilde{\boldsymbol{a}}\cup\tilde{\boldsymbol{c}} = \tilde{\boldsymbol{b}}\cup\tilde{\boldsymbol{c}} = S\downdownarrows p$. If either $\tilde{\boldsymbol{a}} \subseteq \tilde{\boldsymbol{c}}$ or $\tilde{\boldsymbol{b}} \subseteq \tilde{\boldsymbol{c}}$, then, by the maximality statements on both $\tilde{\boldsymbol{a}}$ and $\tilde{\boldsymbol{b}}$, we get $\tilde{\boldsymbol{a}} = \tilde{\boldsymbol{c}} = \tilde{\boldsymbol{b}}$, thus $\boldsymbol{a} = \boldsymbol{b}$, in contradiction with $\boldsymbol{a}\,D\,\boldsymbol{b}$. Hence, there are $a_0 \in \tilde{\boldsymbol{a}} \setminus \tilde{\boldsymbol{c}}$ and $b_0 \in \tilde{\boldsymbol{b}} \setminus \tilde{\boldsymbol{c}}$. For each $a \in \tilde{\boldsymbol{a}}$, the element $a \vee a_0$ belongs to $\tilde{\boldsymbol{a}} \setminus \tilde{\boldsymbol{c}} = \tilde{\boldsymbol{b}} \setminus \tilde{\boldsymbol{c}}$, thus $a \in \tilde{\boldsymbol{b}}$, so $\tilde{\boldsymbol{a}} \subseteq \tilde{\boldsymbol{b}}$. Likewise, using b_0, we get $\tilde{\boldsymbol{b}} \subseteq \tilde{\boldsymbol{a}}$, therefore $\boldsymbol{a} = \boldsymbol{b}$, a contradiction. □

Example 12.3. Theorem 12.2 implies trivially that $\mathrm{Reg}\,S$ is semidistributive, for any finite join-semilattice S. We show in the present example that this result cannot be extended to infinite semilattices.

Shiryaev characterized in [38] the semilattices with semidistributive lattice of subsemilattices. His results were extended to lattices of various kinds of subsemilattices, and various versions of semidistributivity, in Adaricheva [1]. We show here, *via* a straightforward modification of one of Shiryaev's constructions, that there is a distributive lattice Δ whose lattice of all regular closed join-subsemilattices is not semidistributive.

Endow $\Delta = \omega^{\mathrm{op}} \times \{0, 1\}$ with the componentwise ordering, and set $a_n = (n, 1)$ and $b_n = (n, 0)$, for each $n < \omega$. The lattice Δ is represented in Figure 12.1. Of course, Δ is a distributive lattice; we shall view it as a join-semilattice.

Proposition 12.4. *Every regular closed subset of Δ is clopen* (*i.e.,* $\mathrm{Clop}\,\Delta = \mathrm{Reg}\,\Delta$)*. Furthermore,* $\mathrm{Reg}\,\Delta$ *has nonzero elements $\boldsymbol{a}$, $\boldsymbol{b}_0$, and $\boldsymbol{b}_1$ such that* $\boldsymbol{a}\wedge\boldsymbol{b}_0 =$

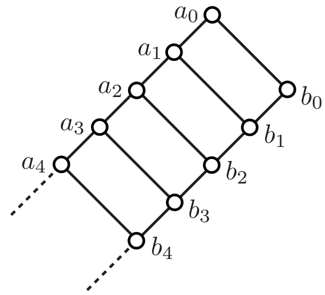

FIGURE 12.1. The lattice Δ

$\boldsymbol{a} \wedge \boldsymbol{b}_1 = \varnothing$ and $\boldsymbol{a} \subseteq \boldsymbol{b}_0 \cup \boldsymbol{b}_1$. In particular, $\operatorname{Reg}\Delta$ is neither semidistributive nor pseudocomplemented.

Proof. We first observe that the nontrivial irredundant joins in Δ are exactly those of the form

$$a_m = a_n \vee b_m\,, \quad \text{with } m < n < \omega\,.$$

Let $\boldsymbol{u}$ be an open subset of Δ, we shall prove that $\operatorname{cl}(\boldsymbol{u})$ is open as well. By the observation above, it suffices to prove that whenever $m < n < \omega$, $a_m \in \operatorname{cl}(\boldsymbol{u})$ implies that either $b_m \in \operatorname{cl}(\boldsymbol{u})$ or $a_n \in \operatorname{cl}(\boldsymbol{u})$. Since $\boldsymbol{u}$ is open, that conclusion is obvious if $a_m \in \boldsymbol{u}$. Now suppose that $a_m \in \operatorname{cl}(\boldsymbol{u}) \setminus \boldsymbol{u}$. Since $\boldsymbol{a}_m$ is obtained as a nontrivial irredundant join of elements of $\boldsymbol{u}$, there exists an integer $k > m$ such that $\{a_k, b_m\} \subseteq \boldsymbol{u}$. In particular, $b_m \in \boldsymbol{u}$ and the desired conclusion holds again. This completes the proof that $\operatorname{Reg}\Delta = \operatorname{Clop}\Delta$.

Now we set

$$\begin{aligned} \boldsymbol{a} &= \{a_n \mid n < \omega\}\,, \\ \boldsymbol{b}_0 &= \{a_{2k} \mid k < \omega\} \cup \{b_{2k} \mid k < \omega\}\,, \\ \boldsymbol{b}_1 &= \{a_{2k+1} \mid k < \omega\} \cup \{b_{2k+1} \mid k < \omega\}\,. \end{aligned}$$

It is straightforward to verify that $\boldsymbol{a}$, $\boldsymbol{b}_0$, and $\boldsymbol{b}_1$ are all clopen and that $\boldsymbol{a} \subseteq \boldsymbol{b}_0 \cup \boldsymbol{b}_1$. Furthermore, for each $i \in \{0, 1\}$, the subset

$$\boldsymbol{a} \cap \boldsymbol{b}_i = \{a_{2k+i} \mid k < \omega\}$$

has empty interior (for $a_{2k+i} = a_{2k+i+1} \vee b_{2k+i}$), that is, $\boldsymbol{a} \wedge \boldsymbol{b}_i = \varnothing$. □

13. Completely join-irreducible regular closed sets in semilattices

We observed in Lemma 12.1 that for any join-semilattice S, every completely join-irreducible element of $\operatorname{Reg} S$ is a minimal neighborhood. By invoking the structure theorem of minimal neighborhoods in $\operatorname{Reg} S$ (viz. Theorem 9.1), we shall obtain, in this section, a complete description of the completely join-irreducible elements of $\operatorname{Reg} S$.

We start with an easy lemma, which will make it possible to identify the possible top elements of completely join-irreducible elements of $\operatorname{Reg} S$. We denote by $\operatorname{Id} S$ the lattice of all ideals of S (the empty set included), under set inclusion.

Lemma 13.1. *The following statements are equivalent, for any element p in a join-semilattice S and any positive integer n.*

(i) *$S\mathbin{\downarrow}p$ has at most n lower covers in* $\operatorname{Id}S$.
(ii) *There are ideals $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ of S such that $S \mathbin{\downarrow\downarrow} p = \bigcup_{1\leq i\leq n} \boldsymbol{a}_i$.*
(iii) *There are lower covers $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ of $S\mathbin{\downarrow}p$ in $\operatorname{Id}S$ such that $S \mathbin{\downarrow\downarrow} p = \bigcup_{1\leq i\leq n} \boldsymbol{a}_i$.*
(iv) *There is no $(n+1)$-element subset W of $S \mathbin{\downarrow\downarrow} p$ such that $p = u \vee v$ for all distinct $u, v \in W$.*

Proof. (ii)⇒(iv). Let W be an $(n+1)$-element subset of $S \mathbin{\downarrow\downarrow} p$ such that $p = u \vee v$ for all distinct $u, v \in W$. Every element of W belongs to some $\boldsymbol{a}_i$, thus there are $i \in [n]$ and distinct $u, v \in W$ such that $u, v \in \boldsymbol{a}_i$. Hence $p = u \vee v$ belongs to $\boldsymbol{a}_i$, a contradiction.

(iv)⇒(iii). Let $W \subseteq S \mathbin{\downarrow\downarrow} p$ such that $p = u \vee v$ for all distinct $u, v \in W$ (we say that W is *anti-orthogonal*), of maximal cardinality, necessarily at most n, with respect to that property. The set $\boldsymbol{a}_u = \{x \in S \mid x \vee u < p\}$ is a lower subset of $S \mathbin{\downarrow\downarrow} p$, for each $u \in W$. If $\boldsymbol{a}_u$ is not an ideal, then there are $x, y \in \boldsymbol{a}_u$ such that $p = x \vee y \vee u$, and then $W' = \{x \vee u, y \vee u\} \cup (W \setminus \{u\})$ is anti-orthogonal with $\operatorname{card} W' > \operatorname{card} W$, a contradiction; hence $\boldsymbol{a}_u$ is an ideal of $S\mathbin{\downarrow}p$. Let $\boldsymbol{b}$ be an ideal of S with $\boldsymbol{a}_u \subsetneqq \boldsymbol{b} \subseteq S\mathbin{\downarrow}p$. Every $x \in \boldsymbol{b} \setminus \boldsymbol{a}_u$ satisfies $p = x \vee u$. Since $x \in \boldsymbol{b}$ and $u \in \boldsymbol{a}_u \subseteq \boldsymbol{b}$, we get $p \in \boldsymbol{b}$, so $\boldsymbol{b} = S\mathbin{\downarrow}p$, thus completing the proof that $\boldsymbol{a}_u \prec S\mathbin{\downarrow}p$. Finally, it follows from the maximality assumption on W that $S \mathbin{\downarrow\downarrow} p = \bigcup_{u\in W} \boldsymbol{a}_u$.

(iii)⇒(ii) is trivial, so (ii)–(iv) are equivalent. Trivially, (iii) implies (i). Finally, suppose that (i) holds. By Zorn's Lemma, every $x \in S \mathbin{\downarrow\downarrow} p$ is contained in some lower cover of $S\mathbin{\downarrow}p$; hence (iii) holds. □

Referring to the canonical join-embedding $S \hookrightarrow \operatorname{Id}S$, $p \mapsto S\mathbin{\downarrow}p$, we shall often identify p and $S\mathbin{\downarrow}p$ and thus state (i) above by saying that "p has at most n lower covers in the ideal lattice of S". Since $\varnothing$ is an ideal, $S\mathbin{\downarrow}p$ has always a lower cover in $\operatorname{Id}S$.

Theorem 13.2. *For any join-semilattice S, the completely join-irreducible members of $\operatorname{Reg}S$ are exactly the set differences $(S\mathbin{\downarrow}p) \setminus \boldsymbol{a}'$, for $p \in S$ with at most two lower covers in the ideal lattice of S, one of them being $\boldsymbol{a}'$.*

Proof. It follows from Lemma 12.1 that every completely join-irreducible member $\boldsymbol{a}$ of $\operatorname{Reg}S$ is a minimal neighborhood of some $p \in S$. By Theorem 9.1, there exists a lower cover $\boldsymbol{a}'$ of $S\mathbin{\downarrow}p$ in $\operatorname{Id}S$ such that $\boldsymbol{a} = (S\mathbin{\downarrow}p) \setminus \boldsymbol{a}'$. Furthermore, it follows from Lemma 11.1 that $\boldsymbol{a}_* = \boldsymbol{a} \setminus \{p\}$, so $\boldsymbol{a}'' = S\mathbin{\downarrow}\boldsymbol{a}_*$ is a proper ideal of $S\mathbin{\downarrow}p$. Since $\boldsymbol{a}' \cup \boldsymbol{a}''$ contains $\boldsymbol{a}' \cup \boldsymbol{a}_* = S \mathbin{\downarrow\downarrow} p$, we get $\boldsymbol{a}' \cup \boldsymbol{a}'' = S \mathbin{\downarrow\downarrow} p$. By Lemma 13.1, it follows that $S\mathbin{\downarrow}p$ has at most two lower covers in $\operatorname{Id}S$.

Conversely, let $p \in S$ with at least one, but at most two, lower covers $\boldsymbol{a}'$ and $\boldsymbol{a}''$ in $\operatorname{Id}S$. The set $\boldsymbol{a} = (S\mathbin{\downarrow}p) \setminus \boldsymbol{a}'$ is clopen. Moreover, it follows from Lemma 13.1 that $S \mathbin{\downarrow\downarrow} p = \boldsymbol{a}' \cup \boldsymbol{a}''$. Now $\boldsymbol{a} \setminus \{p\}$ is a lower subset of $\boldsymbol{a}$, thus it is open. Further, $\boldsymbol{a} \setminus \{p\}$ is contained in $\boldsymbol{a}''$, thus p does not belong to its closure; since $\boldsymbol{a}$ is closed, it follows that $\boldsymbol{a} \setminus \{p\}$ is closed, so it is clopen. Let $\boldsymbol{b} \in \operatorname{Reg}S$ such that $p \in \boldsymbol{b} \subseteq \boldsymbol{a}$. Since $p \in \boldsymbol{b} = \operatorname{cl}\operatorname{int}(\boldsymbol{b})$ and p is not a join of elements of $\boldsymbol{a} \setminus \{p\}$, we get $p \in \operatorname{int}(\boldsymbol{b})$. From $\boldsymbol{a}' \prec S\mathbin{\downarrow}p$ it follows that for each $x \in \boldsymbol{a} \setminus \{p\}$, there exists $a \in \boldsymbol{a}'$ such that $p = x \vee a$. Since $p \in \operatorname{int}(\boldsymbol{b})$ and $a \notin \boldsymbol{b}$, it follows that $x \in \boldsymbol{b}$. Therefore, $\boldsymbol{a} \setminus \{p\} \subseteq \boldsymbol{b}$, so $\boldsymbol{b} = \boldsymbol{a}$, thus completing the proof that $\boldsymbol{a} \setminus \{p\}$ is the unique lower cover of $\boldsymbol{a}$ in $\operatorname{Reg}S$. □

14. Boundedness of lattices of regular closed subsets from graphs

Let G be a graph. We denote by $\boldsymbol{\delta}_G^+$ the poset of all connected subsets of G, ordered by set inclusion, and we set $\boldsymbol{\delta}_G = \boldsymbol{\delta}_G^+ \setminus \{\varnothing\}$. A nonempty finite subset $\boldsymbol{x}$ of $\boldsymbol{\delta}_G$ is a *partition* of an element $X \in \boldsymbol{\delta}_G$, in notation $X = \bigsqcup \boldsymbol{x}$, if X is the disjoint union of all members of $\boldsymbol{x}$. In case $\boldsymbol{x} = \{X_1, \dots, X_n\}$, we shall sometimes write $X = X_1 \sqcup \dots \sqcup X_n$ instead of $X = \bigsqcup \boldsymbol{x}$.

For any $\boldsymbol{x} \subseteq \boldsymbol{\delta}_G$, let $\mathrm{cl}(\boldsymbol{x})$ be the *closure of $\boldsymbol{x}$ under disjoint unions*, that is,

$$\mathrm{cl}(\boldsymbol{x}) = \left\{ X \in \boldsymbol{\delta}_G \mid (\exists \boldsymbol{y} \subseteq \boldsymbol{x})\bigl(X = \bigsqcup \boldsymbol{y}\bigr) \right\} .$$

Dually, we denote by $\mathrm{int}(\boldsymbol{x})$ the interior of $\boldsymbol{x}$, that is, the largest open subset of $\boldsymbol{x}$.

It is straightforward to verify that cl is an algebraic closure operator on $\boldsymbol{\delta}_G$. With respect to that closure operator, a subset $\boldsymbol{a}$ of $\boldsymbol{\delta}_G$ is closed iff for any partition $X = X_1 \sqcup \dots \sqcup X_n$ in $\boldsymbol{\delta}_G$, $\{X_1, \dots, X_n\} \subseteq \boldsymbol{a}$ implies that $X \in \boldsymbol{a}$. Dually, $\boldsymbol{a}$ is open iff for any partition $X = X_1 \sqcup \dots \sqcup X_n$ in $\boldsymbol{\delta}_G$, $X \in \boldsymbol{a}$ implies that $X_i \in \boldsymbol{a}$ for some i. In both statements, it is sufficient to take $n = 2$ (for whenever $X = X_1 \sqcup \dots \sqcup X_n$, there exists $i > 1$ such that $X_1 \cup X_i$ is connected, and then $X = (X_1 \sqcup X_i) \sqcup \bigsqcup_{j \notin \{1,i\}} X_j$). In our arguments about graphs, we shall often allow, by convention, the empty set in partitions, thus letting $X = \varnothing \sqcup X_1 \sqcup \dots \sqcup X_n$ simply mean that $X = X_1 \sqcup \dots \sqcup X_n$. We shall call $(\boldsymbol{\delta}_G, \mathrm{cl})$ the *closure space canonically associated to the graph G*.

For $P \in \boldsymbol{\delta}_G$, a nonempty subset $\boldsymbol{x} \subseteq \boldsymbol{\delta}_G$ belongs to $\mathcal{M}(P)$ iff P is the disjoint union of a nonempty finite subset of $\boldsymbol{x}$, but of no proper subset of $\boldsymbol{x}$. Hence $\boldsymbol{x} \in \mathcal{M}(P)$ iff $\boldsymbol{x}$ is finite and $P = \bigsqcup \boldsymbol{x}$, and we get the following.

Proposition 14.1. *The closure space $(\boldsymbol{\delta}_G, \mathrm{cl})$ has semilattice type, for every graph G.*

Observe that $(\boldsymbol{\delta}_G, \subseteq)$ might not be a join-semilattice, for example if $G = \mathcal{C}_4$. On the other hand, if G is a block graph, then $(\boldsymbol{\delta}_G, \subseteq)$ is a join-semilattice iff G is connected.

Definition 14.2. The *permutohedron* (resp., *extended permutohedron*) on G is the set $\mathsf{P}(G)$ (resp., $\mathsf{R}(G)$) of all clopen (resp., regular closed) subsets of $\boldsymbol{\delta}_G$, ordered by set inclusion. That is, $\mathsf{P}(G) = \mathrm{Clop}(\boldsymbol{\delta}_G, \mathrm{cl})$ and $\mathsf{R}(G) = \mathrm{Reg}(\boldsymbol{\delta}_G, \mathrm{cl})$.

In particular, $\mathsf{R}(G)$ is always a lattice. We will see in Section 15 in which case $\mathsf{P}(G)$ is a lattice.

Example 14.3. The Dynkin diagram G_n of the symmetric group $\mathfrak{S}_n$ consists of all transpositions $\sigma_i = \begin{pmatrix} i & i+1 \end{pmatrix}$, where $1 \leq i < n$, with $\sigma_i \sim \sigma_j$ iff $i - j = \pm 1$.

Observe that there is a bijection between the connected subgraphs of G_n and the pairs (i, j) with $1 \leq i < j \leq n$, whereas a set of connected subsets is closed iff the corresponding pairs form a transitive relation. Hence, $\mathsf{P}(G_n)$ is isomorphic to the lattice of permutations on n elements, that is, to the classical permutohedron $\mathsf{P}(n)$.

We define the collection of all *cuts*, respectively *proper cuts*, of a connected subset H in a graph G as

$$\mathrm{Cuts}(H) = \{X \subseteq H \text{ nonempty} \mid X \text{ and } H \setminus X \text{ are both connected}\} ,$$
$$\mathrm{Cuts}_*(H) = \mathrm{Cuts}(H) \setminus \{H\} .$$

The following lemma says that any completely join-irreducible element of $\mathsf{R}(G)$ is "open on cuts".

Lemma 14.4. *Let $\boldsymbol{a}$ be a completely join-irreducible element of $\mathsf{R}(G)$, with largest element H. Then $\boldsymbol{a} \cap \operatorname{Cuts}(H)$ is contained in $\operatorname{int}(\boldsymbol{a})$.*

Proof. Let $X = X_1 \sqcup \cdots \sqcup X_n$ with $X \in \boldsymbol{a} \cap \operatorname{Cuts}(H)$, we must prove that $X_i \in \boldsymbol{a}$ for some i. If $X = H$ then this follows from $H \in \operatorname{int}(\boldsymbol{a})$ (cf. Lemma 11.1). Suppose from now on that $X \neq H$. The complement $Y = H \setminus X$ is connected. Furthermore, $Y \notin \boldsymbol{a}$ (otherwise X and Y would both belong to $\boldsymbol{a} \setminus \{H\} = \boldsymbol{a}_*$, so $H \in \boldsymbol{a}_*$, a contradiction) and $H = Y \sqcup \bigsqcup_{1 \leq i \leq n} X_i$ belongs to $\operatorname{int}(\boldsymbol{a})$ (cf. Lemma 11.1), thus $X_i \in \boldsymbol{a}$ for some i. □

For subsets U and V in a graph G, we set

$$U \simeq V \quad \text{if} \quad \big(\exists (u,v) \in U \times V\big)(\text{either } u = v \text{ or } u \sim v)\,, \tag{14.1}$$

$$U \sim V \quad \text{if} \quad \Big(U \cap V = \varnothing \text{ and } \big(\exists (u,v) \in U \times V\big)(u \sim v)\Big)\,. \tag{14.2}$$

Hence, $U \sim V$ iff $U \simeq V$ and $U \cap V = \varnothing$. Moreover, if $U, V \in \boldsymbol{\delta}_G$, then $U \simeq V$ iff $U \cup V$ is connected. We denote by $\operatorname{CC}(X)$ the set of all connected components of a subset X of G. We omit the straightforward proof of the following lemma.

Lemma 14.5. *The following statements hold, for all $X, H \in \boldsymbol{\delta}_G$ with $X \subseteq H$ and all $U, V \in \operatorname{CC}(H \setminus X)$:*

(i) *U is a cut of H and $U \sim X$;*
(ii) *if $U \neq V$, then $U \not\simeq V$.*

If $X \subseteq H$ in $\boldsymbol{\delta}_G$, let $X \leq^{\oplus} H$ hold if $\operatorname{CC}(H \setminus X)$ is finite.

Lemma 14.6. *Let G be a graph, let $\boldsymbol{a}$ be a completely join-irreducible element of $\mathsf{R}(G)$ with greatest element H, and let $X \in \partial\boldsymbol{a}$. Then $X \leq^{\oplus} H$ and $\operatorname{CC}(H \setminus X) \cap \boldsymbol{a} = \varnothing$.*

Proof. By the final statement of Lemma 11.1, there exists $\boldsymbol{x} \in \mathcal{M}(H)$ such that $\boldsymbol{x} \cap \boldsymbol{a} = \{X\}$. From $H = X \sqcup \bigsqcup(Z \mid Z \in \boldsymbol{x} \setminus \{X\})$ it follows that $X \leq^{\oplus} H$. Now let $Y \in \operatorname{CC}(H \setminus X)$ and suppose that $Y \in \boldsymbol{a}$. Since Y is a cut of H (cf. Lemma 14.5) and by Lemma 14.4, $Y \in \operatorname{int}(\boldsymbol{a})$. Furthermore, $Y = \bigsqcup \boldsymbol{y}$ for some $\boldsymbol{y} \subseteq \boldsymbol{x} \setminus \{X\}$, thus $\boldsymbol{y} \cap \boldsymbol{a} \neq \varnothing$, and thus $(\boldsymbol{x} \setminus \{X\}) \cap \boldsymbol{a} \neq \varnothing$, a contradiction. □

The following lemma means that in the finite case, the join-irreducible members of $\mathsf{R}(G)$ are determined by their proper cuts. This result will be extended to the infinite case, with a noticeably harder proof, in Corollary 16.9.

Lemma 14.7. *Let G be a finite graph, let $\boldsymbol{a}$ and $\boldsymbol{b}$ be join-irreducible elements of $\mathsf{R}(G)$ with the same largest element H. If $\boldsymbol{a} \cap \operatorname{Cuts}_*(H) = \boldsymbol{b} \cap \operatorname{Cuts}_*(H)$, then $\boldsymbol{a} = \boldsymbol{b}$.*

Proof. By symmetry, it suffices to prove that $\boldsymbol{a} \subseteq \boldsymbol{b}$. Since $\boldsymbol{a} = \operatorname{cl}(\partial\boldsymbol{a})$ (cf. Edelman and Jamison [12, Theorem 2.1]), it suffices to prove that every $X \in \partial\boldsymbol{a}$ belongs to $\boldsymbol{b}$. If $X = H$ this is obvious, so suppose that $X \neq H$. It follows from Lemma 14.6 that $X \leq^{\oplus} H$ and $\operatorname{CC}(H \setminus X) \cap \boldsymbol{a} = \varnothing$. Since every element of $\operatorname{CC}(H \setminus X)$ is a (proper) cut of H and by assumption, it follows that $\operatorname{CC}(H \setminus X) \cap \boldsymbol{b} = \varnothing$. Since $H \in \operatorname{int}(\boldsymbol{b})$ (cf. Lemma 11.1) and $H = X \sqcup \bigsqcup(Y \mid Y \in \operatorname{CC}(H \setminus X))$, it follows that $X \in \boldsymbol{b}$, as desired. □

Lemma 14.8. *Let G be a graph, let $\boldsymbol{a}$ and $\boldsymbol{c}$ be completely join-irreducible elements of $\mathsf{R}(G)$, with the same largest element H, such that $\boldsymbol{a}^{\perp} \searrow \boldsymbol{c}$. Then $\boldsymbol{a} \cap \operatorname{Cuts}_*(H)$ and $\boldsymbol{c} \cap \operatorname{Cuts}_*(H)$ are complementary in $\operatorname{Cuts}_*(H)$.*

Proof. The statement $\operatorname{int}(\boldsymbol{a})\cap\boldsymbol{c}=\{H\}$ is established in the Claim within the proof of Lemma 11.2. By Lemma 14.4, it follows that $\boldsymbol{a}\cap\boldsymbol{c}\cap\operatorname{Cuts}(H)=\{H\}$.

Now let $X\in\operatorname{Cuts}(H)\setminus\boldsymbol{a}$, we must prove that $X\in\boldsymbol{c}$. Necessarily, $X\neq H$. From $H=X\sqcup(H\setminus X)$, $X\notin\boldsymbol{a}$, and $H\in\operatorname{int}(\boldsymbol{a})$ it follows that $H\setminus X\in\operatorname{int}(\boldsymbol{a})$. Since $\operatorname{int}(\boldsymbol{a})\cap\boldsymbol{c}=\{H\}$, it follows that $H\setminus X\notin\boldsymbol{c}$. From $H=X\sqcup(H\setminus X)$, $H\setminus X\notin\boldsymbol{c}$, and $H\in\operatorname{int}(\boldsymbol{c})$ it follows that $X\in\boldsymbol{c}$. □

Theorem 14.9. *The extended permutohedron* $\mathsf{R}(G)=\operatorname{Reg}(\boldsymbol{\delta}_G,\mathrm{cl})$ *on a finite graph* G *is a bounded homomorphic image of a free lattice.*

Proof. As in the proof of Theorem 11.3, it is sufficient to prove that $\boldsymbol{a}\mathbin{D}\boldsymbol{b}$ implies that $\max\boldsymbol{a}>\max\boldsymbol{b}$, for all join-irreducible elements $\boldsymbol{a}$ and $\boldsymbol{b}$ of $\mathsf{R}(G)$. By Lemma 2.1, there exists a meet-irreducible $\boldsymbol{u}\in\mathsf{R}(G)$ such that $\boldsymbol{a}\nearrow\boldsymbol{u}\searrow\boldsymbol{b}$. The element $\boldsymbol{c}=\boldsymbol{u}^{\perp}$ is join-irreducible and it follows from Lemma 11.2 that $\max\boldsymbol{a}\geq\max\boldsymbol{c}\geq\max\boldsymbol{b}$. Suppose that $\max\boldsymbol{a}=\max\boldsymbol{c}=\max\boldsymbol{b}$ and denote that element by H. It follows from Lemma 14.8 that $\bigl(\boldsymbol{a}\cap\operatorname{Cuts}_*(H),\boldsymbol{c}\cap\operatorname{Cuts}_*(H)\bigr)$ and $\bigl(\boldsymbol{c}\cap\operatorname{Cuts}_*(H),\boldsymbol{b}\cap\operatorname{Cuts}_*(H)\bigr)$ are both complementary pairs, within $\operatorname{Cuts}_*(H)$, of proper cuts. It follows that $\boldsymbol{a}\cap\operatorname{Cuts}_*(H)=\boldsymbol{b}\cap\operatorname{Cuts}_*(H)$, so, by Lemma 14.7, $\boldsymbol{a}=\boldsymbol{b}$, in contradiction with $\boldsymbol{a}\mathbin{D}\boldsymbol{b}$. □

Example 14.10. The conclusion of Theorem 14.9 implies, in particular, that $\mathsf{R}(G)$ is semidistributive for any finite graph G. We show here that this conclusion cannot be extended to infinite graphs. The infinite path $\mathcal{P}_\omega=\{0,1,2,\dots\}$, with graph incidence defined by $i\sim j$ if $i-j=\pm1$, is an infinite tree. The subsets $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ of $\boldsymbol{\delta}_{\mathcal{P}_\omega}$ defined by

$$\begin{aligned}
\boldsymbol{a}&=\{[2m,\infty[\;\mid m<\omega\}\cup\{[2m,2n]\mid m\leq n<\omega\}\,,\\
\boldsymbol{b}&=\{[2m+1,\infty[\;\mid m<\omega\}\cup\{[2m+1,2n+1]\mid m\leq n<\omega\}\,,\\
\boldsymbol{c}&=\{[n,\infty[\;\mid n<\omega\}
\end{aligned}$$

are all clopen. Furthermore,

$$\begin{aligned}
\boldsymbol{a}\cap\boldsymbol{c}&=\{[2m,\infty[\;\mid m<\omega\}\,,\\
\boldsymbol{b}\cap\boldsymbol{c}&=\{[2m+1,\infty[\;\mid m<\omega\}
\end{aligned}$$

have both empty interior, so $\boldsymbol{a}\wedge\boldsymbol{c}=\boldsymbol{b}\wedge\boldsymbol{c}=\varnothing$. On the other hand, $\boldsymbol{c}\subseteq\boldsymbol{a}\cup\boldsymbol{b}$, thus $(\boldsymbol{a}\vee\boldsymbol{b})\wedge\boldsymbol{c}=\boldsymbol{c}\neq\varnothing$. Therefore, $\mathsf{R}(\mathcal{P}_\omega)$ (which, by Theorem 15.1, turns out to be identical to $\mathsf{P}(\mathcal{P}_\omega)$) is neither pseudocomplemented nor meet-semidistributive (so it is not join-semidistributive either).

15. Graphs whose permutohedron is a lattice

The main goal of this section is a characterization of those graphs G such that the permutohedron $\mathsf{P}(G)$ $(=\operatorname{Clop}(\boldsymbol{\delta}_G,\mathrm{cl}))$ is a lattice. Observe that unlike Theorem 17.3, the statement of Theorem 15.1 does not require any finiteness assumption on G.

Theorem 15.1. *The following are equivalent, for any graph* G*:*

(i) $\mathsf{P}(G)$ *is a lattice.*
(ii) *The closure of any open subset of* $\boldsymbol{\delta}_G$ *is open* (*i.e.,* $\mathsf{P}(G)=\mathsf{R}(G)$).
(iii) G *is a block graph without* 4*-cliques.*

Proof. (ii)⇒(i) is trivial.

(i)⇒(iii). Suppose that G is not a block graph without 4-cliques; we shall prove that $\mathsf{P}(G)$ is not a lattice. It is easy to construct, for each graph P in the collection $\Gamma = \{\mathcal{K}_4, \mathcal{D}\} \cup \{\mathcal{C}_n \mid 4 \leq n < \omega\}$ (cf. Figure 2.1), nonempty connected subsets P_0, P_1, P_2, and P_3 of P such that

$$P = P_0 \sqcup P_2 = P_1 \sqcup P_3 \text{ while } P_i \cap P_{i+1} \neq \varnothing \text{ for all } i < 4 \tag{15.1}$$

(with indices reduced modulo 4). By assumption, one of the members of Γ embeds into G as an induced subgraph. This yields nonempty connected subsets P_i, for $0 \leq i \leq 3$, and P of G satisfying (15.1).

It is easy to see that the set $\boldsymbol{a}_i = \{X \in \boldsymbol{\delta}_G \mid X \subseteq P_i \text{ and } X \cap P_{i+1} \neq \varnothing\}$ is clopen in $\boldsymbol{\delta}_G$, for each $i < 4$. We claim that $\boldsymbol{a}_i \cap \boldsymbol{a}_j = \varnothing$ whenever $i \in \{0, 2\}$ and $j \in \{1, 3\}$. Indeed, suppose otherwise and let $Z \in \boldsymbol{a}_i \cap \boldsymbol{a}_j$. From $Z \subseteq P_i$ and $Z \cap P_{j+1} \neq \varnothing$ it follows that $P_i \cap P_{j+1} \neq \varnothing$, so $i - j \in \{0, 1, 2\}$, and so $i - j = 1$. Likewise, $j - i = 1$; a contradiction.

It follows that $\boldsymbol{a}_i \subseteq \boldsymbol{\delta}_G \setminus \boldsymbol{a}_j$ for all $i \in \{0, 2\}$ and all $j \in \{1, 3\}$. Suppose that there exists $\boldsymbol{b} \in \mathsf{P}(G)$ such that $\boldsymbol{a}_i \subseteq \boldsymbol{b} \subseteq \boldsymbol{\delta}_G \setminus \boldsymbol{a}_j$. From $P_i \in \boldsymbol{a}_i$ and $P = P_0 \sqcup P_2$ it follows (using the closedness of $\boldsymbol{b}$) that $P \in \boldsymbol{b}$. Since $P = P_1 \sqcup P_3$ and $\boldsymbol{b}$ is open, either $P_1 \in \boldsymbol{b}$ or $P_3 \in \boldsymbol{b}$. In the first case, we get $P_1 \notin \boldsymbol{a}_1$ from $\boldsymbol{b} \subseteq \boldsymbol{\delta}_G \setminus \boldsymbol{a}_1$, while in the second case we get $P_3 \notin \boldsymbol{a}_3$; a contradiction in both cases.

(iii)⇒(ii). Suppose that G is a block graph without 4-cliques, and suppose that there exists an open set $\boldsymbol{u} \subseteq \boldsymbol{\delta}_G$ such that $\mathrm{cl}(\boldsymbol{u})$ is not open. This means that there are $P \in \mathrm{cl}(\boldsymbol{u})$ and a partition $\boldsymbol{y}$ of P such that

$$\boldsymbol{y} \cap \mathrm{cl}(\boldsymbol{u}) = \varnothing\,. \tag{15.2}$$

On the other hand, from $P \in \mathrm{cl}(\boldsymbol{u})$ it follows that there exists a partition $\boldsymbol{x}$ of P such that

$$\boldsymbol{x} \subseteq \boldsymbol{u}\,. \tag{15.3}$$

In particular, from (15.2) and (15.3) it follows that $\boldsymbol{x} \cap \boldsymbol{y} = \varnothing$. Moreover, as G is a block graph, the intersection of any two connected subsets of G is connected, hence, as $P = \bigsqcup \boldsymbol{x} = \bigsqcup \boldsymbol{y}$, we get the following decompositions in $\boldsymbol{\delta}_G$:

$$X = \bigsqcup (X \cap Y \mid X \cap Y \neq \varnothing\,,\ Y \in \boldsymbol{y}) \qquad \text{(for each } X \in \boldsymbol{x})\,, \tag{15.4}$$

$$Y = \bigsqcup (X \cap Y \mid X \cap Y \neq \varnothing\,,\ X \in \boldsymbol{x}) \qquad \text{(for each } Y \in \boldsymbol{y})\,. \tag{15.5}$$

For each $X \in \boldsymbol{x}$, it follows from $X \in \boldsymbol{u}$ (cf. (15.3)) and the openness of $\boldsymbol{u}$ that there exists $s(X) \in \boldsymbol{y}$ such that $X \cap s(X) \in \boldsymbol{u}$. On the other hand, each $Y \in \boldsymbol{y}$ belongs to the complement of $\mathrm{cl}(\boldsymbol{u})$ (cf. (15.2)), thus there exists $s(Y) \in \boldsymbol{x}$ such that $s(Y) \cap Y$ is nonempty and does not belong to $\boldsymbol{u}$. From this it is easy to deduce that

$$Z \cap s(Z) \neq \varnothing \text{ and } s^2(Z) \neq Z\,, \quad \text{for each } Z \in \boldsymbol{x} \cup \boldsymbol{y}\,. \tag{15.6}$$

Consider the graph with vertex set $\boldsymbol{z} = \boldsymbol{x} \cup \boldsymbol{y}$, and incidence relation $\asymp$ defined by $X \asymp Y$ iff $X \neq Y$ and $X \cap Y \neq \varnothing$, for all $X, Y \in \boldsymbol{z}$. Since $\boldsymbol{x}$ and $\boldsymbol{y}$ are both partitions of P, the graph $\boldsymbol{z}$ is bipartite. Let n be a positive integer, minimal with the property that there exists $Z \in \boldsymbol{z}$ such that $s^n(Z) = Z$. It follows from (15.6) that $n \geq 3$. Further, by the minimality assumption on n, all sets Z, $s(Z)$, $\dots$, $s^{n-1}(Z)$ are pairwise distinct. Since $s^k(Z) \asymp s^{k+1}(Z)$ for each k, it follows that the graph $(\boldsymbol{z}, \asymp)$ has an induced cycle of length ≥ 3. Since this graph is bipartite, the cycle above has the form $\vec{P} = (P_0, P_1, \dots, P_{2n-1})$, for some integer $n \geq 2$.

Pick $g_i \in P_i \cap P_{i+1}$, for each $i < 2n$ (indices are reduced modulo $2n$). Since g_i and g_{i+1} both belong to the connected set P_{i+1}, they are joined by a path $\vec{g}_i$ contained in P_i. By joining the $\vec{g}_i$ together, we obtain a path $\vec{g}$ (not induced *a priori*) containing all the g_i as vertices.

Choose the $\vec{g}_i$ in such a way that the length N of $\vec{g}$ is as small as possible. Since the g_i are pairwise distinct (because $\vec{P}$ is an induced path), $N \geq 2n$.

Since $\vec{P}$ is an induced cycle in $(\boldsymbol{z}, \asymp)$, the (ranges of) the paths $\vec{g}_i$ and $\vec{g}_j$ meet iff $i - j \in \{0, 1, -1\}$, for all $i, j < 2n$. Moreover, it follows from the minimality assumption on N that $\vec{g}_i \cap \vec{g}_{i+1} = \{g_{i+1}\}$, for each $i < 2n$. Therefore, the range of the path $\vec{g}$ is biconnected, so, as G is a block graph, $\vec{g}$ is a clique, and so, by assumption, $N \leq 3$, in contradiction with $N \geq 2n$. □

Example 15.2. (Observe the similarity with Example 9.4.) It is an easy exercise to verify that $\mathsf{P}(\mathcal{K}_2) = \mathsf{R}(\mathcal{K}_2)$ is isomorphic to the permutohedron on three letters $\mathsf{P}(3)$, which is the six-element "benzene lattice".

On the other hand, the lattice $\mathsf{P}(\mathcal{K}_3) = \mathsf{R}(\mathcal{K}_3)$ has apparently not been met until now.

Denote by a, b, c the vertices of the graph $\mathcal{K}_3$. The lattice $\mathsf{P}(\mathcal{K}_3)$ is represented on the right hand side of Figure 15.1, by using the following labeling convention:

$$\{\{a\}\} \mapsto a\,, \quad \{\{a,b\},\{a\}\} \mapsto a^2b\,, \quad \{\{a,b\},\{a\},\{b\}\} \mapsto a^2b^2\,,$$
$$\{\{a,b\},\{a,c\},\{a\}\} \mapsto a^3bc$$

(the "variables" a, b, c being thought of as pairwise commuting, so for example $a^2b = ba^2$), then $\overline{\varnothing} = \boldsymbol{\delta}_{\mathcal{K}_3}$, $\overline{a}^2\overline{b}^2 = \boldsymbol{\delta}_{\mathcal{K}_3} \setminus (a^2b^2)$, and so on.

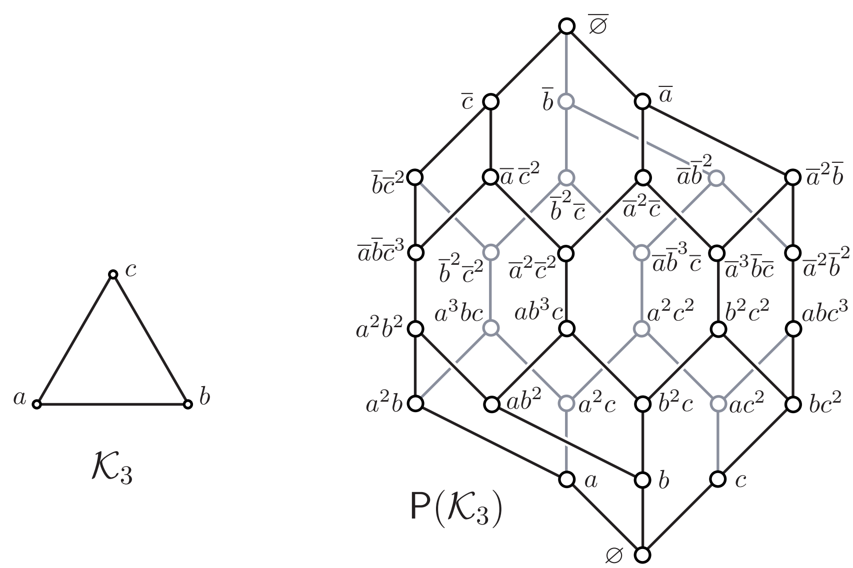

FIGURE 15.1. The permutohedron on the graph $\mathcal{K}_3$

While we prove in [36] that every open subset of a transitive binary relation is a union of clopen subsets, the open subset $\boldsymbol{u} = \{a, b, c, abc\}$ of $\boldsymbol{\delta}_{\mathcal{K}_3}$ is not a union of clopen subsets.

The join-irreducible elements of $\mathsf{P}(\mathcal{K}_3)$ are a, a^2b, $\overline{a}\overline{b}\overline{c}^3$, $\overline{a}^2\overline{b}^2$, and cyclically. They are all closed under intersection, and they never contain all the members of a nontrivial partition.

By Theorem 15.1, the permutohedron $\mathsf{P}(\mathcal{K}_4)$ is not a lattice. Brute force computation shows that $\operatorname{card}\mathsf{P}(\mathcal{K}_4) = 370$ while $\operatorname{card}\mathsf{R}(\mathcal{K}_4) = 382$. Every join-irreducible element of $\mathsf{R}(\mathcal{K}_4)$ belongs to $\mathsf{P}(\mathcal{K}_4)$. Labeling the vertices of $\mathcal{K}_4$ as a, b, c, d, we get the join-irreducible element $\{b, c, ab, ac, bc, abc, bcd, abcd\}$ in $\operatorname{Reg}\mathcal{K}_4$. It contains ab and ac but not their intersection a. It contains all entries of the partition $bc = b \sqcup c$.

Variants of $\mathsf{P}(G)$ and $\mathsf{R}(G)$, with the collection of all connected subsets of G replaced by other alignments, are studied in more detail in Santocanale and Wehrung [37].

16. Completely join-irreducible regular closed sets in graphs

While $\operatorname{Reg} S$ is always the Dedekind-MacNeille completion of $\operatorname{Clop} S$, for any join-semilattice S (cf. Corollary 9.2), the situation for graphs is more complex. In this section we shall give a convenient description of the completely join-irreducible members of $\mathsf{R}(G)$, in terms of so-called *pseudo-ultrafilters* on members of $\boldsymbol{\delta}_G$, for an arbitrary graph G. This will imply that the completely join-irreducible elements are determined by the proper cuts of their top element, thus extending Lemma 14.7 to the infinite case (cf. Corollary 16.9). In addition, this will yield a large class of graphs G for which every completely join-irreducible member of $\mathsf{R}(G)$ is clopen (cf. Theorem 16.10 and Corollary 16.11).

In this section we will constantly refer to the restrictions to $\boldsymbol{\delta}_G$ of the binary relations $\simeq$ and $\sim$ introduced in (14.1) and (14.2). From Lemma 16.1 to Proposition 16.7 we shall fix a graph G and a nonempty connected subset H of G.

Lemma 16.1. *Let $X, Y, Z \in \boldsymbol{\delta}_H$ with $Z = X \sqcup Y$. Denote by $\widehat{X}$ the unique member of $\operatorname{CC}(H \setminus X)$ containing Y, and define $\widehat{Y}$ similarly with X and Y interchanged. The following statements hold:*

(i) *Every member of $\operatorname{CC}(H \setminus X) \setminus \{\widehat{X}\}$ is contained in $\widehat{Y}$, and symmetrically with $(X, \widehat{X})$ and $(Y, \widehat{Y})$ interchanged.*
(ii) *$H = \widehat{X} \cup \widehat{Y}$.*
(iii) $\operatorname{CC}(H \setminus Z) = \operatorname{CC}(\widehat{X} \cap \widehat{Y}) \cup \big(\operatorname{CC}(H \setminus X) \setminus \{\widehat{X}\}\big) \cup \big(\operatorname{CC}(H \setminus Y) \setminus \{\widehat{Y}\}\big)$.
(iv) *$T \sim X$ and $T \sim Y$, for any $T \in \operatorname{CC}(\widehat{X} \cap \widehat{Y})$.*

Proof. Let $U \in \operatorname{CC}(H \setminus X) \setminus \{\widehat{X}\}$. Suppose first that $U \simeq Y$ (cf. (14.1)). Since $U \cup Y$ is connected, disjoint from X, and contains Y, it is contained in $\widehat{X}$, hence $U = \widehat{X}$, a contradiction. Hence $U \not\simeq Y$, so $U \subseteq H \setminus Y$. Since $X \subseteq H \setminus Y$ and $U \sim X$ (cf. Lemma 14.5), it follows that $U \subseteq \widehat{Y}$, thus completing the proof of (i).

Now set $X' = \bigcup\big(\operatorname{CC}(H \setminus \{X\}) \setminus \{\widehat{X}\}\big)$, and define Y' symmetrically. It follows from (i) that $X \cup X' \subseteq \widehat{Y}$. Since $X' \cup \widehat{X} = \bigcup \operatorname{CC}(H \setminus \{X\}) = H \setminus X$, it follows that $H = X \cup X' \cup \widehat{X} \subseteq \widehat{X} \cup \widehat{Y}$ and (ii) follows.

As a further consequence of (i), $(X \cup X') \cap (Y \cup Y') \subseteq \widehat{Y} \cap (Y \cup Y') = \varnothing$. Since $H = X \cup X' \cup \widehat{X} = Y \cup Y' \cup \widehat{Y}$, it follows that $H = X \cup X' \cup Y \cup Y' \cup (\widehat{X} \cap \widehat{Y})$ (disjoint union), so the union of the right hand side of (iii) is $H \setminus Z$. Furthermore, every element of the right hand side of (iii) is, by definition, nonempty and connected.

Finally, it follows easily from (i) together with Lemma 14.5 that any two distinct members U and V of the right hand side of (iii) satisfy $U \not\sim V$; (iii) follows.

(iv). Since $\widehat{X}$ is connected and contains $T \cup Y$, there exists a path γ, within $\widehat{X}$, from an element $t \in T$ to an element of Y. We may assume that the successor y of t in γ does not belong to T. Recall now that y belongs to $H = Y \cup \bigcup \mathrm{CC}(H \setminus Y)$. If $y \in Y'$ for some $Y' \in \mathrm{CC}(H \setminus Y) \setminus \{\widehat{Y}\}$, then we get, from $t \in \widehat{Y}$, that $\widehat{Y} \sim Y'$, a contradiction. If $y \in \widehat{Y}$ then $T \sim W$ for some $W \in \mathrm{CC}(\widehat{X} \cap \widehat{Y})$ distinct of T, a contradiction. Hence, $y \in Y$, so $T \sim Y$. Symmetrically, $T \sim X$. □

Definition 16.2. A *pseudo-ultrafilter* on H is a subset $\mu \subseteq \mathrm{Cuts}(H)$ such that $H \in \mu$ and whenever X, Y, Z are cuts of H such that $Z = X \sqcup Y$,

(i) $X \in \mu$ and $Y \in \mu$ implies that $Z \in \mu$;
(ii) $X \notin \mu$ and $Y \notin \mu$ implies that $Z \notin \mu$;
(iii) $X \in \mu$ iff $H \setminus X \notin \mu$, whenever X is a proper cut of H.

Observe that H is necessarily the largest element of μ. We leave to the reader the straightforward proof of the following lemma.

Lemma 16.3. *If μ is a pseudo-ultrafilter on H, then so is the* conjugate pseudo-ultrafilter $\tilde{\mu} = \big(\mathrm{Cuts}(H) \setminus \mu\big) \cup \{H\}$.

Given a pseudo-ultrafilter μ on H, we define

$$\begin{aligned} \mathtt{j}(\mu) &= \{X \in \boldsymbol{\delta}_H \mid X \leq^{\oplus} H \text{ and } \mathrm{CC}(H \setminus X) \cap \mu = \varnothing\}\,, \\ \mathtt{j}_*(\mu) &= \mathtt{j}(\mu) \setminus \{H\}\,. \end{aligned}$$

We shall fix, until Proposition 16.7, a pseudo-ultrafilter μ on H. It is obvious that $\mathtt{j}(\mu) \cap \mathrm{Cuts}(H) = \mu$. This observation is extended in the following lemma.

Lemma 16.4. $\mu = \mathrm{cl}\big(\mathtt{j}(\mu)\big) \cap \mathrm{Cuts}(H)$.

Proof. We prove the nontrivial containment. We must prove that if $X = \bigsqcup_{i<m} X_i$, with each $X_i \in \mathtt{j}(\mu)$ and X a cut, then $X \in \mu$. If $X = H$ the conclusion is trivial, so we suppose that $X \neq H$. The complement $Y = H \setminus X$ is a proper cut of H. We argue by induction on m. For $m = 1$ the proof is straightforward, as $\mathtt{j}(\mu) \cap \mathrm{Cuts}(H) \subseteq \mu$; let us suppose therefore that $m \geq 2$.

Claim 1. *$X_i \in \mu$, for each $i < m$.*

Proof of Claim. Suppose that X_i is not a cut, denote by Y' the unique connected component of $H \setminus X_i$ containing Y, and let $V \in \mathrm{CC}(H \setminus X_i) \setminus \{Y'\}$. Since V is both a cut and a (disjoint) union of some members of $\{X_j \mid j \neq i\}$, it follows from the induction hypothesis that $V \in \mu$. On the other hand, from $X_i \in \mathtt{j}(\mu)$ together with the definition of $\mathtt{j}(\mu)$, it follows that $V \notin \mu$, a contradiction.

Since X_i is a cut and $\mathtt{j}(\mu) \cap \mathrm{Cuts}(H) \subseteq \mu$, the conclusion follows. □ Claim 1.

By way of contradiction, we suppose next that $X \notin \mu$, so that the cut $Y = H \setminus X$ belongs to μ. Since $H = Y \sqcup \bigsqcup_{i<m} X_i$ is connected, there exists $i_0 < m$ such that $X_{i_0} \sim Y$ (cf. (14.2)); so $X_{i_0} \cup Y = X_{i_0} \sqcup Y$.

Claim 2. *The set $X_{i_0} \sqcup Y$ is a cut of H.*

Proof of Claim. Suppose otherwise and let U, V be distinct connected components of $H \setminus (X_{i_0} \cup Y)$. It follows from Claim 1 that $H \setminus X_{i_0}$ is connected, thus there exists a path γ, within that set, between an element of U and an element of V.

The path γ meets necessarily Y. Since Y is connected, we may assume that γ enters and exits Y exactly once. If p (resp., q) denotes the entry (resp., exit) point of γ in Y, then the predecessor of p in γ belongs to U and the successor of q in γ belongs to V. It follows that $U \sim Y$ and $V \sim Y$ (cf. (14.2)); hence $W \sim Y$ for each $W \in \mathrm{CC}\bigl(H \setminus (X_{i_0} \cup Y)\bigr)$. Since Y is also a cut, a similar proof yields that $W \sim X_{i_0}$ for each $W \in \mathrm{CC}\bigl(H \setminus (X_{i_0} \cup Y)\bigr)$. Now every such W is a cut, and also a disjoint union of members of $\{X_k \mid k \neq i\}$, hence, by the induction hypothesis, $W \in \mu$. Fix such a W. Then $X_{i_0} \sqcup W$ is a cut, whose complement in H is $Y \sqcup \bigsqcup\Bigl(\mathrm{CC}\bigl(H \setminus (X_{i_0} \cup Y)\bigr) \setminus \{W\}\Bigr)$. Since the cardinality of $\mathrm{CC}\bigl(H \setminus (X_{i_0} \cup Y)\bigr)$ is smaller than m, it follows from the induction hypothesis that $H \setminus (X_{i_0} \sqcup W)$ belongs to μ. Further, as X_{i_0}, W, $X_{i_0} \sqcup W$ are cuts and $X_{i_0}, W \in \mu$, it follows from the definition of a pseudo-ultrafilter that $X_{i_0} \sqcup W \in \mu$. Therefore, we obtain two complementary cuts both belonging to μ, a contradiction. □ Claim 2.

Since $H \setminus (X_{i_0} \sqcup Y)$ is a disjoint union of members of $\{X_j \mid j \neq i_0\}$, it follows from the induction hypothesis together with Claim 2 that $H \setminus (X_{i_0} \sqcup Y) \in \mu$. As X_{i_0}, Y, $X_{i_0} \sqcup Y$ are cuts and $X_{i_0}, Y \in \mu$, we deduce that $X_{i_0} \sqcup Y \in \mu$, in contradiction with μ being a pseudo-ultrafilter. This ends the proof of Lemma 16.4. □

Lemma 16.5. $H \notin \mathrm{cl}\bigl(\mathtt{j}_*(\mu)\bigr)$.

Proof. Suppose, otherwise, that $H = \bigsqcup_{1 \leq i \leq n} X_i$, with $n \geq 2$ and each $X_i \in \mathtt{j}(\mu)$. Define a binary relation $\sim$ on $[n]$ by letting $i \sim j$ hold iff $X_i \sim X_j$. There exists $i \in [n]$ such that $\{i\}$ is a cut of $([n], \sim)$ (e.g., take i at maximum $\sim$-distance from 1). Then X_i is a cut of H, so it belongs to μ. By Lemma 16.4, $H \setminus X_i = \bigsqcup_{j \neq i} X_j$ also belongs to μ, a contradiction. □

Lemma 16.6. *The set* $\mathtt{j}(\mu)$ *is open.*

Proof. Let $X, Y, Z \in \boldsymbol{\delta}_G$ such that $Z = X \sqcup Y$ and $Z \in \mathtt{j}(\mu)$. From $Z \leq^{\oplus} H$ it follows immediately that $X \leq^{\oplus} H$ and $Y \leq^{\oplus} H$. By Lemma 16.1, we can write

$$\mathrm{CC}(H \setminus X) = \{X_i \mid 0 \leq i \leq m\}\,, \tag{16.1}$$

$$\mathrm{CC}(H \setminus Y) = \{Y_j \mid 0 \leq j \leq n\} \tag{16.2}$$

without repetitions (e.g., $i \mapsto X_i$ is one-to-one) and with $X \subseteq Y_0$ and $Y \subseteq X_0$,

$$\mathrm{CC}(X_0 \cap Y_0) = \{Z_k \mid k < \ell\} \qquad \text{without repetitions}\,, \tag{16.3}$$

$$\mathrm{CC}(H \setminus Z) = \{Z_k \mid k < \ell\} \cup \{X_i \mid 1 \leq i \leq m\} \cup \{Y_j \mid 1 \leq j \leq n\}\,, \tag{16.4}$$

for natural numbers m, n, ℓ. By (16.4), the assumption $Z \in \mathtt{j}(\mu)$ means that all $Z_k \notin \mu$ while $X_i \notin \mu$ whenever $i > 0$ and $Y_j \notin \mu$ whenever $j > 0$.

Now suppose that $X, Y \notin \mathtt{j}(\mu)$. By the paragraph above together with (16.1) and (16.2), this means that $X_0, Y_0 \in \mu$. Since μ is a pseudo-ultrafilter, $H \setminus X_0 \notin \mu$ and $H \setminus Y_0 \notin \mu$. Since those two proper cuts are disjoint (cf. Lemma 16.1(ii)), we get a partition $H = (H \setminus X_0) \sqcup (H \setminus Y_0) \sqcup \bigsqcup_{k<\ell} Z_k$ with all members belonging to the conjugate pseudo-ultrafilter $\tilde{\mu}$ (cf. Lemma 16.3). By Lemma 16.5 (applied to $\tilde{\mu}$), this is a contradiction. □

We make cash of our previous observations with the following result.

Proposition 16.7. *The set* $\mathrm{cl}\bigl(\mathtt{j}(\mu)\bigr)$ *is completely join-irreducible in* $\mathsf{R}(G)$, *with lower cover* $\mathrm{cl}\bigl(\mathtt{j}(\mu)\bigr) \setminus \{H\} = \mathrm{cl}\bigl(\mathtt{j}_*(\mu)\bigr)$.

Proof. It follows from Lemma 16.6 that $\mathrm{cl}\big(\mathtt{j}(\mu)\big)$ is regular closed. Furthermore, it follows from Lemma 16.5 that $\mathrm{cl}\big(\mathtt{j}(\mu)\big)\setminus\{H\}$ is closed; the equality $\mathrm{cl}\big(\mathtt{j}(\mu)\big)\setminus\{H\} = \mathrm{cl}\big(\mathtt{j}_*(\mu)\big)$ follows. Since $\mathtt{j}_*(\mu)$ is a lower subset of the open set $\mathtt{j}(\mu)$, it is open as well; hence $\mathrm{cl}\big(\mathtt{j}_*(\mu)\big)$ is regular closed.

It remains to prove that every regular closed subset $\boldsymbol{a}$ of $\mathrm{cl}\big(\mathtt{j}(\mu)\big)$ with $H \in \boldsymbol{a}$ contains $\mathtt{j}(\mu)$ (and thus is equal to $\mathrm{cl}\big(\mathtt{j}(\mu)\big)$). If $H \notin \mathrm{int}(\boldsymbol{a})$, then, since H belongs to $\boldsymbol{a} = \mathrm{cl}\,\mathrm{int}(\boldsymbol{a})$, there is a partition of the form $H = \bigsqcup_{i<m} X_i$ with $m \geq 2$ and each $X_i \in \boldsymbol{a}$, thus each $X_i \in \mathrm{cl}\big(\mathtt{j}_*(\mu)\big)$, and thus $H \in \mathrm{cl}\big(\mathtt{j}_*(\mu)\big)$, a contradiction. Hence $H \in \mathrm{int}(\boldsymbol{a})$. Now let $X \in \mathtt{j}(\mu)$ and write $\mathrm{CC}(H \setminus X) = \{X_i \mid i < n\}$. Then each X_i is a cut and $X_i \notin \mu$, thus $X_i \notin \mathrm{cl}\big(\mathtt{j}(\mu)\big)$ by Lemma 16.4, and thus $X_i \notin \boldsymbol{a}$. Since $H \in \mathrm{int}(\boldsymbol{a})$ and $H = X \sqcup \bigsqcup_{i<n} X_i$, it follows that $X \in \boldsymbol{a}$, as desired. □

Now we are ready to prove the main result of this section.

Theorem 16.8. *Let G be a graph. Then the completely join-irreducible elements of $\mathsf{R}(G)$ are exactly the sets $\mathrm{cl}\big(\mathtt{j}(\mu)\big)$, for pseudo-ultrafilters μ.*

Proof. One direction is provided by Proposition 16.7, so that it suffices to prove that every completely join-irreducible element $\boldsymbol{a}$ of $\mathsf{R}(G)$ has the form $\mathrm{cl}\big(\mathtt{j}(\mu)\big)$.

It follows from Lemma 11.1 that $\boldsymbol{a}$ has a largest element H, and $H \in \mathrm{int}(\boldsymbol{a})$. Since $\boldsymbol{a}$ is closed, the set $\mu = \boldsymbol{a} \cap \mathrm{Cuts}(H)$ satisfies item (i) of Definition 16.2.

Let X, Y, and Z be cuts of H such that $Z = X \sqcup Y$ and $Z \in \mu$. It follows from Lemma 14.4 that $Z \in \mathrm{int}(\boldsymbol{a})$, so either $X \in \boldsymbol{a}$ or $Y \in \boldsymbol{a}$, that is, either $X \in \mu$ or $Y \in \mu$, thus completing the proof of item (ii) of Definition 16.2.

Since $\boldsymbol{a}_* = \boldsymbol{a} \setminus \{H\}$ is closed and $H \in \mathrm{int}(\boldsymbol{a})$, item (iii) of Definition 16.2 is also satisfied. Therefore, μ is a pseudo-ultrafilter.

We claim that $\mathtt{j}(\mu) \subseteq \boldsymbol{a}$. Let $X \in \mathtt{j}(\mu)$ and write $\mathrm{CC}(H \setminus X) = \{X_i \mid i < n\}$. Then each X_i is a cut of H and $X_i \notin \mu$, so $X_i \notin \boldsymbol{a}$. Since $H \in \mathrm{int}(\boldsymbol{a})$ and $H = X \sqcup \bigsqcup_{i<n} X_i$, it follows that $X \in \boldsymbol{a}$, as desired.

By Lemma 16.6, $\mathrm{cl}\big(\mathtt{j}(\mu)\big)$ is regular closed. This set contains H as an element, and, by the paragraph above, it is contained in $\boldsymbol{a}$. Hence $\mathrm{cl}\big(\mathtt{j}(\mu)\big)$ is not contained in $\boldsymbol{a}_* = \boldsymbol{a} \setminus \{H\}$, and hence $\mathrm{cl}\big(\mathtt{j}(\mu)\big) = \boldsymbol{a}$. □

We obtain the following strengthening of Lemma 14.7.

Corollary 16.9. *Let G be a graph, let $\boldsymbol{a}$ and $\boldsymbol{b}$ be completely join-irreducible elements of $\mathsf{R}(G)$ with the same largest element H. If $\boldsymbol{a} \cap \mathrm{Cuts}_*(H) \subseteq \boldsymbol{b} \cap \mathrm{Cuts}_*(H)$, then $\boldsymbol{a} = \boldsymbol{b}$.*

Proof. By Theorem 16.8, the sets $\alpha = \boldsymbol{a} \cap \mathrm{Cuts}(H)$ and $\beta = \boldsymbol{b} \cap \mathrm{Cuts}(H)$ are pseudo-ultrafilters on H, $\boldsymbol{a} = \mathrm{cl}\big(\mathtt{j}(\alpha)\big)$, and $\boldsymbol{b} = \mathrm{cl}\big(\mathtt{j}(\beta)\big)$. By assumption, $\alpha \subseteq \beta$. Any $X \in \beta \setminus \alpha$ is a proper cut of H and $H \setminus X \in \alpha \subseteq \beta$, which contradicts $X \in \beta$; so $\alpha = \beta$, and the desired conclusion follows. □

As we will see in Theorem 18.1, a completely join-irreducible element of $\mathsf{R}(G)$ may not be clopen. However, the following result provides a large class of graphs for which every completely join-irreducible regular closed set is clopen.

A graph G is *contractible* to a graph H if H can be obtained from G by contracting some edges; that is, there is a surjective map $\varphi\colon G \twoheadrightarrow H$ with connected fibers such that for all distinct $x, y \in H$, $\varphi^{-1}\{x\} \sim \varphi^{-1}\{y\}$ iff $x \sim y$. If $H = \mathcal{D}$, the diamond (cf. Figure 2.1), we say that G is *diamond-contractible*.

Theorem 16.10. *The following are equivalent, for any graph G:*

(i) $\mathtt{j}(\mu)$ *is clopen, for any pseudo-ultrafilter μ on a member of $\boldsymbol{\delta}_G$.*
(ii) *G has no diamond-contractible induced subgraph.*

Furthermore, if (ii) *holds, then every completely join-irreducible element of* $\mathsf{R}(G)$ *is clopen.*

Proof. Suppose first that G has a diamond-contractible induced subgraph H. There exists a partition $H = X \sqcup Y \sqcup U \sqcup V$ in $\boldsymbol{\delta}_H$ such that (X, Y, U, V) forms a diamond in $(\boldsymbol{\delta}_H, \sim)$, with diagonal $\{X, Y\}$. Pick $v \in V$. The set

$$\mu = \{Z \in \boldsymbol{\delta}_H \mid v \notin Z\} \cup \{H\}$$

is a pseudo-ultrafilter on H, with $X, Y, U \in \mu$ (so $X, Y, U \in \mathtt{j}(\mu)$) and $V \notin \mu$. Since $\mathrm{CC}\big(H \setminus (X \cup Y)\big) = \{U, V\}$ meets μ, $X \sqcup Y \notin \mathtt{j}(\mu)$, so $\mathtt{j}(\mu)$ is not closed.

Conversely, suppose that G has no diamond-contractible induced subgraph, let μ be a pseudo-ultrafilter on $H \in \boldsymbol{\delta}_G$, and let $Z = X \sqcup Y$ in $\boldsymbol{\delta}_H$ with $X, Y \in \mathtt{j}(\mu)$. We can write

$$\mathrm{CC}(H \setminus X) = \{X_i \mid 0 \le i \le m\} \quad \text{and} \quad \mathrm{CC}(H \setminus Y) = \{Y_j \mid 0 \le j \le n\}\,,$$

without repetitions, with all $X_i \notin \mu$ and all $Y_j \notin \mu$, and in such a way that $X \subseteq Y_0$ and $Y \subseteq X_0$.

Claim. *The set $X_0 \cap Y_0$ is either empty, or a cut. Furthermore, $X_0 \cap Y_0 \notin \mu$.*

Proof of Claim. By Lemma 16.1(iv), for any distinct $U, V \in \mathrm{CC}(X_0 \cap Y_0)$, the quadruple (X, Y, U, V) forms a diamond in $\boldsymbol{\delta}_H$, with diagonal $\{X, Y\}$. Hence the (connected) induced subgraph $H' = X \cup Y \cup U \cup V$ is diamond-contractible, a contradiction. Therefore, $X_0 \cap Y_0$ is connected.

Since X_0 and Y_0 are both proper cuts, so are $H \setminus X_0$ and $H \setminus Y_0$. Moreover, $X \subseteq H \setminus X_0$, $Y \subseteq H \setminus Y_0$, and $X \sim Y$, thus $H \setminus X_0 \simeq H \setminus Y_0$, and thus $H \setminus (X_0 \cap Y_0) = (H \setminus X_0) \cup (H \setminus Y_0)$ is connected.

From $X_0, Y_0 \notin \mu$ it follows that both $H \setminus X_0$ and $H \setminus Y_0$ belong to μ. But those two sets are disjoint (cf. Lemma 16.1(ii)), hence their union, namely $H \setminus (X_0 \cap Y_0)$, belongs to μ; whence $X_0 \cap Y_0 \notin \mu$. □ Claim.

Now it follows from Lemma 16.1(iii) that the set

$$\mathrm{CC}(H \setminus Z) = \big(\{X_0 \cap Y_0\} \setminus \{\varnothing\}\big) \cup \{X_i \mid 1 \le i \le m\} \cup \{Y_j \mid 1 \le j \le n\}$$

is disjoint from μ; that is, $Z \in \mathtt{j}(\mu)$.

If (ii) holds, then it follows from the equivalence above together with Theorems 16.6 and 16.8 that every completely join-irreducible element of $\mathsf{R}(G)$ is clopen. □

In particular, it is easy to verify that if G is either a block graph or a cycle, then no induced subgraph of G is diamond-contractible. Therefore, by putting together Theorem 16.10 and Lemma 4.1, we obtain the following result.

Corollary 16.11. *Let G be a finite graph. If G is either a block graph or a cycle, then $\mathsf{R}(G)$ is the Dedekind-MacNeille completion of $\mathsf{P}(G)$.*

Remark 16.12. The statement that all $\mathtt{j}(\mu)$ are clopen, although it implies that every completely join-irreducible element of $\mathsf{R}(G)$ is clopen, is not equivalent to that statement: as a matter of fact, $\mathrm{cl}(\mathsf{j}(\mu))$ might be open while properly containing $\mathsf{j}(\mu)$. For example, for the diamond graph $\mathcal{D}$, every completely join-irreducible element in $\mathsf{R}(\mathcal{D})$ is clopen, but not every $\mathsf{j}(\mu)$ is clopen (or even regular open).

17. Lattices of clopen sets for poset and semilattice type

Some of the results that we have established in earlier sections, in the particular cases of semilattices ($\operatorname{Reg} S$ and $\operatorname{Clop} S$) or graphs ($\mathsf{R}(G)$ and $\mathsf{P}(G)$), about $\operatorname{Reg}(P,\varphi)$ being the Dedekind-MacNeille completion of $\operatorname{Clop}(P,\varphi)$, can be extended to arbitrary closure spaces of semilattice type; some others cannot. In this section we shall survey some of the statements that can be extended, and give counterexamples to some of those that cannot be extended.

The following crucial lemma expresses the abundance of clopen subsets in any well-founded closure space of semilattice type.

Lemma 17.1. *Let (P,φ) be a well-founded closure space of semilattice type. Let $\boldsymbol{a}$ and $\boldsymbol{u}$ be subsets of P with $\boldsymbol{a}$ clopen and $\boldsymbol{u}$ open. If $\boldsymbol{u} \not\subseteq \boldsymbol{a}$, then $(\boldsymbol{a}\downarrow p) \cup \{p\}$ is clopen for any minimal element p of $\boldsymbol{u} \setminus \boldsymbol{a}$.*

Proof. For any $q \in \varphi\big((\boldsymbol{a}\downarrow p) \cup \{p\}\big)$, there exists $\boldsymbol{x} \in \mathcal{M}(q)$ with $\boldsymbol{x} \subseteq (\boldsymbol{a}\downarrow p) \cup \{p\}$. If $\boldsymbol{x} \subseteq \boldsymbol{a}\downarrow p$, then, as $\boldsymbol{a}\downarrow p$ is closed (cf. Lemma 10.2), $q \in \boldsymbol{a}\downarrow p$. If $\boldsymbol{x} \not\subseteq \boldsymbol{a}\downarrow p$, then $p \in \boldsymbol{x}$, thus (as $q = \bigvee \boldsymbol{x}$ and $\boldsymbol{x} \subseteq P\downarrow p$) $q = p$. In both cases, $q \in (\boldsymbol{a}\downarrow p) \cup \{p\}$. Hence $(\boldsymbol{a}\downarrow p) \cup \{p\}$ is closed.

Now let $q \in (\boldsymbol{a}\downarrow p) \cup \{p\}$ and let $\boldsymbol{y} \in \mathcal{M}(q)$ be nontrivial. From $q = \bigvee \boldsymbol{y}$ it follows that $\boldsymbol{y} \subseteq P\downarrow q$. If $q \in \boldsymbol{a}\downarrow p$, then, as $\boldsymbol{a}$ is open, $\boldsymbol{y} \cap \boldsymbol{a} \neq \varnothing$, so $\boldsymbol{y} \cap (\boldsymbol{a}\downarrow p) \neq \varnothing$. Suppose now that $q = p$. As $\boldsymbol{y}$ is a nontrivial covering of p and $p = \bigvee \boldsymbol{y}$, every element of $\boldsymbol{y}$ is smaller than p, thus, by the minimality assumption on p, we get $\boldsymbol{y} \cap \boldsymbol{u} \subseteq \boldsymbol{a}$, so $\boldsymbol{y} \cap \boldsymbol{u} \subseteq \boldsymbol{a}\downarrow p$. As $\boldsymbol{u}$ is open, $p \in \boldsymbol{u}$, and $\boldsymbol{y} \in \mathcal{M}(p)$, we get $\boldsymbol{y} \cap \boldsymbol{u} \neq \varnothing$; whence $\boldsymbol{y} \cap (\boldsymbol{a}\downarrow p) \neq \varnothing$. Hence $(\boldsymbol{a}\downarrow p) \cup \{p\}$ is open. □

Theorem 17.2. *Let (P,φ) be a well-founded closure space of semilattice type. Then the poset $\operatorname{Clop}(P,\varphi)$ is tight in $\operatorname{Reg}(P,\varphi)$.*

Proof. Let $(\boldsymbol{a}_i \mid i \in I)$ be a family of clopen subsets of P, having a meet $\boldsymbol{a}$ in $\operatorname{Clop}(P,\varphi)$. It is obvious that $\boldsymbol{a}$ is contained in the open set $\boldsymbol{u} = \breve{\varphi}\big(\bigcap(\boldsymbol{a}_i \mid i \in I)\big)$. Suppose that the containment is proper. By Lemma 17.1, there exists $p \in \boldsymbol{u} \setminus \boldsymbol{a}$ such that $(\boldsymbol{a}\downarrow p) \cup \{p\}$ is clopen. From $p \in \boldsymbol{u}$ it follows that $p \in \boldsymbol{a}_i$ for each i, thus $(\boldsymbol{a}\downarrow p) \cup \{p\} \subseteq \boldsymbol{a}_i$ for each i, and thus, by the definition of $\boldsymbol{a}$, we get $(\boldsymbol{a}\downarrow p) \cup \{p\} \subseteq \boldsymbol{a}$, so $p \in \boldsymbol{a}$, a contradiction. Therefore, $\boldsymbol{u} = \boldsymbol{a}$ is clopen, so the meet of the $\boldsymbol{a}_i$ in $\operatorname{Reg}(P,\varphi)$, which is equal to $\varphi(\boldsymbol{u})$ (cf. Lemma 3.3), is equal to $\boldsymbol{a}$ as well. □

The analogue of Theorem 17.2 for regular closed subsets in a transitive binary relation holds as well, see Santocanale and Wehrung [36]. This is also the case for semilattices, see Corollary 9.2. For a precursor of those results, for permutohedra on posets, see Pouzet *et al.* [33, Lemma 11].

Theorem 17.3. *Let (P,φ) be a well-founded closure space of semilattice type. Then $\operatorname{Clop}(P,\varphi)$ is a lattice iff $\operatorname{Clop}(P,\varphi) = \operatorname{Reg}(P,\varphi)$.*

Proof. We prove the nontrivial direction. Suppose that $\operatorname{Clop}(P,\varphi)$ is a lattice. Given a regular open subset $\boldsymbol{u}$ of P, we must prove that $\boldsymbol{u}$ is closed (thus clopen). As φ is an algebraic closure operator, $\operatorname{Clop}(P,\varphi)$ is closed under directed unions, thus it follows from Zorn's Lemma that the set of all clopen subsets of $\boldsymbol{u}$ has a maximal element, say $\boldsymbol{a}$.

Suppose that $\boldsymbol{a}$ is properly contained in $\boldsymbol{u}$. Since P is well-founded and by Lemma 17.1, there exists $p \in \boldsymbol{u} \setminus \boldsymbol{a}$ such that $\boldsymbol{b} = (\boldsymbol{a} \downarrow p) \cup \{p\}$ is clopen. By assumption, the pair $\{\boldsymbol{a}, \boldsymbol{b}\}$ has a join $\boldsymbol{d}$ in $\operatorname{Clop}(P, \varphi)$. Furthermore, it follows from Theorem 17.2 that $\boldsymbol{d} = \varphi(\boldsymbol{a} \cup \boldsymbol{b})$, whence $\boldsymbol{d} \subseteq \varphi(\boldsymbol{u})$, so $\boldsymbol{d} = \check{\varphi}(\boldsymbol{d}) \subseteq \check{\varphi}\varphi(\boldsymbol{u}) = \boldsymbol{u}$. Since $\boldsymbol{a} \subseteq \boldsymbol{d}$ and $\boldsymbol{d}$ is clopen, it follows from the maximality statement on $\boldsymbol{a}$ that $\boldsymbol{a} = \boldsymbol{d}$, thus $p \in \boldsymbol{a}$, a contradiction. □

Example 17.4. *A finite closure system (P, φ) of semilattice type with a non open, join-irreducible element of* $\operatorname{Reg}(P, \varphi)$.

Note. By Theorem 17.3, for such an example, $\operatorname{Clop}(P, \varphi)$ cannot be a lattice. By Lemma 4.1, $\operatorname{Reg}(P, \varphi)$ is not the Dedekind-MacNeille completion of $\operatorname{Clop}(P, \varphi)$.

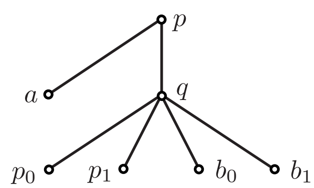

Figure 17.1. The poset P of Example 17.4

Proof. Consider the poset P represented in Figure 17.1, and say that a subset $\boldsymbol{x}$ of P is *closed* if

$$\begin{aligned} \{a, p_i\} \subseteq \boldsymbol{x} &\Rightarrow p \in \boldsymbol{x}, && \text{for each } i \in \{0,1\}, \\ \{p_0, p_1\} \subseteq \boldsymbol{x} &\Rightarrow q \in \boldsymbol{x}, \\ \{b_0, b_1\} \subseteq \boldsymbol{x} &\Rightarrow q \in \boldsymbol{x}. \end{aligned}$$

Denote by φ the corresponding closure operator. The nontrivial coverings in (P, φ) are given by

$$\{a, p_0\}, \{a, p_1\} \in \mathcal{M}(p) \text{ and } \{p_0, p_1\}, \{b_0, b_1\} \in \mathcal{M}(q).$$

It follows that (P, φ) is a finite closure space of semilattice type.

Set $\boldsymbol{a} = \{p, p_0, p_1, q\}$. Then $\check{\varphi}(\boldsymbol{a}) = \{p, p_0, p_1\}$ and $\varphi\check{\varphi}(\boldsymbol{a}) = \boldsymbol{a}$, so $\boldsymbol{a}$ is regular closed. Moreover, $\boldsymbol{a} \setminus \{p\} = \{p_0, p_1, q\}$ is regular closed (and not open), and every regular closed proper subset of $\boldsymbol{a}$ is contained in $\boldsymbol{a} \setminus \{p\}$. Hence $\boldsymbol{a}$ is join-irreducible in $\operatorname{Reg}(P, \varphi)$ and $\boldsymbol{a}_* = \boldsymbol{a} \setminus \{p\}$. The regular closed set $\boldsymbol{a}$ is not open, as $q \in \boldsymbol{a}$ while $b_0 \notin \boldsymbol{a}$ and $b_1 \notin \boldsymbol{a}$. □

For a related counterexample, arising from the context of graphs ($\mathsf{R}(G)$ and $\mathsf{P}(G)$), see Section 18.

The following modification of Example 17.4 shows that Theorem 17.2 cannot be extended to the non well-founded case.

Example 17.5. *A closure space (P, φ) of semilattice type, with clopen subsets $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$ with nonempty meet in $\operatorname{Reg}(P, \varphi)$ and with empty meet in $\operatorname{Clop}(P, \varphi)$.*

Proof. Denote by P the poset represented on the left hand side of Figure 17.2. A detail of P is shown on the right hand side of Figure 17.2. The index ξ ranges over the set $\mathbb{S}$ of all finite sequences of elements of $\{0,1\}$, and $\xi 0$ (resp., $\xi 1$) stands for the concatenation of ξ and 0 (resp., 1). We leave a formal definition of P to the reader.

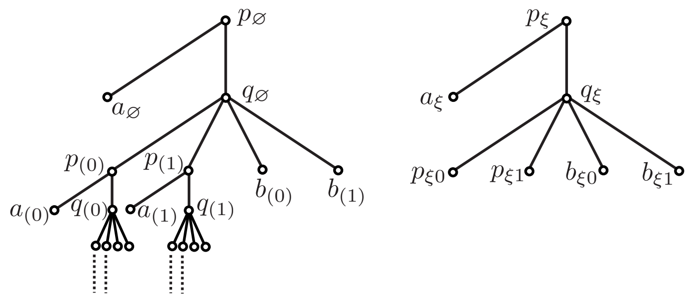

FIGURE 17.2. The poset P of Example 17.5

Now a subset $\boldsymbol{x}$ of P is closed if

$$\begin{aligned}
\{a_\xi, p_{\xi i}\} &\subseteq \boldsymbol{x} \ \Rightarrow\ p_\xi \in \boldsymbol{x}\,, && \text{for each } \xi \in \mathbb{S} \text{ and each } i \in \{0,1\}\,,\\
\{p_{\xi 0}, p_{\xi 1}\} &\subseteq \boldsymbol{x} \ \Rightarrow\ q_\xi \in \boldsymbol{x}\,, && \text{for each } \xi \in \mathbb{S}\,,\\
\{b_{\xi 0}, b_{\xi 1}\} &\subseteq \boldsymbol{x} \ \Rightarrow\ q_\xi \in \boldsymbol{x}\,, && \text{for each } \xi \in \mathbb{S}\,.
\end{aligned}$$

We denote by φ the corresponding closure operator. We set $\boldsymbol{p} = \{p_\xi \mid \xi \in \mathbb{S}\}$, $\boldsymbol{q} = \{q_\xi \mid \xi \in \mathbb{S}\}$, and $\boldsymbol{a} = \boldsymbol{p} \cup \boldsymbol{q}$. Then $\check{\varphi}(\boldsymbol{a}) = \boldsymbol{p}$ and $\varphi\check{\varphi}(\boldsymbol{a}) = \boldsymbol{a}$, so $\boldsymbol{a}$ is regular closed.

We claim that $\boldsymbol{a}$ has no nonempty clopen subset. For let $\boldsymbol{x} \subseteq \boldsymbol{a}$ be clopen. If $q_\xi \in \boldsymbol{x}$ for some $\xi \in \mathbb{S}$, then (as $\{b_{\xi 0}, b_{\xi 1}\} \in \mathcal{M}(q_\xi)$) $\{b_{\xi 0}, b_{\xi 1}\}$ meets $\boldsymbol{x}$, thus it meets $\boldsymbol{a}$, a contradiction; whence $\boldsymbol{x} \subseteq \boldsymbol{p}$. If $p_\xi \in \boldsymbol{x}$ for some $\xi \in \mathbb{S}$, then (as $\{a_\xi, p_{\xi i}\} \in \mathcal{M}(p_\xi)$ for each $i < 2$) $\{p_{\xi 0}, p_{\xi 1}\} \subseteq \boldsymbol{x}$, thus $q_\xi \in \boldsymbol{x}$, a contradiction. This proves our claim.

Now the sets $\boldsymbol{a}_i = \boldsymbol{a} \cup \{b_{\xi i} \mid \xi \in \mathbb{S}\}$, for $i \in \{0,1\}$, are both clopen and they intersect in $\boldsymbol{a}$. By the paragraph above, the meet of $\{\boldsymbol{a}_0, \boldsymbol{a}_1\}$ in $\operatorname{Clop}(P,\varphi)$ is the empty set. However, the meet of $\{\boldsymbol{a}_0, \boldsymbol{a}_1\}$ in $\operatorname{Reg}(P,\varphi)$ is $\boldsymbol{a}$. □

Note. For the closure space (P,φ) of Example 17.5, the poset $\operatorname{Clop}(P,\varphi)$ is not a lattice. Indeed, for each $i < 2$, the subset $\boldsymbol{b}_i = (P{\downarrow}\{p_{(0)}, p_{(1)}\}) \cup \{q_\varnothing, b_{(i)}\}$ is clopen, $\boldsymbol{a}_i \subseteq \boldsymbol{b}_j$ for all $i,j < 2$, and there is no clopen subset $\boldsymbol{c}$ of P such that $\boldsymbol{a}_i \subseteq \boldsymbol{c} \subseteq \boldsymbol{b}_j$ for all $i,j < 2$.

The following example shows that the lattice of all regular closed subsets of a closure space of semilattice type may not be spatial.

Example 17.6. *An infinite closure space (P,φ) of semilattice type such that*

(i) *P is a join-semilattice with largest element.*
(ii) $\operatorname{Reg}(P,\varphi) = \operatorname{Clop}(P,\varphi)$.
(iii) *The poset* $\operatorname{Clop} P$ *(cf. Example 7.10) is not a lattice.*
(iv) *None of the lattices* $\operatorname{Reg} P$ *(cf. Example 7.10) and* $\operatorname{Reg}(P,\varphi)$ *has any completely join-irreducible element.*

Proof. We denote by P the set of all finite sequences of elements of $\{0,1,2\}$. For $p,q \in P$, let $p \leq q$ hold if q is a prefix of p. Observe, in particular, that P is a join-semilattice, with largest element the empty sequence $\varnothing$. The join of any subset $\boldsymbol{x}$ of P is the longest common prefix for all elements of $\boldsymbol{x}$.

Say that a subset $\boldsymbol{x}$ of P is closed if

$$\{pi, pj\} \subseteq \boldsymbol{x} \ \Rightarrow\ p \in \boldsymbol{x}\,, \quad \text{for all } p \in P \text{ and all distinct } i,j \in \{0,1,2\}\,,$$

and denote by φ the associated closure operator. It is easy to verify that (P,φ) is a closure space of semilattice type.

Claim 1. *Let $p \in P$ and let $\boldsymbol{a}_i \subseteq P\downarrow pi$ be closed, for each $i \in \{0,1,2\}$. Set*

$$X = \{i < 3 \mid pi \in \boldsymbol{a}_i\}\,.$$

If $\bigcup_{i<3} \boldsymbol{a}_i$ is not closed, then $\operatorname{card} X \geq 2$.

Proof of Claim. If $\boldsymbol{a} = \bigcup_{i<3} \boldsymbol{a}_i$ is not closed, then there are $q \in P$ and $i \neq j$ such that $\{qi, qj\} \subseteq \boldsymbol{a}$ but $q \notin \boldsymbol{a}$. If $\{qi,qj\} \subseteq \boldsymbol{a}_k$, for some $k<3$, then, as $\boldsymbol{a}_k$ is closed, we get $q \in \boldsymbol{a}_k \subseteq \boldsymbol{a}$, a contradiction. It follows that there are $i' \neq j'$ such that $qi \in \boldsymbol{a}_{i'}$ and $qj \in \boldsymbol{a}_{j'}$. From $\boldsymbol{a}_{i'} \subseteq P\downarrow pi'$ it follows that qi extends pi', thus, as it also extends q, the finite sequences q and pi' are comparable (with respect to $\leq$). Likewise, q and pj' are comparable. Since $i' \neq j'$, it follows that p extends q. Since qi extends pi' and qj extends pj', it follows that $p = q$, $i = i'$, and $j = j'$. Therefore, $pi \in \boldsymbol{a}_i$ and $pj \in \boldsymbol{a}_j$, so $\{i,j\} \subseteq X$. □ Claim 1.

Claim 2. *If $\boldsymbol{a}$ is closed, then $\check{\varphi}(\boldsymbol{a})$ is closed, for any $\boldsymbol{a} \subseteq P$.*

Proof of Claim. Let $p \in P$ and let $i \neq j$ in $\{0,1,2\}$ such that $\{pi,pj\} \subseteq \check{\varphi}(\boldsymbol{a})$, we must prove that $p \in \check{\varphi}(\boldsymbol{a})$. From $\{pi,pj\} \subseteq \check{\varphi}(\boldsymbol{a}) \subseteq \boldsymbol{a}$, together with $\boldsymbol{a}$ being closed, it follows that $p \in \boldsymbol{a}$. Suppose that $p \notin \check{\varphi}(\boldsymbol{a})$. There exists $\boldsymbol{x} \in \mathcal{M}(p)$ such that $\boldsymbol{x} \cap \boldsymbol{a} = \varnothing$; from $p \in \boldsymbol{a}$ it follows that $\boldsymbol{x} \subseteq P \downdownarrows p$, so $\boldsymbol{x} = \bigcup_{k<3}(\boldsymbol{x} \downarrow pk)$. The set $\boldsymbol{a}_k = \varphi(\boldsymbol{x}\downarrow pk)$ is a closed subset of $P\downarrow pk$, for each $k<3$. If $\boldsymbol{b} = \bigcup_{k<3} \boldsymbol{a}_k$ is closed, then, as $\boldsymbol{x} \subseteq \boldsymbol{b}$ and $p \in \varphi(\boldsymbol{x})$, we get $p \in \boldsymbol{b}$, so $p \leq pk$ for some $k<3$, a contradiction. Hence $\boldsymbol{b}$ is not closed, so, by Claim 1 above, there are distinct $i' \neq j'$ in $\{0,1,2\}$ such that $pi' \in \boldsymbol{a}_{i'}$ and $pj' \in \boldsymbol{a}_{j'}$. Pick $k \in \{i,j\} \cap \{i',j'\}$. Then $pk \in \check{\varphi}(\boldsymbol{a})$ and $pk \in \varphi(\boldsymbol{x}\downarrow pk)$, hence $(\boldsymbol{x}\downarrow pk) \cap \boldsymbol{a} \neq \varnothing$, and hence $\boldsymbol{x} \cap \boldsymbol{a} \neq \varnothing$, thus completing the proof that $p \in \check{\varphi}(\boldsymbol{a})$. □ Claim 2.

It follows from Claim 2 that $\operatorname{Reg}(P,\varphi) = \operatorname{Clop}(P,\varphi)$.

Suppose that $\operatorname{Reg} P$ (that is, the lattice of all regular closed subsets over the closure space (P,cl) defined in Example 7.10) has a completely join-irreducible element $\boldsymbol{a}$. By Corollary 9.2, $\boldsymbol{a}$ is clopen. Since (P,cl) is a closure space of semilattice type, it follows from Lemma 11.1 that $\boldsymbol{a}$ has a largest element p, with $\boldsymbol{a}_* = \boldsymbol{a}\setminus\{p\}$. For all $i \neq j$ in $\{0,1,2\}$, it follows from $p \in \check{\varphi}(\boldsymbol{a})$ and $p \in \mathrm{cl}\{pi,pj\}$ that $\{pi,pj\} \cap \boldsymbol{a} \neq \varnothing$. Since this holds for every possible choice of $\{i,j\}$, it follows that there are distinct $i,j \in \{0,1,2\}$ such that $\{pi,pj\} \subseteq \boldsymbol{a}$, so $\{pi,pj\} \subseteq \boldsymbol{a} \setminus \{p\} = \boldsymbol{a}_*$, and so, since $\boldsymbol{a}_*$ is closed, $p \in \boldsymbol{a}_*$, a contradiction.

An argument similar to the one of the paragraph above shows that $\operatorname{Reg}(P,\varphi)$ has no completely join-irreducible element either.

The subsets $\boldsymbol{a}_i = P\downarrow\{i0\}$, $\boldsymbol{b}_0 = (P\downarrow\{0,10,2\}) \cup \{\varnothing\}$, and $\boldsymbol{b}_1 = (P\downarrow\{0,1\}) \cup \{\varnothing\}$ are all clopen, with $\boldsymbol{a}_i \subseteq \boldsymbol{b}_j$ for all $i,j<2$. However, suppose that there exists a clopen subset $\boldsymbol{c} \subseteq P$ such that $\boldsymbol{a}_i \subseteq \boldsymbol{c} \subseteq \boldsymbol{b}_j$ for all $i,j<2$. Since $\{00,10\} \subseteq \boldsymbol{a}_0 \cup \boldsymbol{a}_1 \subseteq \boldsymbol{c}$ and $\boldsymbol{c}$ is closed, $\varnothing = 00 \vee 10$ belongs to $\boldsymbol{c}$. Since $\varnothing = 1 \vee 2$ and $\boldsymbol{c}$ is open, it follows that $\{1,2\}$ meets $\boldsymbol{c}$, thus it meets $\boldsymbol{b}_0 \cap \boldsymbol{b}_1$, a contradiction as $1 \notin \boldsymbol{b}_0$ and $2 \notin \boldsymbol{b}_1$. Therefore, $\operatorname{Clop} P$ is not a lattice. □

Our next example shows that Theorem 17.3 cannot be extended from closure spaces of semilattice type to closure spaces of poset type. That is, in poset type, even if $\operatorname{Clop}(P,\varphi)$ is a lattice, it may not be a sublattice of $\operatorname{Reg}(P,\varphi)$.

Example 17.7. Let $P = \{a_0, a_1, a, \top\}$ be the poset represented on the left hand side of Figure 17.3, and set

$$\varphi(\boldsymbol{x}) = \begin{cases} \boldsymbol{x} \cup \{\top\}, & \text{if either } \{a_0,a_1\} \subseteq \boldsymbol{x} \text{ or } a \in \boldsymbol{x}, \\ \boldsymbol{x}, & \text{otherwise} \end{cases}, \qquad \text{for each } \boldsymbol{x} \subseteq P.$$

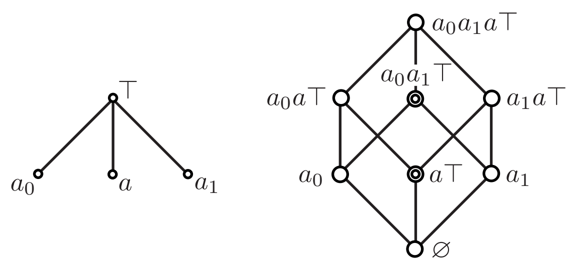

FIGURE 17.3. The poset P and the containment $\operatorname{Clop}(P,\varphi) \subsetneqq \operatorname{Reg}(P,\varphi)$

It is straightforward to verify that φ is a closure operator on P. Furthermore, the nontrivial minimal coverings in (P,φ) are exactly those given by the relations $\top \in \varphi(\{a_0,a_1\})$ and $\top \in \varphi(\{a\})$. Since a_0, a_1, and a are all smaller than $\top$, it follows that (P,φ) is a closure space of poset type.

We represent the eight-element Boolean lattice $L = \operatorname{Reg}(P,\varphi)$ on the right hand side of Figure 17.3 (the labeling is given by $\{a_0\} \mapsto a_0$, $\{a_0,a,\top\} \mapsto a_0a\top$, and so on). The poset $K = \operatorname{Clop}(P,\varphi)$ is the six-element "benzene lattice". The two elements of $L \setminus K$ (viz. $\{a,\top\}$ and $\{a_0,a_1,\top\}$) are marked by doubled circles on the right hand side of Figure 17.3.

Observe that $\{a_0\}$ and $\{a_1\}$ are both clopen, and that

$$\begin{aligned} \{a_0\} \vee \{a_1\} &= \{a_0,a_1,\top\} && \text{in } \operatorname{Reg}(P,\varphi), \\ \{a_0\} \vee \{a_1\} &= \{a_0,a_1,a,\top\} && \text{in } \operatorname{Clop}(P,\varphi). \end{aligned}$$

In particular, $\operatorname{Clop}(P,\varphi)$ is not a sublattice of $\operatorname{Reg}(P,\varphi)$.

18. A non-clopen join-irreducible regular open set in a finite graph

The present section will be devoted to the description of a counterexample showing that Corollary 16.11 cannot be extended to arbitrary finite graphs. We shall denote by H the graph denoted, in the online database http://www.graphclasses.org/, by $\mathcal{K}_{3,3} - e$, labeled as on Figure 18.1.

We skip the braces and commas in denoting the subsets of H, and we set

$$\begin{aligned} \boldsymbol{v} = \{&1,3,5,01,03,12,34,013,123,125,134,145,235,345, \\ &0123,0134,0145,0235,1235,1345,12345,01345,01235,012345\}. \end{aligned} \tag{18.1}$$

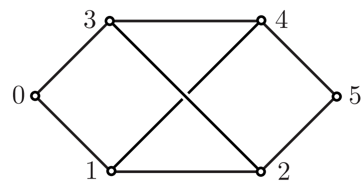

Figure 18.1. The graph H

The elements of $\boldsymbol{v}$ are surrounded by boxes in Table 1. Not every nonempty subset of H belongs to $\boldsymbol{\delta}_H$ (e.g., 02), in which case we mark it as such (e.g., $\underset{\notin\boldsymbol{\delta}_H}{02}$). The subset $1234 = 12 \sqcup 34$ belongs to $\mathrm{cl}(\boldsymbol{v}) \setminus \boldsymbol{v}$.

In order to facilitate the verification of the proof of Theorem 18.1, we list the elements of $\boldsymbol{v}^{\mathsf{c}} = \boldsymbol{\delta}_H \setminus \boldsymbol{v}$:

$$\begin{aligned}\boldsymbol{v}^{\mathsf{c}} = \{&0, 2, 4, 14, 23, 25, 45, 012, 014, 023, 034, 124, 234, 245,\\ &0124, 0125, 0234, 0345, 1234, 1245, 2345, 01234, 01245, 02345\}\,.\end{aligned} \tag{18.2}$$

Theorem 18.1. *The set $\boldsymbol{v}$ is a minimal open, and not closed, neighborhood of H, with $\mathrm{cl}(\boldsymbol{v}) = \boldsymbol{v} \cup \{1234\}$. Furthermore, $\boldsymbol{v}$ is join-irreducible in $\mathrm{Reg}_{\mathrm{op}}(\boldsymbol{\delta}_H, \mathrm{cl})$.*

Proof. We first verify, by using (18.2), that $\boldsymbol{v}^{\mathsf{c}}$ is closed. In order to do this, it is sufficient to verify that whenever X, Y, Z are nonempty connected subsets of H such that $Z = X \sqcup Y$, then $\{X, Y\} \subseteq \boldsymbol{v}^{\mathsf{c}}$ implies that $Z \in \boldsymbol{v}^{\mathsf{c}}$. We thus obtain that $\boldsymbol{v}$ is open.

Likewise, by using (18.1), we verify that $\boldsymbol{v} \cup \{1234\}$ is closed. Since $1234 = 12 \sqcup 34$ belongs to $\mathrm{cl}(\boldsymbol{v})$, it follows that $\mathrm{cl}(\boldsymbol{v}) = \boldsymbol{v} \cup \{1234\}$. Moreover, $1234 = 14 \sqcup 23$ in $\boldsymbol{\delta}_H$ with $14, 23 \notin \mathrm{cl}(\boldsymbol{v})$, thus $1234 \notin \mathrm{int}\,\mathrm{cl}(\boldsymbol{v})$, and thus, since $\boldsymbol{v}$ is open, it follows that $\mathrm{int}\,\mathrm{cl}(\boldsymbol{v}) = \boldsymbol{v}$, that is, $\boldsymbol{v}$ is regular open.

By definition, $H = 012345$ belongs to $\boldsymbol{v}$. We verify, by using Proposition 8.3, that $\boldsymbol{v}$ is a minimal neighborhood of H. For each $X \in \boldsymbol{v}$, we need to find $\boldsymbol{x} \in \mathcal{M}(H)$ such that $\boldsymbol{x} \cap \boldsymbol{v} = \{X\}$. If $X = H$, take $\boldsymbol{x} = \{H\}$. If $X = 123$, take $\boldsymbol{x} = \{123, 0, 45\}$. If $X = 134$, take $\boldsymbol{x} = \{134, 0, 25\}$. If $X = 1235$, take $\boldsymbol{x} = \{1235, 0, 4\}$. If $X = 1345$, take $\boldsymbol{x} = \{1345, 0, 2\}$. In all other cases, $H \setminus X$ belongs to $\boldsymbol{v}^{\mathsf{c}}$, so we can take $\boldsymbol{x} = \{X, H \setminus X\}$.

Since $\boldsymbol{v}$ is a minimal neighborhood of H, it is, *a fortiori*, a minimal element of the set of all regular open neighborhoods of H. In order to verify that $\boldsymbol{v}$ is join-irreducible in $\mathrm{Reg}_{\mathrm{op}}(\boldsymbol{\delta}_H, \mathrm{cl})$, it suffices to verify that H is irreducible in $\boldsymbol{v}$, that is, that there is no partition of the form $H = X \sqcup Y$ with $X, Y \in \mathrm{cl}(\boldsymbol{v})$. This can be easily checked on Table 1. $\square$

Even without invoking Proposition 8.3, it is easy to verify directly that $\boldsymbol{v}$ contains no clopen neighborhood of H. Suppose, to the contrary, that $\boldsymbol{a}$ is such a clopen neighborhood. Since $H = 12 \sqcup 0345$ with $H \in \boldsymbol{a}$ and $0345 \notin \boldsymbol{v}$ (thus $0345 \notin \boldsymbol{a}$), it follows from the openness of $\boldsymbol{a}$ that $12 \in \boldsymbol{a}$. Likewise, $34 \in \boldsymbol{a}$. Since $\boldsymbol{a}$ is closed, it follows that $1234 = 12 \sqcup 34$ belongs to $\boldsymbol{a}$, thus to $\boldsymbol{v}$, a contradiction.

Corollary 18.2. *The extended permutohedron $\mathsf{R}(H)$ is not the Dedekind-MacNeille completion of the permutohedron $\mathsf{P}(H)$.*

0	**1**	2	**3**	4	**5**
12345	02345	**01345**	01245	**01235**	01234

01	02 $\notin\boldsymbol{\delta}_H$	**03**	04 $\notin\boldsymbol{\delta}_H$	05 $\notin\boldsymbol{\delta}_H$
2345	**1345**	1245	**1235**	1234
12	13 $\notin\boldsymbol{\delta}_H$	14	15 $\notin\boldsymbol{\delta}_H$	23
0345	0245 $\notin\boldsymbol{\delta}_H$	**0235**	0234	**0145**
24 $\notin\boldsymbol{\delta}_H$	25	**34**	35 $\notin\boldsymbol{\delta}_H$	45
0135 $\notin\boldsymbol{\delta}_H$	**0134**	0125	0124	**0123**
012	**013**	014	015 $\notin\boldsymbol{\delta}_H$	023
345	245	**235**	234	**145**
024 $\notin\boldsymbol{\delta}_H$	025 $\notin\boldsymbol{\delta}_H$	034	035 $\notin\boldsymbol{\delta}_H$	045 $\notin\boldsymbol{\delta}_H$
135 $\notin\boldsymbol{\delta}_H$	**134**	**125**	124	**123**

TABLE 1. Nonempty proper members of $\boldsymbol{\delta}_H$; members of $\boldsymbol{v}$ boxed

19. A non-clopen minimal regular open neighborhood

For any positive integer n, the complete graph $\mathcal{K}_n$ is a block graph, hence $\mathsf{R}(\mathcal{K}_n)$ is the Dedekind-MacNeille completion of $\mathsf{P}(\mathcal{K}_n)$. (This follows from Corollary 16.11; however, invoking Lemma 14.4 is even easier.) While the corresponding results for transitive binary relations (cf. Santocanale and Wehrung [36]) and for join-semilattices (cf. Corollary 9.2) are obtained *via* the stronger result that every *open* set is a union of clopen sets, we shall prove in this section that for n large enough (i.e., $n \geq 7$), not even every *regular open* subset of $\boldsymbol{\delta}_{\mathcal{K}_n}$ is a union of clopen sets.

In what follows, we shall label the vertices of $G = \mathcal{K}_7$ from 0 to 6, and describe the construction of a regular open subset of $\boldsymbol{\delta}_G$ that is not a set-theoretical union of clopen sets. It can be proved, after quite lengthy calculations, that 7 is the smallest integer with that property: for each $n \leq 6$, every regular open subset of $\boldsymbol{\delta}_{\mathcal{K}_n}$ is a set-theoretical union of clopen sets.

Theorem 19.1. *There exists a minimal neighborhood $\boldsymbol{u}$ of $G = \mathcal{K}_7$, which is, in addition, regular open, and such that*

$$X \in \boldsymbol{u} \;\Leftrightarrow\; G \setminus X \notin \boldsymbol{u}\,, \quad \textit{for any } X \subseteq G\,, \tag{19.1}$$

together with $Q_0, Q_1, Q_2 \in \boldsymbol{u}$ such that $G = Q_0 \sqcup Q_1 \sqcup Q_2$. In particular, $\boldsymbol{u}$ contains no clopen neighborhood of G.

Proof. As in Section 18, we skip the braces in denoting subsets of G, so, for instance, 134 is short for $\{1, 3, 4\}$. The Q_i are defined by

$$Q_0 = 012\,, \quad Q_1 = 34\,, \quad Q_2 = 56\,.$$

We group complementary pairs of nonempty subsets of G on Table 2, and we box and boldface, on that table, the elements of $\boldsymbol{u} \setminus \{0123456\}$.

0	1	2	**3**	4	**5**	6
123456	**023456**	**013456**	012456	**012356**	012346	**012345**
01	**02**	**03**	**04**	**05**	**06**	12
23456	13456	12456	12356	12346	12345	**03456**
13	14	**15**	16	**23**	24	25
02456	**02356**	02346*	**02345**	01456*	**01356**	**01346**
26	**34**	**35**	36	45	46	**56**
01345	01256*	01246	**01245**	**01236**	**01235**	01234*
012	**013**	**014**	**015**	016	**023**	024
3456*	2456	2356*	2346	**2345**	1456	**1356**
025	**026**	**034**	**035**	**036**	**045**	**046**
1346	1345*	1256	1246	1245	1236	1235*
056	123	124	125	126	134	**135**
1234	**0456**	**0356**	**0346**	**0345**	**0256**	0246
136	145	146	156	234	**235**	236
0245	**0236**	**0235**	**0234**	**0156**	0146	**0145**
245	246	256	**345**	346	**356**	456
0136	**0135**	**0134**	0126	**0125**	0124	**0123**

TABLE 2. Proper subsets of G; members of $\boldsymbol{u}$ boxed, members of $\operatorname{cl}(\boldsymbol{u}) \setminus \boldsymbol{u}$ marked by asterisks

Hence, $\boldsymbol{u} = \{0, 023456, 013456, 3, \dots, 356, 0123, 0123456\}$. It has 64 elements. It is obvious, on the table, to see that $Q_i \in \boldsymbol{u}$ for each $i \in \{0, 1, 2\}$. It is an elementary, although quite horrendous, task to verify that $\boldsymbol{u}$ is open and that the subset

$$\boldsymbol{a} = \boldsymbol{u} \cup \{01234, 1235, 1345, 02346, 01256, 2356, 01456, 3456\}$$

is closed. Each element of $\boldsymbol{a} \setminus \boldsymbol{u}$ is marked by an asterisk on Table 2. Each of the decompositions

$$\begin{aligned} 01234 &= 012 \sqcup 34\,, & 1235 &= 15 \sqcup 23\,,\\ 1345 &= 15 \sqcup 34\,, & 02346 &= 23 \sqcup 046\,,\\ 01256 &= 012 \sqcup 56\,, & 2356 &= 23 \sqcup 56\,,\\ 01456 &= 014 \sqcup 56\,, & 3456 &= 34 \sqcup 56 \end{aligned}$$

yields a partition of an element of $\boldsymbol{a} \setminus \boldsymbol{u}$ into elements of $\boldsymbol{u}$; whence $\boldsymbol{a} = \operatorname{cl}(\boldsymbol{u})$. On the other hand, each of the decompositions

$$\begin{aligned} 01234 &= 13 \sqcup 024\,, & 1235 &= 13 \sqcup 25\,,\\ 1345 &= 13 \sqcup 45\,, & 02346 &= 36 \sqcup 024\,,\\ 01256 &= 25 \sqcup 016\,, & 2356 &= 25 \sqcup 36\,,\\ 01456 &= 45 \sqcup 016\,, & 3456 &= 36 \sqcup 45 \end{aligned}$$

yields a partition of an element of $\boldsymbol{a} \setminus \boldsymbol{u}$ into members not belonging to $\operatorname{cl}(\boldsymbol{u})$; whence $\boldsymbol{u} = \operatorname{int}(\operatorname{cl}(\boldsymbol{u}))$, that is, $\boldsymbol{u}$ is regular open.

Since there are no complementary pairs of elements of $\boldsymbol{u}$, it follows from Proposition 8.3 that $\boldsymbol{u}$ is a minimal neighborhood of G in $\boldsymbol{\delta}_G$.

Since $\mathrm{cl}(\boldsymbol{u}) = \boldsymbol{a} \neq \boldsymbol{u}$, the subset $\boldsymbol{u}$ is not clopen. Since it is a minimal neighborhood of G, it contains no clopen neighborhood of G. □

The last statement of Theorem 19.1 can be proved directly, as follows. Let $\boldsymbol{a} \subseteq \boldsymbol{u}$ be clopen and suppose that $G \in \boldsymbol{a}$. From $Q_1 \sqcup Q_2 \notin \boldsymbol{u}$ it follows that $Q_1 \sqcup Q_2 \notin \boldsymbol{a}$, thus, as $G = Q_0 \sqcup Q_1 \sqcup Q_2$ and $\boldsymbol{a}$ is open, we get $Q_0 \in \boldsymbol{a}$. Likewise, $Q_1 \in \boldsymbol{a}$, so, as $\boldsymbol{a}$ is closed, $Q_0 \sqcup Q_1 \in \boldsymbol{a}$, that is, $01234 \in \boldsymbol{a}$, so $01234 \in \boldsymbol{u}$, a contradiction.

Remark 19.2. It is much easier to find, even in $\mathcal{K}_3$, an open set which is not a set-theoretical union of clopen sets: just take $\boldsymbol{u} = \{0, 1, 2, 012\}$.

20. Open problems

Problem 1. Is there a nontrivial lattice-theoretical identity (resp., quasi-identity) that holds in the Dedekind-MacNeille completion of the poset of regions of any central hyperplane arrangement? How about hyperplane arrangements in fixed dimension?

Any identity solving the first part of Problem 1 would, in particular, hold in any permutohedron $\mathsf{P}(n)$, which would solve another problem in Santocanale and Wehrung [35]. There is a nontrivial quasi-identity, in the language of lattices *with zero*, holding in the Dedekind-MacNeille completion L of any central hyperplane arrangement, namely pseudocomplementedness (cf. Corollary 6.4). However, we do not know about quasi-identities only in the language $(\vee, \wedge)$—we do not even know whether the quasi-identity (RSD_1), introduced in Section 10, holds in L. For a related example, see Example 10.7.

Our next problem asks for converses to Theorems 14.9 and 12.2.

Problem 2. Can every finite ortholattice, which is a bounded homomorphic image of a free lattice, be embedded into $\mathsf{R}(G)$ for some finite graph G (resp., into $\mathrm{Reg}\, S$ for some finite join-semilattice S)?

In Example 10.7, we find a finite convex geometry whose lattice of regular closed subsets contains a copy of L_4 (cf. Figure 10.2), thus fails the quasi-identity (RSD_1) introduced in Section 10. However, this example also contains a copy of L_1. This suggests the following problem.

Problem 3. Let (P, φ) be a finite convex geometry. If $\mathrm{Reg}(P, \varphi)$ fails semidistributivity, does it necessarily contain a copy of L_1?

By Theorem 10.4, Problem 3 has a positive answer for (P, φ) of poset type.

Several results of the present paper state the boundedness of lattices of regular closed subsets of certain closure spaces. Permutohedra (on finite chains) are such lattices (cf. Caspard [6]). The latter result was extended in Caspard, Le Conte de Poly-Barbut, and Morvan [7] to all finite *Coxeter lattices* (i.e., finite Coxeter groups with weak Bruhat ordering). Now to every finite Coxeter group is associated a so-called *Dynkin diagram*, which is a *tree*.

Problem 4. Relate an arbitrary finite Coxeter lattice L to the permutohedron on the corresponding Dynkin diagram (or a related graph). Can L be described as $\mathrm{Reg}(P, \varphi)$, for a suitable closure system (P, φ) of semilattice type?

For type A the answer to Problem 4 is well-known, as we just get the usual permutohedra. For other types, the situation looks noticeably more complicated. For example, let $\mathcal{D}_4$ be the graph arising from the Dynkin diagram of type D_4. Thus $\mathcal{D}_4$ is a star with three leaves (and one center). The lattice $\mathsf{P}(\mathcal{D}_4)$, represented on the left hand side of Figure 20.1, has 160 elements, while the Coxeter group arising from that diagram, whose weak Bruhat ordering is represented on the right hand side of Figure 20.1, has 192 elements. It can be shown that the smaller lattice is a homomorphic image of the larger one, but that the smaller lattice does not embed into the larger one.

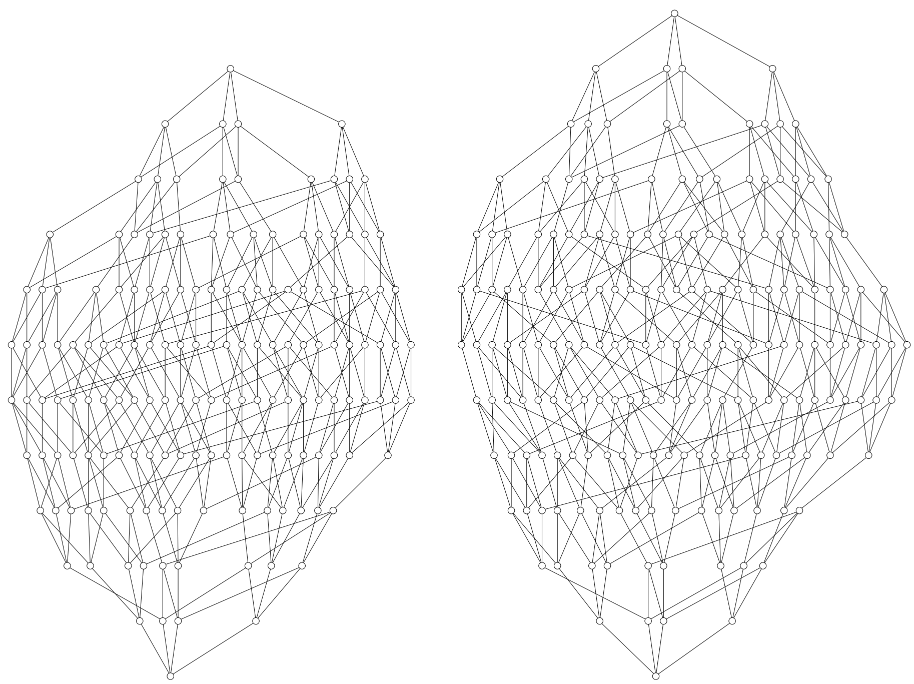

FIGURE 20.1. The lattice $\mathsf{P}(\mathcal{D}_4)$ and the Coxeter lattice of type D_4

Problem 5. Let G be an infinite graph. Is every element of $\mathsf{R}(G)$ a join of completely join-irreducible (resp., clopen) elements of $\mathsf{R}(G)$?

A counterexample to the analogue of Problem 5 for $\operatorname{Reg} S$, for a join-semilattice S, is given by Example 17.6. On the other hand, the analogue of Problem 5 for regular closed subsets of transitive binary relations has a positive answer (cf. Santocanale and Wehrung [36]). We do not even know the answer to Problem 5 for $G = \mathcal{K}_\omega$, the complete graph on a countably infinite vertex set. As evidence towards the negative, see Theorem 19.1.

Problem 6. Let G be a graph. If a set $\{\boldsymbol{a}_i \mid i \in I\}$ of clopen subsets of $\boldsymbol{\delta}_G$ has a join in $\mathsf{P}(G)$, is this join necessarily equal to $\operatorname{cl}\bigl(\bigcup(\boldsymbol{a}_i \mid i \in I)\bigr)$?

The finite case of Problem 6 is settled by Theorem 17.2.

Problem 7. Can one remove the well-foundedness assumption from the statement of Theorem 17.3? That is, is $\mathrm{Clop}(P,\varphi)$ a lattice iff $\mathrm{Clop}(P,\varphi) = \mathrm{Reg}(P,\varphi)$, for any closure space (P,φ) of semilattice type?

Example 17.5 suggests a negative answer to Problem 7, while Corollary 9.3 (dealing with join-semilattices) and Theorem 15.1 (dealing with graphs) both suggest a positive answer to Problem 7.

Problem 8. Is there a nontrivial lattice identity that holds in $\mathsf{R}(G)$ for every finite graph G (resp., in $\mathrm{Reg}\,S$ for every finite join-semilattice S)?

Some ideas about Problem 8 may be found in Santocanale and Wehrung [35].

Problem 9. Let G be an induced subgraph of a graph H. If $\mathsf{R}(H)$ is the Dedekind-MacNeille completion of $\mathsf{P}(H)$, is $\mathsf{R}(G)$ the Dedekind-MacNeille completion of $\mathsf{P}(G)$?

21. Acknowledgment

The authors are grateful to William McCune for his `Prover9-Mace4` program [31], to Ralph Freese for his lattice drawing program [15], and to the authors of the Graphviz framework, available online at http://www.graphviz.org/. Part of this work was completed while the second author was visiting the CIRM in March 2012. Excellent conditions provided by the host institution are greatly appreciated.

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