

Initial sequences and Waldschmidt constants of planar point configurations

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Abstract

The purpose of this work is to extend the classification of planar point configurations with low Waldschmidt constants initiated in [9] and continued in [17] for all values less than $5/2$. As a consequence we prove a conjecture of Dumnicki, Szemberg and Tutał-Gasińska concerning initial sequences with low first differences.

Keywords symbolic power, point configurations, Waldschmidt constants, Chudnovsky conjecture

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1 Introduction

In recent years it has become evident that adopting an asymptotic perspective in algebraic geometry and in commutative algebra often leads to much more regular and clear patterns than obtained when considering isolated phenomena. Examples for the fruitfulness of this approach range from questions about base loci of linear series [11], through the growth of higher cohomology groups [5] to asymptotic syzygies and Betti numbers [10]. In the present note we are primarily interested in Waldschmidt constants. These invariants were introduced by Waldschmidt in [19] and the term “Waldschmidt constant” was coined some 30 years latter by Dumnicki and Harbourne. The interest in Waldschmidt constants stems partly from their applications in describing the geometry of the effective cone of algebraic varieties. They are also quite interesting invariants studied in their own right, see e.g. [1], [2], [7], [15].

To begin with, recall that for a homogeneous ideal I in the ring of polynomials $\mathbb{C}[x_0, \dots, x_n]$ its *initial degree* $\alpha(I)$ is defined as the least integer t such that the homogeneous part of degree t is non-zero. If I is a radical ideal in $\mathbb{C}[x_0, \dots, x_n]$ and m is a positive integer, then the m -th symbolic power $I^{(m)}$ of I equals the m -th differential power of I due to the celebrated Nagata-Zariski Theorem (see [18, Corollary 2.9]). Thus $I^{(m)}$ is the homogeneous ideal consisting of all polynomials vanishing to order at least m along the subvariety $\text{Zeroes}(I) \subset \mathbb{P}^n$. Combining these two notions we introduce the following object.

Definition 1.1 (Initial sequence). Let I be a radical ideal in $\mathbb{C}[x_0, \dots, x_n]$. The *initial sequence* of I is the sequence

$$\alpha(I), \alpha(I^{(2)}), \alpha(I^{(3)}), \dots$$

It is easy to see that the initial sequence is strictly growing and subadditive, i.e. the inequality

$$\alpha(I^{(m+n)}) \leq \alpha(I^{(m)}) + \alpha(I^{(n)})$$

holds for all $m, n \geq 1$. Fekete's Lemma (see [13]) implies then the existence of the following limit.

Definition 1.2 (Waldschmidt constant). The *Waldschmidt constant* of a homogeneous ideal $I \subset \mathbb{C}[x_0, \dots, x_n]$ is the real number

$$\hat{\alpha}(I) = \lim_{m \rightarrow \infty} \frac{\alpha(I^{(m)})}{m}.$$

It follows also from [13] that $\hat{\alpha}(I)$ can be alternatively computed as the infimum

$$\hat{\alpha}(I) = \inf \frac{\alpha(I^{(m)})}{m}.$$

Using the classification types of [14] for $n \leq 8$ points in \mathbb{P}^2 , Harbourne determined the Waldschmidt constant $\hat{\alpha}(I_Z)$ for a reduced scheme Z of each type. Harbourne's unpublished list is available online at [16]. In the present work we extend this classification a little bit. Our first main result shows that the geometry prohibits too small Waldschmidt constants of more than 9 points.

Theorem A. *Let Z be a set of $s \geq 9$ points in \mathbb{P}^2 and let I be its radical ideal. If*

$$\hat{\alpha}(I) < \frac{5}{2},$$

then Z

- *is contained in a conic and consequently $\hat{\alpha}(I) \leq 2$, or;*
- *is a set of nine points arranged as indicated in Figure 8 and $\hat{\alpha}(I) = \frac{17}{7}$.*

This result in a more detailed form is proved below as Theorem 2.3. We apply Theorem A to complete the works of Dumnicki, Szemberg and Tutaj-Gasińska [8] and [9] concerning properties of initial sequences of points in \mathbb{P}^2 . Our second main result is the proof of Conjecture 4.6 from [8].

Theorem B. *Let Z be a finite set of points in \mathbb{P}^2 and let I be its radical ideal. Assume that for some integer m we have*

$$\alpha(I^{(m+k)}) = \alpha(I^{(m)}) + 2k$$

for $k = 1, 2, 3, 4$. Then Z is contained in a conic.

Remark 1.3. It is natural to study the question if a certain number of differences in the initial sequence of Z equal to some integer d implies that Z is contained in a curve of degree d for arbitrary d . It has been proved to be the case for $d = 1$ in [8, Corollary 3.5]. On the other hand [8, Example 4.14] and [9, Proposition 4.2] show that for $d \geq 3$ the analogous property does not hold. Our result completes thus the picture for $d = 2$. It would be interesting to understand better the role of the rationality of lines and conics on \mathbb{P}^2 with regard to this property.

Throughout the paper we work over the field \mathbb{C} of complex numbers.

2 Initial sequences and their properties

Waldschmidt constants of radical ideals of points in complex projective spaces were studied by Chudnovsky in [3]. In particular he observed that the initial degree $\alpha(I)$ of an ideal I (the first term in the initial sequence) and its asymptotic cousin $\widehat{\alpha}(I)$ are surprisingly closely related and asked if the following statement is true.

Conjecture 2.1 (Chudnovsky). *Let I be a radical ideal of a finite set of points in \mathbb{P}^n . Then the inequality*

$$\frac{\alpha(I) + n - 1}{n} \leq \frac{\alpha(I^{(m)})}{m}$$

holds for all $m \geq 1$. In particular

$$\frac{\alpha(I) + n - 1}{n} \leq \widehat{\alpha}(I).$$

This conjecture was proved for $n = 2$ by Chudnovsky in [3] using some variants of Schwarz Lemma and Bombieri results on plurisubharmonic functions. Some special cases have been investigated by Demailly in [6]. Esnault and Viehweg using methods of complex projective geometry have proved the following useful result, see [12, Inégalité A].

Theorem 2.2 (Esnault – Viehweg). *Let I be a radical ideal of a finite set of points in \mathbb{P}^n with $n \geq 2$. Let $k \leq m$ be two integers. Then*

$$\frac{\alpha(I^{(k)}) + 1}{k + n - 1} \leq \frac{\alpha(I^{(m)})}{m}, \quad (1)$$

in particular

$$\frac{\alpha(I^{(k)}) + 1}{k + n - 1} \leq \widehat{\alpha}(I).$$

In the aforementioned paper [3] Chudnovsky announced (in the text right after Theorem 8) the feasibility of enumerating all Waldschmidt constants of all configurations of up to 9 points in \mathbb{P}^2 . In Appendix 1 of that paper, he also listed some point configurations and corresponding constants. The complete list of all configurations of up to 8 points was found by Geramita, Harbourne and Migliore in [14]. They computed all possible Hilbert functions of fat point subschemes supported on up to 8 points in \mathbb{P}^2 . We denote by $\mathcal{H}(s, n)$ the configuration number n for s points from Tables 1, 3 and 5 in [14].

For 9 or more points there seem to be infinitely many possible values of Waldschmidt constants so that a complete classification is not possible. Here we show some restrictions on possible values of Waldschmidt constants. The following Theorem implies immediately Theorem A. Note that configurations of points with $\widehat{\alpha}(I) < 9/4$ have been classified in [9, Main Theorem].

Theorem 2.3 (Configurations of points with $\widehat{\alpha}(I) < 5/2$). *Let I be the radical ideal of a set Z of s points in \mathbb{P}^2 . If*

$$\widehat{\alpha}(I) < \frac{5}{2},$$

then one of the following cases holds:

- a) $\widehat{\alpha}(I) = 1$ and Z is contained in a line,
- b) $\widehat{\alpha}(I) = 2 - \frac{1}{s-1}$ and exactly $s - 1$ points are collinear,

c) $\hat{\alpha}(I) = 2$ and

- Z is contained in a conic, or
- Z is the configuration $\mathcal{H}(6, 10)$ of six points indicated in Figure 1,

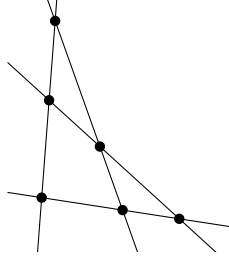


Figure 1: $\mathcal{H}(6, 10)$, $\hat{\alpha}(I) = 2$

d) $\hat{\alpha}(I) = 9/4$ and Z is the configuration $\mathcal{H}(6, 9)$ of six points indicated in Figure 2,

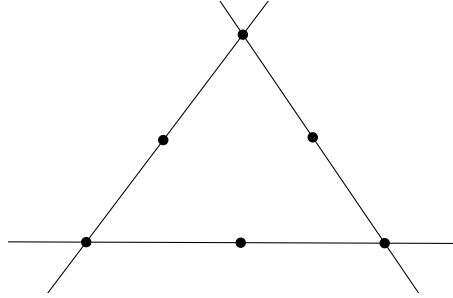


Figure 2: $\mathcal{H}(6, 9)$, $\hat{\alpha}(I) = \frac{9}{4}$

e) $\hat{\alpha}(I) = 16/7$ and Z is the configuration $\mathcal{H}(7, 17)$ of seven points indicated in Figure 3,

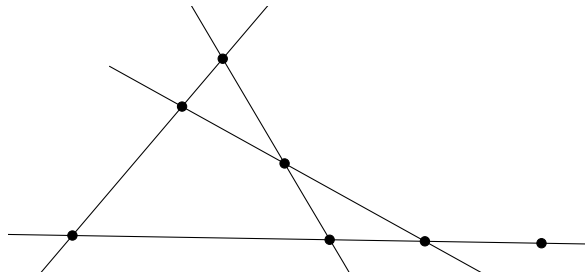
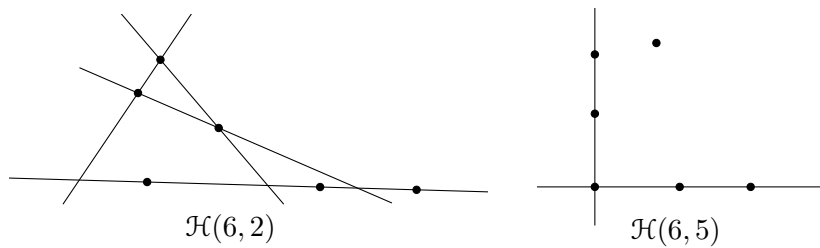


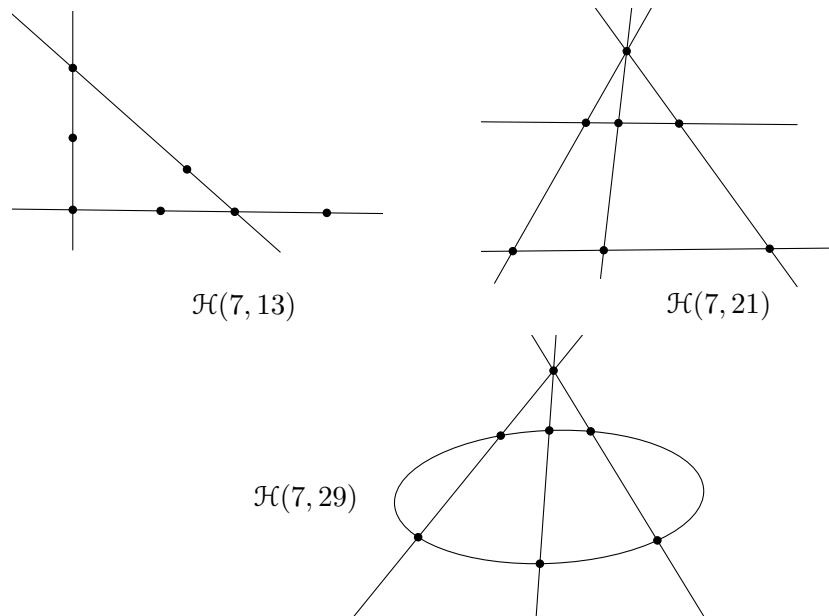
Figure 3: $\mathcal{H}(7, 17)$, $\hat{\alpha}(I) = \frac{16}{7}$

f) $\hat{\alpha}(I) = 7/3$ and

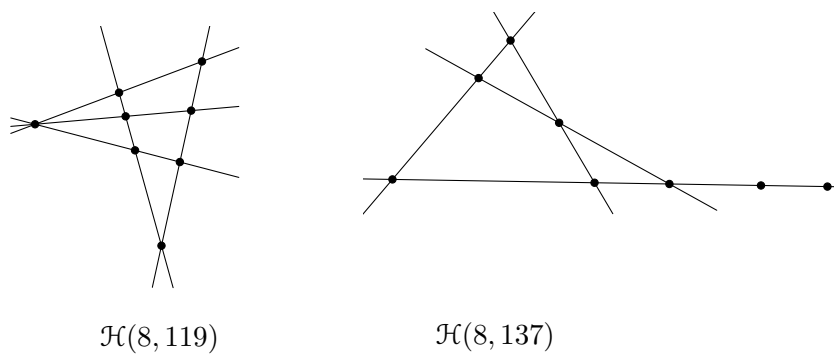
- Z is one of the configurations $\mathcal{H}(6, 2)$ or $\mathcal{H}(6, 5)$ of six points indicated in Figure 4, or

Figure 4: $\hat{\alpha}(I) = \frac{7}{3}$

– Z is one of the configurations $\mathcal{H}(7, 13)$, $\mathcal{H}(7, 21)$ or $\mathcal{H}(7, 29)$ of seven points indicated in Figure 5, or

Figure 5: $\hat{\alpha}(I) = \frac{7}{3}$

– Z is one of the configurations $\mathcal{H}(8, 119)$ or $\mathcal{H}(8, 137)$ of eight points indicated in Figure 6,

Figure 6: $\hat{\alpha}(I) = \frac{7}{3}$

g) $\hat{\alpha}(I) = 12/5$ and Z is a set of six points not contained in a conic and no three points are collinear,

h) $\hat{\alpha}(I) = 17/7$ and

- Z is the configuration $\mathcal{H}(8, 136)$ of eight points indicated in Figure 7, or

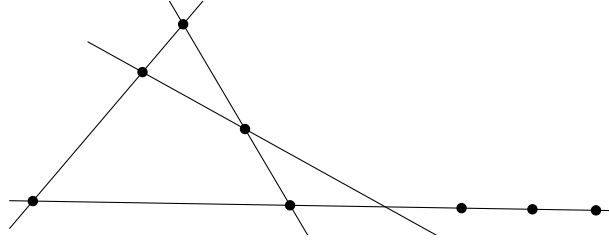


Figure 7: $\mathcal{H}(8, 136)$, $\hat{\alpha}(I) = \frac{17}{7}$

- Z is a configuration of nine points indicated in Figure 8.

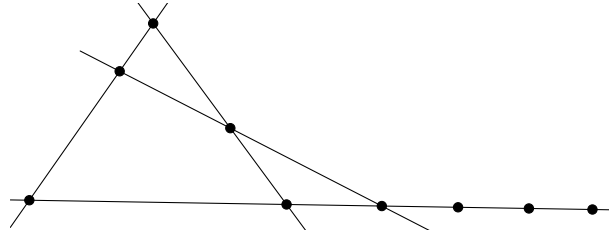


Figure 8: $\hat{\alpha}(I) = \frac{17}{7}$

Proof. It suffices to show that the conditions

$$\hat{\alpha}(I) < \frac{5}{2} \quad \text{and} \quad s \geq 10,$$

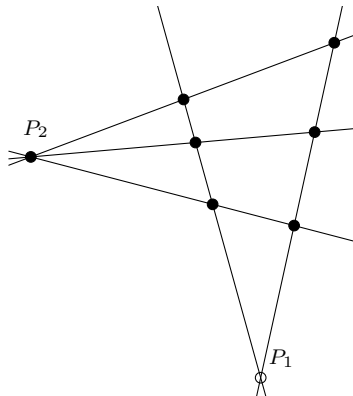
imply $\hat{\alpha}(I) \leq 2$, and

$$\frac{7}{3} \leq \hat{\alpha}(I) < \frac{5}{2} \quad \text{and} \quad s = 9,$$

imply $\hat{\alpha}(I) = \frac{17}{7}$ and the configuration is exactly that indicated in Figure 8. Indeed, the rest follows from looking up the list in [16].

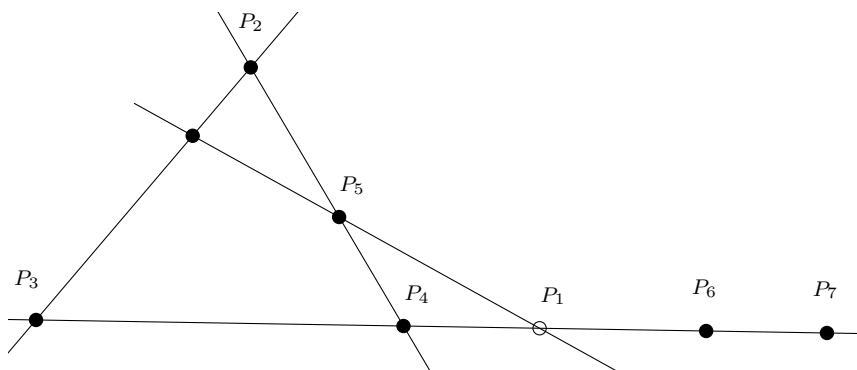
Let $Z = \{P_1, \dots, P_8, P_9, \dots, P_s\}$ be a configuration of $s \geq 9$ points. Assume that Z is not contained in a conic (as otherwise obviously $\hat{\alpha}(I) \leq 2$). Since Z is not contained in a conic, there is also a subset $W \subseteq Z$ of 8 points not contained in a conic. Renumbering the points if necessary, we may assume that $W = \{P_1, \dots, P_8\}$. Then $\hat{\alpha}(I_W) = \frac{7}{3}$ or $\hat{\alpha}(I_W) = \frac{17}{7}$ and W is one of the configurations $\mathcal{H}(8, 119)$, $\mathcal{H}(8, 136)$ or $\mathcal{H}(8, 137)$.

Case $\mathcal{H}(8, 119)$. In this case W is indicated in Figure 9.

Figure 9: Case $\mathcal{H}(8, 119)$

The idea now is to replace W by $W' = \{P_2, \dots, P_9\}$. Since $U = \{P_2, \dots, P_8\}$ is not contained in a conic, neither is W' , so that it must be $\widehat{\alpha}(I_{W'}) = \frac{7}{3}$ and W' is again one of the configurations $\mathcal{H}(8, 119)$, $\mathcal{H}(8, 136)$ or $\mathcal{H}(8, 137)$. Since U has no 4 collinear points, $W' = U \cup \{P_9\}$ cannot have 5 collinear points, so that W' must be again of the type $\mathcal{H}(8, 119)$. Since the triple point of the configuration is fixed in P_2 , it is elementary to check that the only way to complete U to the $\mathcal{H}(8, 119)$ configuration is by adding the point P_1 , a contradiction.

Case $\mathcal{H}(8, 137)$. In this case W is indicated in Figure 10.

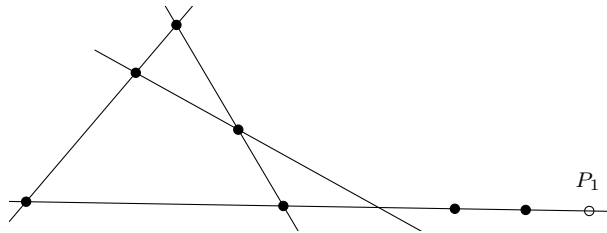
Figure 10: Case $\mathcal{H}(8, 137)$

Now we replace W by $W' = \{P_2, \dots, P_9\}$. Since $U = \{P_2, \dots, P_8\}$ is not contained in a conic, neither is W' . So again W' must be one of the configurations $\mathcal{H}(8, 119)$, $\mathcal{H}(8, 136)$ or $\mathcal{H}(8, 137)$. Assume that W' is of type $\mathcal{H}(8, 119)$. Since U contains only one set of four collinear points, the point P_9 must lie either on the line P_2P_3 or on the line P_2P_4 . Since these cases are equivalent we may assume, without loss of generality, that P_9 lies on the line P_2P_3 . But P_9 must also lie on two lines, each containing three points. There is only one such possibility: the point P_9 has to lie on both lines, P_5P_6 and P_5P_7 , a contradiction.

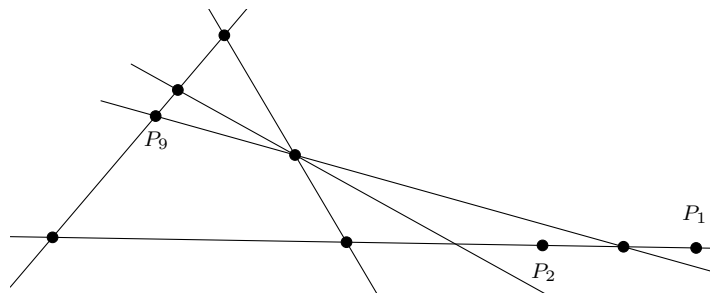
Now we consider the case that W' is of type $\mathcal{H}(8, 137)$. Then P_9 must be the intersection point of the line P_3P_4 and a line passing through two other points. That must be again the point P_1 . A contradiction.

Hence W' is of the type $\mathcal{H}(8, 136)$. Then the point P_9 must be on the line P_6P_7 . Thus the set $Z' = W' \cup \{P_1\}$ is indicated in Figure 8. We will complete the argument in the Case of 9 points below.

Case $\mathcal{H}(8, 136)$. In this situation W is presented in Figure 11.

Figure 11: Case $\mathcal{H}(8, 136)$

We proceed analogously and show that there are only two possibilities. The set $Z' = W' \cup \{P_9\}$ is either as in Figure 8, or as in Figure 12.

Figure 12: The set Z'

If Z' is the configuration indicated in Figure 12, then $Z'' = \{P_1, P_3, \dots, P_9\}$ is the configuration $\mathcal{H}(8, 118)$ and its Waldschmidt constant is $\hat{\alpha}(I_{Z''}) = \frac{5}{2} > \hat{\alpha}(I_Z)$, which is a contradiction.

Now we turn to the final case.

Case of 9 points. From what we have done so far, it follows that the configuration presented in Figure 8 is the only configuration of 9 points with $2 < \hat{\alpha}(I) < \frac{5}{2}$. It is not hard to check that in this case $\hat{\alpha}(I) = \frac{17}{7}$. We verified hand calculations with Singular, [4].

Finally, we claim that there is no configuration of $s \geq 10$ points with $2 < \hat{\alpha}(I) < \frac{5}{2}$. Indeed, assume that such a configuration of 10 points exists. Removing the point P_{10} we obtain a configuration indicated in Figure 13. Its Waldschmidt constant is equal $\frac{17}{7}$. Removing further the point P_1 we obtain $\mathcal{H}(8, 136)$ with the Waldschmidt constant equal to $\frac{17}{7}$. Adding back the point P_{10} we obtain a configuration of 9 points with $2 < \hat{\alpha}(I) < \frac{5}{2}$. The argument above shows that it must be $P_{10} = P_1$, a contradiction.

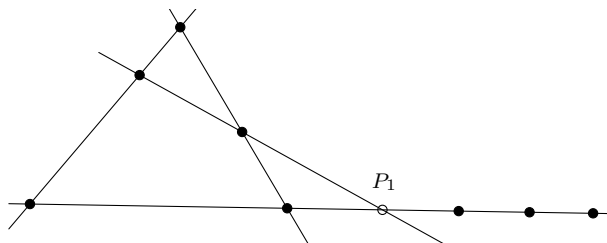


Figure 13: Case of 9 points

□

3 Initial sequences with first differences equal 2

Now we are in the position to show how Theorem 2.3 applies towards the proof of Theorem B.

Proof of Theorem B. Assume to the contrary, that there exists a planar point configuration $Z = \{P_1, \dots, P_s\}$ with radical ideal $I = I_Z$ such that

$$\alpha(I^{(m+i)}) - \alpha(I^{(m+(i-1))}) = 2$$

for $i = 1, 2, 3, 4$ and such that Z is not contained in a conic. Then the inequality (1) immediately implies that

$$\widehat{\alpha}(I) \leq \frac{7}{3}.$$

By Theorem A it must be then $s \leq 8$. On the other hand it must be $s \geq 6$. Hence we must be in one of the cases listed in Theorem 2.3. It follows from the discussion in [14, Remark 5.11] that the sequence $\alpha(I^{(m)})$ in all these cases is determined by a finite number of initial terms. With some more effort one can in fact verify that in each of these cases the sequence of first differences

$$\alpha(I^{(m)}) - \alpha(I^{(m-1)}), \quad \text{with } \alpha(I^{(0)}) = 0$$

of the initial sequence of I is periodic with the period $\pi = (a_1, \dots, a_\ell)$ of length ℓ of one of the following shapes:

- $\mathcal{H}(6, 10) : \ell = 2, \pi = (3, 1),$
- $\mathcal{H}(6, 2), \mathcal{H}(6, 5), \mathcal{H}(7, 13), \mathcal{H}(7, 21), \mathcal{H}(7, 29), \mathcal{H}(8, 119), \mathcal{H}(8, 137) : \ell = 3, \pi = (3, 2, 2),$
- $\mathcal{H}(6, 9) : \ell = 4, \pi = (3, 2, 2, 2),$
- $\mathcal{H}(7, 17) : \ell = 7, \pi = (3, 2, 2, 3, 2, 2, 2).$

We omit easy but tedious calculations. Since in none of the listed cases a subsequence of 4 consecutive 2's appears, this contradicts the initial assumption and completes the proof of Theorem B. \square

Remark 3.1. Note that the configuration $\mathcal{H}(6, 9)$ shows that three consecutive differences of 2 are possible without forcing Z to be contained in a conic. This was already observed in the discussion in [8, Lemma 4.12].

The bound $\widehat{\alpha}(I) < \frac{5}{2}$ taken on in Theorem 2.3 might appear a bit incidental. However, already the list in [16] exhibits a considerable number of cases with $\widehat{\alpha}(I) = \frac{5}{2}$. Moreover, a number of experiments we have run for $s \geq 9$ points show that it might be significantly more difficult to chart all configuration of points with Waldschmidt constants $\widehat{\alpha}(I) \geq \frac{5}{2}$ and the problem might require some new methods. On the other hand, in the view of results obtained so far it is tempting to conclude this note with the following question.

Problem 3.2. Is the set of Waldschmidt constants of all planar point configurations well-ordered?

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