PRODUCTS OF SEVERAL COMMUTATORS IN A LIE NILPOTENT ASSOCIATIVE ALGEBRA

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ABSTRACT. Let F be a field of characteristic $\neq 2, 3$ and let A be a unital associative F-algebra. Define a left-normed commutator $[a_1, a_2, \dots, a_n]$ $(a_i \in A)$ recursively by $[a_1, a_2] = a_1 a_2 - a_2 a_1$, $[a_1, \dots, a_{n-1}, a_n] = [[a_1, \dots, a_{n-1}], a_n]$ $(n \geq 3)$. For $n \geq 2$, let $T^{(n)}(A)$ be the two-sided ideal in A generated by all commutators $[a_1, a_2, \dots, a_n]$ $(a_i \in A)$. Define $T^{(1)}(A) = A$.

Let k, ℓ be integers such that $k > 0$, $0 \leq \ell \leq k$. Let $m_1, \dots, m_k$ be positive integers such that ℓ of them are odd and $k - \ell$ of them are even. Let $N_{k,\ell} = \sum_{i=1}^{k} m_i - 2k + \ell + 2$. The aim of the present note is to show that, for any positive integers $m_1, \dots, m_k$, in general, $T^{(m_1)}(A) \dots T^{(m_k)}(A) \nsubseteq T^{(1+N_{k,\ell})}(A)$. It is known that if $\ell < k$ (that is, if at least one of m_i is even) then $T^{(m_1)}(A) \dots T^{(m_k)}(A) \subseteq T^{(N_{k,\ell})}(A)$ for each A so our result cannot be improved if $\ell < k$.

Let $N_k = \sum_{i=1}^{k} m_i - k + 1$. Recently Dangovski has proved that if $m_1, \dots, m_k$ are any positive integers then, in general, $T^{(m_1)}(A) \dots T^{(m_k)}(A) \nsubseteq T^{(1+N_k)}(A)$. Since $N_{k,\ell} = N_k - (k - \ell - 1)$, Dangovski's result is stronger than ours if $\ell = k$ and is weaker than ours if $\ell \leq k - 2$; if $\ell = k - 1$ then $N_k = N_{k,k-1}$ so both results coincide. It is known that if $\ell = k$ (that is, if all m_i are odd) then, for each A, $T^{(m_1)}(A) \dots T^{(m_k)}(A) \subseteq T^{(N_k)}(A)$ so in this case Dangovski's result cannot be improved.

2010 AMS MSC Classification: 16R10, 16R40
Keywords: polynomial identity, product of ideals, commutators

1. Introduction

Let R be an arbitrary unital associative and commutative ring and let A be a unital associative algebra over R. Define a left-normed commutator $[a_1, a_2, \dots, a_n]$ $(a_i \in A)$ recursively by $[a_1, a_2] = a_1 a_2 - a_2 a_1$, $[a_1, \dots, a_{n-1}, a_n] = [[a_1, \dots, a_{n-1}], a_n]$ $(n \geq 3)$. For $n \geq 2$, let $T^{(n)}(A)$ be the two-sided ideal in A generated by all commutators $[a_1, a_2, \dots, a_n]$ $(a_i \in A)$. Define $T^{(1)}(A) = A$. Clearly, we have

$$A = T^{(1)}(A) \supseteq T^{(2)}(A) \supseteq T^{(3)}(A) \supseteq \dots \supseteq T^{(n)}(A) \supseteq \dots .$$

We are concerned with the following.

Problem 1. *Let $k \geq 2$ and let $m_1, \dots, m_k$ be positive integers. Find the maximal integer $N = N(R, m_1, \dots, m_k)$ such that, for each R-algebra A,*

$$T^{(m_1)}(A) \dots T^{(m_k)}(A) \subseteq T^{(N)}(A).$$

Let $X = \{x_1, x_2, \dots\}$ be an infinite countable set and let $R\langle X\rangle$ be the free unital associative algebra over R freely generated by X. Define $T^{(n)} = T^{(n)}(R\langle X\rangle)$.

Problem 2. *Let $k \geq 2$ and let $m_1, \dots, m_k$ be positive integers. Find the maximal integer $N = N(R, m_1, \dots, m_k)$ such that*

$$T^{(m_1)} \dots T^{(m_k)} \subseteq T^{(N)}.$$

It is easy to check that Problem 1 is equivalent to Problem 2, and the integer N in both problems is the same.

Problem 2 and some other similar questions have been recently studied by Dangovski [6] (using different terminology). The work of Dangovski was motivated by the results of Etingof, Kim and Ma [9] and Bapat and Jordan [2], which in turn were motivated by the pioneering article by Feigin and Shoikhet [10].

The following assertion was proved by Latyshev [17, Lemma 1] in 1965 (Latyshev's paper was published in Russian) and independently rediscovered by Gupta and Levin [15, Theorem 3.2] in 1983.

Theorem 1.1 (see [15, 17])**.** *Let R be an arbitrary unital associative and commutative ring and let A be an associative R-algebra. Let $m, n \in \mathbb{Z}$, $m, n \geq 1$. Then*

$$T^{(m)}(A)\, T^{(n)}(A) \subseteq T^{(m+n-2)}(A).$$

Latyshev [17] has actually proved that $T^{(m)}\, T^{(n)} \subseteq T^{(m+n-2)}$ in $R\langle X\rangle$; this assertion is equivalent to Theorem 1.1.

Note that, for a unital associative ring R, we have $\frac{1}{6} \in R$ if and only if $2(=1+1)$ and 3 are invertible in R. The theorem below was proved by Sharma and Srivastava [19, Theorem 2.8] in 1990 and independently rediscovered (with different proofs) by Bapat and Jordan [2, Corollary 1.4] in 2013 and by Grishin and Pchelintsev [12, Theorem 1] in 2015.

Theorem 1.2 (see [2, 12, 19])**.** *Let R be an arbitrary unital associative and commutative ring such that $\frac{1}{6} \in R$ and let A be an associative R-algebra. Let $m, n \in \mathbb{Z}$, $m, n > 1$ and at least one of the numbers m, n is odd. Then*
$$T^{(m)}(A)\; T^{(n)}(A) \subseteq T^{(m+n-1)}(A).$$

Note that Grishin and Pchelintsev [12] have actually proved that $T^{(m)}\; T^{(n)} \subseteq T^{(m+n-1)}$; this result is equivalent to Theorem 1.2.

Let $N_k = \sum_{i=1}^{k} m_i - k + 1$. The proposition below follows immediately from Theorem 1.2.

Proposition 1.3. *Let R be an arbitrary unital associative and commutative ring such that $\frac{1}{6} \in R$ and let A be an associative R-algebra. Let $k > 0$ be an integer and let $m_i > 0$ $(i = 1, \dots, k)$ be* ***odd*** *integers. Then*
$$T^{(m_1)}(A) \dots T^{(m_k)}(A) \subseteq T^{(N_k)}(A).$$

Let $N_{k,\ell} = \sum_{i=1}^{k} m_i - 2k + \ell + 2 = N_k - (k - \ell - 1)$. One can deduce from Theorems 1.1 and 1.2 the following proposition (see Dangovski [6, Section 6]).

Proposition 1.4 (see [6])**.** *Let R be an arbitrary unital associative and commutative ring such that $\frac{1}{6} \in R$ and let A be an associative R-algebra. Let k, ℓ be integers such that $0 \le \ell < k$. Let $m_i \ge 2$ $(i = 1, \dots, k)$ be integers such that ℓ of them are odd and $(k - \ell) > 0$ of them are even. Then*
$$T^{(m_1)}(A) \dots T^{(m_k)}(A) \subseteq T^{(N_{k,\ell})}(A). \tag{1}$$

We prove Proposition 1.4 in Section 2 in order to have the paper more self-contained.

Recently Dangovski [6, Proposition 2.2] has proved a result that can be reformulated as follows.

Theorem 1.5 (see [6])**.** *Let F be a field and let k be a positive integer. Let $m_1, \dots, m_k$ be positive integers and let N_k be as above. Then there exists an associative F-algebra A such that*
$$T^{(m_1)}(A) \dots T^{(m_k)}(A) \nsubseteq T^{(1+N_k)}(A). \tag{2}$$

One can deduce from Theorem 1.5 the following.

Corollary 1.6. *Let R be an arbitrary unital associative and commutative ring and let $k, m_1, \dots, m_k, N_k$ be as in Theorem 1.5. Then there exists an associative R-algebra A such that (2) holds.*

Proof. Suppose that R is not a field. Let M be a maximal ideal of R (by Zorn's lemma, such an ideal M exists). Then $F = R/M$ is a field and the F-algebra A of Theorem 1.5 can be viewed in a natural way as an R-algebra (with $r \cdot a$ defined by $r \cdot a = (r + M) \cdot a$ for $r \in R, a \in A$). Since A satisfies (2), the result follows. ☐

Let N be the integer defined in Problems 1 and 2. If $\frac{1}{6} \in R$ and all the integers $m_1, \dots, m_k$ are *odd* then $N = N_k$. Indeed, it follows from Proposition 1.3 and Corollary 1.6 that in this case we always have
$$T^{(m_1)}(A) \dots T^{(m_k)}(A) \subseteq T^{(N_k)}(A)$$
and, in general,
$$T^{(m_1)}(A) \dots T^{(m_k)}(A) \nsubseteq T^{(1+N_k)}(A).$$

Suppose that ℓ of the integers $m_1, \dots, m_k$ are odd $(\ell < k)$ and $(k - \ell) > 0$ of them are even. Let $\frac{1}{6} \in R$. Then, by Proposition 1.4, $N_{k,\ell} \le N$ and, by Corollary 1.6, $N \le N_k$. If $\ell = k - 1$ (that is, $k - 1$ of the integers $m_1, \dots, m_k$ are odd and one of them is even) then $N_{k,k-1} = N_k$ so $N = N_k$. However, if $0 \le \ell < k - 1$ then $N_{k,\ell} = N_k - (k - \ell - 1) < N_k$ so one can only deduce from the results above that $N_{k,\ell} \le N \le N_k$.

Our main result is as follows.

Theorem 1.7. *Let F be a field. Let k, ℓ be integers, $0 \le \ell \le k$. Let $m_1, \dots, m_k$ be positive integers such that ℓ of them are odd and $k - \ell$ of them are even and let $N_{k,\ell}$ be as above. Then there exists a unital associative F-algebra A such that*
$$T^{(m_1)}(A) \dots T^{(m_k)}(A) \nsubseteq T^{(1+N_{k,\ell})}(A). \tag{3}$$

In a particular case when $k = 2$ and m_1, m_2 are even Theorem 1.7 has been recently proved by Grishin and Pchelintsev [12] and independently by the authors of the present article [8]. In another particular case when $m_1 = m_2 = \dots = m_{k-1} = 2$ and m_k is even this theorem has been proved by Grishin, Tsybulya and Shokola [13, Theorem 3].

The proof of the following result is similar to that of Corollary 1.6.

Corollary 1.8. *Let R be an arbitrary unital associative and commutative ring and let k, ℓ, $m_1, \dots, m_k$, $N_{k,\ell}$ be as in Theorem 1.7. Then there exists an associative R-algebra A such that (3) holds.*

It follows that if $\frac{1}{6} \in R$ and at least one of the integers m_i is even then $N = N_{k,\ell}$ because, by Proposition 1.4 and Corollary 1.8, in this case we always have

$$T^{(m_1)}(A) \dots T^{(m_k)}(A) \subseteq T^{(N_{k,\ell})}(A)$$

but, in general,

$$T^{(m_1)}(A) \dots T^{(m_k)}(A) \nsubseteq T^{(1+N_{k,\ell})}(A).$$

Thus, the solution of Problems 1 and 2 (for R that contains $\frac{1}{6}$) is as follows. Let R be a unital associative and commutative ring such that $\frac{1}{6} \in R$ and let $k, m_1, \dots, m_k$ be positive integers. Then

$$N = \begin{cases} N_k = \sum_{i=1}^{k} m_i - k + 1 & \text{if all integers } m_i \text{ are odd (Dangovski [6]);} \\ & \\ N_{k,\ell} = \sum_{i=1}^{k} m_i - 2k + \ell + 2 & \begin{array}{l}\text{if } \ell < k \text{ of the integers } m_i \text{ are odd and} \\ k - \ell \text{ of them are even.}\end{array} \end{cases}$$

Recall that an associative algebra A is Lie nilpotent of class at most c if $[u_1, \dots, u_c, u_{c+1}] = 0$ for all $u_i \in A$. Theorem 1.7 follows immediately from the following result.

Theorem 1.9. *Under the hypotheses of Theorem 1.7, there exists a unital associative F-algebra A such that the following two conditions are satisfied:*
i) $T^{(1+N_{k,\ell})}(A) = 0$, that is, the algebra A is Lie nilpotent of class at most $N_{k,\ell}$;
ii) there are $v_{ij} \in A$ such that

$$[v_{11}, \dots, v_{1m_1}] \dots [v_{k1}, \dots, v_{km_k}] \neq 0.$$

To prove Theorem 1.9 we use the same algebra A that was used in [8, Theorem 1.4].

Remarks. 1. Both Theorem 1.5 and Theorem 1.7 are valid for arbitrary k-tuples $m_1, m_2, \dots, m_k$ of positive integers. However, if $\ell = k$ (that is, if all m_i are odd) then Theorem 1.5 gives a stronger result than Theorem 1.7 because $N_{k,k} = N_k + 1 > N_k$ and therefore $T^{(1+N_{k,k})}(A) \subset T^{(1+N_k)}(A)$. If $\ell = k-1$ (that is, if one of the integers $m_1, m_2, \dots, m_k$ is even and $k-1$ of them are odd) then $N_{k,k-1} = N_k$ so the results of Theorem 1.5 and Theorem 1.7 coincide; and if $\ell < k-1$ (that is, if two or more of the integers $m_1, m_2, \dots, m_k$ are even) then $N_{k,\ell} = N_k - (k - \ell - 1) < N_k$ so Theorem 1.5 gives a weaker result than Theorem 1.7.

2. The proofs of Theorem 1.2 given in [2], [12] and [19] are valid for algebras over an associative and commutative unital ring R such that $\frac{1}{6} \in R$. However, the proof given in [2] can be slightly modified to become also valid over any R such that $\frac{1}{3} \in R$ (see [1, Remark 3.9] for explanation). Moreover, for some specific m and n Theorem 1.2 holds over an arbitrary ring R: for instance, $T^{(3)}(A)T^{(3)}(A) \subset T^{(5)}(A)$ for any algebra A over any associative and commutative unital ring R (see [5, Lemma 2.1]). However, in general Theorem 1.2 fails over $\mathbb{Z}$ and over a field of characteristic 3: it was shown in [7, 16] that in this case $T^{(3)}T^{(2)} \nsubseteq T^{(4)}$ and moreover, $T^{(3)}\big(T^{(2)}\big)^{\ell} \nsubseteq T^{(4)}$ for all $\ell \geq 1$.

3. In 1978 Volichenko proved Theorem 1.2 for $m = 3$ and arbitrary n in the preprint [20] written in Russian. In 1985 Levin and Sehgal [18] independently rediscovered Volichenko's result. More recently Etingof, Kim and Ma [9] and Gordienko [11] have independently proved this theorem for small m and n; these authors were unaware of the results of [18, 20].

2. Proofs of Proposition 1.4 and Theorem 1.9

Proof of Proposition 1.4. Induction on k. If $k = 1$ then $\ell = 0$ so $N_{1,0} = m_1$ and (1) holds.

Suppose that $k > 1$ and for all products of less than k terms $T^{(m_i)}(A)$ the proposition has already been proved. We split the proof in 3 cases.

Case 1. Suppose that m_k is odd. Then for some i such that $1 \le i < k$ the number m_i is even so we can apply the induction hypothesis to the product $T^{(m_1)}(A) \dots T^{(m_{k-1})}(A)$. By this hypothesis,

$$T^{(m_1)}(A) \dots T^{(m_{k-1})}(A) \subseteq T^{(N')}(A)$$

where $N' = \sum_{i=1}^{k-1} m_i - 2(k-1) + (\ell - 1) + 2 = \sum_{i=1}^{k-1} m_i - 2k + \ell + 3$. By Theorem 1.2,

$$T^{(N')}(A)\, T^{(m_k)}(A) \subseteq T^{(N'+m_k-1)}(A) = T^{(N_{k,\ell})}(A)$$

since $N' + m_k - 1 = \sum_{i=1}^{k} m_i - 2k + \ell + 2 = N_{k,\ell}$. Thus, in this case (1) holds, as required.

Case 2. Suppose that m_k is even and, for some i such that $1 \le i < k$, m_i is also even. Then we can apply the induction hypothesis to the product $T^{(m_1)}(A) \dots T^{(m_{k-1})}(A)$ so

$$T^{(m_1)}(A) \dots T^{(m_{k-1})}(A) \subseteq T^{(N'')}(A)$$

where $N'' = \sum_{i=1}^{k-1} m_i - 2(k-1) + \ell + 2 = \sum_{i=1}^{k-1} m_i - 2k + \ell + 4$. By Theorem 1.1,

$$T^{(N'')}(A)\, T^{(m_k)}(A) \subseteq T^{(N''+m_k-2)}(A) = T^{(N_{k,\ell})}(A)$$

since $N'' + m_k - 2 = \sum_{i=1}^{k} m_i - 2k + \ell + 2 = N_{k,\ell}$. Hence, in this case (1) holds, as required.

Case 3. Suppose that m_k is even and all m_i for $1 \le i < k$ are odd. Applying Theorem 1.2 $k-1$ times, we get

$$T^{(m_1)}(A) \dots T^{(m_{k-1})}(A)\, T^{(m_k)}(A) \subseteq T^{(m_1 + \dots + m_k - k + 1)}(A) = T^{(N_{k,k-1})}(A)$$

since $\sum_{i=1}^{k} m_i - k + 1 = \sum_{i=1}^{k} m_i - 2k + (k-1) + 2 = N_{k,k-1}$. Thus, in this case (1) also holds.

The proof of Proposition 1.4 is completed. $\square$

The proof of Theorem 1.9 below is a modification of the proof of [8, Theorem 1.4]. First we need some auxiliary results.

Let G and H be unital associative algebras over a field F such that $[g_1, g_2, g_3] = 0$, $[h_1, h_2, h_3] = 0$ for all $g_i \in G$, $h_j \in H$. Note that each commutator $[g_1, g_2]$ ($g_i \in G$) is central in G, that is, $[g_1, g_2]g = g[g_1, g_2]$ for each $g \in G$. Similarly, each commutator $[h_1, h_2]$ ($h_j \in H$) is central in H. The following lemma has been proved in [8, Lemma 2.1] by induction on n.

Lemma 2.1 (see [8])**.** *Let*

$$c_\ell = [g_1 \otimes h_1, g_2 \otimes h_2, \dots, g_\ell \otimes h_\ell]$$

where $\ell \ge 2, g_i \in G, h_j \in H$. Then

$$\begin{aligned} c_2 &= [g_1, g_2] \otimes h_1 h_2 + g_2 g_1 \otimes [h_1, h_2], \\ c_{2n} &= [g_1, g_2][g_3, g_4] \dots [g_{2n-1}, g_{2n}] \otimes [h_1 h_2, h_3][h_4, h_5] \dots [h_{2n-2}, h_{2n-1}] h_{2n} \\ &\quad + [g_2 g_1, g_3][g_4, g_5] \dots [g_{2n-2}, g_{2n-1}] g_{2n} \otimes [h_1, h_2][h_3, h_4] \dots [h_{2n-1}, h_{2n}] \\ &\quad (n > 1), \\ c_{2n+1} &= [g_1, g_2][g_3, g_4] \dots [g_{2n-1}, g_{2n}] g_{2n+1} \otimes [h_1 h_2, h_3][h_4, h_5] \dots [h_{2n}, h_{2n+1}] \\ &\quad + [g_2 g_1, g_3][g_4, g_5] \dots [g_{2n}, g_{2n+1}] \otimes [h_1, h_2][h_3, h_4] \dots [h_{2n-1}, h_{2n}] h_{2n+1} \\ &\quad (n \ge 1). \end{aligned}$$

Corollary 2.2 (see [8])**.** *Suppose that*

$$[f_1, f_2] \dots [f_{2n-1}, f_{2n}] = 0 \qquad \textit{for all} \quad f_j \in H. \tag{4}$$

Then for all $u_i \in G \otimes H$ we have

$$[u_1, u_2, \dots, u_{2n+1}] = 0.$$

Proof. It follows from (4) and Lemma 2.1 that $[g_1 \otimes h_1, g_2 \otimes h_2, \dots, g_{2n+1} \otimes h_{2n+1}] = 0$ for all $g_i \in G$, $h_j \in H$. Since each $u_i \in G \otimes H$ is a sum of products of the form $g \otimes h$ ($g \in G$, $h \in H$), we have $[u_1, u_2, \dots, u_{2n+1}] = 0$ for all $u_i \in G \otimes H$, as required. $\square$

The following assertion follows immediately from Lemma 2.1.

Corollary 2.3. *Let $v_1 = g_1 \otimes 1$, $v_i = g_i \otimes h_i$ ($i = 2, \dots, 2m' - 1$), $v_{2m'} = g_{2m'} \otimes 1$ and let $w_1 = g'_1 \otimes 1$, $w_j = g'_j \otimes h'_j$ ($j = 2, \dots, 2n' + 1$) where $g_i, g'_i \in G$, $h_j, h'_j \in H$. Then*

$$\begin{aligned} [v_1, \dots, v_{2m'}] &= [g_1, g_2] \dots [g_{2m'-1}, g_{2m'}] \otimes\ [h_2, h_3] \dots [h_{2m'-2}, h_{2m'-1}], \\ [w_1, \dots, w_{2n'+1}] &= [g'_1, g'_2] \dots [g'_{2n'-1}, g'_{2n'}] g_{2n'+1} \otimes\ [h'_2, h'_3] \dots [h'_{2n'}, h'_{2n'+1}]. \end{aligned}$$

Proof of Theorem 1.9. Two cases are to be considered: the case when $char\, F \neq 2$ and the case when $char\, F = 2$.

Case 1. Suppose that F is a field of characteristic $\neq 2$. Let E be the unital infinite-dimensional Grassmann (or exterior) algebra over F. Then E is generated by the elements e_i $(i = 1, 2, \dots)$ such that $e_i e_j = -e_j e_i$, $e_i^2 = 0$ for all i, j and the set

$$\mathcal{B} = \{e_{i_1} e_{i_2} \dots e_{i_k} \mid k \geq 0,\ i_1 < i_2 < \dots < i_k\}$$

forms a basis of E over F. It is well known and easy to check that $[g_1, g_2, g_3] = 0$ for all $g_i \in E$.

Recall that the r-generated unital Grassmann algebra E_r is the unital subalgebra of E generated by $e_1, e_2, \dots, e_r$. Note that $[h_1, h_2, h_3] = 0$ for all $h_j \in E_r$.

Take $A = E \otimes E_r$ where $r = \sum_{i=1}^{k} m_i - 2k + \ell = N_{k,\ell} - 2$. It is easy to check that r is an even integer. We can apply Lemma 2.1 and Corollaries 2.2 and 2.3 for $G = E$, $H = E_r$.

Note that $[f_1, f_2] \dots [f_{r+1}, f_{r+2}] = 0$ for all $f_i \in E_r$. Indeed, for all $f, f' \in E_r$ the commutator $[f, f']$ belongs to the linear span of the set $\{e_{i_1} \dots e_{i_{2\ell}} \mid \ell \geq 1, 1 \leq i_s \leq r\}$. Hence, $[f_1, f_2] \dots [f_{r+1}, f_{r+2}]$ belongs to the linear span of the set $\{e_{i_1} \dots e_{i_{2\ell}} \mid \ell \geq (r+2)/2, 1 \leq i_s \leq r\}$. Since $2\ell \geq r + 2 > r$, each product $e_{i_1} \dots e_{i_{2\ell}}$ above contains equal terms $e_{i_s} = e_{i_{s'}}$ $(s < s')$ and, therefore, is equal to 0. Thus, $[f_1, f_2] \dots [f_{r+1}, f_{r+2}] = 0$, as claimed.

Since $N_{k,\ell} = r + 2$, we have $[f_1, f_2] \dots [f_{(N_{k,\ell}-1)}, f_{N_{k,\ell}}] = 0$ for all $f_i \in E_r$. Hence, by Corollary 2.2, we have $[u_1, \dots, u_{(1+N_{k,\ell})}] = 0$ for all $u_i \in A = E \otimes E_r$, that is, $T^{(1+N_{k,\ell})}(A) = 0$, as required.

Now it suffices to find elements $v_{ij} \in A$ such that

$$[v_{11}, \dots, v_{1m_1}] \dots [v_{k1}, \dots, v_{km_k}] \neq 0. \tag{5}$$

Let

$$\mathcal{P} = \{(i, j) \mid 1 \leq i \leq k;\ 1 \leq j \leq m_i\}.$$

Note that v_{ij} appears in (5) if and only if $(i, j) \in \mathcal{P}$. Let $\mathcal{N} = \sum_{i=1}^{k} m_i$ and let $\mu : \mathcal{P} \to \{1, 2, \dots, \mathcal{N}\}$ be a bijection. Define

$$e_{ij} = e_{\mu(i,j)} \qquad \big((i, j) \in \mathcal{P}\big).$$

Note that

$$\prod_{(i,j) \in \mathcal{P}} e_{ij} = (-1)^{\delta} e_1 e_2 \dots e_{\mathcal{N}} \tag{6}$$

for some $\delta \in \{0, 1\}$.

Let $\mathcal{P}' \subset \mathcal{P}$,

$$\mathcal{P}' = \{(i', j') \mid 1 \leq i' \leq k;\ 2 \leq j' \leq m_i - 1 \text{ if } m_i \text{ is even};\ 2 \leq j' \leq m_i \text{ if } m_i \text{ is odd}\}.$$

Let $\mu' : \mathcal{P}' \to \{1, 2, \dots, \sum_{i=1}^{k} m_i - 2k + \ell\} = \{1, 2, \dots, r\}$ be a bijection. Define

$$e'_{i'j'} = e_{\mu'(i',j')} \qquad \big((i', j') \in \mathcal{P}'\big).$$

Note that

$$\prod_{(i',j') \in \mathcal{P}'} e_{i'j'} = (-1)^{\delta'} e_1 e_2 \dots e_r \tag{7}$$

for some $\delta' \in \{0, 1\}$.

Define

$$\begin{aligned} v_{i1} &= e_{i1} \otimes 1; \\ v_{ij} &= e_{ij} \otimes e'_{ij} \qquad (1 \leq i \leq k;\ 2 \leq j \leq m_i - 1); \\ v_{im_i} &= \begin{cases} e_{im_i} \otimes 1 & \text{if } m_i \text{ is even}; \\ e_{im_i} \otimes e'_{im_i} & \text{if } m_i \text{ is odd}. \end{cases} \end{aligned}$$

If m_i is even then, by Corollary 2.3,

$$\begin{aligned} &[v_{i1}, v_{i2}, \dots, v_{im_i}] \\ &= [e_{i1}, e_{i2}][e_{i3}, e_{i4}] \dots [e_{i(m_i-1)}, e_{im_i}] \otimes [e'_{i2}, e'_{i3}][e'_{i4}, e'_{i5}] \dots [e'_{i(m_i-2)}, e'_{i(m_i-1)}]. \end{aligned}$$

Note that $e_{st} e_{s't'} = -e_{s't'} e_{st}$ for all s, s', t, t' so $[e_{st}, e_{s't'}] = 2 e_{st} e_{s't'}$. It follows that if m_i is even then

$$[v_{i1}, v_{i2}, \dots, v_{im_i}] = 2^{m_i - 1} e_{i1} e_{i2} \dots e_{im_i} \otimes e'_{i2} e'_{i3} \dots e'_{i(m_i-1)}.$$

If m_i is odd then, by Corollary 2.3,

$$\begin{aligned}&[v_{i1}, v_{i2}, \dots, v_{im_i}]\\ &= [e_{i1}, e_{i2}][e_{i3}, e_{i4}] \dots [e_{i(m_i-2)}, e_{i(m_i-1)}]e_{im_i} \otimes [e'_{i2}, e'_{i3}][e'_{i4}, e'_{i5}] \dots [e'_{i(m_i-1)}, e'_{im_i}]\\ &= 2^{m_i-1} e_{i1}e_{i2} \dots e_{i(m_i-1)}e_{im_i} \otimes e'_{i2}e'_{i3} \dots e'_{i(m_i-1)}e'_{im_i}.\end{aligned}$$

It follows that

$$[v_{11}, \dots, v_{1m_1}] \dots [v_{k1}, \dots, v_{km_k}] = 2^{N_k-1} \prod_{i=1}^{k} \prod_{j=1}^{m_i} e_{ij} \otimes \prod_{i'=1}^{k} \prod_{j'=2}^{m'_{i'}} e'_{i'j'}$$

where

$$m'_{i'} = \begin{cases} m_{i'} - 1 & \text{if } m_{i'} \text{ is even;} \\ m_{i'} & \text{if } m_{i'} \text{ is odd,} \end{cases}$$

that is,

$$[v_{11}, \dots, v_{1m_1}] \dots [v_{k1}, \dots, v_{km_k}] = 2^{N_k-1} \prod_{(i,j)\in\mathcal{P}} e_{ij} \otimes \prod_{(i',j')\in\mathcal{P}'} e'_{i'j'}.$$

By (6) and (7), we have

$$[v_{11}, \dots, v_{1m_1}] \dots [v_{k1}, \dots, v_{km_k}] = (-1)^{\delta+\delta'} 2^{N_k-1}\ e_1e_2 \dots e_{\mathcal{N}} \otimes e_1e_2 \dots e_r \neq 0,$$

as required.

Case 2. Suppose that F is a field of characteristic 2. Let $\mathcal{G}$ be the group given by the presentation

$$\mathcal{G} = \langle y_1, y_2, \dots \mid y_i^2 = 1,\ \big((y_i, y_j), y_k\big) = 1\ (i, j, k = 1, 2, \dots)\rangle$$

where $(a, b) = a^{-1}b^{-1}ab$. Then it is easy to check that $\mathcal{G}$ is a nilpotent group of class 2 so $(a, b)c = c(a, b)$ for all $a, b, c \in \mathcal{G}$ and, therefore, $(a, bc) = (a, c)c^{-1}(a, b)c = (a, b)(a, c)$ (see [8] for more details). It is clear that the quotient group $\mathcal{G}/\mathcal{G}'$ is an elementary abelian 2-group so $b^2 \in \mathcal{G}' \subseteq Z(\mathcal{G})$ for all $b \in \mathcal{G}$. It follows that $(a, b^2) = 1$ so $(a, b)^2 = (a, b^2) = 1$, that is, $(a, b) = (a, b)^{-1}$. Since $(b, a) = (a, b)^{-1}$, we have $(a, b) = (b, a)$ for all $a, b \in \mathcal{G}$.

Let $(<)$ be an arbitrary linear order on the set $\{(i, j) \mid i, j \in \mathbb{Z},\ 0 < i < j\}$. The following lemma is well known and easy to check.

Lemma 2.4. *Let $a \in \mathcal{G}$. Then a can be written in a unique way in the form*

$$a = y_{i_1} \dots y_{i_q}(y_{j_1}, y_{j_2}) \dots (y_{j_{2q'-1}}, y_{j_{2q'}}) \tag{8}$$

$$\textit{where}\ \ q, q' \geq 0;\ i_1 < \dots < i_q,\ j_{2s-1} < j_{2s} \textit{ for all } s,$$

$$(j_{2s-1}, j_{2s}) < (j_{2s'-1}, j_{2s'}) \textit{ if }\ s < s'.$$

Let $F\mathcal{G}$ be the group algebra of $\mathcal{G}$ over F. Let $d_{ij} = (y_i, y_j) + 1 \in F\mathcal{G}$. Note that $d_{ij} = d_{ji}$ and $d_{ii} = 0$ for all i, j.

Let I be the two-sided ideal of $F\mathcal{G}$ generated by the set

$$S = \{d_{i_1i_2}d_{i_3i_4} + d_{i_1i_3}d_{i_2i_4} \mid i_1, i_2, i_3, i_4 = 1, 2 \dots\}.$$

The following two lemmas are well known (see, for instance, [14, Lemma 2.1], [15, Example 3.8]); their proofs can also be found in [8].

Lemma 2.5. *For all $u_1, u_2, u_3 \in F\mathcal{G}$, we have $[u_1, u_2, u_3] \in I$.*

Lemma 2.6. *For all $\ell > 0$, we have*

$$\big((y_1, y_2) + 1\big)\big((y_3, y_4) + 1\big) \dots \big((y_{2\ell-1}, y_{2\ell}) + 1\big) \notin I.$$

Since the ideal I is invariant under all permutations of the set $\{y_1, y_2, \dots\}$ of generators of the group $\mathcal{G}$, we have the following.

Corollary 2.7. *Let $\ell > 0$. Then $\big((y_{i_1}, y_{i_2})+1\big) \dots \big((y_{i_{2\ell-1}}, y_{i_{2\ell}})+1\big) \notin I$ if all integers $i_1, i_2, \dots, i_{2\ell}$ are distinct.*

Now we are in a position to complete the proof of Theorem 1.9. Recall that $r = \sum_{i=1}^{k} m_i - 2k + \ell = N_{k,\ell} - 2$ is an even integer. Let $\mathcal{G}_r$ be the subgroup of $\mathcal{G}$ generated by $y_1, \dots, y_r$; let $I_r = I \cap F\mathcal{G}_r$. Take $G = F\mathcal{G}/I$, $H = F\mathcal{G}_r/I_r$. Take $A = G \otimes H$. By Lemma 2.5, we can apply Lemma 2.1 and Corollaries 2.2 and 2.3 to A.

We claim that $[f_1,f_2]\dots[f_{r+1},f_{r+2}]\in I_r$ for all $f_i\in F\mathcal{G}_r$. Indeed, we may assume without loss of generality that $f_i\in\mathcal{G}_r$ for all i. Since

$$\begin{aligned}[f_{2s-1},f_{2s}]&=f_{2s-1}f_{2s}+f_{2s}f_{2s-1}\\ &=f_{2s-1}f_{2s}\big((f_{2s},f_{2s-1})+1\big)=f_{2s-1}f_{2s}\big((f_{2s-1},f_{2s})+1\big)\end{aligned}$$

(recall that F is a field of characteristic 2), we have

$$[f_1,f_2]\dots[f_{r+1},f_{r+2}]=f_1f_2\dots f_{r+2}\big((f_1,f_2)+1\big)\dots\big((f_{r+1},f_{r+2})+1\big).$$

It is clear that, for each s, $(f_{2s-1},f_{2s})=\prod_t c_{i_{st}j_{st}}$ for some commutators $c_{i_{st}j_{st}}=(y_{i_{st}},y_{j_{st}})$. Let $d_{i_{st}j_{st}}=c_{i_{st}j_{st}}+1$; then $c_{i_{st}j_{st}}=d_{i_{st}j_{st}}+1$. We have

$$\begin{aligned}(f_{2s-1},f_{2s})+1&=\prod_t c_{i_{st}j_{st}}+1=\Big(\prod_t(d_{i_{st}j_{st}}+1)\Big)+1\\ &=\prod_t d_{i_{st}j_{st}}+\dots+\sum_{t<t'}d_{i_{st}j_{st}}d_{i_{st'}j_{st'}}+\sum_t d_{i_{st}j_{st}}.\end{aligned}$$

It follows that the product $\big((f_1,f_2)+1\big)\dots\big((f_{r+1},f_{r+2})+1\big)$ can be written as a sum of products of the form

$$d_{q_1q_2}\dots d_{q_{2\ell-1}q_{2\ell}}=\big((y_{q_1},y_{q_2})+1\big)\dots\big((y_{q_{2\ell-1}},y_{q_{2\ell}})+1\big) \tag{9}$$

where $2\ell\ge r+2>r$. Hence, in the product (9) we have $q_t=q_{t'}$ for some $t<t'$.

Note that $d_{j_1j_3}d_{j_2j_3}\in I$ for all j_1,j_2,j_3 because $d_{j_1j_3}d_{j_2j_3}=d_{j_1j_3}d_{j_2j_3}+d_{j_1j_2}d_{j_3j_3}\in S$. Since $d_{ij}=d_{ji}$ for all i,j, we have $d_{i_1i_2}d_{i_3i_4}\in I$ if any two of the indices i_1,i_2,i_3,i_4 coincide. It follows that each product (9) belongs to $I_r=I\cap F\mathcal{G}_r$ and so does the product $\big((f_1,f_2)+1\big)\dots\big((f_{r+1},f_{r+2})+1\big)$. Hence, $[f_1,f_2]\dots[f_{r+1},f_{r+2}]\in I_r$, as claimed. Since $N_{k,\ell}=r+2$, we have $[f_1,f_2]\dots[f_{(N_{k,\ell}-1)},f_{N_{k,\ell}}]\in I_r$ for al $f_i\in F\mathcal{G}_r$.

For any $u\in F\mathcal{G}$, let $\bar u=u+I\in G=F\mathcal{G}/I$. Since one can view the algebra $H=F\mathcal{G}_r/I_r$ as a subalgebra of $G=F\mathcal{G}/I$, we also write $\bar u=u+I_r\in H=F\mathcal{G}_r/I_r$ for $u\in F\mathcal{G}_r$.

By the observation above, $[\bar f_1,\bar f_2]\dots[\bar f_{(N_{k,\ell}-1)},\bar f_{N_{k,\ell}}]=0$ for all $\bar f_i\in H$. Hence, by Corollary 2.2, we have $[u_1,\dots,u_{(1+N_{k,\ell})}]=0$ for all $u_i\in A=G\otimes H$, that is, $T^{(1+N_{k,\ell})}(A)=0$, as required.

Let $\mathcal{P}$, $\mathcal{P}'$, μ and μ' be as in Case 1. Recall that $\mathcal{N}=\sum_{i=1}^k m_i$. Define

$$y_{ij}=y_{\mu(i,j)}\quad\big((i,j)\in\mathcal{P}\big),\qquad\qquad y'_{i'j'}=y_{\mu'(i',j')}\quad\big((i',j')\in\mathcal{P}'\big).$$

Define

$$\begin{aligned}v_{i1}&=\ \bar y_{i1}\otimes 1;\\ v_{ij}&=\ \bar y_{ij}\otimes\bar y'_{ij}\qquad(1\le i\le k;\ 2\le j\le m_i-1);\\ v_{im_i}&=\ \begin{cases}\bar y_{im_i}\otimes 1 & \text{if } m_i \text{ is even;}\\ \bar y_{im_i}\otimes\bar y'_{im_i} & \text{if } m_i \text{ is odd.}\end{cases}\end{aligned}$$

If m_i is even then, by Corollary 2.3,

$$\begin{aligned}&[v_{i1},v_{i2},\dots,v_{im_i}]\\ &=[\bar y_{i1},\bar y_{i2}][\bar y_{i3},\bar y_{i4}]\dots[\bar y_{i(m_i-1)},\bar y_{im_i}]\otimes[\bar y'_{i2},\bar y'_{i3}][\bar y'_{i4},\bar y'_{i5}]\dots[\bar y'_{i(m_i-2)},\bar y'_{i(m_i-1)}]\\ &=\bar y_{i1}\bar y_{i2}\bar y_{i3}\dots\bar y_{im_i}\big((\bar y_{i1},\bar y_{i2})+1\big)\big((\bar y_{i3},\bar y_{i4})+1\big)\dots\big((\bar y_{i(m_i-1)},\bar y_{im_i})+1\big)\\ &\otimes\bar y'_{i2}\bar y'_{i3}\dots\bar y'_{i(m_i-1)}\big((\bar y'_{i2},\bar y'_{i3})+1\big)\big((\bar y'_{i4},\bar y'_{i5})+1\big)\dots\big((\bar y'_{i(m_i-2)},\bar y'_{i(m_i-1)})+1\big)\end{aligned}$$

If m_i is odd then, by the same corollary,

$$\begin{aligned}&[v_{i1},v_{i2},\dots,v_{im_i}]\\ &=[\bar y_{i1},\bar y_{i2}][\bar y_{i3},\bar y_{i4}]\dots[\bar y_{i(m_i-2)},\bar y_{i(m_i-1)}]\bar y_{im_i}\otimes[\bar y'_{i2},\bar y'_{i3}][\bar y'_{i4},\bar y'_{i5}]\dots[\bar y'_{i(m_i-1)},\bar y'_{im_i}]\\ &=\bar y_{i1}\bar y_{i2}\bar y_{i3}\dots\bar y_{i(m_i-1)}\bar y_{im_i}\big((\bar y_{i1},\bar y_{i2})+1\big)\big((\bar y_{i3},\bar y_{i4})+1\big)\dots\big((\bar y_{i(m_i-2)},\bar y_{i(m_i-1)})+1\big)\\ &\otimes\bar y'_{i2}\bar y'_{i3}\dots\bar y'_{i(m_i-1)}\bar y'_{im_i}\big((\bar y'_{i2},\bar y'_{i3})+1\big)\big((\bar y'_{i4},\bar y'_{i5})+1\big)\dots\big((\bar y'_{i(m_i-1)},\bar y'_{im_i})+1\big)\end{aligned}$$

It follows that

$$[v_{11},\dots,v_{1m_1}]\dots[v_{k1},\dots,v_{km_k}]=\bar y\ Q\otimes\bar y'\ Q'$$

where

$$\bar{y} = \prod_{i=1}^{k} \prod_{j=1}^{m_i} \bar{y}_{ij}, \qquad \bar{y}' = \prod_{i'=1}^{k} \prod_{j'=2}^{m'_{i'}} \bar{y}'_{i'j'},$$

$$m'_{i'} = \begin{cases} m_{i'} - 1 & \text{if } m_{i'} \text{ is even;} \\ m_{i'} & \text{if } m_{i'} \text{ is odd,} \end{cases}$$

$$Q = \prod_{i=1}^{k} \prod_{j=1}^{\left[\frac{m_i}{2}\right]} \big((\bar{y}_{i(2j-1)}, \bar{y}_{i(2j)}) + 1 \big), \qquad Q' = \prod_{i'=1}^{k} \prod_{j'=1}^{\left[\frac{m'_{i'}-1}{2}\right]} \big((\bar{y}'_{i'(2j')}, \bar{y}'_{i'(2j'+1)}) + 1 \big).$$

Since μ is injective, all elements $y_{i(2j-1)}, y_{i(2j)}$ $(i = 1, 2, \dots, k;\ j = 1, 2, \dots, \left[\frac{m_i}{2}\right])$ that appear in Q are distinct elements of the set $\{y_1, y_2, \dots\}$. Hence, by Corollary 2.7, we have $Q \neq 0$ in $G = F\mathcal{G}/I$. Similarly, $Q' \neq 0$ in $H = F\mathcal{G}_r/I_r$. Since $\bar{y}$ and $\bar{y}'$ are invertible elements of G and H, respectively, we have $\bar{y}\, Q \otimes \bar{y}'\, Q' \neq 0$, that is,

$$[v_{11}, \dots, v_{1m_1}] \dots [v_{k1}, \dots, v_{km_k}] \neq 0,$$

as required.

This completes the proof of Theorem 1.9. $\square$

Remark. Recall that in the proof of Theorem 1.9 we use the same algebra A that was used in the proof of [8, Theorem 1.4]. Note that in both proofs one can choose the algebra A different from one used in our proofs. For example, let F be any field and let $r = m + n - 4 = 2(m' + n' - 2)$. Let $A = F\langle X\rangle/T^{(3)} \otimes F\langle X_r\rangle/T_r^{(3)}$ where $X_r = \{x_1, \dots, x_r\}$ and $T_r^{(3)} = T^{(3)}\big(F\langle X_r\rangle\big) = T^{(3)} \cap F\langle X_r\rangle$. Then A satisfies the conditions i) and ii) of Theorem 1.9; one can check this using a description of a basis of $F\langle X\rangle/T^{(3)}$ over F. Such a description can be deduced, for instance, from [3, Proposition 3.2] or found (if $char\ F \neq 2$) in [4, Proposition 9].

Our choice of the algebra A in the proof of Theorem 1.9 was made with a purpose to have the paper self-contained.

Acknowledgments

This work was partially supported by CNPq grant 310331/2015-3 and by RFBR grant 15-01-05823. We thank Victor Petrogradsky for pointing out the references [18, 19].

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