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Equational definability of (complementary) central elements

Mariana V. Badano* and Diego J. Vaggione†

Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba
Ciudad Universitaria, Córdoba 5000, Argentina
**mbadano@famaf.unc.edu.ar*
†vaggione@famaf.unc.edu.ar

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For a variety with weak existentially definable factor congruences, we characterize when the properties "$\vec{e}$ is a central element" and "$\vec{e}$ and $\vec{f}$ are complementary central elements" are definable by $(\forall \bigwedge p = q)$-formulas and by $(\bigwedge p = q)$-formulas.

Keywords: Central element; Boolean factor congruences; decomposition operation.

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1. Introduction

A *variety with* $\vec{0}$ *and* $\vec{1}$ is a variety $\mathcal{V}$ for which there are 0-ary terms $0_1, \ldots, 0_N$, $1_1, \ldots, 1_N$ such that[a]

$$\mathcal{V} \vDash \vec{0} = \vec{1} \to x = y,$$

where $\vec{0} = (0_1, \ldots, 0_N)$ and $\vec{1} = (1_1, \ldots, 1_N)$. Classical examples of this type of varieties are the variety $\mathcal{S}$ of bounded join semilattices and the variety $\mathcal{R}$ of rings with identity (in both cases $N = 1$). If $\vec{a} \in A^N$ and $\vec{b} \in B^N$, then we use $[\vec{a}, \vec{b}]$ to denote the N-tuple $((a_1, b_1), \ldots, (a_N, b_N)) \in (A \times B)^N$. If $\mathbf{A} \in \mathcal{V}$, then we say that $\vec{e} \in A^N$ is a *central element* of $\mathbf{A}$, if there exists an isomorphism $\mathbf{A} \to \mathbf{A}_1 \times \mathbf{A}_2$ such that $\vec{e} \to [\vec{0}, \vec{1}]$. Also, we say that $\vec{e}$ and $\vec{f}$ are a *pair of complementary central elements* of $\mathbf{A}$, if there exists an isomorphism $\mathbf{A} \to \mathbf{A}_1 \times \mathbf{A}_2$ such that $\vec{e} \to [\vec{0}, \vec{1}]$ and $\vec{f} \to [\vec{1}, \vec{0}]$. Consider the following property.

(L) There is a first-order formula $\lambda(\vec{z}, x, y)$ such that for every $\mathbf{A}, \mathbf{B} \in \mathcal{V}$,

$$\mathbf{A} \times \mathbf{B} \vDash \lambda([\vec{0}, \vec{1}], (a, b), (a', b')) \quad \text{if and only if } a = a'.$$

[a] When the language has a constant symbol, this condition is equivalent to the fact that no nontrivial algebra has a trivial subalgebra (see [16]).

If $\mathcal{V} = \mathcal{S}$, we can take $\lambda = x \vee z_1 = y \vee z_1$, in order to satisfy (L) and if $\mathcal{V} = \mathcal{R}$ we can take $\lambda = x.(1 - z_1) = y.(1 - z_1)$. We note that if (L) holds and θ is any factor congruence of an algebra $\mathbf{A} \in \mathcal{V}$, then taking[b]

$$\vec{e} = \text{unique } \vec{u} \in A^N \quad \text{such that } \vec{u} \equiv \vec{0}(\theta) \text{ and } \vec{u} \equiv \vec{1}(\delta),$$

where δ is any factor complement of θ, we have that

$$\theta = \{(a, b) \in A^2 : \mathbf{A} \vDash \lambda(\vec{e}, a, b)\}.$$

So, condition (L) says that *every* factor congruence θ can be defined by λ parameterized with an adequate central element (which we will see is uniquely determined by θ). Since $\mathbf{A} \times \mathbf{B}$ is isomorphic to $\mathbf{B} \times \mathbf{A}$ via the map $(a, b) \to (b, a)$ it is trivial that a formula λ satisfying (L) also satisfies

$$\mathbf{A} \times \mathbf{B} \vDash \lambda([\vec{1}, \vec{0}], (a, b), (a', b')) \quad \text{if and only if } b = b'$$

for any $\mathbf{A}, \mathbf{B} \in \mathcal{V}$. Since in general $\vec{0}$ and $\vec{1}$ are not interchangeable, it is not obvious that (L) is equivalent to the following condition.

(R) There is a first-order formula $\rho(\vec{z}, x, y)$ such that for every $\mathbf{A}, \mathbf{B} \in \mathcal{V}$,

$$\mathbf{A} \times \mathbf{B} \vDash \rho([\vec{0}, \vec{1}], (a, b), (a', b')) \quad \text{if and only if } b = b'.$$

If $\mathcal{V} = \mathcal{S}$, the reader can easily check that

$$\rho = \forall u (x \vee u \vee z_1 = y \vee u \vee z_1 \to x \vee u = y \vee u)$$

satisfies (R). Moreover, in [1] it is proved that for the variety $\mathcal{S}$ there is no positive nor existential formula satisfying (R), which says that the above ρ is as better as possible in the sense of its complexity. So, for $\mathcal{S}$ the better options are $\lambda = x \vee z_1 = y \vee z_1$ for property (L) and the above ρ for property (R).

A third definability condition is the following:

(W) There is a first-order formula $\omega(\vec{z}, \vec{w}, x, y)$ such that for every $\mathbf{A}, \mathbf{B} \in \mathcal{V}$,

$$\mathbf{A} \times \mathbf{B} \vDash \omega([\vec{0}, \vec{1}], [\vec{1}, \vec{0}], (a, b), (a', b')) \quad \text{if and only if } a = a'.$$

Of course (W) is implied by (L) and (R) taking $\omega(\vec{z}, \vec{w}, x, y) = \lambda(\vec{z}, x, y)$ and $\omega(\vec{z}, \vec{w}, x, y) = \rho(\vec{w}, x, y)$, respectively. Further, we note that, as was shown above for (L), (W) guarantees that every factor congruence can be defined by ω parameterized with an adequate pair of complementary central elements.

In [13], it is proved the following

Theorem 1.1. *For a variety $\mathcal{V}$ with $\vec{0}$ and $\vec{1}$ properties (L), (R) and (W) are equivalent to that $\mathcal{V}$ have Boolean factor congruences, i.e. the set of factor congruences of any algebra in $\mathcal{V}$ is a distributive sublattice of its congruence lattice. Moreover the formulas in (L), (R) and (W) can be chosen to be preserved by direct products and direct factors.*

[b]We write $\vec{a} \equiv \vec{b}(\theta)$ to express that $(a_i, b_i) \in \theta$, $i = 1, \dots, N$.

When some of the equivalent conditions (L), (R) or (W) holds, we say that $\mathcal{V}$ has *definable factor congruences* (DFC). As we exemplified with the semilattice case, the equivalence of (L), (R) and (W) does not preserve the complexity of the defining formula. So, several definitions arise, which we state now (we abbreviate with EDFC (respectively, ExDFC) the phrase "equationally (respectively existentially) DFC"). We say that $\mathcal{V}$ has *left* (*respectively right*, *weak*) EDFC if (L) (respectively (R), (W)) holds with λ (respectively ρ, ω) a conjunction of equations. We say that $\mathcal{V}$ has *twice* EDFC if $\mathcal{V}$ has left and right EDFC. In an analogous fashion, we define the concepts of *left*, *right*, *weak* and *twice* ExDFC.

If $\mathcal{V}$ is a variety with $\vec{0}$ and $\vec{1}$, we use $Z(\mathbf{A})$ to denote the set of central elements of $\mathbf{A}$ and we write $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ to express that $\vec{e}$ and $\vec{f}$ are a pair of complementary central elements of $\mathbf{A}$. We say that a set of first-order formulas $\{\varphi_r(\vec{z}) : r \in R\}$ *defines the property* $\vec{e} \in Z(\mathbf{A})$ *in* $\mathcal{V}$ if for every $\mathbf{A} \in \mathcal{V}$ and $\vec{e} \in A^N$, we have that $\vec{e} \in Z(\mathbf{A})$ if and only if $\mathbf{A} \vDash \varphi_r(\vec{e})$, for every $r \in R$. We say that a set $\{\varphi_r(\vec{z}, \vec{w}) : r \in R\}$ *defines the property* $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ *in* $\mathcal{V}$ if for every $\mathbf{A} \in \mathcal{V}$ and $\vec{e}, \vec{f} \in A^N$, we have that $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ if and only if $\mathbf{A} \vDash \varphi_r(\vec{e}, \vec{f})$, for every $r \in R$. In [13], it is proved that if $\mathcal{V}$ has DFC, then the properties $\vec{e} \in Z(\mathbf{A})$ and $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ are definable in $\mathcal{V}$ by sets of first-order formulas. Moreover the formulas can be chosen to be preserved by direct products and direct factors.

Most classical types of varieties have either EDFC or ExDFC. For example, we have the following:

(1) If $\mathcal{V}$ is a variety with $\vec{0}$ and $\vec{1}$ which has the Fraser–Horn property, then $\mathcal{V}$ has twice ExDFC and the properties $\vec{e} \in Z(\mathbf{A})$ and $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ are definable in $\mathcal{V}$ by sets of $(\forall \exists \bigwedge p = q)$-formulas (see [16]).
(2) If $\mathcal{V}$ is a congruence modular variety with $\vec{0}$ and $\vec{1}$, then $\mathcal{V}$ has twice EDFC and the properties $\vec{e} \in Z(\mathbf{A})$ and $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ are definable in $\mathcal{V}$ by sets of $(\forall \exists \bigwedge p = q)$-formulas (see [15]).
(3) If $\mathcal{V}$ is a congruence permutable variety with $\vec{0}$ and $\vec{1}$, then $\mathcal{V}$ has twice EDFC and the properties $\vec{e} \in Z(\mathbf{A})$ and $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ are definable in $\mathcal{V}$ by sets of $(\forall \bigwedge p = q)$-formulas (see [1]).
(4) If $\mathcal{V}$ is a variety of bounded lattice expansions, then $\mathcal{V}$ has twice EDFC and the properties $\vec{e} \in Z(\mathbf{A})$ and $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ are definable in $\mathcal{V}$ by a sets of $(\forall \bigwedge p = q)$-formulas (folklore).
(5) If $\mathcal{V}$ is a variety of bounded join semilattice expansions, then $\mathcal{V}$ has left EDFC. There is a set Σ_1 of $\forall \exists$-formulas and a set Σ_2 of $\exists \forall$-formulas such that $\Sigma_1 \cup \Sigma_2$ defines $\vec{e} \in Z(\mathbf{A})$ in $\mathcal{V}$, and $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ is definable by a set of $\forall \exists$-formulas (see [1]).

In this paper, we characterize for a variety $\mathcal{V}$ with weak ExDFC each one of the following conditions:

— The property $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ is definable in $\mathcal{V}$ by a set of $(\forall \bigwedge p = q)$-formulas.
— The property $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ is definable in $\mathcal{V}$ by a set of $(\bigwedge p = q)$-formulas.

— The property $\vec{e} \in Z(\mathbf{A})$ is definable in $\mathcal{V}$ by a set of $(\forall \bigwedge p = q)$-formulas.
— The property $\vec{e} \in Z(\mathbf{A})$ is definable in $\mathcal{V}$ by a set of $(\bigwedge p = q)$-formulas.

2. Notation and Basic Results

As usual, $\mathbb{I}(\mathcal{K})$, $\mathbb{S}(\mathcal{K})$ and $\mathbb{P}_u(\mathcal{K})$ denote the classes of isomorphic images, substructures and ultraproducts of elements of $\mathcal{K}$. If $\mathcal{V}$ is a variety, we use $\mathcal{V}_{\mathrm{SI}}$ (respectively $\mathcal{V}_{\mathrm{DI}}$) to denote the class of subdirectly irreducible (respectively directly indecomposable) members of $\mathcal{V}$. If $\mathbf{A}, \mathbf{B}$ are algebras, we write $\mathbf{A}, \leq \mathbf{B}$ to express that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$. By $\mathrm{Con}(\mathbf{A})$, we denote the congruence lattice of $\mathbf{A}$. As usual, the join operation of $\mathrm{Con}(\mathbf{A})$ is denoted by $\vee$. We use $\nabla^{\mathbf{A}}$ to denote the universal congruence on $\mathbf{A}$ and $\Delta^{\mathbf{A}}$ to denote the trivial congruence on $\mathbf{A}$. If $\vec{a}, \vec{b} \in A^n$, we will use $\theta^{\mathbf{A}}(\vec{a}, \vec{b})$ to denote the congruence generated by the set $\{(a_k, b_k) : 1 \leq k \leq n\}$. If $\vec{a}, \vec{b} \in A^n$ and $\theta \in \mathrm{Con}(\mathbf{A})$, we write $\vec{a} \equiv \vec{b}(\theta)$ to express that $(a_i, b_i) \in \theta$, $i = 1, \ldots, n$. We use $\mathrm{FC}(\mathbf{A})$ to denote the set of factor congruences of $\mathbf{A}$. A variety $\mathcal{V}$ has *Boolean factor congruences* if for every $\mathbf{A} \in \mathcal{V}$, the set $\mathrm{FC}(\mathbf{A})$ is a distributive sublattice of $\mathrm{Con}(\mathbf{A})$. If $\theta \in \mathrm{FC}(\mathbf{A})$, we use θ^* to denote the factor complement of θ. Observe that in a variety with Boolean factor congruences $(\mathrm{FC}(\mathbf{A}), \vee, \cap, ^*, \Delta^{\mathbf{A}}, \nabla^{\mathbf{A}})$ is a Boolean algebra.

A *decomposition operation on* A is a homomorphism $d\colon A \times A \to A$ satisfying:

$$d(x, x) = x$$

$$d(d(x, y), z) = d(x, z) = d(x, d(y, z)).$$

We observe that d is a homomorphism if and only if for every n-ary function symbol F, the following identity holds in $\mathbf{A}$

$$F(d(x_1, y_1), \ldots, d(x_n, y_n)) = d(F(\vec{x}), F(\vec{y})).$$

Given a pair (θ, δ) of complementary factor congruences, we have associated a decomposition operation defined by

$$d_{(\theta,\delta)}(a, b) = \text{the unique } c \in A \quad \text{such that } (c, a) \in \theta \text{ and } (c, b) \in \delta.$$

Reciprocally, given a decomposition operation d, the relations

$$\theta_d = \{(x, y) : d(x, y) = y\} \quad \text{and} \quad \delta_d = \{(x, y) : d(x, y) = x\}$$

are a pair of complementary factor congruences. The maps $(\theta, \delta) \to d_{(\theta,\delta)}$ and $d \to (\theta_d, \delta_d)$ are mutually inverse [11]. For algebras $\mathbf{A}, \mathbf{B}$ we use $d^{\mathbf{A}\times\mathbf{B}}$ to denote the *canonical decomposition operation on* $\mathbf{A} \times \mathbf{B}$, i.e. $d^{\mathbf{A}\times\mathbf{B}}((a_1, b_1), (a_2, b_2)) = (a_1, b_2)$.

Given a set of variables X, we use $T(X)$ to denote the set of terms with variables in X. If $X = \{x_1, \ldots, x_n\}$, then we use $T(x_1, \ldots, x_n)$ instead of $T(\{x_1, \ldots, x_n\})$.

2.1. *Basic facts on varieties with* DFC

Let $\mathcal{V}$ be a variety with $\vec{0}$ and $\vec{1}$ and suppose $\mathcal{V}$ has DFC, i.e. $\mathcal{V}$ satisfies the equivalent conditions (L), (R) and (W) from the introduction. We use $Z(\mathbf{A})$ to denote the set of central elements of an algebra $\mathbf{A} \in \mathcal{V}$. It is obvious from the definition that $\vec{e} \in Z(\mathbf{A})$ if and only if there is a pair of complementary factor congruences (θ, δ) satisfying

$$(*) \quad \vec{e} \equiv \vec{0}(\theta) \quad \text{and} \quad \vec{e} \equiv \vec{1}(\delta).$$

Note that, (L) and (R) imply that the central element $\vec{e}$ determines an unique pair of complementary factor congruences satisfying (*), since $\lambda(\vec{e}, \_, \_)$ defines θ and $\rho(\vec{e}, \_, \_)$ defines δ. We denote by $(\theta^{\mathbf{A}}_{\vec{0}\vec{e}}, \theta^{\mathbf{A}}_{\vec{1}\vec{e}})$ this pair. Thus $Z(\mathbf{A})$ is naturally identified with the set of pairs of complementary factor congruences of $\mathbf{A}$. Since $\mathcal{V}$ has Boolean factor congruences (Theorem 1 in the introduction), factor complements are unique and hence, we obtain the following fundamental result

Theorem 2.1. *The maps*

$$\begin{array}{ll} Z(\mathbf{A}) \to \mathrm{FC}(\mathbf{A}) & \mathrm{FC}(\mathbf{A}) \to Z(\mathbf{A}) \\ \vec{e} \to \theta^{\mathbf{A}}_{\vec{0}\vec{e}} & \theta \to \textit{unique } \vec{e} \textit{ satisfying} \\ & \qquad \vec{e} \equiv \vec{0}(\theta) \quad \textit{and} \quad \vec{e} \equiv \vec{1}(\theta^*) \end{array}$$

are mutually inverse bijections.

Thus, we can define

$$\vec{e} \vee^{\mathbf{Z}(\mathbf{A})} \vec{f} = \text{only } \vec{g} \in Z(\mathbf{A}) \text{ satisfying } \theta^{\mathbf{A}}_{\vec{0}\vec{g}} = \theta^{\mathbf{A}}_{\vec{0}\vec{e}} \vee \theta^{\mathbf{A}}_{\vec{0}\vec{f}}$$

$$\vec{e} \wedge^{\mathbf{Z}(\mathbf{A})} \vec{f} = \text{only } \vec{g} \in Z(\mathbf{A}) \text{ satisfying } \theta^{\mathbf{A}}_{\vec{0}\vec{g}} = \theta^{\mathbf{A}}_{\vec{0}\vec{e}} \cap \theta^{\mathbf{A}}_{\vec{0}\vec{f}}$$

$$c^{\mathbf{Z}(\mathbf{A})}(\vec{e}) = \text{only } \vec{g} \in Z(\mathbf{A}) \text{ satisfying } \theta^{\mathbf{A}}_{\vec{0}\vec{g}} = (\theta^{\mathbf{A}}_{\vec{0}\vec{e}})^*$$

to obtain a Boolean algebra $\mathbf{Z}(\mathbf{A}) = (Z(\mathbf{A}), \vee^{\mathbf{Z}(\mathbf{A})}, \wedge^{\mathbf{Z}(\mathbf{A})}, c^{\mathbf{Z}(\mathbf{A})}, \vec{0}, \vec{1})$, which is naturally isomorphic to $(\mathrm{FC}(\mathbf{A}), \vee, \cap, ^*, \Delta^{\mathbf{A}}, \nabla^{\mathbf{A}})$. When no confusion is possible, we will write $\vec{e} \vee \vec{f}$ in place of $\vec{e} \vee^{\mathbf{Z}(\mathbf{A})} \vec{f}$, $c(\vec{e})$ instead of $c^{\mathbf{Z}(\mathbf{A})}(\vec{e})$, etc. We also note that

— $\theta^{\mathbf{A}}_{\vec{0}\vec{e}} = \{(a, b) : \mathbf{A} \vDash \lambda(\vec{e}, a, b)\}$
— $\theta^{\mathbf{A}}_{\vec{1}\vec{e}} = \{(a, b) : \mathbf{A} \vDash \rho(\vec{e}, a, b)\}$
— $\theta^{\mathbf{A}}_{\vec{0}\vec{e}} = \{(a, b) : \mathbf{A} \vDash \omega(\vec{e}, c(\vec{e}), a, b)\}$
— $\theta^{\mathbf{A}}_{\vec{1}\vec{e}} = \{(a, b) : \mathbf{A} \vDash \omega(c(\vec{e}), \vec{e}, a, b)\}$,

where λ, ρ and ω are any formulas satisfying (L), (R) and (W), respectively. Other basic properties, which we will use without making explicit mention, are

— $\vec{e} \leq^{Z(\mathbf{A})} \vec{f}$ if and only if $\theta^{\mathbf{A}}_{\vec{0}\vec{e}} \subseteq \theta^{\mathbf{A}}_{\vec{0}\vec{f}}$
— $\theta^{\mathbf{A}}_{\vec{0}\vec{e}} = \theta^{\mathbf{A}}_{\vec{1}c(\vec{e})}$
— $\theta^{\mathbf{A}}_{\vec{0}\vec{0}} = \Delta^{\mathbf{A}}$ and $\theta^{\mathbf{A}}_{\vec{0}\vec{1}} = \nabla^{\mathbf{A}}$
— The map

$$Z(\mathbf{A}_1) \times Z(\mathbf{A}_2) \to Z(\mathbf{A}_1 \times \mathbf{A}_2)$$

$$(\vec{e}_1, \vec{e}_2) \to [\vec{e}_1, \vec{e}_2]$$

is a Boolean algebra isomorphism.

— $\theta^{\mathbf{A}_1 \times \mathbf{A}_2}_{[\vec{0},\vec{0}][\vec{e}_1,\vec{e}_2]} = \theta^{\mathbf{A}_1}_{\vec{0}\vec{e}_1} \times \theta^{\mathbf{A}_n}_{\vec{0}\vec{e}_2}$
— $\theta^{\mathbf{A}}(\vec{0},\vec{e}) \subseteq \theta^{\mathbf{A}}_{\vec{0}\vec{e}}$ and $\theta^{\mathbf{A}}(\vec{1},\vec{e}) \subseteq \theta^{\mathbf{A}}_{\vec{1}\vec{e}}$.[c]

If $\mathcal{V}$ is a variety with DFC, we say that a formula $\varphi(\vec{z}, x, y)$ *defines* $\theta^{\mathbf{A}}_{\vec{0}\vec{e}}$ *in* $\mathcal{V}$ (respectively *defines* $\theta^{\mathbf{A}}_{\vec{1}\vec{e}}$ *in* $\mathcal{V}$) if for every $\mathbf{A} \in \mathcal{V}$ and $\vec{e} \in Z(\mathbf{A})$, we have that $\theta^{\mathbf{A}}_{\vec{0}\vec{e}} = \{(a,b) \in A^2 : \mathbf{A} \vDash \varphi(\vec{e}, a, b)\}$ (respectively $\theta^{\mathbf{A}}_{\vec{1}\vec{e}} = \{(a,b) \in A^2 : \mathbf{A} \vDash \varphi(\vec{e}, a, b)\}$), i.e. φ satisfies (L) (respectively (R)) of the introduction. Analogously, we say that a formula $\varphi(z_1, \ldots, z_N, w_1, \ldots, w_N, x, y)$ *defines* $\theta^{\mathbf{A}}_{\vec{0}\vec{e}}$ *in* $\mathcal{V}$ if for every $\mathbf{A} \in \mathcal{V}$ and $\vec{e} \in Z(\mathbf{A})$ we have that $\theta^{\mathbf{A}}_{\vec{0}\vec{e}} = \{(a,b) \in A^2 : \mathbf{A} \vDash \varphi(\vec{e}, c(\vec{e}), a, b)\}$, i.e. φ satisfies (W) of the introduction.

3. Varieties with a (Short) Decomposition Term

Let $\mathcal{V}$ be a variety with $\vec{0}$ and $\vec{1}$. We say that a term $U(\vec{z}, \vec{w}, x, y)$ is a *decomposition term* for $\mathcal{V}$ if the following identities hold in $\mathcal{V}$

$$U(\vec{0}, \vec{1}, x, y) = x$$

$$U(\vec{1}, \vec{0}, x, y) = y.$$

We say that $u(\vec{z}, x, y)$ is a *short decomposition term* for $\mathcal{V}$ if the following identities hold in $\mathcal{V}$

$$u(\vec{0}, x, y) = x$$

$$u(\vec{1}, x, y) = y.$$

We note that, if $u(\vec{z}, x, y)$ is a short decomposition term for $\mathcal{V}$, then $U(\vec{z}, \vec{w}, x, y) = u(\vec{z}, x, y)$ is a decomposition term for $\mathcal{V}$. For the case $N = 1$, varieties with a decomposition term have been studied in [6] under the name of separator varieties. In [14], varieties with a decomposition term are called Pierce varieties and they are used to give a characterization of those varieties with the Fraser–Horn property in which every Pierce stalk is directly indecomposable. In [8, 9, 7] and [4] varieties in which there is a short decomposition term with $N = 1$ are called Church varieties and they are studied in connection with the variety of lambda abstraction algebras. For the case of $N = 1$, the short decomposition term is studied in [10] as a modelization of the if-then-else instruction of programming languages.

It is obvious from the defining identities that if $U(\vec{z}, \vec{w}, x, y)$ (respectively $u(\vec{z}, x, y)$) is a decomposition (respectively short decomposition) term for $\mathcal{V}$, then for every $\mathbf{A}, \mathbf{B} \in \mathcal{V}$, we have that $U^{\mathbf{A} \times \mathbf{B}}([\vec{0}, \vec{1}], [\vec{1}, \vec{0}], x, y) = d^{\mathbf{A} \times \mathbf{B}}(x, y)$, for every $x, y \in A \times B$ (respectively $u^{\mathbf{A} \times \mathbf{B}}([\vec{0}, \vec{1}], x, y) = d^{\mathbf{A} \times \mathbf{B}}(x, y)$, for every $x, y \in A \times B$). Thus if $\mathcal{V}$ has a decomposition term $U(\vec{z}, \vec{w}, x, y)$, then the formula $\omega(\vec{z}, \vec{w}, x, y) = U(\vec{z}, \vec{w}, x, y) = y$ satisfies (W) of the introduction and hence $\mathcal{V}$ has weak EDFC. However (1) of the following proposition says that $\mathcal{V}$ have indeed twice EDFC. This

[c] In [13], we give an example to show that in general the congruences $\theta^{\mathbf{A}}_{\vec{0}\vec{e}}$ and $\theta^{\mathbf{A}}_{\vec{1}\vec{e}}$ fail to be finitely generated.

is a key result in the sequel. In order to emphasize the associated semantic of a decomposition term, we will write $U_{\vec{z}\vec{w}}(x, y)$ in place of $U(\vec{z}, \vec{w}, x, y)$ and $u_{\vec{z}}(x, y)$ in place of $u(\vec{z}, x, y)$. Given any $(2N+2)$-ary term $U(\vec{z}, \vec{w}, x, y)$ and an n-ary function symbol F, we define

$$\begin{aligned}
D^U(\vec{z}, \vec{w}, x, y, z) &= U_{\vec{z}\vec{w}}(x, x) = x \\
&\quad \wedge U_{\vec{z}\vec{w}}(x, U_{\vec{z}\vec{w}}(y, z)) = U_{\vec{z}\vec{w}}(x, z) = U_{\vec{z}\vec{w}}(U_{\vec{z}\vec{w}}(x, y), z) \\
P^U_F(\vec{z}, \vec{w}, \vec{x}, \vec{y}) &= F(U_{\vec{z}\vec{w}}(x_1, y_1), \dots, U_{\vec{z}\vec{w}}(x_n, y_n)) = U_{\vec{z}\vec{w}}(F(\vec{x}), F(\vec{y})) \\
B^U(\vec{z}, \vec{w}) &= \bigwedge_{i=1}^{N} z_i = U_{\vec{z}\vec{w}}(0_i, 1_i) \wedge w_i = U_{\vec{z}\vec{w}}(1_i, 0_i).
\end{aligned}$$

Analogously, for any $(N+2)$-ary term $u(\vec{z}, x, y)$ and an n-ary function symbol F, we define

$$\begin{aligned}
D^u(\vec{z}, x, y, z) &= u_{\vec{z}}(x, x) = x \\
&\quad \wedge u_{\vec{z}}(x, u_{\vec{z}}(y, z)) = u_{\vec{z}}(x, z) = u_{\vec{z}}(u_{\vec{z}}(x, y), z) \\
P^u_F(\vec{z}, \vec{x}, \vec{y}) &= F(u_{\vec{z}}(x_1, y_1), \dots, u_{\vec{z}}(x_n, y_n)) = u_{\vec{z}}(F(\vec{x}), F(\vec{y})) \\
B^u(\vec{z}) &= \wedge_{i=1}^{N} z_i = u_{\vec{z}}(0_i, 1_i).
\end{aligned}$$

Proposition 3.1. *Let $\mathcal{V}$ be a variety with $\vec{0}$ and $\vec{1}$ and let $U(\vec{z}, \vec{w}, x, y)$ be a decomposition term for $\mathcal{V}$. Let $\mathcal{L}$ be the language of $\mathcal{V}$.*

(1) *The formula $\lambda(\vec{z}, x, y) = U(\vec{z}, \vec{1}, x, y) = U(\vec{1}, \vec{z}, x, y)$ satisfies* (*L*) *of the introduction and the formula $\rho(\vec{z}, x, y) = U(\vec{z}, \vec{0}, x, y) = U(\vec{0}, \vec{z}, x, y)$ satisfies* (*R*) *of the introduction. Thus $\mathcal{V}$ has twice* EDFC.

(2) *The set $\{\forall\, xyz\; D^U(\vec{z}, \vec{w}, x, y, z), B(\vec{z}, \vec{w})\} \cup \{\forall\, \vec{x}\vec{y}\; P^U_F(\vec{z}, \vec{w}, \vec{x}, \vec{y}) : F \in \mathcal{L}\}$ defines $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ in $\mathcal{V}$.*

(3) *Let $\kappa(\vec{z}, \vec{w}) = \forall\, xyz\;\; D^U(\vec{z}, \vec{w}, x, y, z) \wedge B^U(\vec{z}, \vec{w}) \wedge \pi(\vec{z}, \vec{w})$, where $\pi(\vec{z}, \vec{w})$ is $\forall\, \vec{x}\vec{y}xx'yy'\;\; U_{\vec{x}\vec{y}}(U_{\vec{z}\vec{w}}(x, x'), U_{\vec{z}\vec{w}}(y, y')) = U_{\vec{z}\vec{w}}(U_{\vec{x}\vec{y}}(x, y), U_{\vec{x}\vec{y}}(x', y'))$. Then the set $\{\exists\, \vec{w}\;\; (\kappa(\vec{z}, \vec{w}) \wedge \forall\, \vec{x}, \vec{y}\; P^U_F(\vec{z}, \vec{w}, \vec{x}, \vec{y})) : F \in \mathcal{L}\}$ defines $\vec{e} \in Z(\mathbf{A})$ in $\mathcal{V}$.*

(4) *Let $C(\vec{z}, \vec{u}) = \bigwedge_{i=1}^{N} U_{\vec{z}\vec{1}}(u_i, 1_i) = U_{\vec{1}\vec{z}}(u_i, 1_i) \wedge U_{\vec{z}\vec{0}}(u_i, 0_i) = U_{\vec{0}\vec{z}}(u_i, 0_i)$. The set consisting of the following formulas*

$$\begin{gathered}
\exists\, \vec{u}\; C(\vec{z}, \vec{u}) \\
\forall\, x, y, z, \vec{u}(C(\vec{z}, \vec{u}) \to D^U(\vec{z}, \vec{u}, x, y, z)) \\
\forall\, \vec{x}, \vec{y}, \vec{u}(C(\vec{z}, \vec{u}) \to P^U_F(\vec{z}, \vec{u}, \vec{x}, \vec{y})),\; F \in \mathcal{L} \\
\forall\, \vec{u}(C(\vec{z}, \vec{u}) \to B^U(\vec{z}, \vec{u}))
\end{gathered}$$

defines $\vec{e} \in Z(\mathbf{A})$ in $\mathcal{V}$.

(5) *If the language of $\mathcal{V}$ is finite, then the class $\mathcal{V}_{\mathrm{DI}}$ is a $\forall\exists$-class.*

(6) *When the above equivalent conditions hold, the sets of formulas of* (2), (3) *and* (4) *can be replaced by a finite subset of itself if and only if there is a finite set*

of formulas defining $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ in $\mathcal{V}$ if and only if $\mathcal{V}_{\mathrm{DI}}$ is a first-order class if and only if $\mathbb{P}_u(\mathcal{V}_{\mathrm{SI}}) \subseteq \mathcal{V}_{\mathrm{DI}}$.

Proof. (1) It is easy to check.

(2) Let $\mathbf{A} \in \mathcal{V}$ and $\vec{e}, \vec{f} \in A^N$. Note that the axioms

$$\forall x, y, z \; D^U(\vec{e}, \vec{f}, x, y, z) \quad \forall \vec{x}, \vec{y} \; P_F^U(\vec{e}, \vec{f}, \vec{x}, \vec{y}), \quad F \in \mathcal{L}$$

say that $d(x, y) = U^{\mathbf{A}}_{\vec{e}\vec{f}}(x, y)$ is a decomposition operation on $\mathbf{A}$. The axiom $B^U(\vec{e}, \vec{f})$ says that $\vec{e} = [\vec{0}, \vec{1}]$ and $\vec{f} = [\vec{1}, \vec{0}]$ via the isomorphism

$$A/\theta_d \times A/\delta_d \to A$$

$$(x/\theta_d, y/\delta_d) \to d(x, y).$$

(3) It is easy to check that if $\vec{e} \in Z(\mathbf{A})$, then $\mathbf{A} \vDash \kappa(\vec{e}, \vec{f}) \wedge \forall \vec{x}, \vec{y} \; P_F^U(\vec{e}, \vec{f}, \vec{x}, \vec{y})$, for every $F \in \mathcal{L}$, where $\vec{f}$ is the complement of $\vec{e}$. To prove the converse, suppose that $\vec{e} \in A^N$ is such that

(a) $\mathbf{A} \vDash \exists \vec{w} \; (\kappa(\vec{e}, \vec{w}) \wedge \forall \vec{x}, \vec{y} \; P_F^U(\vec{e}, \vec{w}, \vec{x}, \vec{y}))$, $F \in \mathcal{L}$.

We will prove that $\vec{e} \in Z(\mathbf{A})$. We first prove that

(b) If $\mathbf{A} \vDash \kappa(\vec{e}, \vec{f}) \wedge \kappa(\vec{e}, \vec{g})$, then $\vec{f} = \vec{g}$.

Since $\mathbf{A} \vDash \forall x, y, z \; D^U(\vec{e}, \vec{f}, x, y, z)$, we have that $d(x, y) = U^{\mathbf{A}}_{\vec{e},\vec{f}}(x, y)$ is a decomposition operation on the set A. Since $\mathbf{A} \vDash \pi(\vec{e}, \vec{f})$, we have that θ_d, δ_d are preserved by the binary operation $U^{\mathbf{A}}_{\vec{e},\vec{g}}(x, y)$. Thus we have that

$$\begin{aligned}
g_i &= U^{\mathbf{A}}_{\vec{e},\vec{g}}(1_i, 0_i) \quad (\text{since } \mathbf{A} \vDash B^U(\vec{e}, \vec{g})) \\
&\equiv U^{\mathbf{A}}_{\vec{e},\vec{g}}(f_i, e_i)(\theta_d) \quad (\text{since } 1_i \equiv f_i(\theta_d) \text{ and } 0_i \equiv e_i(\theta_d)) \\
&= U^{\mathbf{A}}_{\vec{e},\vec{g}}(f_i, U^{\mathbf{A}}_{\vec{e},\vec{g}}(0_i, 1_i)) \quad (\text{since } \mathbf{A} \vDash B^U(\vec{e}, \vec{g})) \\
&\equiv U^{\mathbf{A}}_{\vec{e},\vec{g}}(f_i, U^{\mathbf{A}}_{\vec{e},\vec{g}}(e_i, f_i))(\theta_d) \quad (\text{since } 1_i \equiv f_i(\theta_d) \text{ and } 0_i \equiv e_i(\theta_d)) \\
&= U^{\mathbf{A}}_{\vec{e},\vec{g}}(f_i, f_i) \quad (\text{since } \mathbf{A} \vDash \forall x, y, z \; D^U(\vec{e}, \vec{g}, x, y, z)) \\
&= f_i \quad (\text{since } \mathbf{A} \vDash \forall x, y, z \; D^U(\vec{e}, \vec{g}, x, y, z))
\end{aligned}$$

and similarly, we obtain that $g_i \equiv f_i(\delta_d)$. Hence $(g_i, f_i) \in \theta_d \cap \delta_d = \Delta^{\mathbf{A}}$ which proves (a). Now, (a) and (b) imply that there is $\vec{f} \in A^N$ such that $\mathbf{A} \vDash \kappa(\vec{e}, \vec{f}) \wedge \forall \vec{x}, \vec{y} \; P_F^U(\vec{e}, \vec{f}, \vec{x}, \vec{y})$, for every $F \in \mathcal{L}$, which by (2) says that $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ and hence $\vec{e} \in Z(\mathbf{A})$.

(4) By (1) $\lambda(\vec{z}, x, y) = U(\vec{z}, \vec{1}, x, y) = U(\vec{1}, \vec{z}, x, y)$ satisfies (L) of the introduction and the formula $\rho(\vec{z}, x, y) = U(\vec{z}, \vec{0}, x, y) = U(\vec{0}, \vec{z}, x, y)$ satisfies (R) of the introduction. Thus we have that, when $\vec{z}$ is central, the formula $C(\vec{z}, \vec{w})$ says $\vec{w} =$ complement of $\vec{z}$ in $\mathbf{Z}(\mathbf{A})$.

(5) Note that $\mathbf{A} \in \mathcal{V}_{\mathrm{DI}}$ if and only if $\vec{0} \neq \vec{1}$ and for every $\vec{e}, \vec{f} \in A^N$, $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ implies $\vec{e} \in \{\vec{0}, \vec{1}\}$, which by (2) can be expressed with a $\forall\exists$-sentence, when the language is finite.

(6) Let $\Sigma = \{\forall xyz\ D^U(\vec{z}, \vec{w}, x, y, z), B^U(\vec{z}, \vec{w})\} \cup \{\forall \vec{x}\vec{y}\ P^U_F(\vec{z}, \vec{w}, \vec{x}, \vec{y}) : F \in \mathcal{L}\}$. Suppose that $\mathbb{P}_u(\mathcal{V}_{\mathrm{SI}}) \subseteq \mathcal{V}_{\mathrm{DI}}$. We will prove that, there is a finite subset of Σ which defines $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ in $\mathcal{V}$. Note that

$$\mathbb{P}_u(\mathcal{V}_{\mathrm{SI}}) \vDash \bigwedge \Sigma \rightarrow (\vec{z} = \vec{0} \wedge \vec{w} = \vec{1}) \vee (\vec{z} = \vec{1} \wedge \vec{w} = \vec{0}).$$

By compactness, there is a finite subset $\Sigma_0 \subseteq \Sigma$ such that

$$\mathbb{P}_u(\mathcal{V}_{\mathrm{SI}}) \vDash \bigwedge \Sigma_0 \rightarrow (\vec{z} = \vec{0} \wedge \vec{w} = \vec{1}) \vee (\vec{z} = \vec{1} \wedge \vec{w} = \vec{0}).$$

We will prove that Σ_0 defines $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ in $\mathcal{V}$. Let $\mathbf{A} \in \mathcal{V}$ and suppose that $\mathbf{A} \vDash \bigwedge \Sigma_0(\vec{e}, \vec{f})$. We will see that $\vec{e} \diamond_{\mathbf{A}} \vec{f}$. We can suppose that $\mathbf{A} \leq \Pi\{\mathbf{A}_i : i \in I\}$ is a subdirect product, where each $\mathbf{A}_i$ is subdirectly irreducible. Since $\mathbf{A}_i \vDash \bigwedge \Sigma_0(\vec{e}(i), \vec{f}(i))$ we have that $\{\vec{e}(i), \vec{f}(i)\} = \{\vec{0}, \vec{1}\}$, for every $i \in I$. Thus $\vec{e}(i) \diamond_{\mathbf{A}_i} \vec{f}(i)$, for every $i \in I$ and hence (2) says that $\vec{e} \diamond_{\mathbf{A}} \vec{f}$.

The other implications of (6) are straightforward. □

Other basic properties of a variety with a decomposition term are detailed in the following:

Proposition 3.2. *Let $\mathcal{V}$ be a variety with $\vec{0}$ and $\vec{1}$ and let $U(\vec{z}, \vec{w}, x, y)$ be a decomposition term for $\mathcal{V}$.*

(1) $\theta^{\mathbf{A}}_{\vec{0}\vec{e}} = \theta^{\mathbf{A}}(\vec{0}, \vec{e})$, *for every* $\vec{e} \in Z(\mathbf{A})$.

(2) *For* $\vec{e}, \vec{f} \in Z(\mathbf{A})$, *the following are equivalent*

  (a) $\vec{e} \leq \vec{f}$

  (b) $\vec{e} \equiv \vec{0}(\theta^{\mathbf{A}}_{\vec{0}\vec{f}})$

  (c) $U^{\mathbf{A}}(\vec{f}, \vec{1}, 0_i, e_i) = U^{\mathbf{A}}(\vec{1}, \vec{f}, 0_i, e_i)$, $i = 1, \ldots, N$

(3) $Z(\Pi\{\mathbf{A}_i : i \in I\}) = \{\vec{e} \in (\Pi\{\mathbf{A}_i : i \in I\})^N : \vec{e}(i) \in Z(\mathbf{A}_i),\ i \in I\}$ *and the map*

$$Z(\Pi\{\mathbf{A}_i : i \in I\}) \rightarrow \Pi\{Z(\mathbf{A}_i) : i \in I\}$$

$$\vec{e} \rightarrow (\vec{e}(i))_{i \in I}$$

is a Boolean algebra isomorphism.

(4) *Let*

$$J(\vec{z}, \vec{w}, \vec{u}) = \bigwedge_{i=1}^{N} U(\vec{z}, \vec{1}, u_i, w_i) = U(\vec{1}, \vec{z}, u_i, w_i) \wedge U(\vec{z}, \vec{0}, u_i, 1_i) = U(\vec{0}, \vec{z}, u_i, 1_i)$$

$$M(\vec{z}, \vec{w}, \vec{u}) = \bigwedge_{i=1}^{N} U(\vec{z}, \vec{1}, u_i, 0_i) = U(\vec{1}, \vec{z}, u_i, 0_i) \wedge U(\vec{z}, \vec{0}, u_i, w_i) = U(\vec{0}, \vec{z}, u_i, w_i)$$

$$C(\vec{z},\vec{u}) = \bigwedge_{i=1}^{N} U(\vec{z},\vec{1},u_i,1_i) = U(\vec{1},\vec{z},u_i,1_i) \wedge U(\vec{z},\vec{0},u_i,0_i) = U(\vec{0},\vec{z},u_i,0_i)$$

$$O(\vec{z},\vec{w}) = \bigwedge_{i=1}^{N} U(\vec{w},\vec{1},0_i,z_i) = U(\vec{1},\vec{w},0_i,z_i).$$

For $\vec{e},\vec{f} \in Z(\mathbf{A})$, we have

$$\vec{e} \vee \vec{f} = \textit{only } \vec{g} \in A^N \quad \textit{such that } \mathbf{A} \vDash J(\vec{e},\vec{f},\vec{g})$$

$$\vec{e} \wedge \vec{f} = \textit{only } \vec{g} \in A^N \quad \textit{such that } \mathbf{A} \vDash M(\vec{e},\vec{f},\vec{g})$$

$$c(\vec{e}) = \textit{only } \vec{g} \in A^N \quad \textit{such that } \mathbf{A} \vDash C(\vec{e},\vec{g})$$

$$\vec{e} \leq \vec{f} \quad \textit{if and only if } \mathbf{A} \vDash O(\vec{e},\vec{f}).$$

(5) *If $\mathbf{A} \leq \Pi\{\mathbf{A}_i : i \in I\}$ is a subdirect product, then for $\vec{e},\vec{f} \in A^N$, we have that $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ if and only if $\vec{e}(i) \diamond_{\mathbf{A}_i} \vec{f}(i)$, for every $i \in I$. Moreover*

$$Z(\mathbf{A}) = \{\vec{e} \in Z(\Pi\{\mathbf{A}_i : i \in I\}) : \vec{e} \textit{ and } c^{\mathbf{Z}(\Pi\{\mathbf{A}_i : i\in I\})}(\vec{e}) \textit{ are in } A^N\}$$

$$\mathbf{Z}(\mathbf{A}) \leq \mathbf{Z}(\Pi\{\mathbf{A}_i : i \in I\}).$$

(6) *If $\sigma : \mathbf{A} \to \mathbf{B}$ is a onto homomorphism and $\vec{e} \diamond_{\mathbf{A}} \vec{f}$, then $\sigma(\vec{e}) \diamond_{\mathbf{B}} \sigma(\vec{f})$. The map*

$$Z(\mathbf{A}) \to Z(\mathbf{B})$$

$$\vec{e} \to \sigma(\vec{e})$$

is a (non-necessarily onto) Boolean algebra homomorphism.

Proof. (1) Since $\vec{e} \equiv \vec{0}(\theta^{\mathbf{A}}_{\vec{0}\vec{e}})$, we have that $\theta^{\mathbf{A}}(\vec{0},\vec{e}) \subseteq \theta^{\mathbf{A}}_{\vec{0}\vec{e}}$. Suppose $(a,b) \in \theta^{\mathbf{A}}_{\vec{0}\vec{e}}$. By (1) we have that $U^{\mathbf{A}}(\vec{e},\vec{1},a,b) = U^{\mathbf{A}}(\vec{1},\vec{e},a,b)$ and hence we have

$$\begin{aligned} a &= U^{\mathbf{A}}(\vec{0},\vec{1},a,b) \\ &\equiv U^{\mathbf{A}}(\vec{e},\vec{1},a,b)(\theta^{\mathbf{A}}_{\vec{0}\vec{e}}) \\ &= U^{\mathbf{A}}(\vec{1},\vec{e},a,b) \\ &\equiv U^{\mathbf{A}}(\vec{1},\vec{0},a,b)(\theta^{\mathbf{A}}_{\vec{0}\vec{e}}) \\ &= b. \end{aligned}$$

(2) By definition of $\mathbf{Z}(\mathbf{A})$ we have that $\vec{e} \leq \vec{f}$ if and only if $\theta^{\mathbf{A}}_{\vec{0}\vec{e}} \subseteq \theta^{\mathbf{A}}_{\vec{0}\vec{f}}$ and so, (a)⇔(b) follows from (1). (b)⇔(c) follows from (1) of Proposition 3.1.

(3) Since formulas of the form $\forall \bigwedge p = q$ are preserved by direct products and by direct factors, (2) of Proposition 3.1 says that $\vec{e} \diamond_{\Pi\{\mathbf{A}_i : i\in I\}} \vec{f}$ if and only if $\vec{e}(i) \diamond_{\mathbf{A}_i} \vec{f}(i)$, for every $i \in I$. Thus

$$Z(\Pi\{\mathbf{A}_i : i \in I\}) = \{\vec{e} \in (\Pi\{\mathbf{A}_i : i \in I\})^N : \vec{e}(i) \in Z(\mathbf{A}_i),\ i \in I\}.$$

To see that $\vec{e} \to (\vec{e}(i))_{i\in I}$ is a Boolean algebra isomorphism note that, by (a)⇔(c) of (2), this map and its inverse preserve order.

(4) Let $\vec{e}, \vec{f} \in Z(\mathbf{A})$. That $\vec{e} \leq \vec{f}$ if and only if $\mathbf{A} \vDash O(\vec{e}, \vec{f})$ follows from (a)$\Leftrightarrow$(c) of (2). We note that $J(\vec{e}, \vec{f}, \vec{u})$ says that $\vec{u} \equiv \vec{f}(\theta^{\mathbf{A}}_{\vec{0}\vec{e}})$ and $\vec{u} \equiv \vec{1}(\theta^{\mathbf{A}}_{\vec{1}\vec{e}})$ and so there exists exactly one $\vec{u}$ such that $\mathbf{A} \vDash J(\vec{e}, \vec{f}, \vec{u})$. In order to prove that, such a $\vec{u}$ is $\vec{e} \vee \vec{f}$ we can suppose that $\mathbf{A} = \mathbf{A}_1 \times \mathbf{A}_2$, $\vec{e} = [\vec{0}, \vec{1}]$ and $\vec{f} = [\vec{f_1}, \vec{f_2}]$. By (3), $\vec{f_i} \in Z(\mathbf{A}_i)$, $i = 1, 2$ and $\vec{e} \vee \vec{f} = [\vec{0} \vee \vec{f_1}, \vec{1} \vee \vec{f_2}] = [\vec{f_1}, \vec{1}]$ which says that $\vec{e} \vee \vec{f} \equiv \vec{f}(\theta^{\mathbf{A}}_{\vec{0}\vec{e}})$ and $\vec{e} \vee \vec{f} \equiv \vec{1}(\theta^{\mathbf{A}}_{\vec{1}\vec{e}})$. The other equalities can be proved in a similar fashion.

(5) The first part can be proved in a similar manner as in the proof of (3). That $\mathbf{Z}(\mathbf{A}) \leq \mathbf{Z}(\Pi\{\mathbf{A}_i : i \in I\})$ follows from (4).

(6) By (2) of Proposition 3.1 we have that $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ implies $\sigma(\vec{e}) \diamond_{\mathbf{B}} \sigma(\vec{f})$. To see that the map $\vec{e} \to \sigma(\vec{e})$ is a Boolean algebra homomorphism use (4). $\square$

Of course, if $\mathcal{V}$ has a short decomposition term, then $\mathcal{V}$ has the properties of the above two propositions. We give below some additional properties, which are guaranteed by the fact that the decomposition term is short.

Proposition 3.3. *Let $\mathcal{V}$ be a variety with $\vec{0}$ and $\vec{1}$ and let $u(\vec{z}, x, y)$ be a short decomposition term for $\mathcal{V}$. Let $\mathcal{L}$ be the language of $\mathcal{V}$.*

(1) *The formula $\lambda(\vec{z}, x, y) = u(\vec{z}, x, y) = y$ satisfies* (L) *of the introduction and the formula $\rho(\vec{z}, x, y) = u(\vec{z}, x, y) = x$ satisfies* (R) *of the introduction. Thus $\mathcal{V}$ has twice* EDFC.

(2) *The set $\{\forall xyz\ D^u(\vec{z}, x, y, z), B^u(\vec{z})\} \cup \{\forall \vec{x}\vec{y}\ P^u_F(\vec{z}, \vec{x}, \vec{y}) : F \in \mathcal{L}\}$ defines $\vec{e} \in Z(\mathbf{A})$ in $\mathcal{V}$.*

(3) $\{\forall xyz\ D^u(\vec{z}, x, y, z), B^u(\vec{z}), \bigwedge_{i=1}^N w_i = u_{\vec{z}}(1_i, 0_i)\} \cup \{\forall \vec{x}\vec{y}\ P^u_F(\vec{z}, \vec{x}, \vec{y}) : F \in \mathcal{L}\}$ *defines $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ in $\mathcal{V}$.*

(4) *When the above equivalent conditions hold, the sets of formulas of* (2) *and* (3) *can be replaced by a finite subset of itself if and only if there is a finite set of formulas defining $\vec{e} \in Z(\mathbf{A})$ in $\mathcal{V}$ if and only if $\mathcal{V}_{\mathrm{DI}}$ is a first order class if and only if $\mathbb{P}_u(\mathcal{V}_{\mathrm{SI}}) \subseteq \mathcal{V}_{\mathrm{DI}}$.*

(5) *If $\mathbf{A} \in \mathcal{V}$ and $\vec{e}, \vec{f} \in Z(\mathbf{A})$ then for $1 \leq i \leq N$*

$$(c(\vec{e}))_i = u^{\mathbf{A}}(\vec{e}, 1_i, 0_i)$$

$$(\vec{e} \vee \vec{f})_i = u^{\mathbf{A}}(\vec{e}, f_i, 1_i)$$

$$(\vec{e} \wedge \vec{f})_i = u^{\mathbf{A}}(\vec{e}, 0_i, f_i).$$

(6) *If $\mathbf{A} \leq \Pi\{\mathbf{A}_i : i \in I\}$ is subdirect, then $Z(\mathbf{A}) = Z(\Pi\{\mathbf{A}_i : i \in I\}) \cap A^N$.*

(2), (3) and (5) of the above proposition are proved in [7] for the case $N = 1$. We conclude this section by showing that the existence of a decomposition term for $\mathcal{V}$ does not imply the existence of an N-tuple of terms representing any of the operations $c^{\mathbf{Z}(\mathbf{A})}$, $\wedge^{\mathbf{Z}(\mathbf{A})}$ or $\vee^{\mathbf{Z}(\mathbf{A})}$ in $\mathcal{V}$. So the definability of each one of the operations of $\mathbf{Z}(\mathbf{A})$, given in (4) of Proposition 3.2, is optimum. First, we note that the variety of bounded distributive lattices is an example of a variety, where there is no term defining $c^{\mathbf{Z}(\mathbf{A})}$. Next, let $\mathbf{3} = (\{0, a, 1\}, U, 0, 1)$, where

$$U(z, w, x, y) = \begin{cases} x & \text{if } z = 0 \text{ and } w = 1 \\ y & \text{if } z = 1 \text{ and } w = 0 \\ a & \text{otherwise.} \end{cases}$$

Let $\mathcal{V}$ be the variety generated by $\mathbf{3}$. It is clear that $\mathcal{V} \vDash 0 = 1 \to x = y$ and that U is a decomposition term for $\mathcal{V}$. Let $\mathbf{A} = \mathbf{3} \times \mathbf{3} \times \mathbf{3}$ and let $e = (0, 1, 1)$, $g = (1, 1, 0)$. Note that $e, f \in Z(\mathbf{A}) = Z(\mathbf{3}) \times Z(\mathbf{3}) \times Z(\mathbf{3})$ and that $e \wedge g = (0, 1, 0)$. Let $\mathbf{S}$ be the subalgebra of $\mathbf{A}$ generated by e and g. Using the Universal algebra calculator we obtain that

$$S = A - \{(0, 1, 0), (1, 0, a), (1, 0, 0), (1, 0, 1), (0, 0, 1), (a, 0, 1)\}$$

and hence $e \wedge g \notin S$. Of course this says that there is no term representing the operation $\wedge^{\mathbf{Z}(\mathbf{A})}$ in $\mathcal{V}$. The case of $\vee^{\mathbf{Z}(\mathbf{A})}$ is similar.

4. $(\forall \bigwedge p = q)$-Definability of $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ and $\vec{e} \in Z(\mathbf{A})$

Theorem 4.1. *Let $\mathcal{V}$ be a variety with $\vec{0}$ and $\vec{1}$ and suppose that $\mathcal{V}$ has weak* ExDFC. *The following are equivalent*:

(1) *There exists a set of $(\forall \bigwedge p = q)$-formulas which defines $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ in $\mathcal{V}$.*
(2) *There exists a set of universal formulas which defines $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ in $\mathcal{V}$.*
(3) *If $\mathbf{S} \leq \mathbf{A}$ and $\vec{e}, \vec{f} \in S^N$, then $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ implies $\vec{e} \diamond_{\mathbf{S}} \vec{f}$.*
(4) *$\mathcal{V}$ has a decomposition term.*
(5) *If $\mathbf{S} \leq \mathbf{A} \times \mathbf{B}$ and $[\vec{0}, \vec{1}], [\vec{1}, \vec{0}] \in S^N$, then $\mathbf{S} = \mathbf{S}_1 \times \mathbf{S}_2$ for some algebras $\mathbf{S}_1, \mathbf{S}_2$.*

Proof. (1)⇒(2)⇒(3). Trivial.

(3)⇒(4). Let $\omega(\vec{z}, \vec{w}, x, y)$ be an existential formula defining $\theta^{\mathbf{A}}_{\vec{0}\vec{e}}$ in $\mathcal{V}$. Let $\mathbf{F}(x, y)$ be the $\mathcal{V}$-free algebra freely generated by $\{x, y\}$. Let $\mathbf{S}$ be the subalgebra of $\mathbf{F}(x, y) \times \mathbf{F}(x, y)$ generated by the set

$$\{(0_1, 1_1), \ldots, (0_N, 1_N), (1_1, 0_1), \ldots, (1_N, 0_N), (x, x), (y, y)\}.$$

By (3) we have that $[\vec{0}, \vec{1}] \diamond_{\mathbf{S}} [\vec{1}, \vec{0}]$. Since ω is existential, we have that $\mathbf{S} \vDash \omega([\vec{0}, \vec{1}], [\vec{1}, \vec{0}], s, t)$ implies $\mathbf{F}(x, y) \times \mathbf{F}(x, y) \vDash \omega([\vec{0}, \vec{1}], [\vec{1}, \vec{0}], s, t)$, for every $s, t \in S$. Thus we have that

$$\theta^{\mathbf{S}}_{[\vec{0},\vec{0}][\vec{0},\vec{1}]} \subseteq \theta^{\mathbf{F}(x,y) \times \mathbf{F}(x,y)}_{[\vec{0},\vec{0}][\vec{0},\vec{1}]} = \ker \pi_1$$

and similarly, we obtain that

$$\theta^{\mathbf{S}}_{[\vec{0},\vec{0}][\vec{1},\vec{0}]} \subseteq \theta^{\mathbf{F}(x,y) \times \mathbf{F}(x,y)}_{[\vec{0},\vec{0}][\vec{1},\vec{0}]} = \ker \pi_2.$$

Since $((x, x), (y, y)) \in \theta^{\mathbf{S}}_{[\vec{0},\vec{0}][\vec{0},\vec{1}]} \circ \theta^{\mathbf{S}}_{[\vec{0},\vec{0}][\vec{1},\vec{0}]} = \nabla^{\mathbf{S}}$ there is an element $s \in S$ such that

$$(s, (x, x)) \in \theta^{\mathbf{S}}_{[\vec{0},\vec{0}][\vec{0},\vec{1}]} \quad \text{and} \quad (s, (y, y)) \in \theta^{\mathbf{S}}_{[\vec{0},\vec{0}][\vec{1},\vec{0}]}.$$

Thus $s = (x, y) \in S$ and hence there is a term $U(\vec{z}, \vec{w}, x, y)$ such that

$$(x, y) = U^{\mathbf{F}(x,y)\times\mathbf{F}(x,y)}([\vec{0}, \vec{1}], [\vec{1}, \vec{0}], (x, x), (y, y))$$

from which, we obtain that U is a decomposition term for $\mathcal{V}$.

(4)⇒(5). Use that

$$U^{\mathbf{S}}([\vec{0}, \vec{1}], [\vec{1}, \vec{0}], (a_1, b_1), (a_2, b_2)) = d^{\mathbf{A}\times\mathbf{B}}((a_1, b_1), (a_2, b_2)) = (a_1, b_2)$$

for every $(a_1, b_1), (a_2, b_2) \in S$.

(5)⇒(1). Assume (5) holds. Note that (3) holds. Thus by (3)⇒(4) $\mathcal{V}$ has a decomposition term and hence (2) of Proposition 3.1 implies (1). □

The characterization of when the property $\vec{e} \in Z(\mathbf{A})$ is definable by $(\forall \bigwedge p = q)$-formulas is more difficult to prove. First some lemmas.

Lemma 4.2. *Let $\mathcal{V}$ be a variety with $\vec{0}$ and $\vec{1}$ with weak* ExDFC*. Then there is a positive formula satisfying* (*W*) *of the introduction.*

Proof. The proof is completely similar to that of [12, Theorem 2]. □

Lemma 4.3. *Let $\mathcal{V}$ be a variety with $\vec{0}$ and $\vec{1}$ for which there is a positive formula satisfying* (*W*) *of the introduction. Then, $\theta^{\mathbf{A}}_{\vec{0}\vec{e}} = \theta^{\mathbf{A}}(\vec{0}, \vec{e}) \vee \theta^{\mathbf{A}}(\vec{1}, c(\vec{e}))$ and $\theta^{\mathbf{A}}_{\vec{1}\vec{e}} = \theta^{\mathbf{A}}(\vec{1}, \vec{e}) \vee \theta^{\mathbf{A}}(\vec{0}, c(\vec{e}))$, whenever $\vec{e} \in Z(\mathbf{A})$.*

Proof. The proof is completely similar to that of Claim 2 of [13, Proposition 18]. □

Lemma 4.4. *Let $\gamma : \mathbf{A} \to \mathbf{B}$ be a homomorphism and let $\vec{a}, \vec{b} \in A^n$. Then $(a, b) \in \theta^{\mathbf{A}}(\vec{a}, \vec{b})$ implies $(\gamma(a), \gamma(b)) \in \theta^{\mathbf{B}}(\gamma(\vec{a}), \gamma(\vec{b}))$.*

Theorem 4.5. *Let $\mathcal{V}$ be a variety with $\vec{0}$ and $\vec{1}$ and suppose that $\mathcal{V}$ has weak* ExDFC. *The following are equivalent*:

(1) *There exists a set of $(\forall \bigwedge p = q)$-formulas which defines $\vec{e} \in Z(\mathbf{A})$ in $\mathcal{V}$.*
(2) *There exists a set of universal formulas which defines $\vec{e} \in Z(\mathbf{A})$ in $\mathcal{V}$.*
(3) *If $\mathbf{S} \leq \mathbf{A}$ and $\vec{e} \in S^N$, then $\vec{e} \in Z(\mathbf{A})$ implies $\vec{e} \in Z(\mathbf{S})$.*
(4) *$\mathcal{V}$ has a short decomposition term.*
(5) *If $\mathbf{S} \leq \mathbf{A}_1 \times \mathbf{A}_2$ and $[\vec{0}, \vec{1}] \in S^N$, then $\mathbf{S} = \mathbf{S}_1 \times \mathbf{S}_2$ for some algebras $\mathbf{S}_1, \mathbf{S}_2$.*

Proof. (1)⇒(2)⇒(3). Trivial.

(3)⇒(4). First, we will prove that the complement of a central element is definable by a N-tuple of terms. Let X be an infinite set of variables and let $\mathbf{F}(X)$ be the $\mathcal{V}$-free algebra freely generated by X. Let $\mathbf{S}$ be the subalgebra of $\mathbf{F}(X) \times \mathbf{F}(X)$ generated by the set $\{(0_1, 1_1), \ldots, (0_N, 1_N)\} \cup \{(x, x)\}_{x \in X}$. Since $[\vec{0}, \vec{1}] \in Z(\mathbf{A})$, (3) implies that $[\vec{0}, \vec{1}] \in Z(\mathbf{S})$. Therefore there are terms $t_1, \ldots, t_N$ and $x_1, \ldots, x_n \in X$

such that $\vec{t}^{\mathbf{S}}([\vec{0},\vec{1}],(x_1,x_1),\ldots,(x_n,x_n)) = [\vec{t}(\vec{0},\vec{x}),\vec{t}(\vec{1},\vec{x})]$ is the complement of $[\vec{0},\vec{1}]$ in $\mathbf{Z}(\mathbf{S})$. Let $\vec{e} = [\vec{0},\vec{1}]$. By Lemmas 4.2 and 4.3 we have that

$$\theta^{\mathbf{S}}_{\vec{0}\vec{e}} = \theta^{\mathbf{S}}(\vec{0},\vec{e}) \vee \theta^{\mathbf{S}}(\vec{1},[\vec{t}(\vec{0},\vec{x}),\vec{t}(\vec{1},\vec{x})])$$
$$\theta^{\mathbf{S}}_{\vec{1}\vec{e}} = \theta^{\mathbf{S}}(\vec{1},\vec{e}) \vee \theta^{\mathbf{S}}(\vec{0},[\vec{t}(\vec{0},\vec{x}),\vec{t}(\vec{1},\vec{x})]).$$

Now let $x, y \in X$ be such that $x, y \neq x_1,\ldots,x_n$. Since $\theta^{\mathbf{S}}_{\vec{0}\vec{e}} \circ \theta^{\mathbf{S}}_{\vec{1}\vec{e}} = \nabla$, there is a term u and $y_1,\ldots,y_m \in X - \{x_1,\ldots,x_n,x,y\}$ such that

$$((x,x), u^{\mathbf{S}}(\vec{e},(x,x),(y,y),[\vec{x},\vec{x}],[\vec{y},\vec{y}])) \in \theta^{\mathbf{S}}_{\vec{0}\vec{e}}$$
$$((y,y), u^{\mathbf{S}}(\vec{e},(x,x),(y,y),[\vec{x},\vec{x}],[\vec{y},\vec{y}])) \in \theta^{\mathbf{S}}_{\vec{1}\vec{e}}.$$

Thus we have that

$$((x,x),(u(\vec{0},x,y,\vec{x},\vec{y}),u(\vec{1},x,y,\vec{x},\vec{y}))) \in \theta^{\mathbf{S}}(\vec{0},\vec{e}) \vee \theta^{\mathbf{S}}(\vec{1},[\vec{t}(\vec{0},\vec{x}),\vec{t}(\vec{1},\vec{x})])$$
$$((y,y),(u(\vec{0},x,y,\vec{x},\vec{y}),u(\vec{1},x,y,\vec{x},\vec{y}))) \in \theta^{\mathbf{S}}(\vec{1},\vec{e}) \vee \theta^{\mathbf{S}}(\vec{0},[\vec{t}(\vec{0},\vec{x}),\vec{t}(\vec{1},\vec{x})]).$$

Fix $i \in \{1,\ldots,N\}$. Let $\sigma_1,\sigma_2,\sigma_3 : \mathbf{F}(X) \to \mathbf{F}(X)$ be homomorphisms such that

$$\sigma_1(x) = \sigma_2(x) = \sigma_2(y) = 1_i$$
$$\sigma_1(y) = \sigma_3(y) = \sigma_3(x) = 0_i$$
$$\sigma_l(v) = v, \quad \text{for every } v \in X - \{x,y\} \text{ and } l = 1,2,3.$$

Let $\gamma : \mathbf{S} \to \mathbf{F}(X) \times \mathbf{F}(X)$ be given by $\gamma(a,b) = (\sigma_1(a),\sigma_2(b))$. Note that, γ is a homomorphism and $\mathrm{Im}\gamma \subseteq S$. By Lemma 4.4 we have that

$$(\gamma(x,x),\gamma(u(\vec{0},x,y,\vec{x},\vec{y}),u(\vec{1},x,y,\vec{x},\vec{y}))) \in \theta^{\mathbf{S}}(\vec{0},\gamma(\vec{e})) \vee \theta^{\mathbf{S}}(\vec{1},\gamma[\vec{t}(\vec{0},\vec{x}),\vec{t}(\vec{1},\vec{x})])$$
$$(\gamma(y,y),\gamma(u(\vec{0},x,y,\vec{x},\vec{y}),u(\vec{1},x,y,\vec{x},\vec{y}))) \in \theta^{\mathbf{S}}(\vec{1},\gamma(\vec{e})) \vee \theta^{\mathbf{S}}(\vec{0},\gamma[\vec{t}(\vec{0},\vec{x}),\vec{t}(\vec{1},\vec{x})]).$$

Thus we obtain

$$((1_i,1_i),(u(\vec{0},1_i,0_i,\vec{x},\vec{y}),u(\vec{1},1_i,1_i,\vec{x},\vec{y}))) \in \theta^{\mathbf{S}}(\vec{0},\vec{e}) \vee \theta^{\mathbf{S}}(\vec{1},[\vec{t}(\vec{0},\vec{x}),\vec{t}(\vec{1},\vec{x})])$$
$$((0_i,1_i),(u(\vec{0},1_i,0_i,\vec{x},\vec{y}),u(\vec{1},1_i,1_i,\vec{x},\vec{y}))) \in \theta^{\mathbf{S}}(\vec{1},\vec{e}) \vee \theta^{\mathbf{S}}(\vec{0},[\vec{t}(\vec{0},\vec{x}),\vec{t}(\vec{1},\vec{x})]).$$

But

$$((1_i,1_i),(1_i,1_i)) \in \theta^{\mathbf{S}}(\vec{0},\vec{e}) \vee \theta^{\mathbf{S}}(\vec{1},[\vec{t}(\vec{0},\vec{x}),\vec{t}(\vec{1},\vec{x})])$$
$$((0_i,1_i),(1_i,1_i)) \in \theta^{\mathbf{S}}(\vec{1},\vec{e}) \vee \theta^{\mathbf{S}}(\vec{0},[\vec{t}(\vec{0},\vec{x}),\vec{t}(\vec{1},\vec{x})])$$

which since $\theta^{\mathbf{S}}_{\vec{0}\vec{e}} \cap \theta^{\mathbf{S}}_{\vec{1}\vec{e}} = \Delta$ says that

(a) $(u(\vec{0},1_i,0_i,\vec{x},\vec{y}),u(\vec{1},1_i,1_i,\vec{x},\vec{y})) = (1_i,1_i)$.

Similarly, taking $\gamma(a,b)=(\sigma_1(a),\sigma_1(b))$ and applying Lemma 4.4, we obtain

$$((1_i,1_i),(u(\vec{0},1_i,0_i,\vec{x},\vec{y}),u(\vec{1},1_i,0_i,\vec{x},\vec{y}))) \in \theta^{\mathbf{S}}(\vec{0},\vec{e}) \vee \theta^{\mathbf{S}}(\vec{1},[\vec{t}(\vec{0},\vec{x}),\vec{t}(\vec{1},\vec{x})])$$

$$((0_i,0_i),(u(\vec{0},1_i,0_i,\vec{x},\vec{y}),u(\vec{1},1_i,0_i,\vec{x},\vec{y}))) \in \theta^{\mathbf{S}}(\vec{1},\vec{e}) \vee \theta^{\mathbf{S}}(\vec{0},[\vec{t}(\vec{0},\vec{x}),\vec{t}(\vec{1},\vec{x})]).$$

But

$$((1_i,1_i),(t_i(\vec{0},\vec{x}),t_i(\vec{1},\vec{x}))) \in \theta^{\mathbf{S}}(\vec{0},\vec{e}) \vee \theta^{\mathbf{S}}(\vec{1},[\vec{t}(\vec{0},\vec{x}),\vec{t}(\vec{1},\vec{x})])$$

$$((0_i,0_i),(t_i(\vec{0},\vec{x}),t_i(\vec{1},\vec{x}))) \in \theta^{\mathbf{S}}(\vec{1},\vec{e}) \vee \theta^{\mathbf{S}}(\vec{0},[\vec{t}(\vec{0},\vec{x}),\vec{t}(\vec{1},\vec{x})])$$

which says that

(b) $(u(\vec{0},1_i,0_i,\vec{x},\vec{y}),u(\vec{1},1_i,0_i,\vec{x},\vec{y}))=(t_i(\vec{0},\vec{x}),t_i(\vec{1},\vec{x}))$.

Taking $\gamma(a,b)=(\sigma_3(a),\sigma_1(b))$ we obtain

(c) $(u(\vec{0},0_i,0_i,\vec{x},\vec{y}),u(\vec{1},1_i,0_i,\vec{x},\vec{y}))=(0_i,0_i)$.

Combining (a), (b) and (c) we obtain that $\mathcal{V}\vDash \vec{t}(\vec{0},\vec{x})=\vec{1}\wedge\vec{t}(\vec{1},\vec{x})=\vec{0}$. Thus taking $\vec{C}(\vec{z})=\vec{t}(\vec{z},0_1,\ldots,0_1)\in T(\vec{z})^N$, we obtain that $\mathcal{V}\vDash\vec{C}(\vec{0})=\vec{1}\wedge\vec{C}(\vec{1})=\vec{0}$. We note that $\vec{C}([\vec{0},\vec{1}])=[\vec{1},\vec{0}]$ and hence $\vec{C}(\vec{e})=c^{\mathbf{Z}(\mathbf{A})}(\vec{e})$, for every $\vec{e}\in Z(\mathbf{A})$.

Using $\vec{C}$ and that (3) holds we can easily prove that (3) of Theorem 4.1 holds. Thus, Theorem 4.1 says that there is a decomposition term $U(\vec{z},\vec{w},x,y)$ for $\mathcal{V}$. We note that $u(\vec{z},x,y)=U(\vec{z},\vec{C}(\vec{z}),x,y)$ is a short decomposition term for $\mathcal{V}$ and hence (4) holds.

(4)⇒(5)⇒(1). It is similar to the proof of (4)⇒(5)⇒(1) of Theorem 4.1. □

4.1. *Applications to unital sesquishells*

Following Knoebel [6, VII], we say that $\mathcal{V}$ is a *variety of unital sesquishells* if there are two constants $0,1$ and a binary term $\times$ such that $\mathcal{V}$ satisfies the following identities

$$0\times x=0\times y$$

$$1\times x=x.$$

It is clear that, $\mathcal{V}\vDash 0=1\to x=y$ and it is easy to check that $\mathcal{V}$ has right EDFC via the formula $\psi(z,x,y)=z\times x=z\times y$. Hence $\mathcal{V}$ has weak EDFC via the formula $\omega(z,w,x,y)=w\times x=w\times y$. In [1] it is proved that the variety of bounded meet semilattices has not left EDFC and hence not every variety of unital sesquishells has left EDFC. Unital sesquishells are the most general type of algebras with 0 and 1 studied in [6] and Problem VII.3.5 of this book ask for a characterization of those varieties of unital sesquishells for which there exists a set of $(\forall\bigwedge p=q)$-formulas which defines $e\diamond_{\mathbf{A}} f$. Theorem 4.1 produces the following solution to this problem.

Proposition 4.6. *Let $\mathcal{V}$ be a variety of unital sesquishells. The following are equivalent*

(1) *There exists a set of $(\forall \bigwedge p = q)$-formulas which defines $e \diamond_{\mathbf{A}} f$ (respectively $e \in Z(\mathbf{A})$) in $\mathcal{V}$.*
(2) *There exists a set of universal formulas which defines $e \diamond_{\mathbf{A}} f$ (respectively $e \in Z(\mathbf{A})$) in $\mathcal{V}$.*
(3) *If $\mathbf{S} \leq \mathbf{A}$ and $e, f \in S^N$ (respectively $e \in S^N$), then $e \diamond_{\mathbf{A}} f$ (respectively $e \in Z(\mathbf{A})$) implies $e \diamond_{\mathbf{S}} f$ (respectively $e \in Z(\mathbf{S})$).*
(4) *$\mathcal{V}$ has a decomposition (respectively short decomposition) term.*
(5) *If $\mathbf{S} \leq \mathbf{A}_1 \times \mathbf{A}_2$ and $(0,1),(1,0) \in S^N$ (respectively $(0,1) \in S^N$), then $\mathbf{S} = \mathbf{S}_1 \times \mathbf{S}_2$ for some subalgebras $\mathbf{S}_1, \mathbf{S}_2$.*

Proof. Since $\mathcal{V}$ has weak EDFC we can apply Theorems 4.1 and 4.5. □

5. $(\bigwedge p = q)$-Definability of $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ and $\vec{e} \in Z(\mathbf{A})$

Theorem 5.1. *Let $\mathcal{V}$ be a variety with $\vec{0}$ and $\vec{1}$ and suppose that $\mathcal{V}$ has weak* ExDFC. *Let $\mathcal{L}$ be the language of $\mathcal{V}$. The following are equivalent:*

(1) *There exists a set of $(\bigwedge p = q)$-formulas which defines $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ in $\mathcal{V}$.*
(2) *There exists a set of open formulas which defines $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ in $\mathcal{V}$.*
(3) *If $\mathbf{S}_1 \leq \mathbf{A}_1, \mathbf{S}_2 \leq \mathbf{A}_2, \vec{e} \diamond_{\mathbf{A}_1} \vec{f}, \vec{e}, \vec{f} \in S_1^N$ and $\sigma : S_1 \to S_2$ is an isomorphism, then $\sigma(\vec{e}) \diamond_{\mathbf{A}_2} \sigma(\vec{f})$.*
(4) *$\mathcal{V}$ satisfies the equivalent conditions of Theorem 4.1 and if $\mathbf{A} \times \mathbf{B} \leq \mathbf{E}$, then there is an isomorphism $\sigma : E \to A_1 \times B_1$, with $\mathbf{A} \leq \mathbf{A}_1$, $\mathbf{B} \leq \mathbf{B}_1$ such that the following diagram commutes*

$$\begin{tikzcd} A \times B \arrow[r, hook] \arrow[d] & E \arrow[dl, "\sigma"] \\ A_1 \times B_1 & \end{tikzcd} .$$

(5) *$\mathcal{V}$ satisfies the equivalent conditions of Theorem 4.1 and if $\mathbf{S} \leq \mathbf{A}$, then $\vec{e} \diamond_{\mathbf{S}} \vec{f}$ implies $\vec{e} \diamond_{\mathbf{A}} \vec{f}$.*
(6) *$\mathcal{V}$ satisfies the equivalent conditions of Theorem 4.1 and $\mathbb{S}(\mathcal{V}_{\mathrm{SI}}) \subseteq \mathcal{V}_{\mathrm{DI}}$.*
(7) *$\mathcal{V}$ has a decomposition term $U(\vec{z}, \vec{w}, x, y)$ and the set formed by*

$$D^U(\vec{z}, \vec{w}, r(\vec{z}, \vec{w}), s(\vec{z}, \vec{w}), t(\vec{z}, \vec{w})) \quad B^U(\vec{z}, \vec{w})$$

$$P_F^U(\vec{z}, \vec{w}, \vec{r}(\vec{z}, \vec{w}), \vec{s}(\vec{z}, \vec{w}))$$

with $F \in \mathcal{L}$ and $r, s, t, r_1, \ldots, r_n, s_1, \ldots, s_n \in T(\vec{z}, \vec{w})$, defines $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ in $\mathcal{V}$.
(8) *$\mathcal{V}$ has a decomposition term $U(\vec{z}, \vec{w}, x, y)$ and the set formed by*

$$D^U(\vec{z}, \vec{w}, r(\vec{z}, \vec{w}), s(\vec{z}, \vec{w}), t(\vec{z}, \vec{w})) \quad B^U(\vec{z}, \vec{w})$$

$$v(U_{\vec{z},\vec{w}}(r(\vec{z}, \vec{w}), s(\vec{z}, \vec{w})), \vec{z}, \vec{w}) = U_{\vec{z},\vec{w}}(v(r(\vec{z}, \vec{w}), \vec{z}, \vec{w}), v(s(\vec{z}, \vec{w}), \vec{z}, \vec{w}))$$

with $r, s, t \in T(\vec{z}, \vec{w}), v \in T(x, \vec{z}, \vec{w})$, defines $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ in $\mathcal{V}$.

When the above equivalent conditions hold, the sets of formulas of (1), (2), (7) *and* (8) *can be replaced by a finite subset of itself if and only if there is a finite set of formulas defining* $\vec{e}\diamond_{\mathbf{A}}\vec{f}$ *in* $\mathcal{V}$ *if and only if* $\mathbb{P}_u(\mathcal{V}_{\mathrm{SI}})\subseteq\mathcal{V}_{\mathrm{DI}}$ *if and only if* $\mathcal{V}_{\mathrm{DI}}$ *is a first order class.*

Proof. (1)⇒(2)⇒(3). Trivial.

(3)⇒(4). It is easy to check that $\mathcal{V}$ satisfies (3) of Theorem 4.1. So, we have that $\mathcal{V}$ satisfies the equivalent conditions of Theorem 4.1, in particular we have that $\mathcal{V}$ has a decomposition term $U(\vec{z},\vec{w},x,y)$. Suppose that $\mathbf{A}\times\mathbf{B}\leq\mathbf{E}$. Taking $\mathbf{S}_1=\mathbf{A}_1=\mathbf{A}\times\mathbf{B}=\mathbf{S}_2$, $\mathbf{A}_2=\mathbf{E}$, $\sigma=id_{S_1}$ and $\vec{e}=[\vec{0},\vec{1}]$, $\vec{f}=[\vec{1},\vec{0}]$, we have that (3) implies that $[\vec{0},\vec{1}]\diamond_{\mathbf{E}}[\vec{1},\vec{0}]$. Note that $d(x,y)=U^{\mathbf{E}}_{[\vec{0},\vec{1}],[\vec{1},\vec{0}]}(x,y)$ is a decomposition operation on E whose restriction to $A\times B$ is $d^{\mathbf{A}\times\mathbf{B}}$. Let $\sigma_1: A\to E/\theta_d$, be given by $\sigma_1(a)=(a,b)/\theta_d$, with $b\in B$ and define $\sigma_2: B\to E/\delta_d$, by $\sigma_2(b)=(a,b)/\delta_d$, with $a\in A$. It is easy to check that σ_1 and σ_2 are embeddings. Thus if we denote $\tilde{\mathbf{A}}_1=\mathbf{E}/\theta_d$ and $\tilde{\mathbf{B}}_1=\mathbf{E}/\delta_d$, we obtain that the following diagram commutes

$$\begin{array}{ccc} A\times B & \hookrightarrow & E \\ \downarrow\gamma & \swarrow\tilde{\sigma}, & \\ \tilde{A}_1\times\tilde{B}_1 & & \end{array}$$

where $\gamma(a,b)=(\sigma_1(a),\sigma_2(b))$ and $\tilde{\sigma}(x)=(x/\theta_d,x/\delta_d)$. Now it is easy to define $\mathbf{A}_1,\mathbf{B}_1$ and σ in such a manner that (4) holds.

(4)⇒(5). Trivial.

(5)⇒(6). Suppose that $\mathbf{S}\leq\mathbf{A}\in\mathcal{V}_{\mathrm{SI}}$. If $\vec{e}\diamond_{\mathbf{S}}\vec{f}$, then by (5) we have that $\vec{e}\diamond_{\mathbf{A}}\vec{f}$ and hence $\vec{e}\in\{\vec{0},\vec{1}\}$, since $\mathbf{A}\in\mathcal{V}_{\mathrm{SI}}\subseteq\mathcal{V}_{\mathrm{DI}}$. So $\mathbf{S}\in\mathcal{V}_{\mathrm{DI}}$.

(6)⇒(7). By (6) $\mathcal{V}$ has a decomposition term $U(\vec{z},\vec{w},x,y)$. Let $\vec{e},\vec{f}\in A^N$. By (2) of Proposition 3.1, if $\vec{e}\diamond_{\mathbf{A}}\vec{f}$, then

(a) $\mathbf{A}\vDash D^U(\vec{e},\vec{f},r(\vec{e},\vec{f}),s(\vec{e},\vec{f}),t(\vec{e},\vec{f}))\wedge B^U(\vec{e},\vec{f})\wedge P^U_F(\vec{e},\vec{f},\vec{r}(\vec{e},\vec{f}),\vec{s}(\vec{e},\vec{f}))$, for every $F\in\mathcal{L}$, $r,s,t,r_1,\dots,r_n,s_1,\dots,s_n\in T(\vec{z},\vec{w})$.

Conversely, we will prove that if (a) holds, then $\vec{e}\diamond_{\mathbf{A}}\vec{f}$. So suppose that (a) holds. We can suppose that $\mathbf{A}\subseteq\Pi\{\mathbf{A}_i : i\in I\}$ is a subdirect product with each factor in $\mathcal{V}_{\mathrm{SI}}$. Let $\mathbf{S}_i$ be the subalgebra of $\mathbf{A}_i$ generated by $\{e_1(i),\dots,e_N(i),f_1(i),\dots,f_N(i)\}$. Since $\mathbb{S}(\mathcal{V}_{\mathrm{SI}})\subseteq\mathcal{V}_{\mathrm{DI}}$, we have that $\mathbf{S}_i\in\mathcal{V}_{\mathrm{DI}}$. By (a) and (2) of Proposition 3.1 we have that $\vec{e}(i)\diamond_{\mathbf{S}_i}\vec{f}(i)$, where $\vec{e}(i)=(e_1(i),\dots,e_N(i))$ and $\vec{f}(i)=(f_1(i),\dots,f_N(i))$. Since each $\mathbf{S}_i\in\mathcal{V}_{\mathrm{DI}}$ we have that, for every $i\in I$, either $\vec{e}(i)=\vec{0}$ and $\vec{f}(i)=\vec{1}$ or $\vec{e}(i)=\vec{1}$ and $\vec{f}(i)=\vec{0}$. By (5) of Proposition 3.1 we have that $\vec{e}\diamond_{\mathbf{A}}\vec{f}$.

(7)⇒(8). Suppose that $\vec{e},\vec{f}\in A^N$ are such that

(b) $\mathbf{A}\vDash D^U(\vec{e},\vec{f},r(\vec{e},\vec{f}),s(\vec{e},\vec{f}),t(\vec{e},\vec{f}))\wedge B^U(\vec{e},\vec{f})\wedge$
$$\wedge v(U_{\vec{e},\vec{f}}(r(\vec{e},\vec{f}),s(\vec{e},\vec{f})),\vec{e},\vec{f}) = U_{\vec{e},\vec{f}}(v(r(\vec{e},\vec{f}),\vec{e},\vec{f}),v(s(\vec{e},\vec{f}),\vec{e},\vec{f}))$$
for every $r,s,t\in T(\vec{z},\vec{w})$, $v\in T(x,\vec{z},\vec{w})$.

We will prove that $\vec{e} \diamond_{\mathbf{A}} \vec{f}$. By (7) it suffices to prove that $\vec{e} \diamond_{\mathbf{S}} \vec{f}$ where $\mathbf{S}$ is the subalgebra generated by $\{e_1, \dots, e_N, f_1, \dots, f_N\}$. By (b) we have that $\mathbf{A} \vDash \forall x, y, z\, D^U(\vec{e}, \vec{f}, x, y, z) \wedge B^U(\vec{e}, \vec{f})$ and hence $d(x, y) = U^{\mathbf{S}}_{\vec{e}\vec{f}}(x, y)$ is a decomposition operation on the set S which identifies $\vec{e}$ with $[\vec{0}, \vec{1}]$ and $\vec{f}$ with $[\vec{1}, \vec{0}]$. Thus, we only need to prove that $\mathbf{S} \vDash \forall \vec{x}\vec{y}\, P^U_F(\vec{e}, \vec{f}, \vec{x}, \vec{y})$, for every $F \in \mathcal{L}$, or equivalently

(c) θ_d and δ_d preserve $F^{\mathbf{S}}$, for every $F \in \mathcal{L}$.

Since $\mathbf{S} \vDash \forall xy\, v(U_{\vec{e},\vec{f}}(x, y), \vec{e}, \vec{f}) = U_{\vec{e},\vec{f}}(v(x, \vec{e}, \vec{f}), v(y, \vec{e}, \vec{f}))$, for every $v \in T(x, \vec{z}, \vec{w})$, we have that θ_d and δ_d preserve the function $x \to v^{\mathbf{S}}(x, \vec{e}, \vec{f})$, for every $v \in T(x, \vec{z}, \vec{w})$. So θ_d and δ_d preserve every unary polynomial on $\mathbf{S}$ which says that (c) holds. □

Corollary 5.2. *If $\mathcal{V}$ is a locally finite variety satisfying the equivalent conditions of the above theorem, then the class $\mathcal{V}_{\mathrm{DI}}$ is a universal class.*

Proof. Note that by (8) there exists a finite set of $(\bigwedge p = q)$-formulas which defines $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ in $\mathcal{V}$. Since $\mathbf{A} \in \mathcal{V}_{\mathrm{DI}}$ if and only if $\vec{0} \neq \vec{1}$ and for every $\vec{e}, \vec{f} \in A^N$, $\vec{e} \diamond_{\mathbf{A}} \vec{f}$ implies $\vec{e} \in \{\vec{0}, \vec{1}\}$, we can axiomatize the class $\mathcal{V}_{\mathrm{DI}}$ with a universal sentence. □

For the case of a short decomposition term, we have the following analogous to the above theorem. The proof is similar.

Theorem 5.3. *Let $\mathcal{V}$ be a variety with $\vec{0}$ and $\vec{1}$ and suppose that $\mathcal{V}$ has weak* ExDFC. *Let $\mathcal{L}$ be the language of $\mathcal{V}$. The following are equivalent:*

(1) *There exists a set of $(\bigwedge p = q)$-formulas which defines $\vec{e} \in Z(\mathbf{A})$ in $\mathcal{V}$.*
(2) *There exists a set of open formulas which defines $\vec{e} \in Z(\mathbf{A})$ in $\mathcal{V}$.*
(3) *If $\mathbf{S}_1 \leq \mathbf{A}_1, \mathbf{S}_2 \leq \mathbf{A}_2$, $\vec{e} \in Z(\mathbf{A}_1)$, $\vec{e} \in S_1^N$ and $\sigma : S_1 \to S_2$ is an isomorphism, then $\sigma(\vec{e}) \in Z(\mathbf{A}_2)$.*
(4) *$\mathcal{V}$ satisfies the equivalent conditions of Theorem* 4.5 *and if $\mathbf{A} \times \mathbf{B} \leq \mathbf{E}$, then there is an isomorphism $\sigma : E \to A_1 \times B_1$, with $\mathbf{A} \leq \mathbf{A}_1$, $\mathbf{B} \leq \mathbf{B}_1$ such that the following diagram commutes*

$$\begin{array}{ccc} A \times B & \hookrightarrow & E \\ \downarrow & \swarrow_{\sigma} & \\ A_1 \times B_1 & & \end{array}.$$

(5) *$\mathcal{V}$ satisfies the equivalent conditions of Theorem* 4.5 *and if $\mathbf{S} \leq \mathbf{A}$, then $\vec{e} \in Z(\mathbf{S})$ implies $\vec{e} \in Z(\mathbf{A})$.*
(6) *$\mathcal{V}$ satisfies the equivalent conditions of Theorem* 4.5 *and $\mathbb{S}(\mathcal{V}_{\mathrm{SI}}) \subseteq \mathcal{V}_{\mathrm{DI}}$.*
(7) *$\mathcal{V}$ has a short decomposition term $u(\vec{z}, x, y)$ and the set formed by all the equations of the form*

$$D^u(\vec{z}, r(\vec{z}), s(\vec{z}), t(\vec{z})) \quad B^u(\vec{z}) \quad P^u_F(\vec{z}, \vec{r}(\vec{z}), \vec{s}(\vec{z}))$$

with $F \in \mathcal{L}$ and $r, s, t, r_1, \dots, r_n, s_1, \dots, s_n \in T(\vec{z})$, defines $\vec{e} \in Z(\mathbf{A})$ in $\mathcal{V}$.

(8) *$\mathcal{V}$ has a short decomposition term $u(\vec{z}, x, y)$ and the set formed by all the equations of the form*

$$D^u(\vec{z}, r(\vec{z}), s(\vec{z}), t(\vec{z})) \quad B^u(\vec{z})$$

$$v(u_{\vec{z}}(r(\vec{z}), s(\vec{z})), \vec{z}) = u_{\vec{z}}(v(r(\vec{z}), \vec{z}), v(s(\vec{z}), \vec{z}))$$

with $r, s, t \in T(\vec{z}), v \in T(x, \vec{z})$, defines $\vec{e} \in Z(\mathbf{A})$ in $\mathcal{V}$.

When the above equivalent conditions hold, the sets of formulas of (1), (2), (7) *and* (8) *can be replaced by a finite subset of itself if and only if there is a finite set of formulas defining $\vec{e} \in Z(\mathbf{A})$ in $\mathcal{V}$ if and only if $\mathbb{P}_u(\mathcal{V}_{\mathrm{SI}}) \subseteq \mathcal{V}_{\mathrm{DI}}$ if and only if $\mathcal{V}_{\mathrm{DI}}$ is a first-order class.*

5.1. *Equational characterization*

We will show how the fact that a variety satisfies the equivalent conditions of the above theorem can be expressed by means of certain equations. First a lemma.

Lemma 5.4. *Let $\mathcal{V}$ be a variety and let $r, s, r_1, \ldots, r_m, s_1, \ldots, s_m \in T(x_1, \ldots, x_n)$. Then $\mathcal{V} \vDash \vec{r} = \vec{s} \to r = s$ if and only if there exist $(m+n)$-ary terms $p_1(\vec{u}, \vec{x}), \ldots, p_k(\vec{u}, \vec{x})$, with k odd such that the following identities hold in $\mathcal{V}$*

$$r(\vec{x}) = p_1(\vec{r}(\vec{x}), \vec{x})$$

$$p_i(\vec{s}(\vec{x}), \vec{x}) = p_{i+1}(\vec{s}(\vec{x}), \vec{x}), \ i \text{ odd}$$

$$p_i(\vec{r}(\vec{x}), \vec{x}) = p_{i+1}(\vec{r}(\vec{x}), \vec{x}), \ i \text{ even}$$

$$p_k(\vec{s}(\vec{x}), \vec{x}) = s.$$

Proof. Combine [1, Lemmas 1 and 2]. □

Suppose that $\mathcal{V}$ is a variety satisfying (8) of the above theorem. For $R \subseteq T(\vec{z})$ and $S \subseteq T(x, \vec{z})$ let $\zeta_{R,S}$ denote the formula

$$\zeta_{R,S}(\vec{z}) = \bigwedge_{\substack{r,s,t,\in R \\ v \in S}} D^u(\vec{z}, r, s, t) \wedge B^u(\vec{z}) \wedge v(u_{\vec{z}}(r, s), \vec{z}) = u_{\vec{z}}(v(r, \vec{z}), v(s, \vec{z})).$$

We note that, for each $F \in \mathcal{L}$ we have that

$$\mathcal{V} \vDash \zeta_{T(\vec{z}), T(x, \vec{z})}(\vec{z}) \to (D^u(\vec{z}, x, y, z) \wedge B^u(\vec{z}) \wedge P^u_F(\vec{z}, \vec{x}, \vec{y})).$$

Thus by compactness we have that (8) of the above theorem is equivalent to the following condition.

(9) There exist terms $u, 0_1, \ldots, 0_N, 1_1, \ldots, 1_N$ such that $\mathcal{V}$ satisfies the identities $u(\vec{0}, x, y) = x$ and $u(\vec{1}, x, y) = y$ and for each $F \in \mathcal{L}$, there are finite subsets $R^F \subseteq T(\vec{z})$ and $S^F \subseteq T(x, \vec{z})$ such that

$$\mathcal{V} \vDash \zeta_{R^F, S^F}(\vec{z}) \to (D^u(\vec{z}, x, y, z) \wedge B^u(\vec{z}) \wedge P^u_F(\vec{z}, \vec{x}, \vec{y})).$$

Since the sentences in (9) are finite conjunctions of quasi-identities, we can use the above lemma to replace in (9) each quasi-identity by a conjunction of identities.

5.2. *A locally finite example*

We note that, under the equivalent conditions of Theorem 5.3, local finiteness implies that $\vec{e}\diamond_{\mathbf{A}}\vec{f}$ is finitely axiomatizable (use (8) of Theorem 5.3). An interesting question is to what extent, besides the addition of conditions of Theorem 5.3, local finiteness implies that $\vec{e}\diamond_{\mathbf{A}}\vec{f}$ is finitely axiomatizable. We will give an example to show that in general even under very strong structural assumptions on the variety, local finiteness do not imply that $\vec{e}\diamond_{\mathbf{A}}\vec{f}$ be finitely axiomatizable.

Given a bounded chain $\mathbf{C} = (C, \vee, \wedge, 0, 1)$ define

$$x\ i^{\mathbf{C}}\ y = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{L}$ be the language of bounded lattices expanded by adding a new binary function symbol $\Rightarrow$. That is $\mathcal{L} = \{\vee, \wedge, \Rightarrow, 0, 1\}$. Let $\mathcal{P}$ be the variety generated by the following class of $\mathcal{L}$-algebras

$$\mathcal{S} = \{(\mathbf{C}, i^{\mathbf{C}}) : \mathbf{C} \text{ is a bounded chain}\}.$$

The variety $\mathcal{P}$ is equivalent to the variety of P-algebras introduced by Epstein and Horn in [5]. Define

$$x \Leftrightarrow y = (x \Rightarrow y) \wedge (y \Rightarrow x)$$

$$\neg x = x \Rightarrow 0.$$

We note that the term $d(x, y, z) = (x \Leftrightarrow y) \wedge z) \vee (\neg(x \Leftrightarrow y) \wedge x)$ is the ternary discriminator on each member of $\mathcal{S}$. Thus $\mathcal{P}$ is a discriminator variety whose class of simple members is $\mathcal{S}$ (see [3] for basics on discriminator varieties). Since the members of $\mathcal{P}$ are lattice expansions, we have that $\mathcal{P}$ satisfies (L) and (R) via the formulas $\lambda(z, x, y) = x \vee z = y \vee z$ and $\rho(z, x, y) = x \wedge z = y \wedge z$. Thus $\mathcal{P}$ has twice EDFC. Let $\mathbf{A} \in \mathcal{P}$. We will prove that $Z(\mathbf{A}) = \{e \in A : e \text{ is complemented}\}$. Of course, every member of $Z(\mathbf{A})$ is complemented. Let e be a complemented element of $\mathbf{A}$, we will prove that $e \in Z(\mathbf{A})$. Since $\mathcal{P}$ is a discriminator variety, we can suppose that $\mathbf{A} \leq \Pi\{\mathbf{A}_i : i \in I\}$ is a Boolean product, where each $\mathbf{A}_i$ is simple. Since each $\mathbf{A}_i$ is a chain, we have that $e(i) \in \{0, 1\}$, for every $i \in I$. Let $N = \{i \in I : e(i) = 0\}$. Note that $I - N = \{i \in I : e(i) = 1\}$. For a subset $J \subseteq I$, let $\pi_J : A \to \Pi\{A_i : i \in J\}$ be the canonical projection. Since $\mathbf{A}$ is a Boolean product and N is clopen, we have that $\mathbf{A}$ is isomorphic to $\pi_N(\mathbf{A}) \times \pi_{I-N}(\mathbf{A})$ and note that this isomorphism identifies e with $(0, 1)$. Thus, we have proved that $e \in Z(\mathbf{A})$. Since $\neg 0 = 1$ and $\neg 1 = 0$, we have that $\neg e$ is the complement of the central element e. Since $x \wedge \neg x = 0$ is an identity of $\mathcal{P}$ (it can be easily checked on the simple members) we have that the property $e \in Z(\mathbf{A})$ can be axiomatized in $\mathcal{P}$ by the single formula $z \vee \neg z = 1$. Also, we note that the term $u(z, x, y) = (x \wedge \neg z) \vee (y \wedge z)$ is a short decomposition term for $\mathcal{P}$.

Lemma 5.5. *Let* $\mathbf{A}, \mathbf{B}$ *be simple algebras and suppose that* $\mathbf{A} \times \mathbf{B}$ *is congruence distributive. Let* $\pi_1 : A \times B \to A$ *and* $\pi_2 : A \times B \to B$ *be the canonical projections. Then* $\mathrm{Con}(\mathbf{A} \times \mathbf{B}) = \{\Delta, \ker \pi_1, \ker \pi_2, \nabla\}$.

Proof. Since $\mathbf{A} \times \mathbf{B}$ is congruence distributive, every congruence on $\mathbf{A} \times \mathbf{B}$ factorizes. So $\mathbf{A} \times \mathbf{B}$ has four congruences since $\mathbf{A}$ and $\mathbf{B}$ have two congruences each. $\square$

Lemma 5.6. *Let* $\mathbf{C}, \mathbf{D} \in \mathcal{S}$. *If* $\mathbf{A} \leq \mathbf{C} \times \mathbf{D}$, *then either* $\mathbf{A} \in \mathcal{S}$ *or* $\mathbf{A} = \mathbf{C}' \times \mathbf{D}'$, *for some* $\mathbf{C}' \leq \mathbf{C}$ *and* $\mathbf{D}' \leq \mathbf{D}$.

Proof. Since $\mathcal{P}$ is congruence permutable, it follows from [3, IV.10.2]. However, we will give an elementary proof. Let $\pi_1 : C \times D \to C$ be the canonical projection. Suppose that $(c, d), (c, d') \in A$ imply that $d = d'$. Then the restriction of π_1 to A is an embedding and hence $\mathbf{A} \in \mathcal{S}$. So suppose that there are $(c, d), (c, d') \in A$, with $d \neq d'$. Then

$$(0, 1) = (\neg(c \Leftrightarrow^{\mathbf{C}} c), \neg(d \Leftrightarrow^{\mathbf{D}} d')) = \neg((c, d) \Leftrightarrow^{\mathbf{A}} (c, d')) \in A$$

which by (5) of Theorem 4.5 says that $\mathbf{A} = \mathbf{C}' \times \mathbf{D}'$, for some $\mathbf{C}' \leq \mathbf{C}$ and $\mathbf{D}' \leq \mathbf{D}$. $\square$

For each $k \geq 2$, we use $\mathbf{C}_k$ to denote the simple member of $\mathcal{P}$ whose lattice reduct is a bounded chain of k elements. Let $\mathcal{L}_E = \{\vee, \wedge, \Rightarrow, 0, 1\} \cup \{F_2, F_3, \ldots\}$ where for each $n \geq 2$, F_n is an n-ary function symbol. For $k \geq 2$ let $\mathbf{G}_k = (\mathbf{C}_k \times \mathbf{C}_k, f_2, f_3, \ldots)$, where for each $n \neq k$

$$f_n(x_1, \ldots, x_n) = 1, \quad \text{for every } x_1, \ldots, x_n \in C_k \times C_k$$

and

$$f_k(x_1, \ldots, x_k) = \begin{cases} 0 & \text{if } x_i \neq x_j \text{ for every } i \neq j \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 5.7. *If* $\mathbf{B} \leq \mathbf{G}_k$, *then either* $\mathbf{B}$ *is simple or the product of two simple algebras.*

Proof. If the reduct of $\mathbf{B}$ to $\{\vee, \wedge, \Rightarrow, 0, 1\}$ is simple, then $\mathbf{B}$ is simple. If the reduct of $\mathbf{B}$ to $\{\vee, \wedge, \Rightarrow, 0, 1\}$ is not simple, then Lemma 5.6 says that $\mathbf{B} = (\mathbf{C} \times \mathbf{C}', F_2^{\mathbf{B}}, F_3^{\mathbf{B}}, \ldots)$, where $\mathbf{C}, \mathbf{C}' \leq \mathbf{C}_k$. If $|C \times C'| < k$, then $\mathrm{Im} F_i^{\mathbf{B}} = \{1^{\mathbf{B}}\}$, for every $i \geq 2$ and hence $\mathbf{B}$ is the product of two simple algebras. So suppose that

$|C \times C'| \geq k$. We will prove that $\mathbf{B}$ is simple. Let $\pi_1 : C \times C' \to C$ and $\pi_2 : C \times C' \to C'$ be the canonical projections. Note that

$$F_k^{\mathbf{B}}(x_1, \ldots, x_k) = \begin{cases} 0 & \text{if } x_i \neq x_j \text{ for every } i \neq j \\ 1 & \text{otherwise.} \end{cases}$$

By Lemma 5.5, we only need to prove that $\ker \pi_1, \ker \pi_2 \notin \mathrm{Con}(\mathbf{B})$. Take $a_1, a_2, \ldots\ldots, a_k \in C \times C'$ such that $a_i \neq a_j$, for every $i \neq j$ and $(a_1, a_2) \in \ker \pi_1$. We have that

$$F_k^{\mathbf{B}}(a_1, a_2, a_3, \ldots, a_k) = 0$$

$$F_k^{\mathbf{B}}(a_1, a_1, a_3, \ldots, a_k) = 1$$

and hence

$$(F_k^{\mathbf{B}}(a_1, a_2, a_3, \ldots, a_k), F_k^{\mathbf{B}}(a_1, a_1, a_3, \ldots, a_k)) \notin \ker \pi_1.$$

Thus $\ker \pi_1 \notin \mathrm{Con}(\mathbf{B})$ and in a similar manner, we can prove that $\ker \pi_2 \notin \mathrm{Con}(\mathbf{B})$. $\square$

Lemma 5.8 ([2]). *Suppose that $\mathcal{V}$ is a variety generated by a class which for every $n \geq 0$ has finitely many isomorphism types of n-generated subalgebras and each one is finite. Then $\mathcal{V}$ is locally finite.*

Proposition 5.9. *Let $\mathcal{P}_E$ be the variety generated by the class $\mathcal{G} = \{\mathbf{G}_k : k \geq 2\}$.*

(1) *$\mathcal{P}_E$ is arithmetical.*
(2) *$\mathcal{P}_E$ is semisimple.*
(3) *$\mathcal{P}_E$ is locally finite.*
(4) *The term $u(z, x, y) = (x \wedge \neg z) \vee (y \wedge z)$ is a short decomposition term for $\mathcal{P}_E$.*
(5) *Let $\alpha_n(z) = \forall \vec{x} \quad F_n(x_1 \vee z, \ldots, x_n \vee z) \vee z = F_n(\vec{x}) \vee z$. The set*

$$\{z \vee \neg z = 1\} \cup \{\alpha_n(z), \alpha_n(\neg z) : n \geq 2\}$$

defines $e \in Z(\mathbf{A})$ in $\mathcal{P}_E$.

(6) *There is no finite set of formulas defining the property $e \in Z(\mathbf{A})$ in $\mathcal{P}_E$ and hence the class of directly indecomposables of $\mathcal{P}_E$ is not a first-order class.*

Proof. (1) $\mathcal{P}_E$ is arithmetical since $\mathcal{P}$ is arithmetical.

(2) First, we will prove that

(*) $\mathbb{P}_u(\mathcal{G}) \subseteq \mathcal{G} \cup \mathbb{I}\{(\mathbf{C} \times \mathbf{C}', f_2, f_3, \ldots) : \mathbf{C}, \mathbf{C}' \in \mathcal{S}$ and $\mathrm{Im} f_n = \{1\}$, for every $n \geq 2\}$.

Let $\{\mathbf{G}_{k_i} : i \in I\} \subseteq \mathcal{G}$. Let u be an ultrafilter on I and let $\mathbf{A} = \prod_{i \in I} \mathbf{G}_{k_i}/u$. Since for any algebras $\mathbf{B}_i, \mathbf{C}_i$, $i \in I$, we have that $\Pi\{\mathbf{B}_i \times \mathbf{C}_i : i \in I\}/u$ is naturally isomorphic to $\Pi\{\mathbf{B}_i : i \in I\}/u \times \Pi\{\mathbf{C}_i : i \in I\}/u$, the reduct of $\mathbf{A}$ to the language $\{\vee, \wedge, \Rightarrow, 0, 1\}$ is of the form $\mathbf{C} \times \mathbf{C}'$ with $\mathbf{C}, \mathbf{C}' \in \mathcal{S}$. Suppose that $\mathrm{Im} F_n^{\mathbf{A}} \neq \{1\}$, for some $n \geq 2$. Then there are $a_1, \ldots, a_n \in A$ such that $F_n^{\mathbf{A}}(a_1, \ldots, a_n) \neq 1$. Thus there is $J \in u$ such that $F_n^{\mathbf{G}_{k_i}}(a_1(i), \ldots, a_n(i)) \neq 1$, for every $i \in J$. But this implies that $k_i = n$ for every $i \in J$ and so we have that $\mathbf{A} \cong \mathbf{G}_n$.

Next, we will use (*) to prove that

(**) every member of $\mathbb{SP}_u(\mathcal{G})$ is simple or the product of two simple algebras.

First, suppose that $\mathbf{B}$ is a subalgebra of $(\mathbf{C} \times \mathbf{C}', f_2, f_3, \ldots)$, where $\mathbf{C}$, $\mathbf{C}' \in \mathcal{S}$ and $\mathrm{Im} f_i = \{1\}$, for every $i \geq 2$. Then by Lemma 5.6, we have that is $\mathbf{B}$ simple or the product of two simple algebras. If $\mathbf{B} \leq \mathbf{G}_k$ for some $k \geq 2$, then Lemma 5.7 says that $\mathbf{B}$ is simple or the product of two simple algebras.

Now (**) and Lemma 5.5 say that every subdirectly irreducible homomorphic image of a member of $\mathbb{SP}_{\mathcal{U}}(\mathcal{G})$ is simple, which by Jónsson's lemma says that $\mathcal{P}_E$ is semisimple.

(3) In order to apply Lemma 5.8, we need to prove that for every $n \geq 0$, the class $\mathcal{G}$ has finitely many isomorphism types of n-generated subalgebras and each one is finite. This follows from the following observations.

— By Lemma 5.8, we have that $\mathcal{P}$ is locally finite since $\mathcal{P}$ is generated by the class of all chains.
— If S is a subuniverse of $\mathbf{C}_k \times \mathbf{C}_k$, then S is a subuniverse of $\mathbf{G}_k$.
— If $\mathbf{B} \in \mathbb{S}(\mathcal{G})$, then $\mathrm{Im} F_m^{\mathbf{B}} = \{1\}$, for every $m > |B|$.

(4) As it was observed above the term $(x \wedge \neg z) \vee (y \wedge z)$ is a short decomposition term for $\mathcal{P}$.

(5) Routine.

(6) Suppose there is a finite set of formulas defining the property $e \in Z(\mathbf{A})$ in $\mathcal{P}_E$. Then by compactness we have that there is a natural number K such that the set $\Sigma = \{z \vee \neg z = 1\} \cup \{\alpha_n(z), \alpha_n(\neg z) : K \geq n \geq 2\}$ defines $e \in Z(\mathbf{A})$ in $\mathcal{P}_E$. But $\mathrm{Im} F_n^{\mathbf{G}_{K+1}} = \{1\}$, for every $K \geq n \geq 2$ which implies that $\mathbf{G}_{K+1} \vDash \psi((0,1))$, for every $\psi \in \Sigma$. Thus $(0,1) \in Z(\mathbf{G}_{K+1})$, which is absurd since $\mathbf{G}_{K+1}$ is simple. $\square$

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