On κ-reducibility of pseudovarieties of the form $\mathbf{V} * \mathbf{D}$

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Abstract

This paper deals with the reducibility property of semidirect products of the form $\mathbf{V} * \mathbf{D}$ relatively to graph equation systems, where $\mathbf{D}$ denotes the pseudovariety of definite semigroups. We show that, if the pseudovariety $\mathbf{V}$ is reducible with respect to the canonical signature κ consisting of the multiplication and the $(\omega - 1)$-power, then $\mathbf{V} * \mathbf{D}$ is also reducible with respect to κ.

Keywords. Pseudovariety, definite semigroup, semidirect product, implicit signature, graph equations, reducibility.

1 Introduction

A semigroup (resp. monoid) *pseudovariety* is a class of finite semigroups (resp. monoids) closed under taking subsemigroups (resp. submonoids), homomorphic images and finite direct products. It is said to be *decidable* if there is an algorithm to test membership of a finite semigroup (resp. monoid) in that pseudovariety. The semidirect product of pseudovarieties has been getting much attention, mainly due to the Krohn-Rhodes decomposition theorem [18]. In turn, the pseudovarieties of the form $\mathbf{V}*\mathbf{D}$, where $\mathbf{D}$ is the pseudovariety of all finite semigroups whose idempotents are right zeros, are among the most studied semidirect products [23, 25, 3, 1, 4]. For a pseudovariety $\mathbf{V}$ of monoids, $\mathbf{LV}$ denotes the pseudovariety of all finite semigroups S such that $eSe \in \mathbf{V}$ for all idempotents e of S. We know from [17, 23, 24, 25] that $\mathbf{V} * \mathbf{D}$ is contained in $\mathbf{LV}$ and that $\mathbf{V} * \mathbf{D} = \mathbf{LV}$ if and only if $\mathbf{V}$ is *local* in the sense of Tilson [25]. In particular, the equalities $\mathbf{Sl} * \mathbf{D} = \mathbf{LSl}$ and $\mathbf{G} * \mathbf{D} = \mathbf{LG}$ hold for the pseudovarieties $\mathbf{Sl}$ of semilattices and $\mathbf{G}$ of groups.

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It is known that the semidirect product operator does not preserve decidability of pseudovarieties [20, 11]. The notion of *tameness* was introduced by Almeida and Steinberg [7, 8] as a tool for proving decidability of semidirect products. The fundamental property for tameness is *reducibility*. This property was originally formulated in terms of graph equation systems and latter extended to any system of equations [2, 21]. It is parameterized by an *implicit signature* σ (a set of implicit operations on semigroups containing the multiplication), and we speak of σ-reducibility. For short, given an equation system Σ with rational constraints, a pseudovariety $\mathbf{V}$ is σ-reducible relatively to Σ when the existence of a solution of Σ by implicit operations over $\mathbf{V}$ implies the existence of a solution of Σ by σ-words over $\mathbf{V}$ and satisfying the same constraints. The pseudovariety $\mathbf{V}$ is said to be *σ-reducible* if it is σ-reducible with respect to every finite graph equation system. The implicit signature which is most commonly encountered in the literature is the *canonical signature* $\kappa = \{ab, a^{\omega-1}\}$ consisting of the multiplication and the $(\omega-1)$-power. For instance, the pseudovarieties $\mathbf{D}$ [9], $\mathbf{G}$ [10, 8], $\mathbf{J}$ [1, 2] of all finite $\mathscr{J}$-trivial semigroups, $\mathbf{LSl}$ [16] and $\mathbf{R}$ [6] of all finite $\mathscr{R}$-trivial semigroups are κ-reducible.

In this paper, we study the κ-reducibility property of semidirect products of the form $\mathbf{V}*\mathbf{D}$. This research is essentially inspired by the papers [15, 16] (see also [13] where a stronger form of κ-reducibility was established for $\mathbf{LSl}$). We prove that, if $\mathbf{V}$ is κ-reducible then $\mathbf{V}*\mathbf{D}$ is κ-reducible. In particular, this gives a new and simpler proof (though with the same basic idea) of the κ-reducibility of $\mathbf{LSl}$ and establishes the κ-reducibility of the pseudovarieties $\mathbf{LG}$, $\mathbf{J}*\mathbf{D}$ and $\mathbf{R}*\mathbf{D}$. Combined with the recent proof that the κ-word problem for $\mathbf{LG}$ is decidable [14], this shows that $\mathbf{LG}$ is κ-tame, a problem proposed by Almeida a few years ago. This also extends part of our work in the paper [15], where we proved that under mild hypotheses on an implicit signature σ, if $\mathbf{V}$ is σ-reducible relatively to *pointlike* systems of equations (i.e., systems of equations of the form $x_1 = \cdots = x_n$) then $\mathbf{V}*\mathbf{D}$ is pointlike σ-reducible as well. As in [15], we use results from [5], where various kinds of σ-reducibility of semidirect products with an order-computable pseudovariety were considered. More specifically, we know from [5] that a pseudovariety of the form $\mathbf{V}*\mathbf{D}_k$ is κ-reducible when $\mathbf{V}$ is κ-reducible, where $\mathbf{D}_k$ is the order-computable pseudovariety defined by the identity $yx_1\cdots x_k = x_1\cdots x_k$. As $\mathbf{V}*\mathbf{D} = \bigcup_k \mathbf{V}*\mathbf{D}_k$, we utilize this result as a way to achieve our property concerning the pseudovarieties $\mathbf{V}*\mathbf{D}$. The method used in this paper is similar to that of [15]. However, some significant changes, inspired by [16], had to be introduced in order to deal with the much more intricate graph equation systems.

2 Preliminaries

The reader is referred to the standard bibliography on finite semigroups, namely [1, 21], for general background and undefined terminology. For basic definitions and results about combinatorics on words, the reader may wish to consult [19].

2.1 Words and pseudowords

Throughout this paper, A denotes a finite non-empty set called an *alphabet*. The *free semigroup* and the *free monoid* generated by A are denoted respectively by A^+ and A^*. The empty word is represented by 1 and the length of a word $w \in A^*$ is denoted by $|w|$. A word is called *primitive* if it cannot be written in the form u^n with $n > 1$. Two words u and v are said to be *conjugate* if $u = w_1w_2$ and $v = w_2w_1$ for some words $w_1, w_2 \in A^*$. A *Lyndon word* is a primitive word which is minimal in its conjugacy class, for the lexicographic order on A^+.

A *left-infinite* word on A is a sequence $w = (a_n)_n$ of letters of A indexed by $-\mathbb{N}$ also written $w = \cdots a_{-2}a_{-1}$. The set of all left-infinite words on A will be denoted by $A^{-\mathbb{N}}$ and we put $A^{-\infty} = A^+ \cup A^{-\mathbb{N}}$. The set $A^{-\infty}$ is endowed with a semigroup structure by defining a product as follows: if $w, z \in A^+$, then wz is already defined; left-infinite words are right zeros; finally, if $w = \cdots a_{-2}a_{-1}$ is a left-infinite word and $z = b_1b_2 \cdots b_n$ is a finite word, then wz is the left-infinite word $wz = \cdots a_{-2}a_{-1}b_1b_2 \cdots b_n$. A left-infinite word w of the form $u^{-\infty}v = \cdots uuuv$, with $u \in A^+$ and $v \in A^*$, is said to be *ultimately periodic*. In case $v = 1$, the word w is named *periodic*. For a periodic word $w = u^{-\infty}$, if u is a primitive word, then it will be called the *root* of w and its length $|u|$ will be said to be the *period* of w.

For a pseudovariety $\mathbf{V}$ of semigroups, we denote by $\overline{\Omega}_A\mathbf{V}$ the relatively free pro-$\mathbf{V}$ semigroup generated by the set A: for each pro-$\mathbf{V}$ semigroup S and each function $\varphi : A \to S$, there is a unique continuous homomorphism $\overline{\varphi} : \overline{\Omega}_A\mathbf{V} \to S$ extending φ. The elements of $\overline{\Omega}_A\mathbf{V}$ are called *pseudowords* (or *implicit operations*) over $\mathbf{V}$. A pseudovariety $\mathbf{V}$ is called *order-computable* when the subsemigroup $\Omega_A\mathbf{V}$ of $\overline{\Omega}_A\mathbf{V}$ generated by A is finite, in which case $\Omega_A\mathbf{V} = \overline{\Omega}_A\mathbf{V}$, and effectively computable. Recall that, for the pseudovariety $\mathbf{S}$ of all finite semigroups, $\Omega_A\mathbf{S}$ is (identified with) the free semigroup A^+. The elements of $\overline{\Omega}_A\mathbf{S} \setminus A^+$ will then be called *infinite pseudowords*.

A *pseudoidentity* is a formal equality $\pi = \rho$ of pseudowords $\pi, \rho \in \overline{\Omega}_A\mathbf{S}$ over $\mathbf{S}$. We say that $\mathbf{V}$ *satisfies* the pseudoidentity $\pi = \rho$, and write $\mathbf{V} \models \pi = \rho$, if $\varphi\pi = \varphi\rho$ for every continuous homomorphism $\varphi : \overline{\Omega}_A\mathbf{S} \to S$ into a semigroup $S \in \mathbf{V}$, which is equivalent to saying that $p_\mathbf{V}\pi = p_\mathbf{V}\rho$ for the *natural projection* $p_\mathbf{V} : \overline{\Omega}_A\mathbf{S} \to \overline{\Omega}_A\mathbf{V}$.

2.2 Pseudoidentities over $\mathbf{V} * \mathbf{D}_k$

For a positive integer k, let $\mathbf{D}_k$ be the pseudovariety of all finite semigroups satisfying the identity $yx_1 \cdots x_k = x_1 \cdots x_k$. Denote by A^k the set of words over A with length k and by A_k the set $\{w \in A^+ : |w| \leq k\}$ of non-empty words over A with length at most k. We notice that $\Omega_A\mathbf{D}_k$ may be identified with the semigroup whose support set is A_k and whose multiplication is given by $u \cdot v = \mathtt{t}_k(uv)$, where $\mathtt{t}_kw$ denotes the longest suffix of length at most k of a given (finite or left-infinite) word w. Then, the $\mathbf{D}_k$ are order-computable pseudovarieties such that $\mathbf{D} = \bigcup_k \mathbf{D}_k$. Moreover, it is well-known that $\overline{\Omega}_A\mathbf{D}$ is isomorphic to the semigroup $A^{-\infty}$.

For each pseudoword $\pi \in \overline{\Omega}_A\mathbf{S}$, we denote by $\mathtt{t}_k\pi$ the unique smallest word (of A_k) such that $\mathbf{D}_k \models \pi = \mathtt{t}_k\pi$. Simetrically, we denote by $\mathtt{i}_k\pi$ the unique smallest word (of A_k) such that $\mathbf{K}_k \models \pi = \mathtt{i}_k\pi$, where $\mathbf{K}_k$ is the dual pseudovariety of $\mathbf{D}_k$ defined by the identity $x_1 \cdots x_k y = x_1 \cdots x_k$. Let Φ_k be the function $A^+ \to (A^{k+1})^*$ that sends each word $w \in A^+$ to the sequence of factors of length $k+1$ of w, in the order they occur in w. We still denote by Φ_k (see [3] and [1, Lemma 10.6.11]) its unique continuous extension $\overline{\Omega}_A\mathbf{S} \to (\overline{\Omega}_{A^{k+1}}\mathbf{S})^1$. This function Φ_k is a *k-superposition homomorphism*, with the meaning that it verifies the conditions:

i) $\Phi_k w = 1$ for every $w \in A_k$;

ii) $\Phi_k(\pi\rho) = \Phi_k\pi\Phi_k\big((\mathtt{t}_k\pi)\rho\big) = \Phi_k\big(\pi(\mathtt{i}_k\rho)\big)\Phi_k\rho$ for every $\pi, \rho \in \overline{\Omega}_A\mathbf{S}$.

Throughout the paper, $\mathbf{V}$ denotes a non-locally trivial pseudovariety of semigroups. For any pseudowords $\pi, \rho \in \overline{\Omega}_A\mathbf{S}$, it is known from [1, Theorem 10.6.12] that

$$\mathbf{V} * \mathbf{D}_k \models \pi = \rho \quad\Longleftrightarrow\quad \mathtt{i}_k\pi = \mathtt{i}_k\rho,\ \mathtt{t}_k\pi = \mathtt{t}_k\rho \text{ and } \mathbf{V} \models \Phi_k\pi = \Phi_k\rho. \tag{2.1}$$

2.3 Implicit signatures and σ-reducibility

By an *implicit signature* we mean a set σ of pseudowords (over $\mathbf{S}$) containing the multiplication. In particular, we represent by κ the implicit signature $\{ab, a^{\omega-1}\}$, usually called the *canonical signature*. Every profinite semigroup has a natural structure of a σ-algebra, via the natural interpretation of pseudowords on profinite semigroups. The σ-subalgebra of $\overline{\Omega}_A\mathbf{S}$ generated by A is denoted by $\Omega_A^\sigma\mathbf{S}$. It is freely generated by A in the variety of σ-algebras generated by the pseudovariety $\mathbf{S}$ and its elements are called *σ-words* (over $\mathbf{S}$). To a (directed multi)graph $\Gamma = V(\Gamma) \uplus E(\Gamma)$, with vertex set $V(\Gamma)$, edge set $E(\Gamma)$, and two binary operations $\alpha, \omega : E(\Gamma) \to V(\Gamma)$ defining, respectively, the *beginning* and the *end* vertices of each edge, we associate the system Σ_Γ of all equations of the form $(\alpha\mathsf{e})\,\mathsf{e} = \omega\mathsf{e}$, with $\mathsf{e} \in E(\Gamma)$. Let S be a finite A-generated semigroup, $\delta : \overline{\Omega}_A\mathbf{S} \to S$ be the continuous homomorphism respecting the choice of generators and $\varphi : \Gamma \to S^1$ be an evaluation mapping such that $\varphi E(\Gamma) \subseteq S$. We say that a mapping $\eta : \Gamma \to (\overline{\Omega}_A\mathbf{S})^1$ is a *$\mathbf{V}$-solution* of Σ_Γ with respect to (φ, δ) when $\delta\eta = \varphi$ and $\mathbf{V} \models \overline{\eta}u = \overline{\eta}v$ for all $(u = v) \in \Sigma_\Gamma$. Furthermore, if $\eta\Gamma \subseteq (\Omega_A^\sigma\mathbf{S})^1$ for an implicit signature σ, then η is called a *$(\mathbf{V}, \sigma)$-solution*. The pseudovariety $\mathbf{V}$ is said to be *σ-reducible* relatively to the system Σ_Γ if the existence of a $\mathbf{V}$-solution of Σ_Γ with respect to a pair (φ, δ) entails the existence of a $(\mathbf{V}, \sigma)$-solution of Σ_Γ with respect to the same pair (φ, δ). We say that $\mathbf{V}$ is *σ-reducible*, if it is σ-reducible relatively to Σ_Γ for all finite graphs Γ.

3 κ-reducibility of $\mathbf{V} * \mathbf{D}$

Let $\mathbf{V}$ be a given κ-reducible non-locally trivial pseudovariety. The purpose of this paper is to prove the κ-reducibility of the pseudovariety $\mathbf{V} * \mathbf{D}$. So, we fix a finite graph Γ and a finite A-generated semigroup S and consider a $\mathbf{V} * \mathbf{D}$-solution $\eta : \Gamma \to (\overline{\Omega}_A\mathbf{S})^1$ of the system Σ_Γ with respect to a pair (φ, δ), where $\varphi : \Gamma \to S^1$ is an evaluation mapping such that $\varphi E(\Gamma) \subseteq S$ and $\delta : \overline{\Omega}_A\mathbf{S} \to S$ is a continuous homomorphism respecting the choice of generators. We have to construct a $(\mathbf{V} * \mathbf{D}, \kappa)$-solution $\eta' : \Gamma \to (\Omega_A^\kappa\mathbf{S})^1$ of Σ_Γ with respect to the same pair (φ, δ).

3.1 Initial considerations

Suppose that $\mathsf{g} \in \Gamma$ is such that $\eta\mathsf{g} = u$ with $u \in A^*$. Since η and η' are supposed to be $\mathbf{V}*\mathbf{D}$-solutions of the system Σ_Γ with respect to (φ, δ), we must have $\delta\eta = \varphi = \delta\eta'$ and so, in particular, $\delta\eta'\mathsf{g} = \delta u$. As the homomorphism $\delta : \overline{\Omega}_A\mathbf{S} \to S$ is arbitrarily fixed, it may happen that the equality $\delta\eta'\mathsf{g} = \delta u$ holds only when $\eta'\mathsf{g} = u$. In that case we would be obliged to define $\eta'\mathsf{g} = u$. Since we want to describe an algorithm to define η' that should work for any given graph and solution, we will then construct a solution η' verifying the following condition:

$$\forall \mathsf{g} \in \Gamma, \quad (\eta\mathsf{g} \in A^* \Longrightarrow \eta'\mathsf{g} = \eta\mathsf{g}). \qquad \mathscr{C}_1(\Gamma, \eta, \eta')$$

Suppose next that a vertex $\mathsf{v} \in V(\Gamma)$ is such that $\mathbf{D} \models \eta\mathsf{v} = u^\omega$ with $u \in A^+$, that is, suppose that $p_{\mathbf{D}}\eta\mathsf{v} = u^{-\infty}$. Because Γ is an arbitrary graph, it could include, for instance, an edge e such that $\alpha\mathsf{e} = \omega\mathsf{e} = \mathsf{v}$ and the labeling η could be such that $\eta\mathsf{e} = u$. Since $\mathbf{D}$ is a subpseudovariety of $\mathbf{V} * \mathbf{D}$, η is a $\mathbf{D}$-solution of Σ_Γ with respect to (φ, δ). Hence, as by condition $\mathscr{C}_1(\Gamma, \eta, \eta')$ we want to preserve finite labels, it would follow in that case that $\mathbf{D} \models (\eta'\mathsf{v})u = \eta'\mathsf{v}$ and, thus, that $\mathbf{D} \models \eta'\mathsf{v} = u^\omega = \eta\mathsf{v}$. This observation suggests that we should preserve the projection into $\overline{\Omega}_A\mathbf{D}$ of labelings of vertices v such that $p_{\mathbf{D}}\eta\mathsf{v} = u^{-\infty}$ with $u \in A^+$. More generally, we will construct the $(\mathbf{V} * \mathbf{D}, \kappa)$-solution η' in such a way that the following condition holds:

$$\forall \mathsf{v} \in V(\Gamma), \quad (p_{\mathbf{D}}\eta\mathsf{v} = u^{-\infty}z \text{ with } u \in A^+ \text{ and } z \in A^* \Longrightarrow p_{\mathbf{D}}\eta'\mathsf{v} = p_{\mathbf{D}}\eta\mathsf{v}). \qquad \mathscr{C}_2(\Gamma, \eta, \eta')$$

Let $\ell_\eta = \max\{|u| : u \in A^* \text{ and } \eta\mathsf{g} = u \text{ for some } \mathsf{g} \in \Gamma\}$ be the maximum length of finite labels under η of elements of Γ. To be able to make some reductions on the graph Γ and solution η, described in Section 3.2, we want η' to verify the extra condition below, where $L \geq \ell_\eta$ is a non-negative integer to be specified later, on Section 3.3:

$$\forall \mathsf{v} \in V(\Gamma), \quad (\eta\mathsf{v} = u\pi \text{ with } u \in A_L \Longrightarrow \eta'\mathsf{v} = u\pi' \text{ with } \delta\pi = \delta\pi'). \qquad \mathscr{C}_3(\Gamma, \eta, \eta')$$

3.2 Simplifications on the solution η

We begin this section by reducing to the case in which all vertices of Γ are labeled by infinite pseudowords under η. Suppose first that there is an edge $\mathsf{v} \xrightarrow{\mathsf{e}} \mathsf{w}$ such that $\eta\mathsf{v} = u_\mathsf{v}$ and $\eta\mathsf{e} = u_\mathsf{e}$ with $u_\mathsf{v} \in A^*$ and $u_\mathsf{e} \in A^+$, so that $\eta\mathsf{w} = u_\mathsf{v}u_\mathsf{e}$. Drop the edge e and consider the restrictions η_1 and φ_1, of η and φ respectively, to the graph $\Gamma_1 = \Gamma \setminus \{\mathsf{e}\}$. Then η_1 is a $\mathbf{V}*\mathbf{D}$-solution of the system Σ_{Γ_1} with respect to the pair (φ_1, δ). Assume that there is a $(\mathbf{V} * \mathbf{D}, \kappa)$-solution η_1' of Σ_{Γ_1} with respect to (φ_1, δ) verifying condition $\mathscr{C}_1(\Gamma_1, \eta_1, \eta_1')$. Then $\eta_1'\mathsf{v} = u_\mathsf{v}$ and $\eta_1'\mathsf{w} = u_\mathsf{v}u_\mathsf{e}$. Let η' be the extension of η_1' to Γ obtained by letting $\eta'\mathsf{e} = u_\mathsf{e}$. Then η' is a $(\mathbf{V}*\mathbf{D}, \kappa)$-solution of Σ_Γ with respect to (φ, δ). By induction on the number of edges labeled by finite words under η beginning in vertices also labeled by finite words under η, we may therefore assume that there are no such edges in Γ.

Now, we remove all vertices v of Γ labeled by finite words under η such that v is not the beginning of an edge, thus obtaining a graph Γ_1. As above, if η_1' is a $(\mathbf{V} * \mathbf{D}, \kappa)$-solution of

Σ_{Γ_1}, then we build a $(\mathbf{V}*\mathbf{D},\kappa)$-solution η' of Σ_Γ by letting η' coincide with η'_1 on Γ_1 and letting $\eta'\mathsf{v}=\eta\mathsf{v}$ for each vertex $\mathsf{v}\in\Gamma\setminus\Gamma_1$. So, we may assume that all vertices of Γ labeled by finite words under η are the beginning of some edge.

Suppose next that $\mathsf{v}\xrightarrow{\mathsf{e}}\mathsf{w}$ is an edge such that $\eta\mathsf{v}=u$ and $\eta\mathsf{e}=\pi$ with $u\in A^*$ and $\pi\in\overline{\Omega}_A\mathbf{S}\setminus A^+$. Notice that, since it is an infinite pseudoword, π can be written as $\pi=\pi_1\pi_2$ with both π_1 and π_2 being infinite pseudowords. Drop the edge e (and the vertex v in case e is the only edge beginning in v) and let v_1 be a new vertex and $\mathsf{v}_1\xrightarrow{\mathsf{e}_1}\mathsf{w}$ be a new edge thus obtaining a new graph Γ_1. Let η_1 and φ_1 be the labelings of Γ_1 defined as follows:

- η_1 and φ_1 coincide, respectively, with η and φ on $\Gamma'=\Gamma_1\cap\Gamma$;
- $\eta_1\mathsf{v}_1=u\pi_1$, $\eta_1\mathsf{e}_1=\pi_2$, $\varphi_1\mathsf{v}_1=\delta\eta_1\mathsf{v}_1$ and $\varphi_1\mathsf{e}_1=\delta\eta_1\mathsf{e}_1$.

Then η_1 is a $\mathbf{V}*\mathbf{D}$-solution of the system Σ_{Γ_1} with respect to the pair (φ_1,δ). Assume that there is a $(\mathbf{V}*\mathbf{D},\kappa)$-solution η'_1 of Σ_{Γ_1} with respect to (φ_1,δ) verifying conditions $\mathscr{C}_1(\Gamma_1,\eta_1,\eta'_1)$ and $\mathscr{C}_3(\Gamma_1,\eta_1,\eta'_1)$. In particular, since L is chosen to be greater than ℓ_η, $\eta'_1\mathsf{v}_1=u\pi'_1$ with $\delta\pi_1=\delta\pi'_1$. Let η' be the extension of $\eta'_{1|\Gamma'}$ to Γ obtained by letting $\eta'\mathsf{e}=\pi'_1(\eta'_1\mathsf{e}_1)$ (and $\eta'\mathsf{v}=u$ in case $\mathsf{v}\notin\Gamma'$). As one can easily verify, η' is a $(\mathbf{V}*\mathbf{D},\kappa)$-solution of Σ_Γ with respect to (φ,δ). By induction on the number of edges beginning in vertices labeled by finite words under η, we may therefore assume that all vertices of Γ are labeled by infinite pseudowords under η.

Suppose at last that an edge $\mathsf{e}\in\Gamma$ is labeled under η by a finite word $u=a_1\cdots a_n$, where $n>1$ and $a_i\in A$. Denote $\mathsf{v}_0=\alpha\mathsf{e}$ and $\mathsf{v}_n=\omega\mathsf{e}$. In this case, we drop the edge e and, for each $i\in\{1,\ldots,n-1\}$, we add a new vertex v_i and a new edge $\mathsf{v}_{i-1}\xrightarrow{\mathsf{e}_i}\mathsf{v}_i$ to the graph Γ. Let Γ_1 be the graph thus obtained and let η_1 and φ_1 be the labelings of Γ_1 defined as follows:

- η_1 and φ_1 coincide, respectively, with η and φ on $\Gamma'=\Gamma\setminus\{\mathsf{e}\}$;
- for each $i\in\{1,\ldots,n-1\}$, $\eta_1\mathsf{v}_i=(\eta\mathsf{v}_0)a_1\cdots a_i$, $\eta_1\mathsf{e}_i=a_i$, $\varphi_1\mathsf{v}_i=\delta\eta_1\mathsf{v}_i$ and $\varphi_1\mathsf{e}_i=\delta\eta_1\mathsf{e}_i$.

Hence, η_1 is a $\mathbf{V}*\mathbf{D}$-solution of the system Σ_{Γ_1} with respect to the pair (φ_1,δ). Suppose there exists a $(\mathbf{V}*\mathbf{D},\kappa)$-solution η'_1 of Σ_{Γ_1} with respect to (φ_1,δ) verifying condition $\mathscr{C}_1(\Gamma_1,\eta_1,\eta'_1)$. Let η' be the extension of $\eta'_{1|\Gamma'}$ to Γ obtained by letting $\eta'\mathsf{e}=u$. Then η' is a $(\mathbf{V}*\mathbf{D},\kappa)$-solution of Σ_Γ with respect to (φ,δ). By induction on the number of edges labeled by finite words under η, we may further assume that each edge of Γ labeled by a finite word under η is, in fact, labeled by a letter of the alphabet.

3.3 Borders of the solution η

The main objective of this section is to define a certain class of finite words, called *borders of the solution* η. Since the equations (of Σ_Γ) we have to deal with are of the form $(\alpha\mathsf{e})\,\mathsf{e}=\omega\mathsf{e}$, these borders will serve to signalize the transition from a vertex $\alpha\mathsf{e}$ to the edge e.

For each vertex v of Γ, denote by $\mathbf{d}_\mathsf{v}\in A^{-\mathbb{N}}$ the projection $p_\mathbf{D}\eta\mathsf{v}$ of $\eta\mathsf{v}$ into $\overline{\Omega}_A\mathbf{D}$ and let $D_\eta=\{\mathbf{d}_\mathsf{v}\mid\mathsf{v}\in V(\Gamma)\}$. We say that two left-infinite words $v_1,v_2\in A^{-\mathbb{N}}$ are *confinal* if they have a common prefix $y \in A^{-\mathbb{N}}$, that is, if $v_1 = yz_1$ and $v_2 = yz_2$ for some words $z_1, z_2 \in A^*$. As one easily verifies, the relation $\propto$ defined, for each $\mathbf{d}_{\mathsf{v}_1}, \mathbf{d}_{\mathsf{v}_2} \in D_\eta$, by

$$\mathbf{d}_{\mathsf{v}_1} \propto \mathbf{d}_{\mathsf{v}_2} \quad \text{if and only if} \quad \mathbf{d}_{\mathsf{v}_1} \text{ and } \mathbf{d}_{\mathsf{v}_2} \text{ are confinal}$$

is an equivalence on D_η. For each $\propto$-class Δ, we fix a word $y_\Delta \in A^{-\mathbb{N}}$ and words $z_{\mathsf{v}} \in A^*$, for each vertex v with $\mathbf{d}_{\mathsf{v}} \in \Delta$, such that

$$\mathbf{d}_{\mathsf{v}} = y_\Delta z_{\mathsf{v}}.$$

Moreover, when $\mathbf{d}_{\mathsf{v}}$ is ultimately periodic, we choose y_Δ of the form $u^{-\infty}$, with u a Lyndon word, and fix z_{v} not having u as a prefix. The word u and its length $|u|$ will be said to be, respectively, a *root* and a *period* of the solution η. Without loss of generality, we assume that η has at least one root. Indeed, otherwise we could easily modify the graph and the solution in order to include one and then recover a $(\mathbf{V} * \mathbf{D}, \kappa)$-solution of Σ_Γ from a $(\mathbf{V} * \mathbf{D}, \kappa)$-solution of the new system, as done in Section 3.2. For instance, we could add a new edge $\mathsf{v} \xrightarrow{\mathsf{e}} \mathsf{w}$ to the graph Γ, where v is any fixed vertex of Γ and w is a new vertex. Next, it would suffice to extend η to a new labeling η_1 by letting $\eta_1(\mathsf{e}) = a^\omega$ and $\eta_1(\mathsf{w}) = \eta(\mathsf{v})a^\omega$.

We fix a few of the integers that will be used in the construction of the $(\mathbf{V} * \mathbf{D}, \kappa)$-solution η'. They depend only on the mapping η and on the semigroup S.

Definition 3.1 (constants n_S, p_η, L, E and Q) *We let:*

- *n_S be the* exponent *of S which, as one recalls, is the least integer such that s^{n_S} is idempotent for every element s of the finite A-generated semigroup S;*
- *$p_\eta = \mathrm{lcm}\{|u| : u \in A^+$ is a root of $\eta\}$;*
- *$L = \max\{\ell_\eta, |z_{\mathsf{v}}| : \mathsf{v} \in V(\Gamma)\}$;*
- *E be an integer such that $E \geq n_S p_\eta$ and, for each word $w \in A^E$, there is a factor $e \in A^+$ of w for which δe is an idempotent of S. Notice that, for each root u of η, $|u^{n_S}| \leq E$ and $\delta(u^{n_S})$ is an idempotent of S;*
- *$Q = L + E$.*

For each positive integer m, we denote by B_m the set

$$B_m = \{\mathtt{t}_m y_\Delta \in A^m \mid \Delta \text{ is a } \propto\text{-class}\}.$$

If $y_\Delta = u^{-\infty}$ is a periodic left-infinite word, then the element $y = \mathtt{t}_m y_\Delta$ of B_m will be said to be *periodic* (with *root* u and *period* $|u|$). For words $y_1, y_2 \in B_m$, we define the *gap between y_1 and y_2* as the positive integer

$$g(y_1, y_2) = \min\{|u| \in \mathbb{N} : u \in A^+ \text{ and, for some } v \in A^+,\ y_1 u = v y_2 \text{ or } y_2 u = v y_1\},$$

and notice that $g(y_1, y_2) = g(y_2, y_1) \leq m$.

Proposition 3.2 *Consider the constant Q introduced in Definition 3.1. There exists $q_Q \in \mathbb{N}$ such that for all integers $m \geq q_Q$ the following conditions hold:*

(a) If y_1 and y_2 are distinct elements of B_m, then $g(y_1, y_2) > Q$;

(b) If y is a non-periodic element of B_m, then $g(y, y) > Q$.

Proof. Suppose that, for every $q_Q \in \mathbb{N}$ there is an integer $m \geq q_Q$ and elements $y_{m,1}$ and $y_{m,2}$ of B_m such that $g(y_{m,1}, y_{m,2}) \leq Q$. Hence, there exist a strictly increasing sequence $(m_i)_i$ of positive integers and an integer $r \in \{1, \ldots, Q\}$ such that $\big(g(y_{m_i,1}, y_{m_i,2})\big)_i$ is constant and equal to r. Moreover, since the graph Γ is finite, we may assume that $y_{m_i,1} = \mathtt{t}_{m_i} y_{\Delta_1}$ and $y_{m_i,2} = \mathtt{t}_{m_i} y_{\Delta_2}$ for every i and some $\propto$-classes Δ_1 and Δ_2. It then follows that $y_{\Delta_1} u = y_{\Delta_2}$ or $y_{\Delta_2} u = y_{\Delta_1}$ for some word $u \in A^r$. Hence, y_{Δ_1} and y_{Δ_2} are confinal left-infinite words, whence Δ_1 and Δ_2 are the same $\propto$-class Δ. Therefore, for every m, $y_{m,1}$ and $y_{m,2}$ have the same length and are suffixes of the word y_Δ and, so, $y_{m,1}$ and $y_{m,2}$ are the same word. This proves already (a). Now, notice that $y_\Delta u = y_\Delta$, meaning that y_Δ is the periodic left-infinite word $u^{-\infty}$. This shows (b) and completes the proof of the proposition. ∎

We now fix two more integers.

Definition 3.3 (constants M and k) *We let:*

- *M be an integer such that M is a multiple of p_η and M is greater than or equal to the integer q_Q of Proposition 3.2, and notice that $M > Q$;*
- *$k = M + Q$.*

The elements of the set B_M will be called the *borders of the solution* η. We remark that the borders of η are finite words of length M such that, by Proposition 3.2, for any two distinct occurrences of borders y_1 and y_2 in a finite word, either these occurrences have a gap of size at least Q between them, or y_1 and y_2 are the same periodic border y. In this case, y is a power of its root u, since M is a multiple of the period $|u|$, and $g(y, y)$ is $|u|$.

3.4 Getting a $(\mathbf{V} * \mathbf{D}_k, \kappa)$-solution

Let k be the constant defined by Definition 3.3. As $\mathbf{V} * \mathbf{D}_k$ is a subpseudovariety of $\mathbf{V} * \mathbf{D}$, η is a $\mathbf{V} * \mathbf{D}_k$-solution of Σ_Γ with respect to (φ, δ). The given pseudovariety $\mathbf{V}$ was assumed to be κ-reducible. So, by [5, Corollary 6.5], $\mathbf{V} * \mathbf{D}_k$ is κ-reducible too. Therefore, there is a $(\mathbf{V} * \mathbf{D}_k, \kappa)$-solution $\eta'_k : \Gamma \to (\Omega^\kappa_A \mathbf{S})^1$ of Σ_Γ with respect to the same pair (φ, δ). Moreover, as observed in [6, Remark 3.4], one can constrain the values $\eta'_k \mathtt{g}$ of each $\mathtt{g} \in \Gamma$ with respect to properties which can be tested in a finite semigroup. Since the prefixes and the suffixes of length at most k can be tested in the finite semigroup $\Omega_A \mathbf{K}_k \times \Omega_A \mathbf{D}_k$, we may assume further that $\eta'_k \mathtt{g}$ and $\eta \mathtt{g}$ have the same prefixes and the same suffixes of length at most k. We then denote

$$\mathtt{i}_\mathtt{g} = \mathtt{i}_k \eta'_k \mathtt{g} = \mathtt{i}_k \eta \mathtt{g} \quad \text{and} \quad \mathtt{t}_\mathtt{g} = \mathtt{t}_k \eta'_k \mathtt{g} = \mathtt{t}_k \eta \mathtt{g},$$

for each $\mathtt{g} \in \Gamma$. Notice that, by the simplifications introduced in Section 3.2, if $\eta\mathtt{g}$ is a finite word, then $\mathtt{g}$ is an edge and $\eta\mathtt{g}$ is a letter $a_\mathtt{g}$ and so $\mathtt{i_g} = \mathtt{t_g} = a_\mathtt{g}$. Otherwise, $\mathtt{i_g}$ and $\mathtt{t_g}$ are length k words. In particular, condition $\mathscr{C}_1(\Gamma, \eta, \eta'_k)$ holds. That is, $\eta'_k\mathtt{e} = \eta\mathtt{e}$ for every edge $\mathtt{e}$ such that $\eta\mathtt{e}$ is a finite word. On the other hand, Lemma 2.3 (ii) of [12], which is stated only for edges, can be extended easily to vertices, so that $\eta'_k\mathtt{g}$ can be assumed to be an infinite pseudoword for every $\mathtt{g} \in \Gamma$ such that $\eta\mathtt{g}$ is infinite. Thus, in particular, $\eta'_k\mathtt{v}$ is an infinite pseudoword for all vertices $\mathtt{v}$.

Notice that, for each vertex $\mathtt{v}$, there exists a border $y_\mathtt{v}$ of η such that the finite word $y_\mathtt{v} z_\mathtt{v}$ is a suffix of $\eta\mathtt{v}$. On the other hand, by Definitions 3.1 and 3.3, $|z_\mathtt{v}| \leq L < Q$ and $k = M + Q$. So, as $|y_\mathtt{v}| = M$,

$$\mathtt{t_v} = x_\mathtt{v} y_\mathtt{v} z_\mathtt{v} \quad \text{and} \quad \eta'_k\mathtt{v} = \pi_\mathtt{v}\mathtt{t_v} \tag{3.1}$$

for some infinite κ-word $\pi_\mathtt{v}$ and some word $x_\mathtt{v} \in A^+$ with $|x_\mathtt{v}| = Q - |z_\mathtt{v}|$.

3.5 Basic transformations

The objective of this section is to introduce the basic steps that will allow to transform the $(\mathbf{V} * \mathbf{D}_k, \kappa)$-solution η'_k into a $(\mathbf{V} * \mathbf{D}, \kappa)$-solution η'. The process of construction of η' from η'_k is close to the one used in [15] to handle with systems of pointlike equations. Both procedures are supported by (basic) transformations of the form

$$a_1 \cdots a_k \mapsto a_1 \cdots a_j (a_i \cdots a_j)^\omega a_{j+1} \cdots a_k,$$

which replace words of length k by κ-words. Those procedures differ in the way the indices $i \leq j$ are determined. In the pointlike case, the only condition that a basic transformation had to comply with was that j had to be minimum such that the value of the word $a_1 \cdots a_k$ under δ is preserved. In the present case, the basic transformations have to preserve the value under δ as well, but the equations $(\alpha\mathtt{e})\mathtt{e} = \omega\mathtt{e}$ impose an extra restriction that is not required by pointlike equations. Indeed, we need η' to verify, in particular, $\delta\eta'\alpha\mathtt{e} = \delta\eta'_k\alpha\mathtt{e} (= \delta\eta\alpha\mathtt{e})$ and $\delta\eta'\mathtt{e} = \delta\eta'_k\mathtt{e} (= \delta\eta\mathtt{e})$. So, somewhat informally, for a word $a_1 \cdots a_k$ that has an occurrence overlapping both the factors $\eta'_k\alpha\mathtt{e}$ and $\eta'_k\mathtt{e}$ of the pseudoword $(\eta'_k\alpha\mathtt{e})(\eta'_k\mathtt{e})$, the introduction of the factor $(a_i \cdots a_j)^\omega$ by the basic transformation should be done either in $\eta'_k\alpha\mathtt{e}$ or in $\eta'_k\mathtt{e}$, and not in both simultaneously. The borders of the solution η were introduced to help us to deal with this extra restriction. Informally speaking, the borders will be used to detect the "passage" from the labeling under η'_k of a vertex $\alpha\mathtt{e}$ to the labeling of the edge $\mathtt{e}$ and to avoid that the introduction of $(a_i \cdots a_j)^\omega$ affect the labelings under δ of $\eta'_k\alpha\mathtt{e}$ or $\eta'_k\mathtt{e}$.

Consider an arbitrary word $w = a_1 \cdots a_n \in A^+$. An integer $m \in \{M, \ldots, n\}$ will be called a *bound of* w if the factor $w_{[m]} = a_{m'} \cdots a_m$ of w is a border, where $m' = m - M + 1$. The bound m will be said to be *periodic* or *non-periodic* according to the border $w_{[m]}$ is periodic or not. If w admits bounds, then there is a maximum one that we name the *last bound of* w. In this case, if ℓ is the last bound of w, then the border $w_{[\ell]}$ will be called the *last border* of w. Notice that, by Proposition 3.2 and the choice of M, if m_1 and m_2 are two bounds of w with $m_1 < m_2$, then either $m_2 - m_1 > Q$ or $w_{[m_1]}$ and $w_{[m_2]}$ are the same periodic border.

Let $w = a_1 \cdots a_k \in A^+$ be a word of length k. Notice that, since $k = M + Q$, if w has a non-periodic last bound ℓ, then ℓ is the unique bound of w. We split the word w in two parts, $\mathtt{l}_w$ (the *left-hand of* w) and $\mathtt{r}_w$ (the *right-hand of* w), by setting

$$\mathtt{l}_w = a_1 \cdots a_s \quad \text{and} \quad \mathtt{r}_w = a_{s+1} \cdots a_k$$

where s (the *splitting point of* w) is defined as follows: if w has a last bound ℓ then $s = \ell$; otherwise $s = k$. In case w has a periodic last bound ℓ, the splitting point s will be said to be *periodic*. Then, s is not periodic in two situations: either w has a non-periodic last border or w has not a last border. The factorization

$$w = \mathtt{l}_w \mathtt{r}_w$$

will be called the *splitting factorization of* w. We have $s \geq M > Q \geq E$. So, by definition of E, there exist integers i and j such that $s - E < i < j \leq s$ and the factor $e = a_i \cdots a_j$ of $\mathtt{l}_w$ verifies $\delta e = (\delta e)^2$. We begin by fixing the maximum such j and, for that j, we fix next an integer i and a word $\mathtt{e}_w = a_i \cdots a_j$, called the *essential factor of* w, as follows. Notice that, if the splitting point s is periodic and u is the root of the last border of w, then $\delta(u^{n_S})$ is idempotent and the left-hand of w is of the form $\mathtt{l}_w = \mathtt{l}'_w u^{n_S}$. Hence, in this case, $j = s$ and we let $\mathtt{e}_w = u^{n_S}$, thus defining i as $j - n_S|u| + 1$. Suppose now that the splitting point is not periodic. In this case we let i be the maximum integer such that $\delta(a_i \cdots a_j)$ is idempotent. The word w can be factorized as $w = \mathtt{l}'_w \mathtt{e}_w \mathtt{l}''_w \mathtt{r}_w$, where $\mathtt{l}'_w = a_1 \cdots a_{i-1}$. We then denote by $\widehat{w}$ the following κ-word

$$\widehat{w} = \mathtt{l}'_w \mathtt{e}_w \mathtt{e}_w^{\omega} \mathtt{l}''_w \mathtt{r}_w = a_1 \cdots a_j (a_i \cdots a_j)^{\omega} a_{j+1} \cdots a_k$$

and notice that $\delta \widehat{w} = \delta w$. Moreover $|\mathtt{e}_w \mathtt{l}''_w| \leq E$ and so $|\mathtt{l}'_w| \geq M - E > Q - E = L$. It is also convenient to introduce two κ-words derived from $\widehat{w}$

$$\lambda_k w = a_1 \cdots a_j (a_i \cdots a_j)^{\omega}, \quad \varrho_k w = (a_i \cdots a_j)^{\omega} a_{j+1} \cdots a_k. \tag{3.2}$$

This defines two mappings $\lambda_k, \varrho_k : A^k \to \Omega_A^{\kappa}\mathbf{S}$ that can be extended to $\overline{\Omega}_A\mathbf{S}$ as done in [15]. Although they are not formally the same mappings used in that paper, because of the different choice of the integers i and j, we keep the same notation since the selection process of those integers is absolutely irrelevant for the purpose of the mappings. That is, with the above adjustment the mappings maintain the properties stated in [15].

The next lemma presents a property of the $\widehat{\ }$-operation that is fundamental to our purposes.

Lemma 3.4 *For a word $w = a_1 \cdots a_{k+1} \in A^+$ of length $k + 1$, let $w_1 = a_1 \cdots a_k$ and $w_2 = a_2 \cdots a_{k+1}$ be the two factors of w of length k. If $\widehat{w}_1 = a_1 \cdots a_{j_1} (a_{i_1} \cdots a_{j_1})^{\omega} a_{j_1+1} \cdots a_k$ and $\widehat{w}_2 = a_2 \cdots a_{j_2} (a_{i_2} \cdots a_{j_2})^{\omega} a_{j_2+1} \cdots a_{k+1}$, then $a_1 \mathtt{l}_{w_2} = \mathtt{l}_{w_1} x$ for some word $x \in A^*$. In particular $j_1 \leq j_2$.*

Proof. Write $w_2 = b_1 \cdots b_k$ with $b_i = a_{i+1}$. Let s_1 and s_2 be the splitting points of w_1 and w_2 respectively, whence $\mathtt{l}_{w_1} = a_1 \cdots a_{s_1}$ and $\mathtt{l}_{w_2} = b_1 \cdots b_{s_2} = a_2 \cdots a_{s_2+1}$. To prove that there exists a word x such that $a_1 \mathtt{l}_{w_2} = \mathtt{l}_{w_1} x$, we have to show that $s_1 \leq s_2 + 1$. Under this hypothesis, we then deduce that $a_{i_1} \cdots a_{j_1}$ is an occurrence of the essential factor $\mathtt{e}_{w_1}$ in $\mathtt{l}_{w_2}$ which proves that $j_1 \leq j_2$.

Assume first that w_1 has a last bound ℓ_1, in which case $s_1 = \ell_1$. By definition, $\ell_1 \geq M$. If $\ell_1 > M$, then the last border of w_1 occurs in w_2, one position to the left relatively to w_1. Hence $\ell_1 - 1$ is a bound of w_2 and, so, w_2 has a last bound ℓ_2 such that $\ell_2 \geq \ell_1 - 1$. It follows in this case that $s_2 = \ell_2$ and $s_1 \leq s_2 + 1$. Suppose now that $\ell_1 = M$. Since $s_2 \geq M$ by definition, the condition $s_1 \leq s_2 + 1$ holds trivially in this case. Suppose now that w_1 has not a last bound. Then $s_1 = k$. Moreover, either w_2 does not have a last bound or k is its last bound. In both circumstances $s_2 = k$, whence $s_1 = s_2 \leq s_2 + 1$. This concludes the proof of the lemma. ∎

In the conditions of the above lemma and as in [15], we define $\psi_k : (\overline{\Omega}_{A^{k+1}}\mathbf{S})^1 \to (\overline{\Omega}_A\mathbf{S})^1$ as the only continuous monoid homomorphism which extends the mapping

$$\begin{aligned} A^{k+1} &\to \Omega_A^{\kappa}\mathbf{S} \\ a_1 \cdots a_{k+1} &\mapsto (a_{i_1} \cdots a_{j_1})^{\omega} a_{j_1+1} \cdots a_{j_2} (a_{i_2} \cdots a_{j_2})^{\omega} \end{aligned}$$

and let $\theta_k = \psi_k \Phi_k$. The function $\theta_k : \overline{\Omega}_A\mathbf{S} \to (\overline{\Omega}_A\mathbf{S})^1$ is a continuous k-superposition homomorphism since it is the composition of the continuous k-superposition homomorphism Φ_k with the continuous homomorphism ψ_k. We remark that a word $w = a_1 \cdots a_n$ of length $n > k$ has precisely $r = n - k + 1$ factors of length k and

$$\begin{aligned} \theta_k(w) &= \psi_k(a_1 \cdots a_{k+1}, a_2 \cdots a_{k+2}, \ldots, a_{r-1} \cdots a_n) \\ &= \psi_k(a_1 \cdots a_{k+1}) \psi_k(a_2 \cdots a_{k+2}) \cdots \psi_k(a_{r-1} \cdots a_n) \\ &= (e_1^{\omega} f_1 e_2^{\omega})(e_2^{\omega} f_2 e_3^{\omega}) \cdots (e_{r-1}^{\omega} f_{r-1} e_r^{\omega}) \\ &= e_1^{\omega} f_1 e_2^{\omega} f_2 \cdots e_{r-1}^{\omega} f_{r-1} e_r^{\omega} \end{aligned}$$

where, for each $p \in \{1, \ldots, r\}$, e_p is the essential factor $\mathtt{e}_{w_p} = a_{i_p} \cdots a_{j_p}$ of the word $w_p = a_p \cdots a_{k+p-1}$ and $f_p = a_{j_p+1} \cdots a_{j_{p+1}}$ $(p \neq r)$. Above, for each $p \in \{2, \ldots, r-1\}$, we have replaced each expression $e_p^{\omega} e_p^{\omega}$ with e_p^{ω} since, indeed, these expressions represent the same κ-word. More generally, one can certainly replace an expression of the form $x^{\omega} x^n x^{\omega}$ with $x^{\omega} x^n$. Using this reduction rule as long as possible, $\theta_k(w)$ can be written as

$$\theta_k(w) = e_{n_1}^{\omega} \bar{f}_1 e_{n_2}^{\omega} \bar{f}_2 \cdots e_{n_q}^{\omega} \bar{f}_q,$$

called the *reduced form of* $\theta_k(w)$, where $q \in \{1, \ldots, r\}$, $1 = n_1 < n_2 < \cdots < n_q \leq r$, $\bar{f}_p = f_{n_p} \cdots f_{n_{p+1}-1}$ (for $p \in \{1, \ldots, q-1\}$) and $\bar{f}_q$ is $f_{n_q} \cdots f_{r-1}$ if $n_q \neq r$ and it is the empty word otherwise.

3.6 Definition of the $(\mathbf{V} * \mathbf{D}, \kappa)$-solution η'

We are now in conditions to describe the procedure to transform the $(\mathbf{V} * \mathbf{D}_k, \kappa)$-solution η'_k into the $(\mathbf{V} * \mathbf{D}, \kappa)$-solution η'. The mapping $\eta' : \Gamma \to (\Omega^\kappa_A \mathbf{S})^1$ is defined, for each $\mathtt{g} \in \Gamma$, as

$$\eta'\mathtt{g} = (\tau_1\mathtt{g})(\tau_2\mathtt{g})(\tau_3\mathtt{g}),$$

where, for each $i \in \{1, 2, 3\}$, $\tau_i : \Gamma \to (\Omega^\kappa_A \mathbf{S})^1$ is a function defined as follows.

First of all, we let

$$\tau_2 = \theta_k \eta'_k.$$

That τ_2 is well-defined, that is, that $\tau_2\mathtt{g}$ is indeed a κ-word for every $\mathtt{g} \in \Gamma$, follows from the fact that $\eta'_k\mathtt{g}$ is a κ-word and θ_k transforms κ-words into κ-words (see [15]). Next, for each vertex $\mathtt{v}$, consider the length k words $\mathtt{i}_\mathtt{v} = \mathtt{i}_k\eta'_k\mathtt{v} = \mathtt{i}_k\eta\mathtt{v}$ and $\mathtt{t}_\mathtt{v} = \mathtt{t}_k\eta'_k\mathtt{v} = \mathtt{t}_k\eta\mathtt{v}$. We let

$$\tau_1\mathtt{v} = \lambda_k\mathtt{i}_\mathtt{v} \quad \textit{and} \quad \tau_3\mathtt{v} = \varrho_k\mathtt{t}_\mathtt{v},$$

where the mappings λ_k and ϱ_k were defined in (3.2). Note that, by (3.1), $\mathtt{t}_\mathtt{v} = x_\mathtt{v}y_\mathtt{v}z_\mathtt{v}$. Moreover, the occurrence of $y_\mathtt{v}$ shown in this factorization is the last occurrence of a border in $\mathtt{t}_\mathtt{v}$. Hence, the right-hand $\mathtt{r}_{\mathtt{t}_\mathtt{v}}$ of $\mathtt{t}_\mathtt{v}$ is precisely $z_\mathtt{v}$. Therefore, one has

$$\tau_1\mathtt{v} = \lambda_k\mathtt{i}_\mathtt{v} = \mathtt{l}'_{\mathtt{i}_\mathtt{v}}\mathtt{e}_{\mathtt{i}_\mathtt{v}}\mathtt{e}^\omega_{\mathtt{i}_\mathtt{v}} \quad \text{and} \quad \tau_3\mathtt{v} = \varrho_k\mathtt{t}_\mathtt{v} = \mathtt{e}^\omega_{\mathtt{t}_\mathtt{v}}\mathtt{l}''_{\mathtt{t}_\mathtt{v}}z_\mathtt{v}.$$

Consider now an arbitrary edge $\mathtt{e}$. Suppose that $\eta\mathtt{e}$ is a finite word. Then, $\eta\mathtt{e}$ is a letter $a_\mathtt{e}$ and $\eta'_k\mathtt{e}$ is also $a_\mathtt{e}$ in this case. Then $\tau_2\mathtt{e} = \theta_k a_\mathtt{e} = 1$ because θ_k is a k-superposition homomorphism. Since we want $\eta'\mathtt{e}$ to be $a_\mathtt{e}$, we then define, for instance,

$$\tau_1\mathtt{e} = a_\mathtt{e} \quad \text{and} \quad \tau_3\mathtt{e} = 1.$$

Suppose at last that $\eta\mathtt{e}$ (and so also $\eta'_k\mathtt{e}$) is an infinite pseudoword. We let

$$\tau_3\mathtt{e} = \varrho_k\mathtt{t}_\mathtt{e}$$

and notice that $\tau_3\mathtt{e} = \tau_3\omega\mathtt{e}$. Indeed, as η'_k is a $\mathbf{V} * \mathbf{D}_k$-solution of Σ_Γ, it follows from (2.1) that $\mathtt{t}_\mathtt{e} = \mathtt{t}_k\eta'_k\mathtt{e} = \mathtt{t}_k\eta'_k\omega\mathtt{e} = \mathtt{t}_{\omega\mathtt{e}}$. The definition of $\tau_1\mathtt{e}$ is more elaborate. Let $\mathtt{v}$ be the vertex $\alpha\mathtt{e}$ and consider the word $\mathtt{t}_\mathtt{v}\mathtt{i}_\mathtt{e} = a_1 \cdots a_{2k}$. This word has $r = k + 1$ factors of length k. Suppose that $\theta_k(\mathtt{t}_\mathtt{v}\mathtt{i}_\mathtt{e})$ is $e_1^\omega f_1 e_2^\omega f_2 \cdots e_{r-1}^\omega f_{r-1} e_r^\omega$ and consider its reduced form

$$\theta_k(\mathtt{t}_\mathtt{v}\mathtt{i}_\mathtt{e}) = e_1^\omega \bar{f}_1 e_{n_2}^\omega \bar{f}_2 \cdots e_{n_q}^\omega \bar{f}_q.$$

Notice that $\mathtt{t}_\mathtt{v}\mathtt{i}_\mathtt{e} = \bar{f}_0\bar{f}_1 \cdots \bar{f}_q\bar{f}_{q+1}$ for some words $\bar{f}_0, \bar{f}_{q+1} \in A^*$. Hence, there is a (unique) index $m \in \{1, \ldots, q\}$ such that $\mathtt{t}_\mathtt{v} = \bar{f}_0\bar{f}_1 \cdots \bar{f}_{m-1}\bar{f}'_m$ and $\bar{f}_m = \bar{f}'_m\bar{f}''_m$ with $\bar{f}'_m \in A^*$ and $\bar{f}''_m \in A^+$. Then $\theta_k(\mathtt{t}_\mathtt{v}\mathtt{i}_\mathtt{e}) = \beta_1\beta_2$, where $\beta_1 = e_1^\omega \bar{f}_1 e_{n_2}^\omega \bar{f}_2 \cdots e_{n_m}^\omega \bar{f}'_m$ and $\beta_2 = \bar{f}''_m e_{n_{m+1}}^\omega \bar{f}_{m+1} \cdots e_{n_q}^\omega \bar{f}_q$ and we let

$$\tau_1\mathtt{e} = \beta_2 = \bar{f}''_m e_{n_{m+1}}^\omega \bar{f}_{m+1} \cdots e_{n_q}^\omega \bar{f}_q.$$

Note that the word $\beta'_2 = \bar{f}''_m \bar{f}_{m+1} \cdots \bar{f}_q$ is $a_{k+1} \cdots a_{j_r}$, whence $\beta'_2 e_r^\omega = \lambda_k\mathtt{i}_\mathtt{e}$.

The next lemma is a key result that justifies the definition of the $\widehat{\ }$-operation.

Lemma 3.5 *Let* e *be an edge such that* $\eta\mathsf{e}$ *is infinite. Then, with the above notation,* $\beta_1 = \tau_3\mathsf{v}$ *and so* $\theta_k(\mathtt{t_v i_e}) = (\tau_3\mathsf{v})(\tau_1\mathsf{e})$*. Moreover,* $\delta\tau_1\mathsf{e} = \delta\lambda_k\mathtt{i_e}$*.*

Proof. We begin by recalling that $\mathtt{t_v i_e} = a_1 \cdots a_{2k}$ and

$$\theta_k(\mathtt{t_v i_e}) = e_1^\omega f_1 e_2^\omega f_2 \cdots e_{r-1}^\omega f_{r-1} e_r^\omega = e_1^\omega \bar{f}_1 e_{n_2}^\omega \bar{f}_2 \cdots e_{n_q}^\omega \bar{f}_q,$$

where e_p is the essential factor $\mathsf{e}_{w_p} = a_{i_p} \cdots a_{j_p}$ of the word $w_p = a_p \cdots a_{k+p-1}$ and $f_p = a_{j_p+1} \cdots a_{j_{p+1}}$ for each p. Note also that $\lambda_k\mathtt{i_e} = \beta_2' e_r^\omega$, e_r is a suffix of β_2' and δe_r is idempotent. So, to prove the equality $\delta\tau_1\mathsf{e} = \delta\lambda_k\mathtt{i_e}$ it suffices to show that $\delta\tau_1\mathsf{e} = \delta\beta_2'$. We know from (3.1) that $\mathtt{t_v} = x_\mathsf{v} y_\mathsf{v} z_\mathsf{v}$ with $1 \leq |x_\mathsf{v}| \leq Q$. So, $x_\mathsf{v} = a_1 \cdots a_{h-1}$, $y_\mathsf{v} = a_h \cdots a_{M+h-1}$ and $z_\mathsf{v} = a_{M+h} \cdots a_k$ for some $h \in \{2, \ldots, Q+1\}$. There are two cases to verify.

Case 1. y_v is a non-periodic border. Consider the factor $w_h = a_h \cdots a_{k+h-1}$ of $\mathtt{t_v i_e}$. By the choice of M and k, the prefix y_v is the only occurrence of a border in w_h. Hence, M is the last bound of w_h and, so, its splitting point. It follows that $w_h = y_\mathsf{v} \cdot z_\mathsf{v} a_{k+1} \cdots a_{k+h-1}$ is the splitting factorization of w_h. Therefore, as one can verify for an arbitrary $p \in \{1, \ldots, h\}$, there is only one occurrence of a border in w_p, precisely y_v, and the splitting factorization of w_p is

$$w_p = a_p \cdots a_{h-1} y_\mathsf{v} \cdot z_\mathsf{v} a_{k+1} \cdots a_{k+p-1},$$

whence $e_p = e_1$ with $j_p = j_1 \leq M + h - 1$ and, so, $f_p = 1$ for $p < h$. So, the prefix $e_1^\omega f_1 e_2^\omega \cdots f_{h-1} e_h^\omega$ of $\theta_k(\mathtt{t_v i_e})$ reduces to e_1^ω. Consider now the factor $w_{h+1} = a_{h+1} \cdots a_{k+h}$. Hence, either w_{h+1} does not have a last bound or k is its last bound. In both situations, the splitting point of w_{h+1} is k and its splitting factorization is $w_{h+1} = w_{h+1} \cdot 1$. Therefore, one deduces from Lemma 3.4 that, for every $p \in \{h+1, \ldots, r\}$, the occurrence $a_{i_p} \cdots a_{j_p}$ of the essential factor e_{w_p} in w_p is, in fact, an occurrence in the suffix $w' = a_{k+h-E} \cdots a_{2k} = a_{M+L+h} \cdots a_{2k}$ of $\mathtt{t_v i_e}$. Since $|x_\mathsf{v} y_\mathsf{v}| = M+h-1$ and $|z_\mathsf{v}| \leq L$, it follows that $k = |x_\mathsf{v} y_\mathsf{v} z_\mathsf{v}| < M + L + h$, whence w' is a suffix of $\mathtt{i_e}$ and so $k < i_p < j_p$ for all $p \in \{h+1, \ldots, r\}$. This means, in particular, that the ω-power e_{h+1}^ω is introduced at the suffix $\mathtt{i_e}$ of $\mathtt{t_v i_e}$. Hence $\beta_1 = e_1^\omega f_1 e_2^\omega \cdots f_{h-1} e_h^\omega a_{j_h+1} \cdots a_k$ and its reduced form is $e_1^\omega a_{j_1+1} \cdots a_k = \tau_3\mathsf{v}$, which proves that β_1 and $\tau_3\mathsf{v}$ are the same κ-word. Moreover, from $k < i_p$, one deduces that the word e_p is a suffix of $a_{k+1} \cdots a_{j_p}$, which proves that $\delta\tau_1\mathsf{e} = \delta\beta_2'$.

Case 2. y_v is a periodic border. Let u be the root of y_v. Then, since M was fixed as a multiple of $|u|$, $y_\mathsf{v} = u^{M_u}$ where $M_u = \frac{M}{|u|}$. If the prefix y_v is the only occurrence of a border in w_h, then one deduces the lemma as in Case 1 above. So, we assume that there is another occurrence of a border y in w_h. Hence, by Proposition 3.2 and the choice of M and k, y is precisely y_v. Furthermore, since u is a Lyndon word and $k = M + Q$ with $Q < M$, $w_h = y_\mathsf{v} u^d w_h'$ for some positive integer d and some word $w_h' \in A^*$ such that u is not a prefix of w_h'. Notice that, since u is not a prefix of z_v by definition of this word, z_v is a proper prefix of u. On the other hand $w_h = u^d y_\mathsf{v} w_h'$ and the occurrence of y_v shown in this factorization is the last occurrence of y_{v} in w_h. Thus,

$$w_h = u^d y_{\mathsf{v}} \cdot w'_h$$

is the splitting factorization of w_h. Therefore $\widehat{w_h} = u^d y_{\mathsf{v}}(u^{ns})^\omega w'_h$ and $e_h = u^{ns}$. More generally, for any $p \in \{1, \ldots, h\}$, y_{v} is a factor of w_p and it is the only border that occurs in w_p. Hence, the splitting point of w_p is periodic and $e_p = u^{ns}$. Moreover, as one can verify, $j_1 = M + h - 1$ and the prefix $e_1^\omega f_1 e_2^\omega \cdots f_{h-1} e_h^\omega$ of $\theta_k(\mathtt{t_v i_e})$ is $e_1^\omega(u(e_1^\omega)^{|u|})^d$ and so, analogously to Case 1, it reduces to $e_1^\omega u^d$. Since z_{v} is a proper prefix of u and $d \geq 1$, $k < j_h$. This allows already deduce that the reduced form of β_1 is $(u^{ns})^\omega z_{\mathsf{v}} = \tau_3\mathsf{v}$, thus concluding the proof of the first part of the lemma. Now, there are two possible events. Either $m = q$ and $\beta_2 = \bar{f}''_m = \beta'_2$, in which case $\delta\tau_1\mathsf{e} = \delta\beta'_2$ is trivially verified. Or $m \neq q$ and the ω-power $e^\omega_{n_{m+1}}$ was not eliminated in the reduction process of $\theta_k(\mathtt{t_v i_e})$. This means that the splitting point of the word $w_{n_{m+1}}$ is not determined by one of the occurrences of the border y_{v} in the prefix $a_1 \cdots a_{k+h-1}$ of $\mathtt{t_v i_e}$. Then, as in Case 1 above, one deduces that $k < i_p$ for each $p \in \{n_{m+1}, \ldots, r\}$ and, so, that $\delta\tau_1\mathsf{e} = \delta\beta'_2$.

In both cases $\beta_1 = \tau_3\mathsf{v}$ and $\delta\tau_1\mathsf{e} = \delta\lambda_k\mathtt{i_e}$. Hence, the proof of the lemma is complete. ∎

Notice that, as shown in the proof of Lemma 3.5 above, if a vertex v is such that y_{v} is a periodic border with root u, then $\tau_3\mathsf{v} = (u^{ns})^\omega z_{\mathsf{v}}$. So, the definition of the mapping τ_3 on vertices assures condition $\mathscr{C}_2(\Gamma, \eta, \eta')$.

3.7 Proof that η' is a $(\mathbf{V} * \mathbf{D}, \kappa)$-solution

This section will be dedicated to showing that η' is a $(\mathbf{V} * \mathbf{D}, \kappa)$-solution of Σ_Γ with respect to the pair (φ, δ) verifying conditions $\mathscr{C}_1(\Gamma, \eta, \eta')$ and $\mathscr{C}_3(\Gamma, \eta, \eta')$.

We begin by noticing that $\eta'\mathsf{g}$ is a κ-word for every $\mathsf{g} \in \Gamma$. Indeed, as observed above, each $\tau_2\mathsf{g}$ is a κ-word. That both $\tau_1\mathsf{g}$ and $\tau_3\mathsf{g}$ are κ-words too, is easily seen by their definitions. Let us now show the following properties.

Proposition 3.6 *Conditions $\delta\eta' = \varphi$, $\mathscr{C}_1(\Gamma, \eta, \eta')$ and $\mathscr{C}_3(\Gamma, \eta, \eta')$ hold.*

Proof. As η'_k is a $\mathbf{V} * \mathbf{D}_k$-solution of Σ_Γ with respect to (φ, δ) and, so, the equality $\delta\eta'_k = \varphi$ holds, to deduce that $\delta\eta' = \varphi$ holds it suffices to establish the equality $\delta\eta' = \delta\eta'_k$. Consider first a vertex $\mathsf{v} \in \Gamma$. Then $\tau_1\mathsf{v} = \lambda_k\mathtt{i_v} = \mathtt{l}'_{\mathtt{i_v}}\mathtt{e}_{\mathtt{i_v}}\mathtt{e}^\omega_{\mathtt{i_v}}$ and $\tau_3\mathsf{v} = \varrho_k\mathtt{t_v} = \mathtt{e}^\omega_{\mathtt{t_v}}\mathtt{l}''_{\mathtt{t_v}} z_{\mathsf{v}}$. In this case, the equality $\delta\eta'_k\mathsf{v} = \delta\eta'\mathsf{v}$ is a direct application of [15, Proposition 5.3], where the authors proved that

$$\delta\pi = \delta\big((\lambda_k\mathtt{i}_k\pi)(\theta_k\pi)(\varrho_k\mathtt{t}_k\pi)\big) \tag{3.3}$$

for every pseudoword π. Moreover, by definition of the $\widehat{\ }$-operation, $|\mathtt{l}'_{\mathtt{i_v}}| > L$. Therefore, $\eta\mathsf{v}$ and $\eta'\mathsf{v}$ are of the form $\eta\mathsf{v} = u\pi$ and $\eta'\mathsf{v} = u\pi'$ with $u \in A^L$ and $\delta\pi = \delta\pi'$. So, condition $\mathscr{C}_3(\Gamma, \eta, \eta')$ holds.

Consider next an edge $\mathsf{e} \in \Gamma$. If $\eta'_k\mathsf{e}$ is a finite word a_e, then $\eta'\mathsf{e} = (\tau_1\mathsf{e})(\tau_2\mathsf{e})(\tau_3\mathsf{e}) = a_\mathsf{e}\cdot 1\cdot 1 = a_\mathsf{e} = \eta'_k\mathsf{e}$, whence $\delta\eta'\mathsf{e} = \delta\eta'_k\mathsf{e}$ holds trivially. Moreover, since $\eta'_k\mathsf{e} = \eta\mathsf{e}$ in this case and every vertex is labeled under η by an infinite pseudoword, it follows that condition $\mathscr{C}_1(\Gamma, \eta, \eta')$ holds. Suppose at last that $\eta'_k\mathsf{e}$ is infinite and let $\mathsf{v} = \alpha\mathsf{e}$. Then $\tau_3\mathsf{e} = \varrho_k\mathtt{t}_\mathsf{e}$. On the other hand, by Lemma 3.5, $\delta\tau_1\mathsf{e} = \delta\lambda_k\mathtt{i}_\mathsf{e}$. Hence, by (3.3) and since δ is a homomorphism, $\delta\eta'\mathsf{e} = \delta\big((\tau_1\mathsf{e})(\tau_2\mathsf{e})(\tau_3\mathsf{e})\big) = \delta\big((\lambda_k\mathtt{i}_\mathsf{e})(\theta_k\eta'_k\mathsf{e})(\varrho_k\mathtt{t}_\mathsf{e})\big) = \delta\eta'_k\mathsf{e}$. This ends the proof of the proposition. ■

Consider an arbitrary edge $\mathsf{v} \xrightarrow{\mathsf{e}} \mathsf{w}$ of Γ. To achieve the objectives of this section it remains to prove that $\mathbf{V} * \mathbf{D}$ satisfies $(\eta'\mathsf{v})(\eta'\mathsf{e}) = \eta'\mathsf{w}$. Since η'_k is a $\mathbf{V} * \mathbf{D}_k$-solution of Σ_Γ, $\mathbf{V} * \mathbf{D}_k$ satisfies $(\eta'_k\mathsf{v})(\eta'_k\mathsf{e}) = \eta'_k\mathsf{w}$. Hence, by (2.1), $\mathtt{i}_\mathsf{v} = \mathtt{i}_k\big((\eta'_k\mathsf{v})(\eta'_k\mathsf{e})\big) = \mathtt{i}_k(\eta'_k\mathsf{w}) = \mathtt{i}_\mathsf{w}$ and $\mathtt{t}_k\big((\eta'_k\mathsf{v})(\eta'_k\mathsf{e})\big) = \mathtt{t}_k(\eta'_k\mathsf{w}) = \mathtt{t}_\mathsf{w}$. Thus, $\tau_1\mathsf{v} = \lambda_k\mathtt{i}_\mathsf{v} = \mathtt{l}'_{\mathtt{i}_\mathsf{v}}\mathtt{e}_{\mathtt{i}_\mathsf{v}}\mathtt{e}^\omega_{\mathtt{i}_\mathsf{v}} = \mathtt{l}'_{\mathtt{i}_\mathsf{w}}\mathtt{e}_{\mathtt{i}_\mathsf{w}}\mathtt{e}^\omega_{\mathtt{i}_\mathsf{w}} = \lambda_k\mathtt{i}_\mathsf{w} = \tau_1\mathsf{w}$ and $\tau_3\mathsf{w} = \varrho_k\mathtt{t}_\mathsf{w} = \mathtt{e}^\omega_{\mathtt{t}_\mathsf{w}}\mathtt{l}''_{\mathtt{t}_\mathsf{w}}z_\mathsf{w}$. As shown in the proof of [15, Proposition 5.4], it then follows that $\mathbf{V} * \mathbf{D}$ satisfies $\mathtt{e}^\omega_{\mathtt{i}_\mathsf{w}}\theta_k\big((\eta'_k\mathsf{v})(\eta'_k\mathsf{e})\big)\mathtt{e}^\omega_{\mathtt{t}_\mathsf{w}} = \mathtt{e}^\omega_{\mathtt{i}_\mathsf{w}}\theta_k(\eta'_k\mathsf{w})\mathtt{e}^\omega_{\mathtt{t}_\mathsf{w}}$ and, so,

$$\mathbf{V} * \mathbf{D} \models (\tau_1\mathsf{v})\theta_k\big((\eta'_k\mathsf{v})(\eta'_k\mathsf{e})\big)(\tau_3\mathsf{w}) = (\tau_1\mathsf{w})\theta_k(\eta'_k\mathsf{w})(\tau_3\mathsf{w}) = \eta'\mathsf{w}. \tag{3.4}$$

On the other hand, from the fact that θ_k is a k-superposition homomorphism one deduces

$$\theta_k\big((\eta'_k\mathsf{v})(\eta'_k\mathsf{e})\big) = \theta_k(\eta'_k\mathsf{v})\theta_k\big(\mathtt{t}_\mathsf{v}(\eta'_k\mathsf{e})\big) = \theta_k(\eta'_k\mathsf{v})\theta_k(\mathtt{t}_\mathsf{v}\mathtt{i}_\mathsf{e})\theta_k(\eta'_k\mathsf{e}). \tag{3.5}$$

Suppose that $\eta'_k\mathsf{e}$ is an infinite pseudoword. In this case $\mathtt{t}_\mathsf{e} = \mathtt{t}_\mathsf{w}$, whence $\tau_3\mathsf{e} = \tau_3\mathsf{w}$. Moreover, by Lemma 3.5, $\theta_k(\mathtt{t}_\mathsf{v}\mathtt{i}_\mathsf{e}) = (\tau_3\mathsf{v})(\tau_1\mathsf{e})$. Therefore, by conditions (3.4) and (3.5), $\mathbf{V} * \mathbf{D}$ satisfies $(\eta'\mathsf{v})(\eta'\mathsf{e}) = \eta'\mathsf{w}$. Assume now that $\eta'_k\mathsf{e}$ is a finite word, whence $\eta'_k\mathsf{e} = a_\mathsf{e} \in A$ and $\eta'\mathsf{e} = a_\mathsf{e}$. Since η is a $\mathbf{D}$-solution of Σ_Γ, $\mathbf{D} \models (\eta\mathsf{v})a_\mathsf{e} = \eta\mathsf{w}$ and, thus, $\mathbf{d}_\mathsf{v}a_\mathsf{e} = \mathbf{d}_\mathsf{w}$. Hence the left-infinite words $\mathbf{d}_\mathsf{v}$ and $\mathbf{d}_\mathsf{w}$ are confinal and, so, $\propto$-equivalent. Hence $\mathbf{d}_\mathsf{v} = y_\Delta z_\mathsf{v}$, $\mathbf{d}_\mathsf{w} = y_\Delta z_\mathsf{w}$ and $y_\mathsf{v} = y_\mathsf{w} = \mathtt{t}_k y_\Delta$, where Δ is the $\propto$-class of $\mathbf{d}_\mathsf{v}$ and $\mathbf{d}_\mathsf{w}$. It follows that $y_\Delta z_\mathsf{v}a_\mathsf{e} = y_\Delta z_\mathsf{w}$ and $\mathtt{t}_k\big(\mathtt{t}_\mathsf{v}a_\mathsf{e}\big) = \mathtt{t}_\mathsf{w}$. In this case, $\theta_k\big((\eta'_k\mathsf{v})(\eta'_k\mathsf{e})\big) = \theta_k(\eta'_k\mathsf{v})\theta_k(\mathtt{t}_\mathsf{v}a_\mathsf{e})$. On the other hand, $\mathtt{t}_\mathsf{v}a_\mathsf{e} = a_1\cdots a_k a_{k+1} = a_1\mathtt{t}_\mathsf{w}$ is a word of length $k+1$ and, so, $\theta_k(\mathtt{t}_\mathsf{v}a_\mathsf{e}) = \psi_k(\mathtt{t}_\mathsf{v}a_\mathsf{e})$ is of the form

$$\theta_k(\mathtt{t}_\mathsf{v}a_\mathsf{e}) = e_1^\omega f e_2^\omega.$$

The splitting factorizations of $\mathtt{t}_\mathsf{v}$ and $\mathtt{t}_\mathsf{w}$ are, respectively, $\mathtt{t}_\mathsf{v} = x_\mathsf{v}y_\mathsf{v}\cdot z_\mathsf{v}$ and $\mathtt{t}_\mathsf{w} = x_\mathsf{w}y_\mathsf{w}\cdot z_\mathsf{w}$. Since $y_\mathsf{v} = y_\mathsf{w}$, it follows that $e_1 = \mathtt{e}_{\mathtt{t}_\mathsf{v}} = \mathtt{e}_{\mathtt{t}_\mathsf{w}} = e_2$.

Suppose that $z_\mathsf{v}a_\mathsf{e} = z_\mathsf{w}$. In this case it is clear that $f = 1$, so that $\theta_k(\mathtt{t}_\mathsf{v}a_\mathsf{e}) = \mathtt{e}^\omega_{\mathtt{t}_\mathsf{v}}$. Since $\theta_k(\eta'_k\mathsf{v})$ ends with $\mathtt{e}^\omega_{\mathtt{t}_\mathsf{v}}$, it then follows that $\theta_k\big((\eta'_k\mathsf{v})(\eta'_k\mathsf{e})\big) = \theta_k\eta'_k\mathsf{v} = \tau_2\mathsf{v}$. Therefore, $(\tau_1\mathsf{v})\theta_k\big((\eta'_k\mathsf{v})(\eta'_k\mathsf{e})\big)(\tau_3\mathsf{w}) = (\tau_1\mathsf{v})(\tau_2\mathsf{v})(\tau_3\mathsf{w})$. On the other hand,

$$\tau_3\mathsf{w} = \varrho_k\mathtt{t}_\mathsf{w} = \mathtt{e}^\omega_{\mathtt{t}_\mathsf{w}}\mathtt{l}''_{\mathtt{t}_\mathsf{w}}z_\mathsf{w} = \mathtt{e}^\omega_{\mathtt{t}_\mathsf{v}}\mathtt{l}''_{\mathtt{t}_\mathsf{v}}z_\mathsf{v}a_\mathsf{e} = (\tau_3\mathsf{v})a_\mathsf{e}.$$

So, by (3.4), one has that $\mathbf{V} * \mathbf{D}$ satisfies $(\eta'\mathsf{v})a_\mathsf{e} = (\tau_1\mathsf{v})(\tau_2\mathsf{v})(\tau_3\mathsf{v})a_\mathsf{e} = (\tau_1\mathsf{v})(\tau_2\mathsf{v})(\tau_3\mathsf{w}) = \eta'\mathsf{w}$.

Suppose now that $z_\mathsf{v}a_\mathsf{e} \neq z_\mathsf{w}$. In this case, one deduces from the equality $y_\Delta z_\mathsf{v}a_\mathsf{e} = y_\Delta z_\mathsf{w}$, that y_Δ is a periodic left-infinite word. Let u be its root, so that $y_\Delta = u^{-\infty}$, $\mathtt{e}_{\mathtt{t}_\mathsf{v}} = u^{n_S}$ and $\mathtt{l}''_{\mathtt{t}_\mathsf{v}} = \mathtt{l}''_{\mathtt{t}_\mathsf{w}} = 1$. Since, by definition, u is a primitive word which is not a prefix of z_v nor a prefix of

z_{w}, we conclude that $z_{\mathsf{v}}a_{\mathsf{e}} = u$ and $z_{\mathsf{w}} = 1$. In this case $f = u$, whence $\theta_k(\mathtt{t}_{\mathsf{v}}a_{\mathsf{e}}) = \mathtt{e}^{\omega}_{\mathtt{t}_{\mathsf{v}}}u$. Then, $\theta_k\big((\eta'_k\mathsf{v})(\eta'_k\mathsf{e})\big) = (\theta_k\eta'_k\mathsf{v})u = (\tau_2\mathsf{v})u$. Therefore, $(\tau_1\mathsf{v})\theta_k\big((\eta'_k\mathsf{v})(\eta'_k\mathsf{e})\big)(\tau_3\mathsf{w}) = (\tau_1\mathsf{v})(\tau_2\mathsf{v})u(\tau_3\mathsf{w})$. Moreover,

$$u(\tau_3\mathsf{w}) = u\mathtt{e}^{\omega}_{\mathtt{t}_{\mathsf{w}}}\mathtt{l}''_{\mathtt{t}_{\mathsf{w}}}z_{\mathsf{w}} = u(u^{n_S})^{\omega} = (u^{n_S})^{\omega}u = \mathtt{e}^{\omega}_{\mathtt{t}_{\mathsf{v}}}\mathtt{l}''_{\mathtt{t}_{\mathsf{v}}}z_{\mathsf{v}}a_{\mathsf{e}} = (\tau_3\mathsf{v})a_{\mathsf{e}}.$$

Therefore, using (3.4), one deduces as above that $\mathbf{V} * \mathbf{D}$ satisfies $(\eta'\mathsf{v})a_{\mathsf{e}} = \eta'\mathsf{w}$.

We have proved the main theorem of the paper.

Theorem 3.7 *If* $\mathbf{V}$ *is* κ*-reducible, then* $\mathbf{V} * \mathbf{D}$ *is* κ*-reducible.*

This result applies, for instance, to the pseudovarieties **Sl**, **G**, **J** and **R**. Since the κ-word problem for the pseudovariety **LG** of local groups is already solved [14], we obtain the following corollary.

Corollary 3.8 *The pseudovariety* **LG** *is* κ*-tame.*

Final remarks. In this paper we fixed our attention on the canonical signature κ, while in [15] we dealt with a more generic class of signatures σ verifying certain undemanding conditions. Theorem 3.7 is still valid for such generic signatures σ but we preferred to treat only the instance of the signature κ to keep the proofs clearer and a little less technical.

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