# Groups acting on hyperbolic $\Lambda$-metric spaces 

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#### Abstract

In this paper we study group actions on hyperbolic $\Lambda$-metric spaces, where $\Lambda$ is an ordered abelian group. $\Lambda$-metric spaces were first introduced by Morgan and Shalen in their study of hyperbolic structures and then Chiswell, following Gromov's ideas, introduced the notion of hyperbolicty for such spaces. Only the case of 0 -hyperbolic $\Lambda$-metric spaces (that is, $\Lambda$ trees) was systematically studied, while the theory of general hyperbolic $\Lambda$-metric spaces was not developed at all. Hence, one of the goals of the present paper was to fill this gap and translate basic notions and results from the theory of group actions on hyperbolic (in the usual sense) spaces to the case of $\Lambda$-metric spaces for an arbitrary $\Lambda$. The other goal was to show some principal difficulties which arise in this generalization and the ways to deal with them.


## 1 Introduction

In this paper we introduce and study group actions on hyperbolic $\Lambda$-metric spaces. This is a natural development of the theory of groups acting on $\Lambda$ trees. We extend some ideas of Morgan, Shalen, Bass, Chiswell, and Gromov to hyperbolic metric spaces, where the metric takes values in an arbitrary ordered abelian group $\Lambda$.

Motivation. This research stems from several areas. Firstly, it is a very natural generalization of the theory of groups acting on $\Lambda$-trees. It turned out that in the study of group actions on $\Lambda$-trees is convenient sometimes to take a wider look and consider actions on hyperbolic $\Lambda$-metric spaces. This makes results much more general, but also more elegant and sometimes shorter. Secondly, this gives a new approach to general hyperbolicity and a new framework to study groups acting on hyperbolic $\Lambda$-spaces. Thus, Gromov hyperbolic groups can be viewed as $\mathbb{Z}$-hyperbolic, Fuchsian groups as well as Kleinean groups, as $\mathbb{R}$-hyperbolic etc. New interesting classes of $\Lambda$-hyperbolic groups appear as a result of various "limit" constructions. Recall, that limit groups (which are limits of free groups in Gromov-Hausdorff metric) are $\mathbb{Z}^{n}$-free, that is, they act freely on $\mathbb{Z}^{n}$-trees 10, 17, which is one of the crucial properties of these groups. Similarly, limits of torsion-free Gromov hyperbolic groups are $\mathbb{Z}^{n}$-hyperbolic 18. Moreover, various non-standard versions of hyperbolic groups (ultrapowers of
hyperbolic groups and their subgroups) also act nicely on hyperbolic $\tilde{\mathbb{Z}}$-spaces, where $\tilde{\mathbb{Z}}$ is the group of non-standard integers ( $\tilde{\mathbb{Z}}$ is an ultrapower of $\mathbb{Z}$ ). Thirdly, we believe that this framework gives a unified approach to several open problems related to model theory of hyperbolic groups, questions on algebraic structure of subgroups of hyperbolic groups and relatively hyperbolic groups, constructions of effective versions of asymptotic cones of hyperbolic-like groups and some others.

Results. We lay down foundations of the theory in Sections 2 and 3 .
Let $\Lambda$ be an ordered abelian group. In Section 2 we discuss hyperbolic $\Lambda$-metric spaces. In fact, this notion is not new, in 16] Morgan and Shalen defined $\Lambda$-metric space for an arbitrary $\Lambda$, while in [3] Chiswell, following Gromov's ideas, gave a definition of a hyperbolic $\Lambda$-metric space. We show that most of the classical definitions of hyperbolicity remain valid and equivalent in the general case, which gives the base for the whole study. We introduce the notion of a boundary of a hyperbolic $\Lambda$-metric space and establish some of its basic properties which we use throughout the paper. In Section 2.6 we study isometries of hyperbolic $\Lambda$-metric spaces. The results are more technical and proofs are more involved than both in the case of isometries of $\Lambda$-trees and the classical $\mathbb{R}$-hyperbolic spaces, since in the general case one has to accommodate the both of these. The following result (Theorem 2 in Section 2.6) is a crucial result here which gives classification of isometries in the general setting. Let $(X, d)$ be a geodesic $\delta$-hyperbolic $\Lambda$-metric space. Then every minimal isometry of $X$ is either elliptic, or parabolic, or hyperbolic in the case when $\Lambda=2 \Lambda$, and is either elliptic, or parabolic, or hyperbolic, or an inversion when $\Lambda \neq 2 \Lambda$. We have (see Section 2.6) a more detailed description of isometries and their properties in two principle cases, when $\Lambda$ is equal to either $\mathbb{R}^{n}$, or $\mathbb{Z}^{n}$ (both with the right lexicographic order). We conclude Section 2 with examples of hyperbolic $\Lambda$-metric spaces.

In Section 3 we, following ideas of Lyndon and Gromov we introduce group based hyperbolic length functions with values in $\Lambda$. From this view-point Lyndon's length functions are 0-hyperbolic, and our general hyperbolic length functions occur when the Lyndon's 0-hyperbolicity axiom is replaced by a general one that corresponds to the hyperbolicity condition on the Gromov's products (which can be easily expressed in terms of the length functions). Chiswell in [3] showed that groups with Lyndon length functions $l: G \rightarrow \mathbb{R}$ (and an extra axiom) are precisely those ones that act freely on $\mathbb{R}$-trees, and later Morgan and Shalen generalized his construction in 16 to arbitrary $\Lambda$ (we refer the reader to the book [3] for details). In Section 3.1 we show how an action of a group $G$ by isometries on a (hyperbolic) $\Lambda$-metric space naturally induces a (hyperbolic) length function on $G$ with values in $\Lambda$. And in Section 3.2 we prove the converse, thus establishing equivalence of these two approaches. In the end of Section 3 we give examples of groups acting on hyperbolic $\Lambda$-metric spaces. This gives, as in the classical Bass-Serre theory of groups acting on trees, an equivalent approach to study group actions on hyperbolic $\Lambda$-metric spaces.

In Section 4 we consider kernels of hyperbolic length functions. Let $G$ be a
group with a length function $l: G \rightarrow \Lambda$. For a fixed convex subgroup $\Lambda_{0} \leqslant \Lambda$ one can define the $\Lambda_{0}$-kernel of $G$ by $G_{\Lambda_{0}}=\left\{g \in \mid l(g) \in \Lambda_{0}\right\}$, which is a subgroup of $G$. If the hyperbolicity constant $\delta$ is greater then any element in $\Lambda_{0}$ (that is, $\delta \notin \Lambda_{0}$ ) then the restriction of the function $l$ to $G_{\Lambda_{0}}$ becomes $\delta$-hyperbolic, in other words, $l$ does not say much about the $\Lambda_{0}$-kernel. This shows that if $\Lambda$ is not archimedean then all elements in $G$ of length "infinitely smaller" than $\delta$ become invisible for the function $l$, so the $\delta$-hyperbolicity axiom does not impose any restrictions on them. To deal with this on the group level we use the idea of a group which is hyperbolic relative to a subgroup (see below).

It turns out that for a non-Archimedean $\Lambda$ group actions on hyperbolic $\Lambda$ metric spaces can be quite cumbersome, they might have rather strange properties that do not occur in the classical situations. In Section ${ }^{5}$ with introduce several natural types of group actions and the corresponding length functions: regular, complete, free, and proper. The axioms on length functions associated with these action types shed some light on the algebraic structure of the underlying groups. In particular, in Section we consider actions of a finitely generated group $G$ on a geodesic $\delta$-hyperbolic $\mathbb{R}$-metric space ( $X, d$ ) and show (Theorem (9) that if the action is "nice" (regular and proper) then $G$ is weakly hyperbolic (in the sense of Farb, and Osin (19) relative to the kernel of the associated length function. This is an analog of the classical result on hyperbolicity of groups acting "nicely" on hyperbolic metric spaces. We refer the reader to Section 6 for some interesting applications of this result.

In Section 7 we investigate how one can "complete" a given non-geodesic hyperbolic $\mathbb{Z}$-metric space $X$ to a geodesic one, that is, how one can construct a geodesic hyperbolic $\mathbb{Z}$-metric space $\bar{X}$ which $X$ (quasi-)isometrically embeds into. According to Bonk and Schramm, any $\delta$-hyperbolic $\mathbb{Z}$-metric space embeds isometrically into a complete geodesic $\delta$-hyperbolic $\mathbb{R}$-metric space (see [2]), but unfortunately this completion does not have to be a $\mathbb{Z}$-metric space. For a given hyperbolic $\mathbb{Z}$-metric space $X$ we introduce two $\mathbb{Z}$-completions of $X$ which we call $\Gamma_{1}(X)$ and $\Gamma_{2}(X)$. Our constructions will have, compared to Bonk and Schramm's, the disadvantage that the hyperbolicity constant will increase. However, they will have the advantage that isometries, embeddings and quasiisometries of $X$ extend easily and that boundaries are easy to work with.

## 2 Hyperbolic $\Lambda$-metric spaces

### 2.1 Ordered abelian groups

In this section we only mention some definitions and facts that are crucial for understanding of the main concepts of the paper. For details on ordered abelian groups we refer to books [6, (14, (8).

An ordered abelian group is an abelian group $\Lambda$ (with addition denoted by " + ") equipped with a linear order " $\leqslant$ " such that the following axiom holds:
(OA) for all $\alpha, \beta, \gamma \in \Lambda, \alpha \leqslant \beta$ implies $\alpha+\gamma \leqslant \beta+\gamma$.

An abelian group $\Lambda$ is called orderable if there exists a linear order " $\leqslant$ " on $\Lambda$, satisfying the condition (OA) above. In general, $\Lambda$ can be ordered in many different ways. In what follows $\Lambda$ always denotes an ordered abelian group.

If $A$ and $B$ are ordered abelian group then their direct sum $A \oplus B$ can be ordered with the right lexicographic order, where one compares first the right components of two pairs and if they are equal than the left ones, that is, $(a, b) \leqslant(c, d)$ if and only if either $b<d$, or $b=d$ and $a \leqslant c$. Similarly, one can define the left lexicographic order on $A \oplus B$. Throughout the paper we consider only the right lexicographic order. Furthermore, the direct powers $\mathbb{Z}^{n}$ and $\mathbb{R}^{n}$, if not said otherwise, are always considered in the right lexicographic order.

An ordered abelian group $\Lambda$ is called discretely ordered or discrete if $\Lambda$ has a minimal positive element, which we denote by 1 . It will be always clear from the context whether 1 represents a natural number, or the minimal positive element of $\Lambda$. If $\Lambda$ is discrete then for any $\alpha \in \Lambda$ the following hold:
(1) $\alpha+1=\min \{\beta \mid \beta>\alpha\}$,
(2) $\alpha-1=\max \{\beta \mid \beta<\alpha\}$.

Notice, that $\mathbb{Z}^{n}$ is discretely ordered for any $n>0$, but $\mathbb{R}^{n}$ is not.
Sometimes we would like to be able to divide elements of $\Lambda$ by non-zero integers. To this end we fix a canonical order-preserving embedding of $\Lambda$ into an ordered divisible abelian group $\Lambda_{\mathbb{Q}}$ and identify $\Lambda$ with its image in $\Lambda_{\mathbb{Q}}$. The group $\Lambda_{\mathbb{Q}}$ is the tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda$ of two abelian groups (viewed as $\mathbb{Z}$ modules) over $\mathbb{Z}$. One can represent elements of $\Lambda_{\mathbb{Q}}$ by fractions $\frac{\lambda}{m}$, where $\lambda \in$ $\Lambda, m \in \mathbb{Z}, m \neq 0$, and two fractions $\frac{\lambda}{m}$ and $\frac{\mu}{n}$ are equal if and only if $n \lambda=m \mu$. Addition of fractions is defined as usual, and the embedding is given by the map $\lambda \rightarrow \frac{\lambda}{1}$. The order on $\Lambda_{\mathbb{Q}}$ is defined by $\frac{\lambda}{m} \geqslant 0 \Longleftrightarrow m \lambda \geqslant 0$ in $\Lambda$. Obviously, the embedding $\Lambda \rightarrow \Lambda_{\mathbb{Q}}$ preserves the order. It is easy to see that $\mathbb{R}_{\mathbb{Q}}=\mathbb{R}$ and $\mathbb{Z}_{\mathbb{Q}}=\mathbb{Q}$. Furthermore, it is not hard to show that $(A \oplus B)_{\mathbb{Q}} \simeq A_{\mathbb{Q}} \oplus B_{\mathbb{Q}}$, so $\left(\mathbb{R}^{n}\right)_{\mathbb{Q}}=\mathbb{R}^{n}$ and $\left(\mathbb{Z}^{n}\right)_{\mathbb{Q}}=\mathbb{Q}^{n}$. Notice also, that for every $\Lambda$ the group $\mathbb{Z} \oplus \Lambda$ is discrete.

For elements $\alpha, \beta \in \Lambda$ the closed segment $[\alpha, \beta]$ is defined by

$$
[\alpha, \beta]=\{\gamma \in \Lambda \mid \alpha \leqslant \gamma \leqslant \beta\}
$$

Now a subset $C \subset \Lambda$ is called convex if for every $\alpha, \beta \in C$ the set $C$ contains $[\alpha, \beta]$. In particular, a subgroup $C$ of $\Lambda$ is convex if $[0, \beta] \subset C$ for every positive $\beta \in C$. Observe, that the set of all convex subgroups of $\Lambda$ is linearly ordered by inclusion. In the case when $\Lambda=\mathbb{R}^{n}$, or $\Lambda=\mathbb{Z}^{n}$ the convex subgroups form a chain: $0<\Lambda_{1}<\cdots<\Lambda_{n}=\Lambda$, where $\Lambda_{i}=\left\{\left(\lambda_{1}, \ldots, \lambda_{i}, 0, \ldots, 0\right) \mid \lambda_{j} \in\right.$ $\mathbb{R}($ or $\mathbb{Z})\}$. In this case $\Lambda$ has a (unique) minimal non-trivial convex subgroup $\Lambda_{1}$.

For any $a \in \Lambda$ we define $|a|=a$ if $a \geq 0$ and $|a|=-a$ otherwise.

## $2.2 \quad \Lambda$-metric spaces

In (16] Morgan and Shalen defined $\Lambda$-metric spaces for an arbitrary ordered abelian group $\Lambda$.

Let $X$ be a non-empty set and $\Lambda$ an ordered abelian group. A $\Lambda$-metric on $X$ is a mapping $d: X \times X \rightarrow X$ such that:
(LM1) $\forall x, y \in X: d(x, y) \geqslant 0$;
(LM2) $\forall x, y \in X: d(x, y)=0 \Leftrightarrow x=y$;
$(\mathrm{LM} 3) \forall x, y \in X: d(x, y)=d(y, x)$;
(LM4) $\forall x, y, z \in X: d(x, y) \leqslant d(x, z)+d(y, z)$.
A $\Lambda$-metric space is a pair $(X, d)$, where $X$ is a non-empty set and $d$ is a $\Lambda$-metric on $X$. Usually, unless specified otherwise, we always assume that there is no convex subgroup $\Lambda_{0}$ of $\Lambda$ such that $d(x, y) \in \Lambda_{0}$ for every $x, y \in X$ (otherwise we can replace $\Lambda$ by $\Lambda_{0}$ ).

Example 1. For any ordered abelian group $\Lambda$ the map $d(a, b)=|a-b|$ is $a$ metric, so $(\Lambda, d)$ is a $\Lambda$-metric space.

We fix a $\Lambda$-metric space $(X, d)$ and a convex subgroup $\Lambda_{0}$ of $\Lambda$. For any point $x \in X$ the subset

$$
X_{x, \Lambda_{0}}=\left\{y \in X \mid d(x, y) \in \Lambda_{0}\right\}
$$

of $X$ is a $\Lambda_{0}$-metric space with respect to the metric $d_{0}=d_{\left.\right|_{X_{0}}}$, called a $\Lambda_{0}$ metric subspace of $X$.

If $x \in X$ and $\varepsilon \in \Lambda$ is positive then we define the ball of radius $\varepsilon$ centered at $x$ as usual by

$$
B_{\varepsilon}(x)=\{y \in X \mid d(x, y) \leqslant \varepsilon\} .
$$

A subset $Y \subseteq X$ is bounded if $Y \subseteq B_{\varepsilon}(x)$ for some $x \in X$ and $\varepsilon \geqslant 0$. If $\Lambda_{0} \neq \Lambda$ then any $\Lambda_{0}$-metric subspace $X_{x, \Lambda_{0}}$ of $X$ is bounded (it is contained in $B_{\varepsilon}(x)$ for any $0<\varepsilon \in \Lambda \backslash \Lambda_{0}$ ).

If $(X, d)$ and $\left(X^{\prime}, d^{\prime}\right)$ are $\Lambda$-metric spaces, an isometry from $(X, d)$ to $\left(X^{\prime}, d^{\prime}\right)$ is a mapping $f: X \rightarrow X^{\prime}$ such that $d(x, y)=d^{\prime}(f(x), f(y))$ for all $x, y \in X$.

A mapping $f: X \rightarrow X^{\prime}$ is called a $(\lambda, c, L)$-local-quasi-isometry from $(X, d)$ to ( $X^{\prime}, d^{\prime}$ ), where $\lambda \in \mathbb{Z}, c, L \in \Lambda$ are such that $\lambda \geqslant 1, c, L \geqslant 0$, if

$$
\frac{1}{\lambda} d(x, y)-c \leqslant d^{\prime}(f(x), f(y)) \leqslant \lambda d(x, y)+c
$$

for all $x, y \in X$ such that $d(x, y) \leqslant L$. Here, as usual, we understand $\frac{1}{\lambda} d(x, y)$ as an element in $\Lambda_{\mathbb{Q}}$.

Similarly, $f$ is a $(\lambda, c)$-quasi-isometry if the inequalities above hold for any $x, y \in X$ (the condition $d(x, y) \leqslant L$ is dropped).

A segment in a $\Lambda$-metric space $X$ is the image of an isometry $\alpha:[a, b] \rightarrow X$ for some $a, b \in \Lambda$. In this case $\alpha(a), \alpha(b)$ are called the endpoints of the segment. By $[x, y]$ we denote any segment with endpoints $x, y$.

We call a $\Lambda$-metric space $(X, d)$ geodesic if for all $x, y \in X$, there is a segment in $X$ with endpoints $x, y .(X, d)$ is geodesically linear if for all $x, y \in X$, there is a unique segment in $X$ with endpoints $x, y$.

Lemma 1. [3, Lemma 1.2.2] Let $(X, d)$ be a $\Lambda$-metric space.

1. Let $\sigma$ be a segment in $X$ with endpoints $x, z$ and let $\tau$ be a segment in $X$ with endpoints $y, z$.
(a) Suppose that, for all $u \in \sigma$ and $v \in \tau, d(u, v)=d(u, z)+d(z, v)$. Then $\sigma \cup \tau$ is a segment with endpoints $x, y$.
(b) if $\sigma \cap \tau=\{z\}$ and $\sigma \cup \tau$ is a segment, then its endpoints are $x, y$.
2. Assume that $(X, d)$ is geodesically linear. Let $x, y$ and $z \in X$, and let $\sigma$ be the segment with endpoints $x, y$. Then $z \in \sigma$ if and only if $d(x, y)=$ $d(x, z)+d(z, y)$.

### 2.3 Definition of hyperbolic $\Lambda$-metric space

In [3] Chiswell, generalizing Gromov's approach to hyperbolicity 9], introduced hyperbolic $\Lambda$-metric spaces. We briefly discuss this notion below.

Let $(X, d)$ be a $\Lambda$-metric space. Fix a point $v \in X$ and for $x, y \in X$ define the Gromov's product

$$
(x \cdot y)_{v}=\frac{1}{2}(d(x, v)+d(y, v)-d(x, y))
$$

as an element of $\Lambda_{\mathbb{Q}}$. A straightforward computation shows that, if $t$ is another point from $X$ then

$$
(x \cdot y)_{t}=d(t, v)+(x \cdot y)_{v}-(x \cdot t)_{v}-(y \cdot t)_{v}
$$

This and the triangle inequality implies the following result.
Lemma 2. Let $(X, d)$ be a $\Lambda$-metric space. Then the following hold:

1. For any $v, x, y \in X$

$$
0 \leqslant(x \cdot y)_{v} \leqslant \min \{d(x, v), d(y, v)\}
$$

2. If for some $v \in X$ and all $x, y \in X,(x \cdot y)_{v} \in \Lambda$ then for all $v, x, y \in$ $X,(x \cdot y)_{v} \in \Lambda$.

Now, following Gromov (see [9]) one can define a hyperbolic $\Lambda$-metric space.
Definition 1. Let $\delta \in \Lambda$ with $\delta \geqslant 0$. Then $(X, d)$ is $\delta$-hyperbolic with respect to $v$ if, for all $x, y, z \in X$

$$
(x \cdot y)_{v} \geqslant \min \left\{(x \cdot z)_{v},(z \cdot y)_{v}\right\}-\delta
$$

Lemma 3. [3, Lemma 1.2.5] If $(X, d)$ is $\delta$-hyperbolic with respect to $v$, and $t$ is any other point of $X$, then $(X, d)$ is $2 \delta$-hyperbolic with respect to $t$.

In view of Lemma 3, we call a $\Lambda$-metric space $(X, d) \delta$-hyperbolic if it is $\delta$-hyperbolic with respect to any point of $X$.

The definition of $\delta$-hyperbolicity can be reformulated as follows.

Lemma 4. [3, Lemma 1.2.6] The $\Lambda$-metric space $(X, d)$ is $\delta$-hyperbolic if and only if any $x, y, z, t \in X$ satisfy the following 4-point condition:

$$
d(x, y)+d(z, t) \leqslant \max \{d(x, z)+d(y, t), d(y, z)+d(x, t)\}+2 \delta
$$

### 2.4 Geodesic hyperbolic $\Lambda$-metric spaces

Geodesic hyperbolic $\Lambda$-metric spaces have some nice geometric properties, which can be expressed in various forms of "thinness" of geodesic triangles. In this section $(X, d)$ is a geodesic $\Lambda$-metric space.
$\Lambda$-trees give an important class of 0 -hyperbolic geodesic $\Lambda$-metric spaces. They were introduced by Morgan and Shalen in [16. Recall that a $\Lambda$-metric space is a $\Lambda$-tree if it satisfies the following axioms:
(T1) $(X, d)$ is geodesic,
(T2) if two segments of $(X, d)$ intersect in a single point, which is an endpoint of both, then their union is a segment,
(T3) if the intersection of two segments with a common endpoint is also a segment.

If $X$ is a $\Lambda$-tree and $x, y, z \in X$ then $[x, y] \cap[x, z]=[x, w]$ for some $w \in X$. In this case we write $w=Y(y, x, z)$.

The following theorem was proved in [3] (Lemmas 2.1.6 and 2.4.3).
Theorem 1. A geodesic $\Lambda$-metric space $(X, d)$ is a $\Lambda$-tree if and only if it satisfies the following conditions:
(1) for all $x, y, v \in X(x \cdot y)_{v} \in \Lambda$,
(2) $(X, d)$ is 0-hyperbolic.

In particular, if $\Lambda$ is a divisible ordered abelian group (for instance $\Lambda=\mathbb{R}^{n}$ ) then the first condition in the theorem is always satisfied, so in this case $\Lambda$-trees are precisely geodesic 0 -hyperbolic $\Lambda$-metric spaces.

Now we give a characterization of hyperbolic geodesic $\Lambda$-metric spaces in terms of thin triangles.

A $\Lambda$-tripod in a $\Lambda$-metric space is a $\Lambda$-tree spanned by three points (including degenerate cases when the points coincide, or are collinear). Here is an analog of Proposition 2.2 from [7]. The proof is straightforward.

Lemma 5. Let $(X, d)$ be a $\Lambda$-metric space such that for all $x, y, z \in X,(x$. $y)_{z} \in \Lambda$. Then for all $x, y, z \in X$ there exists a $\Lambda$-tripod $T$ and an isometry $\phi:\{x, y, z\} \rightarrow T$, where $T$ is spanned by $\phi(x), \phi(y)$ and $\phi(z)$ such that $(x \cdot y)_{z}$ is equal to the length of the intersection $[\phi(z), \phi(x)] \cap[\phi(z), \phi(y)]$ in $T$.

We call the $\Lambda$-tripod $T$ from the lemma above a comparison $\Lambda$-tripod for the triple $\{x, y, z\}$ and denote $T=T(x, y, z)$.

If $(X, d)$ is geodesic then the isometry $\phi:\{x, y, z\} \rightarrow T(x, y, z)$ extends to an isometry of the geodesic triangle $\Delta(x, y, z)=[x, y] \cup[x, z] \cup[y, z]$ in $(X, d)$ to $T$ whose restriction to $\{x, y, z\}$ is exactly $\phi$ (we denote this extension again by $\phi$ ). Now, $\Delta(x, y, z)$ is called $\delta$-thin for some $\delta \in \Lambda$ if $d(u, v) \leqslant \delta$ for all $u, v \in \Delta(x, y, z)$ such that $\phi(u)=\phi(v)$.

Lemma 6. Let $(X, d)$ be a geodesic $\Lambda$-metric space such that for all $x, y, z \in$ $X,(x \cdot y)_{z} \in \Lambda$. Then
(i) $(x \cdot y)_{z} \leqslant d(z,[x, y])$,
(ii) if $\Delta(x, y, z)$ is $\delta$-thin then $d(z,[x, y]) \leqslant(x \cdot y)_{z}+\delta$

Proof. The proof repeats the one of Lemma 2.17 (7].
Let $p \in[x, y], q \in[x, z], r \in[y, z]$ be such that $\phi(p)=\phi(q)=\phi(r)=$ $Y(\phi(x), \phi(y), \phi(z))$.

Let $w \in[x, y]$ be such that $d(z,[x, y])=d(z, w)$. Then there exists $w^{\prime} \in$ $[x, z] \cup[y, z]$ such that $\phi(w)=\phi\left(w^{\prime}\right)$. Without loss of generality assume that $w^{\prime} \in[x, z]$. Then
$(x \cdot y)_{z} \leqslant d\left(w^{\prime}, z\right)=d(x, z)-d\left(x, w^{\prime}\right)=d(x, z)-d(x, w) \leqslant d(z, w)=d(z,[x, y])$
which proves (i).
Finally, if $\Delta(x, y, z)$ is $\delta$-thin then we have

$$
d(z,[x, y]) \leqslant d(z, q)+d(p, q) \leqslant(x \cdot y)_{z}+\delta .
$$

Proposition 1. Let $(X, d)$ be a geodesic $\Lambda$-metric space such that for all $x, y, z \in$ $X,(x \cdot y)_{z} \in \Lambda$. Consider the following properties of $(X, d)$ :
(H1, $\delta)(X, d)$ is $\delta$-hyperbolic,
$(H 2, \delta) \Delta(x, y, z)$ is $\delta$-thin for any $x, y, z \in X$,
$(H 3, \delta) d(u,[x, z] \cup[y, z]) \leqslant \delta$ for any $x, y, z \in X$ and $u \in[x, y]$.
Then the following implications hold

$$
\begin{gathered}
(H 1, \delta) \Longrightarrow(H 2,4 \delta), \quad(H 2, \delta) \Longrightarrow(H 1,2 \delta) \\
(H 2, \delta) \Longrightarrow(H 3, \delta), \quad(H 3, \delta) \Longrightarrow(H 2,4 \delta) \\
(H 1, \delta) \Longrightarrow(H 3,4 \delta), \quad(H 3, \delta) \Longrightarrow(H 1,8 \delta)
\end{gathered}
$$

Proof. We follow the proof of Proposition 2.21 [7].
$(H 1, \delta) \Longrightarrow(H 2,4 \delta):$
Let $x, y, z \in X$ and $T(x, y, z)$ a comparison $\Lambda$-tripod (denote by $d^{\prime}$ the metric on $T(x, y, z)$ ). Let $u \neq v \in \Delta(x, y, z)$ be such that $\phi(u)=\phi(v)$. We have
to show that $d(u, v) \leqslant 4 \delta$. Without loss of generality we can assume that $u \in[x, y], v \in[x, z]$. If $t=d(x, u)$ then

$$
\begin{gathered}
d^{\prime}(\phi(x), \phi(u))=d^{\prime}(\phi(x), \phi(v))=t \leqslant(y \cdot z)_{x} \\
(u \cdot y)_{x}=(v \cdot z)_{x}=t
\end{gathered}
$$

which implies
$(u \cdot v)_{x} \geqslant \min \left\{(u \cdot y)_{x},(v \cdot y)_{x}\right\}-\delta \geqslant \min \left\{(u \cdot y)_{x},(y \cdot z)_{x},(z \cdot v)_{x}\right\}-2 \delta=t-2 \delta$.
Since $(u \cdot v)_{x}=t-\frac{1}{2} d(u, v)$ then

$$
d(u, v)=2 t-(u \cdot v)_{x} \leqslant 2 t-2(t-2 \delta)=4 \delta
$$

$$
(H 2, \delta) \Longrightarrow(H 1,2 \delta):
$$

Suppose that all triangles in $X$ are $\delta$-thin and let $x_{0}, x_{1}, x_{2}, x_{3} \in X$. We have to show that

$$
\left(x_{1} \cdot x_{2}\right)_{x_{0}} \geqslant \min \left\{\left(x_{1} \cdot x_{3}\right)_{x_{0}},\left(x_{2} \cdot x_{3}\right)_{x_{0}}\right\}-2 \delta
$$

Denote $t=\min \left\{\left(x_{1} \cdot x_{3}\right)_{x_{0}},\left(x_{2} \cdot x_{3}\right)_{x_{0}}\right\}$. If $t \leqslant\left(x_{1} \cdot x_{2}\right)_{x_{0}}$ then there is nothing prove, so let $t>\left(x_{1} \cdot x_{2}\right)_{x_{0}}$.

For $i \in\{1,2,3\}$ let $x_{i}^{\prime} \in\left[x_{0}, x_{i}\right]$ be such that $d\left(x_{0}, x_{i}^{\prime}\right)=t$. Let $\phi_{i, j}, i \neq$ $j \in\{1,2,3\}$ be the isometry of the triangle $\left[x_{0}, x_{i}\right] \cup\left[x_{i}, x_{j}\right] \cup\left[x_{j}, x_{0}\right]$ to the comparison $\Lambda$-tripod $T\left(x_{0}, x_{i}, x_{j}\right)$.

In the case when $i=1,2$ we have $d\left(x_{0}, x_{i}^{\prime}\right)=d\left(x_{0}, x_{3}^{\prime}\right) \leqslant\left(x_{i} \cdot x_{3}\right)_{x_{0}}$ since $\phi_{i, 3}\left(x_{i}^{\prime}\right)=\phi_{i, 3}\left(x_{3}^{\prime}\right)$ and $d\left(x_{3}, x_{i}^{\prime}\right) \leqslant \delta$. Thus we have

$$
d\left(x_{1}^{\prime}, x_{2}^{\prime}\right) \leqslant 2 \delta .
$$

Since $t>\left(x_{1} \cdot x_{2}\right)_{x_{0}}$, there exists $y_{j} \in\left[x_{1}, x_{2}\right]$ such that $\phi_{1,2}\left(x_{j}^{\prime}\right)=\phi_{1,2}\left(y_{j}\right)$ and $d\left(x_{j}^{\prime}, y_{j}\right) \leqslant \delta$. Hence,

$$
\begin{gathered}
2 \delta \geqslant d\left(x_{1}^{\prime}, x_{2}^{\prime}\right) \geqslant d\left(y_{1}, y_{2}\right)-2 \delta=d\left(x_{1}, x_{2}\right)-d\left(x_{1}, y_{1}\right)-d\left(x_{2}, y_{2}\right)-2 \delta \\
=d\left(x_{1}, x_{2}\right)-\left(d\left(x_{1}, x_{0}\right)-d\left(x_{1}^{\prime}, x_{0}\right)\right)-\left(d\left(x_{2}, x_{0}\right)-d\left(x_{2}^{\prime}, x_{0}\right)\right)-2 \delta \\
=2 t-2\left(x_{1} \cdot x_{2}\right)_{x_{0}}-3 \delta .
\end{gathered}
$$

So, $\left(x_{1} \cdot x_{2}\right)_{x_{0}} \geqslant t-2 \delta$.
$(H 2, \delta) \Longrightarrow(H 3, \delta):$ obvious
$(H 3, \delta) \Longrightarrow(H 2,4 \delta):$
Suppose that $(H 2,4 \delta)$ does not hold, that is, there exist $x, y, z \in X$ and $u \in[x, y], v \in[x, z]$ such that $d(x, u)=d(x, v)<(y \cdot z)_{x}$ but $d(u, v)>4 \delta$. By Lemma 6 we have

$$
d(v,[x, y])=\min \{d(v,[x, u]), d(v,[u, y])\} \geqslant \min \left\{(x \cdot u)_{v},(u \cdot y)_{v}\right\}
$$

Next, $2(x \cdot u)_{v}=d(u, v)$ and

$$
\begin{aligned}
& 2(u \cdot y)_{v}=d(u, v)+d(y, v)-(d(x, y)-d(x, u)) \\
& =d(u, v)+(d(y, v)+d(x, v)-d(x, y)) \geqslant d(u, v)
\end{aligned}
$$

hence

$$
d(v,[x, y]) \geqslant \frac{1}{2} d(u, v)>2 \delta
$$

In particular, $d(x, v)>2 \delta$ and there exists $p \in[x, v]$ such that $d(p, v)=\delta$. Now we have

$$
\begin{gathered}
d(p,[x, y]) \geqslant d(v,[x, y])-d(v, p)>\delta \\
d(p,[y, z]) \geqslant d(x,[y, z])-d(x, p) \geqslant(y \cdot z)_{x}-d(x, p)>t-d(x, p) \\
=d(v, x)-d(x, p)=d(p, v)=\delta
\end{gathered}
$$

It follows that $d(p,[x, y] \cup[y, z])>\delta$ which contradicts our assumption.
$(H 1, \delta) \Longrightarrow(H 3,4 \delta)$ : follows from $(H 1, \delta) \Longrightarrow(H 2,4 \delta)$ and $(H 2, \delta) \Longrightarrow$ ( $H 3, \delta$ ).
$(H 3, \delta) \Longrightarrow(H 1,8 \delta)$ : follows from $(H 3, \delta) \Longrightarrow(H 2,4 \delta)$ and $(H 2, \delta) \Longrightarrow$ ( $H 1,2 \delta$ ).

If $(X, d)$ is geodesic then for $x, y, z \in X$ we denote by $\Delta_{I}(x, y, z)$ the geodesic triangle whose vertices are the points $p, q, r$ on the sides of $\Delta(x, y, z)$ which are sent to the center point of $T(x, y, z)$ under the isometry $\phi: \Delta(x, y, z) \rightarrow$ $T(x, y, z)$. From Proposition 1 it follows that if $(X, d)$ is $\delta$-hyperbolic then the length of the sides of $\Delta_{I}(x, y, z)$ is bounded by $4 \delta$.

### 2.5 Boundaries of hyperbolic $\Lambda$-metric spaces

We say that a sequence $\left\{\lambda_{i}\right\}$ of elements of $\Lambda$ converges to infinity, and write

$$
\lim _{i \rightarrow \infty} \lambda_{i}=\infty
$$

if for every $\alpha \in \Lambda$ there is a natural number $n_{\alpha}$ such that $\lambda_{i} \geqslant \alpha$ for every $i \geqslant n_{\alpha}$. Similarly, a double-indexed family $\left\{\lambda_{i j}\right\} \subseteq \Lambda$ converges to infinity, that is,

$$
\lim _{\substack{i \rightarrow \infty \\ j \rightarrow \infty}} \lambda_{i j}=\infty
$$

if for every $\alpha \in \Lambda$ there is a natural number $n_{\alpha}$ such that $\lambda_{i j} \geqslant \alpha$ for every $i, j \geqslant n_{\alpha}$.

Since $\Lambda$ is an arbitrary ordered abelian group, the notion of convergence to infinity can be applied with respect to any convex subgroup $\Lambda_{0} \subseteq \Lambda$ by replacing $\Lambda$ with $\Lambda_{0}$.

Let $(X, d)$ be a $\Lambda$-metric space. Fix a base point $v \in X$. We say that a sequence of points $\left\{x_{i}\right\} \subseteq X$ converges to infinity if

$$
\lim _{\substack{i \rightarrow \infty \\ j \rightarrow \infty}}\left(x_{i} \cdot x_{j}\right)_{v}=\infty .
$$

Observe that if a sequence $\left\{x_{i}\right\}$ converges to infinity with respect to $v \in X$ then it converges to infinity with respect to any other $v^{\prime} \in X$, since (by the triangle inequality)

$$
\begin{equation*}
\left|\left(x_{i} \cdot x_{j}\right)_{v}-\left(x_{i} \cdot x_{j}\right)_{v^{\prime}}\right| \leqslant d\left(v, v^{\prime}\right) \tag{1}
\end{equation*}
$$

for any $x_{i}, x_{j}, v, v^{\prime}$.
Now, we term two convergent to infinity sequences $\left\{x_{i}\right\},\left\{y_{j}\right\} \subseteq X$ close or equivalent (and write $\left\{x_{i}\right\} \sim\left\{y_{i}\right\}$ ) with respect to $v$ if

$$
\lim _{\substack{i \rightarrow \infty \\ j \rightarrow \infty}}\left(x_{i} \cdot y_{j}\right)_{v}=\infty
$$

Notice that (1) shows again that if $\left\{x_{i}\right\},\left\{y_{j}\right\}$ are equivalent with respect to $v$ then they are equivalent with respect to any base point $v^{\prime} \in X$.

Lemma 7. If $(X, d)$ is a hyperbolic $\Lambda$-metric space then the property" "to be close" defines an equivalence relation on the set of all sequences in $(X, d)$ that converge to infinity.

Proof. Reflexivity and symmetry of " $\sim$ " follow immediately from definitions. To prove transitivity consider convergent to infinity sequences $\left\{x_{i}\right\} \sim\left\{y_{j}\right\}$ and $\left\{y_{j}\right\} \sim\left\{z_{k}\right\}$. Suppose that $X$ is $\delta$-hyperbolic for some $\delta \in \Lambda$. Then for a given arbitrary $\alpha \in \Lambda$ choose $n_{\alpha}$ be such that

$$
\left(x_{i} \cdot y_{j}\right)_{v} \geqslant \alpha+\delta, \quad\left(y_{j} \cdot z_{k}\right)_{v} \geqslant \alpha+\delta
$$

for some $v \in X$ and all $i, j, k \geqslant n_{\alpha}$. Hence,

$$
\left(x_{i} \cdot z_{k}\right)_{v} \geq \min \left\{\left(x_{i} \cdot y_{j}\right)_{v},\left(y_{j} \cdot z_{k}\right)_{v}\right\}-\delta \geqslant(\alpha+\delta)-\delta=\alpha
$$

which shows that $\left\{x_{i}\right\}$ and $\left\{z_{k}\right\}$ are $\Lambda$-close.
For a hyperbolic $\Lambda$-metric space $(X, d)$ we define the boundary at infinity of $X$ as the set of equivalence classes of close convergent to infinity sequences in $(X, d)$ and denote it $\partial X$. Observe that if $\Lambda=\mathbb{R}$ then $\partial X$ is the hyperbolic boundary of $(X, d)$. If $a \in \partial X$ and $\left\{x_{i}\right\} \in a$ then we write $x_{i} \rightarrow a$ as $i \rightarrow \infty$.

Now let $\Lambda_{0}$ be a convex non-trivial subgroup of $\Lambda$ and $v \in X$. Take a $\Lambda_{0^{-}}$ subspace $X_{v, \Lambda_{0}}$ with the base point $v$ of $X$, which is a $\Lambda_{0}$-metric space. If $X$ is $\delta$-hyperbolic and $\delta \in \Lambda_{0}$ then $X_{v, \Lambda_{0}}$ is $\delta$-hyperbolic, so the argument above applies and one gets the boundary at infinity $\partial X_{v, \Lambda_{0}}$ of $X_{v, \Lambda_{0}}$, which we call the $\Lambda_{0}$-boundary of $X$ with respect to the base point $v$. Notice that in the case of $X_{v, \Lambda_{0}}$ one can consider sequences not only from $X_{v, \Lambda_{0}}$ but also $\left\{x_{i}\right\}$ such that $x_{i} \in X_{v, \Lambda_{0}}$ for all sufficiently large $i$.

Recall that in the case of hyperbolic $\mathbb{R}$-metric spaces there exists a notion of the Gromov product on the boundary. Similarly, for $x, y \in \partial X$ we define the Gromov product on the $\Lambda$-boundary as follows

$$
(\alpha \cdot \beta)_{v}=\sup _{\substack{x_{i} \rightarrow \alpha \\ y_{j} \rightarrow \beta}} \liminf _{\substack{i \rightarrow \infty \\ j \rightarrow \infty}}\left(x_{i} \cdot y_{j}\right)_{v}
$$

provided the limit exists (it depends on $\Lambda$ ).

Lemma 8. If $\Lambda=\mathbb{R}^{n}$ with the right lexicographic order and $\delta=(d, 0, \ldots, 0)$, then $(\alpha \cdot \beta)_{v}$ exists for any distinct $\alpha, \beta \in \partial X$.

Proof. We define

$$
\left(\left\{x_{i}\right\} \cdot\left\{y_{j}\right\}\right)_{v}=\underset{\substack{i \rightarrow \infty \\ j \rightarrow \infty}}{\liminf _{i}}\left(x_{i} \cdot y_{j}\right)_{v}
$$

We have to prove first that $\left(\left\{x_{i}\right\} \cdot\left\{y_{j}\right\}\right)_{v}$ exists for any two sequences converging at infinity which are not equivalent and then prove that $(\alpha \cdot \beta)_{v}$ exists for any distinct $\alpha, \beta \in \partial X$.

Suppose then that $\left\{x_{i}\right\}$ and $\left\{y_{j}\right\}$ are two sequences converging to infinity. If for any $a \in \mathbb{R}^{n}$ there exist some indices $k_{a}, k_{a}^{\prime}$ such that $\left(x_{k_{a}} \cdot y_{k_{a}^{\prime}}\right)_{v}>a$, then the sub-sequences $\left\{x_{k_{a}}\right\}$ and $\left\{y_{k_{a}^{\prime}}\right\}$, where $a=(0, \ldots, 0, n)$, converge to the same point in $\partial X$, which implies that $\alpha$ and $\beta$ coincide which is a contradiction. Hence, we can assume that there exists some $a \in \mathbb{R}^{n}$ such that $\left(x_{i} \cdot y_{j}\right)_{v} \leqslant a$ for any $i, j$. Since $\left\{x_{i}\right\}$ and $\left\{y_{j}\right\}$ converge to infinity, there exist $m, n$ such that $\left(x_{m} \cdot x_{m+k}\right)_{v},\left(y_{n} \cdot y_{n+k}\right)_{v} \geqslant a$ for any $k \geqslant 0$.

Take $M, N \in \mathbb{N}$ such that $M>m$ and $N>n$. Hence,
$\left(x_{M} \cdot y_{N}\right)_{v} \geqslant \min \left\{\left(x_{M} \cdot x_{m}\right)_{v},\left(x_{m} \cdot y_{N}\right)_{v}\right\}-\delta=\left(x_{m} \cdot y_{N}\right)_{v}-\delta \geqslant\left(x_{m} \cdot y_{n}\right)_{v}-2 \delta$.
A similar argument shows that $\left(x_{m}, y_{n}\right)_{v} \geqslant\left(x_{M} \cdot y_{N}\right)_{v}-2 \delta$.
Suppose now that $\left(x_{m} \cdot y_{n}\right)_{v}=\left(c_{1}, \ldots, c_{n}\right)$. It follows that

$$
\begin{aligned}
\left(c_{1}, \ldots, c_{n-1}, c_{n}-2 d\right) & =\left(x_{m} \cdot y_{n}\right)_{v}-2 \delta \leqslant\left(x_{M} \cdot y_{N}\right)_{v} \leqslant\left(x_{m} \cdot y_{n}\right)_{v}+2 \delta \\
& =\left(c_{1}, \ldots, c_{n-1}, c_{n}+2 d\right)
\end{aligned}
$$

for any $M>m, N>n$. The completeness of $\mathbb{R}$ implies then that $\left(\left\{x_{i}\right\} \cdot\left\{y_{j}\right\}\right)_{v}$ exists for any two non-equivalent sequences which converge at infinity.

Proving existence of $(\alpha \cdot \beta)_{v}$ is quite similar. Let $\left\{x_{i}^{\prime}\right\}$ tend to $\alpha$ and $\left\{y_{j}^{\prime}\right\}$ tend to $\beta$ and $\left(x_{i}^{\prime} \cdot y_{j}^{\prime}\right)_{v}<a$ for any $i, j$. There exist $m, n$ such that $\left(x_{m}\right.$. $\left.x_{m+k}^{\prime}\right)_{v},\left(y_{n} \cdot y_{n+k}^{\prime}\right)_{v}>a$ for any $k \geqslant 0$. We then use the same argument as above to prove that

$$
\left(\left\{x_{i}\right\} \cdot\left\{y_{j}\right\}\right)_{v}-2 \delta \leqslant\left(\left\{x_{i}^{\prime}\right\} \cdot\left\{y_{j}^{\prime}\right\}\right)_{v} \leqslant\left(\left\{x_{i}\right\} \cdot\left\{y_{j}\right\}\right)_{v}+2 \delta,
$$

so the supremum must again exist by completeness of $\mathbb{R}$.

### 2.6 Isometries of $\Lambda$-metric spaces

Let $(X, d)$ be a $\delta$-hyperbolic $\Lambda$-metric space and $\Lambda_{\delta}$ be the minimal convex subgroup of $\Lambda$ containing $\delta$ (that is, for every $\alpha \in \Lambda_{\delta}$ there exists $k \in \mathbb{N}$ such that $\alpha<k \delta)$. If $\delta=0$ then we set $\Lambda_{\delta}$ to be the trivial subgroup of $\Lambda$. Consider the set of all isometric mappings from $X$ to itself which we denote by $\operatorname{Isom}(X)$. The image of $x \in X$ under $\gamma \in \operatorname{Isom}(X)$ we denote by $\gamma x$.

Observe that if $\left\{x_{i}\right\} \subseteq X$ converges to infinity then $\left\{\gamma x_{i}\right\}$ also converges to infinity, and if $\left\{x_{i}\right\},\left\{y_{i}\right\}$ are close then $\left\{\gamma x_{i}\right\},\left\{\gamma y_{i}\right\}$ are also close. This shows
that $\gamma$ extends to the mapping $\partial X \rightarrow \partial X$ which we by abuse of notation again denote $\gamma$.
$\gamma \in \operatorname{Isom}(X)$ is minimal on $X$ if it does not have any invariant $\Lambda_{0}$-subspace of $X$ for some non-trivial proper convex subgroup $\Lambda_{0}$ of $\Lambda$. If $\gamma$ is not minimal then it induces an isometry $\gamma_{0}$ on a $\Lambda_{0}$-subspace $X_{0}$ of $X$ and the definitions below apply to the case when $\Lambda=\Lambda_{0}$ and $X=X_{0}$. Note that if the set $\left\{\gamma^{n} x \mid n \in \mathbb{Z}\right\}$ for some $x \in X$ is bounded by $\alpha \in \Lambda$ and $\Lambda_{0}$ is the minimal convex subgroup of $\Lambda$ containing $\alpha$ then $\gamma$ stabilizes

$$
X_{0}=\left\{y \in X \mid d(x, y) \in \Lambda_{0}\right\}
$$

which is a $\Lambda_{0}$-subspace of $X$ containing $x$. Indeed, for every $y \in X_{0}$ we have

$$
d(y, \gamma y) \leqslant d(x, y)+d(x, \gamma x)+d(\gamma x, \gamma y)=2 d(x, y)+d(x, \gamma x) \in \Lambda_{0}
$$

Hence, $\gamma$ is minimal only if $X_{0}=X$.
An isometry $\gamma: X \rightarrow X$ is called elliptic if there exists $x \in X$ such that the set $\left\{\gamma^{n} x \mid n \in \mathbb{Z}\right\}$ has diameter of at most $K \delta$ for some $K \in \mathbb{N}$. This definition generalizes the standard definition of elliptic isometry in $\mathbb{R}$-metric spaces. Indeed, recall that in a proper geodesic $\mathbb{R}$-metric space an isometry is elliptic if it has a fixed point. Our definition in this case (when $X$ is a proper geodesic $\mathbb{R}$-metric space) also implies a fixed point: since the set $\left\{\gamma^{n} x \mid n \in \mathbb{Z}\right\}$ is bounded and $X$ is proper, there exists a subsequence $\left\{n_{i}\right\}$ such that $\gamma^{n_{i}} x \rightarrow y$ for some $y \in X$ and $\gamma y=y$. But in general, for an arbitrary $\Lambda$-metric space $X$, even if $\Lambda=\mathbb{R}$ our definition does not imply that there exists a fixed point of $\gamma$.

Suppose $\gamma$ is elliptic and $x \in X$ is such that the set $\left\{\gamma^{n} x \mid n \in \mathbb{Z}\right\}$ is bounded by $K \delta$ for some $K \in \mathbb{N}$. Let $X_{0}$ be the $\Lambda_{\delta}$-subspace of $X$ containing $x$. Hence, for every $y \in X_{0}$ and $m, n \in \mathbb{Z}$ there exists $M \in \mathbb{N}$ such that

$$
\begin{gathered}
d\left(\gamma^{n} y, \gamma^{m} y\right) \leqslant d\left(\gamma^{n} y, \gamma^{n} x\right)+d\left(\gamma^{n} x, \gamma^{m} x\right)+d\left(\gamma^{m} x, \gamma^{m} y\right) \\
=2 d(x, y)+d\left(\gamma^{n} x, \gamma^{m} x\right) \leqslant 2 d(x, y)+K \delta \leqslant M \delta
\end{gathered}
$$

In particular, it follows that $\gamma$ fixes $X_{0}$ and it cannot be minimal unless $X=X_{0}$.
The following fact follows immediately from the definition above.
Lemma 9. Let $\Lambda=\mathbb{R}^{n}$ with the right lexicographic order and let $(X, d)$ be a geodesic $\delta$-hyperbolic $\Lambda$-metric space, where $\delta=(d, 0, \ldots, 0)$. Then $\gamma \in$ $\operatorname{Isom}(X)$ is elliptic if and only if for some $x \in X$ the set $\left\{\gamma^{n} x \mid n \in \mathbb{Z}\right\}$ is bounded inside of some $\mathbb{R}$-subspace of $X$.

An isometry $\gamma: X \rightarrow X$ is called parabolic with respect to $x \in X$ if the diameter of the set $\left\{\gamma^{n} x \mid n \in \mathbb{Z}\right\}$ is not bounded by any $\alpha \in \Lambda$ and there exists $a_{x} \in \partial X$ such that for any subsequence of integers $\left\{n_{i}\right\}$ with the property $d\left(x, \gamma^{n_{i}} x\right) \rightarrow \infty$ we have that $\left\{\gamma^{n_{i}} x\right\} \rightarrow a_{x}$.
Lemma 10. Let $(X, d)$ be a $\delta$-hyperbolic $\Lambda$-metric space. If $\gamma \in \operatorname{Isom}(X)$ is parabolic with respect to $x \in X$ then $\gamma$ is parabolic with respect to any other $y \in X$ and $a_{x}=a_{y}$.

Proof. Suppose $\gamma$ is parabolic with respect to $x$ and fix $y \in X$. Notice that for a subsequence of integers $\left\{n_{i}\right\}$ we have

$$
d\left(x, \gamma^{n_{i}} x\right) \rightarrow \infty \Longleftrightarrow d\left(y, \gamma^{n_{i}} y\right) \rightarrow \infty
$$

Indeed, assuming $d\left(x, \gamma^{n_{i}} x\right) \rightarrow \infty$, we get

$$
\begin{gathered}
d\left(y, \gamma^{n_{i}} y\right) \geqslant d\left(x, \gamma^{n_{i}} y\right)-d(x, y) \geqslant d\left(x, \gamma^{n_{i}} x\right)-d\left(\gamma^{n_{i}} x, \gamma^{n_{i}} y\right)-d(x, y) \\
=d\left(x, \gamma^{n_{i}} x\right)-2 d(x, y) \rightarrow \infty
\end{gathered}
$$

and the converse implication can be obtained similarly.
Let $v \in X$ be a base-point. If $d\left(x, \gamma^{n_{i}} x\right) \rightarrow \infty$ then $\left(\gamma^{n_{i}} x \cdot \gamma^{n_{j}} x\right)_{v} \rightarrow \infty$ as $n_{i}, n_{j} \rightarrow \infty$. Next, from hyperbolicity of $X$ we get

$$
\begin{aligned}
& \quad\left(\gamma^{n_{i}} y \cdot \gamma^{n_{j}} y\right)_{v} \geqslant \min \left\{\left(\gamma^{n_{i}} x \cdot \gamma^{n_{i}} y\right)_{v},\left(\gamma^{n_{i}} x \cdot \gamma^{n_{j}} y\right)_{v}\right\}-\delta \\
& \geqslant \min \left\{\left(\gamma^{n_{i}} x \cdot \gamma^{n_{i}} y\right)_{v},\left(\gamma^{n_{i}} x \cdot \gamma^{n_{j}} x\right)_{v},\left(\gamma^{n_{j}} x \cdot \gamma^{n_{j}} y\right)_{v}\right\}-2 \delta
\end{aligned}
$$

which implies that $\left(\gamma^{n_{i}} y \cdot \gamma^{n_{j}} y\right)_{v} \rightarrow \infty$ as $n_{i}, n_{j} \rightarrow \infty$ since

$$
\begin{aligned}
& \left(\gamma^{n_{i}} x \cdot \gamma^{n_{i}} y\right)_{v}=\frac{1}{2}\left(d\left(v, \gamma^{n_{i}} x\right)+d\left(v, \gamma^{n_{i}} y\right)-d\left(\gamma^{n_{i}} x, \gamma^{n_{i}} y\right)\right) \\
& \quad=\frac{1}{2}\left(d\left(v, \gamma^{n_{i}} x\right)+d\left(v, \gamma^{n_{i}} y\right)-d(x, y)\right) \rightarrow \infty \text { as } n_{i} \rightarrow \infty
\end{aligned}
$$

and similarly $\left(\gamma^{n_{j}} x \cdot \gamma^{n_{j}} y\right)_{v} \rightarrow \infty$ as $n_{j} \rightarrow \infty$. It follows that there exists $a_{y} \in \partial X$ such that $\left\{\gamma^{n_{i}} y\right\} \rightarrow a_{y}$ and $\gamma$ is parabolic with respect $y$. Moreover, the fact that $\left(\gamma^{n_{i}} y \cdot \gamma^{n_{j}} y\right)_{v} \rightarrow \infty$ as $n_{i}, n_{j} \rightarrow \infty$ implies that $a_{x}=a_{y}$.

In view of Lemma 10 we say that $\gamma \in \operatorname{Isom}(X)$ is parabolic if it is parabolic with respect to some $x \in X$. Note that a parabolic isometry cannot fix any $\Lambda_{0}$-subspace $X_{0}$ of $X$ for $\Lambda_{0} \subsetneq \Lambda$, so, it is minimal.

Observe that if $\gamma$ is parabolic and a subsequence of integers $\left\{n_{i}\right\}$ is such that $d\left(x, \gamma^{n_{i}} x\right) \rightarrow \infty$ then $\left\{\gamma^{n_{i}} x\right\}$ can be taken as a representative of $a$. But then from $d\left(x, \gamma^{n_{i}+1} x\right) \geqslant d\left(x, \gamma^{n_{i}+1} x\right)-d(x, \gamma x)$ we get $d\left(x, \gamma^{n_{i}+1} x\right) \rightarrow \infty$, so, $\left\{\gamma^{n_{i}+1} x\right\} \rightarrow a$. But at the same time, $\left\{\gamma^{n_{i}+1} x\right\} \rightarrow \gamma a$ and we get $\gamma a=a$.
$\gamma$ is called hyperbolic with respect to $x \in X$ if the diameter of the set $\left\{\gamma^{n} x \mid\right.$ $n \in \mathbb{Z}\}$ is not bounded by any $\alpha \in \Lambda$ and there exist distinct $a_{x}, b_{x} \in \partial X$ such that for any subsequence of natural numbers $\left\{n_{i}\right\}$, if $d\left(x, \gamma^{n_{i}} x\right) \rightarrow \infty$ then $\left\{\gamma^{-n_{i}} x\right\} \rightarrow a_{x}$ and $\left\{\gamma^{n_{i}} x\right\} \rightarrow b_{x}$. Similar to the parabolic case one can show that if $\gamma$ is hyperbolic with respect to $x \in X$ then it is hyperbolic with respect to any other $y \in X$ and $a_{x}=a_{y}, b_{x}=b_{y}$ (up to a permutation). Hence, we say that $\gamma$ is hyperbolic if it is hyperbolic with respect to any $x \in X$ and we denote $a_{-}=a_{x}$ and $a_{+}=b_{x}$. Again, similar to the parabolic case one can easily show that $\gamma a_{-}=a_{-}$and $\gamma a_{+}=a_{+}$.

It is easy to see that if $\gamma$ is elliptic, parabolic, or hyperbolic then so is $\gamma^{k}$ for every fixed $k \in \mathbb{Z}$.

Finally, an isometry $\gamma$ is called an inversion if $\gamma$ does not fix any $\Lambda_{\delta}$-subspace of $X$, but $\gamma^{2}$ fixes a $\Lambda_{\delta}$-subspace of $X$. Obviously, inversions can exist only in the case when $\Lambda \neq \Lambda_{\delta}$. Also, observe that if $\delta=0$ then any $\Lambda_{\delta}$-subspace of $X$ is a single point, hence, in the case when $X$ is a proper geodesic 0 -hyperbolic $\Lambda$-metric space, that is, a $\Lambda$-tree, our definition coincides with the definition of inversion for $\Lambda$-trees.

Observe that if $\Lambda=\mathbb{R}$, a geodesic $\Lambda$-metric space $(X, d)$ is an ordinary hyperbolic space, so, every isometry of $X$ is either elliptic, or parabolic, or hyperbolic (see [ $\lfloor$, Theorem 9.2.1]). If $X$ is not geodesic, or if $\Lambda=\mathbb{Z}$ then the case of an inversion adds.

Next, if $\delta=0$ then a $\Lambda$-metric space $(X, d)$ is a $\Lambda$-tree and classification of its isometries is also known (see [3, Section 3.1]): any isometry of $X$ is either elliptic, or hyperbolic, or an inversion in the case when $\Lambda \neq 2 \Lambda$.

Our next goal is to classify isometries of $\delta$-hyperbolic $\Lambda$-metric spaces.
The following lemma is similar to (4, Lemma 2.2].
Lemma 11. Let $\gamma$ be a minimal isometry of a geodesic $\delta$-hyperbolic $\Lambda$-metric space $(X, d)$. If there exists $x \in X$ such that

$$
d\left(x, \gamma^{2} x\right)>d(x, \gamma x)+3 \delta
$$

then $\gamma$ is hyperbolic in the case when $\Lambda=2 \Lambda$, and $\gamma$ is either hyperbolic, or an inversion if $\Lambda \neq 2 \Lambda$.
Proof. Consider the points $x, \gamma x, \gamma^{2} x$ and $\gamma^{n} x$, where $n \in \mathbb{N}$. By the 4-point condition (see Lemma (1) we have
$d\left(x, \gamma^{2} x\right)+d\left(\gamma x, \gamma^{n} x\right) \leqslant \max \left\{d(x, \gamma x)+d\left(\gamma^{2} x, \gamma^{n} x\right), d\left(x, \gamma^{n} x\right)+d\left(\gamma^{2} x, \gamma x\right)\right\}+2 \delta$ or, if we denote $\alpha_{k}=d\left(\gamma^{k} x, x\right)$ for every $k \in \mathbb{N}$ then

$$
\alpha_{2}+\alpha_{n} \leqslant \max \left\{\alpha_{1}+\alpha_{n-2}, \alpha_{n}+\alpha_{1}\right\}+2 \delta
$$

since $d\left(\gamma^{k} x, \gamma^{m} x\right)=\alpha_{|k-m|}$. In other words we get

$$
\max \left\{\alpha_{n-2}, \alpha_{n}\right\} \geqslant \alpha_{n-1}+\alpha_{2}-\alpha_{1}-2 \delta
$$

and from the assumption $\alpha_{2}>\alpha_{1}+3 \delta$ we obtain

$$
\max \left\{\alpha_{n-2}, \alpha_{n}\right\}>\alpha_{n-1}+\delta
$$

which holds for any $n \in \mathbb{N}$. Next, we prove by induction on $n$ that

$$
\alpha_{n}+\delta<\alpha_{n+1}
$$

If $n=0$ then we have

$$
\alpha_{1}+3 \delta<\alpha_{2} \leqslant 2 \alpha_{1}
$$

and $\alpha_{0}+\delta<\alpha_{1}$. Suppose the inequality holds for $n$, that is, $\alpha_{n+1}>\alpha_{n}+\delta$. Since we have

$$
\max \left\{\alpha_{n+2}, \alpha_{n}\right\}>\alpha_{n+1}+\delta
$$

it implies that $\max \left\{\alpha_{n+2}, \alpha_{n}\right\}=\alpha_{n+2}$ and

$$
\alpha_{n+2}>\alpha_{n+1}+\delta
$$

as required.
In particular, it follows that $\alpha_{n}>n \delta$. Consider two cases.
Case I. There exists $K \in \mathbb{N}$ such that $K \delta>\alpha_{1}=d(x, \gamma x)(\delta$ and $d(x, \gamma x)$ are "comparable").

In this case, the diameter of the set $\left\{\gamma^{n} x \mid n \in \mathbb{Z}\right\}$ is not bounded by any $\alpha \in \Lambda$ (since $\gamma$ is minimal). Moreover, for any $v \in X$ we have

$$
\lim _{\substack{i \rightarrow \infty \\ j \rightarrow \infty}}\left(\gamma^{i} x \cdot \gamma^{j} x\right)_{v}=\infty
$$

and

$$
\lim _{\substack{i \rightarrow \infty \\ j \rightarrow \infty}}\left(\gamma^{-i} x \cdot \gamma^{-j} x\right)_{v}=\infty
$$

since $\gamma$ is minimal and $\delta$ and $d(x, \gamma x)$ are "comparable". It shows that there exist distinct $a_{x}, b_{x} \in \partial X$ such that for any subsequence of natural numbers $\left\{n_{i}\right\}$ we have $\left\{\gamma^{-n_{i}} x\right\} \rightarrow a_{x}$ and $\left\{\gamma^{n_{i}} x\right\} \rightarrow b_{x}$. That is, $\gamma$ is hyperbolic with respect to $x$, hence, hyperbolic.

Case II. $d(x, \gamma x)=\alpha_{1}>K \delta$ for every $K \in \mathbb{N}(d(x, \gamma x)$ is "infinitely large" with respect to $\delta$ ).

Recall that $\Lambda_{\delta}$ is the minimal convex subgroup of $\Lambda$ containing $\delta$. Notice that by the assumption we have $\Lambda_{\delta} \neq \Lambda$. Define an equivalence relation " $\sim$ " on $X$ by setting

$$
y \sim z \quad \Longleftrightarrow \quad d(y, z) \in \Lambda_{\delta}, \text { for any } y, z \in X
$$

Observe that $X_{1}=X / \sim$ is a $\Lambda_{1}$-metric space, where $\Lambda_{1}=\Lambda / \Lambda_{\delta}$, with respect to the metric

$$
d_{1}([y],[z])=d(y, z)+\Lambda_{\delta}
$$

where $[y],[z]$ are the images of $y, z \in X$ in $X_{1}$. Since $X$ is geodesic, from the definition of $X_{1}$ it follows that $X_{1}$ is also geodesic. Moreover, $X_{1}$ is 0-hyperbolic since $\delta \in \Lambda_{\delta}$, and it follows that $X_{1}$ is a $\Lambda_{1}$-tree.

The isometry $\gamma$ of $X$ induces an isometry $\gamma_{1}$ of $X_{1}$ and we have

$$
d_{1}\left([x], \gamma_{1}^{2}[x]\right) \geqslant d_{1}\left([x], \gamma_{1}[x]\right)
$$

Recall that the translation length $l\left(\gamma_{1}\right)$ of $\gamma_{1}$ (see, for example, (3) is defined as

$$
l\left(\gamma_{1}\right)=\min \left\{d_{1}\left([y], \gamma_{1}[y]\right) \mid[y] \in X_{1}\right\}
$$

According to Lemma 3.1.8 3], for any $[y] \in X_{1}$

$$
l\left(\gamma_{1}\right)=\max \left\{d_{1}\left([y], \gamma_{1}^{2}[y]\right)-d_{1}\left([y], \gamma_{1}[y]\right), 0\right\}
$$

and in particular
$l\left(\gamma_{1}\right)=\max \left\{d_{1}\left([x], \gamma_{1}^{2}[x]\right)-d_{1}\left([x], \gamma_{1}[x]\right), 0\right\}=d_{1}\left([x], \gamma_{1}^{2}[x]\right)-d_{1}\left([x], \gamma_{1}[x]\right) \geqslant 0$
If $l\left(\gamma_{1}\right)>0$ then $\gamma_{1}$ is a hyperbolic isometry of $X_{1}$ which fixes a pair of ends $a_{1}, b_{1} \in \partial X_{1}$ of full $\Lambda_{1}$-type (the ends of the axis of $\gamma_{1}$ ). Indeed, if $a_{1}, b_{1}$ are not of full $\Lambda_{1}$-type, then $\gamma_{1}$ fixes a $\Lambda_{2}$-subspace $X_{2}$ of $X_{1}$, where $\Lambda_{2} \neq \Lambda_{1}$. But in this case, $X_{2}$ lifts to a $\Lambda_{3}$-subspace $X_{3}$ of $X$, where $\Lambda_{3} \neq \Lambda$ which is stabilized by $\gamma$ - a contradiction with minimality of $\gamma$. Hence, $a_{1}, b_{1} \in \partial X_{1}$ have preimages $a_{x}, b_{x} \in \partial X$ which are fixed by $\gamma$ and $\gamma$ is hyperbolic.

If $l\left(\gamma_{1}\right)=0$ then $\gamma_{1}$ is either elliptic or an inversion. If $\gamma_{1}$ is elliptic then there exists $[y] \in X_{1}$ such that $\gamma_{1}[y]=[y]$, that is, $\gamma$ stabilizes the $\Lambda_{0}$-subspace of $X$ containing $y$ - a contradiction with minimality of $\gamma$. Suppose $\gamma_{1}$ is an inversion, that is, $\gamma_{1}$ does not fix a point in $X_{1}$ but $\gamma_{1}^{2}$ has a fixed point $[y] \in X_{1}$. Observe that this is possible only if $\Lambda_{1} \neq 2 \Lambda_{1}$ which implies that $\Lambda \neq 2 \Lambda$. Now, $\gamma$ does not fix any $\Lambda_{\delta}$-subspace of $X$, but $\gamma^{2}$ fixes the $\Lambda_{\delta}$-subspace of $X$ containing $y$. Hence, $\gamma$ is an inversion.

The lemma below is similar to [4, Lemma 2.3].
Lemma 12. Let $(X, d)$ be a geodesic $\delta$-hyperbolic $\Lambda$-metric space, where and $\delta>0$. Suppose $\gamma_{1}, \gamma_{2}$ are isometries of $X$ which are neither hyperbolic, nor inversions, and such that for some $x \in X$

$$
d\left(x, \gamma_{1} x\right) \geqslant 2\left(\gamma_{1} x \cdot \gamma_{2} x\right)_{x}+6 \delta, \quad d\left(x, \gamma_{2} x\right) \geqslant 2\left(\gamma_{1} x \cdot \gamma_{2} x\right)_{x}+6 \delta
$$

Then $\gamma_{2} \gamma_{1}$ and $\gamma_{1} \gamma_{2}$ are hyperbolic if $\Lambda=2 \Lambda$, and are either hyperbolic, or inversions if $\Lambda \neq 2 \Lambda$.

Proof. We follow the scheme of proof of [4, Lemma 2.3] and adopt the same terminology: for any isometries $\alpha, \beta$ of $X$ denote $|\alpha-\beta|=d(\alpha x, \beta x)$ and $|\alpha|=d(x, \alpha x)$.

Since $\gamma_{1}, \gamma_{2}$ are neither hyperbolic, nor inversions, by Lemma 11 we have

$$
\left|\gamma_{1}^{2}\right| \leqslant\left|\gamma_{1}\right|+3 \delta,\left|\gamma_{2}^{2}\right| \leqslant\left|\gamma_{2}\right|+3 \delta
$$

Next, from

$$
\left|\gamma_{1}\right| \geqslant 2\left(\gamma_{1} x \cdot \gamma_{2} x\right)_{x}+6 \delta, \quad\left|\gamma_{2}\right| \geqslant 2\left(\gamma_{1} x \cdot \gamma_{2} x\right)_{x}+6 \delta
$$

by definition of Gromov product we get

$$
\left|\gamma_{1}-\gamma_{2}\right| \geqslant\left|\gamma_{1}\right|+6 \delta,\left|\gamma_{1}-\gamma_{2}\right| \geqslant\left|\gamma_{2}\right|+6 \delta
$$

Now we apply the 4-point condition (see Lemma (1) to $x, \gamma_{1} x, \gamma_{1}^{2} x$, and $\left(\gamma_{1} \gamma_{2}\right) x$ :

$$
\left|\gamma_{1}\right|+\left|\gamma_{1}^{2}-\gamma_{1} \gamma_{2}\right| \leqslant \max \left\{\left|\gamma_{1}^{2}\right|+\left|\gamma_{1}-\gamma_{1} \gamma_{2}\right|,\left|\gamma_{1} \gamma_{2}\right|+\left|\gamma_{1}-\gamma_{1}^{2}\right|\right\}+2 \delta
$$

or

$$
\left|\gamma_{1}\right|+\left|\gamma_{1}-\gamma_{2}\right| \leqslant \max \left\{\left|\gamma_{1}^{2}\right|+\left|\gamma_{2}\right|,\left|\gamma_{1} \gamma_{2}\right|+\left|\gamma_{1}\right|\right\}+2 \delta
$$

From $\left|\gamma_{1}^{2}\right| \leqslant\left|\gamma_{1}\right|+3 \delta$ and $\left|\gamma_{1}-\gamma_{2}\right| \geqslant\left|\gamma_{2}\right|+6 \delta$ we obtain

$$
\left|\gamma_{1}\right|+\left|\gamma_{1}-\gamma_{2}\right| \geqslant\left|\gamma_{1}^{2}\right|+\left|\gamma_{2}\right|+3 \delta
$$

which implies that $\left|\gamma_{1}\right|+\left|\gamma_{1}-\gamma_{2}\right|$ cannot not be smaller than $\left|\gamma_{1}^{2}\right|+\left|\gamma_{2}\right|+2 \delta$ and

$$
\max \left\{\left|\gamma_{1}^{2}\right|+\left|\gamma_{2}\right|,\left|\gamma_{1} \gamma_{2}\right|+\left|\gamma_{1}\right|\right\}=\left|\gamma_{1} \gamma_{2}\right|+\left|\gamma_{1}\right|
$$

Hence,

$$
\left|\gamma_{1}\right|+\left|\gamma_{1}-\gamma_{2}\right| \leqslant\left|\gamma_{1} \gamma_{2}\right|+\left|\gamma_{1}\right|+2 \delta
$$

or

$$
\left|\gamma_{1}-\gamma_{2}\right| \leqslant\left|\gamma_{1} \gamma_{2}\right|+2 \delta
$$

Similar argument produces

$$
\left|\gamma_{1}-\gamma_{2}\right| \leqslant\left|\gamma_{2} \gamma_{1}\right|+2 \delta
$$

Combining the above inequalities with $\left|\gamma_{1}-\gamma_{2}\right| \geqslant\left|\gamma_{1}\right|+6 \delta,\left|\gamma_{1}-\gamma_{2}\right| \geqslant\left|\gamma_{2}\right|+6 \delta$ we get

$$
\begin{aligned}
\left|\gamma_{1} \gamma_{2}\right| \geqslant\left|\gamma_{1}\right|+4 \delta, \quad & \left|\gamma_{2} \gamma_{1}\right| \geqslant\left|\gamma_{1}\right|+4 \delta, \quad\left|\gamma_{1} \gamma_{2}\right| \geqslant\left|\gamma_{2}\right|+4 \delta \\
& \left|\gamma_{2} \gamma_{1}\right| \geqslant\left|\gamma_{2}\right|+4 \delta
\end{aligned}
$$

Next we apply the 4 -point condition to $x, \gamma_{1} x,\left(\gamma_{1} \gamma_{2}\right) x$, and $\left(\gamma_{1} \gamma_{2} \gamma_{1}\right) x$ :

$$
\left|\gamma_{2} \gamma_{1}\right|+\left|\gamma_{1} \gamma_{2}\right| \leqslant \max \left\{\left|\gamma_{1} \gamma_{2} \gamma_{1}\right|+\left|\gamma_{2}\right|, 2\left|\gamma_{1}\right|\right\}+2 \delta
$$

But we have

$$
\left|\gamma_{2} \gamma_{1}\right|+\left|\gamma_{1} \gamma_{2}\right| \geqslant 2\left|\gamma_{1}\right|+8 \delta
$$

so

$$
\left|\gamma_{2} \gamma_{1}\right|+\left|\gamma_{1} \gamma_{2}\right| \leqslant\left|\gamma_{1} \gamma_{2} \gamma_{1}\right|+\left|\gamma_{2}\right|+2 \delta
$$

and since $\left|\gamma_{2} \gamma_{1}\right| \geqslant\left|\gamma_{2}\right|+4 \delta$, it follows that

$$
\left|\gamma_{1} \gamma_{2}\right|+2 \delta \leqslant\left|\gamma_{1} \gamma_{2} \gamma_{1}\right|
$$

We combine the above inequality with $\left|\gamma_{1} \gamma_{2}\right| \geqslant\left|\gamma_{1}\right|+4 \delta$ and $\left|\gamma_{1} \gamma_{2}\right| \geqslant\left|\gamma_{2}\right|+4 \delta$ to get

$$
\left|\gamma_{2}\right|+6 \delta \leqslant\left|\gamma_{1} \gamma_{2} \gamma_{1}\right|, \quad\left|\gamma_{1}\right|+6 \delta \leqslant\left|\gamma_{1} \gamma_{2} \gamma_{1}\right|
$$

Apply the 4-point condition to $x$, $\left(\gamma_{1} \gamma_{2}\right) x$, $\left(\gamma_{1} \gamma_{2} \gamma_{1}\right) x$, and $\left(\gamma_{1} \gamma_{2}\right)^{2} x$ :

$$
\left|\gamma_{1} \gamma_{2}\right|+\left|\gamma_{1} \gamma_{2} \gamma_{1}\right| \leqslant \max \left\{\left|\left(\gamma_{1} \gamma_{2}\right)^{2}\right|+\left|\gamma_{1}\right|,\left|\gamma_{2}\right|+\left|\gamma_{1} \gamma_{2}\right|\right\}+2 \delta
$$

From $\left|\gamma_{2}\right|+6 \delta \leqslant\left|\gamma_{1} \gamma_{2} \gamma_{1}\right|$ we get

$$
\left|\gamma_{2}\right|+\left|\gamma_{1} \gamma_{2}\right|+6 \delta \leqslant\left|\gamma_{1} \gamma_{2} \gamma_{1}\right|+\left|\gamma_{1} \gamma_{2}\right|
$$

and it follows that

$$
\max \left\{\left|\left(\gamma_{1} \gamma_{2}\right)^{2}\right|+\left|\gamma_{1}\right|,\left|\gamma_{2}\right|+\left|\gamma_{1} \gamma_{2}\right|\right\}=\left|\left(\gamma_{1} \gamma_{2}\right)^{2}\right|+\left|\gamma_{1}\right|
$$

Hence,

$$
\left|\gamma_{1} \gamma_{2}\right|+\left|\gamma_{1} \gamma_{2} \gamma_{1}\right| \leqslant\left|\left(\gamma_{1} \gamma_{2}\right)^{2}\right|+\left|\gamma_{1}\right|+2 \delta
$$

and from $\left|\gamma_{1}\right|+6 \delta \leqslant\left|\gamma_{1} \gamma_{2} \gamma_{1}\right|$ we obtain

$$
\left|\gamma_{1} \gamma_{2}\right|+4 \delta \leqslant\left|\left(\gamma_{1} \gamma_{2}\right)^{2}\right|
$$

and

$$
\left|\gamma_{1} \gamma_{2}\right|+3 \delta<\left|\left(\gamma_{1} \gamma_{2}\right)^{2}\right|
$$

From Lemma 11 it follows that $\gamma_{1} \gamma_{2}$ is hyperbolic if $\Lambda=2 \Lambda$, and it is either hyperbolic, or an inversion if $\Lambda \neq 2 \Lambda$. The argument for $\gamma_{2} \gamma_{1}$ is similar.

Now, using the above lemmas, we are ready classify minimal isometries of a geodesic $\delta$-hyperbolic $\Lambda$-metric space.

Theorem 2. Let $(X, d)$ be a a geodesic $\delta$-hyperbolic $\Lambda$-metric space. Then every minimal isometry of $X$ is either elliptic, or parabolic, or hyperbolic in the case when $\Lambda=2 \Lambda$, and is either elliptic, or parabolic, or hyperbolic, or an inversion when $\Lambda \neq 2 \Lambda$.

Proof. If $\delta=0$ then $X$ is a $\Lambda$-tree and any isometry of $X$ (not necessarily a minimal one) is either hyperbolic, or elliptic, or an inversion (see, [1] , 3) .

Suppose $\delta>0$ and let $\gamma$ be a minimal isometry of $X$. Suppose $\gamma$ is neither elliptic, nor parabolic. It follows that for any $x \in X$ the diameter of the set $\left\{\gamma^{n} x \mid n \in \mathbb{Z}\right\}$ is not bounded by $K \delta$ for any $K \in \mathbb{N}$.

Next, suppose for any $x \in X$, the diameter of the set $\left\{\gamma^{n} x \mid n \in \mathbb{Z}\right\}$ is bounded by some $\alpha \in \Lambda$. We would like to show that $\gamma$ is an inversion in this case. Indeed, observe that the minimal convex subgroup $\Lambda^{\prime} \subseteq \Lambda$ containing $\alpha$ must coincide with $\Lambda$ (otherwise $\gamma$ stabilizes a $\Lambda^{\prime}$-subspace of $X$ and this is a contradiction with minimality of $\gamma$ ). Next, by our assumption $\Lambda_{\delta} \neq \Lambda$. Define an equivalence relation " $\sim$ " on $X$ by setting

$$
y \sim z \quad \Longleftrightarrow \quad d(y, z) \in \Lambda_{\delta}, \text { for any } y, z \in X
$$

Hence, $X_{1}=X / \sim$ is a $\Lambda_{1}$-metric space, where $\Lambda_{1}=\Lambda / \Lambda_{\delta}$, with respect to the metric

$$
d_{1}([y],[z])=d(y, z)+\Lambda_{\delta}
$$

where $[y],[z]$ are the images of $y, z \in X$ in $X_{1}$. Since $X$ is geodesic, from the definition of $X_{1}$ it follows that $X_{1}$ is also geodesic. Moreover, $X_{1}$ is 0-hyperbolic since $\delta \in \Lambda_{\delta}$, and it follows that $X_{1}$ is a $\Lambda_{1}$-tree.

The isometry $\gamma$ of $X$ induces an isometry $\gamma_{1}$ of $X_{1}$ and the diameter of the set $\left\{\gamma_{1}^{n}[x] \mid n \in \mathbb{Z}\right\}$ is bounded by $\alpha+\Lambda_{\delta}$. Consider the translation length $l\left(\gamma_{1}\right)$ of $\gamma_{1}$. If $l\left(\gamma_{1}\right)>0$ then $\gamma_{1}$ is hyperbolic and the diameter of the set $\left\{\gamma_{1}^{n}[x] \mid n \in \mathbb{Z}\right\}$
cannot be bounded by any $\beta \in \Lambda_{1}$. Hence, $\gamma_{1}$ is either an inversion, or elliptic. If $\gamma_{1}$ is elliptic then it fixes a point $[y] \in X_{1}$ which implies that $\gamma$ stabilizes a $\Lambda_{\delta^{-}}$ subspace of $X$ - a contradiction with minimality of $\gamma$. Hence, $\gamma_{1}$ is an inversion (which is possible only if $\Lambda \neq 2 \Lambda$ ) and it follows that $\gamma$ is also an inversion.

Finally, suppose that for any $x \in X$, the diameter of the set $\left\{\gamma^{n} x \mid n \in \mathbb{Z}\right\}$ is not bounded by any $\alpha \in \Lambda$. It follows that there exists a sequence of integers $\left\{n_{i}\right\}$ such that $d\left(x, \gamma^{n_{i}} x\right) \rightarrow \infty$. Hence, there exists $a \in \partial X$ such that $\left\{\gamma^{n_{i}} x\right\} \rightarrow$ $a$. Since we assume that $\gamma$ is not parabolic, there must be at least one other sequence of integers $\left\{m_{j}\right\}$ such that $d\left(x, \gamma^{m_{j}} x\right) \rightarrow \infty$ and a point $b \in \partial X$ such that $\left\{\gamma^{m_{j}} x\right\} \rightarrow b$ such that $a \neq b$.

Since $a \neq b$, it follows that the Gromov product $(a \cdot b)_{x}$ of $a$ and $b$ is finite. At the same time $d\left(x, \gamma^{n_{i}} x\right) \rightarrow \infty, \quad d\left(x, \gamma^{m_{j}} x\right) \rightarrow \infty$, so, there exist $N \in\left\{n_{i}\right\}$ and $M \in\left\{m_{j}\right\}$ such that $N \neq M$ and

$$
d\left(x, \gamma^{N} x\right) \geqslant 2\left(\gamma^{N} x \cdot \gamma^{M} x\right)_{x}+6 \delta, \quad d\left(x, \gamma^{M} x\right) \geqslant 2\left(\gamma^{N} x \cdot \gamma^{M} x\right)_{x}+6 \delta
$$

By Lemma 12, the isometry $\gamma^{N-M}$ is hyperbolic if $\Lambda=2 \Lambda$, and is either hyperbolic, or an inversion if $\Lambda \neq 2 \Lambda$. Hence, the required statement for $\gamma$ follows.

The theorem above immediately can be applied in the case when $\Lambda=$ $\mathbb{R}^{n}, \mathbb{Z}^{n}$.

Theorem 3. Let $(X, d)$ be a geodesic $\delta$-hyperbolic $\mathbb{R}^{n}$-metric space, where $\mathbb{R}^{n}$ is taken with the right lexicographic order. Then every minimal isometry of $X$ is either elliptic, or parabolic, or hyperbolic.

Theorem 4. Let $(X, d)$ be a geodesic $\delta$-hyperbolic $\mathbb{Z}^{n}$-metric space, where $\mathbb{Z}^{n}$ is taken with the right lexicographic order. Then every minimal isometry of $X$ is either elliptic, or parabolic, or hyperbolic, or an inversion.

Using Theorem we can give a nice characterization of hyperbolic isometries in the case when $\Lambda$ is either $\mathbb{R}^{n}$, or $\mathbb{Z}^{n}$.

Corollary 1. Let $(X, d)$ be a geodesic $\delta$-hyperbolic $\Lambda$-metric space, where $\Lambda$ is either $\mathbb{R}^{n}$, or $\mathbb{Z}^{n}$ with the right lexicographic order and $\delta=\left(\delta_{0}, 0, \ldots, 0\right)$. Let $\gamma$ be a minimal isometry of $X$. If $n>1$ then $\gamma$ can be only either hyperbolic, or an inversion. Moreover, for any $n, \gamma$ is hyperbolic if and only if there exist $x \in X$ and $\lambda, c \in \Lambda$ such that $h t(\lambda)=h t(d(x, \gamma x))$ and $d\left(x, \gamma^{k} x\right) \geqslant k \lambda+c$ for any $k \in \mathbb{N}$.

Proof. If $n=1$ then there exists $x \in X$ such that $k \rightarrow \gamma^{k} x$ is a quasi-isometry (see, for example, $\sharp$ ) of $\mathbb{Z}$ into $X$. Hence, there exist $\lambda, c \in \Lambda$ (which is either $\mathbb{R}$, or $\mathbb{Z}$ in this case) such that $d\left(x, \gamma^{k} x\right) \geqslant k \lambda+c$ for any $k \in \mathbb{N}$.

Suppose $n>1$. By Theorem 2, $\gamma$ is either hyperbolic, or elliptic, or parabolic or an inversion, so, consider all these possibilities. Recall that $\Lambda_{\delta}$ is the minimal convex subgroup of $\Lambda$ containing $\delta$ (in our case, $\Lambda_{\delta}=\mathbb{R}$, or $\Lambda_{\delta}=\mathbb{Z}$ ). Define $X_{1}=X / \sim$, where $x \sim y$ if $d(x, y) \in \Lambda_{\delta}$. Since $X$ is geodesic, $X_{1}$ is a $\Lambda_{1}$-tree
(here, either $\Lambda_{1}=\mathbb{R}^{n-1}$, or $\Lambda_{1}=\mathbb{Z}^{n-1}$ ). Observe that $\gamma$ induces an isometry $\gamma_{1}$ of $X_{1}$ which can be either elliptic, or hyperbolic, or an inversion if $\Lambda_{1}=\mathbb{Z}^{n-1}$.

If $\gamma$ is elliptic then it fixes a $\Lambda_{0}$-subspace - a contradiction with minimality.
If $\gamma$ is parabolic then $\gamma_{1}$ cannot be elliptic because then $\gamma$ is not minimal. $\gamma_{1}$ cannot be hyperbolic because in this case $\gamma_{1}$ fixes two distinct points on the boundary $\partial X_{1}$ (since $\gamma$ is minimal) which can be lifted to two distinct points on the boundary $\partial X$ fixed by $\gamma-$ a contradiction since we assume that $\gamma$ is parabolic. Eventually, if $\gamma_{1}$ is an inversion then $\gamma_{1}$ fixes a point in $X_{1}$, that is, $\gamma^{2}$ fixes a $\Lambda_{\delta}$-subspace of $X$. But since $\gamma$ is parabolic, $\gamma^{2}$ is also parabolic and the diameter of $\left\{\gamma^{2} x\right\}$ is unbounded by any $\alpha \in \Lambda$ for any $x \in X-\mathrm{a}$ contradiction.

Hence, we can conclude that $\gamma$ can be neither elliptic, nor parabolic if $n>1$. That is, it can be only either hyperbolic, or an inversion.

Finally, in the case when $n>1$, the isometry $\gamma_{1}$ is either hyperbolic, or an inversion. Moreover (see [3]), $\gamma_{1}$ is hyperbolic if and only if there exists $x_{1} \in X_{1}$, which belongs to the axis of $\gamma_{1}$, and $\lambda_{1}, c_{1} \in \Lambda_{1}$ such that $d\left(x_{1}, \gamma_{1}^{k} x_{1}\right) \geqslant k \lambda_{1}+c_{1}$ for any $k \in \mathbb{N}$. So, $x_{1}, \lambda_{1}$, and $c_{1}$ can be lifted back to $X$ and $\Lambda$ respectively, and we get the required result for $\gamma$.

Finally, we conclude this section with an investigation of the behavior of non-minimal isometries in the case when $\Lambda$ is either $\mathbb{R}^{n}$, or $\mathbb{Z}^{n}$.

Proposition 2. Let $(X, d)$ be a geodesic $\delta$-hyperbolic $\mathbb{R}^{n}$-metric space, where $\mathbb{R}^{n}$ is taken with the right lexicographic order and $\delta=\left(\delta_{0}, 0, \ldots, 0\right)$. Let $\gamma$ be a non-minimal isometry of $X$ fixing two distinct $\mathbb{R}^{n-i}$-subspaces $X_{0}$ and $X_{1}$, where $i \in[1, n-1]$. Then the action of $\gamma$ on $X_{0}$ and $X_{1}$ is of the same type.

Proof. Since $X$ is geodesic, there exist unique $\alpha_{0} \in \partial X_{0}$ and $\alpha_{1} \in \partial X_{1}$ such that if $\left\{x_{k}\right\} \rightarrow \alpha_{0},\left\{y_{k}\right\} \rightarrow \alpha_{1}$ and $x \in X_{0}, y \in X_{1}$ then

$$
\left(x_{k} \cdot y\right)_{x} \rightarrow \infty, \quad\left(y_{k} \cdot x\right)_{y} \rightarrow \infty \quad \text { with respect to } \mathbb{R}^{n-i}
$$

It follows that $\gamma \alpha_{0}=\alpha_{0}$ and $\gamma \alpha_{1}=\alpha_{1}$.
Suppose $\left.\gamma\right|_{X_{0}}$ is elliptic and $x$ is such that for any $k \in \mathbb{Z}$ we have $d\left(x, \gamma^{k} x\right) \leqslant$ $M \delta$ for some $M \in \mathbb{N}$. If $\left.\gamma\right|_{X_{1}}$ is hyperbolic, it follows that either $\left\{\gamma^{k} y\right\} \rightarrow \alpha_{1}$, or $\left\{\gamma^{-k} y\right\} \rightarrow \alpha_{1}$. Without loss of generality assume that $\left\{\gamma^{k} y\right\} \rightarrow \alpha_{1}$. We have

$$
\begin{gathered}
d(x, y)=d\left(\gamma^{k} y, \gamma^{k} x\right)=d\left(\gamma^{k} y, y\right)+d\left(\gamma^{k} x, y\right)-2\left(\gamma^{k} y \cdot \gamma^{k} x\right)_{y} \\
\leqslant d\left(\gamma^{k} y, y\right)+d(x, y)+d\left(\gamma^{k} x, x\right)-2\left(\gamma^{k} y \cdot \gamma^{k} x\right)_{y} \\
\leqslant d\left(\gamma^{k} y, y\right)+d(x, y)+M \delta-2\left(\gamma^{k} y \cdot \gamma^{k} x\right)_{y}
\end{gathered}
$$

Next, since $X_{0} \neq X_{1}$, we have $\left(\gamma^{k} y \cdot x\right)_{y} \in \mathbb{R}^{n-i}$ and $\left(\gamma^{k} x \cdot x\right)_{y} \in \mathbb{R}^{n-j}$ for some $j<i$. It follows that $\left(\gamma^{k} y \cdot x\right)_{y}<\left(\gamma^{k} x \cdot x\right)_{y}$ and from

$$
\left(\gamma^{k} y \cdot \gamma^{k} x\right)_{y} \geqslant \min \left\{\left(\gamma^{k} y \cdot x\right)_{y},\left(\gamma^{k} x \cdot x\right)_{y}\right\}-\delta
$$

we get

$$
\left(\gamma^{k} y \cdot \gamma^{k} x\right)_{y} \geqslant\left(\gamma^{k} y \cdot x\right)_{y}-\delta
$$

or

$$
-2\left(\gamma^{k} y \cdot \gamma^{k} x\right)_{y} \leqslant-2\left(\gamma^{k} y \cdot x\right)_{y}+2 \delta
$$

Thus,

$$
\begin{aligned}
& d(x, y) \leqslant d\left(\gamma^{k} y, y\right)+d(x, y)+M \delta-2\left(\gamma^{k} y \cdot \gamma^{k} x\right)_{y} \\
& \quad \leqslant d\left(\gamma^{k} y, y\right)+d(x, y)+(M+2) \delta-2\left(\gamma^{k} y \cdot x\right)_{y}
\end{aligned}
$$

and eventually we obtain

$$
\begin{equation*}
0 \leqslant d\left(\gamma^{k} y, y\right)+(M+2) \delta-2\left(\gamma^{k} y \cdot x\right)_{y} \tag{2}
\end{equation*}
$$

for any $k$.
Using a similar argument but starting with $d\left(\gamma^{k} x, y\right) \geqslant d(x, y)-d\left(x, \gamma^{k} x\right)$, we can eventually obtain that

$$
\begin{equation*}
0 \geqslant d\left(\gamma^{k} y, y\right)-(M+2) \delta-2\left(\gamma^{k} y \cdot x\right)_{y} \tag{3}
\end{equation*}
$$

for any $k$. Indeed we have

$$
\begin{gathered}
d(x, y)=d\left(\gamma^{k} y, \gamma^{k} x\right)=d\left(\gamma^{k} y, y\right)+d\left(\gamma^{k} x, y\right)-2\left(\gamma^{k} y \cdot \gamma^{k} x\right)_{y} \\
\geqslant d\left(\gamma^{k} y, y\right)+d(x, y)-d\left(\gamma^{k} x, x\right)-2\left(\gamma^{k} y \cdot \gamma^{k} x\right)_{y} \\
\geqslant d\left(\gamma^{k} y, y\right)+d(x, y)-M \delta-2\left(\gamma^{k} y \cdot \gamma^{k} x\right)_{y}
\end{gathered}
$$

Next, since $X_{0} \neq X_{1}$, we have $\left(\gamma^{k} y \cdot \gamma^{k} x\right)_{y} \in \mathbb{R}^{n-i}$ and $\left(\gamma^{k} x \cdot x\right)_{y} \in \mathbb{R}^{n-j}$ for some $j<i$. It follows that $\left(\gamma^{k} y \cdot \gamma^{k} x\right)_{y}<\left(\gamma^{k} x \cdot x\right)_{y}$ and from

$$
\left(\gamma^{k} y \cdot x\right)_{y} \geqslant \min \left\{\left(\gamma^{k} y \cdot \gamma^{k} x\right)_{y},\left(\gamma^{k} x \cdot x\right)_{y}\right\}-\delta
$$

we get

$$
\left(\gamma^{k} y \cdot x\right)_{y} \geqslant\left(\gamma^{k} y \cdot \gamma^{k} x\right)_{y}-\delta
$$

or

$$
-2\left(\gamma^{k} y \cdot \gamma^{k} x\right)_{y} \geqslant-2\left(\gamma^{k} y \cdot x\right)_{y}-2 \delta
$$

Thus,

$$
\begin{aligned}
& d(x, y) \geqslant d\left(\gamma^{k} y, y\right)+d(x, y)-M \delta-2\left(\gamma^{k} y \cdot \gamma^{k} x\right)_{y} \\
& \quad \geqslant d\left(\gamma^{k} y, y\right)+d(x, y)-(M+2) \delta-2\left(\gamma^{k} y \cdot x\right)_{y}
\end{aligned}
$$

from which we obtain (3).
Now, we assume $\left(\gamma^{k} y \cdot \gamma^{k+1} y\right)_{y} \leqslant\left(\gamma^{k+1} y \cdot x\right)_{y}$ and deduce

$$
d\left(y, \gamma^{k+1} y\right) \leqslant 2 d(y, \gamma y)+(M+2) \delta
$$

From our assumption we get

$$
-2\left(\gamma^{k} y \cdot \gamma^{k+1} y\right)_{y} \geqslant-2\left(\gamma^{k+1} y \cdot x\right)_{y}
$$

Next, we have

$$
d(y, \gamma y)=d\left(\gamma^{k} y, \gamma^{k+1} y\right)=d\left(y, \gamma^{k} y\right)+d\left(y, \gamma^{k+1} y\right)-2\left(\gamma^{k} y \cdot \gamma^{k+1} y\right)_{y}
$$

$$
\geqslant 2 d\left(y, \gamma^{k+1} y\right)-d(y, \gamma y)-2\left(\gamma^{k} y \cdot \gamma^{k+1} y\right)_{y}
$$

where the latter inequality follows from the triangle inequality. We can rewrite the latter inequality in the form

$$
2 d(y, \gamma y) \geqslant 2 d\left(y, \gamma^{k+1} y\right)-2\left(\gamma^{k} y \cdot \gamma^{k+1} y\right)_{y}
$$

so, combining it with $-2\left(\gamma^{k} y \cdot \gamma^{k+1} y\right)_{y} \geqslant-2\left(\gamma^{k+1} y \cdot x\right)_{y}$ we obtain

$$
2 d(y, \gamma y) \geqslant 2 d\left(y, \gamma^{k+1} y\right)-2\left(\gamma^{k+1} y \cdot x\right)_{y}
$$

Eventually, since

$$
d\left(\gamma^{k+1} y, y\right)-2\left(\gamma^{k+1} y \cdot x\right)_{y} \geqslant-(M+2) \delta
$$

(we replaced $k$ by $k+1$ in (2)), it follows that

$$
2 d(y, \gamma y) \geqslant d\left(y, \gamma^{k+1} y\right)-(M+2) \delta
$$

or

$$
d\left(y, \gamma^{k+1} y\right) \leqslant 2 d(y, \gamma y)+(M+2) \delta
$$

Observe that the latter inequality gives a contradiction since we assume that $\gamma$ acts as a hyperbolic isometry on $X_{1}$. It follows that the inequality

$$
\left(\gamma y \cdot \gamma^{k+1} y\right)_{y} \leqslant\left(\gamma^{k+1} y \cdot x\right)_{y}
$$

cannot hold for arbitrarily large $k$ and there exists $N \in \mathbb{N}$ such that $\left(\gamma^{k} y\right.$. $\left.\gamma^{k+1} y\right)_{y}>\left(\gamma^{k+1} y \cdot x\right)_{y}$ for any $k>N$.

It implies that

$$
\left(\gamma^{k} y \cdot x\right)_{y} \geqslant \min \left\{\left(\gamma^{k} y \cdot \gamma^{k+1} y\right)_{y},\left(\gamma^{k+1} y \cdot x\right)_{y}\right\}-\delta=\left(\gamma^{k+1} y \cdot x\right)_{y}-\delta
$$

Therefore, there exists $L \in \mathbb{R}^{n-i}$ such that $\left(\gamma^{k} y \cdot x\right)_{y} \leqslant L+k \delta$ for any $k>N$.
However, since $\gamma$ acts hyperbolically on $X_{1}$, by Corollary 11, there exist $\lambda, c \in$ $\mathbb{R}^{n-i}$ such that $d\left(y, \gamma^{k} y\right)>k \lambda+c$. We can assume that $\lambda>5 \delta$ since we can replace $\gamma$ with $\gamma^{\prime}=\gamma^{i}$ for $i$ large enough so that $\lambda^{\prime}>5 \delta$. However, according to (3) we have

$$
0 \geqslant d\left(\gamma^{k} y, y\right)-(M+2) \delta-2\left(\gamma^{k} y \cdot x\right)_{y} \geqslant d\left(\gamma^{k} y, y\right)-(M+2) \delta-2 L-2 k \delta
$$

which implies that

$$
\begin{gathered}
0>k \lambda+c-(M+2) \delta-2 L-2 k \delta>5 k \delta+c-(M+2) \delta-2 L-2 k \delta \\
=3 k \delta-(2 L+(M+2) \delta-c)
\end{gathered}
$$

Since $2 L+(M+2) \delta-c$ is a constant, we have a contradiction.
The same argument can be used to prove that if $\left.\gamma\right|_{X_{0}}$ is elliptic, then $\left.\gamma\right|_{X_{1}}$ cannot be parabolic since then both $\left\{\gamma^{k} y\right\}$ and $\left\{\gamma^{-k} y\right\}$ converge to the same
point on the boundary while $\left\{\gamma^{k} x \mid k \in \mathbb{Z}\right\}$ stays within a fixed distance from $x$.

We can use a similar argument to show that, if $\left.\gamma\right|_{X_{0}}$ is hyperbolic, then $\left.\gamma\right|_{X_{1}}$ cannot be parabolic. Indeed, if $\left.\gamma\right|_{x_{1}}$ is parabolic then it has a unique fixed point in $\partial X_{1}$ which both $\left\{\gamma^{k} y\right\}$ and $\left\{\gamma^{-k} y\right\}$ converge to. At the same time, $\left.\gamma\right|_{X_{0}}$ has two fixed points in $\partial X_{0}:\left\{\gamma^{k} x\right\}$ converges to one of them and $\left\{\gamma^{-k} x\right\}$ to the other one. Suppose, without loss of generality, that $\left\{\gamma^{k} x\right\}$ converges to $\alpha_{0}$ and $\left\{\gamma^{k} y\right\}$ converges to $\alpha_{1}$. The we have
$d(x, y)=d\left(\gamma^{k} x, \gamma^{k} y\right)=d(x, y)+d\left(x, \gamma^{k} x\right)+d\left(y, \gamma^{k} y\right)-2\left(\gamma^{k} x \cdot y\right)_{x}-2\left(\gamma^{k} y \cdot x\right)_{y}$
which implies that

$$
0=d\left(x, \gamma^{k} x\right)+d\left(y, \gamma^{k} y\right)-2\left(\gamma^{k} x \cdot y\right)_{x}-2\left(\gamma^{k} y \cdot x\right)_{y}
$$

Finally, we use the obtained equality as a analog of (2) and repeat the argument given above for both $\left(\gamma^{k} y \cdot \gamma^{k+1} y\right)_{y}$ and $\left(\gamma^{k} x \cdot \gamma^{k+1} x\right)_{x}$.

Corollary 2. Let $(X, d)$ be a geodesic $\delta$-hyperbolic $\mathbb{Z}^{n}$-metric space, where $\mathbb{Z}^{n}$ is taken with the right lexicographic order and $\delta=\left(\delta_{0}, 0, \ldots, 0\right)$. Let $\gamma$ be a non-minimal isometry of $X$ fixing $\mathbb{Z}^{n-i}$-subspaces $X_{0}$ and $X_{1}$. Then the action of $\gamma$ on $X_{0}$ and $X_{1}$ is of the same type.

Proof. If the actions of $\gamma$ on $X_{0}$ and $X_{1}$ are either elliptic, or hyperbolic, or parabolic then the proof is a straightforward adaptation of that of Proposition 2.

Suppose that $\gamma$ is an inversion on $X_{0}$. Observe that by definition $\gamma$ does not fix any $\mathbb{Z}$-subspace of $X$.

Let $Y$ be the $\mathbb{Z}^{n-1}$-tree obtained by contracting all $\mathbb{Z}$-subspaces of $X$ to points (more precisely, $Y=X / \sim$, where $x \sim y$ if and only if $d(x, y) \in \mathbb{Z}$, and since $X$ is geodesic, $Y$ is a geodesic 0 -hyperbolic $\mathbb{Z}^{n-1}$-metric space). Let $Y_{0}$ and $Y_{1}$ be the subtrees of $Y$ corresponding to $X_{0}$ and $X_{1}$. Observe that $\gamma$ induces an isometry $\gamma_{1}$ of $Y$ such that $\gamma_{1}$ fixes both $Y_{0}$ and $Y_{1}$ and $\gamma_{1}$ acts on $Y_{0}$ as an inversion. Hence, let $a, b \in Y_{0}$ such that $\gamma_{1} a=b, \gamma_{1} b=a$ and take an arbitrary $z \in Y_{1}$. If $[a, b] \cup[b, z]$ is a geodesic in $Y$ then so is $\gamma_{1}[b, z]=\left[a, \gamma_{1} z\right]$ and $b \notin\left[a, \gamma_{1} z\right]$. But the unique geodesic from $a$ to any element of $Y_{1}$ must contain $b$, hence, a contradiction. If $[b, a] \cup[a, z]$ is a geodesic then we get a contradiction in a similar way. Finally, if there is some $c \in[a, b]$ such that $[a, c] \cup[c, z]$ and $[b, c] \cup[c, z]$ are both geodesics then we get a contradiction by using similar considerations with $c$ and $\gamma_{1} c$.

It follows that $\gamma_{1}$ cannot fix distinct $Y_{0}$ and $Y_{1}$ and the same applies to $\gamma$ in $X$.

### 2.7 Examples

Example 2. Let $X$ be a proper, geodesic $\delta$-hyperbolic $\mathbb{R}$-metric space, $* \in X$ and $a, b \in \partial X$ such that $(a \cdot b)_{*}=0$. Let $\left\{X_{i} \mid i \in \mathbb{Z}\right\}$ be a set of copies of $X$ with the copy of $x \in X$ in $X_{i}$ denoted $x_{i}$, and define $Y=\bigcup X_{i}$

Let us define a metric don $Y$. By abuse of notation, we are also going to use $d$ for the metric in $X$. For any $i \in \mathbb{Z}$, define $d\left(x_{i}, y_{i}\right)=(d(x, y), 0)$. If $i<j$ then define

$$
d\left(x_{i}, y_{j}\right)=\left(d(x, *)+d(y, *)-2(x \cdot a)_{*}-2(y \cdot b)_{*},|i-j|\right)
$$

First of all, $(Y, d)$ is a (8, 0$)$-hyperbolic metric space. To see that, take $\rho$ to be a geodesic line joining $a$ and $b$ such that $* \in \rho$ (such a geodesic line exists since $X$ is geodesic). Define $\rho_{i}$ to be the image of $\rho$ in $X_{i}, a_{i}$ and $b_{i}$ to be the images of $a$ and $b$ in $\partial X_{i}$, and $[x, \omega)$ to be a geodesic ray between some $x \in X$ and $\omega \in \partial X$.

For any $x_{i}, y_{j} \in Y$ with $i \neq j$, let

$$
\left[x_{i}, a_{i}\right) \cup \rho_{i+1} \cup \cdots \cup \rho_{j-1} \cup\left(b_{j}, y_{j}\right]
$$

will be a geodesic embedding of $\left[0, d\left(x_{i}, y_{j}\right)\right]$ into $Y$. It is then easy to prove that $(Y, d)$ is $(8 \delta, 0)$-hyperbolic by using a geometric argument.

Let now $\gamma$ be an isometry of $X$ which preserves $a$ and $b$. We would like to extend it to a mapping $\bar{\gamma}: Y \rightarrow Y$ by $\bar{\gamma}\left(x_{i}\right)=(\gamma x)_{i+1}$. For every $x_{i}, y_{j} \in Y$, consider the first component of $d\left(x_{i}, y_{j}\right)$ which we denote by $D(x, y)$. Explicitly,

$$
D(x, y)=d(x, *)+d(y, *)-2(x \cdot a)_{*}-2(y \cdot b)_{*}
$$

It is easy to see that $\bar{\gamma}$ is an isometry of $Y$ if and only if $D(x, y)=D(\gamma x, \gamma y)$ for any $x, y \in X$. We have

$$
\begin{gathered}
D(x, y)=d(x, *)+d(y, *)-2(x \cdot a)_{*}-2(y \cdot b)_{*} \\
=d(x, *)+d(y, *)-\sup _{\substack{a_{i} \rightarrow a \\
b_{i} \rightarrow b}} \lim _{i \rightarrow \infty} d(x, *)+d\left(a_{i}, *\right)-d\left(x, a_{i}\right)+d(y, *)+d\left(b_{i}, *\right)-d\left(y, b_{i}\right) \\
=-\sup _{\substack{a_{i} \rightarrow a \\
b_{i} \rightarrow b}} \lim _{i \rightarrow \infty} d\left(a_{i}, *\right)-d\left(x, a_{i}\right)+d\left(b_{i}, *\right)-d\left(y, b_{i}\right)
\end{gathered}
$$

Recall now that $\gamma$ fixes $a$ and $b$, so we have that $\left\{a_{i} \mid a_{i} \rightarrow a\right\}=\left\{\gamma a_{i} \mid a_{i} \rightarrow a\right\}$ and $\left\{b_{i} \mid b_{i} \rightarrow b\right\}=\left\{\gamma b_{i} \mid b_{i} \rightarrow b\right\}$. Hence,

$$
\begin{aligned}
& D(x, y)- D(\gamma x, \gamma y)=\sup _{\substack{a_{i} \rightarrow a \\
b_{i} \rightarrow b}} \lim _{i \rightarrow \infty}\left(d\left(a_{i}, *\right)-d\left(\gamma x, a_{i}\right)+d\left(b_{i}, *\right)-d\left(\gamma y, b_{i}\right)\right) \\
&-\sup _{\substack{a_{i} \rightarrow a \\
b_{i} \rightarrow b}} \lim _{i \rightarrow \infty}\left(d\left(a_{i}, *\right)-d\left(x, a_{i}\right)+d\left(b_{i}, *\right)-d\left(y, b_{i}\right)\right) \\
&= \sup _{\substack{a_{i} \rightarrow a \\
b_{i} \rightarrow b}} \lim _{i \rightarrow \infty}\left(d\left(\gamma a_{i}, *\right)-d\left(\gamma x, \gamma a_{i}\right)+d\left(\gamma b_{i}, *\right)-d\left(\gamma y, \gamma b_{i}\right)\right) \\
&-\sup _{a_{i} \rightarrow a} \lim _{i \rightarrow \infty}\left(d\left(a_{i}, *\right)-d\left(x, a_{i}\right)+d\left(b_{i}, *\right)-d\left(y, b_{i}\right)\right) \\
& \leqslant \sup _{i} \rightarrow b \\
& \leqslant \lim _{\substack{a_{i} \rightarrow a \rightarrow \infty \\
b_{i} \rightarrow b}} d\left(\gamma a_{i}, *\right)+d\left(\gamma b_{i}, *\right)-d\left(a_{i}, *\right)-d\left(b_{i}, *\right)
\end{aligned}
$$

$$
\begin{gathered}
=\sup _{\substack{a_{i} \rightarrow a \\
b_{i} \rightarrow b}} \lim _{i \rightarrow \infty} d\left(\gamma a_{i}, *\right)+d\left(\gamma b_{i}, *\right)-d\left(\gamma a_{i}, \gamma *\right)-d\left(\gamma b_{i}, \gamma *\right) \\
\leqslant \sup _{\substack{a_{i} \rightarrow a \\
b_{i} \rightarrow b}} \lim _{i \rightarrow \infty} d\left(\gamma a_{i}, *\right)+d\left(\gamma b_{i}, *\right)-d\left(\gamma a_{i}, \gamma b_{i}\right) \\
=2(\gamma a \cdot \gamma b)_{*}=0
\end{gathered}
$$

We use a similar reasoning to prove that $D(\gamma x, \gamma y)-D(x, y) \leqslant 0$, which implies that $D(\gamma x, \gamma y)=D(x, y)$.

Let us reuse the same spaces $X$ and $Y$ but assume $\gamma b=a$ and $\gamma a=b$. Since $\gamma^{2}$ has more fixed points on the boundary than $\gamma$, it is easy to see that $\gamma$ is an elliptic isometry of $X$.

This time, we extend $\gamma$ to $Y$ by using $\bar{\gamma}\left(x_{i}\right)=(\gamma x)_{-i}$. We can use the same argument as above to prove that $\bar{\gamma}$ is an elliptic isometry of $Y$ by using the fact that $\left\{a_{i} \mid a_{i} \rightarrow a\right\}=\left\{\gamma b_{i} \mid b_{i} \rightarrow b\right\}$ and $\left\{b_{i} \mid b_{i} \rightarrow b\right\}=\left\{\gamma a_{i} \mid a_{i} \rightarrow a\right\}$, allowing an analog of the previous computations.

Example 3. Let $X$ be a $\delta$-hyperbolic metric space in $\Lambda_{1}, T$ a $\Lambda_{2}$-tree, $d_{X}$ and $d_{T}$ the associated metrics and $Y \subseteq X$ bounded and $\gamma$ its diameter. Define $d_{Y}: X \times T \rightarrow \Lambda_{1} \oplus \Lambda_{2}$ by

$$
d_{Y}\left(\left(x_{1}, t\right),\left(x_{2}, t\right)\right)=\left(d_{X}\left(x_{1}, x_{2}\right), 0\right)
$$

and

$$
d_{Y}\left(\left(x_{1}, t_{1}\right),\left(x_{2}, t_{2}\right)\right)=\left(d_{X}\left(x_{1}, Y\right)+d_{X}\left(x_{2}, Y\right), d_{T}\left(t_{1}, t_{2}\right)\right)
$$

To see that $d_{Y}$ is a $(\delta+\gamma, 0)$-hyperbolic metric, notice that $\left(X \times T, d_{Y}\right)$ can be embedded into a space where we connect each pair $X \times t_{1}, X \times t_{2}$, where $d_{T}\left(t_{1}, t_{2}\right)=1$ by attaching a copy of $Y \times \Lambda_{2}$ to $Y \times t_{1}$ and $Y \times t_{2}$ and imagining they meet at the end.

In particular, suppose $G$ acts on $X$ and $T$ and there exists an $x \in X$ such that $G x$ is bounded. We can then consolidate both actions into an action on $\left(X \times T, d_{G x}\right)$ which is a $(\delta+\gamma, 0)$-hyperbolic $\Lambda_{1} \oplus \Lambda_{2}$-metric space.

Note that one could use $T$ as a $\delta^{\prime}$-hyperbolic space resulting in $d_{Y}$ being $\left(\delta+\gamma, \delta^{\prime}\right)$-hyperbolic.

## 3 Group actions and hyperbolic length functions

In this section we introduce hyperbolic length functions on groups. This gives an equivalent approach to study group actions on hyperbolic $\Lambda$-metric spaces.

In 15 Lyndon introduced a notion of an abstract length function $l: G \rightarrow \Lambda$ on a group $G$ with values in $\Lambda$. This started the whole study of the group length functions and actions. Following Lyndon we call a function $l: G \rightarrow \Lambda$ a length function if it satisfies the following axioms
$(\Lambda 1) \forall g \in G: l(g) \geqslant 0$ and $l(1)=0$,
$(\Lambda 2) \forall g \in G: l(g)=l\left(g^{-1}\right)$,
$(\Lambda 3) \forall g, h \in G: l(g h) \leqslant l(g)+l(h)$.
Now we introduce the following crucial definition.
A length function $l: G \rightarrow \Lambda$ is called hyperbolic if there is $\delta \in \Lambda$ such that

$$
(\Lambda 4, \delta) \forall f, g, h \in G: c(f, g) \geqslant \min \{c(f, h), c(g, h)\}-\delta,
$$

where $c(g, h)=\frac{1}{2}\left(l_{v}(g)+l_{v}(h)-l_{v}\left(g^{-1} h\right)\right)$.
Usually, a length function satisfying $(\Lambda 4, \delta)$, is called $\delta$-hyperbolic.
Lyndon himself considered a much stronger form of the axiom $(\Lambda 4, \delta)$, the one with $\delta=0$. After him length functions $l: G \rightarrow \Lambda$ are called Lyndon length functions. In our terminology these are 0-hyperbolic length functions. Chiswell in [3] showed that groups with Lyndon length functions $L: G \rightarrow \mathbb{R}$ (and an extra axiom) are precisely those that act freely on $\mathbb{R}$-trees, and later Morgan and Shalen generalized his construction to arbitrary $\Lambda$ 16]. For more details we refer to the book 3 .

In Section 3.1 we show how an action of a group $G$ by isometries on a (hyperbolic) $\Lambda$-metric space induces naturally a (hyperbolic) length function on $G$ with values in $\Lambda$. And in Section 3.2 we prove the converse, thus establishing equivalence of these two approaches.

### 3.1 From actions - to length functions

Let $X=(X, d)$ be a $\Lambda$-metric space. By $\operatorname{Isom}(X)$ we denote the group of bijective isometries of $X$. We say that a group $G$ acts on a $X$ if for any $g \in G$ there is an isometry $\phi_{g} \in \operatorname{Isom}(X)$ such that for any $x \in X$ and any $g, h \in G$ one has $\phi_{g h}(x)=\phi_{g}\left(\phi_{h}(x)\right)$, that is, the map $g \rightarrow \phi_{g}$ is a group homomorphism $G \rightarrow \operatorname{Isom}(X)$. In this case, for every $x \in X$ and $g \in G$ we denote $\phi_{g}(x)$ by $g x$.

If $G$ acts on $(X, d)$ then one can fix a point $v \in X$ and consider a function $l_{v}: G \rightarrow \Lambda$ defined as $l_{v}(g)=d(v, g v)$, called a length function based at $v$. The basic properties of based length functions come from the metric properties of $(X, d)$.

Theorem 5. If a group $G$ acts on a $\Lambda$-metric space $(X, d)$ and $v \in X$ then the length function $l_{v}$ based at $v$ is a length function on $G$ with values in $\Lambda$. Moreover, if $(X, d)$ is $\delta$-hyperbolic with respect to $v$ for some $\delta \in \Lambda$ then $l_{v}$ is $\delta$-hyperbolic.

Proof. ( $\Lambda 1$ ) is obvious since $l_{v}(1)=d(v, v)=0$. Also, ( $\Lambda 2$ ) follows since $d(v, g v)=d\left(g^{-1} v, v\right)$.

Next, since $d(v,(g h) v)=d\left(g^{-1} v, h v\right)$ then by definition of the metric we have

$$
d\left(g^{-1} v, h v\right) \leqslant d\left(g^{-1} v, v\right)+d(v, h v)
$$

and $(\Lambda 3)$ follows from the equality $d\left(g^{-1} v, v\right)=d(v, g v)$.

Finally, assume that $(X, d)$ is $\delta$-hyperbolic. Observe that

$$
\begin{gathered}
c(f, g)=\frac{1}{2}\left(l_{v}(f)+l_{v}(g)-l_{v}\left(f^{-1} g\right)\right)=\frac{1}{2}\left(d(v, f v)+d(v, g v)-d\left(v,\left(f^{-1} g\right) v\right)\right) \\
=\frac{1}{2}(d(v, f v)+d(v, g v)-d(f v, g v))=(f v \cdot g v)_{v}
\end{gathered}
$$

In the same way we have $c(f, h)=(f v \cdot h v)_{v}, c(g, h)=(g v \cdot h v)_{v}$ and since $(X, d)$ is $\delta$-hyperbolic then we have

$$
(f v \cdot g v)_{v} \geqslant \min \left\{(f v \cdot h v)_{v},(g v \cdot h v)_{v}\right\}-\delta,
$$

which proves $(\Lambda 4)$ for $l_{v}$.

### 3.2 From length functions - to actions

Let $l: G \rightarrow \Lambda$ be a length function. The set

$$
\operatorname{ker}(l)=\{g \in G \mid l(g)=0\}
$$

is called the kernel of $l$. It is easy to see that $\operatorname{ker}(l)$ is a subgroup of $G$ (this follows from ( $\Lambda 3$ )).
Lemma 13. Let $l: G \rightarrow \Lambda$ be a length function. Then for any $a \in \operatorname{ker}(l), g \in G$

$$
l(a g)=l(g a)=l(g)
$$

that is, $l$ is a constant function on each coset of $\operatorname{ker}(l)$.
Proof. Let $a \in \operatorname{ker}(l), g \in G$. Then $l(a g) \leqslant l(a)+l(g)=l(g)$ and $l(g)=$ $l\left(a^{-1} a g\right) \leqslant l\left(a^{-1}\right)+l(a g)=l(a g)$, so $l(a g)=l(g)$. A similar argument works for $l(g a)=l(g)$.

Theorem 6. If $l: G \rightarrow \Lambda$ is a length function, then there are a $\Lambda$-metric space $(X, d)$, an action of $G$ on $X$, and a point $v \in X$ such that $l=l_{v}$. Moreover, if $l: G \rightarrow \Lambda$ is $\delta$-hyperbolic then the space $(X, d)$ is also $\delta$-hyperbolic.

Proof. Denote $A=\operatorname{ker}(l)$ and consider the set $X=G / A$ of all left cosets of $A$ in $G$. Define a function $d_{A}: G / A \times G / A \rightarrow \Lambda$ so that $d_{A}(g A, h A)=l\left(g^{-1} h\right)$. Observe that by Lemma 13, $d_{A}$ is well-defined. Indeed, if $g^{\prime}, h^{\prime} \in G$ such that $g^{\prime} A=g A, h^{\prime} A=h A$ then $g^{\prime}=g a_{1}$ and $h^{\prime}=h a_{2}$ and $d_{A}\left(g^{\prime} A, h^{\prime} A\right)=$ $l\left(a_{1}^{-1} g^{-1} h a_{2}\right)=l\left(g^{-1} h\right)=d_{A}(g A, h A)$.

We claim that $\left(X, d_{A}\right)$ is a $\Lambda$-metric space. Indeed, axioms (LM1) and (LM3) of $\Lambda$-metric space are evident. Next, $d_{A}(g A, h A)=l\left(g^{-1} h\right)=0$ if and only if $g^{-1} h \in A$ if and only if $g A=h A$, so (LM2) follows. Finally, the triangle inequality (LM4) follows from the corresponding property of the length function $l$.

Now we show that $\left(X, d_{A}\right)$ is $\delta$-hyperbolic with respect to the point $A$, provided $l$ is $\delta$-hyperbolic. Notice that for $f A, g A \in G / A$ we have

$$
(f A \cdot g A)_{A}=\frac{1}{2}\left(d_{A}(f A, A)+d_{A}(g A, A)-d_{A}(f A, g A)\right)
$$

$$
=\frac{1}{2}\left(l(f)+l(g)-l\left(f^{-1} g\right)\right)=c(f, g)
$$

Hyperbolicity is then a consequence of $G$ having a $\delta$-hyperbolic length function.
Now, $G$ acts on $G / A$ in a natural way, that is, if $g \in G, h A \in G / A$ then $g \cdot(h A)=(g h) A$. This action is isometric since $d_{A}(f A, h A)=l\left(f^{-1} h\right)=$ $l\left(f^{-1} g^{-1} g h\right)=d_{A}(g \cdot(f A), g \cdot(h A))$. Finally, $l(g)=d_{A}(A, g A)=l_{A}(g)$.

Given a group $G$ and a $\delta$-hyperbolic length function $l: G \rightarrow \Lambda$, denote by $\left(X_{l}, d_{l}\right)$ the $\delta$-hyperbolic $\Lambda$-metric space constructed in Theorem 6. Note that the stabilizer of the point $v$ is exactly the kernel of $l=l_{v}$.

In general, given an action of $G$ on an arbitrary $\delta$-hyperbolic $\Lambda$-metric space $(X, d)$ and a point $x \in X$, the stabilizer $G_{x}$ of $x$ is exactly the kernel of the $\delta$-hyperbolic length function $l_{x}$ based at $x$.

We are going to use the notion of the kernel later in Section 6 .

### 3.3 Examples of group actions on hyperbolic $\Lambda$-metric spaces

Here are some examples of groups acting on hyperbolic $\Lambda$-metric spaces for various $\Lambda$.

Example 4. Given a torsion-free word-hyperbolic group $G$ and its generating set $S$, the Cayley graph $X=C a y(G, S)$ with the word metric (with respect to $S$ ) is a $\delta$-hyperbolic $\mathbb{Z}$-metric space for some $\delta \in \mathbb{Z}$. $G$ acts on $X$ by isometries, in particular, no element of $G$ fixed a point in $X$.

Example 5. Since any $\Lambda$-tree is a $\delta$-hyperbolic $\Lambda$-metric space with $\delta=0$, the class of groups acting on hyperbolic $\Lambda$-metric spaces contains all groups acting on $\Lambda$-trees (in particular, all $\Lambda$-free groups).

Example 6. Any subgroup of a group acting on a $\Lambda$-metric space $(X, d)$ also acts on this space. So, the class of groups acting on hyperbolic $\Lambda$-metric spaces is closed under taking subgroups.

Example 7. Given two groups $G_{1}$ and $G_{2}$ acting respectively on $\Lambda$-metric spaces $\left(X_{1}, d_{1}\right)$ and $\left(X_{2}, d_{2}\right)$, there exists a $\Lambda \oplus \mathbb{Z}$-metric space $(X, d)$, where $\Lambda \oplus \mathbb{Z}$ is taken with the right lexicographic order, such that $G_{1} * G_{2}$ acts on $X$. To see this, take some $x_{1} \in X_{1}$ and $x_{2} \in X_{2}$ and define $l(g)=d_{n}\left(x_{n}, g x_{n}\right)$ for $g \in G_{n}$.

Any element $g$ of $G_{1} * G_{2}$ has a unique normal form $g=g_{1} g_{2} \cdots g_{n}$ with $g_{i} \in G_{1} \cup G_{2}$ for any $i$ and if $g_{i} \in G_{1}$, then $g_{i+1} \in G_{2}$ and vice versa. Define then $l(g)=\left(l\left(g_{1}\right)+\cdots+l\left(g_{n}\right), 1\right)$ and $X=\left(G_{1} * G_{2}, d_{l}\right)$. It is easy to show that this is a metric space.

Finally, it is not hard to see that, if both $X_{1}$ and $X_{2}$ are hyperbolic, then so is $X$.

Example 8. Let $H$ be a torsion-free hyperbolic group, $u$ a cyclically reduced word in generators of $H$, and

$$
G=\left\langle H, t \mid t^{-1} u t=u\right\rangle
$$

We would like to construct a $\mathbb{Z}^{2}$-metric space $X$ on which $G$ acts as follows.
Define $T$ as the subset of the boundary of $G$ made up of limits of sequences of the form

$$
\left\{w, w u, w u^{2}, w u^{2}, \ldots\right\} \quad \text { and } \quad\left\{w, w u^{-1}, w u^{-2}, w u^{-3}, \ldots\right\}
$$

where $w$ is an arbitrary element of $G$. Take a rooted tree $Y$ whose vertices have valence $|T|$ and label them by elements of $G / H$ in such a way that the adjacent vertices are the pairs of the form $\left(g H,\left(h t^{ \pm 1} g\right) H\right)$, where $h \in H, g \in G$.

Denote the metric on $Y$ by $d_{Y}$, the word metric on $H$ (with respect to some fixed generating set) $b d_{H}$, and the hyperbolicity constant of $\left(H, d_{H}\right)$ by $\delta$. Note that any element of $G$ either is of the form $h$, if it is in $H$, or is a product of elements of the type $g t^{ \pm 1} h$ with $g \in G$ and $h \in H$ otherwise.

Now, let $X=Y \times H$ and for $\left(g H, h_{1}\right),\left(g H, h_{2}\right) \in Y \times H$ we define the distance $d\left(\left(g_{1} H, h_{1}\right),\left(g_{2} H, h_{2}\right)\right)$ as follows. If $g_{1} H=g_{2} H$ then we set

$$
d\left(\left(g_{1} H, h_{1}\right),\left(g_{2} H, h_{2}\right)\right)=\left(d_{H}\left(h_{1}, h_{2}\right), 0\right)
$$

If $g_{1} H \neq g_{2} H$ then let $e_{1} e_{2} \ldots e_{N}$, where $N=d_{Y}\left(g_{1} H, g_{2} H\right)$, be the path from $g_{1} H$ to $g_{2} H$ in $Y$. Observe that every edge $e_{i}$ is associated with a pair of ends $\left(\alpha\left(e_{i}\right), \omega\left(e_{i}\right)\right)$ in the corresponding copies of $\left(H, d_{H}\right)$. Hence, define

$$
\begin{gathered}
d\left(\left(g_{1} H, h_{1}\right),\left(g_{2} H, h_{2}\right)\right)=\left(d_{H}\left(1, g_{1}\right)-2\left(\left(g H, h_{1}\right) \cdot \alpha\left(e_{1}\right)\right)+d_{H}\left(1, g_{2}\right)\right. \\
\left.-2\left(\left(g H, h_{1}\right) \cdot \omega\left(e_{n}\right)\right)-2 \sum_{i=1}^{N}\left(\omega\left(e_{i}\right) \cdot \alpha\left(e_{i+1}\right)\right), d_{Y}\left(g_{1} H, g_{2} H\right)\right),
\end{gathered}
$$

where"." represents the Gromov product of two ends, or an end and a point in a hyperbolic space (computed in the appropriate copy of $\left(H, d_{H}\right)$ ).

In the light of the alternative given in the definition of a hyperbolic length function, it suffices to prove that $X$ is $(\delta, 0)$-hyperbolic and the action

$$
\left(g t^{ \pm 1} h\right) \cdot\left(g^{\prime} H, h^{\prime}\right)=\left(\left(g t^{ \pm 1} h g^{\prime}\right) H, h^{\prime}\right)
$$

if $g^{\prime} \notin H$, or

$$
\left(g t^{ \pm 1} h\right) \cdot\left(g^{\prime} H, h^{\prime}\right)=\left(\left(g t^{ \pm 1}\right) H, h h^{\prime}\right)
$$

if $g^{\prime} \in H$, is an action by isometries to have that

$$
l\left(g t^{ \pm 1} h\right)=d\left((H, 1),\left(\left(g t^{ \pm 1}\right) H, h\right)\right)
$$

is a $(\delta, 0)$-hyperbolic length function on $G$.

## 4 Kernels and hyperbolicity constants

Let $\Lambda$ be an ordered abelian group and $G$ a group with a length function $l: G \rightarrow$ $\Lambda$ (that is, $l$ satisfies the axioms $(\Lambda 1)-(\Lambda 3))$. Fix a convex subgroup $\Lambda_{0} \leqslant \Lambda$. The order on $\Lambda$ induces an order on the quotient abelian group $\bar{\Lambda}=\Lambda / \Lambda_{0}$,
so $\bar{\Lambda}$ becomes an ordered abelian group with an order preserving the quotient epimorphim $\eta: \Lambda \rightarrow \bar{\Lambda}$.

The $\Lambda_{0}$-kernel of $G$ (with respect to $l: G \rightarrow \Lambda$ ) is the set

$$
G_{\Lambda_{0}}=\left\{g \in G \mid l(g) \in \Lambda_{0}\right\}
$$

By the axiom ( $\Lambda 3$ ), the kernel $G_{\Lambda_{0}}$ is a subgroup of $G$ (though not necessary normal).

Lemma 14. Let $l: G \rightarrow \Lambda$ be a length function, $\Lambda_{0}$ be a convex subgroup of $\Lambda$, and $0<\delta \notin \Lambda_{0}$. Then the restriction $l_{0}$ of the length function $l$ to $G_{\Lambda_{0}}$ is a $\delta$-hyperbolic length function $l_{0}: G_{\Lambda_{0}} \rightarrow \Lambda$.

Proof. Since $\Lambda_{0}$ is convex, it follows that $\delta>\lambda$ for any $\lambda \in \Lambda_{0}$. To prove the result one needs only to show that for any $f, g, h \in G_{\Lambda_{0}}$, the following inequality holds

$$
c(f, g) \geqslant \min \{c(f, h), c(g, h)\}-\delta,
$$

which is, of course, equivalent to (avoiding working in $\mathbb{Q} \otimes \Lambda$ ):

$$
2 c(f, g) \geqslant \min \{2 c(f, h), 2 c(g, h)\}-2 \delta
$$

Since $\Lambda_{0}$ is convex, it follows that $2 c(f, h), 2 c(g, h) \in \Lambda_{0}$, so

$$
\min \{2 c(f, h), 2 c(g, h)\}-2 \delta<0
$$

But $2 c(f, g) \geqslant 0$, which finishes the proof.
The whole idea behind $\delta$-hyperbolic length functions is a generalization of hyperbolic properties from usual metric spaces to $\Lambda$-metric spaces for an arbitrary $\Lambda$. But when $\Lambda$ is not archimedean the choice of $\delta$ becomes very important as the following construction shows.

Let $G$ be a word-hyperbolic group, that is, the geodesic word length function $l=|\cdot|_{S}: G \rightarrow \mathbb{Z}$ with respect to some finite generating set $S$ is $\delta$-hyperbolic for some $\delta \in \mathbb{Z}$.

Consider the group $H=G \times G$ and a $\operatorname{map} l_{H}: H \rightarrow \mathbb{Z}^{2}$ defined by $l_{H}(f, g)=$ $(l(f), l(g))$, where $\mathbb{Z}^{2}$ is considered with the right lexicographic order.

Lemma 15. The function $l_{H}: H \rightarrow \mathbb{Z}^{2}$ is a $\delta_{H}$-hyperbolic length function on $H$, where $\delta_{H}=\left(\delta_{1}, \delta_{1}\right) \in \mathbb{Z}^{2}$ and $\delta_{1} \in \mathbb{Z}$ is such that $\delta_{1}>\delta$.

Proof. It is easy to see that the axioms ( $\Lambda 1$ ) and ( $\Lambda 2$ ) hold immediately.
The triangle inequality $(\Lambda 3)$ is also straightforward. Indeed, let $\mathbf{h}_{1}, \mathbf{h}_{2} \in H$ be such that $\mathbf{h}_{1}=\left(f_{1}, g_{1}\right), \mathbf{h}_{2}=\left(f_{2}, g_{2}\right)$. Then

$$
\begin{aligned}
l_{H}\left(\mathbf{h}_{1} \mathbf{h}_{2}\right)=l_{H}\left(f_{1} f_{2}, g_{1} g_{2}\right)= & \left(l\left(f_{1} f_{2}\right), l\left(g_{1} g_{2}\right)\right) \leqslant\left(l\left(f_{1}\right)+l\left(f_{2}\right), l\left(g_{1}\right)+l\left(g_{2}\right)\right) \\
& =l_{H}\left(\mathbf{h}_{1}\right)+l_{H}\left(\mathbf{h}_{2}\right) .
\end{aligned}
$$

To prove ( $\Lambda 4$ ) let $\mathbf{h}_{1}, \mathbf{h}_{2}, \mathbf{h}_{3} \in H$ be such that $\mathbf{h}_{1}=\left(f_{1}, g_{1}\right), \mathbf{h}_{2}=\left(f_{2}, g_{2}\right)$ and $\mathbf{h}_{3}=\left(f_{3}, g_{3}\right)$. We have to show that

$$
c\left(\mathbf{h}_{1}, \mathbf{h}_{2}\right) \geqslant \min \left\{c\left(\mathbf{h}_{1}, \mathbf{h}_{3}\right), c\left(\mathbf{h}_{2}, \mathbf{h}_{3}\right)\right\}-\delta_{H} .
$$

We have

$$
c\left(\mathbf{h}_{i}, \mathbf{h}_{j}\right)=\frac{1}{2}\left(l\left(\mathbf{h}_{i}\right)+l\left(\mathbf{h}_{j}\right)-l\left(\mathbf{h}_{i}^{-1} \mathbf{h}_{j}\right)\right)=\left(c\left(f_{i}, f_{j}\right), c\left(g_{i}, g_{j}\right)\right) .
$$

Hence, we have to show that

$$
\left(c\left(f_{1}, f_{2}\right), c\left(g_{1}, g_{2}\right)\right) \geqslant \min \left\{\left(c\left(f_{1}, f_{3}\right), c\left(g_{1}, g_{3}\right)\right),\left(c\left(f_{2}, f_{3}\right), c\left(g_{2}, g_{3}\right)\right)\right\}-\left(\delta_{1}, \delta_{1}\right),
$$

where $\delta_{1} \in \mathbb{Z}$ is such that $\delta_{1}>\delta$. Since $G$ is $\delta$-hyperbolic we have

$$
c\left(f_{1}, f_{2}\right) \geqslant \min \left\{c\left(f_{1}, f_{3}\right), c\left(f_{2}, f_{3}\right)\right\}-\delta>\min \left\{c\left(f_{1}, f_{3}\right), c\left(f_{2}, f_{3}\right)\right\}-\delta_{1} .
$$

Without loss of generality suppose that $c\left(g_{1}, g_{3}\right) \leqslant c\left(g_{2}, g_{3}\right)$, so that

$$
c\left(g_{1}, g_{2}\right)>c\left(g_{1}, g_{3}\right)-\delta_{1} .
$$

Thus, we need to prove that

$$
\left(c\left(f_{1}, f_{2}\right), c\left(g_{1}, g_{2}\right)\right) \geqslant\left(\kappa, c\left(g_{1}, g_{3}\right)\right)-\left(\delta_{1}, \delta_{1}\right),
$$

where $\kappa \in \mathbb{Z}^{2}$ is defined as follows

$$
\kappa= \begin{cases}c\left(f_{1}, f_{3}\right) & c\left(g_{1}, g_{3}\right)<c\left(g_{2}, g_{3}\right) \\ \min \left\{c\left(f_{1}, f_{3}\right), c\left(f_{2}, f_{3}\right)\right\} & c\left(g_{1}, g_{3}\right)=c\left(g_{2}, g_{3}\right)\end{cases}
$$

The above inequality holds since $c\left(g_{1}, g_{2}\right)>c\left(g_{1}, g_{3}\right)-\delta_{1}$.

Remark 1. The result of Lemma 1 does not hold for $\delta_{1}=\delta$. Indeed, in the case when $c\left(g_{1}, g_{3}\right)<c\left(g_{2}, g_{3}\right)$ and $c\left(g_{1}, g_{2}\right)=c\left(g_{1}, g_{3}\right)-\delta$ we must have $c\left(f_{1}, f_{2}\right) \geqslant c\left(f_{1}, f_{3}\right)-\delta$ which may not be true.

The construction above shows that if $G$ has a $\delta$-hyperbolic length function $l$ in a non-archimedean $\Lambda$ and if there are many elements of $G$ such that $l(g)<\delta$ then the properties of $G$ may be quite far from properties of word-hyperbolic groups.

## 5 Various types of actions and functions

In this section we study properties of $\delta$-hyperbolic length functions. In particular, we consider new axioms in addition to $(\Lambda 1)-(\Lambda 4, \delta)$ which shed some light on the structure of the underlying group. We also try to characterize the corresponding group actions (which exist in view of Theorem (6).

Below we use the following notation. If $l: G \rightarrow \Lambda$ is a $\delta$-hyperbolic length function and $\alpha \in \Lambda$ then for $g, h \in G$ we write

$$
g h=g \circ_{\alpha} h
$$

if $c\left(g^{-1}, h\right) \leqslant \alpha$. Respectively, $g h=g \circ h$ means that $c\left(g^{-1}, h\right)=0$, that is, $l(g h)=l(g)+l(h)$.

### 5.1 Regularity

Let $l: G \rightarrow \Lambda$ be a $\delta$-hyperbolic length function. We introduce several conditions on $l$, all of which lead to the same property called regularity. Both conditions depend on a parameter $k \in \mathbb{N}$. Below is the first condition on $l$, which we denote ( $R 1, k$ ).
$(R 1, k): \exists k \in \mathbb{N} \quad \forall g, h \in G \quad \exists u \in G$

$$
g=u \circ_{k \delta} g_{1} \& \quad h=u \circ_{k \delta} h_{1} \& g^{-1} h=g_{1}^{-1} \circ_{k \delta} h_{1}
$$

Observe that if $\delta=0$ then $l$ is a Lyndon length function on $G$ and it satisfies ( $R 1, k$ ) for some $k$ (and hence for any) if and only if $l$ is regular (see 11,13 for all definitions and properties of regular Lyndon length functions). For an arbitrary $\delta$, clearly $(R 1, k)$ implies $(R 1, m)$ for any $m>k$.

Here is another condition on $l$, we call it $(R 2, k)$.
$(R 2, k): \exists k \in \mathbb{N} \quad \forall g, h \in G \quad \exists u \in G$

$$
\begin{gathered}
l(u) \leqslant c(g, h)+k \delta \quad \& l\left(u^{-1} g\right) \leqslant c\left(g^{-1}, g^{-1} h\right)+k \delta \\
\& l\left(u^{-1} h\right) \leqslant c\left(h^{-1}, h^{-1} g\right)+k \delta
\end{gathered}
$$

Again, as in the case of $(R 1, k)$, the condition $(R 2, k)$ obviously implies $(R 2, m)$ for any $m>k$.

The following result makes a connection between these conditions.
Lemma 16. Let $G$ be a group and $l: G \rightarrow \Lambda$ be a $\delta$-hyperbolic length function. Then the following implications hold for $l$ :

$$
(R 1, k) \Longrightarrow(R 2, k+1), \quad(R 2, k) \Longrightarrow(R 1, k)
$$

Proof. $(R 1, k) \Longrightarrow(R 2, k+1)$ :
We have $g=u \circ_{k \delta} g_{1}$ which is equivalent to $c\left(u^{-1}, g_{1}\right) \leqslant k \delta$, so $l(u)+l\left(g_{1}\right)-$ $l(g) \leqslant 2 k \delta$. It follows that $2 l(u)-2 c(u, g) \leqslant 2 k \delta$, or $l(u) \leqslant c(u, g)+k \delta$. Using the same argument we get $l(u) \leqslant c(u, h)+k \delta$. Hence,

$$
c(g, h) \geqslant \min \{c(u, g), c(u, h)\}-\delta \geqslant l(u)-(k+1) \delta
$$

and we have $l(u) \leqslant c(g, h)+(k+1) \delta$.
Similarly, from $g^{-1}=g_{1}^{-1} \circ_{k \delta} u^{-1}$ we get

$$
l\left(g_{1}\right)=l\left(g_{1}^{-1}\right) \leqslant c\left(g^{-1}, g_{1}^{-1}\right)+k \delta
$$

from $g^{-1} h=g_{1}^{-1} \circ_{k \delta} h_{1}$ we get

$$
l\left(g_{1}\right)=l\left(g_{1}^{-1}\right) \leqslant c\left(g^{-1}, g^{-1} h\right)+k \delta
$$

which imply

$$
l\left(u^{-1} g\right)=l\left(g_{1}\right) \leqslant c\left(g^{-1}, g^{-1} h\right)+(k+1) \delta
$$

The inequality $l\left(u^{-1} h\right)=l\left(h_{1}\right) \leqslant c\left(h^{-1}, h^{-1} g\right)+(k+1) \delta$ is derived in the same way.
$(R 2, k) \Longrightarrow(R 1, k):$
We have that $l(u) \leqslant c(g, h)+k \delta$ and $l\left(u^{-1} g\right) \leqslant c\left(g^{-1}, g^{-1} h\right)+k \delta$ which imply that $l(u)+l\left(u^{-1} g\right) \leqslant l(g)+2 k \delta$. Therefore,

$$
c\left(u^{-1}, u^{-1} g\right)=\frac{1}{2}\left(l(u)+l\left(u^{-1} g\right)-l(g)\right) \leqslant k \delta
$$

that is, $g=u\left(u^{-1} g\right)=u \circ_{k \delta}\left(u^{-1} g\right)$. Similar argument proves the inequalities $h=u \circ_{k \delta}\left(u^{-1} h\right)$ and $g^{-1} h=\left(u^{-1} g\right)^{-1} \circ_{k \delta}\left(u^{-1} h\right)$.

Definition 2. A $\delta$-hyperbolic length function $l: G \rightarrow \Lambda$ is called regular if it satisfies either $(R 1, k)$, or $(R 2, k)$ for some $k \in \mathbb{N}$.

Observe that Definition 2 agrees with the notion of regularity for Lyndon length functions in the case when $\delta=0$.

Now, we say that the action of $G$ on a $\delta$-hyperbolic space $(X, d)$ is regular if for some $x \in X$ (hence, for any) the length function $l$ based at $x$ is regular. Since in this case for every $g, h \in G$ we have $c(g, h)=(g x \cdot h x)_{x}$ then regularity of the action can be explicitly characterized by the following condition (which is just a reformulation of $(R 2, k)$ in terms of actions):
$\exists k \in \mathbb{N} \quad \forall g, h \in G \quad \exists u \in G$

$$
\begin{gathered}
d(x, u x) \leqslant(g x \cdot h x)_{x}+k \delta \quad \& d\left(x,\left(u^{-1} g\right) x\right) \leqslant\left(g^{-1} x \cdot\left(g^{-1} h\right) x\right)_{x}+k \delta \\
\& d\left(x,\left(u^{-1} h\right) x\right) \leqslant\left(h^{-1} x \cdot\left(h^{-1} g\right) x\right)_{x}+k \delta
\end{gathered}
$$

In the case when $(X, d)$ is geodesic we introduce the following condition on the action of $G$.
$(R A, k)$ : there exists $k \in \mathbb{N}$ such that for any $g, h \in G$ there exists $u \in G$ with the property that $u x$ belongs to the $k \delta$-neighborhood of the interior of $\Delta_{I}(x, g x, h x)$.

Obviously, if $\delta=0$ then the interior of $\Delta_{I}(x, g x, h x)$ is a single point $Y(x, g x, h x)=[x, g x] \cap[x, h x] \cap[g x, h x]$ and $(R A, k)$ is equivalent to the regularity condition for group actions on $\Lambda$-trees (see 12, 11). In general, equivalence of $(R A, k)$ and regularity of the action follows from the lemma below.

Lemma 17. Let $G$ be a group acting on a $\delta$-hyperbolic space $(X, d)$ and $l$ : $G \rightarrow \Lambda a \delta$-hyperbolic length function based at $x \in X$. If $X$ is geodesic then the following implications hold

$$
(R A, k) \Longrightarrow(R 2, k+4), \quad(R 2, k) \Longrightarrow(R A, 3 k+4)
$$

Proof. $(R A, k) \Longrightarrow(R 2, k+4)$ :
Since $X$ is $\delta$-hyperbolic then for any $g, h \in G$, the triangle $\Delta_{I}(x, g x, h x)$ has diameter at most $4 \delta$. Hence, one of its vertices is at a distance $c(g, h)=(g x \cdot h x)_{x}$
from $x$, another vertex is at a distance $c\left(g^{-1}, g^{-1} h\right)$ from $g x$ (one can see this by translating the whole picture by $g^{-1}$ ), and the third one is at a distance $c\left(h^{-1}, h^{-1} g\right)$ from $h x$ (again, one can see this by translating the whole picture by $\left.h^{-1}\right)$. Thus, if $u x$ is in the $k \delta$-neighborhood of the interior of $\Delta_{I}(x, g x, h x)$, then it is in the $(k+4) \delta$-neighborhood of all three of its corners which implies $(R 2, k+4)$.
$(R 2, k) \Longrightarrow(R A, k+4):$
Suppose $(R 2, k)$ holds, so, the action of $G$ on $(X, d)$ (that is, $\left.\left(X_{l}, d_{l}\right)\right)$ is regular, and we want to show that it satisfies $(R A, 3 k+4)$ with respect to the base-point $x \in X$.

First of all, it is easy to see that

$$
(g x \cdot h x)_{x}+\left(g^{-1} x \cdot\left(g^{-1} h\right) x\right)_{x}=d_{l}(x, g x)
$$

Next, we have

$$
d_{l}(x, g x) \leqslant d_{l}(x, u x)+d_{l}(u x, g x) \leqslant(g x \cdot h x)_{x}+k \delta+d_{l}(u x, g x)
$$

from which it follows that

$$
d_{l}(x, g x)-d_{l}(u x, g x) \leqslant d_{l}(x, u x) \leqslant(g x \cdot h x)_{x}+k \delta
$$

which, combined with the inequalities $d_{l}(u x, g x) \leqslant\left(g^{-1} x \cdot\left(g^{-1} h\right) x\right)+k \delta$ and $d_{l}(x, g x)-\left(g^{-1} x \cdot\left(g^{-1} h\right) x\right)_{x}=(g x \cdot h x)_{x}$, implies that

$$
(g x \cdot h x)_{x}-k \delta \leqslant d_{l}(x, u x) \leqslant(g x \cdot h x)_{x}+k \delta
$$

A similar argument shows that

$$
\left(g^{-1} x \cdot\left(g^{-1} h\right) x\right)_{x}-k \delta \leqslant d_{l}(g x, u x) \leqslant\left(g^{-1} x \cdot\left(g^{-1} h\right) x\right)_{x}+k \delta
$$

Consider $\Delta(x, g x, h x)$ and $\Delta(x, g x, u x)$ which are both geodesic triangles in $X$. The point on $[x, g x]$ situated at a distance $(g x \cdot h x)_{x}$ from $x$ is a vertex of $\Delta_{I}(x, g x, h x)$, and the points at a distance $(g x \cdot u x)_{x}$ on $[x, g x]$ and $[x, u x]$ are at a distance at most $4 \delta$ from one another. Next, we have

$$
\begin{gathered}
(u x \cdot g x)_{x}=\frac{1}{2}(d(x, u x)+d(x, g x)-d(u x, g x)) \\
\geqslant \frac{1}{2}\left(\left((g x \cdot h x)_{x}-k \delta\right)+d(x, u x)-\left(\left(g^{-1} x \cdot\left(g^{-1} h\right) x\right)_{x}+k \delta\right)\right) \\
=\frac{1}{2}\left(2(g x \cdot h x)_{x}-2 k \delta\right)=(g x \cdot h x)_{x}-k \delta
\end{gathered}
$$

It follows that a point on $[x, g x]$ at a distance $(g x \cdot u x)_{x}$ from $x$ is at a distance at most $k \delta$ from a vertex of $\Delta_{I}(x, g x, h x)$, a point on $[x, u x]$ at a distance $(g x \cdot u x)_{x}$ from $x$ is at a distance at most $2 k \delta$ from $u x$, so $u x$ is at a distance at most $(3 k+4) \delta$ from one of the vertices of $\Delta_{I}(x, g x, h x)$. Hence, $(R A, 3 k+4)$ holds for the action of $G$ on $X$.

### 5.2 Completeness and hyperbolic groups

A $\delta$-hyperbolic length function $l: G \rightarrow \Lambda$ is called complete if the following axiom holds:
(C) $\forall g \in G \quad \forall \alpha \leqslant l(g) \exists u \in G: g=u \circ g_{1} \quad \& \quad l(u)=\alpha$.

Note that such an element $u$ may not be unique.
Lemma 18. Let $l: G \rightarrow \Lambda$ be a complete $\delta$-hyperbolic length function. Then for all $g, h \in G$, if $g=u \circ g_{1}, h=v \circ h_{1}$, with $l(u)=l(v) \leqslant c(g, h)$, it follows that $l\left(u^{-1} v\right) \leqslant 4 \delta$.

Proof. Consider the triple $\left\{h_{1}, v^{-1}, v^{-1} u g_{1}\right\}$. Then by $\delta$-hyperbolicity of $l$ we have

$$
2 c\left(v^{-1}, h_{1}\right) \geqslant \min \left\{2 c\left(v^{-1} u g_{1}, v^{-1}\right), 2 c\left(h_{1}, v^{-1} u g_{1}\right)\right\}-2 \delta,
$$

so that

$$
0 \geqslant \min \left\{l\left(v^{-1} u g_{1}\right)-l\left(g_{1}\right), l\left(v^{-1} u g_{1}\right)+l\left(h_{1}\right)-l\left(g^{-1} h\right)\right\}-2 \delta .
$$

At the same time we have $l(u)=l(v) \leqslant c(g, h)$, which implies that $l\left(g^{-1} h\right) \leqslant$ $l\left(g_{1}\right)+l\left(h_{1}\right)$ so that

$$
l\left(v^{-1} u g_{1}\right)-l\left(g_{1}\right) \leqslant l\left(v^{-1} u g_{1}\right)+l\left(h_{1}\right)-l\left(g^{-1} h\right),
$$

which gives $0 \geq l\left(v^{-1} u g_{1}\right)-l\left(g_{1}\right)-2 \delta$. Hence

$$
0 \leqslant l\left(v^{-1} u g_{1}\right)-l\left(g_{1}\right) \leqslant 2 \delta .
$$

Now consider the triple $\{u, v, g\}$. We have

$$
2 c(u, v) \geqslant \min \{2 c(u, g), 2 c(v, g)\}-2 \delta .
$$

Next, $2 c(u, g)=2 l(u)$ and $2 c(v, g)=2 l(u)+l\left(g_{1}\right)-l\left(v^{-1} u g_{1}\right)$. But $l\left(g_{1}\right)-$ $l\left(v^{-1} u g_{1}\right) \leqslant 0$ implies $2 c(u, g) \geqslant 2 c(v, g)$, so $2 c(u, v) \geqslant 2 c(v, g)-2 \delta$ and therefore

$$
2 l(u)-l\left(u^{-1} v\right) \geqslant 2 l(u)+l\left(g_{1}\right)-l\left(v^{-1} u g_{1}\right)-2 \delta,
$$

so that we have the desired inequality

$$
l\left(u^{-1} v\right) \leqslant l\left(v^{-1} u g_{1}\right)-l\left(g_{1}\right)+2 \delta \leqslant 4 \delta .
$$

Corollary 3. Let $l: G \rightarrow \Lambda$ be a complete $\delta$-hyperbolic length function. Suppose also that $c(g, h) \in \Lambda$ for all $g, h \in G$. Then $\forall g, h \in G \exists u, v \in G: g=$ $u \circ g_{1} \quad \& \quad h=v \circ h_{1} \quad \& \quad l(u)=l(v)=c(g, h) \quad \& \quad l\left(u^{-1} v\right) \leqslant 4 \delta$.

Proof. From completeness of $l$ it follows that for any $g, h \in G$ there exist $u, v \in G$ such that $g=u \circ g_{1}, h=v \circ h_{1}$, with $l(u)=l(v)=c(g, h)$. Now, by Lemma 18 we have $l\left(u^{-1} v\right) \leqslant 4 \delta$ and the result follows.

Here are examples of groups with complete $\delta$-hyperbolic length functions.
Example 9. Every $\delta$-hyperbolic group has a complete $\mathbb{Z}$-valued $\delta$-hyperbolic length function, which is the geodesic length of its elements.

Example 10. If $G$ has a complete $\delta$-hyperbolic length function with values in $\Lambda$ and $C_{G}(u)$ is a centralizer of $u \in G$, then the group $G^{\prime}=\left\langle G, t \mid\left[C_{G}(u), t\right]=1\right\rangle$ has a complete $\delta$-hyperbolic length function with values in $\Lambda \oplus \mathbb{Z}$ (with the right lexicographic order).

We call a $\Lambda$-metric space $(X, d)$ quasi-geodesic if for every $x, y \in X$ there is a map $\gamma:[0, d(x, y)] \rightarrow X$ such that $\gamma(0)=x, \gamma(d(x, y))=y$, and for any $0 \leqslant \alpha \leqslant \beta \leqslant d(x, y)$ we have

$$
\beta-\alpha \leqslant d(\gamma(\alpha), \gamma(\beta)) \leqslant \beta-\alpha+C
$$

here $C$ is a constant which depends on $X$ only.
Theorem 7. Let $l: G \rightarrow \Lambda$ be a complete $\delta$-hyperbolic length function. Then the $\delta$-hyperbolic $\Lambda$-metric space $\left(X_{l}, d_{l}\right)$ constructed from the pair $(G, l)$ is quasigeodesic.
Proof. Recall that $X_{l}$ is the set $G / A$, where $A=\operatorname{ker} l$, and the metric $d_{l}$ on $X_{l}$ is defined by $d_{l}(g A, h A)=l\left(g^{-1} h\right)$.

Let $x=g A, y=h A \in X_{l}$. Consider a map $\gamma:[0, d(x, y)] \rightarrow X_{l}$ such that $\gamma(0)=x, \gamma(d(x, y))=y$, and for any $0 \leqslant \alpha \leqslant d(x, y)$ we put $\gamma(\alpha)=\left(g u_{1}\right) A$, where $g^{-1} h=u_{1} \circ u_{2}$ and $l\left(u_{1}\right)=\alpha$ (such $u_{1}, u_{2} \in G$ exist by completeness of $l)$. Thus we have $d(\gamma(0), \gamma(\alpha))=d\left(g A,\left(g u_{1}\right) A\right)=l\left(u_{1}\right)=\alpha$.

Now, for any $\alpha$ and $\beta$ such that $0 \leqslant \alpha \leqslant \beta \leqslant d(x, y)$ we have $\gamma(\alpha)=$ $\left(g u_{1}\right) A, \gamma(\beta)=\left(g v_{1}\right) A$, where $g^{-1} h=u_{1} \circ u_{2}=v_{1} \circ v_{2}$ and $l\left(u_{1}\right)=\alpha, l\left(v_{1}\right)=\beta$. Since $\alpha \leqslant \beta$, by Lemma 18, $v_{1}=\left(u_{1} s\right) \circ v_{2}$, where $l(s) \leqslant 4 \delta$. Hence, we have

$$
\beta-\alpha \leqslant d(\gamma(\alpha), \gamma(\beta))=l\left(u_{1}^{-1} v_{1}\right) \leqslant l\left(v_{2}\right)+l(s) \leqslant \beta-\alpha+4 \delta
$$

Theorem 8. A group $G$ is $\delta$-hyperbolic if and only if there exists a $\delta$-hyperbolic length function $l: G \rightarrow \mathbb{Z}$ with the following properties
(a) $l$ is complete,
(b) $|\{g \in G \mid l(g) \leqslant 1\}|<\infty$.

Proof. If $G$ is $\delta$-hyperbolic then its word metric $|\cdot|_{S}: G \rightarrow \mathbb{Z}$ with respect to some finite generating set $S$ is a $\delta$-hyperbolic length function, it is obvious.

Now, suppose on a group $G$ there exists a $\delta$-hyperbolic length function $l$ : $G \rightarrow \mathbb{Z}$ which satisfies the conditions (a) and (b). Denote $S=\{g \in G \mid l(g) \leqslant$ $1\}$. Observe that $S$ is finite and $G=\langle S\rangle$ since by completeness of $l$ every $g \in G$ can be decomposed as a finite product of elements from $S$. Hence, $l$ can be viewed as a word metric $|\cdot|_{S}$ with respect to $S$. Finally, the Cayley graph of $G$ with respect to $S$ is $\delta$-hyperbolic which follows from $\delta$-hyperbolicity of $l$.

### 5.3 Free length functions

A $\delta$-hyperbolic length function $l: G \rightarrow \Lambda$ is called free if
(F) $\forall g \in G: g \neq 1 \rightarrow l\left(g^{2}\right)>l(g)+3 \delta$

Observe that if $\delta=0$, that is, the $\delta$-hyperbolic $\Lambda$-metric space $\left(X_{l}, d_{l}\right)$ is a $\Lambda$-tree then $l$ is a free Lyndon length function.

We say that the action of $G$ on a $\delta$-hyperbolic space $(X, d)$ is free if for some $x \in X$ (hence, for any) the length function $l$ based at $x$ is free. Obviously, if a $\delta$-hyperbolic length function $l: G \rightarrow \Lambda$ is free then $\operatorname{ker}(l)$ is trivial.

Example 11. Every torsion-free $\delta$-hyperbolic group has a free (and complete) $\mathbb{Z}$ valued $\delta$-hyperbolic length function, which is the geodesic length of its elements.

Observe that, in view of Lemma 11, free action implies that every element of $G$ acts as either a hyperbolic isometry, or an inversion. Hence, we say that a group $G$ is $\Lambda$-free if it acts on a $\delta$-hyperbolic $\Lambda$-metric space $(X, d)$ freely and without inversions.

## 6 Proper actions and hyperbolicity relative to the kernel

In this section we consider action of a finitely generated group $G$ on a geodesic $\delta$-hyperbolic $\mathbb{R}$-metric space $(X, d)$ and show that if the action is "nice" (regular and proper) then $G$ is weakly hyperbolic relative to the kernel of the associated length function.

### 6.1 Proper actions

We fix the group $G$ with a finite generating set $S$ and the $\delta$-hyperbolic $\mathbb{R}$-metric space $(X, d)$ with a base-point $x \in X$. As usual we have a $\delta$-hyperbolic length function $l: G \rightarrow \mathbb{R}$ based at $x$ and its kernel $\operatorname{ker}(l)$ is a subgroup of $G$. Recall that $\operatorname{ker}(l)=G_{x}=\operatorname{Stab}_{G}(x)$.

Recall that the action of $G$ on $(X, d)$ satisfies the axiom $(R 2, k)$ if there exists $k \in \mathbb{N}$ such that for all $g, h \in G$ there exists $u \in G$ with the property

$$
\begin{gathered}
d(x, u x) \leqslant(g x \cdot h x)_{x}+k \delta \quad \& d\left(x,\left(u^{-1} g\right) x\right) \leqslant\left(g^{-1} x \cdot\left(g^{-1} h\right) x\right)_{x}+k \delta \\
\& d\left(x,\left(u^{-1} h\right) x\right) \leqslant\left(h^{-1} x \cdot\left(h^{-1} g\right) x\right)_{x}+k \delta
\end{gathered}
$$

See Subsection 5.1 for all the details.
Next, we say the action is proper relative to $x$ if there exists some $\alpha \in \mathbb{R}$ such that $l(g)>\alpha$ for any $g \in G \backslash G_{x}$ and the set

$$
B_{N}=\{g \in G \mid d(x, g x) \leqslant N\}
$$

is bounded for any $N \in \mathbb{N}$ in the Cayley graph $\Gamma\left(G, S \cup G_{x}\right)$ of $G$ relative to $S \cup G_{x}$.

Since $S$ is finite, there exists $N \in \mathbb{N}$ such that $S \subseteq B_{N}$. Define the weighted graph $\Gamma$ so that

$$
V(\Gamma)=G / G_{x}, \quad E(\Gamma)=\left\{\left(g G_{x},(g h) G_{x}\right) \mid g \in G, h \in B_{N}\right\}
$$

and the weight function $w: E(\Gamma) \rightarrow \mathbb{R}$ is defined by $w\left(g G_{x},(g h) G_{x}\right)=l(h)$. Note that $\Gamma$ is connected since $S \subseteq B_{N}$. Next, $\Gamma$ is a metric space with respect to the metric $d_{\Gamma}$ defined by

$$
d_{\Gamma}\left(g G_{x}, h G_{x}\right)=\min \left\{w(p) \mid p \text { is a path in } \Gamma \text { connecting } g G_{x} \text { and } h G_{x}\right\}
$$

Notice that for $g, h \in G$, we have $d(g x, h x) \leqslant N$ if and only if $\left(g G_{x}, h G_{x}\right) \in$ $E(\Gamma)$. It follows that in a geodesic path in $\Gamma$ no two consecutive edges both have weights less than or equal to $\frac{N}{2}$ (by the triangle inequality).

Lemma 19. Suppose the action of $G$ on $(X, d)$ is proper relative to $x$ and that it satisfies $(R 2, k)$ for some $k \in \mathbb{N}$.
(i) If $\left(G_{x}, a G_{x}\right),\left(a G_{x}, b G_{x}\right) \in E(\Gamma)$ and $\left(G_{x}, a G_{x}\right) \cup\left(a G_{x}, b G_{x}\right)$ is a geodesic in $\Gamma$ then $d(x, b x) \geqslant d(x, a x)+d(a x, b x)-2 k \delta$.
(ii) If $\left(G_{x}, a G_{x}\right),\left(a G_{x}, b G_{x}\right),\left(b G_{x}, c G_{x}\right) \in E(\Gamma)$ and $\left(G_{x}, a G_{x}\right) \cup\left(a G_{x}, b G_{x}\right) \cup$ $\left(b G_{x}, c G_{x}\right)$ is a geodesic in $\Gamma$ with $w\left(a G_{x}, b G_{x}\right)<\frac{N}{2}$ then $d(x, c x) \geqslant$ $d(x, a x)+d(a x, b x)+d(b x, c x)-5 k \delta$.

Proof. (i) Consider the geodesic triangle $\Delta_{I}(x, a x, b x)$ with the vertices $p \in$ $[x, a x], q \in[x, b x]$ and $r \in[a x, b x]$. From $(R 2,1)$ it follows that there exists $u \in G$ such that

$$
d(x, u x) \leqslant d(x, p)+k \delta, \quad d(a x, u x) \leqslant d(a x, q)+k \delta, \quad d(b x, u x) \leqslant d(b x, r)+k \delta
$$

By definition of $\Delta_{I}(x, a x, b x)$ we have

$$
d(x, b x)=d(x, q)+d(q, b x)=d(x, a x)+d(a x, b x)-d(p, a x)-d(r, a x)
$$

If $d(p, a x)=d(r, a x)<k \delta$ then there is nothing to prove. Otherwise, we have

$$
d(x, u x) \leqslant d(x, p)+k \delta \leqslant d(x, a x)-k \delta+k \delta=d(x, a x) \leqslant N
$$

and

$$
d(u x, b x) \leqslant d(r, b x)+k \delta \leqslant d(a x, b x)-k \delta+k \delta=d(a x, b x) \leqslant N
$$

so we have that $\left(G_{x}, u G_{x}\right) \cup\left(u G_{x}, b G_{x}\right)$ is a path in $\Gamma$. But then

$$
\begin{gathered}
d(x, a x)+d(a x, b x)=w\left(G_{x}, a G_{x}\right)+w\left(a G_{x}, b G_{x}\right) \leqslant w\left(G_{x}, u G_{x}\right)+w\left(u G_{x}, b G_{x}\right) \\
=d(x, u x)+d(u x, b x) \leqslant d(x, q)+d(q, b x)+2 k \delta=d(x, b x)+2 k \delta
\end{gathered}
$$

(ii) Consider the geodesic triangle $\Delta_{I}(x, b x, c x)$ with the vertices $p \in[x, b x]$, $q \in[x, c x]$ and $r \in[b x, c x]$. Again, from $(R 2, k)$ it follows that there exists $u \in G$ such that

$$
d(x, u x) \leqslant d(x, p)+k \delta, \quad d(b x, u x) \leqslant d(b x, q)+k \delta, \quad d(c x, u x) \leqslant d(c x, r)+k \delta
$$

By definition of $\Delta_{I}(x, a x, c x)$ we have

$$
d(x, c x)=d(x, q)+d(q, c x)=d(x, b x)+d(b x, c x)-d(p, b x)-d(r, b x)
$$

If $d(p, b x)=d(r, b x)<k \delta$ then
$d(x, c x) \geqslant d(x, b x)+d(b x, c x)-2 k \delta \geqslant d(x, a x)+d(a x, b x)-2 k \delta+d(b x, c x)-2 k \delta$, so, there is nothing to prove. So, assume that $d(r, b x) \geqslant k \delta$. Then we have

$$
d(u x, c x) \leqslant d(r, c x)+k \delta \leqslant d(b x, c x)-k \delta+k \delta \leqslant N
$$

that is, $\left(u G_{x}, c G_{x}\right) \in E(\Gamma)$. Note also that, if $d(x, u x) \leqslant N$ then $\left(G_{x}, u G_{x}\right) \in$ $E(\Gamma)$ which implies that

$$
d(x, a x)+d(a x, b x)+d(b x, c x) \leqslant d(x, u x)+d(u x, c x) \leqslant d(x, c x)+2 k \delta
$$

so, again, there would be nothing to prove. Thus we can assume that $d(x, u x)>$ $N$.

Observe that $u x$ is at a distance of at most $k \delta$ from a point on $[x, b x]$. If this point is at a distance of at most $\delta$ from a point on $[a x, b x]$, then $u x$ is at a distance of at most $\frac{N}{2}+(k+1) \delta$ from $a x$. If, on the other hand, it is at a distance of at most $\delta$ from a point on $[x, a x]$, say $t$, then we have

$$
d(x, a x)<d(x, u x) \leqslant d(x, t)+(k+1) \delta,
$$

so, $d(t, a x)<(k+1) \delta$, and thus $d(a x, u x)<(2 k+2) \delta$. In both cases, since we can assume that $N>(2 k+2) \delta$, we have $\left(a G_{x}, u G_{x}\right) \in E(\Gamma)$. It follows that

$$
d(x, a x)+d(a x, b x)+d(b x, c x) \leqslant d(x, a x)+d(a x, u x)+d(u x, c x)
$$

Finally, we have $d(x, b x) \geqslant d(x, a x)+d(a x, b x)-2 k \delta$, so, $(a x \cdot b x)_{x} \leqslant k \delta$. It follows that if $v$ is a point on $[x, b x]$ such that

$$
d(x, a x)-2 k \delta \leqslant d(v, x) \leqslant d(x, a x)+2 k \delta
$$

then $d(a x, v) \leqslant 2 k \delta$. Similarly, if $w$ is a point on $[x, b x]$ such that

$$
d(x, u x)-2 k \delta \leqslant d(w, x) \leqslant d(x, u x)+2 k \delta
$$

then $d(u x, w) \leqslant k \delta$. Since we assume that $d(x, u x)>d(x, a x)$, we have that

$$
d(a x, u x) \leqslant d(x, u x)-d(x, a x)+3 k \delta .
$$

It follows that

$$
\begin{gathered}
d(x, a x)+d(a x, u x)+d(u x, c x) \leqslant d(x, a x)+d(x, u x)-d(x, a x)+3 k \delta+d(u x, c x) \\
=d(x, u x)+d(u x, c x)+3 k \delta \leqslant d(x, c x)+5 k \delta .
\end{gathered}
$$

Now, define a map $\varphi: \Gamma \rightarrow X$ so that

$$
\varphi\left(g G_{x}\right)=g x \text { and } \varphi\left(g G_{x}, h G_{x}\right)=[g x, h x] .
$$

Corollary 4. Suppose the action of $G$ on $(X, d)$ satisfies $(R 2, k)$ for some $k \in \mathbb{N}$. Then $\Gamma$ is hyperbolic.

Proof. Let $\gamma$ be a geodesic in $\Gamma$. By the construction, $\varphi(\gamma)$ is a concatenation of geodesic segments of lengths at most $N$ so that no two consecutive segments both have weights less than or equal to $\frac{N}{2}$. Hence, if $x \in \varphi(\gamma)$ then the intersection $B_{\frac{N}{4}}(x) \cap \varphi(\gamma)$ is contained in the image $\varphi\left(\gamma_{0}\right)$ of a subpath $\gamma_{0}$ of $\gamma$ in $\Gamma$ which contains not more than three edges. Moreover, if $\gamma_{0}$ consists of three edges then the length of the middle one is less than $\frac{N}{2}$. Now, by Lemma $19, \varphi(\gamma)$ is a $\left(1,5 \delta, \frac{N}{4}\right)$-local-quasi-geodesic. It is a known result (see [月, Theorem 3.1.4]) that there exist constants $L(\delta, \lambda, c), \lambda^{\prime}(\delta, \lambda, c)$, and $c^{\prime}(\delta, \lambda, c)$ such that any $(\lambda, c, L)$ -local-quasi-geodesic is a $\left(\lambda^{\prime}, c^{\prime}\right)$-quasi-geodesic.

We can assume that $\frac{N}{4} \geqslant L(\delta, 1,5 \delta)$ since $N$ is independent of $\delta$ and $L=$ $L(\delta)$. Then we use $\lambda=\lambda^{\prime}(\delta, 1,5 \delta)$ and $c=c^{\prime}(\delta, 1,5 \delta)$ and it is immediate that $\Gamma$ is $(\lambda \delta+c)$-hyperbolic.

Lemma 20. If the action of $G$ on $(X, d)$ is proper relative to $x$ then $\Gamma$ is quasiisometric to the Cayley graph $\Gamma\left(G, S \cup G_{x}\right)$ of $G$ relative to $S \cup G_{x}$.

Proof. Denote the metric on $\Gamma\left(G, S \cup G_{x}\right)$ by $d^{\prime}$. Consider the function $\Gamma(G, S \cup$ $\left.G_{x}\right) \rightarrow \Gamma$ defined by $g \rightarrow g G_{x}$. Since $S \cup G_{x} \subseteq B_{N}$, the image of a path from $\Gamma\left(G, S \cup G_{x}\right)$ is a path in $\Gamma$ of the same combinatorial length and we have

$$
d_{\Gamma}\left(g G_{x}, h G_{x}\right) \leqslant N d^{\prime}(g, h) .
$$

Next, suppose that $N^{\prime} \in \mathbb{N}$ is such that $B_{N}$ is contained in the ball of radius $N^{\prime}$ centered at the identity in $\Gamma\left(G, S \cup G_{x}\right)$ (such $N^{\prime}$ exists since the action is proper). Any edge of $\Gamma$ has weight at least $\alpha$ and it lifts to a geodesic path in $\Gamma\left(G, S \cup G_{x}\right)$ of length at most $N^{\prime}$. Since in any geodesic word in $\left(S \cup G_{x}\right)$ at least every other letter is in $S$, we have that the edge together with any elements of $S$ adjacent to it will have length at most $2 N^{\prime}$. Hence,

$$
d^{\prime}(g, h) \leqslant \frac{2 N^{\prime}}{\alpha} d_{\Gamma}\left(g G_{x}, h G_{x}\right)
$$

and the required statement follows.
Theorem 9. Let $G$ be a finitely generated group acting on a $\delta$-hyperbolic $\mathbb{R}$ metric space $(X, d)$ with a base-point $x \in X$. If the action of $G$ on $X$ is proper relative to $x$ and that it satisfies $(R 2, k)$ for some $k \in \mathbb{N}$ then $G$ is weakly hyperbolic relative to $G_{x}$.
Proof. This is a direct consequence of Lemma 20 and Corollary 6 .

### 6.2 Geometric alternative to relatively proper actions

One problem with our definition of a relatively proper action is that it is very hard to detect, especially geometrically. Let us look at an alternative condition which gives the same result but it is more easy to interpret geometrically, though it gives less insight into the relation between the action and the relative Cayley graph.

Let $X$ be a geodesic $\delta$-hyperbolic space on which $G$ acts and choose a basepoint $x$. As before, $B_{n}=\{g \in G \mid d(x, g x) \leqslant n\}$. We say the action of $G$ on $X$ has property $P(n)$ if
(i) there exists $\alpha$ such that $B_{\alpha}=G_{x}$,
(ii) $B_{n}$ generates $G$,
(iii) the set of double cosets $\left\{G_{x} g G_{x} \mid g \in B_{n}\right\}$ is finite.

Proposition 3. Suppose $G$ is finitely generated relatively to $G_{x}$ and the action of $G$ on $X$ has property $P(n)$ for some $n>6144 \log _{2}(154)+768+2288 \delta$. Then $G$ is weakly relatively hyperbolic relative to $G_{x}$.
Proof. It is suffficient to prove that Lemma 20 and Corollary 10 hold in this case. Since Lemma 19 requires only regularity and that $n \geqslant(2 k+2) \delta$, we have nothing to prove there.

Lemma 20 is rather easy to prove in this case. By assumption, $B_{n}$ generates $G$ and that $\left\{G_{x} g G_{x} \mid g \in B_{n}\right\}$ is finite, so we can take $S$ to be a finite set of representatives of $\left\{G_{x} g G_{x} \mid g \in B_{n}\right\}$. Again, since both $S$ and $G_{x}$ are contained in $B_{n}$, paths in $\Gamma\left(G, G_{x} \cup S\right)$ are still paths in $\Gamma$ and the images of its edges have weight at most $N$, so $d_{\Gamma}\left(g G_{x}, h G_{x}\right) \leqslant N d_{\Gamma}\left(G, G_{x} \cup S\right)(g, h)$ for any $g, h$. On the other hand, any edge in $\Gamma$ has weight at least $\alpha$ and the length of any of its preimages in $\Gamma\left(G, G_{x} \cup S\right)$ is at most 3, so

$$
d_{\Gamma\left(G, G_{x} \cup S\right)}(g, h) \leqslant \frac{3}{\alpha} d_{\Gamma}\left(g G_{x}, h G_{x}\right)
$$

Finally, in order for the proof of Corollary to work, it is sufficient to have $\frac{N}{4} \geqslant L(\delta, 1,5 \delta)$, and it can be inferred from the proofs of Lemma 3.1.9 and Theorem 3.1.4 of that $L(\delta, 1,5 \delta)=1536 \log _{2}(154)+192+572 \delta$.

### 6.3 Application to actions on trees

Corollary 5. Let $G$ be a finitely generated relative to some $G_{x}$ group, which acts regularly on an $\mathbb{R}$-tree $T$ with the property $P(n)$ for some $n \in \mathbb{R}, n>0$. Then $G$ is weakly hyperbolic relative to $G_{x}$.
Proof. Define $T^{k}$ to be the metric space $T$ with all distances multiplied by $k$ and take a basepoint $x^{k}$. It is obvious that $G$ acts on $T^{k}$ for any $k$, so we can define $B_{n}^{k}=\left\{g \in G \mid d\left(x^{k}, g x^{k}\right) \leqslant n\right\}$. It is easy to see that $B_{n}^{k}=B_{n k}$, thus, the action of $G$ on $T^{k}$ has property $P(n k)$.

Since $\delta=0$ for all $T^{k}$, it follows that $a \delta^{2}+b \delta+c=c$ for any such action, and for some $k$ we obtain $n k \geqslant c$, as required.

## 7 Completing hyperbolic $\mathbb{Z}$ - and $\mathbb{R}$-metric spaces

In this section we investigate the question of "completing" a given non-geodesic hyperbolic $\mathbb{Z}$-metric space $X$, that is, constructing a geodesic hyperbolic $\mathbb{Z}$ metric space $\bar{X}$ in which $X$ (quasi-)isometrically embeds. Observe that any $\delta$-hyperbolic $\mathbb{Z}$-metric space embeds isometrically into a complete geodesic $\delta$ hyperbolic $\mathbb{R}$-metric space (see [2]) but this completion does not have to be a $\mathbb{Z}$-metric space.

Given a $\delta$-hyperbolic $\mathbb{Z}$-metric space $(X, d)$ which we fix for the rest of this section, below we introduce two $\mathbb{Z}$-completions of $X$ which we call $\Gamma_{1}(X)$ and $\Gamma_{2}(X)$. We shall also define an analogous construction $\beta(X)$ when $X$ is an $\mathbb{R}$-metric space.

Our constructions will have, compared to Bonk and Schramm's, the disadvantage that the hyperbolicity constant will increase. However, they will have the advantage that isometries, embeddings and quasi-isometries of $X$ extend easily and that boundaries are easy to work with.

## $7.1 \quad \Gamma_{1}(X)$

Define a graph $\Gamma_{1}(X)$ as follows: to the set of points of $X$ which we call essential vertices we add new vertices which fill "gaps" between essential vertices.
(1) Define $\Gamma_{1}(X)=X$, that is, all vertices of $\Gamma_{1}(X)$ initially are essential.
(2) For any pair $\{x, y\}$ of essential vertices with the property that there exists no $z \in \Gamma_{1}(X)$ such that $d(x, y)=d(x, z)+d(z, y)$, add to $\Gamma_{1}(X)$ all vertices on a $\mathbb{Z}$-path of distance $d(x, y)$. The added $\mathbb{Z}$-paths we call basic and the new vertices we call auxiliary vertices. Observe that after this step, for every essential vertices of $\Gamma_{1}(X)$ there exists a $\mathbb{Z}$-geodesic segment (composed from auxiliary vertices) connecting them.
(3) For any triple $\{x, y, z\}$ of essential vertices, consider the projection of the triangle $\Delta(x, y, z)$ onto the tripod $T(x, y, z)$. Every two auxiliary vertices of $\Delta(x, y, z)$ which map into the same point of the tripod we connect by a $\mathbb{Z}$-path whose length is the smallest integer larger or equal to $4 \delta$ (that is, we add to $\Gamma_{1}(X)$ all vertices on this path) unless there exists already a path of length less than or equal to $4 \delta$ between them. The added paths we call bridges and the new vertices we call negligible vertices.
(4) We extend the metric $d: X \rightarrow \mathbb{Z}$ to the metric $d: \Gamma_{1}(X) \rightarrow \mathbb{Z}$ as follows:
(a) the distance between two essential vertices is inherited from $X$,
(b) the distance between an auxiliary vertex to the adjacent essential vertices is defined by construction, hence, the distance from an auxiliary vertex to any other either essential, or auxiliary vertex is also defined (as the minimum of lengths of paths connecting them),
(c) the distance from a negligible vertex to the adjacent auxiliary vertices is defined by construction, so, the distance from a negligible vertex to any other vertex of $\Gamma_{1}(X)$ is also defined.

Remark 2. 1. Observe that if $\delta=0$ then the process of building $\Gamma_{1}(X)$ is equivalent to the construction of a $\Lambda$-tree out of a 0-hyperbolic $\Lambda$-metric space (see, for example, [3, Theorem 2.4.4]) since all bridges have length 0 .
2. If $X$ is geodesic then $\Gamma_{1}(X)=X$. Indeed, if $X$ is geodesic then for any essential vertices $x$ and $y$ there is no essential vertex $z$ such that $d(x, y)=d(x, z)+d(y, z)$ only when $d(x, y)=1$. So, no auxiliary vertices are added. Finally, since $X$ is hyperbolic, for any pair of auxiliary vertices there is already a path of length at most $4 \delta$. Hence, no bridges are added and, hence, no negligible vertices are added either.

Remark 3. Note that $\Gamma_{1}(X)$ is unique for a given $X$. Indeed it is easy to see that the steps (1) and (2) above do not depend on the order in which we process pairs of essential vertices. As for the step (3), since bridges are exactly of length $4 \delta$, the only cases where a bridge is not added is one where there already was a path which did not contain bridges, so the order in which we process triples of essential vertices does not matter.

By [2, Theorem 4.1], $X$ isometrically embeds into a complete geodesic $\delta$ hyperbolic $\mathbb{R}$-metric space $\bar{X}$. Denote the metric on $\bar{X}$ by $\bar{d}$. We are going to use $(\bar{X}, \bar{d})$ in our construction below.

Define a map $\varphi: \Gamma_{1}(X) \rightarrow \bar{X}$ as follows. First of all, observe that the set of essential vertices of $\Gamma_{1}(X)$ is naturally identified with $X$, hence, it embeds into $\bar{X}$. Next, for a pair $\{x, y\}$ of essential vertices, the basic path between them in $\Gamma_{1}(X)$ can be mapped to some geodesic segment between $\varphi(x)$ and $\varphi(y)$. Finally, for a pair $\{x, y\}$ of auxiliary vertices (whose images under $\varphi$ are already defined), the bridge between them can also be mapped to a geodesic segment between $\varphi(x)$ and $\varphi(y)$. Observe that $\varphi$ is not unique but it is well-defined by the construction.

Lemma 21. Let $v, w$ be vertices of $\Gamma_{1}(X)$.
(i) If $v$ and $w$ are essential then $d(v, w)=\bar{d}(\varphi(v), \varphi(w))$.
(ii) If $v$ and $w$ are auxiliary then

$$
\bar{d}(\varphi(v), \varphi(w)) \leqslant d(v, w) \leqslant \bar{d}(\varphi(v), \varphi(w))+24 \delta
$$

(iii) If $v$ is essential and $w$ is auxiliary then

$$
\bar{d}(\varphi(v), \varphi(w)) \leqslant d(v, w) \leqslant \bar{d}(\varphi(v), \varphi(w))+8 \delta
$$

Proof. First, notice that for any $v, w \in \Gamma_{1}(X)$ we have

$$
d(v, w) \geqslant \bar{d}(\varphi(v), \varphi(w))
$$

Indeed, any edge of $\Gamma_{1}(X)$ belongs either to a basic path, or to a bridge. Basic paths are embedded isometrically in $\bar{X}$ since $X$ is embedded isometrically in $\bar{X}$. However, $\bar{X}$ is also $\delta$-hyperbolic, so for a pair of auxiliary vertices connected by a bridge in $\Gamma_{1}(X)$, their images in $\bar{X}$ are also at a distance of at most $4 \delta$. It follows that $\varphi$ can only shorten distances.
(i) Obvious.
(iii) By the construction, $w$ is on the geodesic linking two essential vertices $w_{1}$ and $w_{2}$ in $\Gamma_{1}(X)$. Consider the geodesic triangle $\Delta\left(v, w_{1}, w_{2}\right)$. Hence, $w$ is at a distance of at most $4 \delta$ from either $\left[v, w_{1}\right]$, or $\left[v, w_{2}\right]$. Let $\gamma$ be the path $\left[v, w^{\prime}\right] \cup\left[w^{\prime}, w\right]$, where $w^{\prime} \in\left[v, w_{i}\right]$ for $i=1,2$ and let $l(\gamma)$ the length of $\gamma$. Then $\left[v, w^{\prime}\right]$ is isometrically embedded in $\bar{X}$ and $d\left(w, w^{\prime}\right) \leqslant 4 \delta$, so $\left.\varphi\right|_{\gamma}$ is a $(1,4 \delta)$ -quasi-isometry. Furthermore, $\varphi\left(\left[v, w^{\prime}\right]\right)$ is a geodesic and $\bar{d}\left(\varphi(w), \varphi\left(w^{\prime}\right)\right) \leqslant 4 \delta$, so $\varphi(\gamma)$ is a $(1,4 \delta)$-quasi-geodesic. It follows that

$$
d(v, w) \leqslant l(\gamma) \leqslant \bar{d}(\varphi(v), \varphi(w))+8 \delta
$$



Figure 1: Case (iii) in the proof of Lemma 21
(ii) By the construction, there exist essential vertices $v_{1}, v_{2}, w_{1}, w_{2} \in \Gamma_{1}(X)$ such that $v \in\left[v_{1}, v_{2}\right], w \in\left[w_{1}, w_{2}\right]$. Consider the geodesic square $\left\{v_{1}, v_{2}, w_{2}, w_{1}\right\}$ (linked together in the given order). Suppose $v$ is at a distance of $4 \delta$ from $v^{\prime} \in\left[v_{1}, w_{1}\right]$. We can always assume this since $v$ must be at a distance of $4 \delta$ from either $\left[v_{1}, w_{1}\right]$, or $\left[v_{2}, w_{1}\right]$, and we can twist the square to fit this situation.

If $w$ is at a distance $4 \delta$ from $w^{\prime} \in\left[v_{1}, w_{1}\right]$ then set $\gamma=\left[v, v^{\prime}\right] \cup\left[v^{\prime}, w^{\prime}\right] \cup\left[w^{\prime}, w\right]$. Otherwise, $w$ is at a distance of $4 \delta$ from $w^{\prime} \in\left[v_{1}, w_{2}\right]$ and $v^{\prime}$ is at a distance of $4 \delta$ from either $v^{\prime \prime} \in\left[v_{1}, w_{2}\right]$, or $v^{\prime \prime} \in\left[w_{1}, w_{2}\right]$. Hence, we set $\gamma=\left[v, v^{\prime}\right] \cup$ $\left[v^{\prime}, v^{\prime \prime}\right] \cup\left[v^{\prime \prime}, w^{\prime}\right] \cup\left[w^{\prime}, w\right]$ in the former case, and $\gamma=\left[v, v^{\prime}\right] \cup\left[v^{\prime}, v^{\prime \prime}\right] \cup\left[v^{\prime \prime}, w\right]$ in the latter one.

Suppose first that $\gamma=\left[v, v^{\prime}\right] \cup\left[v^{\prime}, w^{\prime}\right] \cup\left[w^{\prime}, w\right]$. Then $\varphi\left(\left[v^{\prime}, w^{\prime}\right]\right)$ is an isometrically embedded geodesic. Next, the other two segments of $\gamma$ have lengths of at most $4 \delta$, and so do their images. It follows that $\left.\varphi\right|_{\gamma}$ is a $(1,8 \delta)$-quasi-isometry


Figure 2: Case (ii) in the proof of Lemma 21
and $\varphi(\gamma)$ is a $(1,8 \delta)$-quasi-geodesic. Hence,

$$
d(v, w) \leqslant l(\gamma) \leqslant \bar{d}(\varphi(v), \varphi(w))+16 \delta
$$

Now assume that $\gamma=\left[v, v^{\prime}\right] \cup\left[v^{\prime}, v^{\prime \prime}\right] \cup\left[v^{\prime \prime}, w^{\prime}\right] \cup\left[w^{\prime}, w\right]$. Then $\varphi\left(\left[v^{\prime \prime}, w^{\prime}\right]\right)$ is an isometrically embedded geodesic. The other three segments of $\gamma$ have lengths of at most $4 \delta$, and so do their images. It follows that $\left.\varphi\right|_{\gamma}$ is a $(1,12 \delta)$-quasiisometry and $\varphi(\gamma)$ is a $(1,12 \delta)$-quasi-geodesic. Hence,

$$
d(v, w) \leqslant l(\gamma) \leqslant \bar{d}(\varphi(v), \varphi(w))+24 \delta
$$

Finally, suppose $\gamma=\left[v, v^{\prime}\right] \cup\left[v^{\prime}, v^{\prime \prime}\right] \cup\left[v^{\prime \prime}, w\right]$. Then $\varphi\left(\left[v^{\prime \prime}, w\right]\right)$ is an isometrically embedded geodesic. The other two segments of $\gamma$ have lengths of at most $4 \delta$, and so do their images. Hence, $\left.\varphi\right|_{\gamma}$ is a $(1,8 \delta)$-quasi-isometry and $\varphi(\gamma)$ is a $(1,8 \delta)$-quasi-geodesic. So

$$
d(v, w) \leqslant l(\gamma) \leqslant \bar{d}(\varphi(v), \varphi(w))+16 \delta
$$

Proposition 4. $\Gamma_{1}(X)$ is $\delta^{\prime}$-hyperbolic with $\delta^{\prime}=29 \delta$.
Proof. By Lemma 21, the restriction of $\varphi$ to any geodesic of $\Gamma_{1}(X)$, whose endpoints are essential or auxiliary vertices, is a (1,24 )-quasi-isometry. Suppose
then that a geodesic has negligible vertices as endpoints. Negligible vertices have always valency 2 and they belong to paths of length of at most $2 \delta+1$ linking auxiliary vertices together. Thus, if $a, b$ are the endpoints, there exist auxiliary vertices $a^{\prime}, b^{\prime}$ such that $d\left(a, a^{\prime}\right), d\left(b, b^{\prime}\right) \leqslant 2 \delta$. It follows that, since $[a, b] \rightarrow\left[a^{\prime}, b^{\prime}\right]$ is a $(1,4 \delta)$-quasi-isometry by Lemma 21 and $\left[a^{\prime}, b^{\prime}\right] \rightarrow \bar{X}$ is a $(1,24 \delta)$-quasi-isometry, then $[a, b] \rightarrow \bar{X}$ is a $(1,28 \delta)$-quasi-isometry. Thus, the embedding of any geodesic of $\Gamma_{1}(X)$ in $\bar{X}$ is a $(1,28 \delta)$-quasi-isometric. It follows that $\varphi$ is a $(1,28 \delta)$-quasi-isometric embedding, and the result follows.

Lemma 22. Let $g$ be an isometry of $X$. Then there exists an unique isometry $\bar{g}$ of $\Gamma_{1}(X)$ such that $\left.\bar{g}\right|_{X}=g$.

Proof. Define $\bar{g}$ simply by mapping basic paths to basic paths (since we know the action on their endpoints) and bridges to bridges. Since $g$ is an isometry of $X$, it preserves the length of basic paths and the sizes of triangles, so it also preserves the presence of bridges.

The uniqueness of $\bar{g}$ is pretty obvious from the construction. Since all auxiliary vertices have valence 3 or 4 , unless they are at distance smaller than or equal to $2 \delta$ from an essential vertex, and all negligible vertices have valence 2 , and are at a distance greater than $2 \delta$ from any essential vertex, it follows that any isometry of $\Gamma_{1}(X)$ which preserves essential vertices must also preserve auxiliary and negligible vertices. Therefore, any extension of $g$ will map basic paths to basic paths and bridges to bridges, and so will be equivalent to $\bar{g}$.

Lemma 23. Let $Y$ be a geodesic $\Delta$-hyperbolic metric space with $X$ isometrically embedded into $Y$. Then $\Gamma_{1}(X)$ is quasi-isometrically embedded into $Y$ and the constants of the quasi-isometry depend only on $\delta$ and $\Delta$

Proof. Denote by $d^{*}$ the metric on $Y$ and let $\psi: X \rightarrow Y$ be the embedding of $X$ into $Y$. We can extend it to $\Gamma_{1}(X)$ in an obvious way by mapping geodesics to geodesics. Basic paths are embedded isometrically, and the images of bridges cannot be longer than $4 \Delta$. If we have a path in $\Gamma_{1}(X)$, it maps to a path of length at least multiplied by $\min \left\{1, \frac{\Delta}{\delta}\right\}$, so $\min \left\{1, \frac{\delta}{\Delta}\right\} d^{*}(\psi(x), \psi(y)) \leqslant d(x, y)$ for any $x, y \in \Gamma_{1}(X)$. On the other hand, if we have a path $\gamma$ which consists of $n$ bridges and only one segment of a basic path, since basic paths are isometrically embedded, we have that $\left.\psi\right|_{\gamma}$ is a $(1, n \cdot \max \{4 \delta, 4(\Delta-\delta)\})$-quasiisometric embedding and $\psi(\gamma)$ is a $(1, n \cdot 4 \Delta)$-quasi-geodesic. Using the same argument as in the proof of Lemma 21, we have that $\psi$ extended to $\Gamma_{1}(X)$ is a $(1,12(\Delta+\max \{\delta, \Delta-\delta\}))$-quasi-isometric embedding.

Lemma 24. Let $Y$ be a $\Delta$-hyperbolic metric space with $X$ quasi-isometric to $Y$. Then $\Gamma_{1}(X)$ is quasi-isometric to $\Gamma_{1}(Y)$ and the constants of the quasi-isometry depend only on $\delta, \Delta$, and the constants of the quasi-isometry between $X$ and $Y$.

Proof. Let $\psi: X \rightarrow Y$ be a $(\lambda, k)$-quasi-isometry. Recall that $\bar{d}$ is the metric on $\Gamma_{1}(X)$ and denote by $\overline{d^{*}}$ the metric on $\Gamma_{1}(Y)$. We can build a map $\Psi: \Gamma_{1}(X) \rightarrow$ $\Gamma_{1}(Y)$ as follows. Define $\Psi(x)=\psi(x)$ for $x \in X$. Next, we can approximate how auxiliary vertices should be mapped based on how far they are from the
endpoints of the basic paths they lie on. In other words, if $z$ is on a basic path $[x, y]$ at a distance of $d$ from $x$, then $\Psi(z)$ should be the auxiliary vertex on $[\Psi(x), \Psi(y)]$ at a distance of $d \cdot \frac{\bar{d}(x, y)}{\overline{d^{*}}(\Psi(x), \Psi(y))}$ from $\Psi(x)$. Finally, we map bridges to bridges approximating how integer distances should be mapped. To see that $\Psi$ is a quasi-isometry, notice that for any $x, y \in X$ we have that $\left.\Psi\right|_{[x, y]}$ is a $(\lambda, k+1)$-quasi-isometry, then reuse the same paths we have above to obtain bounds for $\bar{d}(a, b)$ in terms of $\overline{d^{*}}(\Psi(a), \Psi(b))$ by using the fact that bridges have length $4 \delta$, their images have length $4 \Delta$ and basic paths are quasi-isometrically mapped.

Corollary 6. Let $Y$ be a geodesic $\Delta$-hyperbolic metric space such that $X$ is quasi-isometrically embedded into $Y$. Then $\Gamma_{1}(X)$ is quasi-isometrically embedded into $Y$.

## $7.2 \quad \Gamma_{2}(X)$

The space $\left(\Gamma_{1}(X), d\right)$ introduced in the previous subsection is constructed so that geodesics between any two essential vertices $x$ and $y$ almost never include any other essential vertices. The only exception happens when there exists an essential vertex $z$ in $\Gamma_{1}(X)$ such that $d(x, y)=d(x, z)+d(y, z)$. This property makes $\Gamma_{1}(X)$ an artificially "thinned" weighted complete graph.

The goal of this subsection is to construct another completion $\Gamma_{2}(X)$ of $X$ in the case when $X$ is regular, that is, when the following condition holds
(RS) $\forall x, y, z \in X, \exists v \in X:$

$$
\begin{gathered}
d(x, v)+d(v, y) \leqslant d(x, y)+2 \delta, \quad d(x, v)+d(v, z) \leqslant d(x, z)+2 \delta, \\
d(y, v)+d(v, z) \leqslant d(y, z)+2 \delta .
\end{gathered}
$$

Any point $v$ from the definition above we call a mid-point of $\{x, y, z\}$.
The process of building $\Gamma_{2}(X)$ is considerably more involved but the graph itself appears to be more natural than $\Gamma_{1}(X)$. At first, for $n \in \mathbb{N}$ we build an auxiliary graph $\Gamma_{2}^{n}(X)$ using $\Gamma_{1}(X)$.

Recall that $\delta^{\prime}$ is the hyperbolicity constant for $\Gamma_{1}(X)$ (see Proposition (1). Now, define $\mathcal{H}$ to be the maximal Hausdorff distance between a geodesic and a $\left(4 \delta^{\prime}, 240 \delta^{\prime 3}+108 \delta^{\prime 2}\right)$-quasi-geodesic in $\Gamma_{1}(X)$ and set $B=2 \mathcal{H}+2 \delta^{\prime}$.

Set $\Gamma_{2}^{n}(X)=X$ and call all vertices of $X$ essential.
(1) For any essential vertices $x, y$ with $d(x, y)=n$ add to $\Gamma_{2}^{n}(X)$ a $\mathbb{Z}$-path of length $n$ connecting $x$ and $y$. This path we call basic and its vertices auxiliary.
(2) For a pair of essential vertices $\{x, y\}$ such that $d(x, y)=n$ and another essential vertex $z$, consider all mid-points $v$ for $\{x, y, z\}$. If there exists some $v$ such that $d(x, v), d(y, v) \geqslant 2 \delta$ then we remove the basic path $[x, y]$.
(3) Repeat step (2) for the pair $\{x, y\}$ and all essential vertices $z$.
(4) Repeat steps (2) and (3) for all pairs of essential vertices $\{x, y\}$ with $d(x, y)=n$.
(5) Repeat steps (1)-(4) for all integers smaller than $n$ in descending order.
(6) Finally, if the distance in $\Gamma_{1}(X)$ between an auxiliary vertex $x$ and a basic path $p$, which has not been removed on previous steps, is smaller than $B$, add a $\mathbb{Z}$-path, which we call a bridge and whose vertices we call negligible, connecting $x$ and the closest to it vertex $y$ on $p$ (if there are two such vertices on $p$ then add a bridge for each one) in $\Gamma_{2}^{n}(X)$. The length of the added bridge connecting $x$ and $y$ is equal to $d(x, y)$ in $\Gamma_{1}(X)$. See Figure 3.


Figure 3: Construction of $\Gamma_{2}^{n}(X)$
(7) We extend the metric $d: X \rightarrow \mathbb{Z}$ to the metric $d_{2}^{n}: \Gamma_{2}^{n}(X) \rightarrow \mathbb{Z}$ as follows:
(a) the distance between two essential vertices is inherited from $X$,
(b) the distance between an auxiliary vertex to the adjacent essential vertices is defined by construction, hence, the distance from an auxiliary vertex to any other either essential, or auxiliary vertex is also defined (as the minimum of lengths of paths connecting them),
(c) the distance from a negligible vertex to the adjacent auxiliary vertices is defined by construction, so, the distance from a negligible vertex to any other vertex of $\Gamma_{2}^{n}(X)$ is also defined.

Lemma 25. Suppose for some pair $x, y$ there exists $z$ such that there is a midpoint $v$ of $\{x, y, z\}$ chosen on step (2) above. Then $d(x, v)<d(x, y), d(y, v)<$ $d(x, y)$.

Proof. Suppose on the contrary that $d(y, v) \geqslant d(x, y)$. From regularity of $X$ it follows that $d(x, v) \leqslant 2 \delta$, hence, by construction, $v$ could not be chosen on step 2, a contradiction.

Remark 4. 1. From Lemma 25 it follows that the algorithm of constructing $\Gamma_{2}^{n}(X)$ is correct in the sense that there is no risk of re-adding a geodesic that was previously removed and the process ends since we always split geodesics into geodesics of strictly shorter integer length.
2. Lemma 25 also explains why a geodesic is removed in the process only if there exists a mid-point of $\{x, y, z\}$ which is sufficiently far away from $x$, yand $z$. Unfortunately, it also means that no geodesic shorter than $2 \delta$ is ever removed (since any path through an acceptable mid-point has length of at most $4 \delta$, so it is contained in the $2 \delta$-neighborhood of the two points), so, $\delta$-neighborhoods of points in $\Gamma_{1}(X)$ and $\Gamma_{2}^{n}(X)$ are essentially the same.
3. Finally, from Lemma 25 it follows that $\Gamma_{2}^{n}(X)$ is connected since a basic path is removed only if there are two shorter paths connecting the same vertices which are added later on.

Observe that $\Gamma_{2}^{n}(X), n \in \mathbb{N}$ can be viewed as a graph whose vertices $V\left(\Gamma_{2}^{n}(X)\right)$ are points of $\Gamma_{2}^{n}(X)$ and edges $E\left(\Gamma_{2}^{n}(X)\right)$ are pairs of points at distance 1 from each other.

Lemma 26. If $n<m$ then

$$
V\left(\Gamma_{2}^{n}(X)\right) \subseteq V\left(\Gamma_{2}^{m}(X)\right) \quad \text { and } \quad E\left(\Gamma_{2}^{n}(X)\right) \subseteq E\left(\Gamma_{2}^{m}(X)\right)
$$

Proof. In the process of building $\Gamma_{2}^{m}(X)$ we eventually run steps (1)-(5) for $m$. Thus, we add all the geodesics we would add and remove all the geodesics we would remove in the process of building $\Gamma_{2}^{n}(X)$ (since we always split geodesics into shorter ones). Since the set of geodesics between essential vertices in $\Gamma_{2}^{n}(X)$ is contained in the set of geodesics between essential vertices of $\Gamma_{2}^{m}(X)$, the same relation is going to hold for the sets of bridges in both graphs. The required statement follows.

Observe that Lemma 26 does not imply that $\Gamma_{2}^{n}(X)$ isometrically embeds into $\Gamma_{2}^{m}(X)$. At the same time, it implies that if $X$ is not bounded then the sequence $\left\{\Gamma_{2}^{n}(X)\right\}_{n \in \mathbb{N}}$ converges to a graph $\Gamma_{2}(X)$ whose vertices and edges are defined by

$$
V\left(\Gamma_{2}(X)\right)=\bigcup_{n \in \mathbb{N}} V\left(\Gamma_{2}^{n}(X), \quad E\left(\Gamma_{2}(X)\right)=\bigcup_{n \in \mathbb{N}} E\left(\Gamma_{2}^{n}(X)\right)\right.
$$

If $X$ is bounded and has diameter $d$ then we set $\Gamma_{2}(X)=\Gamma_{2}^{d}(X)$.
Remark 5. If $X$ is geodesic then $\Gamma_{2}(X)=X$. Indeed, if $X$ is geodesic then all basic paths are removed except those of length 1, and all bridges already exist since $X$ is $\delta$-hyperbolic.
Remark 6. Given a specific $X$, then there will exist an unique extension $\Gamma_{2}(X)$. This is less obvious than in the case of $\Gamma_{1}$, but still pretty easy to see. Notice that, in the construction of $\Gamma_{2}^{n}(X)$, steps (1) and (2) do not depend on the order in which we consider pairs of essential vertices, steps (3)-(5) are a repetition of steps (1) and (2), and step (6) depends entirely on the image of the graph built up until then in $\Gamma_{1}(X)$, which we have already shown to be unique.
$\Gamma_{2}(X)$, being an union of uniquely determined extensions of $X$, will itself be an unique extension of $X$.

Suppose $n \in \mathbb{N}$. Let $\mathcal{N}$ be a set which initially contains only $n$ and we apply to all elements of $\mathcal{N}$ the following steps. For each $k \in \mathcal{N}$, if $k>2 \delta$ then replace it by two new numbers $k_{1}$ and $k_{2}$ such that $k_{1}, k_{2}<k$ and $k \leqslant k_{1}+k_{2} \leqslant k+2 \delta$. Continue this splitting process until all elements of $\mathcal{N}$ are smaller than $2 \delta$. Denote by $\tau(n)$ the maximal sum obtained by the above algorithm starting from $n$.
Lemma 27. $\tau(n) \leqslant 4 \delta n$.
Proof. Observe that for $n \leqslant 2 \delta$ we have $\tau(n)=n \leqslant 4 \delta n$. Next, for $n>2 \delta$ there exist $i, j \in \mathbb{N}$ such that $\tau(n)=\tau(i)+\tau(j)$ with $j \leqslant n-i+2 \delta$. We are gong to prove that for $n>2 \delta$

$$
\tau(n) \leqslant 4 \delta n-8 \delta^{2}<4 \delta n
$$

Assume without loss of generality that $j \geq i$.
First of all, for $n=2 \delta+1$, any $n_{1}, n_{2}$ will be smaller than $2 \delta$, so the ideal is to use $n_{1}=n_{2}=2 \delta$ giving us $\tau(2 \delta+1)=4 \delta=4 \delta(2 \delta+1)-8 \delta^{2}$.

Suppose now that $\tau(k) \leq 4 \delta k-8 \delta^{2}$ for any $2 \delta+1 \leqslant k \leqslant n-1$. We have $\tau(n)=\tau(j)+\tau(i)$ with $n-1 \geqslant j \geqslant i$. It is worth noticing that $j \leqslant n-1$, so $j+2 \delta+1 \leqslant n+2 \delta$. Thus, if we can choose $j>2 \delta$, we can choose $i>2 \delta$ as well. Since $\tau$ is an increasing function, if $n>2 \delta+1$ then we can assume that $i, j>2 \delta$. So, we have that

$$
\begin{gathered}
\tau(n)=\tau(i)+\tau(j) \leqslant 4 \delta i+4 \delta j-16 \delta^{2} \leqslant 4 \delta(n-j+2 \delta)+4 \delta j-16 \delta^{2} \\
=4 \delta n-8 \delta^{2} \leqslant 4 \delta n
\end{gathered}
$$

For any $v, w \in \Gamma_{2}(X)$ define $[v, w]_{2}$ to be a geodesic between $v$ and $w$ either in $\Gamma_{2}(X)$ and $\varphi[v, w]_{2}$ its embedding into $\Gamma_{1}(X)$. To simplify notation, let $d_{2}$ represent the graph metric on $\Gamma_{2}(X)$ and $d_{1}$ the metric on $\Gamma_{1}(X)$. Let $k=30 \delta^{\prime 2}$. If $[x, y]_{2}$ contains no other essential vertices so that it coincides with a geodesic in $\Gamma_{1}(X)$, we say that $[x, y]_{2}$ is long if $d(x, y)>k$ and short otherwise.

Lemma 28. (i) Let $[x, y]_{2}$ be long and $z$ be an essential vertex. Then either $\varphi\left([x, y]_{2} \cup[x, z]_{2}\right)$, or $\varphi\left([x, y]_{2} \cup[y, z]_{2}\right)$ is a $(1,6 \delta)$-quasi-geodesic.
(ii) Let $\left[x, x^{\prime}\right]_{2}$ and $\left[y, y^{\prime}\right]_{2}$ be long. Then there exist $\chi \in\left\{x, x^{\prime}\right\}$ and $\psi \in\left\{y, y^{\prime}\right\}$ such that $\varphi\left(\left[x, x^{\prime}\right]_{2} \cup[\chi, \psi]_{2} \cup\left[y, y^{\prime}\right]_{2}\right)$ is a $(1,12 \delta)$-quasi-geodesic.

Proof. (i) Observe that $[x, y]_{2}$ is long and it has not been split into two segments in the process of building the graph. It implies that for any essential vertex $v$, any mid-point of $\{x, y, v\}$ is within a $2 \delta$-ball centered either at $x$, or at $y$. Suppose, without loss of generality, that there exists a mid-point $m$ of $\{x, y, z\}$ in the $2 \delta$-neighborhood of $x$. We have that $\varphi\left([z, m]_{2} \cup[m, y]_{2}\right)$ is a $(1,2 \delta)$ -quasi-geodesic and that $d(x, m) \leqslant 2 \delta$, but this implies that $\varphi\left([z, m]_{2} \cup[m, x]_{2} \cup\right.$ $\left.[x, m]_{2} \cup[m, y]_{2}\right)$ is a $(1,6 \delta)$-quasi-geodesic. Hence, so is $\varphi\left([z, x]_{2} \cup[x, y]_{2}\right)$.
(ii) By the above result, there exists $\chi \in\left\{x, x^{\prime}\right\}$ such that $\varphi\left(\left[y^{\prime}, \chi\right]_{2} \cup\left[x, x^{\prime}\right]_{2}\right)$ is a $(1,6 \delta)$-quasi-geodesic. Suppose, without loss of generality, that $\chi=x^{\prime}$. At the same time, similarly there exists $\psi \in\left\{y, y^{\prime}\right\}$ such that $\varphi\left(\left[x^{\prime}, \psi\right]_{2} \cup\left[y^{\prime}, y\right]_{2}\right)$ is a $(1,6 \delta)$-quasi-geodesic. If $\psi=y$ then the path $\varphi\left(\left[x, x^{\prime}\right]_{2} \cup\left[x^{\prime}, y\right]_{2} \cup\left[y, y^{\prime}\right]_{2}\right)$ is simply the $(1,6 \delta)$ quasi-geodesic $\varphi\left(\left[y^{\prime}, x^{\prime}\right]_{2} \cup\left[x^{\prime}, x\right]_{2}\right)$ with the segment $\varphi\left(\left[y^{\prime}, x^{\prime}\right]_{2}\right)$ replaced by a $(1,6 \delta)$-quasi-geodesic. It follows that the path in question is a $(1,12 \delta)$ quasi-geodesic.

Suppose that $\chi=x$. There exists some $\psi^{\prime}$ such that $\varphi\left(\left[x, \psi^{\prime}\right]_{2} \cup\left[y, y^{\prime}\right]_{2}\right)$ is a $(1,6 \delta)$-quasi-geodesic. If $\psi^{\prime}=y^{\prime}$ then the path $\varphi\left(\left[x, x^{\prime}\right]_{2} \cup\left[x^{\prime}, y^{\prime}\right]_{2} \cup\left[y^{\prime}, y\right]_{2}\right)$ is merely the $(1,6 \delta)$-quasi-geodesic $\varphi\left(\left[x, y^{\prime}\right]_{2} \cup\left[y^{\prime}, y\right]_{2}\right)$ with the segment $\varphi\left(\left[x, y^{\prime}\right]_{2}\right)$ replaced by a $(1,6 \delta)$-quasi-geodesic. It follows that the path in question is a $(1,12 \delta)$-quasi-geodesic.

Suppose that $\chi=x^{\prime}, \psi=y^{\prime}, \psi^{\prime}=y$. Both $\varphi\left(\left[y^{\prime}, y\right]_{2} \cup[y, x]_{2}\right)$ and $\varphi\left(\left[y^{\prime}, x^{\prime}\right]_{2} \cup\left[x^{\prime}, x\right]_{2}\right)$ are (1,6 $)$-quasi-geodesic, so, $c_{y^{\prime}}\left(x^{\prime}, x\right) \geqslant d\left(y^{\prime}, x^{\prime}\right)-3 \delta$ and $c_{y^{\prime}}(y, x) \geqslant d\left(y^{\prime}, y\right)-3 \delta$, which implies that $c_{y^{\prime}}\left(y, x^{\prime}\right) \geqslant \min \left\{d\left(y^{\prime}, y\right), d\left(y^{\prime}, x^{\prime}\right)\right\}-$ $2 \delta^{\prime}-3 \delta \geqslant \min \left\{d\left(y^{\prime}, y\right), d\left(y^{\prime}, x^{\prime}\right)\right\}-5 \delta^{\prime}$. But $\varphi\left(\left[y, y^{\prime}\right]_{2} \cup\left[y^{\prime}, x^{\prime}\right]_{2}\right)$ is a $(1,6 \delta)-$ quasi-geodesic, which means that $d\left(x^{\prime}, y\right) \geqslant d\left(x^{\prime}, y^{\prime}\right)+d\left(y^{\prime}, y\right)-6 \delta$. Now, $d\left(x^{\prime}, y^{\prime}\right)=d\left(x^{\prime}, y^{\prime}\right)+d\left(y^{\prime}, y\right)-2 c_{y^{\prime}}\left(x^{\prime}, y\right)$, so,

$$
3 \delta^{\prime} \geqslant 3 \delta \geqslant c_{y^{\prime}}\left(x^{\prime}, y\right) \geqslant \min \left\{d\left(y^{\prime}, y\right), d\left(y^{\prime}, x^{\prime}\right)\right\}-5 \delta^{\prime}
$$

so,

$$
8 \delta^{\prime} \geqslant c_{y^{\prime}}\left(x^{\prime}, y\right) \geqslant \min \left\{d\left(y^{\prime}, y\right), d\left(y^{\prime}, x^{\prime}\right)\right\}
$$

Since $\left[y, y^{\prime}\right]$ is long, it follows that $d\left(y^{\prime}, x^{\prime}\right) \leqslant 8 \delta^{\prime}$.
Consider $\Delta\left(x^{\prime}, y, x\right)$. We have that $\varphi\left(\left[y^{\prime}, y\right]_{2} \cup[y, x]_{2}\right)$ is a $(1,6 \delta)$-quasigeodesic, so $y$ is at a distance of at most $3 \delta$ from $\Delta_{I}\left(x^{\prime}, y, x\right)$, and so it is at a distance of at most $3 \delta+\delta^{\prime} \leqslant 4 \delta^{\prime}$ from a point $z \in \varphi\left(\left[y^{\prime}, x\right]_{2}\right)$. Furthermore, since $d\left(x^{\prime}, y^{\prime}\right) \leqslant 8 \delta^{\prime}$, letting $z^{\prime}$ be a point on $\varphi\left(\left[y^{\prime}, x^{\prime}\right]_{2} \cup\left[x^{\prime}, x\right]_{2}\right)$ at a distance


Figure 4: Red curves here are $(1,6 \delta)$-quasi-geodesics.
of smaller than $\delta^{\prime}$ from $z$, we get $z^{\prime} \in\left[y^{\prime}, y\right]$ since $30 \delta^{\prime 2} \leqslant d\left(y, y^{\prime}\right) \leqslant d\left(y, z^{\prime}\right)+$ $d\left(z^{\prime}, y^{\prime}\right) \leqslant 5 \delta^{\prime}+d\left(z^{\prime}, y^{\prime}\right)$ and $d\left(x^{\prime}, y^{\prime}\right) \leqslant 13 \delta^{\prime}$. This implies that $\left[x^{\prime}, y\right] \cup[y, x]$ is a $\left(1,10 \delta^{\prime}\right)$-quasi-geodesic. By the alternative definition of regularity that any mid-point of $\left\{x^{\prime}, y, x\right\}$ must be within the $7 \delta^{\prime}$-neighborhood of $\Delta_{I}\left(x^{\prime}, y, x\right)$. But since $\left[x^{\prime}, y\right] \cup[y, x]$ is a $\left(1,10 \delta^{\prime}\right)$-quasi-geodesic, it follows that the vertices of $\Delta_{I}\left(x^{\prime}, y, x\right)$ which lie on $\left[x^{\prime}, y\right]$ and $[x, y]$ are at a distance of at most $5 \delta^{\prime}$ from $y$. So, $\Delta_{I}\left(x^{\prime}, y, x\right)$ must be within the $6 \delta^{\prime}$-ball around $y$, so any mid-point of $\left\{x^{\prime}, y, x\right\}$ is within the $13 \delta^{\prime}$-ball around $y$.

Now, from the fact that is $\left[x, x^{\prime}\right]$ is long it follows that any such mid-point must be within $2 \delta^{\prime}$ of either $x^{\prime}$, or $x$. It cannot be near $x^{\prime}$ since $d\left(x^{\prime} y\right) \geqslant k-8 \delta^{\prime}>$ $10 \delta^{\prime}$, so, any such mid-point must be at a distance of at most $2 \delta^{\prime}$ from $x$, and It follows that $d(x, y) \leqslant 15 \delta^{\prime}$. However, $d_{2}(x, y) \geqslant 2 k>4 \delta^{\prime} \cdot 15 \delta^{\prime} \geqslant 4 \delta^{\prime} d(x, y)$, a contradiction. Thus, the only possible cases occur when there exists one such path which is a $(1,12 \delta)$-quasi-geodesic.

Lemma 29. Let $v, w$ be essential vertices. The embedding of $[v, w]_{2}$ into $\Gamma_{1}$ is $a\left(4 \delta, 48 \delta^{2}+(8 \delta+2) k\right)$-quasi-geodesic.

Proof. By Lemma 27, for any two essential vertices $v$ and $w$ we have $d_{1}(v, w) \leqslant$ $d_{2}(v, w) \leqslant 4 \delta d_{1}(v, w)$. Suppose that $a$ and $b$ are vertices such that $\varphi\left([v, a]_{2} \cup\right.$ $\left.[a, b]_{2} \cup[b, w]_{2}\right)=\varphi\left([v, w]_{2}\right)$. We can assume that there exist no essential vertices between $a$ and $v$, or $b$ and $w$, and that $a \in\left[v, v^{\prime}\right]_{2}, b \in\left[w, w^{\prime}\right]_{2}$ with $v^{\prime}$, $w^{\prime}$ being essential vertices. We have three cases.

Case I. If both $\left[v, v^{\prime}\right]$ and $\left[w, w^{\prime}\right]$ are short then we have

$$
\begin{gathered}
d_{1}(\varphi(a), \varphi(b)) \leqslant d_{2}(a, b)=d_{2}\left(a, v^{\prime}\right)+d_{2}\left(v^{\prime}, w^{\prime}\right)+d_{2}\left(w^{\prime}, b\right) \\
\leqslant d_{1}\left(\varphi(a), v^{\prime}\right)+4 \delta d_{1}\left(v^{\prime}, w^{\prime}\right)+d_{1}\left(w^{\prime}, \varphi(b)\right) \\
\leqslant 4 \delta\left(d_{1}(a, b)+2 k\right)+2 k=4 \delta d_{1}(a, b)+(8 \delta+2) k
\end{gathered}
$$

Case II. If $\left[v, v^{\prime}\right]_{2}$ is short and $\left[w, w^{\prime}\right]_{2}$ is long (without loss of generality) then that there exists $\omega \in\left\{w, w^{\prime}\right\}$ such that $\varphi\left(\left[v^{\prime}, \omega\right]_{2} \cup\left[w, w^{\prime}\right]_{2}\right)$ is a $(1,6 \delta)$ -quasi-geodesic. Thus, we have

$$
\begin{gathered}
d_{2}(a, b) \leqslant d_{2}\left(a, v^{\prime}\right)+d_{2}\left(v^{\prime}, \omega\right)+d_{2}(\omega, b) \leqslant d_{1}\left(a, \varphi\left(v^{\prime}\right)\right)+4 \delta d_{1}\left(v^{\prime}, \omega\right)+d_{1}(\omega, \varphi(b)) \\
\leqslant k+4 \delta\left(d_{1}\left(v^{\prime}, \omega\right)+d_{1}(\omega, \varphi(b))\right) \leqslant k+4 \delta\left(d_{1}\left(v^{\prime}, \varphi(b)\right)+6 \delta\right) \\
\leqslant k+4 \delta(d(a, b)+k+6 \delta)=4 \delta d(a, b)+24 \delta^{2}+(1+4 \delta) k
\end{gathered}
$$

Case III. If both $\left[v, v^{\prime}\right]_{2}$ and $\left[w, w^{\prime}\right]_{2}$ are long then there exist $\phi \in\left\{v, v^{\prime}\right\}$ and $\omega \in\left\{w, w^{\prime}\right\}$ such that $\varphi\left(\left[v, v^{\prime}\right]_{2} \cup[\phi, \omega]_{2} \cup\left[w, w^{\prime}\right]_{2}\right)$ is a (1, 12 $\delta$ )-quasi-geodesic. Thus, we have

$$
\begin{gathered}
d_{2}(a, b) \leqslant d_{2}(a, \phi)+d_{2}(\phi, \omega)+d_{2}(\omega, b) \leqslant d_{1}(\varphi(a), \phi)+4 \delta d_{1}(\phi, \omega)+d_{1}(\omega, \varphi(b)) \\
\leqslant 4 \delta\left(d_{1}(\varphi(a), \phi)+d_{1}(\phi, \omega)+d_{1}(\omega, \varphi(b))\right) \leqslant 4 \delta\left(d_{1}(\varphi(a), \varphi(b))+12 \delta\right) \\
=4 \delta d_{1}(\varphi(a), \varphi(b))+48 \delta^{2}
\end{gathered}
$$

Remark 7. Since for any essential vertices $x, y$, the embedding of $[x, y]_{2}$ into $\Gamma_{1}(X)$ is a $\left(4 \delta, 240 \delta^{\prime 3}+60 \delta^{\prime 2}+48 \delta^{2}\right)$-quasi-geodesic, there exists $\mathcal{H}$ such that the Hausdorff distance between $\varphi[x, y]_{2}$ and $[x, y]_{1}$ is at most $\mathcal{H}$.

Lemma 30. Let $v, w$ be vertices of $\Gamma_{2}(X)$.
(i) If $v$ and $w$ are both auxiliary, then $[v, w]_{2}$ is a $\left(4 \delta, 240 \delta^{\prime 3}+60 \delta^{\prime 2}+48 \delta^{2}+\right.$ $\left.2 \mathcal{H}+\delta^{\prime}+1\right)$-quasi-geodesic.
(ii) If $v$ is essential and $w$ is auxiliary then $[v, w]_{2}$ is a $\left(4 \delta^{\prime}, 240 \delta^{\prime 3}+60 \delta^{\prime 2}+\right.$ $\left.48 \delta^{2}+4 \mathcal{H}+4 \delta^{\prime}+2\right)$-quasi-geodesic.

Proof. (ii) By the construction, $w$ is on the geodesic in $\Gamma_{2}(X)$ connecting two essential vertices $w_{1}$ and $w_{2}$. Consider the geodesic triangle $\Delta\left(v, w_{1}, w_{2}\right)$. We have that $w$ is at a distance of at most $H$ from some point on $\left[w_{1}, w_{2}\right]$ which is at a distance of at most $\delta^{\prime}$ from either $\left[v, w_{1}\right]$, or $\left[v, w_{2}\right]$, which itself is at a distance of at most $\mathcal{H}$ from $\left[v, w_{i}\right]_{2}$. So, $w$ is at a distance of at most $2 \mathcal{H}+\delta^{\prime}$ from either $\left[v, w_{1}\right]_{2}$, or $\left[v, w_{2}\right]_{2}$. By the construction, we have a bridge in $\Gamma_{2}(X)$ of length at most $2 \mathcal{H}+\delta^{\prime}+1$ between $w$ and some auxiliary vertex $w^{\prime}$ on $\left[v, w_{i}\right]_{2}$. It follows that the embedding of $\left[w, w^{\prime}\right]_{1} \cup\left[w^{\prime}, v\right]_{1}$ is a $\left(4 \delta, 240 \delta^{\prime 3}+60 \delta^{\prime 2}+48 \delta^{2}+2 \mathcal{H}+\delta^{\prime}+1\right)$ -quasi-geodesic.


Figure 5: Case (ii) in the proof of Lemma 30. Wavy lines are quasi-geodesics
(i) There exist essential vertices $v_{1}, v_{2}, w_{1}, w_{2}$ such that $v \in\left[v_{1}, v_{2}\right]_{2}, w \in$ [ $\left.w_{1}, w_{2}\right]_{2}$. Consider the geodesic square $\left\{v_{1}, v_{2}, w_{2}, w_{1}\right\}$ (linked together in the given order). Thus, $v$ and $w$ must be within a distance of $\mathcal{H}$ from some points $v^{\prime}$ and $w^{\prime}$ on $\left[v_{1}, v_{2}\right]$ and $\left[w_{1}, w_{2}\right]$ respectively. Furthermore, both $v^{\prime}$ or $w^{\prime}$ must be within a distance of $2 \delta^{\prime}$ from either $\left[v_{1}, w_{1}\right]$, or $\left[v_{2}, w_{2}\right]$. If they are both within $2 \delta^{\prime}$ from the same edge, say $\left[v_{1}, w_{1}\right]$, then they are within $2 \delta^{\prime}+2 \mathcal{H}$ from $\left[v_{1}, w_{1}\right]_{2}$, so that both $v$ and $w$ are within at most $2 \mathcal{H}+2 \delta^{\prime}$ from $\left[v_{1}, w_{1}\right]_{2}$. Hence, there are bridges of length at most $2 \delta^{\prime}+2 \mathcal{H}+1$ between them and that geodesic, thus, $[v, w]_{2}$ is a $\left(4 \delta^{\prime}, 240 \delta^{\prime 3}+108 \delta^{\prime 2}+4 \mathcal{H}+4 \delta^{\prime}+2\right)$-quasi-geodesic.

Suppose on the contrary that, without loss of generality, $v^{\prime}$ is within $2 \delta^{\prime}$ from $\left[v_{1}, w_{1}\right]$, $w^{\prime}$ is within $2 \delta^{\prime}$ from $\left[v_{2}, w_{2}\right]$, and neither one is within $2 \delta^{\prime}$ of the other edge. Consider the triangles $\Delta\left(v_{1}, v_{2}, w_{2}\right)$ and $\Delta\left(v_{1}, w_{1}, w_{2}\right)$. By the
assumption, both $v^{\prime}$ and $w^{\prime}$ must be within $\delta^{\prime}$ from $\left[v_{1}, w_{2}\right]$, which itself is at a Hausdorff distance of at most $\mathcal{H}$ from $\left[v_{1}, w_{2}\right]_{2}$. Thus, there exist bridges of length at most $2 \mathcal{H}+\delta^{\prime}+1$ from $v$ and $w$ to some $v^{\prime \prime}$ and $w^{\prime \prime}$ on $\left[v_{1}, w_{2}\right]_{2}$. It follows that the embedding of $[v, w]_{2}$ is a $\left(4 \delta^{\prime}, 240 \delta^{\prime 3}+108 \delta^{\prime 2}+4 \mathcal{H}+2 \delta^{\prime}+1\right)$ -quasi-geodesic.


Figure 6: Case (i) in the proof of Lemma 30. Wavy lines are quasi-geodesics

Proposition 5. $\Gamma_{2}(X)$ is $\delta^{\prime \prime}$-hyperbolic with $\delta^{\prime \prime}=240 \delta^{\prime 3}+64 \delta^{\prime 2}+48 \delta^{2}+8 \mathcal{H}+$ $8 \delta^{\prime}+2$.

Proof. Let $\varphi: \Gamma_{2}(X) \rightarrow \Gamma_{1}(X)$ be the function sending vertices of $\Gamma_{2}(X)$ to their embedding in $\Gamma_{1}(X)$. We have that the embedding of any geodesic of $\Gamma_{2}(X)$ whose endpoints are either essential, or auxiliary vertices is a $\left(4 \delta, 240 \delta^{\prime 3}+60 \delta^{\prime 2}+\right.$ $\left.48 \delta^{2}+4 \mathcal{H}+4 \delta^{\prime}+2\right)$-quasi-geodesic. Suppose then that a geodesic has negligible vertices as endpoints. Negligible vertices always have valency 2 and they belong to paths of length at most $2 \mathcal{H}+2 \delta^{\prime}+1$ connecting auxiliary vertices. Thus, if $a, b$ are the endpoints, there exist auxiliary vertices $a^{\prime}, b^{\prime}$ such that $[a, b]_{2}=$ $\left[a, a^{\prime}\right]_{2} \cup\left[a^{\prime}, b^{\prime}\right]_{2} \cup\left[b^{\prime}, b\right]_{2}$ with $d\left(a, a^{\prime}\right), d\left(b, b^{\prime}\right) \leqslant 2 \mathcal{H}+2 \delta^{\prime}$. Thus, the embedding of any geodesic of $\Gamma_{2}(X)$ into $\Gamma_{1}$ is a $\left(4 \delta^{\prime}, 240 \delta^{\prime 3}+108 \delta^{\prime 3}+8 \mathcal{H}+8 \delta^{\prime}+2\right)$-quasigeodesic. It follows that $\varphi$ is a $\left(4 \delta, 240 \delta^{\prime 3}+60 \delta^{\prime 2}+48 \delta^{2}+8 \mathcal{H}+8 \delta^{\prime}+2\right)$-quasiisometric embedding, and the result follows.

Remark 8. Since the maximal Hausdorff distance between a geodesic and a $\left(4 \delta, 240 \delta^{\prime 3}+60 \delta^{\prime 2}+48 \delta^{2}\right)$-quasi-geodesic in a $\delta^{\prime}$-hyperbolic space is polynomial in $\delta^{\prime}$ and $\delta$, and remembering that $\delta^{\prime}$ is polynomial in $\delta$, it follows that $\delta^{\prime \prime}$ depends polynomially on $\delta$.

Corollary 7. $\Gamma_{2}(X)$ is quasi-isometrically embedded into $\Gamma_{1}(X)$ and all universal properties of $\Gamma_{1}(X)$ extend to $\Gamma_{2}(X)$.

Lemma 31. Let $g$ be an isometry of $X$. There exists an unique isometry $\bar{g}$ of $\Gamma_{2}(X)$ such that $\left.\bar{g}\right|_{X}=g$

Proof. If $g$ is an isometry of $X$ then it preserves mid-points of triangles. It follows that if $[x, y]$ is removed, so is $g[x, y]$. Finally, since by Lemma 22, $g$ can be extended to $\Gamma_{1}(X)$ and that bridges are placed based upon proximity in $\Gamma_{1}(X)$, then if we have a bridge between $a$ and $b$, we also have a bridge between $g a$ and $g b$ defined by extending the isometry on the essential vertices to the basic paths joining them.

Uniqueness of $\bar{g}$ follows easily from the fact that it is defined entirely by the action of $g$ on $X$ and the fact that the extension of $g$ to $\Gamma_{1}(X)$ is itself unique.

## $7.3 \quad \beta(X)$

If we have a hyperbolic $\mathbb{R}$-metric space, it is possible to define an embedding into a geodesic hyperbolic $\mathbb{R}$-metric space analogous to the construction of $\Gamma_{1}(X)$ in the discrete case. Since the two constructions are very similar, the proofs will often be done by reference to the case of $\Gamma_{1}(X)$. The same terminology will be used to make transferring those proofs easier. Notice that a continuous analog of $\Gamma_{2}(X)$ is not possible. There is no guarantee that our algorithm of breaking down geodesics ever stops since arbitrarily small distances can occur.

Let now $X$ be an $\mathbb{R}$-metric space. Define a band complex $\beta(X)$ as follows:
(1) Define first $\beta(X)=X$ and define these points as essential points.
(2) For any pair $(x, y)$ of essential points, add to our complex a copy of the interval $[0, d(x, y)]$ with endpoints $x$ and $y$ unless there exists some $z$ such that $d(x, y)=d(x, z)+d(z, y)$. Let us call these lines basic paths and the points that are on them auxiliary. Take the completion of the weighted graph so obtained. Observe that after this step, for every essential points of $\beta(X)$ there exists an $\mathbb{R}$-geodesic segment (composed from auxiliary vertices) connecting them.
(3) For any triple $\{x, y, z\}$ of essential vertices, consider the projection of the triangle $\Delta(x, y, z)$ onto the tripod $T(x, y, z)$. Attach bands of length $4 \delta$ to the basic paths linking together the points that are mapped together on the tripod, except for those at the distance less than $2 \delta$ from $x, y$, or $z$. Define the fibers of the bands to be bridges and the points that make them up negligible.
(4) We extend the metric $d: X \rightarrow \mathbb{R}$ to the metric $\widehat{d}: \operatorname{beta}(X) \rightarrow \mathbb{R}$ as follows:
(a) the distance between two essential points is inherited from $X$,
(b) the distance between an auxiliary point to the adjacent essential points is defined by construction, hence, the distance from an auxiliary point to any other either essential, or auxiliary point is also defined (as the minimum of lengths of paths connecting them),
(c) the distance from a negligible point to the adjacent auxiliary points is defined by construction, so, the distance from a negligible point to any other point of $\beta(X)$ is also defined.

Remark 9. As in the case of $\Gamma_{1}$ and $\Gamma_{2}$, every given space $X$ has a unique extension $\beta(X)$. The reasoning is the same as for $\Gamma_{1}$.

Lemma 32. Let $v, w$ be points of $\beta(X)$.
(i) If $v$ and $w$ are essential then $d(v, w)=\widehat{d}(\varphi(v), \varphi(w))$.
(ii) If $v$ and $w$ are auxiliary then

$$
\widehat{d}(\varphi(v), \varphi(w)) \leqslant d(v, w) \leqslant \widehat{d}(\varphi(v), \varphi(w))+24 \delta
$$

(iii) If $v$ is essential and $w$ is auxiliary then

$$
\widehat{d}(\varphi(v), \varphi(w)) \leqslant d(v, w) \leqslant \widehat{d}(\varphi(v), \varphi(w))+8 \delta
$$

Proof. The proof is a straightforward adaptation of the proof of Lemma 21. All the arguments work in the exact same way.

Proposition 6. $\beta(X)$ is $\delta^{\prime}$-hyperbolic with $\delta^{\prime}=29 \delta$.
Proof. Let $\varphi: \beta(X) \rightarrow \bar{X}$ be the function sending points of $X \subseteq \beta(X)$ to their embedding in $\bar{X}$ extended by mapping basic paths and fibers of bands to geodesics of $\bar{X}$.

The embedding of any geodesic of $\beta(X)$ whose endpoints are essential or auxiliary is a $(1,24 \delta)$-quasi-isometry. Suppose then that a geodesic has endpoints which are on bands. Bands are always of length $4 \delta$ and link together essential and auxiliary points. Thus, if $a, b$ are the endpoints, there exist auxiliary points $a^{\prime}, b^{\prime}$ such that $d\left(a, a^{\prime}\right), d\left(b, b^{\prime}\right) \leqslant 2 \delta$. It follows that, since $[a, b] \rightarrow\left[a^{\prime}, b^{\prime}\right]$ is a $(1,4 \delta)$-quasi-isometry and $\varphi:\left[a^{\prime}, b^{\prime}\right] \rightarrow \bar{X}$ is a ( $1,24 \delta$ )-quasi-isometry, $\varphi:[a, b] \rightarrow \bar{X}$ is a $(1,28 \delta)$-quasi-isometry.

Thus, the embedding of any geodesic of $\beta(X)$ into $\bar{X}$ is a $(1,28 \delta)$-quasigeodesic. It follows that $\varphi$ is a (1,28 )-quasi-isometric embedding, and the result follows easily.

Lemma 33. (i) Let $g$ be an isometry of $X$. There exists a unique isometry $\bar{g}$ of $\beta(X)$ such that $\left.\bar{g}\right|_{X}=g$.
(ii) If $\delta=0$, our construction is equivalent to the one given in [3, Theorem 2.4.4].

Proof. Both statements are quite obvious. Define $\bar{g}$ simply by mapping basic paths to basic paths (since we know the action on their endpoints) and bands to bands. Since $g$ is an isometry of $X$, it preserves the length of basic paths and the size of triangles, so that it also preserves the presence of bands. Uniqueness is proven in a way analogous to Lemma 22 .

If $\delta=0$, then bands have length 0 , so we simply identify points in the same way as shown in the proof of [3, Theorem 2.4.4].

Lemma 34. (i) Let $Y$ be a geodesic $\Delta$-hyperbolic metric space with $X$ isometrically embedded in $Y$. Then $\beta(X)$ is quasi-isometrically embedded in $Y$ and the constants of the quasi-isometry depend only on $\delta$ and $\Delta$.
(ii) Let $Y$ be a $\Delta$-hyperbolic metric space with $X \simeq Y$. Then $\beta(X) \simeq \beta(Y)$ and the constants of the quasi-isometry depend only on $\delta, \Delta$, and the constants of the quasi-isometry between them.
(iii) Let $Y$ be a geodesic $\Delta$-hyperbolic metric space and $X$ quasi-isometrically embedded in $Y$. Then $\beta(X)$ is quasi-isometrically embedded in $Y$.

Proof. The proofs are similar to the proofs of Lemma 23 and Lemma 24 .Transfer of the proofs is very straightforward.

### 7.4 Boundaries $\partial \Gamma_{1}(X), \partial \Gamma_{2}(X)$ and $\partial \beta(X)$

Proposition 7. $\partial X=\partial \Gamma_{1}(X)$
Proof. First of all, since $X$ embeds isometrically into $\Gamma_{1}(X)$, it is obvious that $\partial X \subseteq \partial \Gamma_{1}(X)$.

Let then $\left\{x_{n}\right\}$ be a sequence in $\Gamma_{1}(X)$ converging at infinity, representing the point $\alpha \in \partial \Gamma_{1}$. If $x_{n}$ is a negligible vertex, it is at distance at most $2 \delta$ from an auxiliary vertex $x_{n}^{\prime}$, so we can replace $x_{n}$ by $x_{n}^{\prime}$ without changing the behavior of $\left\{x_{n}\right\}$ at infinity. If for any $n$ there would exist $k_{n} \geqslant n$ such that $x_{k_{n}}$ is an essential vertex, we would have that $\left\{x_{k_{n}}\right\}$ is a sequence of essential vertices converging at infinity, so that $\alpha \in \partial X$.

We can therefore assume that all $x_{n}$ are auxiliary vertices. Hence, there exist $x_{n}^{\alpha}$ and $x_{n}^{\omega}$ such that $x_{n} \in\left[x_{n}^{\alpha}, x_{n}^{\omega}\right]$. Let $x_{n}^{\prime} \in\left\{x_{n}^{\alpha}, x_{n}^{\omega}\right\}$ such that $\left(x_{n} \cdot x_{n}^{\prime}\right)$ is maximal. If for any $n$ there exists $k_{n}$ such that $\left(x_{k_{n}} \cdot x_{k_{n}}^{\prime}\right) \geqslant n$, then $\left\{x_{k_{n}}^{\prime}\right\}$ converges at infinity towards the same point as $\left\{x_{k_{n}}\right\}$, so we have that $\alpha \in \partial X$.

Let us then assume there exists some $N$ such that $\left(x_{n} \cdot x_{n}^{\alpha}\right),\left(x_{n} \cdot x_{n}^{\omega}\right) \leqslant N$ for any $n$. Let $x$ be the basepoint. We have

$$
d\left(x_{n}^{\alpha}, x_{n}\right)=d\left(x, x_{n}^{\alpha}\right)+d\left(x, x_{n}\right)-2\left(x_{n} \cdot x_{n}^{\alpha}\right) \geqslant d\left(x, x_{n}\right)+d\left(x, x_{n}^{\alpha}\right)-2 N .
$$

By the same reasoning we have

$$
d\left(x_{n}^{\omega}, x_{n}\right) \geqslant d\left(x, x_{n}^{\omega}\right)+d\left(x, x_{n}\right)-2 N .
$$

However, $x_{n} \in\left[x_{n}^{\alpha}, x_{n}^{\omega}\right]$, so we have that

$$
\begin{aligned}
d\left(x_{n}^{\alpha}, x_{n}^{\omega}\right)=d\left(x_{n}^{\alpha}, x_{n}\right) & +d\left(x_{n}^{\omega}, x_{n}\right) \geqslant d\left(x, x_{n}^{\alpha}\right)+d\left(x, x_{n}^{\omega}\right)+2 d\left(x, x_{n}\right)-4 N \\
\geqslant & d\left(x_{n}^{\alpha}, x_{n}^{\omega}\right)+2 d\left(x, x_{n}\right)-4 N
\end{aligned}
$$

This implies that $d\left(x, x_{n}\right) \leqslant 2 N$ for any $n$ and we have a contradiction with the assumption that $\left\{x_{n}\right\}$ converges at infinity.

Then it follows that for any sequence $\left\{x_{n}\right\}$ of vertices in $\Gamma_{1}(X)$ which converges at infinity, there exists a subsequence $\left\{x_{k_{n}}\right\}$ and a sequence $\left\{x_{k_{n}}^{\prime}\right\}$ in $X$ such that $\left(x_{k_{n}} \cdot x_{k_{n}}^{\prime}\right) \geqslant n$ for any $n$. We have therefore that $\partial \Gamma_{1}(X) \subseteq \partial X$.

Corollary 8. $\partial X=\partial \Gamma_{2}(X)$
Proof. By Lemma 27, $X$ is quasi-isometrically embedded in $\Gamma_{2}(X)$, so we have that $\partial X \subseteq \partial \Gamma_{2}(X)$. At the same time, from the proof of Lemma 5 we have that $\Gamma_{2}(X)$ is quasi-isometrically embedded in $\Gamma_{1}(X)$, so $\partial \Gamma_{2}(X) \subseteq \partial \Gamma_{1}(X)=$ $\partial X$.

Corollary 9. $\partial X=\partial \beta(X)$
Proof. It is again straightforward to use the same rationale as in the proof of Proposition 7. We leave the details to the reader.

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