# Diagonal approximation and the cohomology ring of torus fiber bundles 

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#### Abstract

For a torus bundle $\left(S^{1} \times S^{1}\right) \rightarrow E \rightarrow S^{1}$, we construct a finite free resolution of $\mathbb{Z}$ over $\mathbb{Z}\left[\pi_{1}(E)\right]$ and compute the cohomology groups $H^{*}\left(\pi_{1}(E), \mathbb{Z}\right)$ and $H^{*}\left(\pi_{1}(E), \mathbb{Z}_{p}\right)$ for a prime $p$. We also construct a partial diagonal approximation for the resolution, which allows us to compute the cup product in $H^{*}\left(\pi_{1}(E), \mathbb{Z}\right)$ and $H^{*}\left(\pi_{1}(E), \mathbb{Z}_{p}\right)$.


## 1 Introduction

The calculation of the cohomology of a given group, including its ring structure, is relevant for many applications and is considered an interesting problem in its own right. For finite groups, for example, see 1]. Another example are the manifolds that are $K(\pi, 1)$ spaces, in special many closed 3 -manifolds. These manifolds are all $K(\pi, 1)$ spaces with the exception of the spherical ones and four 3-manifolds that are covered by $S^{2} \times \mathbb{R}$ (see [4), and so the cohomology rings of these manifolds coincide with the cohomology rings of their fundamental groups.

In addition, for a group $\pi$, the efficient calculation of the multiplicative structure given by the cup product in $H^{*}(\pi, M)$ for various coefficients $M$ is most easily accomplished by obtaining a diagonal approximation for a projective resolution of $\mathbb{Z}$ over $\mathbb{Z} \pi$. This was done, for example, by Tomoda and Zvengrowski in [5], where they computed projective resolutions and diagonal approximations for groups that arise as the fundamental groups of some Seifert 3-manifolds. In that case, the groups considered have 4-periodic cohomology and hence their cohomology groups are not isomorphic to the cohomology groups of the corresponding manifolds.

In this paper, we consider the problem of the computing the cohomology rings of the 3 -manifolds known as torus bundles. Since torus bundles are $K(\pi, 1)$ spaces, we compute their cohomology rings by constructing a finite free resolution of $\mathbb{Z}$ over $\mathbb{Z} G$, where $G$ is the fundamental group of the torus bundle, and then we construct a (partial) diagonal approximation for that resolution. Finally, we compute the multiplicative structure of $H^{*}(G, \mathbb{Z})$ and $H^{*}\left(G, \mathbb{Z}_{p}\right)$, where $p$ is prime. Although we only did the calculation for the trivial coefficients $\mathbb{Z}$ and $\mathbb{Z}_{p}$, the methods we employ also allow us to deal with other systems of coefficients, even non trivial ones. We didn't further explore the computation for non trivial coefficients in order to keep the length of the paper to a reasonable size, leaving it for specific applications.

We observe also that J. Hillman has recently determined the cohomology ring of the $S_{o l}{ }^{3}$-manifolds with coefficients $\mathbb{Z}_{2}$ using different methods. See [3].

A torus bundle is the total space $E$ of a fiber bundle with the torus $S^{1} \times S^{1}$ as the fiber and the circle $S^{1}$ as the base space. Given a torus bundle $\left(S^{1} \times S^{1}\right) \rightarrow E \rightarrow S^{1}$, the fundamental group of the total space $E$ fits in an exact sequence of groups

and that implies $\pi_{1}(E) \cong(\mathbb{Z} \oplus \mathbb{Z}) \rtimes_{\theta} \mathbb{Z}$ for some action $\theta: \mathbb{Z} \rightarrow \operatorname{Aut}(\mathbb{Z} \oplus \mathbb{Z}) \cong G L_{2}(\mathbb{Z})$, since $\mathbb{Z}$ is a free group.

Let $\mathbb{Z} \oplus \mathbb{Z}=\langle a, b \mid a b=b a\rangle$ be the fundamental group of the torus and $\mathbb{Z}=\langle t\rangle$ be the fundamental group of the circle, and let the action $\theta$ be such that $\theta(t)(a)=a^{\alpha} b^{\beta}$ and $\theta(t)(b)=a^{\gamma} b^{\delta}$ for $\left[\begin{array}{ll}\alpha & \gamma \\ \beta & \delta\end{array}\right] \in$ $G L_{2}(\mathbb{Z})$. For the elements of $G=\pi_{1}(E)$, we use the notation $G=\left\{\left(a^{m} b^{n}, t^{k}\right): m, n, k \in \mathbb{Z}\right\}$, and we write $\theta=\left[\begin{array}{ll}\alpha & \gamma \\ \beta & \delta\end{array}\right]$.

In Section 2 we construct a finite free resolution $P$ of $\mathbb{Z}$ as a trivial $\mathbb{Z} G$-module and compute the groups $H^{*}(G, \mathbb{Z})$ and $H^{*}\left(G, \mathbb{Z}_{p}\right)$ for $p$ prime. In Section 3 we define a partial diagonal approximation $\Delta: P \rightarrow P \otimes P$, which will be enough to allow us to compute the cup product in $H^{*}(G, \mathbb{Z})$ and $H^{*}\left(G, \mathbb{Z}_{p}\right)$.

## 2 Free resolution of $\mathbb{Z}$ over $\mathbb{Z} G$

Given the exact sequence of groups

$$
1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1
$$

Wall describes in [6] how we can construct a free resolution of $\mathbb{Z}$ over $\mathbb{Z} G$ from free resolutions of $\mathbb{Z}$ over $\mathbb{Z} K$ and $\mathbb{Z} H$ : let

$$
\cdots \longrightarrow B_{r} \longrightarrow \cdots \longrightarrow B_{1} \longrightarrow B_{0} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0
$$

be a free resolution over $\mathbb{Z} K$, and let

$$
\cdots \longrightarrow C_{s} \longrightarrow \cdots \longrightarrow C_{1} \longrightarrow C_{0} \longrightarrow \mathbb{Z} \longrightarrow 0
$$

be a free resolution over $\mathbb{Z} H$, with $C_{s}$ free on $\alpha_{s}$ generators.
Since $\mathbb{Z} G$ is a free $\mathbb{Z} K$-module, $\mathbb{Z} G \otimes_{\mathbb{Z} K} B_{r}$ is a free $\mathbb{Z} G$-module and $1 \otimes \varepsilon: \mathbb{Z} G \otimes_{\mathbb{Z} K} B \rightarrow \mathbb{Z} G \otimes_{\mathbb{Z} K} \mathbb{Z} \cong \mathbb{Z} H$ induces an isomorphism in homology. We define $D_{s}$ as the direct sum of $\alpha_{s}$ copies of $\mathbb{Z} G \otimes_{\mathbb{Z} K} B$, which gives us an augmentation map over the direct sum of $\alpha_{s}$ copies of $\mathbb{Z} H$, that we identify with $C_{s}$ and write $\varepsilon_{s}: D_{s} \rightarrow C_{s}$. If $A_{r, s}$ is the submodule of $D_{s}$ given by the direct sum of $\alpha_{s}$ copies of $\mathbb{Z} G \otimes_{\mathbb{Z} K} B_{r}$, then $A_{r, s}$ is a free $\mathbb{Z} G$-module and $D_{s}$ is the direct sum of the modules $A_{r, s}$. Denoting by $d_{0}$ the differential in each of the complexes $D_{s}$, Wall proves in [6] the following lemma:

Lemma 1 (Wall) There are $\mathbb{Z} G$-homomorphisms $d_{k}: A_{r, s} \rightarrow A_{r+k-1, s-k}$ such that
(i) $d_{1} \varepsilon_{s-1}=\varepsilon_{s} d: A_{0, s} \rightarrow C_{s-1}$, where $d$ denotes the differential in $C$;
(ii) $\sum_{i=0}^{k} d_{i} d_{k-i}=0$ for all $k$, where $d_{k}$ is null if $r=k=0$ or if $s<k$.

Given those homomorphisms, the following theorem is then proved:
Theorem 1 (Wall) Let $A$ be the direct sum of the modules $A_{r, s}$, graded by $\operatorname{dim} A_{r, s}=r+s$, and let $d=\sum d_{k}$. The complex $(A, d)$ is acyclic, and hence gives us a free resolution of $\mathbb{Z}$ over $\mathbb{Z} G$.

Let's apply these ideas for the group extension

$$
1 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow G \longrightarrow \mathbb{Z} \longrightarrow 1
$$

where $G=\pi_{1}(E)$ is the group given in the Introduction. A free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}]$ is given by

$$
\begin{equation*}
0 \longrightarrow Q_{2} \xrightarrow{d_{2}} Q_{1} \xrightarrow{d_{1}} Q_{0} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \tag{1}
\end{equation*}
$$

where

$$
\begin{aligned}
& Q_{0}=\left\langle e_{0}\right\rangle \cong \mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}], \\
& Q_{1}=\left\langle e_{1}^{1}, e_{1}^{2}\right\rangle \cong \mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}] \oplus \mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}], \\
& Q_{2}
\end{aligned}=\left\langle e_{2}\right\rangle \cong \mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}], ~ \$, ~
$$

and the differentials $d_{i}(i=1,2)$ and $\varepsilon$ are given by

$$
\begin{align*}
\varepsilon\left(e_{0}\right) & =1, \\
d_{1}\left(e_{1}^{1}\right) & =(a-1) e_{0}, \\
d_{1}\left(e_{1}^{2}\right) & =(b-1) e_{0},  \tag{2}\\
d_{2}\left(e_{2}\right) & =(1-b) e_{1}^{1}+(a-1) e_{1}^{2} .
\end{align*}
$$

The fact that (1) is indeed a free resolution can be easily seen considering the cell decomposition of the universal cover $\mathbb{R}^{2}$ of the torus. Also, a free resolution of $\mathbb{Z}$ over $\mathbb{Z} \mathbb{Z}$ is given by

$$
\begin{equation*}
0 \longrightarrow \mathbb{Z} \mathbb{Z} \xrightarrow{t-1} \mathbb{Z} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \tag{3}
\end{equation*}
$$

as is shown, for example, in [2]. If we can find the homomorphisms of Lemma 1 then we'll have our desired free resolution of $\mathbb{Z}$ over $\mathbb{Z} G$.

From now on, let

$$
\left[\begin{array}{cc}
m_{1} & m_{2} \\
n_{1} & n_{2}
\end{array}\right]=-\theta^{-1}=\frac{1}{\alpha \delta-\beta \gamma}\left[\begin{array}{cc}
-\delta & \gamma \\
\beta & -\alpha
\end{array}\right]
$$

and let $A, B, C, D, E \in \mathbb{Z} G$ be the elements given by

$$
\begin{align*}
& A= \begin{cases}(1,1)+\sum_{k=1}^{m_{1}}\left(a^{-k \alpha} b^{-k \beta}, t\right), & \text { if } m_{1}>0, \\
(1,1)-\sum_{k=0}^{-m_{1}-1}\left(a^{k \alpha} b^{k \beta}, t\right), & \text { if } m_{1}<0, \\
(1,1), & \text { if } m_{1}=0,\end{cases}  \tag{4}\\
& B= \begin{cases}\sum_{k=0}^{n_{1}-1}\left(a^{1+k \gamma} b^{k \delta}, t\right), & \text { if } n_{1}>0 \\
-\sum_{k=1}^{-n_{1}}\left(a^{1-k \gamma} b^{-k \delta}, t\right), & \text { if } n_{1}<0 \\
0, & \text { if } n_{1}=0,\end{cases}  \tag{5}\\
& C= \begin{cases}\sum_{k=1}^{m_{2}}\left(a^{-k \alpha} b^{-k \beta}, t\right), & \text { if } m_{2}>0, \\
-\sum_{k=0}^{-m_{2}-1}\left(a^{k \alpha} b^{k \beta}, t\right), & \text { if } m_{2}<0, \\
0, & \text { if } m_{2}=0,\end{cases}  \tag{6}\\
& D= \begin{cases}(1,1)+\sum_{k=0}^{n_{2}-1}\left(a^{k \gamma} b^{1+k \delta}, t\right), & \text { if } n_{2}>0, \\
(1,1)-\sum_{k=1}^{-n_{2}}\left(a^{-k \gamma} b^{1-k \delta}, t\right), & \text { if } n_{2}<0, \\
(1,1), & \text { if } n_{2}=0,\end{cases}  \tag{7}\\
& E=-(1,1)+\sum_{(m, n) \in I_{1} \times J_{1}}\left(a^{\alpha m+\gamma n} b^{\beta m+\delta n}, t\right)-\sum_{(m, n) \in I_{2} \times J_{2}}\left(a^{\alpha m+\gamma n} b^{\beta m+\delta n}, t\right), \tag{8}
\end{align*}
$$

where the sets $I_{1}, I_{2}, J_{1}, J_{2}$ vary depending on the signs of $m_{1}, n_{1}, m_{2}$ and $n_{2}$ and are completely described in Table प

Theorem 2 Let

$$
\begin{aligned}
& P_{0}=\langle x\rangle \cong \mathbb{Z} G, \\
& P_{1}=\left\langle y_{1}\right\rangle \oplus\left\langle y_{2}\right\rangle \oplus\left\langle y_{3}\right\rangle \cong \mathbb{Z} G \oplus \mathbb{Z} G \oplus \mathbb{Z} G, \\
& P_{2}=\left\langle z_{1}\right\rangle \oplus\left\langle z_{2}\right\rangle \oplus\left\langle z_{3}\right\rangle \cong \mathbb{Z} G \oplus \mathbb{Z} G \oplus \mathbb{Z} G, \\
& P_{3}=\langle w\rangle \cong \mathbb{Z} G .
\end{aligned}
$$

The sequence

$$
\begin{equation*}
0 \longrightarrow P_{3} \xrightarrow{\partial_{3}} P_{2} \xrightarrow{\partial_{2}} P_{1} \xrightarrow{\partial_{1}} P_{0} \xrightarrow{\varepsilon_{0}} \mathbb{Z} \longrightarrow 0 \tag{9}
\end{equation*}
$$

is a free resolution of $\mathbb{Z}$ over $\mathbb{Z} G$, where the maps $\partial_{i}(i=1,2,3)$ and $\varepsilon_{0}$ are defined by

$$
\begin{aligned}
& \varepsilon_{0}(x)=1, \\
& \partial_{1}\left(y_{1}\right)=[(a, 1)-(1,1)] x, \\
& \partial_{1}\left(y_{2}\right)=[(b, 1)-(1,1)] x, \\
& \partial_{1}\left(y_{3}\right)=[(1, t)-(1,1)] x, \\
& \partial_{2}\left(z_{1}\right)=A y_{1}+B y_{2}+[(a, 1)-(1,1)] y_{3}, \\
& \partial_{2}\left(z_{2}\right)=C y_{1}+D y_{2}+[(b, 1)-(1,1)] y_{3}, \\
& \partial_{2}\left(z_{3}\right)=[(1,1)-(b, 1)] y_{1}+[(a, 1)-(1,1)] y_{2}, \\
& \partial_{3}(w)=[(1,1)-(b, 1)] z_{1}+[(a, 1)-(1,1)] z_{2}+E z_{3} .
\end{aligned}
$$

Proof: Applying the ideas of Wall [6], the modules $A_{r, s}$ are such that $A_{r, s} \neq 0$ only for $0 \leq r \leq 2$ and $0 \leq s \leq 1$, and

$$
\begin{aligned}
& A_{2,1} \cong A_{2,0} \cong A_{0,1} \cong A_{0,0} \cong \mathbb{Z} G, \\
& A_{1,1} \cong A_{1,0} \cong \mathbb{Z} G \oplus \mathbb{Z} G
\end{aligned}
$$

Using Lemma we get the diagram

in which we must determine the maps $\varepsilon_{0}, \varepsilon_{1}, d_{0}$ and $d_{1}$. Having done that, Theorem will give us the
desired resolution. The homomorphisms $\varepsilon_{0}, \varepsilon_{1}$ and $d_{0}$ are easily calculated:

$$
\begin{aligned}
\varepsilon_{0}: & \langle x\rangle \rightarrow \mathbb{Z} \mathbb{Z} \\
& \varepsilon_{0}\left(\left(a^{m} b^{n}, t^{k}\right) x\right)=t^{k}, \\
\varepsilon_{1}: & \left\langle y_{3}\right\rangle \rightarrow \mathbb{Z} \mathbb{Z} \\
& \varepsilon_{1}\left(\left(a^{m} b^{n}, t^{k}\right) y_{3}\right)=t^{k}, \\
d_{0}: & \langle w\rangle \rightarrow\left\langle z_{1}\right\rangle \oplus\left\langle z_{2}\right\rangle \\
& d_{0}(w)=((1,1)-(b, 1)) z_{1}+((a, 1)-(1,1)) z_{2}, \\
d_{0}: & \left\langle z_{3}\right\rangle \rightarrow\left\langle y_{1}\right\rangle \oplus\left\langle y_{2}\right\rangle \\
& d_{0}\left(z_{3}\right)=((1,1)-(b, 1)) y_{1}+((a, 1)-(1,1)) y_{2}, \\
d_{0}: & \left\langle z_{1}\right\rangle \oplus\left\langle z_{2}\right\rangle \rightarrow\left\langle y_{3}\right\rangle \\
& d_{0}\left(z_{1}\right)=((a, 1)-(1,1)) y_{3}, \\
& d_{0}\left(z_{2}\right)=((b, 1)-(1,1)) y_{3}, \\
d_{0}: & \left\langle y_{1}\right\rangle \oplus\left\langle y_{2}\right\rangle \rightarrow\langle x\rangle \\
& d_{0}\left(y_{1}\right)=((a, 1)-(1,1)) x, \\
& d_{0}\left(y_{2}\right)=((b, 1)-(1,1)) x .
\end{aligned}
$$

The three maps $d_{1}$ are computed as follows: the map $d_{1}:\left\langle y_{3}\right\rangle \rightarrow\langle x\rangle$ must satisfy $\varepsilon_{0} \circ d_{1}=(t-1) \circ \varepsilon_{1}$, and that happens if we define $d_{1}\left(y_{3}\right)=((1, t)-(1,1)) x$. For the map $d_{1}:\left\langle z_{1}\right\rangle \oplus\left\langle z_{2}\right\rangle \rightarrow\left\langle y_{1}\right\rangle \oplus\left\langle y_{2}\right\rangle$, consider the diagram

where $d_{0} d_{1}+d_{1} d_{0}=0$. If we write $d_{1}\left(z_{1}\right)=A y_{1}+B y_{2}$ and $d_{1}\left(z_{2}\right)=C y_{1}+D y_{2}$ for some $A, B, C$, $D \in \mathbb{Z} G$, then

$$
\begin{equation*}
\left(d_{0} d_{1}+d_{1} d_{0}\right)\left(z_{1}\right)=0 \Leftrightarrow A((a, 1)-(1,1))+B((b, 1)-(1,1))=(1, t)-(1,1)+(a, 1)-(a, t) \tag{11}
\end{equation*}
$$

In order to solve (11), we write

$$
\begin{aligned}
A & =(1,1)+\sum_{m, n \in \mathbb{Z}} g_{m n}\left(a^{m} b^{n}, t\right) \\
B & =\sum_{m, n \in \mathbb{Z}} h_{m n}\left(a^{m} b^{n}, t\right)
\end{aligned}
$$

so (11) becomes

$$
\begin{equation*}
\sum_{m, n \in \mathbb{Z}}\left(g_{(m-\alpha)(n-\beta)}-g_{m n}+h_{(m-\gamma)(n-\delta)}-h_{m n}\right)\left(a^{m} b^{n}, t\right)=(1, t)-(a, t) . \tag{12}
\end{equation*}
$$

We define a directed graph to solve (12) in the following way: the vertices of the graph are the points of $\mathbb{Z} \times \mathbb{Z}$, and each vertex $(m, n)$ is the origin of two edges, one going to ( $m-\alpha, n-\beta$ ) and the other to $(m-\gamma, n-\delta)$. Given that $\operatorname{det} \theta= \pm 1$, we can draw the graph as a square grid graph with horizontal edges going left and vertical edges going down, where each horizontal edge goes from ( $m, n$ ) to ( $m-\alpha, n-\beta$ ) and each vertical edge goes from $(m, n)$ to $(m-\gamma, n-\delta)$. On each vertex $(m, n)$, we put two labels (yet to be determined): one with the value of $g_{m n}$ and another with the value of $h_{m n}$. Then, on each horizontal edge we put a label with the value of $g_{(m-\alpha)(n-\beta)}-g_{m n}$, and on each vertical edge we put a label with the value of $h_{(m-\gamma)(n-\delta)}-h_{m n}$. Observing that we can write (12) as

$$
g_{(m-\alpha)(n-\beta)}-g_{m n}+h_{(m-\gamma)(n-\delta)}-h_{m n}= \begin{cases}1, & \text { if }(m, n)=(0,0)  \tag{13}\\ -1, & \text { if }(m, n)=(1,0) \\ 0, & \text { otherwise }\end{cases}
$$

we use the graph we just defined (except for labels) to solve it in the following way: first, we find a path (in the undirected graph obtained from the one we defined) between the vertices $(0,0)$ and $(1,0)$. In order to do that, let $m_{1}$ and $n_{1}$ be integers such that

$$
\left\lvert\, \begin{aligned}
& -\alpha m_{1}-\gamma n_{1}=1 \\
& -\beta m_{1}-\delta n_{1}=0 .
\end{aligned}\right.
$$

Hence we can go from $(0,0)$ to $(1,0)$ passing through $\left|m_{1}\right|$ horizontal edges and followed by $\left|n_{1}\right|$ vertical edges. Just to give a concrete case, suppose $m_{1}<0$ and $n_{1}>0$. Then our path looks like

where the top left vertex is $(0,0)$ and the bottom right vertex is $(1,0)$. From (13), we know that the sum of the labes of the edges originated at $(0,0)$ must be 1 , the sum of the labels of the edges originated at $(1,0)$ must be -1 , and the sum of the labels originated from any other vertex must be zero. This can be accomplished with the following labels on the vertices:


For each vertex, the left label is $g_{m n}$ and the right one is $h_{m n}$. For all the vertices that were not drawn, we have $g_{m n}=h_{m n}=0$. From the labels above we read the values of $A$ and $B$ :

$$
\begin{aligned}
& A=(1,1)-\sum_{k=0}^{m_{1}-1}\left(a^{k \alpha} b^{k \beta}, t\right), \\
& B=\sum_{k=0}^{n_{1}-1}\left(a^{1+k \gamma} b^{k \delta}, t\right) .
\end{aligned}
$$

Given any $\theta$, we can compute explicitly the values of $m_{1}, n_{1}$, and then discover the values of $A$ and $B$. The same can be done with the elements $C$ and $D$, and the values we find out for these four variables are the ones given in the statement of the theorem.

Finally, it remains to compute the map $d_{1}:\langle w\rangle \rightarrow\left\langle z_{3}\right\rangle$ in the diagram


If $d_{1}(w)=E z_{3}$ for some $E \in \mathbb{Z} G$, then $\left(d_{0} d_{1}+d_{1} d_{0}\right)(w)=0$ is equivalent to

$$
\left\lvert\, \begin{align*}
& E((1,1)-(b, 1))=((b, 1)-(1,1)) A+((1,1)-(a, 1)) C \\
& E((a, 1)-(1,1))=((b, 1)-(1,1)) B+((1,1)-(a, 1)) D . \tag{14}
\end{align*} .\right.
$$

Given the expressions, for $A, B, C$ and $D$, we guess that $E$ can be written in the form

$$
E=-(1,1)+\sum_{m, n \in \mathbb{Z}} h_{m n}\left(a^{m} b^{n}, t\right)
$$

for some integers $h_{m n}$ yet to be determined. Now the trick to compute the integers $h_{m n}$ is similar to the one we used to find the elements $A, B, C$ and $D$. In order to show the computations in an actual case, let's suppose that $m_{1}<0, n_{1}>0, m_{2}>0$ and $n_{2}<0$ (this case arises when $\alpha, \beta, \gamma$ and $\delta$ are positive and $\operatorname{det} \theta=1$ ). In this case we can write (14) as

$$
\begin{align*}
& ((b, 1)-(1,1))+\sum_{m, n \in \mathbb{Z}}\left(h_{m n}-h_{(m-\gamma)(n-\delta)}\right)\left(a^{m} b^{n}, t\right)=((b, 1)-(1,1))+ \\
+ & \sum_{k=0}^{-m_{1}-1}\left[\left(a^{k \alpha} b^{k \beta}, t\right)-\left(a^{k \alpha} b^{1+k \beta}, t\right)\right]+\sum_{k=1}^{m_{2}}\left[\left(a^{-k \alpha} b^{-k \beta}, t\right)-\left(a^{1-k \alpha} b^{-k \beta}, t\right)\right] \tag{15}
\end{align*}
$$

and

$$
\begin{align*}
& ((1,1)-(a, 1))+\sum_{m, n \in \mathbb{Z}}\left(h_{(m-\alpha)(n-\beta)}-h_{m n}\right)\left(a^{m} b^{n}, t\right)=((1,1)-(a, 1))+ \\
+ & \sum_{k=0}^{n_{1}-1}\left[\left(a^{1+k \gamma} b^{1+k \delta}, t\right)-\left(a^{1+k \gamma} b^{k \delta}, t\right)\right]-\sum_{k=1}^{-n_{2}}\left[\left(a^{-k \gamma} b^{1-k \delta}, t\right)-\left(a^{1-k \gamma} b^{1-k \delta}, t\right)\right] . \tag{16}
\end{align*}
$$

Once again we construct a directed graph with the points of $\mathbb{Z} \times \mathbb{Z}$ as vertices. Each vertex $(m, n)$ is the origin of two edges, one going to $(m-\alpha, n-\beta)$ and another going to $(m+\gamma, n+\delta)$. On the edge with origin at $(m, n)$ and going to $(m-\alpha, n-\beta)$, we put a label with the value of $h_{(m-\alpha)(n-\beta)}-h_{m n}$, and on the edge with origin at $(m-\gamma, n-\delta)$ and going to $(m, n)$, we put a label with the value of $h_{m n}-h_{(m-\gamma)(n-\delta)}$. Those values are available to us and are given by the equations (15) and (16). Our task then consists in putting a label with the value of $h_{m n}$ on each vertex $(m, n)$ in such a way that it is consistent with the labels of the edges of the graph. In order to do so, it is more convenient to draw the graph with horizontal and vertical edges, like we did before, which can be done applying the linear $\operatorname{map} T: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ given by

$$
T(m, n)=\theta^{-1}\left[\begin{array}{c}
m \\
n
\end{array}\right]=-\left[\begin{array}{cc}
m_{1} & m_{2} \\
n_{1} & n_{2}
\end{array}\right]\left[\begin{array}{c}
m \\
n
\end{array}\right]=\left(-m_{1} m-m_{2} n,-n_{1} m-n_{2} n\right)
$$

to the vertices of our graph. Then, in the case we are considering, our graph looks like this:


In the picture above, we did not draw edges with null labels. From the graph above, define the sets $I_{1}$, $J_{1}, I_{2}, J_{2} \subset \mathbb{Z}$ by

$$
\begin{aligned}
& I_{1}=\left[-m_{2},-m_{1}-m_{2}-1\right] \cap \mathbb{Z}, \\
& J_{1}=\left[0,-n_{2}-1\right] \cap \mathbb{Z}, \\
& I_{2}=\left[-m_{1}-m_{2},-m_{1}-1\right] \cap \mathbb{Z}, \\
& J_{2}=\left[-n_{1},-1\right] \cap \mathbb{Z} .
\end{aligned}
$$

If we put the label 1 on the vertices of $I_{1} \times J_{1}$, the label -1 on the vertices $(m, n) \in I_{2} \times J_{2}$ and the label 0 otherwise, then those labels will be consistent with the ones on the edges. Applying now the map $T^{-1}$, we obtain

$$
E=-(1,1)+\sum_{(m, n) \in I_{1} \times J_{1}}\left(a^{\alpha m+\gamma n} b^{\beta m+\delta n}, t\right)-\sum_{(m, n) \in I_{2} \times J_{2}}\left(a^{\alpha m+\gamma n} b^{\beta m+\delta n}, t\right) .
$$

The same construction can be done for all the other cases, and in all of them the element $E$ is given by (8), for varying sets $I_{1}, J_{1}, I_{2}, J_{2}$. A complete description of those sets is given in Table $\mathbb{1}$ Now that we have computed all the maps in the diagram (10), Theorem 1 gives the free resolution of $\mathbb{Z}$ over $\mathbb{Z} G$.

Given the previous theorem, we now proceed to compute the cohomology groups $H^{*}(G, \mathbb{Z})$ and $H^{*}\left(G, \mathbb{Z}_{p}\right)$ for $p$ prime.

Theorem 3 The cohomology groups $H^{*}(G, \mathbb{Z})$, where $\mathbb{Z}$ is the trivial $\mathbb{Z} G$-module, are given by

$$
\begin{aligned}
& H^{0}(G, \mathbb{Z}) \cong \mathbb{Z}, \\
& H^{1}(G, \mathbb{Z}) \cong(\mathbb{Z})^{3-\operatorname{rank}(\theta-I)}, \\
& H^{2}(G, \mathbb{Z}) \cong \begin{cases}\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, & \text { if } \operatorname{rank}(\theta-I)=0, \\
\mathbb{Z}_{\operatorname{gcd}(\beta, \gamma)} \oplus \mathbb{Z} \oplus \mathbb{Z}, & \text { if } \operatorname{rank}(\theta-I)=1 \text { and } \operatorname{det} \theta=1, \\
\mathbb{Z}_{\operatorname{gcd}(\beta, \gamma, 2)} \oplus \mathbb{Z}, & \text { if } \operatorname{rank}(\theta-I)=1 \text { and } \operatorname{det} \theta=-1, \\
\mathbb{Z}_{c_{1}} \oplus \mathbb{Z}_{c_{2}} \oplus \mathbb{Z}, & \text { if } \operatorname{rank}(\theta-I)=2 \text { and } \operatorname{det} \theta=1, \\
\mathbb{Z}_{c_{1}} \oplus \mathbb{Z}_{c_{2}}, & \text { if } \operatorname{rank}(\theta-I)=2 \text { and } \operatorname{det} \theta=-1,\end{cases} \\
& H^{3}(G, \mathbb{Z}) \cong \begin{cases}\mathbb{Z}, & \text { if } \operatorname{det} \theta=1, \\
\mathbb{Z}_{2}, & \text { if } \operatorname{det} \theta=-1,\end{cases} \\
& H^{n}(G, \mathbb{Z}) \cong 0, \\
& \text { if } n \geq 4 .
\end{aligned}
$$

The positive integers $c_{1}$ and $c_{2}$ are such that $c_{1}\left|c_{2}, c_{1} c_{2}=|\operatorname{det}(\theta-I)|\right.$.
Proof: Applying the functor $\operatorname{Hom}_{\mathbb{Z} G}(\ldots, \mathbb{Z})$ to the resolution given in Theorem 2 we get


A quick computation shows that $\partial_{1}^{*}=0$ and $H^{0}(G, \mathbb{Z}) \cong \mathbb{Z}$, with a generator being given by [ $x^{*}$ ]. If $\varepsilon: \mathbb{Z} G \rightarrow \mathbb{Z}$ represents the augmentation map, then the elements $A, B, C$ and $D$ of Theorem 2 given by the equations (4), (5), (6) and (7) are such that $\varepsilon(A)=1+m_{1}, \varepsilon(B)=n_{1}, \varepsilon(C)=m_{2}$ and $\varepsilon(D)=1+n_{2}$. Hence the matrix of $\partial_{2}^{*}: \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ relative to the dual bases of $\left\{y_{1}, y_{2}, y_{3}\right\}$ and $\left\{z_{1}, z_{2}, z_{3}\right\}$ is

$$
\left[\partial_{2}^{*}\right]=\left[\begin{array}{ccc}
1+m_{1} & n_{1} & 0  \tag{17}\\
m_{2} & 1+n_{2} & 0 \\
0 & 0 & 0
\end{array}\right]
$$

Then $H^{1}(G, \mathbb{Z})=\frac{\operatorname{ker} \partial_{2}^{*}}{\operatorname{im} \partial_{1}^{*}} \cong \operatorname{ker} \partial_{2}^{*} \cong(\mathbb{Z})^{3-\operatorname{rank}\left(\left[\partial_{2}^{*}\right]\right)}=(\mathbb{Z})^{3-\operatorname{rank}(\theta-I)}$, since $\operatorname{rank}(\theta-I)=\operatorname{rank}\left(I-\theta^{-1}\right)$ and

$$
I-\theta^{-1}=\left[\begin{array}{cc}
1+m_{1} & m_{2} \\
n_{1} & 1+n_{2}
\end{array}\right]
$$

We can also exhibit explicit generators for $H^{1}(G, \mathbb{Z})$ : if $\operatorname{rank}(\theta-I)=0$, then $H^{1}(G, \mathbb{Z})=\left\langle\left[y_{1}^{*}\right],\left[y_{2}^{*}\right],\left[y_{3}^{*}\right]\right\rangle$. If $\operatorname{rank}(\theta-I)$, then $H^{1}(G, \mathbb{Z})=\left\langle\left[y_{3}^{*}\right]\right\rangle$. Finally, if $\operatorname{rank}(\theta-I)=1$, then one of the generators of $H^{1}(G, \mathbb{Z})$ is $\left[y_{3}^{*}\right]$, while the other generator is obtained in the following way: if $\left(1+m_{1}\right)=n_{1}=0$, then the second generator of $H^{1}(G, \mathbb{Z})$ is given by

$$
\left[-\frac{1+n_{2}}{\operatorname{gcd}\left(m_{2}, 1+n_{2}\right)} y_{1}^{*}+\frac{m_{2}}{\operatorname{gcd}\left(m_{2}, 1+n_{2}\right)} y_{2}^{*}\right],
$$

and if $\left(1+m_{1}\right) \neq 0$ or $n_{1} \neq 0$ we can take

$$
\left[-\frac{n_{1}}{\operatorname{gcd}\left(1+m_{1}, n_{1}\right)} y_{1}^{*}+\frac{1+m_{1}}{\operatorname{gcd}\left(1+m_{1}, n_{1}\right)} y_{2}^{*}\right]
$$

as the second generator of $H^{1}(G, \mathbb{Z})$.
The matrix of $\partial_{3}^{*}$ relative to the dual bases of $\left\{z_{1}, z_{2}, z_{3}\right\}$ and $\{w\}$ is

$$
\left[\partial_{3}^{*}\right]=\left[\begin{array}{lll}
0 & 0 & (-1+\operatorname{det} \theta) \tag{18}
\end{array}\right]
$$

since the element $E$ given by equation (8) is such that $\varepsilon(E)=-1+\left|I_{1} \times J_{1}\right|-\left|I_{2} \times J_{2}\right|$ and the sets $I_{1}$, $J_{1}, I_{2}$ and $J_{2}$ always satisfy (see Table (I)

$$
\begin{equation*}
\left|\left(I_{1} \times J_{1}\right)\right|-\left|\left(I_{2} \times J_{2}\right)\right|=\operatorname{det} \theta= \pm 1 \tag{19}
\end{equation*}
$$

This implies

$$
H^{3}(G, \mathbb{Z}) \cong \begin{cases}\mathbb{Z}, & \text { if } \operatorname{det} \theta=1 \\ \mathbb{Z}_{2}, & \text { if } \operatorname{det} \theta=-1\end{cases}
$$

with $\left[w^{*}\right]$ a generator for $H^{3}(G, \mathbb{Z})$. Finally we proceed to the computation of $H^{2}(G, \mathbb{Z})$. If $\operatorname{det} \theta=1$, then $z_{3}^{*} \in \operatorname{ker} \partial_{3}^{*}$ and

$$
H^{2}(G, \mathbb{Z}) \cong \frac{\left\langle z_{1}^{*}\right\rangle \oplus\left\langle z_{2}^{*}\right\rangle}{\operatorname{im} \partial_{2}^{*}} \oplus \mathbb{Z}
$$

whereas if $\operatorname{det} \theta=-1$ we have

$$
H^{2}(G, \mathbb{Z}) \cong \frac{\left\langle z_{1}^{*}\right\rangle \oplus\left\langle z_{2}^{*}\right\rangle}{\operatorname{im} \partial_{2}^{*}}
$$

In both cases, the group structure of $\frac{\left\langle z_{1}^{*}\right\rangle \oplus\left\langle z_{2}^{*}\right\rangle}{\operatorname{im} \partial_{2}^{*}}$ can be obtained calculating the Smith normal form of $\left(I-\theta^{-1}\right)$. If $\operatorname{rank}\left(I-\theta^{-1}\right)=0$, then $\frac{\left\langle z_{1}^{*}\right\rangle \oplus\left\langle z_{2}^{*}\right\rangle}{\operatorname{im} \partial_{2}^{*}}=\left\langle\left[z_{1}^{*}\right]\right\rangle \oplus\left\langle\left[z_{2}^{*}\right]\right\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. If $\operatorname{rank}\left(I-\theta^{-1}\right)=1$, then $\left(I-\theta^{-1}\right)$ is a non-zero matrix that can be written as

$$
\left[\begin{array}{ll}
r & (p / q) r \\
s & (p / q) s
\end{array}\right] \quad \text { or } \quad\left[\begin{array}{ll}
(p / q) r & r \\
(p / q) s & s
\end{array}\right],
$$

where $r, s, p$ and $q$ are integers such that $q \neq 0$ and $\operatorname{gcd}(p, q)=1$. Since both cases are similar, we analyze the first one. Writing $r=q r^{\prime}$ and $s=q s^{\prime}$, we have $I-\theta^{-1}=\left[\begin{array}{ll}q r^{\prime} & p r^{\prime} \\ q s^{\prime} & p s^{\prime}\end{array}\right]$. If $k, \ell \in \mathbb{Z}$ are such that $p k+q \ell=1$, then

$$
\left(I-\theta^{-1}\right)\left[\begin{array}{cc}
\ell & -p \\
k & q
\end{array}\right]=\left[\begin{array}{ll}
r^{\prime} & 0 \\
s^{\prime} & 0
\end{array}\right]
$$

and the Smith normal form of $\left(I-\theta^{-1}\right)$ is $\left[\begin{array}{cc}\operatorname{gcd}\left(r^{\prime}, s^{\prime}\right) & 0 \\ 0 & 0\end{array}\right]$. Hence $\frac{\left\langle z_{1}^{*}\right\rangle \oplus\left\langle z_{2}^{*}\right\rangle}{\operatorname{im} \partial_{2}^{*}} \cong \mathbb{Z}_{\operatorname{gcd}\left(r^{\prime}, s^{\prime}\right)} \oplus \mathbb{Z}$. Now, if $p \neq 0, \operatorname{gcd}\left(r^{\prime}, s^{\prime}\right)=\operatorname{gcd}\left(p r^{\prime}, q s^{\prime}\right)=\operatorname{gcd}\left(m_{2}, n_{1}\right)=\operatorname{gcd}(\beta, \gamma)$. If $p=0$ and $\operatorname{det} \theta=1$, then $\alpha=\delta=1$ and $\gamma=0$, which implies $r^{\prime}=0$ and $\operatorname{gcd}\left(r^{\prime}, s^{\prime}\right)=\operatorname{gcd}(0, \beta)=\beta=\operatorname{gcd}(\beta, \gamma)$. And, if $p=0$ and $\operatorname{det} \theta=-1$, then $\alpha=-1, \delta=1, \gamma=0$ and $r^{\prime}=2$ (for in this case we take $q=1$ ), so $\operatorname{gcd}\left(r^{\prime}, s^{\prime}\right)=$ $\operatorname{gcd}(\beta, 2)=\operatorname{gcd}(\beta, \gamma, 2)$. When $\operatorname{rank}\left(I-\theta^{-1}\right)=1$ and $\operatorname{det} \theta=-1$, we make the extra observation that $\operatorname{gcd}\left(r^{\prime}, s^{\prime}\right) \in\{1,2\}$. This can be easily seen, as in this case we have

$$
\left[\begin{array}{ll}
q r^{\prime} & p r^{\prime} \\
q s^{\prime} & p s^{\prime}
\end{array}\right]=I-\theta^{-1}=\left[\begin{array}{cc}
1+m_{1} & m_{2} \\
n_{1} & 1+n_{2}
\end{array}\right]=\left[\begin{array}{cc}
1+\delta & -\gamma \\
-\beta & 1+\alpha
\end{array}\right],
$$

and $\operatorname{det}\left(I-\theta^{-1}\right)=0 \Leftrightarrow \alpha+\delta=0 \Leftrightarrow\left(p s^{\prime}-1\right)+\left(q r^{\prime}-1\right)=0 \Leftrightarrow p s^{\prime}+q r^{\prime}=2 \Rightarrow \operatorname{gcd}\left(r^{\prime}, s^{\prime}\right) \mid 2$. Hence, when $\operatorname{det} \theta=-1$ and $\operatorname{rank}(\theta-I)=1$, we can always write $\operatorname{gcd}\left(r^{\prime}, s^{\prime}\right)=\operatorname{gcd}(\beta, \gamma, 2)$.

Finally, if $\operatorname{rank}\left(I-\theta^{-1}\right)=2$, the Smith normal form of $\left(I-\theta^{-1}\right)$ is a matrix $\left[\begin{array}{cc}c_{1} & 0 \\ 0 & c_{2}\end{array}\right]$, with $c_{1}, c_{2}>0$, $c_{1} \mid c_{2}$ and $c_{1} c_{2}=\left|\operatorname{det}\left(I-\theta^{-1}\right)\right|=|\operatorname{det}(\theta-I)|$.

The calculations of the groups $H^{*}\left(G, \mathbb{Z}_{2}\right)$ and $H^{*}\left(G, \mathbb{Z}_{p}\right)$ for and odd prime $p$ are now simple, as the matrices of $\partial_{2}^{*}$ and $\partial_{3}^{*}$ are obtained from the case of $\mathbb{Z}$ coefficients by reducing them mod 2 and mod $p$ : for $\mathbb{Z}_{2}$ coefficients, we observe that $\partial_{3}^{*}=0$, while for $\mathbb{Z}_{p}$ coefficients we observe that $\partial_{3}^{*}=0$ if $\operatorname{det} \theta=1$ and $\partial_{3}^{*}$ is a bijection if $\operatorname{det} \theta=-1$. We then get the following two theorems.

Theorem 4 The cohomology groups $H^{*}\left(G, \mathbb{Z}_{2}\right)$ are given by

$$
\begin{aligned}
& H^{0}\left(G, \mathbb{Z}_{2}\right) \cong \mathbb{Z}_{2}, \\
& H^{1}\left(G, \mathbb{Z}_{2}\right) \cong\left(\mathbb{Z}_{2}\right)^{3-\operatorname{rank}_{\mathbb{Z}_{2}}(\theta-I)}, \\
& H^{2}\left(G, \mathbb{Z}_{2}\right) \cong\left(\mathbb{Z}_{2}\right)^{3-\operatorname{rank}_{\mathbb{Z}_{2}}(\theta-I)}, \\
& H^{3}\left(G, \mathbb{Z}_{2}\right) \cong \mathbb{Z}_{2}, \\
& H^{n}\left(G, \mathbb{Z}_{2}\right) \cong 0, \text { if } n \geq 4 .
\end{aligned}
$$

Theorem 5 Let $p$ be an odd prime. The cohomology groups $H^{*}\left(G, \mathbb{Z}_{p}\right)$, where $\mathbb{Z}_{p}$ is the trivial $\mathbb{Z} G$ module, are given by

$$
\begin{aligned}
& H^{0}\left(G, \mathbb{Z}_{p}\right) \cong \mathbb{Z}_{p}, \\
& H^{1}\left(G, \mathbb{Z}_{p}\right) \cong\left(\mathbb{Z}_{p}\right)^{3-\operatorname{rank}_{\mathbb{Z}_{p}}(\theta-I)}, \\
& H^{2}\left(G, \mathbb{Z}_{p}\right) \cong \begin{cases}\left(\mathbb{Z}_{p}\right)^{3-\operatorname{rank}_{\mathbb{Z}_{p}}(\theta-I)}, & \text { if } \operatorname{det} \theta=1, \\
\left(\mathbb{Z}_{p}\right)^{2-\operatorname{rank}_{\mathbb{Z}_{p}}(\theta-I)}, & \text { if } \operatorname{det} \theta=-1,\end{cases} \\
& H^{3}\left(G, \mathbb{Z}_{p}\right) \cong \begin{cases}\mathbb{Z}_{p}, & \text { if } \operatorname{det} \theta=1, \\
0, & \text { if } \operatorname{det} \theta=-1,\end{cases} \\
& H^{n}\left(G, \mathbb{Z}_{p}\right) \cong 0, \text { if } n \geq 4 .
\end{aligned}
$$

## 3 Diagonal approximation and the cup product

In order to compute the cup product in the cohomology groups $H^{*}(G, \mathbb{Z}), H^{*}\left(G, \mathbb{Z}_{2}\right)$ and $H^{*}\left(G, \mathbb{Z}_{p}\right)$, we seek a diagonal approximation $\Delta: P \rightarrow(P \otimes P)$ for the free resolution $P$ given in Theorem [2 In [5], we find the following two propositions, which can help us determine $\Delta$.

Proposition 1 For a group $G$, let

be a finitely generated free resolution of $\mathbb{Z}$ over $\mathbb{Z} G$, that is, each $C_{n}$ is finitely generated as a $\mathbb{Z} G$-module. If $s$ is a contracting homotopy for the resolution $C$, then a contracting homotopy $\tilde{s}$ for the free resolution $C \otimes C$ of $\mathbb{Z}$ over $\mathbb{Z} G$ is given by

$$
\begin{aligned}
\tilde{s}_{-1}: & \mathbb{Z} \rightarrow C_{0} \otimes C_{0} \\
& \tilde{s}_{-1}(1)=s_{-1}(1) \otimes s_{-1}(1), \\
\tilde{s}_{n}: & (C \otimes C)_{n} \rightarrow(C \otimes C)_{n+1} \\
& \tilde{s}_{n}\left(u_{i} \otimes v_{n-i}\right)=s_{i}\left(u_{i}\right) \otimes v_{n-i}+s_{-1} \varepsilon\left(u_{i}\right) \otimes s_{n-i}\left(v_{n-i}\right), \quad \text { se } n \geq 0,
\end{aligned}
$$

where $s_{-1} \varepsilon: C_{0} \rightarrow C_{0}$ is extended to $s_{-1} \varepsilon=\left\{\left(s_{-1} \varepsilon\right)_{n}: C_{n} \rightarrow C_{n}\right\}$ in such a way that $\left(s_{-1} \varepsilon\right)_{n}=0$ for $n \geq 1$.
Proposition 2 For a group $G$, let

$$
\cdots \longrightarrow C_{n} \xrightarrow{d_{n}} \cdots \longrightarrow C_{1} \xrightarrow{d_{1}} C_{0} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0
$$

be a finitely generated free resolution of $\mathbb{Z}$ over $\mathbb{Z} G$ (i.e., each $C_{n}$ is a finitely generated free $\mathbb{Z} G$-module), and let $s$ be a contracting homotopy for this resolution $C$. If $\tilde{s}$ is the contracting homotopy for the resolution $C \otimes C$ given by Proposition 1, then a diagonal approximation $\Delta: C \rightarrow C \otimes C$ can be defined in the following way: for each $n \geq 0$, the map $\Delta_{n}: C_{n} \rightarrow(C \otimes C)_{n}$ is given in each generator $\rho$ of $C_{n}$ by

$$
\begin{aligned}
& \Delta_{0}=s_{-1} \varepsilon \otimes s_{-1} \varepsilon, \\
& \Delta_{n}(\rho)=\tilde{s}_{n-1} \Delta_{n-1} d_{n}(\rho), \quad \text { if } n \geq 1
\end{aligned}
$$

The two above propositions tell us that if we can manage to find a contracting homotopy for the resolution $P$ given in Theorem 2 then we can construct a diagonal approximation $\Delta$ and then procceed to calculate the cup product in the cohomology ring. The maps $s_{-1}: \mathbb{Z} \rightarrow P_{0}$ and $s_{0}: P_{0} \rightarrow P_{1}$ are easy to define: we take $s_{-1}(1)=x$ and $s_{0}\left(\left(a^{m} b^{n}, t^{k}\right) x\right)=\frac{\partial\left(a^{m}, 1\right)}{\partial(a, 1)} y_{1}+\left(a^{m}, 1\right) \frac{\partial\left(b^{n}, 1\right)}{\partial(b, 1)} y_{2}+\left(a^{m} b^{n}, 1\right) \frac{\partial\left(1, t^{k}\right)}{\partial(1, y)} y_{3}$, where the partial derivatives are the Fox derivatives, and it is immediate to check that $\varepsilon s_{-1}=$ id $\mathbb{Z}_{\mathbb{Z}}$ and $\partial_{1} s_{0}+s_{-1} \varepsilon_{0}=\operatorname{id}_{P_{0}}$. As to the maps $s_{1}$ and $s_{2}$, we don't need their full description to compute the cup
product in the cohomology ring. In fact, we don't need $s_{2}$ at all, as we shall see now: having defined the maps $s_{-1}$ and $s_{0}$, we use Propositions 1 and 2 to compute $\Delta_{0}$ and $\Delta_{1}$. We get

$$
\begin{aligned}
\Delta_{0}: & P_{0} \rightarrow P_{0} \otimes P_{0} \\
& \Delta_{0}(x)=x \otimes x, \\
\Delta_{1}: & P_{1} \rightarrow\left(P_{1} \otimes P_{0}\right) \oplus\left(P_{0} \otimes P_{1}\right) \\
& \Delta_{1}\left(y_{1}\right)=y_{1} \otimes(a, 1) x+x \otimes y_{1}, \\
& \Delta_{1}\left(y_{2}\right)=y_{2} \otimes(b, 1) x+x \otimes y_{2}, \\
& \Delta_{1}\left(y_{3}\right)=y_{3} \otimes(1, t) x+x \otimes y_{3} .
\end{aligned}
$$

Let $\pi_{i j}:(P \otimes P)_{i+j} \rightarrow P_{i} \otimes P_{j}$ denote the projection and $\Delta_{i j}=\pi_{i j} \circ \Delta_{i+j}: P_{i+j} \rightarrow P_{i} \otimes P_{j}$. We observe that, for the computation of $H^{1}(G, \mathbb{Z}) \otimes H^{1}(G, \mathbb{Z}) \breve{\longrightarrow} H^{2}(G, \mathbb{Z} \otimes \mathbb{Z})$, we need to know only the map $\Delta_{11}: P_{2} \rightarrow P_{1} \otimes P_{1}$, and the computation of $H^{1}(G, \mathbb{Z}) \otimes H^{2}(G, \mathbb{Z}) \hookrightarrow H^{3}(G, \mathbb{Z} \otimes \mathbb{Z})$ can be done once we have $\Delta_{12}: P_{3} \rightarrow P_{1} \otimes P_{2}$.

From the resolution of Theorem 2 and Propositions 1 and 2, we can then verify that the maps $\Delta_{11}$ and $\Delta_{12}$ can be calculated if we know how to compute $s_{1}$ for the elements of the following list:

$$
\begin{align*}
& y_{3}, \\
& (a, 1) y_{3}, \\
& (b, 1) y_{3}, \\
& \left(a^{m} b^{n}, 1\right) y_{1},  \tag{20}\\
& \left(a^{m} b^{n}, 1\right) y_{2}, \\
& \left(a^{m} b^{n}, t\right) y_{1}, \\
& \left(a^{m} b^{n}, t\right) y_{2} .
\end{align*}
$$

Before we compute $s_{1}$ for the elements of this list, we make one more observation that will be useful later: if $M$ and $N$ are trivial $\mathbb{Z} G$-modules, $g \in \mathbb{Z} G, m \in M$ and $f \in \operatorname{Hom}_{\mathbb{Z} G}(M, N)$, then

$$
\begin{equation*}
f(g m)=g \cdot f(m)=\varepsilon(g) \cdot f(m) \tag{21}
\end{equation*}
$$

where $\varepsilon: \mathbb{Z} G \rightarrow \mathbb{Z}$ is the augmentation map.
Lemma 2 Let

$$
0 \longrightarrow P_{3} \xrightarrow{\partial_{3}} P_{2} \xrightarrow{\partial_{2}} P_{1} \xrightarrow{\partial_{1}} P_{0} \xrightarrow{\varepsilon_{0}} \mathbb{Z} \longrightarrow 0
$$

be the free resolution of Theorem 2. There is a contracting homotopy sor the resolution $P$ such that

$$
\begin{aligned}
s_{-1}(1)= & x, \\
s_{0}\left(\left(a^{m} b^{n}, t^{k}\right) x\right)= & \frac{\partial\left(a^{m}, 1\right)}{\partial(a, 1)} y_{1}+\left(a^{m}, 1\right) \frac{\partial\left(b^{n}, 1\right)}{\partial(b, 1)} y_{2}+\left(a^{m} b^{n}, 1\right) \frac{\partial\left(1, t^{k}\right)}{\partial(1, y)} y_{3}, \\
s_{1}\left(\left(a^{m} b^{n}, 1\right) y_{1}\right)= & -\left(a^{m}, 1\right) \frac{\partial\left(b^{n}, 1\right)}{\partial(b, 1)} z_{3}, \\
s_{1}\left(\left(a^{m} b^{n}, 1\right) y_{2}\right)= & 0, \\
s_{1}\left(y_{3}\right)= & 0, \\
s_{1}\left((a, 1) y_{3}\right)= & 0, \\
s_{1}\left((b, 1) y_{3}\right)= & 0, \\
s_{1}\left(\left(a^{m} b^{n}, t\right) y_{1}\right)= & -\left(a^{m} b^{n}, 1\right) \frac{\partial\left(a^{\alpha}, 1\right)}{\partial(a, 1)} z_{1}-\left(a^{m+\alpha} b^{n}, 1\right) \frac{\partial\left(b^{\beta}, 1\right)}{\partial(b, 1)} z_{2}+ \\
& +\left(-\left(a^{m}, 1\right) \frac{\partial\left(a^{\alpha}, 1\right)}{\partial(a, 1)} \frac{\partial\left(b^{n}, 1\right)}{\partial(b, 1)}+\sum_{u, v \in \mathbb{Z}} h_{u v}\left(a^{u} b^{v}, t\right)\right) z_{3}, \\
& +\left(-\left(a^{m}, 1\right) \frac{\partial\left(a^{\gamma}, 1\right)}{\partial(a, 1)} \frac{\partial\left(b^{n}, 1\right)}{\partial(b, 1)}+\sum_{u, v \in \mathbb{Z}} q_{u v}\left(a^{u} b^{v}, t\right)\right) z_{3},
\end{aligned}
$$

where the integers $h_{u v}$ e $q_{u v}$ satisfy

$$
\begin{aligned}
& \sum_{u, v \in \mathbb{Z}} h_{u v}=\frac{\alpha \beta}{2}(\gamma+\delta-\operatorname{det} \theta) \\
& \sum_{u, v \in \mathbb{Z}} q_{u v}=\frac{\gamma \delta}{2}(\alpha+\beta-\operatorname{det} \theta)
\end{aligned}
$$

Proof: We'we already defined the maps $s_{-1}$ and $s_{0}$. From $\partial_{2} s_{1}+s_{0} \partial_{1}=\operatorname{id}_{P_{1}}$, it is trivial to verify that we can take $s_{1}\left(y_{3}\right)=s_{1}\left((a, 1) y_{3}\right)=s_{1}\left((b, y) y_{3}\right)=0$. It is also easy to see that we can define $s_{1}\left(\left(a^{m} b^{n}, 1\right) y_{1}\right)=-\left(a^{m}, 1\right) \frac{\partial\left(b^{n}, 1\right)}{\partial(b, 1)} z_{3}$ and $s_{1}\left(\left(a^{m} b^{n}, 1\right) y_{2}\right)=0$.

Let us now write $s_{1}\left(\left(a^{m} b^{n}, t\right) y_{1}\right)=k_{1}^{m, n} z_{1}+k_{2}^{m, n} z_{2}+k_{3}^{m, n} z_{3}$, where $k_{1}^{m, n}, k_{2}^{m, n}, k_{3}^{m, n} \in \mathbb{Z} G$. Substituting in $\partial_{2}\left(s_{1}\left(\left(a^{m} b^{m}, t\right) y_{1}\right)\right)+s_{0}\left(\partial_{1}\left(\left(a^{m} b^{n}, t\right) y_{1}\right)\right)=\left(a^{m} b^{n}, t\right) y_{1}$, we get

$$
\left\lvert\, \begin{align*}
& k_{1}^{m, n} A+k_{2}^{m, n} C+k_{3}^{m, n}[(1,1)-(b, 1)]=\left(a^{m} b^{n}, t\right)-\left(a^{m}, 1\right) \frac{\partial\left(a^{\alpha}, 1\right)}{\partial(a, 1)} \\
& k_{1}^{m, n} B+k_{2}^{m, n} D+k_{3}^{m, n}[(a, 1)-(1,1)]=\left(a^{m}, 1\right) \frac{\partial\left(b^{n}, 1\right)}{\partial(b, 1)}-\left(a^{m+\alpha}, 1\right) \frac{\partial\left(b^{n+\beta}, 1\right)}{\partial(b, 1)},  \tag{22}\\
& k_{1}^{m, n}[(a, 1)-(1,1)]+k_{2}^{m, n}[(b, 1)-(1,1)]=\left(a^{m} b^{n}, 1\right)-\left(a^{m+\alpha} b^{n+\beta}, 1\right) .
\end{align*}\right.
$$

The last equation above is satisfied for $k_{1}^{m, n}=-\left(a^{m} b^{n}, 1\right) \frac{\partial\left(a^{\alpha}, 1\right)}{\partial(a, 1)}$ and $k_{2}^{m, n}=-\left(a^{m+\alpha} b^{n}, 1\right) \frac{\partial\left(b^{\beta}, 1\right)}{\partial(b, 1)}$. We must then find $k_{3}^{m, n}$ such that

$$
\begin{align*}
k_{3}^{m, n}[(1,1)-(b, 1)] & =\left(a^{m} b^{n}, t\right)-\left(a^{m}, 1\right) \frac{\partial\left(a^{\alpha}, 1\right)}{\partial(a, 1)}-k_{1}^{m, n} A-k_{2}^{m, n} C \\
& =\left(a^{m} b^{n}, t\right)-\left(a^{m}, 1\right) \frac{\partial\left(a^{\alpha}, 1\right)}{\partial(a, 1)}+\left(a^{m} b^{n}, 1\right) \frac{\partial\left(a^{\alpha}, 1\right)}{\partial(a, 1)} A+\left(a^{m+\alpha} b^{n}, 1\right) \frac{\partial\left(b^{\beta}, 1\right)}{\partial(b, 1)} C \tag{23}
\end{align*}
$$

and

$$
\begin{align*}
k_{3}^{m, n}[(a, 1)-(1,1)]= & \left(a^{m}, 1\right) \frac{\partial\left(b^{n}, 1\right)}{\partial(b, 1)}-\left(a^{m+\alpha}, 1\right) \frac{\partial\left(b^{n+\beta}, 1\right)}{\partial(b, 1)}-k_{1}^{m, n} B-k_{2}^{m, n} C \\
= & \left(a^{m}, 1\right) \frac{\partial\left(b^{n}, 1\right)}{\partial(b, 1)}-\left(a^{m+\alpha}, 1\right) \frac{\partial\left(b^{n+\beta}, 1\right)}{\partial(b, 1)}+\left(a^{m} b^{n}, 1\right) \frac{\partial\left(a^{\alpha}, 1\right)}{\partial(a, 1)} B+ \\
& +\left(a^{m+\alpha} b^{n}, 1\right) \frac{\partial\left(b^{\beta}, 1\right)}{\partial(b, 1)} D . \tag{24}
\end{align*}
$$

Now, if

$$
k_{3}^{m, n}=\sum_{u, v \in \mathbb{Z}} g_{u v}\left(a^{u} b^{v}, 1\right)+\sum_{u, v \in \mathbb{Z}} h_{u v}\left(a^{u} b^{v}, t\right), \quad g_{u v}, h_{u v} \in \mathbb{Z}
$$

then the equations (23) and (24) are written as

$$
\begin{align*}
& \sum_{u, v \in \mathbb{Z}}\left(g_{u v}-g_{u(v-1)}\right)\left(a^{u} b^{v}, 1\right)=\left(a^{m} b^{n}, 1\right) \frac{\partial\left(a^{\alpha}, 1\right)}{\partial(a, 1)}-\left(a^{m}, 1\right) \frac{\partial\left(a^{\alpha}, 1\right)}{\partial(a, 1)}  \tag{25}\\
& \sum_{u, v \in \mathbb{Z}}\left(g_{(u-1) v}-g_{u v}\right)\left(a^{u} b^{v}, 1\right)=\left(a^{m}, 1\right) \frac{\partial\left(b^{n}, 1\right)}{\partial(b, 1)}-\left(a^{m+\alpha}, 1\right) \frac{\partial\left(b^{n}, 1\right)}{\partial(b, 1)} \tag{26}
\end{align*}
$$

and

$$
\begin{align*}
\sum_{u, v \in \mathbb{Z}}\left(h_{u v}-h_{(u-\gamma)(v-\delta)}\right)\left(a^{u} b^{v}, t\right)= & \left(a^{m} b^{n}, t\right)+\left(a^{m} b^{n}, 1\right) \frac{\partial\left(a^{\alpha}, 1\right)}{\partial(a, 1)}(A-(1,1))+ \\
& +\left(a^{m+\alpha} b^{n}, 1\right) \frac{\partial\left(b^{\beta}, 1\right)}{\partial(b, 1)} C  \tag{27}\\
\sum_{u, v \in \mathbb{Z}}\left(h_{(u-\alpha)(v-\beta)}-h_{u v}\right)\left(a^{u} b^{v}, t\right)= & \left(a^{m} b^{n}, 1\right) \frac{\partial\left(a^{\alpha}, 1\right)}{\partial(a, 1)} B+\left(a^{m+\alpha} b^{n}, 1\right) \frac{\partial\left(b^{\beta}, 1\right)}{\partial(b, 1)}(D-(1,1)) . \tag{28}
\end{align*}
$$

We can solve (25) and (26) in a way that is similar to the one we used to find the element $E$ given by equation (8). We get

$$
\begin{equation*}
\sum_{u, v \in \mathbb{Z}} g_{u v}\left(a^{u} b^{v}, 1\right)=-\left(a^{m}, 1\right) \frac{\partial\left(a^{\alpha}, 1\right)}{\partial(a, 1)} \frac{\partial\left(b^{n}, 1\right)}{\partial(b, 1)} . \tag{29}
\end{equation*}
$$

In order to solve (27) and (28), we first observe that explicit expressions for the elements

$$
A, B, C, D, \frac{\partial\left(a^{\alpha}, 1\right)}{\partial(a, 1)}, \frac{\partial\left(b^{\beta}, 1\right)}{\partial(b, 1)}
$$

depend on the signs of $\alpha, \beta, \gamma, \delta$ and $\operatorname{det} \theta$, so we have many cases to consider. As all of them are similar, we show how to solve (27) and (28) when $\alpha, \beta, \gamma$ and $\delta$ are positive and $\operatorname{det} \theta=1$. In this case we have

$$
\begin{aligned}
A-(1,1) & =-\sum_{k=0}^{\delta-1}\left(a^{k \alpha} b^{k \beta}, t\right), \\
B & =\sum_{k=0}^{\beta-1}\left(a^{1+k \gamma} b^{k \delta}, t\right), \\
C & =\sum_{k=1}^{\gamma}\left(a^{-k \alpha} b^{-k \beta}, t\right), \\
D-(1,1) & =-\sum_{k=1}^{\alpha}\left(a^{-k \gamma} b^{1-k \delta}, t\right),
\end{aligned}
$$

and the equations (27) and (28) can be written as

$$
\begin{align*}
& \sum_{u, v \in \mathbb{Z}}\left(h_{u v}-h_{(u-\gamma)(v-\delta)}\right)\left(a^{u} b^{v}, t\right)= \\
& \quad=\left(a^{m} b^{n}, 1\right)\left[(1, t)-\sum_{j=0}^{\alpha-1} \sum_{k=0}^{\delta-1}\left(a^{j+k \alpha} b^{k \beta}, t\right)+\sum_{j=0}^{\beta-1} \sum_{k=1}^{\gamma}\left(a^{\alpha-k \alpha} b^{j-k \beta}, t\right)\right] \tag{30}
\end{align*}
$$

and

$$
\begin{align*}
\sum_{u, v \in \mathbb{Z}}\left(h_{(u-\alpha)(v-\beta)}-h_{u v}\right) & \left(a^{u} b^{v}, t\right)= \\
& =\left(a^{m} b^{n}, 1\right)\left[\sum_{j=0}^{\alpha-1} \sum_{k=0}^{\beta-1}\left(a^{1+j+k \gamma} b^{k \delta}, t\right)-\sum_{j=0}^{\beta-1} \sum_{k=1}^{\alpha}\left(a^{\alpha-k \gamma} b^{1+j-k \delta}, t\right)\right] . \tag{31}
\end{align*}
$$

Just like we did to compute the value of the variable $E$ in the proof of Theorem 2 once more we define a directed graph, with the set $\mathbb{Z} \times \mathbb{Z}$ as vertices and edges going from each $(m, n)$ to $(m-\alpha, n-\beta)$ and $(m+\gamma, n+\delta)$. On the edge from $(u-\gamma, v-\delta)$ to $(u, v)$, we put a label with the value of $h_{u v}-h_{(u-\gamma)(v-\delta)}$, which is given by equation (30) and on the edge from $(u, v)$ to $(u-\alpha, v-\beta)$, we put a label with the value of $h_{(u-\alpha)(v-\beta)}-h_{u v}$, which is given by (31). In order to draw the graph, we apply again the map $T: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ defined by

$$
\left[\begin{array}{l}
u \\
v
\end{array}\right]=T(x, y)=\left[\begin{array}{ll}
\alpha & \gamma \\
\beta & \delta
\end{array}\right]^{-1}\left[\begin{array}{l}
x \\
y
\end{array}\right],
$$

and we call the new coordinates, obtained after applying $T, u$ and $v$.
In the case we are analyzing, let's take a look at the right side of (31). As we are interested in calculating the cup product with trivial coefficients $\mathbb{Z}, \mathbb{Z}_{2}$ and $\mathbb{Z}_{p}$, all we need is the value of $\varepsilon\left(k_{3}^{m, n}\right)=$ $\sum g_{u v}+\sum h_{u v}$, where $\varepsilon: \mathbb{Z} G \rightarrow \mathbb{Z}$ is the augmentation map. Since (29) already gives us $\sum g_{u v}$, all we need is $\sum h_{u v}$, and for that we can drop the term $\left(a^{m} b^{n}, 1\right)$ in the right side of (31) and focus on

$$
\sum_{j=0}^{\alpha-1} \sum_{k=0}^{\beta-1}\left(a^{1+j+k \gamma} b^{k \delta}, t\right)-\sum_{j=0}^{\beta-1} \sum_{k=1}^{\alpha}\left(a^{\alpha-k \gamma} b^{1+j-k \delta}, t\right) .
$$

The sum $\sum_{j=0}^{\alpha-1} \sum_{k=0}^{\beta-1}\left(a^{1+j+k \gamma} b^{k \delta}, t\right)$ provides us with edges with label 1. For a fixed value of $j \in$ $\{0, \ldots, \alpha-1\}$, we get the following "block" of edges (after applying $T$ ):


Now, as we let $j$ vary between 0 and $\alpha-1$, other blocks of edges with label 1 appear, and are arranged according to the following pattern, where each white rectangle represents a block of edges for a fixed value of $j$ :


If we do the same with the sum $-\sum_{j=0}^{\beta-1} \sum_{k=1}^{\alpha}\left(a^{\alpha-k \gamma} b^{1+j-k \delta}, t\right)$, which will give us edges labeled with -1 , we get a picture of the horizontal edges of our graph (where the gray rectangles represent the edges of label -1 ):


In order to label each vertex in a way that is consistent with the labels on the edges, it is enough to put the label 1 in every vertex between a gray block and a white one, and put the label 0 on every other
vertex. Since we can compute the exact location of the blocks in our graph, we can then write

$$
\begin{aligned}
\sum_{u, v \in \mathbb{Z}} h_{u v} & =\left(\begin{array}{l}
\text { sum of the } u \text {-coordinates of } \\
\text { the origins of vertices with } \\
\text { label } 1
\end{array}\right)-\left(\begin{array}{l}
\text { sum of the } u \text {-coordinates of } \\
\text { the origins of vertices with } \\
\text { label }-1
\end{array}\right) \\
& =\frac{(\alpha \delta+\delta) \alpha \beta}{2}-\frac{(\alpha \delta-\gamma+1) \alpha \beta}{2}=\frac{\alpha \beta}{2}(\gamma+\delta-1) .
\end{aligned}
$$

One thing is missing: we need to check that our labels on the vertices are also consistent with equation (30), but fortunately they are.

If we now analyze all the remaining cases, we get

$$
\sum_{u, v \in \mathbb{Z}} h_{u v}= \begin{cases}\frac{\alpha \beta}{2}(\gamma+\delta-1), & \text { if } \operatorname{det} \theta=1 \\ \frac{\alpha \beta}{2}(\gamma+\delta+1), & \text { if } \operatorname{det} \theta=-1\end{cases}
$$

Also, the computation of $s_{1}\left(\left(a^{m} b^{n}, t\right) y_{2}\right)$ is analogous to that of $s_{1}\left(\left(a^{m} b^{n}, t\right) y_{1}\right)$, and what we get is exactly what is in the statement of this lemma.

Remark: If we are interested in computing the cup product with non trivial coefficients, then we can still construct the graphs described in the lemma for a given action $\theta$ and explicitly calculate $s_{1}\left(\left(a^{m} b^{n}, t\right) y_{1}\right)$ and $s_{1}\left(\left(a^{m} b^{n}, t\right) y_{2}\right)$.

Now we proceed to calculate the cup product $H^{p}(G, \mathbb{Z}) \otimes H^{q}(G, \mathbb{Z}) \breve{\longrightarrow} H^{p+q}(G, \mathbb{Z})$. Just to simplify the notation, let's write the elements $A, B, C, D$ and $E \in \mathbb{Z} G$ as

$$
A=\sum \pm A_{k}, \quad B=\sum \pm B_{k}, \quad C=\sum \pm C_{k}, \quad D=\sum \pm D_{k}, \quad E=\sum \pm E_{k} .
$$

Then, using Propositions 1 and 2 we can write

$$
\begin{align*}
\Delta_{2}\left(z_{1}\right)= & \tilde{s}_{1} \Delta_{1} \varphi_{2}\left(z_{1}\right)=\tilde{s}_{1}\left(A \Delta_{1}\left(y_{1}\right)+B \Delta_{1}\left(y_{2}\right)+[(a, 1)-(1,1)] \Delta_{1}\left(y_{3}\right)\right) \\
= & \tilde{s}_{1}\left(A\left[y_{1} \otimes(a, 1) x+x \otimes y_{1}\right]+B\left[y_{2} \otimes(b, 1) x+x \otimes y_{2}\right]+\right. \\
& \left.\quad+(a, 1) y_{3} \otimes(a, t) x+(a, 1) x \otimes(a, 1) y_{3}-y_{3} \otimes(1, t) x-x \otimes y_{3}\right), \\
\Delta_{2}\left(z_{2}\right)= & \tilde{s}_{1} \Delta_{1} \varphi_{2}\left(z_{2}\right)=\tilde{s}_{1}\left(C \Delta_{1}\left(y_{1}\right)+D \Delta_{1}\left(y_{2}\right)+[(b, 1)-(1,1)] \Delta_{1}\left(y_{3}\right)\right) \\
= & \tilde{s}_{1}\left(C\left[y_{1} \otimes(a, 1) x+x \otimes y_{1}\right]+D\left[y_{2} \otimes(b, 1) x+x \otimes y_{2}\right]+\right.  \tag{32}\\
& \left.\quad+(b, 1) y_{3} \otimes(b, t) x+(b, 1) x \otimes(b, 1) y_{3}-y_{3} \otimes(1, t) x-x \otimes y_{3}\right), \\
& \quad \Delta_{2}\left(z_{3}\right)= \\
= & \tilde{s}_{1} \Delta_{1} \varphi_{2}\left(z_{3}\right)=\tilde{s}_{1}\left(\Delta_{1}\left(y_{1}\right)-(b, 1) \Delta_{1}\left(y_{1}\right)+(a, 1) \Delta_{1}\left(y_{2}\right)-\Delta_{1}\left(y_{2}\right)\right) \\
= & s_{1}\left(y_{1}\right) \otimes(a, 1) x+x \otimes s_{1}\left(y_{1}\right)-s_{1}\left((b, 1) y_{1}\right) \otimes(a b, 1) x-s_{0}((b, 1) x) \otimes(b, 1) y_{1} \\
& -x \otimes s_{1}\left((b, 1) y_{1}\right)+s_{1}\left((a, 1) y_{2}\right) \otimes(a b, 1) x+s_{0}((a, 1) x) \otimes(a, 1) y_{2} \\
& +x \otimes s_{1}\left((a, 1) y_{2}\right)-s_{1}\left(y_{2}\right) \otimes(b, 1) x-x \otimes s_{1}\left(y_{2}\right),
\end{align*}
$$

and we get

$$
\begin{align*}
\Delta_{11}\left(z_{1}\right) & =\pi_{11}\left(\Delta_{2}\left(z_{1}\right)\right)=\pi_{11} \circ \tilde{s}_{1}\left(A\left(x \otimes y_{1}\right)+B\left(x \otimes y_{2}\right)+(a, 1) x \otimes(a, 1) y_{3}-x \otimes y_{3}\right) \\
& = \pm \sum s_{0}\left(A_{k} x\right) \otimes A_{k} y_{1} \pm \sum s_{0}\left(B_{k} x\right) \otimes B_{k} y_{2}+y_{1} \otimes(a, 1) y_{3}, \\
\Delta_{11}\left(z_{2}\right) & =\pi_{11}\left(\Delta_{2}\left(z_{2}\right)\right)=\pi_{11} \circ \tilde{s}_{1}\left(C\left(x \otimes y_{1}\right)+D\left(x \otimes y_{2}\right)+(b, 1) x \otimes(b, 1) y_{3}-x \otimes y_{3}\right)  \tag{33}\\
& = \pm \sum s_{0}\left(C_{k} x\right) \otimes C_{k} y_{1} \pm \sum s_{0}\left(D_{k} x\right) \otimes D_{k} y_{2}+y_{2} \otimes(b, 1) y_{3}, \\
\Delta_{11}\left(z_{3}\right) & =\pi_{11}\left(\Delta_{2}\left(z_{3}\right)\right)=-s_{0}((b, 1) x) \otimes(b, 1) y_{1}+s_{0}((a, 1) x) \otimes(a, 1) y_{2} \\
& =y_{1} \otimes(a, 1) y_{2}-y_{2} \otimes(b, 1) y_{1} .
\end{align*}
$$

Now, if $u, v \in \operatorname{Hom}_{\mathbb{Z} G}\left(P_{1}, \mathbb{Z}\right)$, equation (33) implies that the product $[u] \smile[v] \in H^{2}(G, \mathbb{Z})$ is represented by a map $(u \smile v) \in \operatorname{Hom}_{\mathbb{Z} G}\left(P_{2}, \mathbb{Z}\right)$ such that

$$
\begin{aligned}
(u \smile v)\left(z_{1}\right)= & (u \times v) \Delta_{11}\left(z_{1}\right)= \\
= & \left(-\frac{\alpha m_{1}\left(m_{1}+1\right)}{2}\right) u\left(y_{1}\right) \otimes v\left(y_{1}\right)+\left(-\frac{\beta m_{1}\left(m_{1}+1\right)}{2}\right) u\left(y_{2}\right) \otimes v\left(y_{1}\right)+m_{1} u\left(y_{3}\right) \otimes v\left(y_{1}\right)+ \\
& +\left(n_{1}+\frac{\gamma n_{1}\left(n_{1}-1\right)}{2}\right) u\left(y_{1}\right) \otimes v\left(y_{2}\right)+\left(\frac{\delta n_{1}\left(n_{1}-1\right)}{2}\right) u\left(y_{2}\right) \otimes v\left(y_{2}\right)+n_{1} u\left(y_{3}\right) \otimes v\left(y_{2}\right)+ \\
& +u\left(y_{1}\right) \otimes v\left(y_{3}\right), \\
(u \smile v)\left(z_{2}\right)= & (u \times v) \Delta_{11}\left(z_{2}\right)= \\
= & \left(-\frac{\alpha m_{2}\left(m_{2}+1\right)}{2}\right) u\left(y_{1}\right) \otimes v\left(y_{1}\right)+\left(-\frac{\beta m_{2}\left(m_{2}+1\right)}{2}\right) u\left(y_{2}\right) \otimes v\left(y_{1}\right)+m_{2} u\left(y_{3}\right) \otimes v\left(y_{1}\right)+ \\
& +\left(\frac{\gamma n_{2}\left(n_{2}-1\right)}{2}\right) u\left(y_{1}\right) \otimes v\left(y_{2}\right)+\left(n_{2}+\frac{\delta n_{2}\left(n_{2}-1\right)}{2}\right) u\left(y_{2}\right) \otimes v\left(y_{2}\right)+n_{2} u\left(y_{3}\right) \otimes v\left(y_{2}\right)+ \\
& +u\left(y_{2}\right) \otimes v\left(y_{3}\right), \\
(u \smile v)\left(z_{3}\right)= & u\left(y_{1}\right) \otimes v\left(y_{2}\right)-u\left(y_{2}\right) \otimes v\left(y_{1}\right) .
\end{aligned}
$$

Even though the expressions for $A, B, C$, and $D$ depend on the signs of $m_{1}, n_{1}, m_{2}$ and $n_{2}$, the above equations hold in every case.

Using once again Propositions 1 and 2 we also write

$$
\begin{align*}
\Delta_{3}(w) & =\tilde{s}_{2} \Delta_{2} \varphi_{3}(w)=\tilde{s}_{2} \Delta_{2}\left([(1,1)-(b, 1)] z_{1}+[(a, 1)-(1,1)] z_{2}+E z_{3}\right) \\
& =\tilde{s}_{2}\left([(1,1)-(b, 1)] \Delta_{2}\left(z_{1}\right)+[(a, 1)-(1,1)] \Delta_{2}\left(z_{2}\right)+E \Delta_{2}\left(z_{3}\right)\right) \tag{35}
\end{align*}
$$

A term belonging to $P_{1} \otimes P_{2}$ in $\Delta_{3}(w)$ arises when we evaluate $\tilde{s}_{2}$ at an element of $P_{0} \otimes P_{2}$ (this follows from Proposition (1). Hence we have $\Delta_{12}(w)=\pi_{12} \circ \Delta_{3}(w)=\pi_{12} \circ \tilde{s}_{2} \circ \Delta_{02} \circ \partial_{3}(w)$, and looking at (32), we get

$$
\begin{align*}
\Delta_{12}(w)= & \pi_{12} \circ \tilde{s}_{2}\left(\Delta_{02}\left(z_{1}\right)-(b, 1) \Delta_{02}\left(z_{1}\right)+(a, 1) \Delta_{02}\left(z_{2}\right)-\Delta_{02}\left(z_{2}\right)+E \Delta_{02}\left(z_{3}\right)\right) \\
= & \pm \sum y_{1} \otimes(a, 1) s_{1}\left(C_{k} y_{1}\right) \pm \sum y_{1} \otimes(a, 1) s_{1}\left(D_{k} y_{2}\right)  \tag{36}\\
& \mp \sum y_{2} \otimes(b, 1) s_{1}\left(A_{k} y_{1}\right) \mp \sum y_{2} \otimes(b, 1) s_{1}\left(B_{k} y_{2}\right) \\
& +\sum \pm s_{0}\left(E_{k} x\right) \otimes E_{k} z_{3}
\end{align*}
$$

If $u \in \operatorname{Hom}_{\mathbb{Z}}\left(P_{1}, \mathbb{Z}\right)$ and $v \in \operatorname{Hom}_{\mathbb{Z} G}\left(P_{2}, \mathbb{Z}\right)$, then $[u] \smile[v] \in H^{3}(G, \mathbb{Z})$ is represented by a map $u \smile v \in \operatorname{Hom}_{\mathbb{Z} G}\left(P_{3}, \mathbb{Z}\right)$ such that $(u \smile v)(w)=(u \times v) \Delta_{12}(w)$. Using the above expression for $\Delta_{12}(w)$, we can then write

$$
\begin{aligned}
(u \times v)\left(\sum \pm s_{0}\left(E_{k} x\right) \otimes E_{k} z_{3}\right)= & \sum_{(m, n) \in I_{1} \times J_{1}}(\alpha m+\gamma n) u\left(y_{1}\right) \otimes v\left(z_{3}\right)+\sum_{(m, n) \in I_{1} \times J_{1}}(\beta m+\delta n) u\left(y_{2}\right) \otimes v\left(z_{3}\right) \\
& +\sum_{(m, n) \in I_{1} \times J_{1}} u\left(y_{3}\right) \otimes v\left(z_{3}\right) \\
& -\sum_{(m, n) \in I_{2} \times J_{2}}(\alpha m+\gamma n) u\left(y_{1}\right) \otimes v\left(z_{3}\right)-\sum_{(m, n) \in I_{2} \times J_{2}}(\beta m+\delta n) u\left(y_{2}\right) \otimes v\left(z_{3}\right) \\
& -\sum_{(m, n) \in I_{2} \times J_{2}} u\left(y_{3}\right) \otimes v\left(z_{3}\right) \\
= & \left(\alpha\left|J_{1}\right| \sum_{m \in I_{1}} m+\gamma\left|I_{1}\right| \sum_{n \in J_{1}} n-\alpha\left|J_{2}\right| \sum_{m \in I_{2}} m-\gamma\left|I_{2}\right| \sum_{n \in J_{2}} n\right) u\left(y_{1}\right) \otimes v\left(z_{3}\right) \\
& +\left(\beta\left|J_{1}\right| \sum_{m \in I_{1}} m+\delta\left|I_{1}\right| \sum_{n \in J_{1}} n-\beta\left|J_{2}\right| \sum_{m \in I_{2}} m-\delta\left|I_{2}\right| \sum_{n \in J_{2}} n\right) u\left(y_{2}\right) \otimes v\left(z_{3}\right) \\
& +(\operatorname{det} \theta) u\left(y_{3}\right) \otimes v\left(z_{3}\right),
\end{aligned}
$$

and similar calculations show that

$$
\begin{aligned}
(u \times v)\left( \pm \sum y_{1} \otimes(a, 1) s_{1}\left(C_{k} y_{1}\right)\right)= & \left(-\alpha m_{2}\right) u\left(y_{1}\right) \otimes v\left(z_{1}\right)+\left(-\beta m_{2}\right) u\left(y_{1}\right) \otimes v\left(z_{2}\right) \\
& +\left(\frac{\alpha \beta m_{2}}{2}\left(\gamma+\delta+m_{2}+1-\operatorname{det} \theta\right)\right) u\left(y_{1}\right) \otimes v\left(z_{3}\right), \\
(u \times v)\left( \pm \sum y_{1} \otimes(a, 1) s_{1}\left(D_{k} y_{2}\right)\right)= & \left(-\gamma n_{2}\right) u\left(y_{1}\right) \otimes v\left(z_{1}\right)+\left(-\delta n_{2}\right) u\left(y_{1}\right) \otimes v\left(z_{2}\right) \\
& +\left(-\gamma n_{2}-\frac{\gamma \delta n_{2}}{2}\left(-\alpha-\beta+n_{2}-1+\operatorname{det} \theta\right)\right) u\left(y_{1}\right) \otimes v\left(z_{3}\right), \\
(u \times v)\left(\mp \sum y_{2} \otimes(b, 1) s_{1}\left(A_{k} y_{1}\right)\right)= & \left(\alpha m_{1}\right) u\left(y_{2}\right) \otimes v\left(z_{1}\right)+\left(\beta m_{1}\right) u\left(y_{2}\right) \otimes v\left(z_{2}\right) \\
& +\left(\frac{-\alpha \beta m_{1}}{2}\left(\gamma+\delta+m_{1}+1-\operatorname{det} \theta\right)\right) u\left(y_{2}\right) \otimes v\left(z_{3}\right), \\
(u \times v)\left(\mp \sum y_{2} \otimes(b, 1) s_{1}\left(B_{k} y_{2}\right)\right)= & \left(\gamma n_{1}\right) u\left(y_{2}\right) \otimes v\left(z_{1}\right)+\left(\delta n_{1}\right) u\left(y_{2}\right) \otimes v\left(z_{2}\right) \\
& +\left(\frac{\gamma \delta n_{1}}{2}\left(-\alpha-\beta+n_{1}-1+\operatorname{det} \theta\right)\right) u\left(y_{2}\right) \otimes v\left(z_{3}\right) .
\end{aligned}
$$

Substituting those results in (36), we are left with

$$
\begin{align*}
(u \smile v)(w)= & (u \times v) \Delta_{12}(w) \\
= & u\left(y_{1}\right) \otimes v\left(z_{2}\right)-u\left(y_{2}\right) \otimes v\left(z_{1}\right)+ \\
+ & {\left[\frac{\alpha \beta m_{2}}{2}\left(\gamma+\delta+m_{2}+1-\operatorname{det} \theta\right)-\gamma n_{2}-\frac{\gamma \delta n_{2}}{2}\left(-\alpha-\beta+n_{2}-1+\operatorname{det} \theta\right)+\right.} \\
& \left.\quad \alpha\left|J_{1}\right| \sum_{m \in I_{1}} m+\gamma\left|I_{1}\right| \sum_{n \in J_{1}} n-\alpha\left|J_{2}\right| \sum_{m \in I_{2}} m-\gamma\left|I_{2}\right| \sum_{n \in J_{2}} n\right] u\left(y_{1}\right) \otimes v\left(z_{3}\right)+ \\
+ & {\left[\frac{-\alpha \beta m_{1}}{2}\left(\gamma+\delta+m_{1}+1-\operatorname{det} \theta\right)+\frac{\gamma \delta n_{1}}{2}\left(-\alpha-\beta+n_{1}-1+\operatorname{det} \theta\right)+\right.} \\
& \left.\beta\left|J_{1}\right| \sum_{m \in I_{1}} m+\delta\left|I_{1}\right| \sum_{n \in J_{1}} n-\beta\left|J_{2}\right| \sum_{m \in I_{2}} m-\delta\left|I_{2}\right| \sum_{n \in J_{2}} n\right] u\left(y_{2}\right) \otimes v\left(z_{3}\right)+ \\
+ & (\operatorname{det} \theta) u\left(y_{3}\right) \otimes v\left(z_{3}\right) . \tag{37}
\end{align*}
$$

The above equation can be simplified: first, we note that

$$
\begin{aligned}
\frac{\alpha \beta m_{2}}{2}\left(\gamma+\delta+m_{2}\right. & +1-\operatorname{det} \theta)-\gamma n_{2}-\frac{\gamma \delta n_{2}}{2}\left(-\alpha-\beta+n_{2}-1+\operatorname{det} \theta\right)= \\
& = \begin{cases}0, & \text { if } \operatorname{det} \theta=1 \\
\alpha \gamma(\delta-\beta-1), & \text { if } \operatorname{det} \theta=-1\end{cases}
\end{aligned}
$$

and

$$
\begin{gathered}
\frac{-\alpha \beta m_{1}}{2}\left(\gamma+\delta+m_{1}+1-\operatorname{det} \theta\right)+\frac{\gamma \delta n_{1}}{2}\left(-\alpha-\beta+n_{1}-1+\operatorname{det} \theta\right)= \\
= \begin{cases}0, & \text { if } \operatorname{det} \theta=1 \\
\beta \delta(\gamma-\alpha+1), & \text { if } \operatorname{det} \theta=-1\end{cases}
\end{gathered}
$$

We also have the following:
Lemma 3 Let $S: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ be the function defined by

$$
S(x, y)=x\left|J_{1}\right| \sum_{m \in I_{1}} m+y\left|I_{1}\right| \sum_{n \in J_{1}} n-x\left|J_{2}\right| \sum_{m \in I_{2}} m-y\left|I_{2}\right| \sum_{n \in J_{2}} n
$$

where the sets $I_{1}, J_{1}, I_{2}$ and $J_{2}$ are described in Table प. Then

$$
S(\alpha, \gamma)= \begin{cases}\frac{1-\alpha-\gamma-\alpha \gamma}{2}, & \text { if } \operatorname{det} \theta=1 \\ \frac{-1+\alpha+\gamma-\alpha \gamma}{2}, & \text { if } \operatorname{det} \theta=-1\end{cases}
$$

and

$$
S(\beta, \delta)= \begin{cases}\frac{1-\beta-\delta+\beta \delta}{2}, & \text { if } \operatorname{det} \theta=1 \\ \frac{-1+\beta+\delta+\beta \delta}{2}, & \text { if } \operatorname{det} \theta=-1\end{cases}
$$

Proof: The proof is nothing but a straightforward calculation using the descriptions of the sets $I_{1}, J_{1}$, $I_{2}$ and $J_{2}$ and remembering that

$$
\left[\begin{array}{cc}
m_{1} & m_{2} \\
n_{1} & n_{2}
\end{array}\right]=-\left[\begin{array}{cc}
\alpha & \gamma \\
\beta & \delta
\end{array}\right]^{-1}=\frac{1}{\alpha \delta-\beta \gamma}\left[\begin{array}{cc}
-\delta & \gamma \\
\beta & -\alpha
\end{array}\right]
$$

With those two observations, equation (37) can then be written, if $\operatorname{det} \theta=1$, as

$$
\begin{align*}
(u \smile v)(w)= & u\left(y_{1}\right) \otimes v\left(z_{2}\right)-u\left(y_{2}\right) \otimes v\left(z_{1}\right)+\left(\frac{1-\alpha-\gamma-\alpha \gamma}{2}\right) u\left(y_{1}\right) \otimes v\left(z_{3}\right)+ \\
& \left(\frac{1-\beta-\delta+\beta \delta}{2}\right) u\left(y_{2}\right) \otimes v\left(z_{3}\right)+u\left(y_{3}\right) \otimes v\left(z_{3}\right) \tag{38}
\end{align*}
$$

and, if $\operatorname{det} \theta=-1$, we can write

$$
\begin{align*}
(u \smile v)(w)= & u\left(y_{1}\right) \otimes v\left(z_{2}\right)-u\left(y_{2}\right) \otimes v\left(z_{1}\right)+ \\
& {\left[\alpha \gamma(\delta-\beta-1)+\left(\frac{-1+\alpha+\gamma-\alpha \gamma}{2}\right)\right] u\left(y_{1}\right) \otimes v\left(z_{3}\right)+}  \tag{39}\\
& {\left[\beta \delta(\gamma-\alpha+1)+\left(\frac{-1+\beta+\delta+\beta \delta}{2}\right)\right] u\left(y_{2}\right) \otimes v\left(z_{3}\right)-u\left(y_{3}\right) \otimes v\left(z_{3}\right) . }
\end{align*}
$$

We are now finally in a position where we can compute the products

$$
H^{p}(G, \mathbb{Z}) \otimes H^{q}(G, \mathbb{Z}) \breve{\hookrightarrow} H^{p+q}(G, \mathbb{Z} \otimes \mathbb{Z}) \cong H^{p+q}(G, \mathbb{Z})
$$

Theorem 6 The cohomology ring $H^{*}(G, \mathbb{Z})$ is given by:

1. If $\operatorname{rank}(\theta-I)=0$, then

$$
H^{*}(G, \mathbb{Z}) \cong \frac{\mathbb{Z}\left[\zeta_{1}, \zeta_{2}, \zeta_{3}\right]}{\left(\zeta_{1}^{2}=\zeta_{2}^{2}=\zeta_{3}^{2}=0\right)}
$$

where $\operatorname{dim}\left(\zeta_{1}\right)=\operatorname{dim}\left(\zeta_{2}\right)=\operatorname{dim}\left(\zeta_{3}\right)=1$.
2. If $\operatorname{rank}(\theta-I)=1$ and $\operatorname{det} \theta=1$, then

$$
H^{*}(G, \mathbb{Z}) \cong \frac{\mathbb{Z}\left[\zeta_{1}, \zeta_{2}, \xi_{2}, \xi_{3}\right]}{\left(\begin{array}{c}
\zeta_{1}^{2}=\zeta_{2}^{2}=0, \operatorname{gcd}(\beta, \gamma) \zeta_{1} \zeta_{2}=0, \zeta_{2} \xi_{2}=0, \zeta_{2} \xi_{3}=\zeta_{1} \xi_{2}, \\
\zeta_{1} \xi_{3}=\frac{1}{2} \sqrt{\frac{|\beta|}{\operatorname{gcd}(\beta, \gamma)}}(1-\alpha-\alpha \gamma-\beta \gamma) \zeta_{1} \xi_{2} \\
\xi_{2}^{2}=\xi_{3}^{2}=\xi_{2} \xi_{3}=0
\end{array}\right)},
$$

where $\operatorname{dim}\left(\zeta_{1}\right)=\operatorname{dim}\left(\zeta_{2}\right)=1$ and $\operatorname{dim}\left(\xi_{2}\right)=\operatorname{dim}\left(\xi_{3}\right)=2$.
3. If $\operatorname{rank}(\theta-I)=1$, $\operatorname{det} \theta=-1$ and $\operatorname{gcd}(\beta, \gamma, 2)=1$, then

$$
H^{*}(G, \mathbb{Z}) \cong \frac{\mathbb{Z}\left[\zeta_{1}, \zeta_{2}, \xi\right]}{\binom{\zeta_{1}^{2}=\zeta_{2}^{2}=0, \zeta_{1} \zeta_{2}=2 \xi,}{\zeta_{2} \xi=0,2 \zeta_{1} \xi=0, \xi^{2}=0}},
$$

where $\operatorname{dim}\left(\zeta_{1}\right)=\operatorname{dim}\left(\zeta_{2}\right)=1$ and $\operatorname{dim}(\xi)=2$.
4. If $\operatorname{rank}(\theta-I)=1$, $\operatorname{det} \theta=-1$ and $\operatorname{gcd}(\beta, \gamma, 2)=2$, then

$$
H^{*}(G, \mathbb{Z}) \cong \frac{\mathbb{Z}\left[\zeta_{1}, \zeta_{2}, \xi_{1}, \xi_{2}\right]}{\left(\begin{array}{c}
\zeta_{1}^{2}=\zeta_{2}^{2}=0, \zeta_{1} \zeta_{2}=m \xi_{1}+\xi_{2}, 2 \xi_{1}=0 \\
\zeta_{1} \xi_{2}=\left(\frac{k r^{\prime}-\ell s^{\prime}}{2}\right) \zeta_{1} \xi_{1}, \zeta_{2} \xi_{1}=0 \\
\zeta_{2} \xi_{2}=0, \xi_{1}^{2}=\xi_{2}^{2}=\xi_{1} \xi_{2}=0
\end{array}\right)}
$$

where $\operatorname{dim}\left(\zeta_{1}\right)=\operatorname{dim}\left(\zeta_{2}\right)=1, \operatorname{dim}\left(\xi_{1}\right)=\operatorname{dim}\left(\xi_{2}\right)=2$. Also, the integers $k$ and $\ell$ are such that $p k+q \ell=1$, where $p$ and $q$ are relatively prime integers satisfying $\frac{p}{q}=\frac{\gamma}{1+\delta}=\frac{1+\alpha}{\beta}$. Finally, the integers $r^{\prime}, s^{\prime}$ and $m$ are such that $p s^{\prime}+q r^{\prime}=2, m=\frac{2 k-s^{\prime}}{2 q}=\frac{r^{\prime}-2 \ell}{2 p}$.
5. If $\operatorname{rank}(\theta-I)=2 e \operatorname{det} \theta=1$, then

$$
H^{*}(G, \mathbb{Z}) \cong \frac{\mathbb{Z}\left[\zeta, \xi_{1}, \xi_{2}, \xi_{3}\right]}{\binom{\zeta^{2}=0, c_{1} \xi_{1}=c_{2} \xi_{2}=0, \zeta \xi_{1}=\zeta \xi_{2}=0,}{\xi_{1}^{2}=\xi_{2}^{2}=\xi_{3}^{2}=0, \xi_{1} \xi_{2}=\xi_{1} \xi_{3}=\xi_{2} \xi_{3}=0}},
$$

where $\operatorname{dim}(\zeta)=1$ and $\operatorname{dim}\left(\xi_{1}\right)=\operatorname{dim}\left(\xi_{2}\right)=\operatorname{dim}\left(\xi_{3}\right)=2$. Also, the integers $c_{1}$ and $c_{2}$ are such that $c_{1} \mid c_{2}$ and $c_{1} c_{2}=|\operatorname{det}(\theta-I)|$.
6. If $\operatorname{rank}(\theta-I)=2$ and $\operatorname{det} \theta=-1$, then

$$
H^{*}(G, \mathbb{Z}) \cong \frac{\mathbb{Z}\left[\zeta, \xi_{1}, \xi_{2}, \chi\right]}{\binom{\zeta^{2}=0, c_{1} \xi_{1}=c_{2} \xi_{2}=0, \zeta \xi_{1}=0, \zeta \xi_{2}=0,}{\zeta \chi=0, \xi_{1}^{2}=\xi_{2}^{2}=0, \xi_{1} \chi=0, \xi_{2} \chi=0, \chi^{2}=0}},
$$

where $\operatorname{dim}(\zeta)=1, \operatorname{dim}\left(\xi_{1}\right)=\operatorname{dim}\left(\xi_{2}\right)=2, \operatorname{dim}(\chi)=3$. Also, the integers $c_{1}$ and $c_{2}$ are such that $c_{1} \mid c_{2}$ and $c_{1} c_{2}=|\operatorname{det}(\theta-I)|$.

Proof: First case: if $\operatorname{rank}(\theta-I)=0$, then

$$
\begin{aligned}
H^{1}(G, \mathbb{Z}) & =\left\langle\left[y_{1}^{*}\right]\right\rangle \oplus\left\langle\left[y_{2}^{*}\right]\right\rangle \oplus\left\langle\left[y_{3}^{*}\right]\right\rangle \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \\
H^{2}(G, \mathbb{Z}) & =\left\langle\left[z_{1}^{*}\right]\right\rangle \oplus\left\langle\left[z_{2}^{*}\right]\right\rangle \oplus\left\langle\left[z_{3}^{*}\right]\right\rangle \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \\
H^{3}(G, \mathbb{Z}) & =\left\langle\left[w^{*}\right]\right\rangle \cong \mathbb{Z}
\end{aligned}
$$

and equations (34) and (38) give us the cohomology ring $H^{*}(G, \mathbb{Z})$ exactly like in the statement of the theorem, with $\left[y_{j}^{*}\right]=\zeta_{j}$.

Second case: $\operatorname{rank}(\theta-I)=1$ and $\operatorname{det} \theta=1$. Just like in the proof of Theorem 3 suppose $1+m_{1} \neq 0$ oor $n_{1} \neq 0$ (the case where $1+m_{1}=n_{1}=0$ is analogous). We have

$$
I-\theta^{-1}=\left[\begin{array}{cc}
1+m_{1} & m_{2} \\
n_{1} & 1+n_{2}
\end{array}\right]=\left[\begin{array}{cc}
q r^{\prime} & p r^{\prime} \\
q s^{\prime} & p s^{\prime}
\end{array}\right]
$$

where the integers $p$ and $q$ satisfy $\operatorname{gcd}(p, q)=1$ and $p, q \neq 0$. The generators of $H^{1}(G, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ are $[u]$ and $\left[y_{3}^{*}\right]$, where

$$
u=-\frac{q s^{\prime}}{\operatorname{gcd}\left(q r^{\prime}, q s^{\prime}\right)} y_{1}^{*}+\frac{q r^{\prime}}{\operatorname{gcd}\left(q r^{\prime}, q s^{\prime}\right)} y_{2}^{*}=-\frac{s^{\prime}}{\operatorname{gcd}\left(r^{\prime}, s^{\prime}\right)} y_{1}^{*}+\frac{r^{\prime}}{\operatorname{gcd}\left(r^{\prime}, s^{\prime}\right)} y_{2}^{*} .
$$

We can say even more: we have $\operatorname{det}\left(I-\theta^{-1}\right)=0 \Leftrightarrow \alpha+\delta=2$. But $\alpha+\delta=2 \Leftrightarrow\left(1-q r^{\prime}\right)+(1-$ $\left.p s^{\prime}\right)=2 \Leftrightarrow q r^{\prime}+p s^{\prime}=0$. Hence $p \mid r^{\prime}$ and $q \mid s^{\prime}$. Writing $r^{\prime}=p r^{\prime \prime}$ and $s^{\prime}=q s^{\prime \prime}$, we then get $p q r^{\prime \prime}+p q s^{\prime \prime}=0 \Leftrightarrow s^{\prime \prime}=-r^{\prime \prime}$. We can then write

$$
I-\theta^{-1}=\left[\begin{array}{cc}
1+m_{1} & m_{2}  \tag{40}\\
n_{1} & 1+n_{2}
\end{array}\right]=\left[\begin{array}{cc}
(1-\delta) & \gamma \\
\beta & (1-\alpha)
\end{array}\right]=\left[\begin{array}{cc}
p q r^{\prime \prime} & p^{2} r^{\prime \prime} \\
-q^{2} r^{\prime \prime} & -p q r^{\prime \prime}
\end{array}\right]
$$

and also $\operatorname{gcd}(\beta, \gamma)=\operatorname{gcd}\left(r^{\prime}, s^{\prime}\right)=\left|r^{\prime \prime}\right|$, so

$$
u=-\frac{s^{\prime}}{\operatorname{gcd}\left(r^{\prime}, s^{\prime}\right)} y_{1}^{*}+\frac{r^{\prime}}{\operatorname{gcd}\left(r^{\prime}, s^{\prime}\right)} y_{2}^{*}=q y_{1}^{*}+p y_{2}^{*}
$$

Using equation (34), we obtain

$$
\begin{aligned}
(u \smile u)\left(z_{1}\right)= & -\frac{\left(1+p q r^{\prime \prime}\right)\left(p q r^{\prime \prime}\right)\left(p q r^{\prime \prime}-1\right) q^{2}}{2}+\frac{\left(q^{2} r^{\prime \prime}\right)\left(p q r^{\prime \prime}\right)\left(p q r^{\prime \prime}-1\right) p q}{2}+ \\
& +\left(-q^{2} r^{\prime \prime}+\frac{\left(p^{2} r^{\prime \prime}\right)\left(-q^{2} r^{\prime \prime}\right)\left(-q^{2} r^{\prime \prime}-1\right)}{2}\right) p q+\frac{\left(1-p q r^{\prime \prime}\right)\left(-q^{2} r^{\prime \prime}\right)\left(-q^{2} r^{\prime \prime}-1\right) p^{2}}{2} \\
= & \frac{p q^{2}(p-q) r^{\prime \prime}}{2}, \\
(u \smile u)\left(z_{2}\right)= & \frac{p^{2} q(p-q) r^{\prime \prime}}{2}, \\
(u \smile u)\left(z_{3}\right)= & 0 .
\end{aligned}
$$

Thus $(u \smile u)=\frac{p q(p-q) r^{\prime \prime}}{2}\left(q z_{1}^{*}+p z_{2}^{*}\right)$. We know that $H^{2}(G, \mathbb{Z}) \cong \mathbb{Z}_{\operatorname{gcd}(\beta, \gamma)} \oplus \mathbb{Z} \oplus \mathbb{Z} \cong \mathbb{Z}_{r^{\prime \prime}} \oplus \mathbb{Z} \oplus \mathbb{Z}$. The generator of one of the $\mathbb{Z}$ factors in $H^{2}(G, \mathbb{Z})$ is $\left[z_{3}^{*}\right]$. The other two generators are obtained from the Smith normal form of

$$
I-\theta^{-1}=\left[\begin{array}{cc}
1+m_{1} & m_{2} \\
n_{1} & 1+n_{2}
\end{array}\right]=\left[\begin{array}{cc}
p q r^{\prime \prime} & p^{2} r^{\prime \prime} \\
-q^{2} r^{\prime \prime} & -p q r^{\prime \prime}
\end{array}\right]
$$

Let $k$ and $\ell$ be integers such that $p k+q \ell=1$. We have $\left[\begin{array}{cc}\ell & -p \\ k & q\end{array}\right] \in G L_{2}(\mathbb{Z})$ and

$$
\left[\begin{array}{cc}
\ell & -p \\
k & q
\end{array}\right]^{-1}=\left[\begin{array}{cc}
q & p \\
-k & \ell
\end{array}\right]
$$

which gives us

$$
\left(I-\theta^{-1}\right)\left[\begin{array}{cc}
\ell & -p \\
k & q
\end{array}\right]=\left[\begin{array}{cc}
p r^{\prime \prime} & 0 \\
-q r^{\prime \prime} & 0
\end{array}\right] \quad \text { and } \quad\left[\begin{array}{cc}
\ell & -p \\
k & q
\end{array}\right]^{-1}\left[\begin{array}{l}
z_{1}^{*} \\
z_{2}^{*}
\end{array}\right]=\left[\begin{array}{c}
q z_{1}^{*}+p z_{2}^{*} \\
-k z_{1}^{*}+\ell z_{2}^{*}
\end{array}\right]
$$

Hence the generator of the factor $\mathbb{Z}_{r^{\prime \prime}}$ of $H^{2}(G, \mathbb{Z})$ is the class of the map $q z_{1}^{*}+p z_{2}^{*}$, and the generator of the other factor $\mathbb{Z}$ of $H^{2}(G, \mathbb{Z})$ is the class of $-k z_{1}^{*}+\ell z_{2}^{*}$. As $p q(p-q)$ is even, we get

$$
(u \smile u)=\left(\frac{p q(p-q)}{2}\right) \cdot r^{\prime \prime}\left(q z_{1}^{*}+p z_{2}^{*}\right)
$$

which means that $[u]^{2}=0$. We also have $\left[y_{3}^{*}\right]^{2}=0$. Let us compute $[u] \smile\left[y_{3}^{*}\right]$ : using (34), we obtain

$$
\begin{aligned}
& \left(u \smile y_{3}^{*}\right)\left(z_{1}\right)=q, \\
& \left(u \smile y_{3}^{*}\right)\left(z_{2}\right)=p, \\
& \left(u \smile y_{3}^{*}\right)\left(z_{3}\right)=0,
\end{aligned}
$$

so $[u] \smile\left[y_{3}^{*}\right]=\left[q z_{1}^{*}+p z_{2}^{*}\right]$.

Now, to compute the products $H^{1}(G, \mathbb{Z}) \otimes H^{2}(G, \mathbb{Z}) \hookrightarrow H^{3}(G, \mathbb{Z})$, we use (38), that gives us

$$
\begin{aligned}
& {[u] \smile\left[q z_{1}^{*}+p z_{2}^{*}\right]=0,} \\
& {[u] \smile\left[-k z_{1}^{*}+\ell z_{2}^{*}\right]=\left[w^{*}\right],} \\
& {\left[y_{3}^{*}\right] \smile\left[q z_{1}^{*}+p z_{2}^{*}\right]=0,} \\
& {\left[y_{3}^{*}\right] \smile\left[-k z_{1}^{*}+\ell z_{2}^{*}\right]=0,} \\
& {\left[y_{3}^{*}\right] \smile\left[z_{3}^{*}\right]=\left[w^{*}\right] .}
\end{aligned}
$$

We are then left with the computation of $[u] \smile\left[z_{3}^{*}\right]$. Equation (38) gives us

$$
\begin{equation*}
\left(u \smile z_{3}^{*}\right)(w)=\left(\frac{1-\alpha-\gamma-\alpha \gamma}{2}\right) q+\left(\frac{1-\beta-\delta+\beta \delta}{2}\right) p \tag{41}
\end{equation*}
$$

Assuming that $q \geq 0$, it follows from (40) that

$$
r^{\prime \prime}=-\frac{\beta}{|\beta|} \operatorname{gcd}(\beta, \gamma)=\frac{\gamma}{|\gamma|} \operatorname{gcd}(\beta, \gamma) \quad \text { and } \quad q=\sqrt{\frac{-\beta}{r^{\prime \prime}}}=\sqrt{\frac{|\beta|}{\operatorname{gcd}(\beta, \gamma)}}
$$

We also have

$$
\frac{p}{q}=\frac{1-\alpha}{\beta} \Leftrightarrow p=\sqrt{\frac{|\beta|}{\operatorname{gcd}(\beta, \gamma)}} \cdot \frac{1-\alpha}{\beta}
$$

and if we substitute the values of $p$ and $q$ in (41), we get

$$
\begin{equation*}
[u] \smile\left[z_{3}^{*}\right]=\frac{1}{2} \sqrt{\frac{|\beta|}{\operatorname{gcd}(\beta, \gamma)}}\left(1-\alpha-\gamma-\alpha \gamma+\frac{(1-\beta)(1-\delta)(1-\alpha)}{\beta}\right)\left[w^{*}\right] . \tag{42}
\end{equation*}
$$

But $\operatorname{det} \theta=1$ and $\alpha+\delta=2$, so $\alpha(2-\alpha)-\beta \gamma=1 \Leftrightarrow-\beta \gamma=(\alpha-1)^{2}$ and $1-\delta=\alpha-1$. Substituting in (42), we are left with

$$
\begin{equation*}
[u] \smile\left[z_{3}^{*}\right]=\frac{1}{2} \sqrt{\frac{|\beta|}{\operatorname{gcd}(\beta, \gamma)}}(1-\alpha-\alpha \gamma-\beta \gamma)\left[w^{*}\right] . \tag{43}
\end{equation*}
$$

We get the statement defining $\zeta_{1}=[u], \zeta_{2}=\left[y_{3}^{*}\right], \xi_{2}=\left[-k z_{1}^{*}+\ell z_{2}^{*}\right], \xi_{3}=\left[z_{3}^{*}\right]$ and $\chi=\left[w^{*}\right]$. Observe that $\xi_{1}=\left[q z_{1}^{*}+p z_{2}^{*}\right]$ is equal to $\zeta_{1} \smile \zeta_{2}$.

Let us see now what happens if $\operatorname{rank}(\theta-I)=1$ and $\operatorname{det} \theta=-1$. Again we assume $1+m_{1} \neq 0$ or $n_{1} \neq 0$ and write

$$
I-\theta^{-1}=\left[\begin{array}{cc}
1+m_{1} & m_{2} \\
n_{1} & 1+n_{2}
\end{array}\right]=\left[\begin{array}{cc}
q r^{\prime} & p r^{\prime} \\
q s^{\prime} & p s^{\prime}
\end{array}\right]
$$

where the integers $p$ and $q$ satisfy $\operatorname{gcd}(p, q)=1$ and $p, q \neq 0$. In this case, we have $\operatorname{det}\left(I-\theta^{-1}\right)=0 \Leftrightarrow \alpha+$ $\delta=0$. Hence $\left(1+m_{1}\right)+\left(1+n_{2}\right)=(1+\delta)+(1+\alpha)=2 \Leftrightarrow q r^{\prime}+p s^{\prime}=2$, so $\operatorname{gcd}(\beta, \gamma, 2)=\operatorname{gcd}\left(r^{\prime}, s^{\prime}\right) \in\{1,2\}$. We separate the analysis in two subcases.

Third case: First we see what happens when $\operatorname{gcd}(\beta, \gamma, 2)=\operatorname{gcd}\left(r^{\prime}, s^{\prime}\right)=1$. If that is the case, then

$$
\begin{aligned}
H^{1}(G, \mathbb{Z}) & =\langle[u]\rangle \oplus\left\langle\left[y_{3}^{*}\right]\right\rangle \cong \mathbb{Z} \oplus \mathbb{Z} \\
H^{2}(G, \mathbb{Z}) & =\left\langle\left[-k z_{1}^{*}+\ell z_{2}^{*}\right]\right\rangle \cong \mathbb{Z} \\
H^{3}(G, \mathbb{Z}) & =\left\langle\left[w^{*}\right] \mid 2 \cdot\left[w^{*}\right]=0\right\rangle \cong \mathbb{Z}_{2}
\end{aligned}
$$

where $u=-s^{\prime} y_{1}^{*}+r^{\prime} y_{2}^{*}$ and the integers $k$ and $\ell$ satisfy $p k+q \ell=1$. Looking at the Smith normal form of $\left(I-\theta^{-1}\right)$ like we did in the previous case, we get that $\left[q z_{1}^{*}+p z_{2}^{*}\right]=0$ in $H^{2}(G, \mathbb{Z})$. Using equation (34), we obtain

$$
(u \smile u)=\frac{r^{\prime} s^{\prime}\left(-r^{\prime}-s^{\prime}\right)}{2}\left(q z_{1}^{*}+p z_{2}^{*}\right)
$$

that implies $[u]^{2}=0$. We also have $\left[y_{3}^{*}\right]^{2}=0$ and $\left(u \smile y_{3}^{*}\right)=-s^{\prime} z_{1}^{*}+r^{\prime} z_{2}^{*}$.

As $p k+q \ell=1 \Leftrightarrow p(2 k)+q(2 \ell)=2$ and $p s^{\prime}+q r^{\prime}=2$, there is an integer $m$ such that

$$
\left\lvert\, \begin{aligned}
& s^{\prime}=2 k-q m \\
& r^{\prime}=2 \ell+p m
\end{aligned} .\right.
$$

We know that $m$ is odd, since $\operatorname{gcd}\left(r^{\prime}, s^{\prime}\right)=1$. Hence

$$
[u] \smile\left[y_{3}^{*}\right]=\left[-2 k z_{1}^{*}+2 \ell z_{2}^{*}\right]+\left[m q z_{1}^{*}+m p z_{2}^{*}\right]=2\left[-k z_{1}^{*}+\ell z_{2}^{*}\right] .
$$

As to the products $H^{1}(G, \mathbb{Z}) \otimes H^{2}(G, \mathbb{Z}) \hookrightarrow H^{3}(G, \mathbb{Z})$, we use equation (39) to get that $\left[y_{3}^{*}\right] \smile$ $\left[-k z_{1}^{*}+\ell z_{2}^{*}\right]=0$, and we also have

$$
\begin{aligned}
\left(u \smile\left(-k z_{1}^{*}+\ell z_{2}^{*}\right)\right) & =\left(r^{\prime} k-s^{\prime} \ell\right) w^{*} \\
& =((2 \ell+p m) k-(2 k-q m) \ell) w^{*} \\
& =m w^{*},
\end{aligned}
$$

and, since $m$ is odd, $[u] \smile\left[-k z_{1}^{*}+\ell z_{2}^{*}\right]=\left[w^{*}\right]$. We get the statement of the theorem in this case by letting $\zeta_{1}=[u], \zeta_{2}=\left[y_{3}^{*}\right]$ and $\xi=\left[-k z_{1}^{*}+\ell z_{2}^{*}\right]$.

Fourth case: Let us now analyze the case $\operatorname{rank}(\theta-I)=1, \operatorname{det} \theta=-1$ and $\operatorname{gcd}(\beta, \gamma, 2)=\operatorname{gcd}\left(r^{\prime}, s^{\prime}\right)=2$. We have

$$
\begin{aligned}
H^{1}(G, \mathbb{Z}) & =\langle[u]\rangle \oplus\left\langle\left[y_{3}^{*}\right]\right\rangle \cong \mathbb{Z} \oplus \mathbb{Z} \\
H^{2}(G, \mathbb{Z}) & =\left\langle\left[q z_{1}^{*}+p z_{2}^{*}\right]\right\rangle \oplus\left\langle\left[-k z_{1}^{*}+\ell z_{2}^{*}\right]\right\rangle \cong \mathbb{Z}_{2} \oplus \mathbb{Z}, \\
H^{3}(G, \mathbb{Z}) & =\left\langle\left[w^{*}\right]\right\rangle \cong \mathbb{Z}_{2}
\end{aligned}
$$

where $u=-\frac{s^{\prime}}{2} y_{1}^{*}+\frac{r^{\prime}}{2} y_{2}^{*}$ and the integers $k$ and $\ell$ such that $p k+q \ell=1$. Using (34), we get

$$
(u \smile u)=\frac{r^{\prime} s^{\prime}\left(-r^{\prime}-s^{\prime}\right)}{8}\left(q z_{1}^{*}+p z_{2}^{*}\right)
$$

But $r^{\prime}$ and $s^{\prime}$ are even, so $r^{\prime} s^{\prime}\left(-r^{\prime}-s^{\prime}\right) \equiv 0(\bmod 16)$, which implies $[u]^{2}=0$. We also have $\left[y_{3}\right]^{2}=0$ and

$$
\left(u \smile y_{3}^{*}\right)=-\frac{s^{\prime}}{2} z_{1}^{*}+\frac{r^{\prime}}{2} z_{2}^{*}
$$

There is an integer $m$ such that

$$
\left\lvert\, \begin{aligned}
& s^{\prime}=2 k-q m, \\
& r^{\prime}=2 \ell+p m .
\end{aligned}\right.
$$

In this case $m$ is even, $m=2 m^{\prime}$, hence

$$
\left(u \smile y_{3}^{*}\right)=\left(-k+q m^{\prime}\right) z_{1}^{*}+\left(\ell+p m^{\prime}\right) z_{2}^{*}=\left(-k z_{1}^{*}+\ell z_{2}^{*}\right)+m^{\prime}\left(q z_{1}^{*}+p z_{2}^{*}\right),
$$

from where it follows that $[u] \smile\left[y_{3}^{*}\right]=m^{\prime}\left[q z_{1}^{*}+p z_{2}^{*}\right]+\left[-k z_{1}^{*}+\ell z_{2}^{*}\right]$, and the integer $m^{\prime}$ may be even or odd.

We still have $[u] \smile\left[q z_{1}^{*}+p z_{2}^{*}\right]$ and $[u] \smile\left[-k z_{1}^{*}+\ell z_{2}^{*}\right]$ left to compute, since equation (39) gives us at once that $\left[y_{3}^{*}\right] \smile\left[q z_{1}^{*}+p z_{2}^{*}\right]=0$ and $\left[y_{3}^{*}\right] \smile\left[-k z_{1}^{*}+\ell z_{2}^{*}\right]=0$. We have

$$
\left(u \smile\left(q z_{1}^{*}+p z_{2}^{*}\right)\right)=\frac{-s^{\prime} p-r^{\prime} q}{2} w^{*}=-w^{*}
$$

so $[u] \smile\left[q z_{1}^{*}+p z_{2}^{*}\right]=\left[w^{*}\right]$. Finally,

$$
\left(u \smile\left(-k z_{1}^{*}+\ell z_{2}^{*}\right)\right)=\frac{k r^{\prime}-\ell s^{\prime}}{2} w^{*}
$$

hence $[u] \smile\left[-k z_{1}^{*}+\ell z_{2}^{*}\right]=\frac{k r^{\prime}-\ell s^{\prime}}{2}\left[w^{*}\right]$, where the integer $\frac{k r^{\prime}-\ell s^{\prime}}{2}$ may be even or odd. We get the statement of the theorem in this case setting $\zeta_{1}=[u], \zeta_{2}=\left[y_{3}^{*}\right], \xi_{1}=\left[q z_{1}^{*}+p z_{2}^{*}\right], \xi_{2}=\left[-k z_{1}^{*}+\ell z_{2}^{*}\right]$.

Fifth case: $\operatorname{rank}(\theta-I)=2$ e $\operatorname{det} \theta=1$. In this case we have

$$
\begin{aligned}
& H^{1}(G, \mathbb{Z})=\left\langle\left[y_{3}^{*}\right]\right\rangle \cong \mathbb{Z} \\
& H^{2}(G, \mathbb{Z})=\left\langle[u] \mid c_{1} \cdot[u]=0\right\rangle \oplus\left\langle[v] \mid c_{2} \cdot[v]=0\right\rangle \oplus\left\langle\left[z_{3}^{*}\right]\right\rangle \cong \mathbb{Z}_{c_{1}} \oplus \mathbb{Z}_{c_{2}} \oplus \mathbb{Z}, \\
& H^{3}(G, \mathbb{Z})=\left\langle\left[w^{*}\right]\right\rangle \cong \mathbb{Z},
\end{aligned}
$$

From (34), we get $\left[y_{3}^{*}\right]^{2}=0$. The generators of the factors $\mathbb{Z}_{c_{1}}$ and $\mathbb{Z}_{c_{2}}$ of $H^{2}(G, \mathbb{Z})$ are the classes of maps $u$ and $v$ that are linear combinations of $z_{1}^{*}$ and $z_{2}^{*}$, hence equation (38) shows us immediately that $\left[y_{3}^{*}\right] \smile[u]=\left[y_{3}^{*}\right] \smile[v]=0$ and $\left[y_{3}^{*}\right] \smile\left[z_{3}^{*}\right]=\left[w^{*}\right]$. We get the statement of the theorem in this case setting $\zeta=\left[y_{3}^{*}\right], \xi_{1}=[u], \xi_{2}=[v], \xi_{3}=\left[z_{3}^{*}\right]$.

Sixth case: $\operatorname{rank}(\theta-I)=2$ e $\operatorname{det} \theta=-1$. In this case we have

$$
\begin{aligned}
H^{1}(G, \mathbb{Z}) & =\left\langle\left[y_{3}^{*}\right]\right\rangle \cong \mathbb{Z} \\
H^{2}(G, \mathbb{Z}) & =\langle[u]\rangle \oplus\langle[v]\rangle \cong \mathbb{Z}_{c_{1}} \oplus \mathbb{Z}_{c_{2}} \\
H^{3}(G, \mathbb{Z}) & =\left\langle\left[w^{*}\right]\right\rangle \cong \mathbb{Z}_{2}
\end{aligned}
$$

This case is similar to the last one. We have $\left[y_{3}^{*}\right]^{2}=0$ and $\left[y_{3}^{*}\right] \smile[u]=\left[y_{3}^{*}\right] \smile[v]=0$. We get the statement defining $\zeta=\left[y_{3}^{*}\right], \xi_{1}=[u], \xi_{2}=[v], \chi=\left[w^{*}\right]$.

Once we've computed the cup product in $H^{*}(G, \mathbb{Z})$, doing the same in $H^{*}\left(G, \mathbb{Z}_{2}\right)$ and $H^{*}\left(G, \mathbb{Z}_{p}\right)$ is easy, since essentially all we have to do is reduce everything modulo 2 and $p$, respectively. We summarize the results we get in the next two theorems.
Theorem 7 The cohomology ring $H^{*}\left(G, \mathbb{Z}_{2}\right)$ is given by

1. If $\theta \equiv\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right](\bmod 2)$, then

$$
H^{*}\left(G, \mathbb{Z}_{2}\right) \cong \frac{\mathbb{Z}_{2}\left[\zeta_{1}, \zeta_{2}, \zeta_{3}\right]}{\binom{\zeta_{1}^{2}=\left(\frac{1+m_{1}}{2}\right) \zeta_{1} \zeta_{3}+\frac{m_{2}}{2} \zeta_{2} \zeta_{3}}{\zeta_{2}^{2}=\frac{n_{1}}{2} \zeta_{1} \zeta_{3}+\left(1+\frac{n_{2}-1}{2}\right) \zeta_{2} \zeta_{3}, \zeta_{3}^{2}=0}},
$$

where $\operatorname{dim}\left(\zeta_{1}\right)=\operatorname{dim}\left(\zeta_{2}\right)=\operatorname{dim}\left(\zeta_{3}\right)=1$.
2. If $\theta \equiv\left[\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right](\bmod 2)$, then

$$
H^{*}\left(G, \mathbb{Z}_{2}\right) \cong \frac{\mathbb{Z}_{2}\left[\zeta_{1}, \zeta_{2}, \xi_{1}, \xi_{2}\right]}{\binom{\zeta_{1}^{2}=\left(1+\frac{\alpha+\delta+m_{2}+n_{1}}{2}\right) \xi_{1}, \zeta_{2}^{2}=0, \zeta_{1} \zeta_{2}=0, \zeta_{1} \xi_{1}=\zeta_{2} \xi_{2},}{\zeta_{1} \xi_{2}=(S(\alpha, \gamma)+S(\beta, \delta)) \zeta_{1} \xi_{1}, \xi_{1}^{2}=0, \xi_{2}^{2}=0, \xi_{1} \xi_{2}=0}}
$$

where $\operatorname{dim}\left(\zeta_{1}\right)=\operatorname{dim}\left(\zeta_{2}\right)=1$ and $\operatorname{dim}\left(\xi_{1}\right)=\operatorname{dim}\left(\xi_{2}\right)=2$.
3. If $\theta \equiv\left[\begin{array}{ll}1 & 1 \\ 1 & 0\end{array}\right](\bmod 2)$, then

$$
H^{*}\left(G, \mathbb{Z}_{2}\right) \cong \frac{\mathbb{Z}_{2}[\zeta, \xi]}{\left(\zeta^{2}=0, \xi^{2}=0\right)}
$$

where $\operatorname{dim}(\zeta)=1$ and $\operatorname{dim}(\xi)=2$.
4. If $\theta \equiv\left[\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right](\bmod 2)$, then

$$
H^{*}\left(G, \mathbb{Z}_{2}\right) \cong \frac{\mathbb{Z}_{2}\left[\zeta_{1}, \zeta_{2}, \xi_{1}, \xi_{2}\right]}{\binom{\zeta_{1}^{2}=\frac{\gamma}{2} \xi_{1}, \zeta_{2}^{2}=0, \zeta_{1} \zeta_{2}=0, \zeta_{1} \xi_{2}=S(\alpha, \gamma) \zeta_{1} \xi_{1},}{\zeta_{2} \xi_{1}=0, \zeta_{2} \xi_{2}=\zeta_{1} \xi_{1}, \xi_{1}^{2}=0, \xi_{2}^{2}=0, \xi_{1} \xi_{2}=0}}
$$

where $\operatorname{dim}\left(\zeta_{1}\right)=\operatorname{dim}\left(\zeta_{2}\right)=1$ and $\operatorname{dim}\left(\xi_{1}\right)=\operatorname{dim}\left(\xi_{2}\right)=2$.
5. If $\theta \equiv\left[\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right](\bmod 2)$, then

$$
H^{*}\left(G, \mathbb{Z}_{2}\right) \cong \frac{\mathbb{Z}_{2}\left[\zeta_{1}, \zeta_{2}, \xi_{1}, \xi_{2}\right]}{\binom{\zeta_{1}^{2}=\frac{\beta}{2} \xi_{1}, \zeta_{2}^{2}=0, \zeta_{1} \zeta_{2}=0, \zeta_{1} \xi_{2}=S(\beta, \delta) \zeta_{1} \xi_{1},}{\zeta_{2} \xi_{1}=0, \zeta_{2} \xi_{2}=\zeta_{1} \xi_{1}, \xi_{1}^{2}=0, \xi_{2}^{2}=0, \xi_{1} \xi_{2}=0}}
$$

where $\operatorname{dim}\left(\zeta_{1}\right)=\operatorname{dim}\left(\zeta_{2}\right)=1$ and $\operatorname{dim}\left(\xi_{1}\right)=\operatorname{dim}\left(\xi_{2}\right)=2$.
6. If $\theta \equiv\left[\begin{array}{ll}0 & 1 \\ 1 & 1\end{array}\right](\bmod 2)$, then

$$
H^{*}\left(G, \mathbb{Z}_{2}\right) \cong \frac{\mathbb{Z}_{2}[\zeta, \xi]}{\left(\zeta^{2}=0, \xi^{2}=0\right)}
$$

where $\operatorname{dim}(\zeta)=1$ and $\operatorname{dim}(\xi)=2$.
Theorem 8 The cohomology ring $H^{*}\left(G, \mathbb{Z}_{p}\right)$, for $p$ an odd prime, is given by

1. If $\operatorname{rank}_{\mathbb{Z}_{p}}(\theta-I)=0$ and $\operatorname{det} \theta=1$, then

$$
H^{*}\left(G, \mathbb{Z}_{p}\right) \cong \frac{\mathbb{Z}_{p}\left[\zeta_{1}, \zeta_{2}, \zeta_{3}\right]}{\left(\zeta_{1}^{2}=\zeta_{2}^{2}=\zeta_{3}^{2}=0\right)}
$$

where $\operatorname{dim}\left(\zeta_{1}\right)=\operatorname{dim}\left(\zeta_{2}\right)=\operatorname{dim}\left(\zeta_{3}\right)=1$.
2. If $\operatorname{rank}_{\mathbb{Z}_{p}}(\theta-I)=0$ and $\operatorname{det} \theta=-1$, then

$$
H^{*}\left(G, \mathbb{Z}_{p}\right) \cong \frac{\mathbb{Z}_{p}\left[\zeta_{1}, \zeta_{2}, \zeta_{3}\right]}{\left(\zeta_{1}^{2}=\zeta_{2}^{2}=\zeta_{3}^{2}=\zeta_{1} \zeta_{2}=0\right)}
$$

where $\operatorname{dim}\left(\zeta_{1}\right)=\operatorname{dim}\left(\zeta_{2}\right)=\operatorname{dim}\left(\zeta_{3}\right)=1$.
3. If $\operatorname{rank}_{\mathbb{Z}_{p}}(\theta-I)=1$ and $\operatorname{det} \theta=1$, then

$$
\begin{aligned}
& H^{1}\left(G, \mathbb{Z}_{p}\right)=\langle[u]\rangle \oplus\left\langle\left[y_{3}^{*}\right]\right\rangle \cong \mathbb{Z}_{p} \oplus \mathbb{Z}_{p} \\
& H^{2}\left(G, \mathbb{Z}_{p}\right)=\frac{\left\langle z_{1}^{*}\right\rangle \oplus\left\langle z_{2}^{*}\right\rangle}{\operatorname{im} \partial_{2}^{*}} \oplus\left\langle\left[z_{3}^{*}\right]\right\rangle \cong \mathbb{Z}_{p} \oplus \mathbb{Z}_{p} \\
& H^{3}\left(G, \mathbb{Z}_{p}\right)=\left\langle\left[w^{*}\right]\right\rangle \cong \mathbb{Z}_{p}
\end{aligned}
$$

where $u$ is described in the following way: we have

$$
\left[\begin{array}{cc}
m_{1} & m_{2} \\
n_{1} & n_{2}
\end{array}\right]=-\theta^{-1}=\left[\begin{array}{cc}
-\delta & \gamma \\
\beta & -\alpha
\end{array}\right] .
$$

If $\left(1+m_{1}\right) \not \equiv 0(\bmod p)$ or $n_{1} \not \equiv 0(\bmod p)$, we take $u=-n_{1} y_{1}^{*}+\left(1+m_{1}\right) y_{2}^{*}$ and, if $\left(1+m_{1}\right) \equiv$ $n_{1} \equiv 0(\bmod p)$, we take $u=\left(1+n_{2}\right) y_{1}^{*}-m_{2} y_{2}^{*}$. Assuming, without loss of generality, that $u=-n_{1} y_{1}^{*}+\left(1+m_{1}\right) y_{2}^{*}$, and taking the class of $m_{1} z_{1}^{*}+n_{1} z_{2}^{*}$ as the generator of $\frac{\left\langle z_{1}^{*}\right\rangle \oplus\left\langle z_{2}^{*}\right\rangle}{\operatorname{im} \partial_{2}^{*}} \cong \mathbb{Z}_{p}$, we have

$$
H^{*}\left(G, \mathbb{Z}_{p}\right) \cong \frac{\mathbb{Z}_{p}\left[\zeta_{1}, \zeta_{2}, \xi_{1}, \xi_{2}\right]}{\left(\begin{array}{c}
\zeta_{1}^{2}=0, \zeta_{2}^{2}=0, \zeta_{1} \zeta_{2}=\lambda \xi_{1}, \zeta_{2} \xi_{1}=0, \\
\zeta_{1} \xi_{1}=\left(-n_{1}^{2}-\left(1+m_{1}\right) m_{1}\right) \zeta_{2} \xi_{2} \\
\zeta_{1} \xi_{2}=\left(\frac{(1-\alpha-\gamma-\alpha \gamma)}{2}\left(-n_{1}\right)+\frac{(1-\beta-\delta+\beta \delta)}{2}\left(1+m_{1}\right)\right) \zeta_{2} \xi_{2}, \\
\xi_{1}^{2}=0, \xi_{2}^{2}=0, \xi_{1} \xi_{2}=0
\end{array}\right)}
$$

where $\zeta_{1}=[u], \zeta_{2}=\left[y_{3}^{*}\right], \xi_{1}=\left[m_{1} z_{1}^{*}+n_{1} z_{2}^{*}\right], \xi_{2}=\left[z_{3}^{*}\right]$, and $\lambda \in \mathbb{Z}_{p}$ is such that $\lambda \xi_{1}=\left[-n_{1} z_{1}^{*}+\right.$ $\left.\left(1+m_{1}\right) z_{2}^{*}\right]$.
4. If $\operatorname{rank}_{\mathbb{Z}_{p}}(\theta-I)=1$ and $\operatorname{det} \theta=-1$, then

$$
\begin{aligned}
& H^{1}\left(G, \mathbb{Z}_{p}\right)=\langle[u]\rangle \oplus\left\langle\left[y_{3}^{*}\right]\right\rangle \cong \mathbb{Z}_{p} \oplus \mathbb{Z}_{p}, \\
& H^{2}\left(G, \mathbb{Z}_{p}\right)=\frac{\left\langle z_{1}^{*}\right\rangle \oplus\left\langle z_{2}^{*}\right\rangle}{\operatorname{im} \partial_{2}^{*}} \cong \mathbb{Z}_{p}, \\
& H^{3}\left(G, \mathbb{Z}_{p}\right)=0,
\end{aligned}
$$

where $u$ where is described like in the previous case and assuming, without loss of generality, that $u=-n_{1} y_{1}^{*}+\left(1+m_{1}\right) y_{2}^{*}$, and once again taking the class of $m_{1} z_{1}^{*}+n_{1} z_{2}^{*}$ as the generator of $\frac{\left\langle z_{1}^{*}\right\rangle \oplus\left\langle z_{2}^{*}\right\rangle}{\operatorname{im} \partial_{2}^{*}} \cong \mathbb{Z}_{p}$, we have

$$
H^{*}\left(G, \mathbb{Z}_{p}\right) \cong \frac{\mathbb{Z}_{p}\left[\zeta_{1}, \zeta_{2}, \xi\right]}{\binom{\zeta_{1}^{2}=0, \zeta_{2}^{2}=0, \zeta_{1} \zeta_{2}=\lambda \xi}{\zeta_{1} \xi=0, \zeta_{2} \xi=0, \xi^{2}=0}},
$$

where $\zeta_{1}=[u], \zeta_{2}=\left[y_{3}^{*}\right], \xi=\left[m_{1} z_{1}^{*}+n_{1} z_{2}^{*}\right]$, and $\lambda \in \mathbb{Z}_{p}$ is such that $\lambda \xi=\left[-n_{1} z_{1}^{*}+\left(1+m_{1}\right) z_{2}^{*}\right]$.
5. If $\operatorname{rank}_{\mathbb{Z}_{p}}(\theta-I)=2$ and $\operatorname{det} \theta=1$, then

$$
H^{*}\left(G, \mathbb{Z}_{p}\right) \cong \frac{\mathbb{Z}_{p}[\zeta, \xi]}{\left(\zeta^{2}=0, \xi^{2}=0\right)}
$$

where $\operatorname{dim}(\zeta)=1$ and $\operatorname{dim}(\xi)=2$.
6. If $\operatorname{rank}_{\mathbb{Z}_{p}}(\theta-I)=2$ and $\operatorname{det} \theta=-1$, then

$$
H^{*}\left(G, \mathbb{Z}_{p}\right) \cong \frac{\mathbb{Z}_{p}[\zeta]}{\left(\zeta^{2}=0\right)}
$$

where $\operatorname{dim}(\zeta)=1$.

## A Appendix

The following table describes the sets $I_{1}, J_{1}, I_{2}, J_{2}$ that define the element $E \in \mathbb{Z} G$ in Theorem 2 depending on the signs of $m_{1}, n_{1}, m_{2}$ and $n_{2}$. The first four columns indicate the sign (or the value) of the corresponding variable. The symbol $\mathbb{Z}$ in one of the first four columns indicate that the value of the corresponding variable doesn't matter for the computation of the sets $I_{1}, J_{1}, I_{2}, J_{2}$.

Table I: The sets $I_{1}, J_{1}, I_{2}$ and $J_{2}$

| $m_{1}$ | $n_{1}$ | $m_{2}$ | $n_{2}$ | $I_{1}, J_{1}, I_{2}, J_{2}$ | Remark |
| :---: | :---: | :---: | :---: | :--- | :--- |
| + | + | + | + | $I_{1}=\left[-m_{1}-m_{2},-m_{2}-1\right] \cap \mathbb{Z}$, <br> $J_{1}=\left[-n_{2},-n_{1}-1\right] \cap \mathbb{Z}$, <br> $I_{2}=\left[-m_{2},-m_{1}-1\right] \cap \mathbb{Z}$, <br> $J_{2}=\left[-n_{1},-1\right] \cap \mathbb{Z}$ | if $m_{1}<m_{2}, n_{1}<n_{2}$ |
|  |  |  |  |  |  |
| + | + | + | + | $I_{1}=\left[-m_{1},-m_{2}-1\right] \cap \mathbb{Z}$, <br> $J_{1}=\left[-n_{2},-1\right] \cap \mathbb{Z}$, <br> $I_{2}=\left[-m_{1}-m_{2},-m_{1}-1\right] \cap \mathbb{Z}$, <br> $J_{2}=\left[-n_{1},-n_{2}-1\right] \cap \mathbb{Z}$ |  |
|  |  |  |  | - | - |
| - | - | - | $I_{1}=\left[-m_{2},-m_{1}-m_{2}-1\right] \cap \mathbb{Z}$, <br> $J_{1}=\left[-n_{1},-n_{2}-1\right] \cap \mathbb{Z}$, <br> $I_{2}=\left[-m_{1},-m_{2}-1\right] \cap \mathbb{Z}$, <br> $J_{2}=\left[0,-n_{1}-1\right] \cap \mathbb{Z}$ | if $m_{1}>m_{2}, n_{1}>n_{2}$ |  |
| - | - | - | - | $I_{1}=\left[-m_{2},-m_{1}-1\right] \cap \mathbb{Z}$, <br> $J_{1}=\left[0,-n_{2}-1\right] \cap \mathbb{Z}$, <br> $I_{2}=\left[-m_{1},-m_{1}-m_{2}-1\right] \cap \mathbb{Z}$, <br> $J_{2}=\left[-n_{2},-n_{1}-1\right] \cap \mathbb{Z}$ | if $m_{1}>m_{2}, n_{1}>n_{2}$ |
|  |  |  |  |  |  |


| - | + | + | - | $\begin{aligned} \hline I_{1} & =\left[-m_{2},-m_{1}-m_{2}-1\right] \cap \mathbb{Z}, \\ J_{1} & =\left[0,-n_{2}-1\right] \cap \mathbb{Z}, \\ I_{2} & =\left[-m_{1}-m_{2},-m_{1}-1\right] \cap \mathbb{Z}, \\ J_{2} & =\left[-n_{1},-1\right] \cap \mathbb{Z} \end{aligned}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| + | - | - | $+$ | $\begin{aligned} & I_{1}=\left[-m_{1}-m_{2},-m_{2}-1\right] \cap \mathbb{Z}, \\ & J_{1}=\left[-n_{2},-1\right] \cap \mathbb{Z}, \\ & I_{2}=\left[-m_{1},-m_{1}-m_{2}-1\right] \cap \mathbb{Z}, \\ & J_{2}=\left[0,-n_{1}-1\right] \cap \mathbb{Z} \end{aligned}$ |  |
| + | - | + | - | $\begin{aligned} & I_{1}=\left[-m_{2},-m_{1}-1\right] \cap \mathbb{Z}, \\ & J_{1}=\left[0,-n_{1}-1\right] \cap \mathbb{Z}, \\ & I_{2}=\left[-m_{1}-m_{2},-m_{2}-1\right] \cap \mathbb{Z}, \\ & J_{2}=\left[-n_{1},-n_{2}-1\right] \cap \mathbb{Z} \\ & \hline \end{aligned}$ | if $m_{1}<m_{2}, n_{1}>n_{2}$ |
| + | - | + | - | $\begin{aligned} & I_{1}=\left[-m_{1}-m_{2},-m_{1}-1\right] \cap \mathbb{Z}, \\ & J_{1}=\left[-n_{2},-n_{1}-1\right] \cap \mathbb{Z}, \\ & I_{2}=\left[-m_{1},-m_{2}-1\right] \cap \mathbb{Z}, \\ & J_{2}=\left[0,-n_{2}-1\right] \cap \mathbb{Z} \end{aligned}$ | if $m_{1}>m_{2}, n_{1}<n_{2}$ |
| - | + | - | + | $\begin{aligned} \hline I_{1} & =\left[-m_{1},-m_{1}-m_{2}-1\right] \cap \mathbb{Z}, \\ J_{1} & =\left[-n_{1},-n_{2}-1\right] \cap \mathbb{Z}, \\ I_{2} & =\left[-m_{2},-m_{1}-1\right] \cap \mathbb{Z}, \\ J_{2} & =\left[-n_{2},-1\right] \cap \mathbb{Z} \end{aligned}$ | if $m_{1}<m_{2}, n_{1}>n_{2}$ |
| - | + | - | + | $\begin{aligned} & I_{1}=\left[-m_{1},-m_{2}-1\right] \cap \mathbb{Z}, \\ & J_{1}=\left[-n_{1},-1\right] \cap \mathbb{Z}, \\ & I_{2}=\left[-m_{2},-m_{1}-m_{2}-1\right] \cap \mathbb{Z}, \\ & J_{2}=\left[-n_{2},-n_{1}-1\right] \cap \mathbb{Z} \end{aligned}$ | if $m_{1}>m_{2}, n_{1}<n_{2}$ |
| + | + | - | - | $\begin{aligned} & I_{1}=\left[-m_{1},-m_{1}-m_{2}-1\right] \cap \mathbb{Z}, \\ & J_{1}=\left[-n_{1},-1\right] \cap \mathbb{Z}, \\ & I_{2}=\left[-m_{1}-m_{2},-m_{2}-1\right] \cap \mathbb{Z}, \\ & J_{2}=\left[0,-n_{2}-1\right] \cap \mathbb{Z} \end{aligned}$ |  |
| - | - | + | + | $\begin{aligned} & I_{1}=\left[-m_{1}-m_{2},-m_{1}-1\right] \cap \mathbb{Z}, \\ & J_{1}=\left[0,-n_{1}-1\right] \cap \mathbb{Z}, \\ & I_{2}=\left[-m_{2},-m_{1}-m_{2}-1\right] \cap \mathbb{Z}, \\ & J_{2}=\left[-n_{2},-1\right] \cap \mathbb{Z} \end{aligned}$ |  |
| 0 | 1 | 1 | $\mathbb{Z}$ | $\begin{aligned} & I_{1}=\emptyset, \\ & J_{1}=\emptyset, \\ & I_{2}=\{-1\}, \\ & J_{2}=\{-1\} \\ & \hline \end{aligned}$ |  |
| 0 | -1 | -1 | $\mathbb{Z}$ | $\begin{aligned} & I_{1}=\emptyset, \\ & J_{1}=\emptyset, \\ & I_{2}=\{0\}, \\ & J_{2}=\{0\} \\ & \hline \end{aligned}$ |  |
| 0 | 1 | -1 | $\mathbb{Z}$ | $\begin{aligned} & I_{1}=\{0\}, \\ & J_{1}=\{-1\}, \\ & I_{2}=\emptyset \\ & J_{2}=\emptyset \end{aligned}$ |  |
| 0 | -1 | 1 | $\mathbb{Z}$ | $\begin{aligned} & I_{1}=\{-1\}, \\ & J_{1}=\{0\}, \\ & I_{2}=\emptyset, \\ & J_{2}=\emptyset \\ & \hline \end{aligned}$ |  |
| 1 | 0 | $\mathbb{Z}$ | 1 | $\begin{aligned} & I_{1}=\left\{-m_{2}-1\right\}, \\ & J_{1}=\{-1\}, \\ & I_{2}=\emptyset, \\ & J_{2}=\emptyset \\ & \hline \end{aligned}$ |  |


| -1 | 0 | $\mathbb{Z}$ | -1 | $\begin{aligned} & I_{1}=\left\{-m_{2}\right\}, \\ & J_{1}=\{0\}, \\ & I_{2}=\emptyset, \\ & J_{2}=\emptyset \end{aligned}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 0 | $\mathbb{Z}$ | -1 | $\begin{aligned} & I_{1}=\emptyset, \\ & J_{1}=\emptyset, \\ & I_{2}=\left\{-m_{2}-1\right\}, \\ & J_{2}=\{0\} \\ & \hline \end{aligned}$ |  |
| -1 | 0 | $\mathbb{Z}$ | 1 | $\begin{aligned} I_{1} & =\emptyset \\ J_{1} & =\emptyset, \\ I_{2} & =\left\{-m_{2}\right\}, \\ J_{2} & =\{-1\} \end{aligned}$ |  |
| 1 | $\mathbb{Z}$ | 0 | 1 | $\begin{aligned} & I_{1}=\{-1\}, \\ & J_{1}=\{-1\}, \\ & I_{2}=\emptyset \\ & J_{2}=\emptyset \end{aligned}$ |  |
| -1 | $\mathbb{Z}$ | 0 | -1 | $\begin{aligned} & I_{1}=\{0\}, \\ & J_{1}=\{0\}, \\ & I_{2}=\emptyset, \\ & J_{2}=\emptyset \end{aligned}$ |  |
| 1 | $\mathbb{Z}$ | 0 | -1 | $\begin{aligned} & I_{1}=\emptyset, \\ & J_{1}=\emptyset, \\ & I_{2}=\{-1\}, \\ & J_{2}=\{0\} \end{aligned}$ |  |
| -1 | $\mathbb{Z}$ | 0 | 1 | $\begin{aligned} & I_{1}=\emptyset, \\ & J_{1}=\emptyset, \\ & I_{2}=\{0\}, \\ & J_{2}=\{-1\} \end{aligned}$ |  |
| $\mathbb{Z}$ | 1 | 1 | 0 | $\begin{aligned} & I_{1}=\emptyset, \\ & J_{1}=\emptyset, \\ & I_{2}=\left\{-m_{1}-1\right\}, \\ & J_{2}=\{-1\} \\ & \hline \end{aligned}$ |  |
| $\mathbb{Z}$ | -1 | -1 | 0 | $\begin{aligned} & I_{1}=\emptyset, \\ & J_{1}=\emptyset, \\ & I_{2}=\left\{-m_{1}\right\}, \\ & J_{2}=\{0\} \\ & \hline \end{aligned}$ |  |
| $\mathbb{Z}$ | 1 | -1 | 0 | $\begin{aligned} & I_{1}=\left\{-m_{1}\right\}, \\ & J_{1}=\{-1\}, \\ & I_{2}=\emptyset \\ & J_{2}=\emptyset \end{aligned}$ |  |
| $\mathbb{Z}$ | -1 | 1 | 0 | $\begin{aligned} & I_{1}=\left\{-m_{1}-1\right\}, \\ & J_{1}=\{0\}, \\ & I_{2}=\emptyset, \\ & J_{2}=\emptyset \end{aligned}$ |  |

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