QUASI-ISOMETRIC MAPS BETWEEN DIRECT PRODUCTS OF HYPERBOLIC SPACES

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ABSTRACT. We give conditions under which a quasi-isometric map between direct products of hyperbolic spaces splits as a direct product up to bounded distance and permutation of factors. This is a variation on a result due to Kapovich, Kleiner and Leeb.

1. Introduction

In this paper, we consider a quasi-isometric embedding of a finite direct product of Gromov hyperbolic spaces into another such product with the same number of factors. We show that such a map respects the product structure up to bounded distance, and permutation of factors. For this, we need to assume that each factor in the domain is "bushy" in the sense that every point is a bounded distance from the centre of an ideal (quasi)geodesic triangle. Without this, there are obvious counterexamples. For example, one can map the euclidean plane, $\mathbb{R} \times \mathbb{R}$, to itself sending rays from the origin to logarithmic spirals. One can, however, give a variation of this statement allowing for $\mathbb{R}$-factors.

If one replaces "quasi-isometric embedding" with "quasi-isometry", then these statements follow from [KaKL], using work in [KlL]. The arguments here are related, but are more direct for these particular kinds of spaces.

Recall that a *geodesic space* is a metric space in which any two points are connected by a geodesic path. We use $N(.;r)$ to denote r-neighbourhood. A map, $\phi : X \longrightarrow Y$ (not necessarily continuous or injective) is a *quasi-isometric embedding* if there exist $k_0 > 0$, $k_1, t_0, t_1 \geq 0$ such that for all $x, y \in X$, we have $k_0\rho(x,y) - t_0 \leq \rho'(\phi x, \phi y) \leq k_1\rho(x,y) + t_1$. For more background to this, and to Gromov hyperbolic spaces, see for example, [GhH].

Let Λ be a Gromov hyperbolic space [Gr1]. Let $\partial\Lambda$ be the Gromov boundary. Given $x, y, z \in \partial\Lambda$ write $\mu(x,y,z) \in \Lambda$ for the centre of any geodesic triangle in Λ (i.e. a bounded distance from all three sides).

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Up to bounded distance in Λ, this extends to a natural map $\mu : (\Lambda \cup \partial\Lambda)^3 \longrightarrow \Lambda \cup \partial\Lambda$, where $\mu(x, y, z) \in \Lambda$ if $x, y, z \in \partial\Lambda$ are all distinct.

Definition. We say that Λ is *bushy* if every point of Λ is a bounded distance from $\mu(x, y, z)$ for some distinct $x, y, z \in \partial\Lambda$.

Of course, this implicitly implies a constant of "bushiness". (This terminology arises from the case of quasitrees.)

We show:

Theorem 1.1. *Suppose that for $i = 1, \dots, n$, we have hyperbolic spaces, Λ_i and Λ'_i, with each Λ_i bushy. Let $L = \prod_{i=1}^n \Lambda_i$ and $L' = \prod_{i=1}^n \Lambda'_i$. Suppose that $\phi : L \longrightarrow L'$ is a quasi-isometric embedding. Then there is a permutation $\omega : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$ and maps $\phi_i : \Lambda_i \longrightarrow \Lambda'_{\omega(i)}$ such that for all $(x_1, \dots, x_n) \in L$, the distance between $\phi(x_1, \dots, x_n)$ and $(\phi_{\omega^{-1}(1)}(x_{\omega^{-1}(1)}), \dots, \phi_{\omega^{-1}(n)}(x_{\omega^{-1}(n)}))$ is bounded above in terms of the constants of the hypotheses (namely, hyperbolicity, bushiness, and quasi-isometry).*

For definiteness, we take the l^2 metric on the products (though of course, any quasi-isometrically equivalent geodesic metric, such as any l^p metric, would serve for our purposes). It is easily seen that each of the maps ϕ_i is necessarily a quasi-isometric embedding. Moreover, ϕ is a quasi-isometry (i.e. has cobounded image) if and only if each of the ϕ_i is a quasi-isometry. Theorem 1.1 for quasi-isometries is a consequence of the main result of [KaKL]. (There it is stated for "periodic" hyperbolic spaces, but one can easily check that bushiness is all that is needed for their argument.) In particular, the quasi-isometry type of a product of bushy hyperbolic spaces determines the quasi-isometry type of its factors up to permutation.

In fact, one can allow for factors quasi-isometric to $\mathbb{R}$, to give a more general result:

Theorem 1.2. *Suppose that for $i = 1, \dots, n$, we have hyperbolic spaces, Λ_i and Λ'_i. Suppose that $1 \leq q \leq p \leq n$, such that for all $i \leq p$, Λ_i is bushy and for all $j > q$, Λ'_j is quasi-isometric to the real line. Let $L = \prod_{i=1}^n \Lambda_i$ and $L' = \prod_{i=1}^n \Lambda'_i$. Suppose that $\phi : L \longrightarrow L'$ is a quasi-isometric embedding. Then $p = q$. Moreover, after permuting the indices $1, \dots, p$, there are quasi-isometric emdeddings, $\phi_i : \Lambda_i \longrightarrow \Lambda'_i$ for $i \leq p$, and a quasi-isometry, $\phi_u : \prod_{i=p+1}^n \Lambda_i \longrightarrow \prod_{i=p+1}^n \Lambda'_i$, for each $u \in \prod_{i=1}^p \Lambda_i$ such that for all $x \in L$, $\phi(x)$ is a bounded distance from $(\psi(u), \phi_u(v))$, where u, v are respectively projections of x to the first p and last $n - p$ coordinates, and where ψ is a direct product of the maps ϕ_i for $i \leq p$. Again, the bound depends only on the parameters of the hypotheses.*

Note that, in general, the assumption that $q \leq p$ is required — for example one can quasi-isometrically fold $\Lambda \times \mathbb{R}$ into $\Lambda \times \Lambda$ for any hyperbolic space containing a quasigeodesic ray. (See the discussion in Section 2.) Again, for quasi-isometries, the statement follows from [KaKL].

Our proof uses the fact that the asymptotic cone of a (bushy) hyperbolic space is a (universal) $\mathbb{R}$-tree. We will use a result analogous to the main result regarding a continuous embedding of one product of $\mathbb{R}$-trees in another (see Proposition 2.1 here).

Note that we are not assuming that our hyperbolic spaces are proper. Thus, $\partial\Lambda$ can be viewed as the set of parallel classes of quasigeodesic rays (not necessarily geodesic rays). However, to simplify the exposition it will be convenient to assume that any hyperbolic space, Λ, has the "visibility" property, namely that any two distinct points of $\Lambda \cup \partial\Lambda$ are connected by a geodesic. In general, this is true for uniform quasigeodesics, and so our arguments are easily reinterpreted in the general case.

We will also say that a geodesic metric space is *taut* if every point is a bounded distance from a bi-infinite geodesic. Thus, for a hyperbolic space (with the visibility property), bushy implies taut. Also, taut implies that any two points are simultaneously a bounded distance from some bi-infinite geodesic. For if $x_1, x_2 \in \Lambda$, then x_i is a bounded distance from a bi-infinite geodesic with endpoints $a_i, b_i \in \partial\Lambda$ say. By hyperbolicity, the union of all bi-infinite geodesics with endpoints in $\{a_1, a_2, b_1, b_2\}$ is a bounded Hausdorff distance from a uniformly quasi-isometrically embedded tree in Λ (with at most five edges). It follows that x_1, x_2 both lie a bounded distance from some bi-infinite path in this tree, hence also a bounded distance from any bi-infinite geodesic in Λ connecting its endpoints.

We remark that one can find products of hyperbolic spaces in various naturally occurring spaces. In particular, one motivation for this work was to study the quasi-isometric rigidity of the mapping class group [B1] and Teichmüller space [B2, B3].

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2. $\mathbb{R}$-trees

This section is aimed at proving Proposition 2.1 below. This will be used in Section 4 to show that product flats are coarsely respected by quasi-isometric embeddings. We need the following definition.

Definition. We say that an $\mathbb{R}$-tree is *furry* if every point has valence at least 3.

We note that a complete and furry $\mathbb{R}$-tree is geodesically complete; that is, any two points are contained in a bi-infinite geodesic. (To see this, note that, by Zorn's Lemma, any two points in the tree are contained in some maximal real interval. This interval must be infinite in each direction. Otherwise, by completeness, it must have an endpoint. This endpoint would have valence at least 2, allowing us to extend the interval, and contradict maximality.)

Proposition 2.1. *Suppose that for $i = 1, \dots, n$, we have complete $\mathbb{R}$-trees Δ_i, Δ'_i. Suppose that we have $1 \leq q \leq p \leq n$ such that Δ_i is furry for all $i \leq p$ and Δ'_i is isometric to $\mathbb{R}$ for $i > q$. Let $D = \prod_{i=1}^n \Delta_i$ and $D' = \prod_{i=1}^n \Delta'_i$. Suppose that $f : D \longrightarrow D'$ is a continuous injective map with $f(D)$ closed in D'. Then $p = q$, and there is a permutation $\omega : \{1, \dots, p\} \longrightarrow \{1, \dots, p\}$, such that if $x, y \in D$ with $\pi_i x = \pi_i y$ for some $i \leq p$, then $\pi'_{\omega(i)} f(x) = \pi'_{\omega(i)} f(y)$. Here π_i and π'_j are respectively the coordinate projections to Δ_i and to Δ'_j.*

After permuting indices, we can assume that ω is the identity. Write $x \in D$ as $x = (u, v)$ with $u \in \prod_{i=1}^p \Delta_i$ and $v \in \mathbb{R}^{n-p} \equiv \prod_{j=p+1}^n \Delta_i$. Then it follows that we can write $f(x) = (g(u), h_u(v))$, where g splits as a product, $g_1 \times \cdots \times g_p$, with $g_i : \Delta_i \longrightarrow \Delta'_i$ injective, and where each $h_u : \mathbb{R}^{n-p} \longrightarrow \mathbb{R}^{n-p}$ is a homeomorphism (since it is injective with closed range).

Note that the statement does not necessarily hold of $p < q$. In fact, if Δ is any tree containing at least one ray, then $\Delta \times \mathbb{R}$ embeds in $\Delta \times \Delta$ in a manner that does not split as a product. For example, let $e \in \partial\Delta$ be any ideal boundary point. Given $x \in \Delta$ and $t \in [0, \infty)$, write $x + t$ for the point in the ray with basepoint $x = x + 0$ and with ideal point e, such that $x + t$ is a distance t from x. If $(x, t) \in \Delta \times \mathbb{R}$, set $h(x, t) = (x, x + t)$ if $t \geq 0$, and $h(x, t) = (x + (-t), x)$ if $t \leq 0$. Thus $h : \Delta \times \mathbb{R} \longrightarrow \Delta \times \Delta$ continuous, injective with closed image, but does not split as a product.

(We remark that a similar construction works for a hyperbolic space with non-empty boundary, to give a quasi-isometric embedding of $\Lambda \times \mathbb{R}$ into $\Lambda \times \Lambda$ which folds.)

To prove Proposition 2.1, we first establish some regularity for the map f. We use the following, cf. [KlL, KaKL, B1] etc. By a *cube* in D, we mean a subset of the form $\prod_i I_i$ where $I_i \subseteq \Delta_i$ is a compact interval. It is an r-*cube* if exactly r of the I_i are non-trivial. Note that if two n-cubes meet precisely in a codimension-1 face, then their union is also an n-cube. We say that a subset, $Q \subseteq D$ is *cubulated* if it is a locally finite union of cubes. This implies easily that it has the local structure of a cube complex, which we refer to as a "local cubulation" — see [B1] for a more general discussion. In this section, we will put the l^2 metric on the product, so that D is a CAT(0) space.

The following is a consequence of the results in [KlL] (or [B1]), but in this context the argument can be simplified. (One can find related arguments in Section 6 of [KlL]. There, the authors use singular homology, whereas we will use Čech homology.)

Lemma 2.2. *If $\Phi \subseteq D$ is a closed subset homeomorphic to $\mathbb{R}^n$, then Φ is cubulated.*

Proof. Let $B \subseteq \Phi$ be any closed subset homeomorphic to a topological n-ball. Let $B' \subseteq \Phi$ be another such ball containing B in its relative interior. Thus, $N(B;\epsilon) \cap \Phi \subseteq B'$ for some $\epsilon > 0$. Triangulate $\partial B'$ so that every simplex has diameter less than ϵ. Let $V \subseteq \partial B'$ be the set of vertices. Let $\delta_i \subseteq \Delta_i$ be the convex hull of $\pi_i V$. This is a finite simplicial tree. Let $\delta = \prod_i \delta_i \subseteq D$. Thus, δ is compact, convex and cubulated. Since $V \subseteq \delta$ we have $\partial B' \subseteq N(\delta;\epsilon)$. Let $\psi : D \longrightarrow \delta$ be nearest-point projection, and let $\theta = \psi | B'$. Now $\theta | \partial B'$ is homotopic to the inclusion map via a homotopy whose trajectories all have length at most ϵ. In particular the image of the homotopy does not meet B. Combining our map $\theta : B' \longrightarrow \delta$ with the homotopy of $\theta | \partial B'$, we get a continuous map, $\theta' : B' \longrightarrow D$, with $\theta' | \partial B'$ inclusion. Since D is contractible, θ' is homotopic to the inclusion of B' into D, relative to $\partial B'$. We write $Z \subseteq D$ for the image of this homotopy.

We claim that $B \subseteq \theta' B'$. For suppose to the contrary that $p \in B' \setminus \theta' B'$. Let N be an open neighbourhood of p in B' whose closure is homeomorphic to a closed n-ball disjoint from $\theta' B'$. Thus, $H_n(B', B' \setminus N) \cong H_{n-1}(\partial B') \cong \mathbb{Z}_2$, where "$H$" denotes Čech homology with $\mathbb{Z}_2$ coefficients. The image of $H_n(B', B' \setminus N)$ in $H_n(Z, (B' \cup \theta' B') \setminus N)$ is trivial (since it corresponds to the image of $H_{n-1}(\partial B')$ induced by the identity map, hence also by θ'). The natural map, $H_n(B', B' \setminus N) \longrightarrow H_n(B' \cup \theta' B', (B' \cup \theta' B') \setminus N)$ is an isomorphism, by excision. Since $Z \subseteq D$ has topological dimension n, $H_{n+1}(Z, B' \cup \theta' B')$ is trivial. (This is why we use Čech homology.) Thus, the exact sequence of triples tells us that the natural map $H_n(B' \cup \theta' B', (B' \cup \theta' B') \setminus N) \longrightarrow H_n(Z, (B' \cup \theta' B') \setminus N)$ is injective. (Note, we need to observe that B', $\theta' B'$ and Z are all compact, and that we are using field coefficients, so that the exactness property of Čech homology holds, see [ES].) Composing, we get that the natural map $H_n(B', B' \setminus N) \longrightarrow H_n(Z, (B' \cup \theta' B') \setminus N)$ is injective, contradicting the earlier statement, hence proving the claim.

By construction, the homotopy part of θ' does not meet B, so it follows in fact that $B \subseteq \theta B' \subseteq \delta$.

Now suppose that $K \subseteq \Phi$ is any compact subset. Let $B \supseteq B_0 \supseteq K$ be closed topological n-balls in Φ, with $B_0 \cap \partial B = \emptyset$. By the previous paragraph, B lies inside a finite union, δ, of cubes in D. After subdivision, we can assume that any cube meeting B_0 is disjoint from ∂B. Let Υ be the union of those n-cubes, P, which meet B_0 and whose interiors, $I(P)$, meet B. For any such P, $I(P)$ is open in δ, so $B \cap I(P) = (B \setminus \partial B) \cap I(P)$ is open in $B \setminus \partial B \cong \mathbb{R}^n$. In particular, $B \cap I(P)$ is homeomorphic to an open subset of $\mathbb{R}^n$, and hence, by Invariance of Domain, is open in $I(P) \cong \mathbb{R}^n$. But $B \cap I(P)$ is also closed in $I(P)$, so by connectedness of $I(P)$ it follows that $I(P) \subseteq B$, so $P \subseteq B$. This holds for all such P, so $\Upsilon \subseteq B$.

We also have $B_0 \subseteq \Upsilon$ (since by a simple dimension argument, if B_0 meets a lower dimensional cube of δ, it must also meet the interior of an incident n-dimensional cube of δ). Thus, $K \subseteq \Upsilon$. In other words, any compact subset of Φ lies inside a finite union of cubes in Φ, and it follows that Φ is cubulated. $\square$

Given $x \in \Phi$, we define the link, $L(\Phi, x)$, of x in Φ as the metric r-sphere about x, with the induced path-metric rescaled by a factor of $1/r$. Given that Φ is locally cubulated, we see easily that this construction gives rise to the same metric space for all sufficiently small $r > 0$. It is a CAT(1) space. In fact, it has a canonical structure as a combinatorial CAT(1) complex, built out of spherical simplices with all dihedral angles equal to $\pi/2$. To see this, note that after subdivision, we could take x to be a vertex of a local cubulation of Φ. The simplicial structure of $L(\Phi, x)$ is then the usual combinatorial link of the cubulation.

A simple argument, inducting on dimension, shows that $L(\Phi, x)$ must have at least 2^n $(n-1)$-cells. (Note that the link of any simplex in $L(\Phi, x)$ is a homology sphere, and so, in particular cannot be contractible. In particular, one can find two vertices of $L(\Phi, x)$ at distance π apart. By induction their links in turn have at least 2^{n-1} $(n-2)$-cells, and so give rise to two disjoint sets of 2^{n-1} $(n-1)$-cells in $L(\Phi, x)$.) Moreover, the inductive argument also shows that if $L(\Phi, x)$ has exactly 2^n $(n-1)$-cells, then it is a round sphere triangulated as a cross polytope. In this case, we say that x is *regular*. This is equivalent to saying that x lies in the interior of an n-cube in Φ (that is to say, an isometrically embedded euclidian n-cube in Φ). Note that the set of singular points has dimension at most $n-2$.

By a *product flat* in Φ, we mean a subset of the form $\Phi = \prod_{i=1}^n \gamma_i$, where $\gamma_i \subseteq \Delta_i$ is a bi-infinite geodesic. In this case, every point of Φ is regular. Given $x \in \Phi$, and $i \in \{1, \ldots, n\}$, let $\Theta_i(x) = \Phi \cap \pi_i^{-1}\pi_i x$, i.e. the ith codimension-1 coordinate plane through x. This determines a great sphere, $\Sigma_i = \Sigma_i(\Phi, x) \subseteq L(\Phi, x)$. Let $C_i = C_i(\Phi, x)$ be the set of $(n-2)$-cells in Σ_i.

Note that, if $i \leq p$, then since Δ_i is complete and furry, we can find bi-infinite geodesics, β_i^0 and β_i^1 with $\gamma_i \cap \beta_i^0 \cap \beta_i^1 = \{\pi_i x\}$. Then $\Theta_i(x) = \Phi \cap \Psi_i^0 \cap \Psi_i^1$, where $\Psi_i^0 = \prod_{j=1}^{i-1} \gamma_j \times \beta_i^0 \times \prod_{j=i+1}^{n} \gamma_j$ and $\Psi_i^1 = \prod_{j=1}^{i-1} \gamma_j \times \beta_i^1 \times \prod_{j=i+1}^{n} \gamma_j$ are product flats. Note that $\Xi_i(x) = \Phi \cup \Psi_i^0 \cup \Psi_i^1$ is the product of a tree, $\tau_i = \gamma_i \cup \beta_i^0 \cup \beta_i^1$, with $\prod_{j \neq i} \gamma_j \cong \mathbb{R}^{n-1}$, and that $\pi_i x$ has valence at least 3 in τ_i.

We now introduce the map $f : D \longrightarrow D'$. By Invariance of Domain, $f|\Phi$ is a homeomorphism onto its range, $\Phi' = f\Phi$, which by Lemma 2.2 is cubulated. In particular, if $x \in \Phi$, we have a link, $L'(\Phi', x')$, where $x' = fx$. Moreover, for $i \leq p$, $\Theta_i'(x') = f\Theta_i(x)$ is also cubulated (since it is the intersection of three cubulated sets: $\Theta_i'(x') = f\Phi \cap f\Psi_i^0 \cap f\Psi_i^1$). It therefore determines a subcomplex, $\Sigma_i' = \Sigma_i'(\Phi', x') \subseteq L'(\Phi', x')$. Let $C_i' = C_i'(\Phi', x')$ be the set of $(n-2)$-cells of Σ_i'. Since $\Theta_i'(x')$ is homeomorphic to $\mathbb{R}^{n-1}$, we see that $|C_i'| \geq 2^{n-1}$. Moreover, $C_i' \cap C_j' = \emptyset$ for $i \neq j$.

Suppose, for the moment, that x' is regular in Φ'. Then $L'(\Phi', x')$ is a cross polytope, with great spheres, $\Sigma_i'' \subseteq L'(\Phi', x')$, determined locally by the codimension-1 coordinate subspaces, as in the case of a product flat. Let C_i'' be the set of $(n-2)$-cells in Σ_i''. Then, $|C_i''| = 2^{n-1}$, and $C_i'' \cap C_j''$ if $i \neq j$.

Now, for each $i \leq p$, we claim that $\Sigma_i' \subseteq \bigcup_{j=1}^q \Sigma_j''$. For if not, it would contain a simplex of C_k' for some $k > q$. This determines an $(n-1)$-cube, $P \subseteq \Theta_i'(x) \subseteq \Phi'$ (in the star of x'). Now $\Xi_i(x) \cong \tau_i \times \mathbb{R}^{n-1}$ maps injectively to the cubulated set $f\Xi_i(x) = f\Phi \cup f\Psi_i^0 \cup f\Psi_i^1$. It follows that P must lie in at least 3 n-cubes in $f\Xi_i(x)$, which would imply that $\pi_k x'$ has valence at least 3 in Δ_k', contrary to the assumption that $\Delta_k' \cong \mathbb{R}$. This proves the claim. We therefore have $\bigcup_{i=1}^p C_i' \subseteq \bigcup_{j=1}^q C_j''$. But $|\bigcup_{i=1}^p C_i'| \geq 2^{n-1}p \geq 2^{n-1}q \geq |\bigcup_{j=1}^q C_j''|$. It follows that $p = q$ and that $|C_i'| = 2^{n-1}$ for all $i \leq p$. Thus Σ_i' is a cross polytope. It follows that there is some permutation, ω, of $\{1, \ldots, p\}$ so that $\Sigma_i' = \Sigma_{\omega(i)}''$. We see that x' is regular in $\Theta_{\omega(i)}'(x')$, that is, it is contained in the interior of an $(n-1)$-cube in $\Theta_{\omega(i)}'(x')$.

In summary, we have already shown that $p = q$ (since such regular points certainly exist in Φ'). Moreover, if x lies in a product flat Φ, and fx is regular in $f\Phi$, we have shown that there is a permutation ω of $\{1, \dots, p\}$ and a neighbourhood, U, of x in Φ such that if $y \in U$ with $\pi_i x = \pi_i y$, then $\pi'_{\omega(i)} fx = \pi'_{\omega(i)} fy$. In fact, ω is determined by x and Φ (since, if there were two candidates, say j, k, for $\omega(i)$, the $(n-1)$-dimensional set, $U \cap \pi_i^{-1} \pi_i x$ would get mapped injectively by f into the $(n-2)$-dimensional subset, $(\pi'_j)^{-1} \pi'_j fx \cap (\pi'_k)^{-1} \pi'_k fx$, of D', giving a contradiction).

Now a simple continuity argument shows that if $y \in \Theta_i(x) \subseteq \Phi$ is connected to x by a path $\beta \subseteq \Theta_i(x)$, with $f\beta$ lying entirely within the regular set of $f\Phi$, then $\pi'_{\omega(i)} fx = \pi'_{\omega(i)} fy$, for some permutation ω of $\{1, \dots, p\}$ (since the permutation must be constant along β).

But the same holds without the regularity assumption. For suppose $y \in \Theta_i(x)$. Since the singular set of $f\Phi$ has dimension at most $n-2$, we can find a sequence of pairs in Φ, satisfying the conditions of the previous paragraph, and tending to x, y. After passing to a subsequence, we can assume ω to be constant for this sequence, and so by continuity, we again get $\pi'_{\omega(i)} fx = \pi'_{\omega(i)} fy$. Moreover, by continuity, we see that ω must be constant on Φ.

We have shown that, for any product flat, Φ, there is a permutation ω_Φ of $\{1, \dots, p\}$ such that if $x, y \in \Phi$ with $\pi_i x = \pi_i y$ for $i \le p$, then $\pi'_{\omega_\Phi(i)} fx = \pi'_{\omega_\Phi(i)} fy$.

But any two points of D lie in a common product flat. Therefore to complete the proof of Proposition 2.1, it is sufficient to show that ω_Φ is, in fact, independent of Φ. This follows, for example, on observing that if Φ_1 and Φ_2 are any two product flats, then there is a third product flat Φ_0 such that $\Phi_0 \cap \Phi_1$ and $\Phi_0 \cap \Phi_2$ both contain an n-cube, and one sees that $\omega_{\Phi_1} = \omega_{\Phi_0} = \omega_{\Phi_2}$, again by a simple dimension argument.

This proves Proposition 2.1.

3. Asymptotic cones

Let $\mathcal{Z}$ be a countable set equipped with a non-principal ultrafilter. By a *$\mathcal{Z}$-sequence* we mean a sequence indexed by $\mathcal{Z}$. We will say that a predicate depending on $\zeta \in \mathcal{Z}$ holds *almost always* if the set of ζ for which it holds has measure 1 (i.e. lies in the ultrafilter). Let $(h_\zeta)_\zeta$ be a $\mathcal{Z}$-sequence of positive real numbers with $h_\zeta \to \infty$ (with respect to the ultrafilter). Let (X, ρ) be a metric space. Let (X^∞, ρ^∞) be the asymptotic cone obtained with a $\mathcal{Z}$-sequence of basepoints $e_\zeta \in X$, and rescaling the metric by factors $1/h_\zeta$ (see [VW, Gr2]). Thus, X^∞ is a complete metric space. Fix any $e \in X$. If $(x_\zeta)_\zeta$ is a $\mathcal{Z}$-sequence in X with $\rho(e, x_\zeta)/h_\zeta$ almost always bounded, then $x_\zeta \to x$ for some unique $x \in X^\infty$. If X is a geodesic space, so is (X^∞, ρ^∞).

If $(A_\zeta)_\zeta$ is a $\mathcal{Z}$-sequence of subsets of X, then there is a well defined closed subset $A^\infty \subseteq X^\infty$, with the property that $(\exists x \in A^\infty)(x_\zeta \to x)$ if and only if for all $\epsilon > 0$, we almost always have $\rho(x_\zeta, A_\zeta) \leq \epsilon h_\zeta$. We will also use the notation, $A_\zeta \to A^\infty$, to denote this situation.

Definition. If $A, B, C \subseteq X$, we say that A, B are r-close on C if $A \cap C \subseteq N(B; r)$ and $B \cap C \subseteq N(A; r)$.

If A_ζ, B_ζ are $\mathcal{Z}$-sequences of subsets of X, then $A^\infty = B^\infty$ if and only if for all $\epsilon > 0$ and all $R \geq 0$, A_ζ, B_ζ are almost always (ϵh_ζ)-close on $N(e; Rh_\zeta)$.

If $\phi : X \longrightarrow Y$ is a quasi-isometric embedding of one metric space in another, then ϕ induces a map $\phi^\infty : X^\infty \longrightarrow Y^\infty$ with $\phi^\infty(X^\infty)$ closed in Y^∞, and with ϕ^∞ bilipschitz onto $\phi^\infty(X^\infty)$. If $A \subseteq X$, then one can verify that $\phi^\infty(A^\infty) = (\phi(A))^\infty$.

If Λ is hyperbolic, then Λ^∞ is an $\mathbb{R}$-tree. If Λ is also bushy, then Λ^∞ is the universal complete $2^{\aleph_0}$-regular tree, in particular furry.

If δ_ζ is a sequence of bi-infinite geodesics in Λ, with $\rho(e, \delta_\zeta)$ bounded, then δ^∞ is a bi-infinite geodesic in Λ^∞. Conversely, we note:

Lemma 3.1. *Suppose that Λ is a taut hyperbolic space and that $\delta \subseteq \Lambda^\infty$ is a bi-infinite geodesic. Then there is a $\mathcal{Z}$-sequence, $(\delta_\zeta)_\zeta$, of bi-infinite geodesics in Λ, with $\delta_\zeta \to \delta$.*

Proof. Let a_i, b_i be $\mathbb{N}$-sequences of points in δ tending monotonically out opposite ends of δ. Given $i \in \mathbb{N}$, choose $\mathcal{Z}$-sequences, $a_{i,\zeta}, b_{i,\zeta}$ with $a_{i,\zeta} \to a_i$ and $b_{i,\zeta} \to b_i$. Given $\zeta \in \mathcal{Z}$, let $p = p(\zeta)$ be maximal such that there is some bi-infinite geodesic, $\delta_\zeta \subseteq \Lambda$, with $\rho(a_{i,\zeta}, \delta_\zeta), \rho(b_{i,\zeta}, \delta_\zeta) \leq 2^{-p} h_\zeta$ for all $i \leq p$. (We will see below that δ_ζ almost always exists.) Now $\delta_\zeta \to \beta \subseteq \Lambda^\infty$, where β is a bi-infinite geodesic.

We claim that $\beta = \delta$. Clearly it's enough to show that $\delta \subseteq \beta$, hence enough that $a_i, b_i \in \beta$ for all i. In other words, we want to show that given i and $\epsilon > 0$, we almost always have $\rho(a_{i,\zeta}, \delta_\zeta), \rho(b_{i,\zeta}, \delta_\zeta) \leq \epsilon h_\zeta$. Now if $i < p(\zeta)$, then $\rho(a_{i,\zeta}, \delta_\zeta), \rho(b_{i,\zeta}, \delta_\zeta) \leq 2^{-p(\zeta)} h_\zeta$. It is therefore enough to show that $p(\zeta) \to \infty$, that is, for any $p \in \mathbb{N}$, we almost always have $p(\zeta) \geq p$.

To see this, let $i \leq p$, and let $a_{pi,\zeta}, b_{pi,\zeta}$ be respectively the projections of $a_{i,\zeta}$ and $b_{i,\zeta}$ to the geodesics $[a_{p,\zeta}, b_{p,\zeta}]$. Since $a_i, b_i \in [a_p, b_p]$, we see that $a_{pi,\zeta} \to a_i$ and $b_{pi,\zeta} \to b_i$. Therefore, given any $\eta > 0$, we have $\rho(a_{pi,\zeta}, a_{i,\zeta}), \rho(a_{pi,\zeta}, a_{i,\zeta}) \leq 2^{-p} h_\zeta \eta$ for almost all ζ. In particular, there must be some ζ such that this holds for all $i \leq p$. By tautness, there is a bi-infinite geodesic, δ_ζ, with $\rho(a_{p,\zeta}, \delta_\zeta), \rho(b_{p,\zeta}, \delta_\zeta)$ bounded above (in terms of the tautness and hyperbolicity constants). Thus, $[a_{p,\zeta}, b_{p,\zeta}]$ lies in a bounded neighbourhood of δ_ζ. It follows that, if we choose η sufficiently small in relation to this bound, we have $\rho(a_{i,\zeta},\delta_\zeta), \rho(a_{i,\zeta},\delta_\zeta) \leq 2^{-p}h_\zeta$ for all $i \leq p$. Therefore, $p(\zeta) \geq p$ as required. $\square$

We use hd to denote Hausdorff distance. If $(A_i)_{i\in\mathbb{N}}$ is a sequence of subsets of X, and $A \subseteq X$, we say that A_i *r-converges* to A, if for all bounded $C \subseteq X$, there is some $p \in \mathbb{N}$ such that A_i and A are r-close on C for all $i \geq p$. (Note that if A_i also r-converges on some $A' \subseteq X$, then $\mathrm{hd}(A, A') \leq 2r$.)

If X is taut hyperbolic, and each A_i is a bi-infinite geodesic, then then there is a bi-infinite geodesic, γ, with $\mathrm{hd}(A,\gamma) - r$ bounded above in terms of the hyperbolicity constant.

To see this, write $A_i = [c_i, d_i] \cap X$, for $c_i, d_i \in \partial X$. We can choose c_i, d_i so that the geodesics $[c_i, c_j] \cap X$ and $[d_i, d_j] \cap X$ leave all bounded subsets of X as $i, j \to \infty$. Then, we have $c_i \to c$ and $d_i \to d$ for some $c, d \in \partial X$. Let $\gamma = [c, d] \cap X$. Now, if $x \in \gamma$, then $\rho(x, A_i)$ is bounded above in terms of the hyperbolicity constant for all sufficiently large i, and so $\rho(x, A) - r$ is also bounded above. Similarly, if $x \in A$, and i is sufficiently large, there is some $y \in A_i$ with $\rho(y, A_i) \leq r$. Also, if i is sufficiently large, we know that $\rho(y,\gamma)$ is bounded above in terms of the hyperbolicity constant. This proves the claim.

Suppose now that $X_1, \ldots, X_n$ are taut hyperbolic spaces, and $X = \prod_{i=1}^n X_i$. By a *product flat* in X, we mean a subset of the form $\prod_i \gamma_i$, where each γ_i is a bi-infinite geodesic in X_i. This is consistent with the terminology introduced for products of $\mathbb{R}$-trees in Section 2.

Thus, if $(F_\zeta)_\zeta$ is a $\mathcal{Z}$-sequence of product flats in X, then F^∞ is either empty or a product flat in $X^\infty = \prod_i X_i^\infty$. Conversely, by Lemma 3.1, we see that if Φ is a product flat in X^∞, then there is $\mathcal{Z}$-sequence, $(F_\zeta)_\zeta$, of product flats in X such that $F_\zeta \to \Phi$.

From the earlier observation, we note:

Lemma 3.2. *Suppose that X is a direct product of taut hyperbolic spaces. Suppose that $(F_i)_{i\in\mathbb{N}}$ is a sequence of product flats that r-converges to some set $A \subseteq X$. Then there is some product flat F such that* $\mathrm{hd}(A, F)$ *is finite, and bounded above in terms of r and the constants of hyperbolicity and tautness of the factors.*

We note that we can similarly construct an asymptotic cone, X^∞, of a $\mathcal{Z}$-sequence of spaces $(X_\zeta)_\zeta$ and basepoints, $e_\zeta \in X_\zeta$, together with scaling factors, h_ζ with $h_\zeta \to \infty$. If the spaces are uniformly hyperbolic, then X^∞ will be an $\mathbb{R}$-tree. If they are uniformly bushy, then X^∞ will be the universal $2^{\aleph_0}$-regular tree. If $\phi_\zeta : X_\zeta \longrightarrow Y_\zeta$ are uniformly quasi-isometric embeddings, then we get a bilipschitz map

$\phi^\infty : X^\infty \longrightarrow Y^\infty$. This generalisation is needed to obtain uniform constants in the various results below.

4. Product Flats

Suppose now that L, L', ϕ, are as in the hypotheses of Theorem 1.2. Let $(e_\zeta)_\zeta$ be any $\mathcal{Z}$-sequence of basepoints in L, and let $(h_\zeta)_\zeta$ be any $\mathcal{Z}$-sequence, with $h_\zeta \to \infty$. and let L^∞ and $(L')^\infty$ be the resulting asymptotic cones. We get a map $\phi^\infty : L^\infty \longrightarrow (L')^\infty$, which is bilipschitz onto its range.

Note that ϕ^∞ satisfies the hypotheses of Proposition 2.1. From this, we deduce immediately that $p = q$. Note also that if $\Phi \subseteq L^\infty$ is a product flat, then $\phi^\infty(\Phi)$ is a product flat in $(L')^\infty$.

We make the following elementary observation regarding product flats in any product of trees in the l^2 metric:

Lemma 4.1. *Suppose that Φ, Φ' are product flats in $(L')^\infty$. Suppose that $x \in \Phi'$, and $\lambda \geq 0$. Then there is some $y \in \Phi'$ with $\rho'(x, y) = \lambda\rho'(x, \Phi)$ and with $\rho'(y, \Phi) = (1 + \lambda)\rho'(x, \Phi)$.*

Proof. Let $z \in \Phi$ be the nearest point in Φ to x. Note that the geodesic from z to x extends to a geodesic ray based at x which lies entirely in Φ' beyond the point x. Let y be the point in this ray at distance $(1 + \lambda)\rho'(x, \Phi)$ from z. □

We will apply this below when $\rho'(x, \Phi) = 1$ and $\lambda = 3/2$, so that $\rho'(x, y) = 3/2$ and $\rho'(y, \Phi') = 5/2$.

Lemma 4.2. *Suppose that $\phi : L \longrightarrow L'$ is a quasi-isometric embedding. Then there is some $u \geq 0$ with the following properties. Suppose that $E \subseteq L$ and $F \subseteq L'$ are product flats. Suppose there is some $x \in \phi(E)$ with $t \geq u$, where $t = \rho'(x, F)$. Then there is some $y \in \phi(E)$ with $\rho'(x, y) \leq 2t$ and $\rho'(y, F) \geq 2t$. Furthermore, suppose there is some $x \in F$ with $t \geq u$, where $t = \rho'(x, \phi(E))$. Then there is some $y \in F$ with $\rho'(x, y) \leq 2t$ and $\rho'(y, \phi(E)) \geq 2t$. Moreover, the constant u can be chosen to depend only on the quasi-isometry constants of ϕ and the hyperbolicity and bushiness constants of the bushy hyperbolic factors of L and L'.*

Proof. We suppose the first statement fails. This means that there is an $\mathbb{N}$-sequence of points $(e_i)_{i\in\mathbb{N}}$ and product flats $E_i \subseteq L$ and $F_i \subseteq L'$, with $x_i = \phi(e_i) \in \phi(E_i)$ and with $h_i = \rho'(x_i, F_i) \to \infty$, and with $\rho'(z, F_i) < 2h_i$ for all $z \in \phi(E_i)$ with $\rho(x_i, z) \leq 2h_i$. Now take $\mathcal{Z} = \mathbb{N}$ with any non-principal ultrafilter, and L^∞ and $(L')^\infty$ be the asymptotic cones with scaling factors $(h_\zeta)_\zeta = (h_i)_i$, and with basepoints $(e_i)_i$ and

$(x_i)_i$ respectively. Let $\phi^\infty : L^\infty \longrightarrow (L')^\infty$ be the induced map. Thus, we have limiting product flats, $E^\infty \subseteq L^\infty$, and $F^\infty \subseteq (L')^\infty$. By Proposition 2.1, $\phi(E^\infty)$ is also a product flat. It is the limit of the sets $\phi(E_i)$. Let $x \in \phi^\infty(E^\infty) \subseteq (L')^\infty$ be the limit of $(x_i)_i$. By construction, $(\rho')^\infty(x, F^\infty) = 1$. Therefore, by Lemma 4.1, there is some $y \in \phi^\infty(E^\infty)$ with $(\rho')^\infty(x, y) = 3/2$ and $(\rho')^\infty(y, F^\infty) \geq 5/2$. Now choose any points $y_i \in L'$ with $y_i \to y$. This implies that $\rho'(x_i, y_i)/h_i \to (\rho')^\infty(x, y)$, and so for almost all i we have $\rho'(x_i, y_i) < 2h_i$. Similarly, for almost all i we have $\rho(y_i, F_i) > 2h_i$. In particular, this must hold for some i. Setting $z = y_i$, this contradicts the choice of E_i, F_i and x_i.

The second statement follows by exactly the same argument, interchanging the roles of F and $\phi(E)$.

Finally to see that u only depends on the various parameters of the hypotheses, we need to allow the spaces and maps to vary. That is, we have an $\mathbb{N}$-sequence, $\phi_i : L_i \longrightarrow L'_i$, of uniform quasi-isometric embeddings between products of uniformly bushy hyperbolic spaces and euclidean spaces, all satisfying the hypotheses of Proposition 1.2. If the conclusion of the lemma fails, we can find $E_i \subseteq L_i$, $F_i \subseteq L'_i$ and $x_i \in \phi(E_i)$ etc, as before, and derive the same contradiction on passing to the asymptotic cone. □

Lemma 4.3. *There is some $u \geq 0$ such that if $d \geq r \geq u$ and $E \subseteq L$ and $F \subseteq L'$ are product flats, and $a \in F$ with $\phi(E)$ and F r-close on $N(a; d + 6r)$, then $\phi(E)$ and F are u-close on $N(a; d)$. Moroever u depends only on the parameters L, L' and ϕ.*

Proof. Let u be the constant given by Lemma 4.2. We first show that $\phi(E) \cap N(a; d) \subseteq N(F; u)$. Suppose, for contradiction, that $x \in \phi(E)$ with $\rho'(a, x) \leq d$ and $\rho'(x, F) > u$. Let $t_0 = \rho'(x, F)$, so $u < t_0 \leq r$. By Lemma 4.2, there is some $x_1 \in \phi(E)$ with $\rho(x, x_1) \leq 2t_0$ and with $\rho'(x_1, F) \geq 2t_0$. Let $t_1 = \rho'(x_1, F)$. Thus, $2t_0 \leq t_1 \leq 3t_0$. Clearly $t_1 > u$, so we can apply the same argument with x_1 in place of x_0 to give $x_2 \in \phi(E)$ with $\rho'(x_1, x_2) \leq 2t_1$ and $\rho'(x_2, F) \geq 2t_1$.

We now continue inductively to get a sequence of points $x_n \in \phi(E)$, with $\rho'(x_n, F) = t_n$, $\rho'(x_{n-1}, x_n) \leq 2t_{n-1}$ and $2t_{n-1} \leq t_n \leq 3t_{n-1}$. Note that $\sum_{m=0}^{n-1} t_m \leq (2^{-m} + \cdots + 2^{-1})t_n < t_n$, and so $\rho'(x, x_n) \leq 2t_n$. Now let n be least such that $t_n > r$. Then $t_n \leq 3r$. We get $\rho'(x, x_n) \leq 6r$ so $\rho'(a, x_n) \leq d + 6r$. But $\rho'(x_n, F) > r$ contradicting the assumption that $\phi(E) \cap N(a; d + 6r) \subseteq N(F; r)$.

The statement that $F \cap N(a; d) \subseteq N(\phi(E); u)$ follows similarly, interchanging the roles of $\phi(E)$ and F. □

The following is the main result of this section:

Lemma 4.4. *Suppose L, L', ϕ, are as in the hypotheses of Theorem 1.2. Then there is some $s \geq 0$, depending only on the constants of the hypotheses, such that if $E \subseteq L$ is a product flat, then there is a product flat, $F \subseteq L'$, such that* $\mathrm{hd}(F, \phi(E)) \leq s$.

Proof. Let $\phi^\infty : L^\infty \longrightarrow (L')^\infty$ be the limiting map on any asymptotic cone with fixed basepoints, $e \in E \subseteq L$, and $a = \phi(e) \in L'$. We get a flat, $E^\infty \subseteq L^\infty$ as the limit of E. Then $\phi^\infty(E^\infty)$ is the limit of $\phi(E)$. By Proposition 2.1, this is a product flat in $(L')^\infty$. Let $F^\infty = \phi^\infty(E^\infty)$. As observed at the end of Section 3, there is a sequence of product flats, $(F_\zeta)_\zeta$ in L', with $F_\zeta \longrightarrow F^\infty$. Therefore, given any $\epsilon > 0$ and $R \geq 0$, we have that F_ζ and $\phi(E)$ are (ϵh_ζ)-close on $N(a; Rh_\zeta)$ for almost all ζ. Now set $\epsilon = 1$ and given any $d > 0$, set $R = 6 + (d/u)$, where u is the constant given by Lemma 4.3. Then if $h_\zeta > u$, we have $Rh_\zeta > d + 6h_\zeta$. Thus, for almost all ζ, we see that F_ζ and $\phi(E)$ are h_ζ-close on $N(a; d + 6h_\zeta)$, so by Lemma 4.3, they are also u-close on $N(a; d)$. We therefore see that for any $n \in \mathbb{N}$, there is a product flat, F_n in L', with F_n and $\phi(E)$ u-close on $N(a; n)$. In other words, F_n u-converges on $\phi(E)$. By Lemma 3.2, it follows that $\phi(E)$ is a uniformly bounded Hausdorff distance from a product flat, as claimed. This bound depends only on u, and the constants of L', hence ultimately only on the parameters of L, L' and ϕ, as claimed. □

5. Product structure

Let (X, ρ) be a geodesic space. Given $A, B \subseteq X$ write $A \sim B$ to mean that $\mathrm{hd}(A, B) < \infty$. Clearly, this is an equivalence relation, and we write $\mathcal{B}(X)$ for the set of $\sim$-classes. Let $0 \in \mathcal{B}(X)$ denote the class of non-empty bounded subsets of X. Let $\mathcal{Q}(X) \subseteq \mathcal{B}(X)$ denote the set of $\sim$-classes of images of bi-infinite quasigeodesics.

We say that two sets $A, B \subseteq X$ have *coarse intersection* if there is some $r \geq 0$ such that for all $s \geq r$, $N(A; r) \cap N(B; r) \sim N(A; s) \cap N(B; s)$. Clearly, this depends only on the $\sim$-classes of A and B, and determines an element of $\mathcal{B}(X)$, denoted $A \wedge B$.

For example, in a hyperbolic space, any two bi-infinite geodesics, α, β, have coarse intersection, and $\alpha \wedge \beta$ is bounded, a ray, or a bi-infinite geodesic (the last case arising precisely if $\alpha \sim \beta$).

Note that, if $\phi : X \longrightarrow Y$ is a quasi-isometric embedding, then ϕ determines a map $\mathcal{B}(X) \longrightarrow \mathcal{B}(Y)$, sending $A \subseteq X$ to ϕA. Clearly, $\phi 0 = 0$ and $\phi(\mathcal{Q}(X)) \subseteq \mathcal{Q}(Y)$.

Suppose now that $\Lambda_1, \ldots, \Lambda_n$ are hyperbolic spaces, and that $L = \prod_i \Lambda_i$. We write $\mathcal{F}(L) \subseteq \mathcal{B}(L)$ for the set of $\sim$-classes of product flats. We will refer to an element of $\mathcal{F}(L)$ as a *coarse product flat*. Note that any pair of elements of $\mathcal{F}(L)$ have coarse intersection. (Since this applies to each of the factors.)

By an *ith coordinate line* in L, we mean a subset of the form $\prod_j A_j$, where A_j is a point if $j \neq i$, and a bi-infinite geodesic if $j = i$.

Let $\mathcal{L}_i(L) \subseteq \mathcal{B}(L)$ be the set of $\sim$-classes of ith coordinate lines, and let $\mathcal{L}(L) = \bigcup_{i=1}^{n} \mathcal{L}_i(L)$. We refer to elements of $\mathcal{L}(L)$ as *coarse coordinate lines*. Note that $\mathcal{L}(L) \subseteq \mathcal{Q}(L)$.

We make the following observations, which are simple consequences of properties of bi-infinite geodesics in a hyperbolic space. We omit proofs.

First note that if $l \in \mathcal{L}_i(L)$ and $l' \in \mathcal{L}_j(L)$ with $i \neq j$, then $l \wedge l' = 0$. In fact:

Lemma 5.1. *Suppose that $l \in \mathcal{L}_i(L)$ and $l' \in \mathcal{L}_j(L)$. Then $i = j$ if and only if there is some $l'' \in \mathcal{L}(L)$ with $l \wedge l'' \neq 0$ and $l' \wedge l'' \neq 0$.*

Clearly, in this case, $l'' \in \mathcal{L}_i(L) = \mathcal{L}_j(L)$.

Lemma 5.2. *Suppose that $F, G \in \mathcal{F}(L)$, and $F \wedge G \in \mathcal{Q}(L)$, then $F \wedge G \in \mathcal{L}(L)$.*

Lemma 5.3. *Suppose that each Λ_i is bushy. If $l \in \mathcal{L}(L)$, then there exist $F, G \in \mathcal{F}(L)$ with $l = F \wedge G$.*

Suppose now, that $L = \prod_i \Lambda_i$, $L' = \prod_i \Lambda'_i$ and $\phi : L \longrightarrow L'$ are as in the hypotheses of Theorem 1.1. Write ρ_i and ρ'_i for the metrics on Λ_i and Λ'_i respectively.

By Lemma 4.4, $\phi(\mathcal{F}(L)) \subseteq \mathcal{F}(L')$, and so by Lemmas 5.3 and 5.2, we see that $\phi(\mathcal{L}(L)) \subseteq \mathcal{L}(L')$. Now, by Lemma 5.1, we see that if $l \in \mathcal{L}_i(L)$, $l' \in \mathcal{L}_j(L)$, $\phi l \in \mathcal{L}_{i'}(L')$ and $\phi l' \in \mathcal{L}_{j'}(L')$, then $i = j$ if and only if $i' = j'$. Thus, there is a permutation, $\omega : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$, such that $\phi(\mathcal{L}_i(L)) \subseteq \mathcal{L}_{\omega(i)}(L')$.

To simplify notation, we will take ω to be the identity, so that $\phi(\mathcal{L}_i(L)) \subseteq \mathcal{L}_i(L')$.

We write $\pi_i : L \longrightarrow \Lambda_i$ and $\pi'_i : L' \longrightarrow \Lambda'_i$ for the coordinate projections.

To proceed, we need a more quantitative version of this, namely:

Lemma 5.4. *There is some $t \geq 0$, depending only on the constants of the hypotheses of Theorem 1.1, such that if $\lambda \subseteq L$ is a coordinate line, then there is a coordinate line, $\lambda' \subseteq L'$, with $\mathrm{hd}(\lambda', \phi\lambda) \leq t$.*

Note that, by the above, if λ is an ith coordinate line, then so is λ'.

Lemma 5.4 follows easily from more quantitative versions of Lemmas 5.2 and 5.3. Note that, by bushiness, if $x \in \Lambda_i$, then there are bi-infinite geodesics, $\beta, \gamma \subseteq \Lambda_i$, with $x \in N(\beta; r_1) \cap N(\gamma; r_1) \subseteq N(x; r_2)$, where $r_2 \geq r_1 \geq 0$ depend only on the constants of hyperbolicity and bushiness. It follows that, if λ is any line in Λ, we can find product flats, F, G, with $\lambda \subseteq N(F; r_1) \cap N(G; r_1) \subseteq N(\lambda; r_2)$ (applying this to all factors other than that containing λ). One can then get $\phi\lambda \subseteq N(\phi F; r'_1) \cap N(\phi G; r'_1) \subseteq N(\phi\lambda; r'_2)$, for uniform constants, $r'_2 \geq r'_1 \geq 0$. Now, there are product flats, $F', G' \subseteq L'$, with $\mathrm{hd}(F', \phi F)$ and $\mathrm{hd}(G', \phi G)$ bounded above. We then get a coordinate line, l', with $l' \subseteq N(F'; r''_1) \cap N(G'; r''_2) \subseteq N(l'; r''_2)$, with $r''_2 \geq r''_2 \geq 0$ uniform. Finally, we see that $\mathrm{hd}(l', \phi l)$ is bounded above as required.

Now, any two points in a taut hyperbolic space are a bounded distance from some bi-infinite geodesic. Therefore, if $x, y \in L$ differ only in the ith coordinate, then there is an ith coordinate line, λ, with $x, y \in \lambda$, and so $\phi x, \phi y \in \phi\lambda$. By Lemma 5.4, the diameter of $\pi'_j(\phi\lambda)$ is at most $2t$ for all $j \neq i$. Therefore, $\rho'_j(\pi'_j\phi x, \pi'_j\phi y)$ is bounded for all $j \neq i$. By changing coordinates one at a time, we therefore deduce:

Lemma 5.5. *There is some $h \geq 0$ such that for all $i \in \{1, \ldots, n\}$ and for all $x, y \in L$, with $\pi_i x = \pi_i y$, we have $\rho'_i(\pi'_i\phi x, \pi'_i\phi y) \leq h$.*

It follows that if $\rho_i(\pi_i x, \pi_i y)$ is bounded, so is $\rho'_i(\pi'_i\phi x, \pi'_i\phi y)$ (by considering the point obtained by changing the ith coordinate of y to that of x).

To prove Theorem 1.1, it remains to make the following elementary observation:

Lemma 5.6. *Suppose that $L = \prod_{i=1}^n \Lambda_i$, $L' = \prod_{i=1}^n \Lambda'_i$ are products of geodesic metric spaces, and that $\phi : L \longrightarrow L'$ is a quasi-isometric embedding. Suppose that there is some $h \geq 0$, such that for all $i \in \{1, \ldots, n\}$, and all $x, y \in L$ with $\pi_i x = \pi_i y$, we have $\rho'_i(\pi'_i\phi x, \pi'_i\phi y) \leq h$. Then there are maps (necessarily quasi-isometric embeddings), $\phi_i : \Lambda_i \longrightarrow \Lambda'_i$, such that for all $(x_1, \ldots, x_n) \in X$, $\rho'(\phi(x_1, \ldots, x_n), (\phi_1 x_1, \ldots, \phi_n x_n))$ is bounded above in terms of h and the constants of quasi-isometry of ϕ.*

Proof. Given $x \in \Lambda_i$, choose any $a \in \pi_i^{-1}x \subseteq L$ and set $\phi_i x = \pi'_i\phi a$. The hypotheses tell us that ϕ_i is well defined up to bounded distance and coarsely lipschitz (see the remark after Lemma 5.5). To see that it is a quasi-isometric embedding, suppose $x, y \in \Lambda_i$, and choose any $a \in \pi_i^{-1}x$ and $b \in \pi_i^{-1}y$ with $\pi_j a = \pi_j b$ for all $j \neq i$. Then $\rho_i(x, y) = \rho(a, b)$ and by the above, $\rho'_j(\pi'_j\phi a, \pi'_j\phi b)$ is bounded for all $j \neq i$. Thus, $\rho'(\phi a, \phi b)$ agrees up to an additive constant with $\rho'_i(\pi'_i\phi a, \pi'_i\phi b)$ and hence also with $\rho'_i(\phi_i x, \phi_i y)$. Since $\rho(a, b)$ is linearly bounded above in terms of $\rho'(\phi a, \phi b)$, we see that $\rho_i(x, y)$ is linearly bounded above in terms of $\rho'_i(\phi_i x, \phi_i y)$. $\square$

Putting this together with Lemma 5.5 therefore proves Theorem 1.1.

We can prove Theorem 1.2 by a similar argument. We have already observed in Section 4 that Proposition 2.1 applied to ϕ^∞ tells us that $p = q$.

We now define $\mathcal{L}_i(L)$ and $\mathcal{L}_i(L')$ as before, but this time set $\mathcal{L}(L) = \bigcup_{i=1}^p \mathcal{L}_i(L)$ and $\mathcal{L}(L') = \bigcup_{i=1}^p \mathcal{L}_i(L')$. (In other words, we only consider coordinate lines in the bushy factors.) This is needed for Lemma 5.3 to remain valid in the form stated. Lemmas 5.1 and 5.2 still hold with this definition. We now get a permutation, $\omega : \{1, \dots, p\} \longrightarrow \{1, \dots, p\}$, which we can take to be the identity; and so $\phi(\mathcal{L}_i(L)) \subseteq \mathcal{L}_i(L')$ for all $i \in \{1, \dots, p\}$. Lemma 5.4 holds for the ith coordinate lines with $i \leq p$, and so Lemma 5.5 holds, again restricting i to $\{1, \dots, p\}$. For such i, we get a map $\phi_i : \Lambda_i \longrightarrow \Lambda'_i$, which can be seen to be coarsely lipschitz for the same reason as before. Let $P = \prod_{i=1}^p \Lambda_i$ and $P' = \prod_{i=1}^p \Lambda'_i$. Combining the ϕ_i, we get a coarsely lipschitz product map $\psi : P \longrightarrow P'$. This has the property that if two points of L have the same projection to P, then their ϕ-images have the same projection to P' up to bounded distance.

Now, given $u \in P$, set $Q(u) \subseteq L$ to be the set of points whose first p coordinates are given by u, and define $Q'(\psi u) \subseteq L'$ similarly. Now, $\phi(Q(u))$ lies in a bounded neighbourhood of $Q(\psi u)$, and so $\phi|Q(u)$ is a bounded distance from a quasi-isometric map from $Q(u)$ to $Q'(\psi u)$. But $Q(u) \equiv \prod_{i=p+1}^n \Lambda_i$ and $Q'(\psi u) \equiv \prod_{i=p+1}^n \Lambda'_i$ are both quasi-isometric to $\mathbb{R}^{n-p}$. Now it is well known that a quasi-isometric embedding between euclidean spaces of the same dimension must be a quasi-isometry. This gives us our quasi-isometry $\phi_u : \prod_{i=p+1}^n \Lambda_i \longrightarrow \prod_{i=p+1}^n \Lambda'_i$.

It remains to check that the maps ϕ_i are quasi-isometric embeddings for all $i \in \{1, \dots, p\}$. As before, suppose $x, y \in \Lambda_i$, and choose $a, b \in L$ with $\pi_i a = x$, $\pi_i b = y$ and $\pi_j a = \pi_j b$ for all $j \in \{1, \dots, n\} \setminus \{i\}$. Let $u \in P$ be the projection to the first p coordinates of b, so that $b \in Q(u)$. Thus, ϕb is a bounded distance from $Q'(\psi u)$. Since ϕ_u is a quasi-isometry, we can find some $c \in Q(u)$ such that the final $n - p$ coordinates of ϕc agree with those of ϕa up to bounded distance. We also have $\pi'_j \phi c$ a bounded distance from $\pi'_j \phi b$ and hence also from $\pi'_j \phi a$ for all $j \in \{1, \dots, p\} \setminus \{i\}$. Thus, $\rho'(\phi a, \phi c)$ agrees up to an additive constant with $\rho'_i(\pi'_i \phi a, \pi'_i \phi c)$ and hence with $\rho'_i(\pi'_i \phi a, \pi'_i \phi b)$ and so also with $\rho'_i(\phi_i x, \phi_i y)$. But now, $\rho_i(x, y) = \rho(a, b) \leq \rho(a, c)$, (since a, b differ only in the ith coordinate, and c has the same ith coordinate as b). Also $\rho(a, c)$ is linearly bounded above in terms of $\rho'(\phi a, \phi c)$ and hence in terms of $\rho'_i(\phi_i x, \phi_i y)$. This shows that ϕ_i is a quasi-isometric embedding.

This proves Theorem 1.2.

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