# EXTENDED LETTERPLACE CORRESPONDENCE FOR NONGRADED NONCOMMUTATIVE IDEALS AND RELATED ALGORITHMS 

ROBERTO LA SCALA*


#### Abstract

Let $K\left\langle x_{i}\right\rangle$ be the free associative algebra generated by a finite or a countable number of variables $x_{i}$. The notion of "letterplace correspondence" introduced in 22,23 for the graded (two-sided) ideals of $K\left\langle x_{i}\right\rangle$ is extended in this paper also to the nongraded case. This amounts to the possibility of modelizing nongraded noncommutative presented algebras by means of a class of graded commutative algebras that are invariant under the action of the monoid $\mathbb{N}$ of natural numbers. For such purpose we develop the notion of saturation for the graded ideals of $K\left\langle x_{i}, t\right\rangle$, where $t$ is an extra variable and for their letterplace analogues in the commutative polynomial algebra $K\left[x_{i j}, t_{j}\right]$, where $j$ ranges in $\mathbb{N}$. In particular, one obtains an alternative algorithm for computing inhomogeneous noncommutative Gröbner bases using just homogeneous commutative polynomials. The feasibility of the proposed methods is shown by an experimental implementation developed in the computer algebra system Maple and by using standard routines for the Buchberger algorithm contained in Singular.


## 1. Introduction

Many structures and models in mathematics and physics are based on noncommutative associative algebras that are given by a presentation with a finite or a countable number of generators. It is sufficient to mention the role of Hecke algebras or Temperley-Lieb ones in statistical mechanics and noncommutative geometry [11, 21], as well as the relevance of more classical enveloping algebras [13] or relatively free algebras defined for PI-algebras [15, 18]. A systematic way to control the consequences of the defining relations of a presented algebra consists in considering a well-ordering on the monomials of the free associative algebra (tensor algebra) which is compatible with multiplication and in computing what is modernly called a "Gröbner basis" or a "Gröbner-Shirshov basis". In fact, if it is possible to describe such a basis for the two-sided ideal of the relations satisfied by the generators of the associative algebra then a monomial linear basis is given for it, that is, one has some kind of generalization of the Poincaré-Birkhoff-Witt theorem.

Among the founding contributions to the theory of Gröbner bases for associative algebras one has to mention of course Bruno Buchberger [9 for the commutative case and the fundamental papers [2, 19, 29, 34, 35] for the non-commutative one. For nonassociative algebras one finds the roots of this algorithmic theory in the

[^0]pioneeristic work of Anatolii Shirshov [32, 33]. To explore some history and the wide range of applications of the modern theory of noncommutative Gröbner bases one can see, for instance, 4, 5, 6.

Starting with the papers [22, 23], through a substantial development of the concept of letterplace embedding contained in [14, a new approach for the theory and computation of noncommutative Gröbner bases has been proposed. The basic idea is to define a bijective correspondence between all graded two-sided ideals of the free associative algebra and a class of multigraded invariant ideals of a commutative polynomial algebra in double-indexed (letter-place) variables where shift operators act over the place indices. Such bijection provides also a correspondence between the homogeneous Gröbner bases of these ideals. It follows that the notion of Gröbner basis in the commutative and noncommutative case and the related algorithms can be considered as special instances of a general theory of Gröbner bases for commutative ideals that are invariant under the action of suitable algebra endomorphisms [8, 17, 23, 24]. Since the endomorphisms acting on the letterplace algebra are just shift operators, note that these results contribute also to the theory of algebras of finite difference polynomials [10, 26].

The goal of the present paper is to complete the work initiated in [22, 23] by proposing an extension of the letterplace correspondence to the nongraded case. This is obtained by analyzing in detail the concept of saturation for nongraded ideals of the free associative algebra and for their letterplace analogues. Note that the homogenization and saturation processes for the noncommutative case were previously introduced in [28, 30, 36] (see also [27]). From the extended letterplace correspondence one obtains an alternative algorithm to compute inhomogeneous noncommutative Gröbner bases by using homogeneous polynomials in commutative variables. In fact, these methods can be easily implemented in any commutative computer algebra system. Then, one has that the theory and methods for commutative and noncommutative Gröbner bases are unified whenever they are homogeneous or not. The feasibily of the proposed algorithms is shown in practice by means of an experimental implementation and a test set consisting of relevant classes of noncommutative algebras.

In Section 2 we describe the bijective correspondence between all (two-sided) ideals of the free associative algebra $F=K\langle X\rangle$ and the class of saturated graded ideals of the algebra $\bar{F}=K\langle\bar{X}\rangle$, where $\bar{X}=X \cup\{t\}$. If $\mathbb{N}=\{n \in \mathbb{Z} \mid n \geq 0\}$ and $\mathbb{N}^{*}=\mathbb{N} \backslash\{0\}$, for the letterplace algebras $P=K\left[X \times \mathbb{N}^{*}\right]$ and $\bar{P}=K\left[\bar{X} \times \mathbb{N}^{*}\right]$ we introduce the action of the monoid $(\mathbb{N},+)$ on the place indices of the variables and also a multigrading based on such indices. Then, one obtains a bijection between all $\mathbb{N}$-invariant ideals of $P$ and the class of saturated multigraded $\mathbb{N}$-ideals of $\bar{P}$. In Section 3 we review some key results proved in [22, 23]. Precisely, the letterplace ideals of $P$ are defined as $\mathbb{N}$-ideals generated by elements that are multilinear with respect to the place multigrading. Then, we introduce the letterplace correspondence as a bijection between all graded ideals of $F$ and the class of letterplace ideals of $P$. Note that under this correspondence a saturated ideal of $\bar{F}$ does not map into a saturated ideal of $\bar{P}$. It is necessary therefore to introduce the notion of $L$-saturation for letterplace ideals as a saturation property that involves only multilinear elements. By composing the above ideal correspondences, we finally obtain the extended letterplace correspondence which maps all ideals of $F$ into the class of $L$-saturated letterplace ideals of $\bar{P}$.

To develop effective methods for the $L$-saturation, in Section 4 we review the notion of monomial $\mathbb{N}$-ordering of $P$ and the constructin of an important class of such orderings that we call place $\mathbb{N}$-orderings. Then, we prove that they induce the graded right lexicographic ordering of the free associative algebra $F$. We review finally the theory of Gröbner $\mathbb{N}$-bases for ideals of $P$ that are invariant under shift operators and the related letterplace algorithm that computes homogeneous noncommutative Gröbner bases by using just elements of the commutative algebra $P$. In Section 5 we solve the problem of computing $L$-saturations of letterplace ideals by using Gröbner L-bases that are Gröbner $\mathbb{N}$-bases restricted to multilinear elements. The monomial orderings of $\bar{P}$ suitable for this task are place $\mathbb{N}$-orderings which are of elimination for the extra variables $t(j)$. As a byproduct one obtains finally a letterplace algorithm for computing inhomogeneous noncommutative Gröbner bases using homogeneous polynomials of $\bar{P}$. This method is illustrated in a detailed simple example in Section 6 and it is experimented in Section 7 for classes of presented associative algebras that are of interest in different areas of algebra. The experiments are performed by means of an implementation developed in the language of Maple and also by using standard routines for the Buchberger algorithm that are implemented in Singular [12. Conclusions about the letterplace approach to noncommutative computations and further developments of it are finally discussed in Section 8.

## 2. Homogenized and saturated ideals

We start studying the notion of homogenization and saturation for ideals of the free associative algebra. These concepts have been introduced essentially in [28, 30, 36 but we intend to clarify why commutators naturally arise in such constructions. Denote by $F=K\langle X\rangle$ the free associative algebra freely generated by a finite or a countable set $X=\left\{x_{1}, x_{2}, \ldots\right\}$. Clearly, one has the algebra grading $F=\bigoplus_{d \in \mathbb{N}} F_{d}$ where $F_{d}$ is the subspace of homogeneous polynomials of total degree $d$. Let $t$ be a new variable disjoint by $X$. Define $\bar{X}=X \cup\{t\}, \bar{F}=K\langle\bar{X}\rangle$. Consider the algebra endomorphism $\varphi: \bar{F} \rightarrow \bar{F}$ such that $x_{i} \mapsto x_{i}$ and $t \mapsto 1$ for all $i \geq 1$. Clearly $\varphi^{2}=\varphi$ and $F=\varphi(\bar{F})$. Then, the map $\varphi$ defines a bijective correspondence between all two-sided ideals of $F$ and two-sided ideals of $\bar{F}$ containing $\operatorname{ker} \varphi=\langle t-1\rangle$. In what follows, all the ideals of the algebras $F, \bar{F}$ are assumed two-sided ones.

Definition 2.1. Denote by $C$ the largest graded ideal contained in $\operatorname{ker} \varphi$, that is, the ideal $C$ is generated by all homogeneous elements $f \in \bar{F}$ such that $\varphi(f)=0$.

Proposition 2.2. The ideal $C \subset \bar{F}$ is generated by the commutators $\left[x_{i}, t\right]=$ $x_{i} t-t x_{i}$, for any $i \geq 1$.

Proof. Let $f \in \bar{F}$ be a homogeneous element such that $\varphi(f)=0$. Since the commutators $\left[x_{i}, t\right]$ clearly belongs to $C$, we have to prove that $f$ is congruent to 0 modulo them. In fact, it is clear that $f$ is congruent to a homogeneous element $f^{\prime}=t^{d^{\prime}} \sum_{k} f_{k} t^{d-k}$ where $d^{\prime} \geq 0$ and $f_{k} \in F$ is homogeneous of degree $k$, for any $k$. Then $0=\varphi(f)=\varphi\left(f^{\prime}\right)=\sum_{k} f_{k}$ and hence $f_{k}=0$ for all $k$. We conclude that $f^{\prime}=0$.

We want now to define a bijective correspondence between all ideals of $F$ and some class of graded ideals of $\bar{F}$ containing $C$.

Definition 2.3. Let $I$ be any ideal of $F$. We define $I^{*} \subset \bar{F}$ to be the largest graded ideal contained in the preimage $\varphi^{-1}(I)$, that is, the ideal $I^{*}$ is generated by all homogeneous elements in $\varphi^{-1}(I)$. Clearly $C=0^{*} \subset I^{*}$. We call $I^{*}$ the homogenization of the ideal I.
Definition 2.4. Let $f \in F, f \neq 0$ and denote by $f=\sum_{k} f_{k}$ the decomposition of $f$ in its homogeneous components. We denote $\operatorname{deg}(f)=d=\max \{k\}$ and define $f^{*}=$ $\sum_{k} f_{k} t^{d-k}$. We call $\operatorname{deg}(f)$ the top degree of $f$ and $f^{*}$ its homogenization. Clearly $f^{*} \in \bar{F}$ is a homogeneous element such that $\operatorname{deg}\left(f^{*}\right)=\operatorname{deg}(f)$ and $\varphi\left(f^{*}\right)=f$.

Theorem 2.5. Let $I$ be an ideal of $F$. Then $I^{*}=\left\langle f^{*} \mid f \in I, f \neq 0\right\rangle+C$.
Proof. Denote $J=\left\langle f^{*} \mid f \in I, f \neq 0\right\rangle+C$. Clearly $J$ is a graded ideal of $\bar{F}$ such that $\varphi(J) \subset I$ and hence $J \subset I^{*}$. Let $g \in I^{*}$ be a homogeneous element and define $f=\varphi(g) \in I$. If $f=0$ then $g \in C \subset J$. Otherwise, denote $d=\operatorname{deg}(f)$ and $d^{\prime}=\operatorname{deg}(g)$. Since clearly $d^{\prime} \geq d$ one has that $g$ is congruent modulo $C$ to the element $t^{d^{\prime}-d} f^{*}$ and hence $g \in J$.

If $I \subset F$ is an ideal one has clearly that $\varphi\left(I^{*}\right)=I$. Moreover, if $J \subset \bar{F}$ is a graded ideal containing $C$ then in general $J \subset \varphi(J)^{*}$.
Definition 2.6. Let $J \subset \bar{F}$ be a graded ideal which contains $C$. Define $\operatorname{Sat}(J)=$ $\varphi(J)^{*}=\left\langle\varphi(f)^{*}\right| f \in J, f \notin C, f$ homogeneous $\rangle+C$. Then $\operatorname{Sat}(J) \subset \bar{F}$ is a graded ideal containing $J$ that we call the saturation of $J$.
Definition 2.7. Let $J \subset \bar{F}$ be a graded ideal containing $C$. We say that $J$ is saturated if $J$ coincides with its saturation $\operatorname{Sat}(J)$, that is, for any homogeneous element $f \in J, f \notin C$ one has that $\varphi(f)^{*} \in J$. If $I$ is an ideal of $F$ then its homogenization $I^{*}$ is clearly a saturated ideal.

Note that in [28] an equivalent definition of saturated ideal is named dh-closed. Then, a bijective correspondence is given between all ideals of $F$ and the saturated graded ideals of $\bar{F}$ containing $C$. One can characterize such ideals in the following way.
Theorem 2.8. Let $J \subset \bar{F}$ be a graded ideal containing $C$. Then $J$ is saturated if and only if $t f \in J$ with $f \in \bar{F}$ implies that $f \in J$.

Proof. Suppose that $J$ is saturated and let $t g \in J$ with $g \in \bar{F}$. Since $J$ is graded, we can assume that $g$ is homogeneous. Put $f=\varphi(g)=\varphi(t g)$. If $f=0$ then $g \in C \subset J$. Otherwise, since $J$ is saturated and $t g \in J$ we obtain that $f^{*} \in J$. Moreover, one has clearly that $g$ is congruent modulo $C$ to an element $t^{d} f^{*} \in J$ for some $d \geq 0$ and hence $g \in J$. Suppose now that $t g \in J$ implies $g \in J$ and let $g \in J, g \notin C$ be a homogeneous element. If $f=\varphi(g)$ then $g$ is congruent modulo $C \subset J$ to an element $t^{d} f^{*}$. We conclude that $t^{d} f^{*} \in J$ and therefore $f^{*} \in J$.

Corollary 2.9. Let $J \subset \bar{F}$ be a graded ideal containing $C$. Then, we have that $\operatorname{Sat}(J)=\left\{f \in \bar{F} \mid t^{i} f \in J\right.$, for some $\left.i \geq 0\right\}$.
Proof. Denote $J^{\prime}=\left\{f \mid t^{i} f \in J\right.$, for some $\left.i\right\}$. Let $g \in \bar{F}$ and $f \in J^{\prime}$, that is, $t^{i} f \in J$ for some $i$. Clearly $g t^{i} f \in J$ and also $t^{i} g f \in J$ since $C \subset J$. We conclude that $g f \in J^{\prime}$. With similar arguments one proves that $J^{\prime}$ is a graded ideal of $\bar{F}$ containing $J$. Moreover, by Theorem 2.8 it follows immediately that $J^{\prime}$ is a saturated ideal. Finally, we have clearly that $\varphi\left(J^{\prime}\right)=\varphi(J)$ and hence $J^{\prime}=\varphi\left(J^{\prime}\right)^{*}=\varphi(J)^{*}=\operatorname{Sat}(J)$.

We start now considering commutative polynomial algebras with the purpose of defining analogues of the above noncommutative constructions. Denote $\mathbb{N}^{*}=\mathbb{N} \backslash\{0\}$ and consider the product set $X\left(\mathbb{N}^{*}\right)=X \times \mathbb{N}^{*}$. For the elements of this set we make use of the notation $x_{i}(j)=\left(x_{i}, j\right)$, for all $i, j \geq 1$. Define $P=K\left[X\left(\mathbb{N}^{*}\right)\right]$ the polynomial algebra in all commuting variables $x_{i}(j)$. The algebra $P$ is called the letterplace algebra [14. Denote by $\operatorname{End}(P)$ the monoid of all algebra endomorphisms of $P$. A monoid homomorphism $\rho: \mathbb{N} \rightarrow \operatorname{End}(P)$ is defined by putting $\rho(k)\left(x_{i}(j)\right)=$ $x_{i}(k+j)$, for all $k \geq 0$ and for any $i, j \geq 1$. We say therefore that $P$ is an $\mathbb{N}$ algebra. In fact, $P$ is a free $\mathbb{N}$-algebra generated by the set $X(1)=\left\{x_{i}(1) \mid i \geq 1\right\}$ (see [24, 17]). We make use of the notation $k \cdot f=\rho(k)(f)$, for all $k \geq 0$ and for any $f \in P$. Note finally that $\rho$ together with all $\rho(k)$ are injective maps. An ideal $I \subset P$ is called an $\mathbb{N}$-invariant ideal or an $\mathbb{N}$-ideal if $\mathbb{N} \cdot I \subset I$. Clearly, we have the algebra grading $P=\bigoplus_{d \in \mathbb{N}} P_{d}$ where $P_{d}$ is the subspace of homogeneous polynomials of total degree $d$. The algebra $P$ has another natural multigrading defined as follows. If $m=x_{i_{1}}\left(j_{1}\right) \cdots x_{i_{d}}\left(j_{d}\right) \in \operatorname{Mon}(P)$ then we denote $\partial(m)=\mu=$ $\left(\mu_{k}\right)_{k \in \mathbb{N}^{*}}$ where $\mu_{k}=\#\left\{\alpha \mid j_{\alpha}=k\right\}$. If $P_{\mu} \subset P$ is the subspace spanned by all monomials of multidegree $\mu$ then $P=\bigoplus_{\mu} P_{\mu}$ is clearly a multigrading. Note that the multidegrees $\mu=\left(\mu_{k}\right)_{k \in \mathbb{N}^{*}}$ have finite support and one can define $|\mu|=\sum_{k} \mu_{k}$. Then, one has clearly that $P_{d}=\bigoplus_{|\mu|=d} P_{\mu}$, that is, the multihomogeneous elements are also homogeneous ones. Note that the multigrading is compatible with the $\mathbb{N}$ algebra structure on $P$. Precisely, if $\mu=\left(\mu_{k}\right)$ is a multidegree then we denote $i \cdot \mu=\left(\mu_{k-i}\right)_{k \in \mathbb{N}^{*}}$ where we put $\mu_{k-i}=0$ when $k-i<1$. Then, for all $i \geq 0$ and for any multidegree $\mu$ one has that $i \cdot P_{\mu} \subset P_{i \cdot \mu}$.

Define $\bar{P}=K\left[\bar{X}\left(\mathbb{N}^{*}\right)\right]$ and consider the $\mathbb{N}$-algebra endomorphism $\psi: \bar{P} \rightarrow \bar{P}$ such that $x_{i}(1) \mapsto x_{i}(1)$ and $t(1) \mapsto 1$ for all $i \geq 1$. Clearly, the map $\psi$ is idempotent and $P=\psi(\bar{P})$. Moreover, one has that the $\mathbb{N}$-ideal $\operatorname{ker}(\psi)=\langle t(1)-1\rangle_{\mathbb{N}}$ does not contain any multihomogeneous element different from zero. We define now a bijective correspondence between all $\mathbb{N}$-ideals of $P$ and some class of multigraded $\mathbb{N}$-ideals of $\bar{P}$.

Definition 2.10. Let $I$ be any $\mathbb{N}$-ideal of $P$. We define $I^{*} \subset \bar{P}$ to be the largest multigraded $\mathbb{N}$-ideal contained in the preimage $\psi^{-1}(I)$, that is, the ideal $I^{*}$ is generated by all multihomogeneous elements in $\psi^{-1}(I)$. We call $I^{*}$ the multihomogenization of the ideal $I$. Note that $0^{*}=0$.

Definition 2.11. Let $f \in P, f \neq 0$ and denote by $f=\sum_{\mu} f_{\mu}$ the decomposition of $f$ in its multihomogeneous components. We denote $\partial(f)=\nu=\left(\max _{\mu}\left\{\mu_{k}\right\}\right)_{k \in \mathbb{N}^{*}}$ and define $f^{*}=\sum_{\mu} f_{\mu} \prod_{k} t(k)^{\nu_{k}-\mu_{k}}$. We call $\partial(f)$ the top multidegree of $f$ and $f^{*}$ its multihomogenization. Clearly $f^{*} \in \bar{P}$ is a multihomogeneous element such that $\partial\left(f^{*}\right)=\partial(f)$ and $\psi\left(f^{*}\right)=f$. Moreover, one has that $(i \cdot f)^{*}=i \cdot f^{*}$, for all $i \geq 0$.

Theorem 2.12. Let $I$ be an $\mathbb{N}$-ideal of $P$. Then $I^{*}=\left\langle f^{*} \mid f \in I, f \neq 0\right\rangle$.
Proof. Denote $J=\left\langle f^{*} \mid f \in I, f \neq 0\right\rangle$. Clearly $J$ is a multigraded $\mathbb{N}$-ideal of $\bar{P}$ such that $\psi(J) \subset I$ and hence $J \subset I^{*}$. Let $g \in I^{*}$ be a multihomogeneous element and define $f=\psi(g) \in I$. Denote $\mu=\partial(f)$ and $\nu=\partial(g)$. Since clearly $\nu_{k} \geq \mu_{k}$ for all $k$, one has that $g=\prod_{k} t(k)^{\nu_{k}-\mu_{k}} f^{*}$ and hence $g \in J$.

If $I \subset P$ is an $\mathbb{N}$-ideal one has clearly that $\psi\left(I^{*}\right)=I$. Moreover, if $J \subset \bar{P}$ is a multigraded $\mathbb{N}$-ideal then in general $J \subset \psi(J)^{*}$.

Definition 2.13. Let $J \subset \bar{P}$ be a multigraded $\mathbb{N}$-ideal. Define Sat $(J)=\psi(J)^{*}=$ $\left\langle\psi(f)^{*}\right| f \in J, f$ multihomogeneous $\rangle$. Then $\operatorname{Sat}(J) \subset \bar{P}$ is a multigraded $\mathbb{N}$-ideal containing $J$ that we call the saturation of $J$.
Definition 2.14. Let $J \subset \bar{P}$ be a multigraded $\mathbb{N}$-ideal. We say that $J$ is saturated if $J$ coincides with its saturation $\operatorname{Sat}(J)$, that is, if $f \in J$ is a multihomogeneous element then $\psi(f)^{*} \in J$. If $I$ is an $\mathbb{N}$-ideal of $P$ then its multihomogenization $I^{*}$ is clearly a saturated ideal.

Then, a bijective correspondence is given between all $\mathbb{N}$-ideals of $P$ and the saturated multigraded $\mathbb{N}$-ideals of $\bar{P}$. One can characterize such ideals in the following way.

Theorem 2.15. Let $J \subset \bar{P}$ be a multigraded $\mathbb{N}$-ideal. Then $J$ is saturated if and only if $t(j) f \in J$ with $f \in \bar{P}, j \geq 1$ implies that $f \in J$.

Proof. Suppose that $J$ is saturated and let $t(j) g \in J$ with $g \in \bar{F}, j \geq 1$. Since $J$ is multigraded, we can assume that $g$ is multihomogeneous. Put $f=\psi(g)=\psi(t(j) g)$. Since $J$ is saturated and $t(j) g \in J$ we obtain that $f^{*} \in J$. Moreover, one has that $g=\prod_{k} t(k)^{\mu_{k}} f^{*} \in J$ for some multidegree $\mu$ and hence $g \in J$. Suppose now that $t(j) g \in J$ implies $g \in J$ and let $g \in J$ be a multihomogeneous element. If $f=\psi(g)$ then clearly $\prod_{k} t(k)^{\mu_{k}} f^{*}=g \in J$, for some $\mu$. We conclude that $f^{*} \in J$.

Corollary 2.16. Let $J \subset \bar{P}$ be a multigraded $\mathbb{N}$-ideal. Then, we have that $\operatorname{Sat}(J)=$ $\left\{f \in \bar{P} \mid \prod_{k} t(k)^{\mu_{k}} f \in J\right.$, for some multidegree $\left.\mu\right\}$.

Proof. Put $J^{\prime}=\left\{f \mid \prod_{k} t(k)^{\mu_{k}} f \in J\right.$, for some $\left.\mu\right\}$. Let $i \geq 0$ and $f \in J^{\prime}$, that is, $m f \in J$ for some $m=\prod_{k} t(k)^{\mu_{k}}$. Since $J$ is an $\mathbb{N}$-ideal, we have that $(i \cdot m)(i \cdot f)=i \cdot(m f) \in J$ where $(i \cdot m)=\prod_{k} t(i+k)^{\mu_{k}}$. We conclude that $i \cdot f \in J^{\prime}$. With similar arguments one proves that $J^{\prime}$ is a multigraded $\mathbb{N}$-ideal of $\bar{P}$ containing $J$. By Theorem 2.15 we obtain also that $J^{\prime}$ is a saturated ideal. Finally, we have clearly that $\psi\left(J^{\prime}\right)=\psi(J)$ and hence $J^{\prime}=\psi\left(J^{\prime}\right)^{*}=\psi(J)^{*}=\operatorname{Sat}(J)$.

## 3. Letterplace correspondence and $L$-Saturation

Consider the $K$-linear embedding $\iota: F \rightarrow P$ such that $\iota(m)=x_{i_{1}}(1) \cdots x_{i_{d}}(d)$ for all monomials $m=x_{i_{1}} \cdots x_{i_{d}} \in \operatorname{Mon}(F)$. This mapping was introduced in [14. Note that the map $\iota$ preserves the total degree. Then, define $V=\bigoplus_{d} V_{d}$ the graded subspace of $P$ which is the image of map $\iota$. For all $d \geq 0$, denote by $1^{d}$ the multidegree $\mu=\left(\mu_{k}\right)$ such that $\mu_{k}=1$ for $k \leq d$ and $\mu_{k}=0$ otherwise. Clearly one has that $V_{d}=P_{1^{d}}$.
Definition 3.1. Denote by $L=\bigcup_{d} V_{d}$ the set of multihomogeneous elements of $V$. We call such elements the multilinear elements of $P$.

There is a bijective correspondence between all graded ideals of $F$ and some class of multigraded $\mathbb{N}$-ideals of $P$. This class is defined as follows.

Definition 3.2. Let $J$ be an $\mathbb{N}$-ideal of $P$. We call $J$ a letterplace ideal or L-ideal or multilinear $\mathbb{N}$-ideal if $J=\langle J \cap L\rangle_{\mathbb{N}}$, that is, $J$ is $\mathbb{N}$-generated by multilinear elements. Clearly $J$ is a multigraded ideal.

The following key result has been proved in 22

Theorem 3.3. Let $I \subset F$ be a graded ideal and denote $J=\langle\iota(I)\rangle_{\mathbb{N}}$. Then $J \subset P$ is an L-ideal. Conversely, let $J \subset P$ be an L-ideal and denote $I=\iota^{-1}(J \cap V)$. Then $I \subset F$ is a graded ideal. Moreover, the mappings $I \mapsto J$ and $J \mapsto I$ define a bijective correspondence between graded ideals of $F$ and letterplace ideals of $P$. Hence, we call $J$ the letterplace analogue of $I$.

We assume now that the above result is extended to the algebras $\bar{F}, \bar{P}$. Then, we make use of notations $\bar{\iota}: \bar{F} \rightarrow \bar{P}, \bar{V}=\operatorname{Im} \bar{\iota}$ and $\bar{L}=\bigcup_{d} \bar{V}_{d}$. Consider the letterplace analogue $D$ of the ideal $C=0^{*}$. In other words, we have that $D \subset \bar{P}$ is the $\mathbb{N}$-ideal generated by the multilinear elements $\bar{\iota}\left(\left[x_{i}, t\right]\right)=x_{i}(1) t(2)-t(1) x_{i}(2)$, for all $i \geq 1$. Note that $D$ is not a saturated ideal. In fact, the ideal $D$ contains the element $t(1) f$, but not $f=x_{1}(1) x_{2}(2)-x_{2}(1) x_{1}(2)$. Moreover, its saturation $\operatorname{Sat}(D)$ is not an $L$-ideal, that is, this ideal is not generated by multilinear elements. For instance, the element $x_{1}(1) t(3)-t(1) x_{1}(3) \notin \bar{L}$ is contained in $\operatorname{Sat}(D)$. More generally, the letterplace analogue of a saturated ideal of $\bar{F}$ is not saturated and its saturation is not a letterplace ideal. This suggests that one needs a different notion of saturation for such analogues that are in bijective correspondence with all ideals of $F$. To motivate the following definition, note also that if $f$ and $t(j) f$ are multilinear elements then necessarily $j=\operatorname{deg}(f)+1$.
Definition 3.4. Let $J \subset \bar{P}$ be an L-ideal which contains $D$. We say that $J$ is $L$-saturated or multilinearly saturated if $t(d+1) f \in J$ with $f \in \bar{L}$ and $d=\operatorname{deg}(f)$ implies that $f \in J$.

Proposition 3.5. Let $J \subset \bar{P}$ be an L-ideal containing $D$. If we denote $\operatorname{Sat}_{L}(J)=$ $\langle f \in \bar{L}| \prod_{d<k \leq d^{\prime}} t(k) f \in J$, for some $\left.d^{\prime} \geq d=\operatorname{deg}(f)\right\rangle_{\mathbb{N}}$ then $\operatorname{Sat}_{L}(J)$ is an $L$-saturated letterplace ideal containing $J$. We call $\operatorname{Sat}_{L}(J)$ the $L$-saturation or multilinear saturation of $J$ and one has clearly that $\operatorname{Sat}_{L}(J) \subset \operatorname{Sat}(J)$.

Proof. By definition, one has that $J^{\prime}=\operatorname{Sat}_{L}(J)$ is an $L$-ideal that contains $J \supset D$. Denote $m_{l}=\prod_{0<k \leq l} t(k)$ and suppose $g\left(d \cdot m_{1}\right) \in J^{\prime}$ with $g \in \bar{V}_{d}$ and $d \geq 0$. It remains to prove that $g \in J^{\prime}$, that is, the ideal $J^{\prime}$ is $L$-saturated. By definition of $J^{\prime}$ we have that $g\left(d \cdot m_{1}\right)=\sum_{i} f_{i}\left(d_{i} \cdot g_{i}\right)$ with $f_{i} \in \bar{V}_{d_{i}}, g_{i} \in \bar{V}_{d-d_{i}+1}$ and $f_{i}\left(d_{i} \cdot m_{l_{i}}\right) \in J$, for some $l_{i} \geq 0$. If $l=\max \left\{l_{i}\right\}$ then the element $g\left(d \cdot m_{l+1}\right)=g\left(d \cdot m_{1}\right)\left((d+1) \cdot m_{l}\right)$ is congruent modulo $D \subset J$ to $\sum_{i} f_{i}\left(d_{i} \cdot m_{l}\right)\left(\left(d_{i}+l\right) \cdot g_{i}\right) \in J$ and therefore $g \in J^{\prime}$.
Theorem 3.6. Let $I \subset \bar{F}$ be an ideal containing $C$ which is saturated and denote by $J \subset \bar{P}$ the letterplace analogue of $I$ (hence $D \subset J$ ). Then $J$ is an $L$-saturated ideal.

Proof. Assume $g t(d+1) \in J$ with $g \in \bar{V}_{d}$, for some $d \geq 0$. Then, let $f \in \bar{F}_{d}$ such that $\bar{\iota}(f)=g$. We have that $\bar{\iota}(f t)=g t(d+1) \in J \cap \bar{V}$, that is, $f t \in I$ and therefore $f \in I$ since $C \subset I$ is a saturated ideal. We conclude that $g \in J$.

Proposition 3.7. Let $J \subset \bar{P}$ be a letterplace ideal containing $D$ which is Lsaturated and put $I=\bar{\iota}^{-1}(J \cap \bar{V}) \subset \bar{F}$. Clearly $C \subset I$ and one has that $I$ is a saturated ideal.

Proof. It is sufficient to reverse the argument of Theorem 3.6.
We obtain therefore a bijective correspondence between all ideals of $F$ and the class of $L$-saturated letterplace ideals of $\bar{P}$. We call this bijection the extended letterplace correspondence.

Definition 3.8. Let $I$ be any ideal of $F$ and denote by $J \subset \bar{P}$ the letterplace analogue of $I^{*} \subset \bar{F}$ (hence $C \subset I^{*}$ and $D \subset J$ ). We call $J$ the extended letterplace analogue of $I$. Clearly, one has that $J=\left\langle\iota\left(f^{*}\right) \mid f \in I, f \neq 0\right\rangle_{\mathbb{N}}+D$ and $I=$ $\varphi \bar{\iota}^{-1}(J \cap \bar{V})$.

With the notations of the above definition, by Theorem 3.6 we have that $J=$ $\operatorname{Sat}_{L}(J)$. Then, it is natural to ask what is the ideal Sat $(J)$ extending $J$.

Denote by $Q=K[X(1)]$ the polynomial algebra in the variables $x_{i}(1)\left(x_{i} \in X\right)$ and consider the natural algebra epimorphism $\eta: F \rightarrow Q$ such that $x_{i} \mapsto x_{i}(1)$, for all $i \geq 1$. Assume that $\mathbb{N}$ acts on $Q$ in the trivial way, that is, $j \cdot x_{i}(1)=x_{i}(1)$ for any $j \geq 0$. Then, one has the $\mathbb{N}$-algebra epimorphism $\theta: P \rightarrow Q$ such that $x_{i}(j) \mapsto x_{i}(1)$, for all $i, j \geq 1$. The kernel of $\theta$ is clearly the $\mathbb{N}$-ideal $E$ generated by the elements $x_{i}(1)-x_{i}(2)$, for all $i$. Note that $E=\psi(D)$ and hence $E^{*}=\operatorname{Sat}(D)$.

Theorem 3.9. Let $I$ be any ideal of $F$ and put $I^{\prime}=\theta^{-1} \eta(I)$. Clearly $I^{\prime} \subset P$ is an $\mathbb{N}$-ideal containing $E$. Denote by $J \subset \bar{P}$ the extended letterplace analogue of $I$. Then, we have that $\operatorname{Sat}(J)=I^{\prime *}$.

Proof. Since $J$ is a multigraded $\mathbb{N}$-ideal of $\bar{P}$, it is sufficient to show that $\psi(J)=I^{\prime}$. Consider any element $g^{\prime} \in I^{\prime}$. Clearly $g^{\prime}$ is congruent modulo $E=\operatorname{ker} \theta$ to an element $\eta(f) \in Q \subset P$, for some $f \in I$. If $\eta(f)=0$ then $g^{\prime} \in E=\psi(D)$ where $D \subset J$. Otherwise, we have that $f \neq 0$ and one can consider $f^{*} \in I^{*}$ and hence $g=\bar{\iota}\left(f^{*}\right) \in J$. It is clear that $\theta \psi(g)=\eta(f)$, that is, $\psi(g)$ is congruent modulo $E$ to the element $\eta(f)$. Then, $\psi(g)$ is congruent also to $g^{\prime}$, that is, $g^{\prime}=\psi(g)+h$ with $h \in E$. Since $E=\psi(D)$ and $D \subset J$, we conclude that $g^{\prime} \in \psi(J)$. With similar arguments one proves also that $\psi(J) \subset I^{\prime}$.

Assume now one wants to compute the extended letterplace analogue $J \subset \bar{P}$ of any ideal $I \subset F$. If $I$ is given by a generating set $G$, we may form the graded ideal $I^{\prime}=C+\left\langle f^{*} \mid f \in G\right\rangle \subset \bar{F}$ and then its letterplace analogue $J^{\prime} \subset \bar{P}$. One has clearly that $\operatorname{Sat}\left(I^{\prime}\right)=I^{*}$ and $\operatorname{Sat}_{L}\left(J^{\prime}\right)=J$. It is well know that for the commutative case [3, 16] a standard tool to compute saturation consists in performing Gröbner bases with respect to appropriate monomial orderings. Aiming to have a similar method for $L$-saturation, in the next section we review the Gröbner bases theory for letterplace ideals that has been introduced in [22, 23].

## 4. Gröbner $\mathbb{N}$-Bases of letterplace ideals

Since letterplace ideals are a special class of $\mathbb{N}$-ideals, a first step consists in introducing monomial orderings for the polynomial algebra $P$ that are compatible with the action of $\mathbb{N}$. Owing to the Higman's Lemma, one can provide $P=K\left[X\left(\mathbb{N}^{*}\right)\right]$ by monomial orderings even if the set $X\left(\mathbb{N}^{*}\right)$ is infinite. For that purpose, this lemma can be stated in the following way (see for instance [1, Corollary 2.3).

Proposition 4.1. Let $\prec$ be a total ordering on $M=\operatorname{Mon}(P)$ such that
(i) $1 \preceq m$ for all $m \in M$;
(ii) $\prec$ is compatible with multiplication on $M$, that is, if $m \prec n$ then $t m \prec t n$, for any $m, n, t \in M$.
If $\prec$ induces $a$ well-ordering on the variables set $X\left(\mathbb{N}^{*}\right) \subset M$ then $\prec$ is also $a$ well-ordering on $M$ and hence it is a monomial ordering of $P$.

We can easily assign well-orderings to the countable set $X\left(\mathbb{N}^{*}\right)$ which is in bijective correspondence to $\mathbb{N}^{2}$. Note now that the monoid $\mathbb{N}$ stabilizes the variables set $X\left(\mathbb{N}^{*}\right)$ and hence the monomials set $M$. We have then the following notion.
Definition 4.2. Let $\prec$ be a monomial ordering of $P$. We call $\prec a$ (monomial) $\mathbb{N}$-ordering of $P$ if $m \prec n$ implies that $i \cdot m \prec i \cdot n$, for all $m, n \in M$ and $i \geq 0$.

One defines a main class of $\mathbb{N}$-orderings of $P$ in the following way. Denote $P(j)=$ $K\left[x_{i}(j) \mid i \geq 1\right]$ and put $M(j)=\operatorname{Mon}(P(j))$. Clearly $P=\bigotimes_{j \geq 1} P(j)$, that is, all monomials $m \in M$ can be factorized as $m=m_{j_{1}} \cdots m_{j_{k}}$, where $m_{j_{s}} \in M\left(j_{s}\right)$ and $j_{1}>\ldots>j_{k}$. Let $\rho: \mathbb{N} \rightarrow \operatorname{End}(P)$ be the monoid homomorphism corresponding to the action of $\mathbb{N}$ over $P$. For any $j \geq 0$, one has that the map $\rho(j)$ defines an isomorphism between the monoids $M(1), M(j+1)$ and hence between the algebras $P(1), P(j+1)$.
Definition 4.3. Let $\prec$ be any monomial ordering of the subalgebra $P(1) \subset P$ and extend it to all subalgebras $P(j+1)(j \geq 0)$ by the isomorphisms $\rho(j)$. In other words, we put $j \cdot m \prec j \cdot n$ if and only if $m \prec n$, for any $m, n \in M(1)$. Then, for all $m, n \in M, m=m_{j_{1}} \cdots m_{j_{k}}, n=n_{j_{1}} \cdots n_{j_{k}}$ with $j_{1}>\ldots>j_{k}$, we define $m \prec^{\prime} n$ if and only if $m_{j_{s}}=n_{j_{s}}$ and $m_{j_{t}} \prec n_{j_{t}}$, for some $1 \leq t \leq k$ and for all $1 \leq s<t$. By Proposition 3.7 in [24] one has that $\prec^{\prime}$ is a monomial $\mathbb{N}$-ordering that we call place $\mathbb{N}$-ordering of $P$ induced by a monomial ordering of $P(1)$.

Note that if $X$ is finite then $P(1)$ is a polynomial algebra in a finite number of variables whose monomial orderings were classified in 31. If $X$ is infinite, by Proposition 4.1 we have that the algebra $P(1)$ can be endowed anyway with monomial orderings provided that $x_{1}(1) \prec x_{2}(1) \prec \ldots$..

An important feature of the place $\mathbb{N}$-orderings is that they are compatible with some special grading of $P$ which is in turn compatible with the action of $\mathbb{N}$. Denote $\hat{\mathbb{N}}=\{-\infty\} \cup \mathbb{N}$.
Definition 4.4. Let $\mathrm{w}: M \rightarrow \hat{\mathbb{N}}$ be the unique mapping such that
(i) $\mathrm{w}(1)=-\infty$;
(ii) $\mathrm{w}(m n)=\max (\mathrm{w}(m), \mathrm{w}(n))$, for any $m, n \in M$;
(iii) $\mathrm{w}\left(x_{i}(j)\right)=j$, for all $i, j \geq 1$.

We call w the weight function of $P$. If $P_{(i)} \subset P$ is the subspace spanned by all monomials of weight $i$ then $P=\bigoplus_{i \in \hat{\mathbb{N}}} P_{(i)}$ is grading of $P$ over the idempotent commutative monoid ( $\hat{\mathbb{N}}$, max). Clearly, one has that $i \cdot P_{(j)} \subset P_{(i+j)}$, for all $i, j$.
Definition 4.5. Let $\prec$ be a monomial $\mathbb{N}$-ordering of $P$. We say that $\prec$ is a weighted ordering if $\mathrm{w}(m)<\mathrm{w}(n)$ implies that $m \prec n$, for all $m, n \in M$.

By Proposition 5.11 in [24] one has that all place $\mathbb{N}$-orderings are weighted ones. Note also that for multilinear monomials $m \in M \cap L$ one has that $\mathrm{w}(m)=\operatorname{deg}(m)$.
Definition 4.6. Let $\prec$ be a well-ordering of $W=\operatorname{Mon}(F)$. We call $\prec a$ monomial ordering of $F$ if $m \prec n$ implies that $u m v \prec u n v$, for all $m, n, u, v \in W$. In particular, we say that $\prec$ is a graded ordering if $\operatorname{deg}(m)<\operatorname{deg}(n)$ implies that $m \prec n$, for any $m, n \in W$.
Theorem 4.7. Let $\prec$ be a weighted $\mathbb{N}$-ordering of $P$ and define a total ordering $\prec^{\prime}$ of $W$ by putting $m \prec^{\prime} n$ if and only if $\iota(m) \prec \iota(n)$, for all $m, n \in W$. Then, the ordering $\prec^{\prime}$ is a graded monomial ordering of $F$ that we call induced by $\prec$.

Proof. It is clear that $\prec^{\prime}$ is a well-ordering since the same holds for the restriction of $\prec$ to $M \cap L$. Let $m^{\prime}, n^{\prime}, u^{\prime}, v^{\prime} \in W$ and denote by $m, n, u, v \in M \cap L$ their images under the map $\iota$. If $\operatorname{deg}\left(m^{\prime}\right)<\operatorname{deg}\left(n^{\prime}\right)$ then $\mathrm{w}(m)<\mathrm{w}(n)$ and hence $m \prec n$, that is, one has that $m^{\prime} \prec^{\prime} n^{\prime}$. Assume now $m^{\prime} \prec^{\prime} n^{\prime}$. If $\operatorname{deg}\left(m^{\prime}\right)<\operatorname{deg}\left(n^{\prime}\right)$ we have that $\operatorname{deg}\left(u^{\prime} m^{\prime} v^{\prime}\right)<\operatorname{deg}\left(u^{\prime} n^{\prime} v^{\prime}\right)$ and hence $u^{\prime} m^{\prime} v^{\prime} \prec^{\prime} u^{\prime} n^{\prime} v^{\prime}$. If $d^{\prime}=\operatorname{deg}\left(m^{\prime}\right)=\operatorname{deg}\left(n^{\prime}\right)$ and $d=\operatorname{deg}\left(u^{\prime}\right)$ one obtains that $d \cdot m \prec d \cdot n$ since $\prec$ is an $\mathbb{N}$-ordering. We conclude that $\iota\left(u^{\prime} m^{\prime} v^{\prime}\right)=u(d \cdot m)\left(\left(d+d^{\prime}\right) \cdot v\right) \prec u(d \cdot n)\left(\left(d+d^{\prime}\right) \cdot v\right)=\iota\left(u^{\prime} n^{\prime} v^{\prime}\right)$, that is, we have that $u^{\prime} m^{\prime} v^{\prime} \prec^{\prime} u^{\prime} n^{\prime} v^{\prime}$.

The above result implies that a class of graded monomial orderings of $F=K\langle X\rangle$ can be obtained from the class of weighted $\mathbb{N}$-orderings of $P$ by restriction to $L$. In particular, one has the following result.

Corollary 4.8. Let $\prec$ be any monomial ordering of $P(1)$ and extend it to a place $\mathbb{N}$ ordering of $P$. Moreover, denote by $\prec^{\prime}$ the graded monomial ordering of $F$ induced by $\prec$ according to Theorem 4.7. Then $\prec^{\prime}$ is the graded right lexicographic order, that is, for any $m=x_{i_{1}} \cdots x_{i_{k}}, n=x_{j_{1}} \cdots x_{j_{k}} \in W$ one has $m \prec^{\prime} n$ if and only if $k<l$ or $k=l, i_{s}=j_{s}$ and $i_{t}<j_{t}$, for some $1 \leq t \leq k$ and for all $t<s \leq k$.

Proof. Note that if $X$ is an infinite set then necessarily $x_{1}(i) \prec x_{2}(i) \prec \ldots$ and $x_{1} \prec^{\prime} x_{2} \prec^{\prime} \ldots$ because $\prec, \prec^{\prime}$ are well-orderings. Then, one has that $\iota(m)=$ $x_{i_{k}}(k) \cdots x_{i_{1}}(1), \iota(n)=x_{j_{k}}(k) \cdots x_{j_{1}}(1)$ and $\iota(m) \prec \iota(n)$ if and only if $x_{i_{s}}(s)=$ $x_{j_{s}}(s)$ and $x_{i_{t}}(t) \prec x_{j_{t}}(t)$, that is, $i_{s}=j_{s}$ and $i_{t}<j_{t}$, for some $1 \leq t \leq k$ and for all $t<s \leq k$.

We start now introducing Gröbner bases in the context of $\mathbb{N}$-ideals. Fix $\prec$ any $\mathbb{N}$-ordering of $P$. Let $f=\sum_{i} c_{i} m_{i} \in P$ with $m_{i} \in M, c_{i} \in K, c_{i} \neq 0$. We denote $\operatorname{lm}(f)=m_{k}=\max _{\prec}\left\{m_{i}\right\}, \operatorname{lc}(f)=c_{k}$ and $\operatorname{lt}(f)=\operatorname{lc}(f) \operatorname{lm}(f)$. Let $f, g \in P, f, g \neq 0$ and put $\operatorname{lt}(f)=c m, \operatorname{lt}(g)=d n$ with $m, n \in M$ and $c, d \in K$. If $l=\operatorname{lcm}(m, n)$ we define as usual the $S$-polynomial

$$
\operatorname{spoly}(f, g)=(l / c m) f-(l / d n) g
$$

Finally, if $G \subset P$ then we put $\operatorname{lm}(G)=\{\operatorname{lm}(f) \mid f \in G, f \neq 0\}$ and we denote by $\mathrm{LM}(G)$ the ideal of $P$ generated by $\operatorname{lm}(G)$. The following results were proved in 22, 23].
Proposition 4.9. Let $G \subset P$. Then $\operatorname{lm}(\mathbb{N} \cdot G)=\mathbb{N} \cdot \operatorname{lm}(G)$. In particular, if $I$ is an $\mathbb{N}$-ideal of $P$ then $\mathrm{LM}(I)$ is also $\mathbb{N}$-ideal.

Definition 4.10. Let $I \subset P$ be an $\mathbb{N}$-ideal and $G \subset I$. We call $G$ a Gröbner $\mathbb{N}$-basis of $I$ if $\operatorname{lm}(G)$ is an $\mathbb{N}$-basis of $\mathrm{LM}(I)$. In other words, $\mathbb{N} \cdot G$ is a Gröbner basis of $I$ as an ideal of $P$.
Definition 4.11. Let $f \in P, f \neq 0$ and $G \subset P$. If $f=\sum_{i} f_{i} g_{i}$ with $f_{i} \in P, g_{i} \in G$ and $\operatorname{lm}(f) \succeq \operatorname{lm}\left(f_{i}\right) \operatorname{lm}\left(g_{i}\right)$ for all $i$, we say that $f$ has a Gröbner representation with respect to $G$.

Theorem 4.12. Let $G$ be an $\mathbb{N}$-basis of an $\mathbb{N}$-ideal $I \subset P$. Then, $G$ is a Gröbner $\mathbb{N}$ basis of $I$ if and only if for all $f, g \in G, f, g \neq 0$ and for any $i \geq 0$, the $S$-polynomial $\operatorname{spoly}(f, i \cdot g)$ has a Gröbner representation with respect to $\mathbb{N} \cdot G$.

For the sake of completeness, we recall also the notion of Gröbner bases for ideals of the free associative algebra. For any subset $G \subset F$, define $\operatorname{lm}(G)$ and $\operatorname{LM}(G)$ as we have done for $P$.

Definition 4.13. Let $I \subset F$ be an ideal and $G \subset I$. We call $G$ a Gröbner basis of $I$ if $\operatorname{lm}(G)$ is a basis of $\operatorname{LM}(I)$. In other words, for any $f \in I, f \neq 0$ one has that $\operatorname{lm}(f)=u \operatorname{lm}(g) v$, for some $g \in G, g \neq 0$ and $u, v \in W$.

From now on, assume that $P$ is endowed with a weighted $\mathbb{N}$-ordering and $F$ is endowed with the induced graded monomial ordering. By abuse of notation, we will denote both these orderings as $\prec$. We mention finally the following key result proved in [22] for Gröbner $\mathbb{N}$-bases of letterplace ideals.

Theorem 4.14. Let $I \subset F$ be a graded ideal and denote by $J \subset P$ its letterplace analogue. If $G$ is a multihomogeneous Gröbner $\mathbb{N}$-basis of $J$ then $\iota^{-1}(G \cap L)$ is a homogeneous Gröbner basis of I.

This result together with the Theorem 4.12 implies the following algorithm for the computation of homogeneous noncommutative Gröbner bases which is alternative to the classical method developed in [19, 29, 34, 35].

```
Algorithm 4.1 HFREEGBASIS
    Input: \(H\), a homogeneous basis of a graded ideal \(I \subset F\).
    Output: \(\iota^{-1}(G)\), a homogeneous Gröbner basis of \(I\).
    \(G:=\iota(H)\);
    \(B:=\{(f, g) \mid f, g \in G\} ;\)
    while \(B \neq \emptyset\) do
        choose \((f, g) \in B\);
        \(B:=B \backslash\{(f, g)\} ;\)
        for all \(i \geq 0\) s.t. \(\operatorname{gcd}(\operatorname{lm}(f), \operatorname{lm}(i \cdot g)) \neq 1, \operatorname{lcm}(\operatorname{lm}(f), \operatorname{lm}(i \cdot g)) \in L\) do
            \(h:=\operatorname{ReducE}(\operatorname{spoly}(f, i \cdot g), \mathbb{N} \cdot G)\);
            if \(h \neq 0\) then
                    \(B:=B \cup\{(h, h),(h, k),(k, h), \mid k \in G\} ;\)
                    \(G:=G \cup\{h\} ;\)
            end if;
        end for;
    end while;
    return \(\iota^{-1}(G)\).
```

The function REDUCE is given by the following standard routine.

```
Algorithm 4.2 REDUCE
    Input: \(G \subset P\) and \(f \in P\).
    Output: \(h \in P\) such that \(f-h \in\langle G\rangle\) and \(h=0\) or \(\operatorname{lm}(h) \notin \operatorname{LM}(G)\).
    \(h:=f\);
    while \(h \neq 0\) and \(\operatorname{lm}(h) \in \operatorname{LM}(G)\) do
        choose \(g \in G, g \neq 0\) such that \(\operatorname{lm}(g)\) divides \(\operatorname{lm}(h)\);
        \(h:=h-(\operatorname{lt}(h) / \operatorname{lt}(g)) g ;\)
    end while;
    return \(h\).
```

Note that the iteration "for all $i \geq 0$ s.t. ..." in the procedure HFREEGBAsIS runs over a finite number of integers since condition $\operatorname{gcd}(\operatorname{lm}(f), \operatorname{lm}(i \cdot g)) \neq 1$ implies
that $i<\mathrm{w}(f)=\operatorname{deg}(f)$. Moreover, by multihomogeneity of the elements of $P$ involved in the computation, one has that the condition $\operatorname{lcm}(\operatorname{lm}(f), \operatorname{lm}(i \cdot g)) \in L$ is equivalent to require that the element $h=\operatorname{REDUCE}(\operatorname{spoly}(f, i \cdot g), \mathbb{N} \cdot G)$ is multilinear. Note finally that there are clearly a finite number of elements of the infinite set $\operatorname{lm}(\mathbb{N} \cdot G)=\mathbb{N} \cdot \operatorname{lm}(G)$ that may participate to such reduction. Even if one assumes that $X$ is a finite set, owing to non-Noetherianity of the free associative algebra $F=K\langle X\rangle$ or of the polynomial algebra $P=K\left[X\left(\mathbb{N}^{*}\right)\right]$ that has an infinite number of variables, we may have that the ideal $I \subset F$ is finitely generated but the leading monomial ideal $\mathrm{LM}(I)$ is not such, that is, the Gröbner bases of $I$ are infinite sets. In other words, we do not have general termination for the algorithm HFreeGBasis but termination is clearly provided for truncated computations up to some fixed degree, assuming that the ideal $I$ is finitely generated within such degree. For more details about the above algorithm we refer to [22, 23].

## 5. Gröbner $L$-bases and $L$-Saturation

The fact that the letterplace ideals are $\mathbb{N}$-generated by multilinear elements and Theorem 4.14 suggest that for such ideals one needs a notion of Gröbner basis that involves only multilinear elements.

Definition 5.1. Let $J$ be an L-ideal of $P$ and let $H \subset J \cap L$ be a subset of multilinear elements. If $H$ is an $\mathbb{N}$-basis of $J$ then we call $H$ a $L$-basis or multilinear $\mathbb{N}$-basis of $J$.
Definition 5.2. Let $J \subset P$ be an L-ideal and denote $\operatorname{LM}_{L}(J)=\langle\operatorname{lm}(f)| f \in$ $J \cap L\rangle_{\mathbb{N}}$. Let $G \subset J \cap L$ be a subset of multilinear elements. We call $G$ a Gröbner $L$-basis or Gröbner multilinear $\mathbb{N}$-basis of $J$ if $\operatorname{lm}(G)$ is an $\mathbb{N}$-basis of $\operatorname{LM}_{L}(J)$, that is, for all multilinear elements $f \in J \cap L$ one has that $i \cdot \operatorname{lm}(g)$ divides $\operatorname{lm}(f)$, for some $g \in G$ and $i \geq 0$. Clearly, all Gröbner L-bases are also L-bases of letterplace ideals.

If $I$ is a graded ideal of $F$ and $J \subset P$ is its letterplace analogue, by Theorem 4.14 one has that $G \subset J \cap L$ is a Gröbner $L$-basis of $J$ if and only if $\iota^{-1}(G)$ is a homogeneous Gröbner basis of $I$. In this sense, we may say that Gröbner $L$-bases are "letterplace analogues" of homogeneous Gröbner bases of the free associative algebra. Another interesting feature of Gröbner $L$-bases is that they can be obtained as complete multihomogeneous Gröbner $\mathbb{N}$-bases of suitable ideals.

Definition 5.3. Denote $N=\left\langle x_{i}(1) x_{j}(1) \mid i, j \geq 1\right\rangle_{\mathbb{N}} \subset P$. A monomial $m=$ $x_{i_{1}}\left(j_{1}\right) \cdots x_{i_{d}}\left(j_{d}\right) \in M$ is said to be normal modulo $N$ if $j_{1} \neq \ldots \neq j_{d}$. A polynomial $f \in P$ is in normal form modulo $N$ if all its monomials are normal modulo $N$.

Definition 5.4. Let $J \subset P$ be an $\mathbb{N}$-ideal containing $N$ and let $G \subset J$ be a subset of polynomials that are in normal form modulo $N$. We say that $G$ is a Gröbner $\mathbb{N}$-basis of $J$ modulo $N$ if $G \cup\left\{x_{i}(1) x_{j}(1) \mid i, j \geq 1\right\}$ is a Gröbner $\mathbb{N}$-basis of $J$.

Theorem 5.5. Let $J$ be an L-ideal of $P$ and let $G \subset J \cap L$. Then $G$ is a Gröbner L-basis of $J$ if and only if $G$ is a multihomogeneous Gröbner $\mathbb{N}$-basis of $J+N$ modulo $N$.

Proof. It is sufficient to prove that there is a Gröbner $\mathbb{N}$-basis of $J+N$ modulo $N$ whose elements are all multilinear. Then, consider to apply the Buchberger algorithm to an $L$-basis of $J$. By the product criterion and multihomogeneity of
the computation, it is clear that for the monomials $m=\operatorname{lcm}(\operatorname{lm}(f), i \cdot \operatorname{lm}(g))$ where $f, g$ are elements of the current $\mathbb{N}$-basis, one has that either $m$ is multilinear or $m \in N$.

Note that the above result, together with the comments after Definition5.2, provide another insight into the subtle relationships between noncommutative structures and their commutative analogues subjected to the action of the monoid $\mathbb{N}$.

Let us extend now the results of Section 4 and the previous ones to the algebras $\bar{F}, \bar{P}$. In what follows, assume the polynomial algebra $\bar{P}$ be endowed with a place $\mathbb{N}$ ordering which is induced by a monomial ordering of $\bar{P}(1)$ such that $t(1) \prec x_{1}(1) \prec$ $x_{2}(1) \prec \ldots$.. Therefore, the free associative algebra $\bar{F}$ is provided with the graded right lexicographic ordering such that $t \prec x_{1} \prec x_{2} \prec \ldots$.. One obtains immediately the following result.

Proposition 5.6. The elements $\bar{\iota}\left(\left[t, x_{i}\right]\right)=t(1) x_{i}(2)-x_{i}(1) t(2)(i \geq 1)$ are $a$ Gröbner L-basis of the L-ideal $D$, that is, the commutators $\left[t, x_{i}\right]$ are a homogeneous Gröbner basis of the graded ideal C. Then, a multilinear element $f \in \bar{L}$ is said to be in normal form modulo $D$ if it is such with respect to the above Gröbner L-basis.
Definition 5.7. Let $J \subset \bar{P}$ be an L-ideal which contains $D$ and let $G \subset J \cap \bar{L}$ be a subset of multilinear elements that are in normal form modulo $D$. We say that $G$ is a Gröbner $L$-basis of $J$ modulo $D$ if $G \cup\left\{\bar{\iota}\left(\left[t, x_{i}\right]\right) \mid i \geq 1\right\}$ is a Gröbner L-basis of $J$.

A natural characterization of the $L$-saturation of a letterplace ideal containing $D$ is the following one.

Lemma 5.8. Let $J \subset \bar{P}$ be an L-ideal containing D. Then, a Gröbner L-basis of $\operatorname{Sat}_{L}(J)$ modulo $D$ is given by the elements $\psi(f)^{*}$, for all $f \in J \cap \bar{L}$ that are in normal form modulo $D$.
Proof. It is sufficient to note that if $f \in \bar{L}$ is in normal form modulo $D$ then $\psi(f) \in V$ and $g=\psi(f)^{*} \in \bar{L}$. Moreover, it is clear that $f=\prod_{d<k \leq d^{\prime}} t(k) g$ where $\operatorname{deg}(f)=d^{\prime} \geq d=\operatorname{deg}(g)$.
Theorem 5.9. Let $J \subset \bar{P}$ be an L-ideal which contains $D$ and denote by $J^{\prime}=$ $\operatorname{Sat}_{L}(J)$ its $L$-saturation. Moreover, let $G$ be a Gröbner L-basis of $J$ modulo $D$. Then $G^{\prime}=\psi(G)^{*}=\left\{\psi(g)^{*} \mid g \in G\right\}$ is a Gröbner L-basis of $J^{\prime}$ modulo $D$.

Proof. Note that if $f^{\prime} \in \bar{L}$ is in normal form modulo $D$ and $f=\psi\left(f^{\prime}\right) \in V$ then $\operatorname{lm}\left(f^{*}\right)=\operatorname{lm}(f) \in M$ by definition of the monomial ordering of $\bar{P}$. Now, let $f^{\prime} \in J \cap \bar{L}$ be an element in normal form modulo $D$. Hence, there is $g^{\prime} \in G$ and $h \geq 0$ such that $h \cdot \operatorname{lm}\left(g^{\prime}\right)$ divides $\operatorname{lm}\left(f^{\prime}\right)$. Put $f=\psi\left(f^{\prime}\right), g=\psi\left(g^{\prime}\right)$ and $m_{i}=\prod_{0<j \leq i} t(j)$. Then, one has that $f^{\prime}=f^{*}\left(i \cdot m_{j}\right), g^{\prime}=g^{*}\left(k \cdot m_{l}\right)$ where $i=\operatorname{deg}(f), k=\operatorname{deg}(g)$ and $j, l \geq 0$. From $h \cdot \operatorname{lm}\left(g^{\prime}\right)$ divides $\operatorname{lm}\left(f^{\prime}\right)$ if follows that $h+k \leq i$ and hence $k \cdot \operatorname{lm}\left(g^{*}\right)$ divides $\operatorname{lm}\left(f^{*}\right)$. We conclude that $\psi(G)^{*}$ is a Gröbner $L$-basis of $\operatorname{Sat}_{L}(J)$ modulo D.

From the above result one obtains immediately an algorithm for computing Gröbner $L$-bases of $L$-saturated letterplace ideals of $\bar{P}$ containing $D$. This is especially relevant since such bases are in correspondence with homogeneous Gröbner bases of saturated ideals of $\bar{F}$ containing $C$. In fact, the Gröbner bases of any ideal $I \subset F$ are in correspondence with the homogeneous ones of its homogenization $I^{*}$.

Definition 5.10. A homogeneous element $f \in \bar{F}$ is said to be in normal form modulo $C$ if it is such with respect to the Gröbner basis $\left\{\left[t, x_{i}\right] \mid i \geq 1\right\}$. In other words, the multilinear element $\bar{\iota}(f) \in \bar{L}$ is in normal form modulo $D$.

Note that $\bar{\iota}\left(f^{*}\right)=\iota(f)^{*}$, for all $f \in F, f \neq 0$. Moreover, if $f \in \bar{F}$ is a homogeneous element in normal form modulo $C$ then we have also that $\iota(\varphi(f))=\psi(\bar{\iota}(f))$.
Definition 5.11. Let $I \subset \bar{F}$ be a graded ideal which contains $C$ and let $G \subset I$ be a subset of homogeneous elements which are in normal form modulo $C$. We say that $G$ is a Gröbner basis of $I$ modulo $C$ if $G \cup\left\{\left[t, x_{i}\right] \mid i \geq 1\right\}$ is a Gröbner basis of $I$. In other words, $\bar{\iota}(G) \subset \bar{L}$ is a Gröbner L-basis modulo $D$ of the letterplace analogue of $I$.

The following result can be found also in [28, 36].
Theorem 5.12. Let $I \subset F$ be any ideal and let $G$ be any Gröbner basis of $I$. Then $G^{*}=\left\{g^{*} \mid g \in G\right\}$ is a homogeneous Gröbner basis of $I^{*}$ modulo C. Moreover, one has that $\operatorname{lm}\left(G^{*}\right)=\operatorname{lm}(G)$.

Proof. Let $f^{\prime} \in I^{*}$ be a homogeneous element in normal form modulo $C$ and put $f=\varphi\left(f^{\prime}\right)$. Then $f^{\prime}=f^{*} t^{i}$ for some $i \geq 0$ and $\operatorname{lm}(f)=u \operatorname{lm}(g) v$ for some $g \in G$ and $u, v \in W$. Since $\operatorname{lm}\left(f^{*}\right)=\operatorname{lm}(f), \operatorname{lm}\left(g^{*}\right)=\operatorname{lm}(g)$ we conclude that $\operatorname{lm}\left(f^{\prime}\right)=$ $u \operatorname{lm}\left(g^{*}\right) v t^{i}$.

The above result and Theorem 5.9 imply an alternative algorithm to compute Gröbner bases of nongraded noncommutative ideals of the free associative algebra via homogeneous commutative computations in their extended letterplace analogues.

```
Algorithm 5.1 FreeGBasis
    Input: \(H\), a basis of an ideal \(I \subset F\).
    Output: \(\varphi\left(\bar{\iota}^{-1}(G)\right)\), a Gröbner basis of \(I\).
    \(G:=\bar{\iota}\left(H^{*} \cup\left\{\left[t, x_{i}\right] \mid i \geq 1\right\}\right) ;\)
    \(B:=\{(f, g) \mid f, g \in G\}\);
    while \(B \neq \emptyset\) do
        choose \((f, g) \in B\);
        \(B:=B \backslash\{(f, g)\} ;\)
        for all \(i \geq 0\) s.t. \(\operatorname{gcd}(\operatorname{lm}(f), \operatorname{lm}(i \cdot g)) \neq 1, \operatorname{lcm}(\operatorname{lm}(f), \operatorname{lm}(i \cdot g)) \in \bar{L}\) do
            \(h:=\operatorname{ReducE}(\operatorname{spoly}(f, i \cdot g), \mathbb{N} \cdot G) ;\)
                if \(h \neq 0\) then
                    \(h:=\psi(h)^{*}\)
                    \(B:=B \cup\{(h, h),(h, k),(k, h) \mid k \in G\} ;\)
                \(G:=G \cup\{h\} ;\)
            end if;
        end for;
    end while;
    return \(\varphi\left(\bar{l}^{-1}(G)\right)\).
```

Theorem 5.13. The algorithm FreegBasis is correct.
Proof. Let $J \subset \bar{P}$ be the extended letterplace analogue of $I \subset F$. At each step of the procedure FreeGBasis, the set $G$ is clearly an $L$-basis of an ideal $J^{\prime} \subset$ $\bar{P}$ containing $D$ such that $\operatorname{Sat}_{L}\left(J^{\prime}\right)=J$. Moreover, since the elements $\bar{\iota}\left(\left[t, x_{i}\right]\right)$ initially belong to $G$ we have the automatic normalization modulo $D$ of the elements obtained during the computation. Recall now that if $h \in \bar{L}$ is a multilinear element which is in normal form modulo $D$ then $h^{\prime}=\psi(h)^{*}$ divides $h$. This implies that if an S-polynomial can be reduced to zero by adding $h$ to the basis $G$, the same holds if we substitute $h$ with $h^{\prime}$. In case of termination, one has therefore that the set $G$ is a Gröbner $L$-basis of $J^{\prime}$ whose elements satisfy $h=\psi(h)^{*}$. By Theorem 5.9 we conclude that $J^{\prime}$ is $L$-saturated, that is, one has that $J^{\prime}=J$. Then $G^{\prime}=\bar{\iota}^{-1}(G)$ is homogeneous Gröbner basis of $I^{*}$, that is, $\varphi\left(G^{\prime}\right)$ is a Gröbner basis of $I$ by Theorem 5.12 .

Note that the above algorithm has neither general termination nor just termination up to some fixed degree $d$. The reason is that even if all computations are homogeneous, because of the saturation $h=\psi(h)^{*}$ that may decrease the current degree we cannot be sure at some suitable step that we will not get additional elements of degree $\leq d$ in the steps that will follow. This agrees with the well known fact that the word-problem is generally undecidable for nongraded associative algebras even if these are finitely generated. Nevertheless, if an ideal of the free associative algebra has a finite Gröbner basis then the algorithm FreegBasis is able to compute it in a finite number of steps.

Definition 5.14. Let $G \subset F$ be any subset. We call $G$ a minimal Gröbner basis if $\operatorname{lm}(G)$ is a minimal basis of $\mathrm{LM}(G)$, that is, $\operatorname{lm}(f) \neq u \operatorname{lm}(g) v$, for all $f, g \in G, f \neq g$ and for any $u, v \in W$.

By the choice of the monomial ordering of $\bar{F}$ and the property that the elements are kept in normal form modulo $C$ we have clearly that if $G^{\prime}$ is a minimal Gröbner basis of $I^{*}$ modulo $C$ then $\varphi\left(G^{\prime}\right)$ is also a mimimal Gröbner basis of $I$ since $\operatorname{lm}\left(G^{\prime}\right)=$ $\operatorname{lm}\left(\varphi\left(G^{\prime}\right)\right)$. This is the main advantage to compute on the fly the homogenization $I^{*}$ instead of working with any graded ideal $C \subset I^{\prime} \subset \bar{F}$ such that $\varphi\left(I^{\prime}\right)=I$. In fact, the ideal $I^{\prime}$ may have an infinite minimal Gröbner basis even if $I$ has a finite one and more generally this basis has elements in higher degrees than the basis of $I^{*}$. In other words, to compute without saturation is usually very inefficient. Such strategy is described in 36] in the context of classical algorithm and called "rabbit strategy" or "cancellation rule".

Note that actual computations with the algorithm FreeGBasis are performed by bounding the weight of the variables of $P$, that is, in a (Noetherian) polynomial algebra with a finite number of variables. This may result in an incomplete computation because some of the S-polynomials may be not defined owing to this bound. Since the S-polynomials $s=\operatorname{spoly}(f, i \cdot g)$ such that $\operatorname{gcd}(\operatorname{lm}(f), \operatorname{lm}(i \cdot g)) \neq 1$ that are considered in the procedure are multilinear elements, it is clear that $\mathrm{w}(s)=\operatorname{deg}(s) \leq 2 d-1$ where $d=\max \{\operatorname{deg}(f) \mid f \in G\}$ and $G$ is the current basis. We conclude that an actual computation is certified complete if the weight bound for the variables of $P$ is $\geq 2 d-1$, where $d$ is the maximal degree occuring in the output generators.

## 6. An illustrative Example

With the aim of showing a concrete computation with the algorithm FreeGBASIS, we present here a simple application to finitely presented (noncommutative) groups. Consider the symmetric group $S_{3}$ that can be presented, as a Coxeter group, in the following way

$$
S_{3}=\left\langle x, y \mid x^{2}=y^{2}=(x y)^{3}=1\right\rangle
$$

Define the free associative algebra $F=K\langle x, y\rangle$ and consider the elements

$$
f_{1}=x^{2}-1, f_{2}=y^{2}-1, f_{3}=(x y)^{3}-1 \in F
$$

Then, the group algebra $K S_{3}$ is clearly isomorphic to the quotient algebra $F / I$ where $I=\left\langle f_{1}, f_{2}, f_{3}\right\rangle$. A next step is to consider the free commutative $\mathbb{N}$-algebra $\bar{P}=K[x(1), y(1), t(1), x(2), y(2), t(2), \ldots]$ and to encode the noncommutative algebra $F / I$ in the letterplace way, that is, by defining the extended letterplace analogue $J \subset \bar{P}$ of the two-sided ideal $I \subset F$. As explained in the comments at the end of Section 3, we consider therefore the polynomials

$$
\begin{gathered}
d_{1}=\bar{\iota}([t, x])=t(1) x(2)-x(1) t(2), d_{2}=\bar{\iota}([t, y])=t(1) y(2)-y(1) t(2), \\
g_{1}=\bar{\iota}\left(f_{1}^{*}\right)=x(1) x(2)-t(1) t(2), g_{2}=\bar{\iota}\left(f_{2}^{*}\right)=y(1) y(2)-t(1) t(2), \\
g_{3}=\bar{\iota}\left(f_{3}^{*}\right)=x(1) y(2) x(3) y(4) x(5) y(6)-t(1) t(2) t(3) t(4) t(5) t(6) \in \bar{P}
\end{gathered}
$$

and we define the $L$-ideal $J^{\prime}=\left\langle d_{1}, d_{2}, g_{1}, g_{2}, g_{3}\right\rangle_{\mathbb{N}}$. In fact, one has that $J=$ $\operatorname{Sat}_{L}\left(J^{\prime}\right)$ and to perform this ideal operation one needs a Gröbner basis computation. Then, we fix the lexicographic monomial ordering on $\bar{P}$ with

$$
\ldots \succ x(2) \succ y(2) \succ t(2) \succ x(1) \succ y(1) \succ t(1)
$$

which is clearly a place $\mathbb{N}$-ordering inducing the graded right lexicographic ordering on $F$ with $x \succ y$. Then, to compute $\operatorname{Sat}_{L}\left(J^{\prime}\right)$ one has to reduce the multilinear S-polynomials between generators and performing the saturation of new generators arising by such reductions. At the end of computation, whenever $I$ admits a finite Gröbner basis, one obtains a (saturated) Gröbner $L$-basis $G \subset J$, that is, a Gröbner basis $\varphi\left(\bar{\iota}^{-1}(G)\right)$ of $I$, as prescribed by the algorithm FreeGBasis.

First of all, note that no multilinear S-polynomials are defined for the elements $d_{i}$. Moreover, it is easy to see that all multilinear S-polynomials between $d_{i}$ and any saturated element can be reduced to zero. For instance, one has that the S-polynomial

$$
\operatorname{spoly}\left(d_{1}, 1 \cdot g_{1}\right)=-x(1) t(2) x(3)+t(1) t(2) t(3)
$$

is reduced modulo $1 \cdot d_{1}$ to the element $x(1) x(2) t(3)-t(1) t(2) t(3)=g_{1} t(3)$ which clearly goes to zero modulo $g_{1}$.

Consider now the S-polynomial $\operatorname{spoly}\left(g_{1}, 1 \cdot g_{1}\right)=-t(1) t(2) x(3)+x(1) t(2) t(3)$ that can be clearly reduced to zero modulo the set $\mathbb{N} \cdot d_{1}$. In a similar way, one obtains that $\operatorname{spoly}\left(g_{2}, 1 \cdot g_{2}\right)$ reduces to zero. Then, we define the S-polynomial

$$
\operatorname{spoly}\left(g_{3}, 5 \cdot g_{2}\right)=-t(1) t(2) t(3) t(4) t(5) t(6) y(7)+x(1) y(2) x(3) y(4) x(5) t(6) t(7)
$$

which is reduced modulo the set $\mathbb{N} \cdot d_{2}$ to the element

$$
g_{4}^{\prime}=x(1) y(2) x(3) y(4) x(5) t(6) t(7)-y(1) t(2) t(3) t(4) t(5) t(6) t(7)
$$

This polynomial cannot be further reduced by the current $\mathbb{N}$-basis and hence one adds to this set the corresponding saturated element

$$
g_{4}=\psi\left(g_{4}^{\prime}\right)^{*}=x(1) y(2) x(3) y(4) x(5)-y(1) t(2) t(3) t(4) t(5)
$$

Then, we consider $\operatorname{spoly}\left(g_{3}, g_{4}\right)=y(1) t(2) t(3) t(4) t(5) y(6)-t(1) t(2) t(3) t(4) t(5) t(6)$ that can be reduced to zero modulo $\mathbb{N} \cdot\left\{d_{2}, g_{2}\right\}$. Consider now the next S-polynomial

$$
\operatorname{spoly}\left(g_{1}, 1 \cdot g_{4}\right)=-t(1) t(2) y(3) x(4) y(5) x(6)+x(1) y(2) t(3) t(4) t(5) t(6)
$$

By applying the set $\mathbb{N} \cdot\left\{d_{1}, d_{2}\right\}$ one obtains the element

$$
g_{5}^{\prime}=y(1) x(2) y(3) x(4) t(5) t(6)-x(1) y(2) t(3) t(4) t(5) t(6)
$$

and hence its saturation

$$
g_{5}=\psi\left(g_{5}^{\prime}\right)^{*}=y(1) x(2) y(3) x(4)-x(1) y(2) t(3) t(4)
$$

enters the current $\mathbb{N}$-basis. Then, one considers

$$
\operatorname{spoly}\left(g_{2}, 1 \cdot g_{5}\right)=-t(1) t(2) x(3) y(4) x(5)+y(1) x(2) y(3) t(4) t(5)
$$

which is reduced modulo $\mathbb{N} \cdot\left\{d_{1}, d_{2}\right\}$ to the element

$$
g_{6}^{\prime}=x(1) y(2) x(3) t(4) t(5)-y(1) x(2) y(3) t(4) t(5)
$$

and therefore the corresponding saturated element

$$
g_{6}=\psi\left(g_{6}^{\prime}\right)^{*}=x(1) y(2) x(3)-y(1) x(2) y(3)
$$

is appended to the $\mathbb{N}$-basis of the current $L$-ideal. All remaining S-polynomials reduce to zero which means that such ideal is $L$-saturated and therefore coincides with $J=\operatorname{Sat}_{L}\left(J^{\prime}\right)$. Note that the sequence of leading monomials of the polynomials $g_{i}$ is

$$
\operatorname{lm}\left(g_{1}\right)=x(1) x(2), \operatorname{lm}\left(g_{2}\right)=y(1) y(2), \operatorname{lm}\left(g_{3}\right)=x(1) y(2) x(3) y(4) x(5) y(6)
$$

$$
\operatorname{lm}\left(g_{4}\right)=x(1) y(2) x(3) y(4) x(5), \operatorname{lm}\left(g_{5}\right)=y(1) x(2) y(3) x(4), \operatorname{lm}\left(g_{6}\right)=x(1) y(2) x(3)
$$

and one has that $\operatorname{lm}\left(g_{6}\right)$ divides $\operatorname{lm}\left(g_{3}\right), \operatorname{lm}\left(g_{4}\right)$ and $1 \cdot \operatorname{lm}\left(g_{6}\right)$ divides $\operatorname{lm}\left(g_{5}\right)$. We conclude that a minimal Gröbner $L$-basis of the ideal $J$ is given by the set $G=$ $\left\{d_{1}, d_{2}, g_{1}, g_{2}, g_{6}\right\}$. Because $J \subset \bar{P}$ is exactly the extended letterplace analogue of the two-sided ideal $I \subset F$, we obtain that the set $\left\{x^{2}-1, y^{2}-1, x y x-y x y\right\}$ is a minimal Gröbner basis of $I$ with respect to graded right lexicographic ordering. In other words, we have found the canonical presentation

$$
S_{3}=\left\langle x, y \mid x^{2}=y^{2}=1, x y x=y x y\right\rangle
$$

showing that this group is a quotient of the braid group $B_{3}=\langle x, y \mid x y x=y x y\rangle$.

## 7. Implementations and testing

In this section we present an experimental implementation of the algorithm FreegBasis that has been developed in the language of Maple. We have obtained such implementation by modifying the algorithm SigmaGBasis introduced and experimented in [24] for the computation of Gröbner bases for finite difference ideals. Precisely, the letterplace computations are a special case of the ordinary difference ones. The main modifications to obtain FreeGBasis consist in adding the commutators $\left[t, x_{i}\right]$ to the elements introduced by homogenizing the initial noncommutative generators and in encoding all such elements in the letterplace way. Moreover, it is necessary to add to the procedure the "multilinearity criterion", that is, the condition $\operatorname{lcm}(\operatorname{lm}(f), \operatorname{lm}(i \cdot g)) \in \bar{L}$ when considering the S-polynomial $\operatorname{spoly}(f, i \cdot g)$. Finally, one has to implement the saturation of the elements that are obtained by reducing these S-polynomials. Note that according to Theorem5.5, the multilinearity criterion, which is essential to have tractable computations, can be obtained simply by adding the set of monomials $\mathcal{N}=\left\{x_{i}(1) x_{j}(1), t(1)^{2}, t(1) x_{i}(1)\right\}$
to the initial letterplace basis. This option is a useful trick if one wants to obtain the algorithm FreeGBasis by means of a standard implementation of the Buchberger procedure for commutative Gröbner bases.

To the purpose of studying the impact of different strategies used in FreeGBASIS, we have tested also two variants of this algorithm that are indicated in the examples with the suffix noc (no-criterion) and bas (basic). Both these variants make use of the saturation step $h:=\psi(h)^{*}$ since it is well known that mere homogenization of the initial generators is generally inefficient and may lead to an infinite Gröbner basis for the corresponding graded but not saturated ideal even if the input ideal have a finite one [36. The variant noc is obtained simply by suppressing the "shift criterion", that is, all S-polynomials spoly $(i \cdot f, j \cdot g)(i, j \in \mathbb{N})$ are considered for reduction. In the variant bas we suppress also the shifting of the new generators obtained from the reduction of the S-polynomials. In other words, one applies shift operators just to the input letterplace generators. This is correct since the different shifted versions of the generators that are necessary to the reduction process will be created in any case from the S-polynomials provided that the shift criterion is off. Up to the saturation step, the basic version can be obtained therefore by applying the Buchberger algorithm to the set of shifted elements of the initial letterplace basis joined to the set of monomials $\mathcal{N}$. We apply this trick on some examples where no saturation arises, in order to have computing times with standard routines of Singular that estimate approximately the speed-up that one may obtain moving from the Maple interpreter to the kernel of a computer algebra system. Note that an implementation of noncommutative Gröbner bases in the library LETTERPLACE of Singular is currently under development 25].

The monomial $\mathbb{N}$-ordering which is considered for the polynomial algebra $\bar{P}$ is the lexicographic ordering with

$$
\ldots \succ x_{1}(2) \succ \ldots \succ x_{n}(2) \succ t(2) \succ x_{1}(1) \succ \ldots \succ x_{n}(1) \succ t(1)
$$

that is clearly a place $\mathbb{N}$-ordering. Then, one has that the free associative algebra $\bar{F}=F\left\langle x_{1}, \ldots, x_{n}, t\right\rangle$ is endowed with the graded left lexicographic ordering with $x_{1} \succ \ldots \succ x_{n} \succ t$ by means of a reversing letterplace embedding $\iota^{\prime}: y_{1} \cdots y_{d} \mapsto$ $y_{d}(1) \cdots y_{1}(d)$, where $y_{k}=x_{i_{k}}$ or $y_{k}=t$.

The parameters that are considered in the experiments are the number of respectively input generators, output Gröbner generators, elements of a minimal Gröbner basis, pairs (S-polynomials) that are actually reduced and saturations steps. The last parameter is the computing time which is given in the format "minutes:seconds". Attached to the number of elements of a basis, after the letter "d" we indicate the maximum degree of such elements. Note that for the variant bas the input and output numbers count all the shifted versions of the basis elements. Moreover, for all variants we have that the pairs number includes the initial generators since we actually treat them as S-polynomials in order to interreduce. All examples have been computed with Maple 12 running on a server with a four core Intel Xeon at 3.16 GHz and 64 GB RAM.

| Example | gens | gb | min gb | pairs | sats | time |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| g3332d10 | 8 d 8 | 52 d 8 | 29 d 5 | 665 | 33 | $01: 15$ |
| g3332d10-noc | 8 d 8 | 52 d 8 | 29 d 5 | 1904 | 33 | $02: 29$ |
| g3332d10-bas | 60 d 8 | 232 d 8 | 29 d 5 | 1001 | 142 | $00: 35$ |
| g444d10 | 7 d 6 | 95 d 7 | 51 d 5 | 2657 | 40 | $11: 28$ |
| g444d10-noc | 7 d 6 | 95 d 7 | 51 d 5 | 4201 | 40 | $17: 43$ |
| g444d10-bas | 47 d 6 | 578 d 9 | 51 d 5 | 2396 | 342 | $06: 29$ |
| heckeAd15 | 10 d 3 | 27 d 11 | 27 d 11 | 237 | 0 | $00: 49$ |
| heckeAd15-noc | 10 d 3 | 27 d 11 | 27 d 11 | 1657 | 0 | $02: 53$ |
| heckeAd15-bas | 136 d 3 | 950 d 15 | 27 d 11 | 3902 | 0 | $51: 53$ |
| heckeDd15 | 10 d 3 | 16 d 7 | 16 d 7 | 89 | 0 | $00: 13$ |
| heckeDd15-noc | 10 d 3 | 16 d 7 | 16 d 7 | 783 | 0 | $00: 47$ |
| heckeDd15-bas | 137 d 3 | 250 d 11 | 16 d 7 | 1028 | 0 | $00: 57$ |
| heckeEd10 | 21 d 3 | 50 d 10 | 50 d 10 | 396 | 0 | $01: 00$ |
| heckeEd10-noc | 21 d 3 | 50 d 10 | 50 d 10 | 1528 | 0 | $02: 28$ |
| heckeEd10-bas | 184 d 3 | 630 d 10 | 50 d 10 | 2730 | 0 | $11: 29$ |
| lie5d25 | 3 d 2 | 26 d 25 | 26 d 25 | 26 | 0 | $02: 02$ |
| lie5d25-noc | 3 d 2 | 26 d 25 | 26 d 25 | 279 | 0 | $03: 08$ |
| lie5d25-bas | 72 d 2 | 348 d 25 | 26 d 25 | 348 | 0 | $02: 53$ |
| lie7d5 | 10 d 2 | 40 d 3 | 21 d 2 | 181 | 11 | $00: 41$ |
| lie7d5-noc | 10 d 2 | 40 d 3 | 21 d 2 | 368 | 11 | $00: 55$ |
| lie7d5-bas | 40 d 2 | 908 d 5 | 21 d 2 | 1982 | 106 | $>2 \mathrm{~h}$ |
| templieb8d8 | 34 d 3 | 64 d 8 | 64 d 8 | 581 | 0 | $01: 02$ |
| templieb8d8-noc | 34 d 3 | 64 d 8 | 64 d 8 | 1721 | 0 | $02: 18$ |
| templieb8d8-bas | 226 d 3 | 336 d 8 | 64 d 8 | 1879 | 0 | $01: 31$ |
| templieb9d9 | 43 d 3 | 85 d 9 | 85 d 9 | 920 | 0 | $03: 47$ |
| templieb9d9-noc | 43 d 3 | 85 d 9 | 85 d 9 | 3189 | 0 | $09: 41$ |
| templieb9d9-bas | 330 d 3 | 512 d 9 | 85 d 9 | 3418 | 0 | $05: 43$ |

The performance of the different variants of the algorithm FreeGBasis have been studied on a test set based on presentations of relevant classes of noncommutative algebras. The examples $g 3332$ and $g 444$ refer to the presentation of group algebras of presented groups. Precisely, such groups belong to the classes $G(l, m, n, q)=$ $\left\langle r, s \mid r^{l}, s^{m},(r s)^{n},[r, s]^{q}\right\rangle$ and $G(m, n, p)=\left\langle a, b, c \mid a^{m}, b^{n}, c^{p},(a b)^{2},(b c)^{2},(c a)^{2},(a b c)^{2}\right\rangle$. The examples hecke are the presentation of the Hecke algebras defined by the following Coxeter matrices

$$
A=\left(\begin{array}{llll}
1 & 3 & 2 & 3 \\
3 & 1 & 3 & 2 \\
2 & 3 & 1 & 3 \\
3 & 2 & 3 & 1
\end{array}\right) ; D=\left(\begin{array}{llll}
1 & 3 & 2 & 2 \\
3 & 1 & 3 & 3 \\
2 & 3 & 1 & 2 \\
2 & 3 & 2 & 1
\end{array}\right) ; E=\left(\begin{array}{lllll}
1 & 2 & 3 & 2 & 2
\end{array} 2\right.
$$

For the noncommutative polynomials defining the relations of the considered Hecke algebras, the quantity " $q$ " is assumed a parameter. The examples indicated as lie refer to the universal eveloping algebra of two indecomposable nilpotent Lie
algebras, namely

$$
\begin{aligned}
\text { lie5 }: & {\left[x_{1}, x_{2}\right]-x_{3},\left[x_{1}, x_{3}\right]-x_{4},\left[x_{2}, x_{5}\right]-x_{4} ; } \\
\text { lie7 }: & {\left[x_{1}, x_{2}\right]-x_{3},\left[x_{1}, x_{3}\right]-x_{4},\left[x_{1}, x_{4}\right]-x_{5},\left[x_{1}, x_{5}\right]-x_{6}, } \\
& {\left[x_{2}, x_{3}\right]-\frac{1}{2} x_{4}-\frac{1}{4} x_{5}+\frac{1}{8} x_{6}+\frac{1}{2} x_{7},\left[x_{2}, x_{4}\right]-\frac{1}{2} x_{5}-\frac{1}{4} x_{6}, } \\
& {\left[x_{2}, x_{5}\right]-x_{6},\left[x_{2}, x_{7}\right]-\frac{1}{2} x_{5}+\frac{1}{4} x_{6},\left[x_{3}, x_{4}\right]+\frac{1}{2} x_{6},\left[x_{3}, x_{7}\right]-\frac{1}{2} x_{6} . }
\end{aligned}
$$

Finally, the examples templieb8, templieb9 are the defining relations of the Temper-ley-Lieb algebras 21 respectively in 7 and 8 variables. The quantity " $\delta$ " used in the definition of such algebras is considered a parameter. In all the names of the tests, we indicate after the letter " d " the bounded degree within the computation is performed, that is, the maximal weight which is allowed for the variables of $P$.

The experiments show in a sufficiently clear way that the standard version of the algorithm FreeGBasis is generally the most efficient one. In fact, this procedure is able to decrease relevantly the number of S-polynomial reductions that are usually time-consuming. For instance, this emerges in a dramatic way for the example lie 7 . Note that for the examples $g 3332, g 444$ the basic variant results very competitive. This can be explained as the result of a low cost for the S-polynomial reductions (binomial generators) compared to the cost of applying shifting to letterplace polynomials. As previously remarked, a noncommutative Gröbner basis computed up to a fixed weight is certified complete if such bound is $\geq 2 d-1$, where $d$ is the maximal degree of the output generators. This happens for instance for the examples g3332, g444,heckeD and lie7. In particular, one obtains a computational proof that the ideal obtained by homogenizing the relations defining the Hecke algebra of the example hecke $D$ is a saturated one.

The computing times obtained with the implementation of FreegBasis in the language of Maple are useful to evaluate the possible different variants of this algorithm but they are not especially relevant when compared to other implementations of noncommutative Gröbner bases developed in the kernel of highly efficient computer algebra systems. Among these fast implementations, one has to mention the one of Magma [7] that makes use of a noncommutative version of the Faugere's F4 method. To the purpose of estimating the speed-up that may be achieved with a kernel implementation, we have computed the timings of some examples with the basic variant of FreeGBasis obtained by using the function "std" of Singular that implements the Buchberger algorithm. For the examples heckeAd15, heckeEd10 and teli9d9 such computing times are respectively $0.26,0.34$ and 1.01 sec . Keeping into account that the variant bas shows to be the less efficient, we believe that these data, together with all experiments performed in [22, 23, 25], indicate that letterplace approach is feasible for both the homogeneous and inhomogeneous case.

## 8. Conclusion and future directions

The theory and the methods proposed in this paper and in the previous ones [22, 23 proves that commutative and noncommutative Gröbner bases and the related algorithms can be unified in a general theory for Gröbner bases of commutative ideals that are invariant under the action of suitables algebra endomorphisms [8, 17, 23, 24.

We believe that this idea will have not only consequences in the development of new algorithmic methods but also in the reformulation in the letterplace language of structures and problems of noncommutative nature. It is sufficient in fact to
mention that the notion of Gröbner basis is a key ingredient for the description and computation of many fundamental invariants. The experiments shows that the letterplace methods are computationally practicable and hence new noncommutative tasks can be achieved now by commutative computer algebra systems. Future research directions may consist in investigating relationships between commutative and noncommutative invariants based on Gröbner bases and in developing optimized libraries for their computation.

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* Dipartimento di Matematica, Università di Bari, Via Orabona 4, 70125 Bari, Italy E-mail address: roberto.lascala@uniba.it


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