# ON THE TAME AUTHOMORPHISM APPROXIMATION, AUGMENTATION TOPOLOGY OF AUTOMORPHISM GROUPS AND Ind-SCHEMES, AND AUTHOMORPHISMS OF TAME AUTOMORPHISM GROUPS 

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#### Abstract

We study topological properties of Ind-groups $\operatorname{Aut}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ and $\operatorname{Aut}\left(K\left\langle x_{1}, \ldots, x_{n}\right\rangle\right)$ of automorphisms of polynomial and free associative algebras via Ind-schemes, toric varieties, approximations, and singularities.

We obtain a number of properties of $\operatorname{Aut}(\operatorname{Aut}(A))$, where $A$ is the polynomial or free associative algebra over the base field $K$. We prove that all Ind-scheme automorphisms of $\operatorname{Aut}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ are inner for $n \geq 3$, and all Ind-scheme automorphisms of $\operatorname{Aut}\left(K\left\langle x_{1}, \ldots, x_{n}\right\rangle\right)$ are semi-inner.

As an application, we prove that $\operatorname{Aut}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ cannot be embedded into $\operatorname{Aut}\left(K\left\langle x_{1}, \ldots, x_{n}\right\rangle\right)$ by the natural abelianization. In other words, the Automorphism Group Lifting Problem has a negative solution.

We explore close connection between the above results and the Jacobian conjecture, as well as the Kanel-Belov - Kontsevich conjecture, and formulate the Jacobian conjecture for fields of any characteristic.

We make use of results of Bodnarchuk and Rips, and we also consider automorphisms of tame groups preserving the origin and obtain a modification of said results in the tame setting.


Dedicated to prof. B.I.Plotkin who initiated the subject

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## 1. Introduction and main results

1.1. Automorphisms of $K\left[x_{1}, \ldots, x_{n}\right]$ and $K\left\langle x_{1}, \ldots, x_{n}\right\rangle$. Let $K$ be a field. The main objects of this study are the $K$-algebra automorphism groups Aut $K\left[x_{1}, \ldots, x_{n}\right]$ and Aut $K\left\langle x_{1}, \ldots, x_{n}\right\rangle$ of the (commutative) polynomial algebra and the free associative algebra with $n$ generators, respectively. The former is equivalent to the group of all polynomial one-to-one mappings of the affine space $\mathbb{A}_{K}^{n}$. Both groups admit a representation as a colimit of algebraic sets of automorphisms filtered by total degree (with morphisms in the direct system given by closed embeddings) which turns them into topological spaces with Zariski topology compatible with the group structure. The two groups carry a power series topology as well, since every automorphism $\varphi$ may be identified with the $n$-tuple $\left(\varphi\left(x_{1}\right), \ldots, \varphi\left(x_{n}\right)\right)$ of the images of generators. This topology plays an especially important role in the applications, and it turns out - as reflected in the main results of this study - that approximation properties arising from this topology agree well with properties of combinatorial nature.

Ind-groups of polynomial automorphisms play a central part in the study of the Jacobian conjecture of O . Keller as well as a number of problems of similar nature. One outstanding
example is provided by a recent conjecture of Kanel-Belov and Kontsevich (B-KKC), 6, 7, which asks whether the group

$$
\operatorname{Sympl}\left(\mathbb{C}^{2 n}\right) \subset \operatorname{Aut}\left(\mathbb{C}\left[x_{1}, \ldots, x_{2 n}\right]\right)
$$

of complex polynomial automorphisms preserving the standard Poisson bracket

$$
\left\{x_{i}, x_{j}\right\}=\delta_{i, n+j}-\delta_{i+n, j}
$$

is isomorphid to the group of automorphisms of the $n$-th Weyl algebra $W_{n}$

$$
\begin{gathered}
W_{n}(\mathbb{C})=\mathbb{C}\left\langle x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n}\right\rangle / I, \\
I=\left(x_{i} x_{j}-x_{j} x_{i}, y_{i} y_{j}-y_{j} y_{i}, y_{i} x_{j}-x_{j} y_{i}-\delta_{i j}\right) .
\end{gathered}
$$

The physical meaning of Kanel-Belov and Kontsevich conjecture is the invariance of the polynomial symplectomorphism group of the phase space under the procedure of deformation quantization.

The B-KKC was conceived during a successful search for a proof of stable equivalence of the Jacobian conjecture and a well-known conjecture of Dixmier stating that $\operatorname{Aut}\left(W_{n}\right)=$ $\operatorname{End}\left(W_{n}\right)$ over any field of characteristic zero. In the papers [6, 7] a particular family of homomorphisms (in effect, monomorphisms) $\operatorname{Aut}\left(W_{n}(\mathbb{C})\right) \rightarrow \operatorname{Sympl}\left(\mathbb{C}^{2 n}\right)$ was constructed, and a natural question whether those homomorphisms were in fact isomorphisms was raised. The aforementioned morphisms, independently studied by Tsuchimoto to the same end, were in actuality defined as restrictions of morphisms of the saturated model of Weyl algebra over an algebraically closed field of positive characteristic - an object which contains $W_{n}(\mathbb{C})$ as a proper subalgebra. One of the defined morphisms turned out to have a particularly simple form over the subgroup of the so-called tame automorphisms, and it was natural to assume that morphism was the desired B-KK isomorphism (at least for the case of algebraically closed base field). Central to the construction is the notion of infinitely large prime number (in the sense of hyperintegers), which arises as the sequence $\left(p_{m}\right)_{m \in \mathbb{N}}$ of positive characteristics of finite fields comprising the saturated model. This leads to the natural problem (7):

Problem. Prove that the B-KK morphism is independent of the choice of the infinite prime $\left(p_{m}\right)_{m \in \mathbb{N}}$.

A general formulation of this question in the paper [7] goes as follows:

[^1]For a commutative ring $R$ define

$$
R_{\infty}=\lim _{\rightarrow}\left(\prod_{p} R^{\prime} \otimes \mathbb{Z} / p \mathbb{Z} / \bigoplus_{p} R^{\prime} \otimes \mathbb{Z} / p \mathbb{Z}\right)
$$

where the direct limit is taken over the filtered system of all finitely generated subrings $R^{\prime} \subset R$ and the product and the sum are taken over all primes $p$. This larger ring possesses a unique "nonstandard Frobenius" endomorphism Fr : $R_{\infty} \rightarrow R_{\infty}$ given by

$$
\left(a_{p}\right)_{\text {primes } p} \mapsto\left(a_{p}^{p}\right)_{\text {primes } p} .
$$

The Kanel-Belov and Kontsevich construction returns a morphism

$$
\psi_{R}: \operatorname{Aut}\left(W_{n}(R)\right) \rightarrow \operatorname{Sympl} R_{\infty}^{2 n}
$$

such that there exists a unique homomorphism

$$
\phi_{R}: \operatorname{Aut}\left(W_{n}\right)(R) \rightarrow \operatorname{Aut}\left(P_{n}\right)\left(R_{\infty}\right)
$$

obeying $\psi_{R}=\operatorname{Fr}_{*} \circ \phi_{R}$. Here $\operatorname{Fr}_{*}: \operatorname{Aut}\left(P_{n}\right)\left(R_{\infty}\right) \rightarrow \operatorname{Aut}\left(P_{n}\right)\left(R_{\infty}\right)$ is the Ind-group homomorphism induced by the Frobenius endomorphism of the coefficient ring, and $P_{n}$ is the commutative Poisson algebra, i.e. the polynomial algebra in $2 n$ variables equipped with additional Poisson structure (so that $\operatorname{Aut}\left(P_{n}(R)\right)$ is just $\operatorname{Sympl}\left(R^{2 n}\right)$ - the group of Poisson structure-preserving automorphisms).

Question. In the above formulation, does the image of $\phi_{R}$ belong to

$$
\operatorname{Aut}\left(P_{n}\right)(i(R) \otimes \mathbb{Q}),
$$

where $i: R \rightarrow R_{\infty}$ is the tautological inclusion? In other words, does there exist a unique homomorphism

$$
\phi_{R}^{c a n}: \operatorname{Aut}\left(P_{n}\right)(R) \rightarrow \operatorname{Aut}\left(P_{n}\right)(R \otimes \mathbb{Q})
$$

such that $\psi_{R}=\mathrm{Fr}_{*} \circ i_{*} \circ \phi_{R}^{c a n}$.
Comparing the two morphisms $\phi$ and $\varphi$ defined using two different free ultrafilters, we obtain a "loop" element $\phi \varphi^{-1}$ of $\operatorname{Aut}_{\operatorname{Ind}}\left(\operatorname{Aut}\left(W_{n}\right)\right.$ ), (i.e. an automorphism which preserves the structure of infinite dimensional algebraic group). Describing this group would provide a solution to this question.

Some progress toward resolution of the B-KKC independence problem has been made recently in [10, 11], although the general unconditional case is still open.

In the spirit of the above we propose the following
Conjecture. All automorphisms of the Ind-group $\operatorname{Sympl}\left(\mathbb{C}^{2 n}\right)$ are inner.
A similar conjecture may be put forward for $\operatorname{Aut}\left(W_{n}(\mathbb{C})\right)$.

We are focused on the investigation of the group $\operatorname{Aut}\left(\operatorname{Aut}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)\right)$ and the corresponding noncommutative (free associative algebra) case. This way of thinking has its roots in the realm of universal algebra and universal algebraic geometry and was conceived in the pioneering work of Boris Plotkin. A more detailed discussion can be found in [5].

Wild automorphisms and the lifting problem. In 2004, the celebrated Nagata conjecture over a field $K$ of characteristic zero was proved by Shestakov and Umirbaev 30, 31 and a stronger version of the conjecture was proved by Umirbaev and Yu 34. Let $K$ be a field of characteristic zero. Every wild $K[z]$-automorphism (wild $K[z]$-coordinate) of $K[z][x, y]$ is wild viewed as a $K$-automorphism ( $K$-coordinate) of $K[x, y, z]$. In particular, the Nagata automorphism $\left(x-2 y\left(y^{2}+x z\right)-\left(y^{2}+x z\right)^{2} z, y+\left(y^{2}+x z\right) z, z\right)$ (Nagata coordinates $x-2 y\left(y^{2}+x z\right)-\left(y^{2}+x z\right)^{2} z$ and $\left.y+\left(y^{2}+x z\right) z\right)$ are wild. In 34], a related question was raised:

The lifting problem. Can an arbitrary wild automorphism (wild coordinate) of the polynomial algebra $K[x, y, z]$ over a field $K$ be lifted to an automorphism (coordinate) of the free associative algebra $K\langle x, y, z\rangle$ ?

In the paper [8], based on the degree estimate [25, 24], it was proved that any wild $z$ automorphism including the Nagata automorphism cannot be lifted as a $z$-automorphism (moreover, in [9] it is proved that every $z$-automorphism of $K\langle x, y, z\rangle$ is stably tame and becomes tame after adding at most one variable). It means that if every automorphism can be lifted, then it provides an obstruction $z^{\prime}$ to $z$-lifting and the question to estimate such an obstruction is naturally raised.

In view of the above, we may ask the following:

The automorphism group lifting problem. Is $\operatorname{Aut}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ isomorphic to a subgroup of $\operatorname{Aut}\left(K\left\langle x_{1}, \ldots, x_{n}\right\rangle\right)$ under the natural abelianization?

The following examples show this problem is interesting and non-trivial.
Example 1. There is a surjective homomorphism (taking the absolute value) from $\mathbb{C}^{*}$ onto $\mathbb{R}^{+}$. But $\mathbb{R}^{+}$is isomorphic to the subgroup $\mathbb{R}^{+}$of $\mathbb{C}^{*}$ under the homomorphism.

Example 2. There is a surjective homomorphism (taking the determinant) from $\mathrm{GL}_{n}(\mathbb{R})$ onto $\mathbb{R}^{*}$. But obviously $\mathbb{R}^{*}$ is isomorphic to the subgroup $\mathbb{R}^{*} I_{n}$ of $\mathrm{GL}_{n}(\mathbb{R})$.

In this paper we prove that the automorphism group lifting problem has a negative answer.

The lifting problem and the automorphism group lifting problem are closely related to the Kanel-Belov and Kontsevich Conjecture (see Section 3.1).

Consider a symplectomorphism $\varphi: x_{i} \mapsto P_{i}, y_{i} \mapsto Q_{i}$. It can be lifted to some automorphism $\hat{\varphi}$ of the quantized algebra $W_{\hbar}[[\hbar]]$ :

$$
\widehat{\varphi}: x_{i} \mapsto P_{i}+P_{i}^{1} \hbar+\cdots+P_{i}^{m} \hbar^{m} ; y_{i} \mapsto Q_{i}+Q_{i}^{1} \hbar+\cdots+Q_{i}^{m} \hbar^{m}
$$

The point is to choose a lift $\widehat{\varphi}$ in such a way that the degree of all $P_{i}^{m}, Q_{i}^{m}$ would be bounded. If that is true, then the B-KKC follows.
1.2. Main results. The main results of this paper are as follows.

Theorem 1.1. Any Ind-scheme automorphism $\varphi$ of $\operatorname{NAut}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ for $n \geq 3$ is inner, i.e. is a conjugation via some automorphism.

Theorem 1.2. Any Ind-scheme automorphism $\varphi$ of $\operatorname{NAut}\left(K\left\langle x_{1}, \ldots, x_{n}\right\rangle\right)$ for $n \geq 3$ is semi-inner (see definition 1.6).

NAut denotes the group of nice automorphisms, i.e. automorphisms which can be approximated by tame ones (definition 3.1). In characteristic zero case every automorphism is nice.

For the group of automorphisms of a semigroup a number of similar results on settheoretical level was obtained previously by Kanel-Belov, Lipyanski and Berzinsh [4, 5]. All these questions (including Aut(Aut) investigation) take root in the realm of Universal Algebraic Geometry and were proposed by Boris Plotkin. Equivalence of two algebras having the same generalized identities and isomorphism of first order means semi-inner properties of automorphisms (see [4, 5] for details).

Automorphisms of tame automorphism groups. Regarding the tame automorphism group, something can be done on the group- theoretic level. In the paper of H. Kraft and I. Stampfli [23] the automorphism group of the tame automorphism group of the polynomial algebra was thoroughly studied. In that paper, conjugation of elementary automorphisms via translations played an important role. The results of our study are different. We describe the group $\operatorname{Aut}\left(\mathrm{TAut}_{0}\right)$ of the group $\mathrm{TAut}_{0}$ of tame automorphisms preserving the
origin (i.e. taking the augmentation ideal onto an ideal which is a subset of the augmentation ideal). This is technically more difficult, and will be universally and systematically done for both commutative (polynomial algebra) case and noncommutative (free associative algebra) case. We observe a few problems in the shift conjugation approach for the noncommutative (free associative algebra) case, as it was for commutative case in [23]. Any evaluation on a ground field element can return zero, for example in Lie polynomial $[[x, y], z]$. Note that the calculations of $\operatorname{Aut}\left(\mathrm{TAut}_{0}\right)\left(\right.$ resp. $\left.\mathrm{Aut}_{\text {Ind }}\left(\mathrm{TAut}_{0}\right), \mathrm{Aut}_{\text {Ind }}\left(\mathrm{Aut}_{0}\right)\right)$ imply also the same results for $\operatorname{Aut}(T A u t)$ (resp. Aut ${ }_{\text {Ind }}(T A u t)$, Aut $\left._{\text {Ind }}(A u t)\right)$ according to the approach of this article via stabilization by the torus action.

Theorem 1.3. Any automorphism $\varphi$ of $\operatorname{TAut}_{0}\left(K\left[x_{1}, \ldots, x_{n}\right]\right.$ ) (in the group-theoretic sense) for $n \geq 3$ is inner, i.e. is a conjugation via some automorphism.

Theorem 1.4. The group $\operatorname{TAut}_{0}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ is generated by the automorphism

$$
x_{1} \mapsto x_{1}+x_{2} x_{3}, x_{i} \mapsto x_{i}, \quad i \neq 1
$$

and linear substitutions if $\operatorname{Char}(K) \neq 2$ and $n>3$.
Let $G_{N} \subset \operatorname{TAut}\left(K\left[x_{1}, \ldots, x_{n}\right]\right), E_{N} \subset \operatorname{TAut}\left(K\left\langle x_{1}, \ldots, x_{n}\right\rangle\right)$ be tame automorphism subgroups preserving the $N$-th power of the augmentation ideal.

Theorem 1.5. Any automorphism $\varphi$ of $G_{N}$ (in the group-theoretic sense) for $N \geq 3$ is inner, i.e. is given by a conjugation via some automorphism.

Definition 1.6. An anti-automorphism $\Psi$ of a $K$-algebra $B$ is a vector space automorphism such that $\Psi(a b)=\Psi(b) \Psi(a)$. For instance, transposition of matrices is an antiautomorphism. An anti-automorphism of the free associative algebra $A$ is a mirror antiautomorphism if it sends $x_{i} x_{j}$ to $x_{j} x_{i}$ for some fixed $i$ and $j$. If a mirror anti-automorphism $\theta$ acts identical on all generators $x_{i}$, then for any monomial $x_{i_{1}} \cdots x_{i_{k}}$ we have

$$
\theta\left(x_{i_{1}} \cdots x_{i_{k}}\right)=x_{i_{k}} \cdots x_{i_{1}} .
$$

Such an anti-automorphism will be generally referred to as the mirror anti-automorphism.
An automorphism of $\operatorname{Aut}(A)$ is semi-inner if it can be expressed as a composition of an inner automorphism and a conjugation by a mirror anti-automorphism.

Theorem 1.7. a) Any automorphism $\varphi$ of $\operatorname{TAut}_{0}\left(K\left\langle x_{1}, \ldots, x_{n}\right\rangle\right)$ and also
$\operatorname{TAut}\left(K\left\langle x_{1}, \ldots, x_{n}\right\rangle\right)$ (in the group-theoretic sense) for $n \geq 4$ is semi-inner, i.e. is a conjugation via some automorphism and/or mirror anti-automorphism.
b) The same is true for $E_{n}, n \geq 4$.

The case of $\operatorname{TAut}(K\langle x, y, z\rangle)$ is substantially more difficult. We can treat it only on Ind-scheme level, but even then it is the most technical part of the paper (see section 5.2). For the two-variable case a similar proposition is probably false.

Theorem 1.8. a) Let $\operatorname{Char}(K) \neq 2$. Then $\operatorname{Aut}_{\operatorname{Ind}}(\operatorname{TAut}(K\langle x, y, z\rangle)$ ) (resp.
$\operatorname{Aut}_{\mathrm{Ind}}\left(\mathrm{TAut}_{0}(K\langle x, y, z\rangle)\right)$ ) is generated by conjugation by an automorphism or a mirror anti-automorphism.
b) The same is true for $\operatorname{Aut}_{\text {Ind }}\left(E_{3}\right)$.

By TAut we denote the tame automorphism group, Aut $_{\text {Ind }}$ is the group of Ind-scheme automorphisms (see section (2.2).

Approximation allows us to formulate the celebrated Jacobian conjecture for any characteristic.

Lifting of the automorphism groups. In this article we prove that the automorphism group of polynomial algebra over an arbitrary field $K$ cannot be embedded into the automorphism group of free associative algebra induced by the natural abelianization.

Theorem 1.9. Let $K$ be an arbitrary field, $G=\operatorname{Aut}_{0}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ and $n>2$. Then $G$ cannot be isomorphic to any subgroup $H$ of $\operatorname{Aut}\left(K\left\langle x_{1}, \ldots, x_{n}\right\rangle\right)$ induced by the natural abelianization. The same is true for $\operatorname{NAut}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$.

## 2. Varieties of automorphisms

2.1. Elementary and tame automorphisms. Let $P$ be a polynomial that is independent of $x_{i}$ with $i$ fixed. An automorphism

$$
x_{i} \mapsto x_{i}+P, x_{j} \mapsto x_{j} \text { for } i \neq j
$$

is called elementary. The group generated by linear automorphisms and elementary ones for all possible $P$ is called the tame automorphism group (or subgroup) TAut and elements of TAut are tame automorphisms.

### 2.2. Ind-schemes and Ind-groups.

Definition 2.1. An Ind-variety $M$ is the direct limit of algebraic varieties $M=\underset{\longrightarrow}{\lim }\left\{M_{1} \subseteq\right.$ $\left.M_{2} \cdots\right\}$. An Ind-scheme is an Ind-variety which is a group such that the group inversion is a morphism $M_{i} \rightarrow M_{j(i)}$ of algebraic varieties, and the group multiplication induces a morphism from $M_{i} \times M_{j}$ to $M_{k(i, j)}$. A map $\varphi$ is a morphism of an Ind-variety $M$ to
an Ind-variety $N$, if $\varphi\left(M_{i}\right) \subseteq N_{j(i)}$ and the restriction $\varphi$ to $M_{i}$ is a morphism for all $i$. Monomorphisms, epimorphisms and isomorphisms are defined similarly in a natural way.

Example. $M$ is the group of automorphisms of the affine space, and $M_{j}$ are the sets of all automorphisms in $M$ with degree $\leq j$.

There is an interesting
Problem. Investigate growth functions of Ind-varieties. For example, the dimension of varieties of polynomial automorphisms of degree $\leq n$.

Note that coincidence of growth functions of $\operatorname{Aut}\left(W_{n}(\mathbb{C})\right)$ and $\operatorname{Sympl}\left(\mathbb{C}^{2 n}\right)$ would imply the Kanel-Belov - Kontsevich conjecture [7].

Definition 2.2. The ideal $I$ generated by variables $x_{i}$ is called the augmentation ideal. For a fixed positive integer $N>1$, the augmentation subgroup $H_{N}$ is the group of all automorphisms $\varphi$ such that $\varphi\left(x_{i}\right) \equiv x_{i} \bmod I^{N}$. The larger group $\hat{H}_{N} \supset H_{N}$ is the group of automorphisms whose linear part is scalar, and $\varphi\left(x_{i}\right) \equiv \lambda x_{i} \bmod I^{N}(\lambda$ does not depend on $i$ ). We often say an arbitrary element of the group $\hat{H}_{N}$ is an automorphism that is homothety modulo (the $N$-th power of) the augmentation ideal.

## 3. The Jacobian conjecture in any characteristic, Kanel-Belov -

Kontsevich conjecture, and approximation
3.1. Approximation problems and Kanel-Belov - Kontsevich Conjecture. Let us give formulation of the Kanel-Belov - Kontsevich Conjecture:

$$
B-K K C_{n}: \operatorname{Aut}\left(W_{n}\right) \simeq \operatorname{Sympl}\left(\mathbb{C}^{2 n}\right)
$$

A similar conjecture can be stated for endomorphisms

$$
B-K K C_{n}: \operatorname{End}\left(W_{n}\right) \simeq \operatorname{Sympl} \operatorname{End}\left(\mathbb{C}^{2 n}\right) .
$$

If the Jacobian conjecture $J C_{2 n}$ is true, then the respective conjunctions over all $n$ of the two conjectures are equivalent.

It is natural to approximate automorphisms by tame ones. There exists such an approximation up to terms of any order for polynomial automorphisms as well as Weyl algebra automorphisms, symplectomorphisms etc. However, the naive approach fails.

It is known that $\operatorname{Aut}\left(W_{1}\right) \equiv \operatorname{Aut}_{1}(K[x, y])$ where $\operatorname{Aut}_{1}$ stands for the subgroup of automorphisms of Jacobian determinant one. However, considerations from [28] show that Lie algebra of the first group is the algebra of derivations of $W_{1}$ and thus possesses
no identities apart from the ones of the free Lie algebra, another coincidence of the vector fields which diverge to zero, and has polynomial identities. These cannot be isomorphic [6, 7]. In other words, this group has two coordinate system non-smooth with respect to one another (but integral with respect to one another). One system is built from the coefficients of differential operators in a fixed basis of generators, while its counterpart is provided by the coefficients of polynomials, which are images of the basis $\tilde{x}_{i}, \tilde{y}_{i}$.

In the paper [28] functionals on $\mathfrak{m} / \mathfrak{m}^{2}$ were considered in order to define the Lie algebra structure. In the spirit of that we have the following

Conjecture. The natural limit of $\mathfrak{m} / \mathfrak{m}^{2}$ is zero.

It means that the definition of the Lie algebra admits some sort of functoriality problem and it depends on the presentation of (reducible) Ind-scheme.

In his remarkable paper, Yu. Bodnarchuk [16] established Theorem 1.1]by using Shafarevich's results for the tame automorphism subgroup and for the case when the Ind-scheme automorphism is regular in the sense that it sends coordinate functions to coordinate functions. In this case the tame approximation works (as well as for the symplectic case), and the corresponding method is similar to ours. We present it here in order to make the text more self-contained, as well as for the purpose of tackling the noncommutative (that is, the free associative algebra) case. Note that in general, for regular functions, if the Shafarevich-style approximation were valid, then the Kanel-Belov - Kontsevich conjecture would follow directly, which is absurd.

In the sequel, we do not assume regularity in the sense of [16] but only assume that the restriction of a morphism on any subvariety is a morphism again. Note that morphisms of Ind-schemes $\operatorname{Aut}\left(W_{n}\right) \rightarrow \operatorname{Sympl}\left(\mathbb{C}^{2 n}\right)$ have this property, but are not regular in the sense of Bodnarchuk [16].

We use the idea of singularity which allows us to prove the augmentation subgroup structure preservation, so that the approximation works in this case.

Consider the isomorphism $\operatorname{Aut}\left(W_{1}\right) \cong \operatorname{Aut}_{1}(K[x, y])$. It has a strange property. Let us add a small parameter $t$. Then an element arbitrary close to zero with respect to $t^{k}$ does not go to zero arbitrarily, so it is impossible to make tame limit! There is a sequence of convergent product of elementary automorphisms, which is not convergent under this isomorphism. Exactly the same situation happens for $W_{n}$. These effects cause problems in perturbative quantum field theory.
3.2. The Jacobian conjecture in any characteristic. Recall that the Jacobian conjecture in characteristic zero states that any polynomial endomorphism

$$
\varphi: K^{n} \rightarrow K^{n}
$$

with constant Jacobian is globally invertible.
A naive attempt to directly transfer this formulation to positive characteristic fails because of the counterexample $x \mapsto x-x^{p}(p=$ Char $K)$, whose Jacobian is everywhere 1 but which is evidently not invertible. Approximation provides a way to formulate a suitable generalization of the Jacobian conjecture to any characteristic and put it in a framework of other questions.

Definition 3.1. An endomorphism $\varphi \in \operatorname{End}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ is good if for any $m$ there exist $\psi_{m} \in \operatorname{End}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ and $\phi_{m} \in \operatorname{Aut}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ such that

- $\varphi=\psi_{m} \phi_{m}$
- $\psi_{m}\left(x_{i}\right) \equiv x_{i} \bmod \left(x_{1}, \ldots, x_{n}\right)^{m}$.

An automorphism $\varphi \in \operatorname{Aut}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ is nice if for any $m$ there exist $\psi_{m} \in$ $\operatorname{Aut}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ and $\phi_{m} \in \operatorname{TAut}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ such that

- $\varphi=\psi_{m} \phi_{m}$
- $\psi_{m}\left(x_{i}\right) \equiv x_{i} \bmod \left(x_{1}, \ldots, x_{n}\right)^{m}$, i.e. $\psi_{m} \in H_{m}$.

Anick [1] has shown that if $\operatorname{Char}(K)=0$, any automorphism is nice. However, this is unclear in positive characteristic.

Question. Is any automorphism over arbitrary field nice?
Ever good automorphism has Jacobian 1, and all such automorphisms are good - and even nice - when Char $(K)=0$. This observation allows for the following question to be considered a generalization of the Jacobian conjecture to positive characteristic.

The Jacobian conjecture in any characteristic: Is any good endomorphism over arbitrary field an automorphism?

Similar notions can be formulated for the free associative algebra. That justifies the following

Question. Is any automorphism of free associative algebra over arbitrary field nice?

Question (version of free associative positive characteristic case of JC). Is any good endomorphism of the free associative algebra over arbitrary field an automorphism?
3.3. Approximation for the automorphism group of affine spaces. Approximation is the most important tool utilized in this paper. In order to perform it, we have to prove that $\varphi \in \operatorname{Aut}_{\text {Ind }}\left(\operatorname{Aut}_{0}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)\right.$ preserves the structure of the augmentation subgroup.

The proof method utilized in theorems below works for commutative associative and free associative case. It is a problem of considerable interest to develop similar statements for automorphisms of other associative algebras, such as the commutative Poisson algebra (for which the Aut functor returns the group of polynomial symplectomorphisms); however, the situation there is somewhat more difficult.

The following two theorems, for the commutative and the free associative cases, respectively, constitute the foundation of the approximation technique.

Theorem 3.2. Let $\varphi \in \operatorname{Aut}_{\text {Ind }}\left(\operatorname{Aut}_{0}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)\right)$ and let $H_{N} \subset \operatorname{Aut}_{0}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ be the subgroup of automorphisms which are identity modulo the ideal $\left(x_{1}, \ldots, x_{n}\right)^{N}(N>1)$. Then $\varphi\left(H_{N}\right) \subseteq H_{N}$.
 of automorphisms which are identity modulo the ideal $\left(x_{1}, \ldots, x_{n}\right)^{N}$. Then $\varphi\left(H_{N}\right) \subseteq H_{N}$.

Corollary 3.4. In both commutative and free associative cases under the assumptions above one has $\varphi=\mathrm{Id}$.

Proof. Every automorphism can be approximated via the tame ones, i.e. for any $\psi$ and any $N$ there exists a tame automorphism $\psi_{N}^{\prime}$ such that $\psi \psi_{N}^{\prime}{ }^{-1} \in H_{N}$.

The main point therefore is why $\varphi\left(H_{N}\right) \subseteq H_{N}$ whenever $\varphi$ is and Ind-automorphism.

## Proof of Theorem 3.2.

The method of proof is based upon the following useful fact from algebraic geometry:

Lemma 3.5. Let

$$
\varphi: X \rightarrow Y
$$

be a morphism of affine varieties, and let $A(t) \subset X$ be a curve (or rather, a one-parameter family of points) in $X$. Suppose that $A(t)$ does not tend to infinity as $t \rightarrow 0$. Then the image $\varphi A(t)$ under $\varphi$ also does not tend to infinity as $t \rightarrow 0$.

The proof is straightforward and is left to the reader.
We now put the above fact to use. For $t>0$ let

$$
\hat{A}(t): \mathbb{A}_{K}^{n} \rightarrow \mathbb{A}_{K}^{n}
$$

be a one-parameter family of invertible linear transformations of the affine space preserving the origin. To that corresponds a curve $A(t) \subset \operatorname{Aut}_{0}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ of polynomial automorphisms whose points are linear substitutions. Suppose that, as $t$ tends to zero, the $i$-th eigenvalue of $A(t)$ also tends to zero as $t^{k_{i}}, k_{i} \in \mathbb{N}$. Such a family will always exist.

Suppose now that the degrees $\left\{k_{i}, i=1, \ldots n\right\}$ of singularity of eigenvalues at zero are such that for every pair $(i, j)$, if $k_{i} \neq k_{j}$, then there exists a positive integer $m$ such that

$$
\text { either } k_{i} m \leq k_{j} \text { or } k_{j} m \leq k_{i} .
$$

The largest such $m$ we will call the order of $A(t)$ at $t=0$. As $k_{i}$ are all set to be positive integer, the order equals $\frac{k_{\text {max }}}{k_{\text {min }}}$.

Let $M \in \operatorname{Aut}_{0}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ be a polynomial automorphism.
Lemma 3.6. The curve $A(t) M A(t)^{-1}$ has no singularity at zero for any $A(t)$ of order $\leq N$ if and only if $M \in \hat{H}_{N}$, where $\hat{H}_{N}$ is the subgroup of automorphisms which are homothety modulo the augmentation ideal.

Proof. The 'If' part is elementary, for if $M \in \hat{H}_{N}$, the action of $A(t) M A(t)^{-1}$ upon any generator $x_{i}$ (with $i$ fixed) 2 is given by

$$
\begin{array}{r}
A(t) M A(t)^{-1}\left(x_{i}\right)=\lambda x_{i}+t^{-k_{i}} \sum_{l_{1}+\cdots+l_{n}=N} a_{l_{1} \ldots l_{n}} t^{k_{1} l_{1}+\cdots+k_{n} l_{n}} x_{1}^{l_{1}} \cdots x_{n}^{l_{n}}+ \\
+S_{i}\left(t, x_{1}, \ldots, x_{n}\right),
\end{array}
$$

where $\lambda$ is the homothety ratio of (the linear part of) $M$ and $S_{i}$ is polynomial in $x_{1}, \ldots, x_{n}$ of total degree greater than $N$. Now, for any choice of $l_{1}, \ldots, l_{n}$ in the sum, the expression

$$
k_{1} l_{1}+\cdots+k_{n} l_{n}-k_{i} \geq k_{\min } \sum l_{j}-k_{i}=k_{\min } N-k_{i} \geq 0
$$

for every $i$, so whenever $t$ goes to zero, the coefficient will not blow up to infinity. Obviously the same argument applies to higher-degree monomials within $S_{i}$.

[^2]The other direction is slightly less elementary; assuming that $M \notin \hat{H}_{N}$, we need to show that there is a curve $A(t)$ such that conjugation of $M$ by it produces a singularity at zero. We distinguish between two cases.

Case 1. The linear part $\bar{M}$ of $M$ is not a scalar matrix. Then - after a suitable basis change (see the footnote) - it is not a diagonal matrix and has a non-zero entry in the position $(i, j)$. Consider a diagonal matrix $A(t)=D(t)$ such that on all positions on the main diagonal except $j$-th it has $t^{k_{i}}$ and on $j$-th position it has $t^{k_{j}}$. Then $D(t) \bar{M} D^{-1}(t)$ has $(i, j)$ entry with the coefficient $t^{k_{i}-k_{j}}$ and if $k_{j}>k_{i}$ it has a singularity at $t=0$.

Let also $k_{i}<2 k_{j}$. Then the non-linear part of $M$ does not produce singularities and cannot compensate the singularity of the linear part.

Case 2. The linear part $\bar{M}$ of $M$ is a scalar matrix. Then conjugation cannot produce singularities in the linear part and we as before are interested in the smallest non-linear term. Let $M \in H_{N} \backslash H_{N+1}$. Performing a basis change if necessary, we may assume that

$$
\varphi\left(x_{1}\right)=\lambda x_{1}+\delta x_{2}^{N}+S,
$$

where $S$ is a sum of monomials of degree $\geq N$ with coefficients in $K$.
Let $A(t)=D(t)$ be a diagonal matrix of the form $\left(t^{k_{1}}, t^{k_{2}}, t^{k_{1}}, \ldots, t^{k_{1}}\right)$ and let $(N+1)$. $k_{2}>k_{1}>N \cdot k_{2}$. Then in $A^{-1} M A$ the term $\delta x_{2}^{N}$ will be transformed into $\delta x_{2}^{N} t^{N k_{2}-k_{1}}$, and all other terms are multiplied by $t^{l k_{2}+s k_{1}-k_{1}}$ with $(l, s) \neq(1,0)$ and $l, s>0$. In this case $l k_{2}+s k_{1}-k_{1}>0$ and we are done with the proof of Lemma 3.6,

The next lemma is proved by direct computation. Recall that for $m>1$, the group $G_{m}$ is defined as the group of all tame automorphisms preserving the $m$-th power of the augmentation ideal.

## Lemma 3.7.

a) $\left[G_{m}, G_{m}\right] \subset H_{m}, m>2$. There exist elements
$\varphi \in H_{m+k-1} \backslash H_{m+k}, \quad \psi_{1} \in G_{k}, \quad \psi_{2} \in G_{m}$, such that $\varphi=\left[\psi_{1}, \psi_{2}\right]$.
b) $\left[H_{m}, H_{k}\right] \subset H_{m+k-1}$.
c) Let $\varphi \in G_{m} \backslash H_{m}, \psi \in H_{k} \backslash H_{k+1}, k>m$. Then $[\varphi, \psi] \in H_{k} \backslash H_{k+1}$.

Proof. a) Consider elementary automorphisms

$$
\begin{aligned}
& \psi_{1}: x_{1} \mapsto x_{1}+x_{2}^{k}, \quad x_{2} \mapsto x_{2}, \quad x_{i} \mapsto x_{i}, i>2, \\
& \psi_{2}: x_{1} \mapsto x_{1}, \quad x_{2} \mapsto x_{2}+x_{1}^{m}, \quad x_{i} \mapsto x_{i}, i>2 .
\end{aligned}
$$

Set $\varphi=\left[\psi_{1}, \psi_{2}\right]=\psi_{1}^{-1} \psi_{2}^{-1} \psi_{1} \psi_{2}$.
Then

$$
\begin{gathered}
\varphi: x_{1} \mapsto x_{1}-x_{2}^{k}+\left(x_{2}-\left(x_{1}-x_{2}^{k}\right)^{m}\right)^{k}, \\
x_{2} \mapsto x_{2}-\left(x_{1}-x_{2}^{k}\right)^{m}+\left(x_{1}-x_{2}^{k}+\left(x_{2}-\left(x_{1}-x_{2}^{k}\right)^{m}\right)^{k}\right)^{m}, \quad x_{i} \mapsto x_{i}, i>2 .
\end{gathered}
$$

It is easy to see that if either $k$ or $m$ is relatively prime with $\operatorname{Char}(K)$, then not all terms of degree $k+m-1$ vanish. Thus $\varphi \in H_{m+k-1} \backslash H_{m+k}$.

Now suppose that $\operatorname{Char}(K) \nmid m$, then obviously $m-1$ is relatively prime with $\operatorname{Char}(K)$. Consider the mappings

$$
\begin{gathered}
\psi_{1}: x_{1} \mapsto x_{1}+x_{2}^{k}, \quad x_{2} \mapsto x_{2}, \quad x_{i} \mapsto x_{i}, i>2 \\
\psi_{2}: x_{1} \mapsto x_{1}, \quad x_{2} \mapsto x_{2}+x_{1}^{m-1} x_{3}, \quad x_{i} \mapsto x_{i}, i>2 .
\end{gathered}
$$

Set again $\varphi^{\prime}=\left[\psi_{1}, \psi_{2}\right]=\psi_{1}^{-1} \psi_{2}^{-1} \psi_{1} \psi_{2}$. Then $\varphi^{\prime}$ acts as

$$
\begin{gathered}
x_{1} \mapsto x_{1}-x_{2}^{k}+\left(x_{2}-\left(x_{1}-x_{2}^{k}\right)^{m-1} x_{3}\right)^{k}= \\
=x_{1}-k\left(x_{1}-x_{2}^{k}\right)^{m-1} x_{2}^{k-1} x_{3}+S \\
x_{2} \mapsto x_{2}-\left(x_{1}-x_{2}^{k}\right)^{m-1} x_{3}+\left(x_{1}-x_{2}^{k}+\left(x_{2}-\left(x_{1}-x_{2}^{k}\right)^{m-1} x_{3}\right)^{k}\right)^{m-1} x_{3} \\
x_{i} \mapsto x_{i}, \quad i>2
\end{gathered}
$$

here $S$ stands for a sum of terms of degree $\geq m+k$. Again we see that $\varphi \in H_{m+k-1} \backslash H_{m+k}$.
b) Let

$$
\psi_{1}: x_{i} \mapsto x_{i}+f_{i} ; \psi_{2}: x_{i} \mapsto x_{i}+g_{i},
$$

for $i=1, \ldots, n$; here $f_{i}$ and $g_{i}$ do not have monomials of degree less than or equal to $m$ and $k$, respectively. Then, modulo terms of degree $\geq m+k$, we have $\psi_{1} \psi_{2}: x_{i} \mapsto$ $x_{i}+f_{i}+g_{i}+\frac{\partial f_{i}}{\partial x_{j}} g_{j}$, so that modulo terms of degree $\geq m+k-1$ we get $\psi_{1} \psi_{2}: x_{i} \mapsto x_{i}+f_{i}+g_{i}$ and $\psi_{2} \psi_{1}: x_{i} \mapsto x_{i}+f_{i}+g_{i}$. Therefore $\left[\psi_{1}, \psi_{2}\right] \in H_{m+k-1}$.
c) If $\varphi\left(I^{m}\right) \subseteq I^{m}$ and

$$
\psi:\left(x_{1}, \ldots, x_{n}\right) \mapsto\left(x_{1}+g_{1}, \ldots, x_{n}+g_{n}\right)
$$

is such that for some $i_{0}$ the polynomial $g_{i_{0}}$ contains a monomial of total degree $k$ (and all $g_{i}$ do not contain monomials of total degree less than $k$ ), then, by evaluating the composition of automorphisms directly, one sees that the commutator is given by

$$
[\varphi, \psi]:\left(x_{1}, \ldots, x_{n}\right) \mapsto\left(x_{1}+g_{1}+S_{1}, \ldots, x_{n}+g_{n}+S_{n}\right)
$$

with $S_{i}$ containing no monomials of total degree $<k+1$. Then the image of $x_{i_{0}}$ is $x_{i_{0}}$ modulo polynomial of height $k$.

Corollary 3.8. Let $\Psi \in \operatorname{Aut}_{\text {Ind }}\left(\operatorname{NAut}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)\right)$. Then
$\Psi\left(G_{n}\right)=G_{n}, \Psi\left(H_{n}\right)=H_{n}$.

Corollary 3.8 together with Proposition 4.3 of the next section imply Theorem 3.2, for every nice automorphism, by definition, can be approximated by tame ones. Note that in characteristic zero every automorphism is nice (Anick's theorem).

### 3.4. Lifting of automorphism groups.

3.4.1. Lifting of automorphisms from $\operatorname{Aut}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ to $\operatorname{Aut}\left(K\left\langle x_{1}, \ldots, x_{n}\right\rangle\right)$.

Definition 3.9. In the sequel, we call an action of the $n$-dimensional torus $\mathbb{T}^{n}$ on $\mathbb{K}\left\langle x_{1}, \ldots, x_{n}\right\rangle$ (the number of generators coincides with the dimension of the torus) linearizable if it is conjugate to the standard diagonal action given by

$$
\left(\lambda_{1}, \ldots, \lambda_{n}\right):\left(x_{1}, \ldots, x_{n}\right) \mapsto\left(\lambda_{1} x_{1}, \ldots, \lambda_{n} x_{n}\right) .
$$

The following result is a direct free associative analogue of a well-known theorem of Białynicki-Birula [14, 15]. We will make frequent reference of the classical (commutative) case as well, which appears as Theorem 4.1 in the text.

Theorem 3.10. Any effective action of the $n$-torus on $\mathbb{K}\left\langle x_{1}, \ldots, x_{n}\right\rangle$ is linearizable.

The proof is somewhat similar to that of Theorem 4.1, with a few modifications. We will address this issue in the upcoming paper [22].

As a corollary of the above theorem, we get
Proposition 3.11. Let $T^{n}$ denote the standard torus action on $K\left[x_{1}, \ldots, x_{n}\right]$. Let $\widehat{T}^{n}$ denote its lifting to an action on the free associative algebra $K\left\langle x_{1}, \ldots, x_{n}\right\rangle$. Then $\widehat{T}^{n}$ is also given by the standard torus action.

Proof. Consider the roots $\widehat{x_{i}}$ of this action. They are liftings of the coordinates $x_{i}$. We have to prove that they generate the whole associative algebra.

Due to the reducibility of this action, all elements are product of eigenvalues of this action. Hence it is enough to prove that eigenvalues of this action can be presented as a linear combination of this action. This can be done along the lines of Biatynicki-Birula [15. Note that all propositions of the previous section hold for the free associative algebra. Proof of Theorem 3.3 is similar. Hence we have the following

Theorem 3.12. Any Ind-scheme automorphism $\varphi$ of $\operatorname{Aut}\left(K\left\langle x_{1}, \ldots, x_{n}\right\rangle\right)$ for $n \geq 3$ is inner, i.e. is a conjugation by some automorphism.

We therefore see that the group lifting (in the sense of isomorphism induced by the natural abelianization) implies the analogue of Theorem 3.2,

This also implies that any automorphism group lifting, if exists, satisfies the approximation properties.

Proposition 3.13. Suppose

$$
\Psi: \operatorname{Aut}\left(K\left[x_{1}, \ldots, x_{n}\right]\right) \rightarrow \operatorname{Aut}\left(K\left\langle z_{1}, \ldots, z_{n}\right\rangle\right)
$$

is a group homomorphism such that its composition with the natural map $\operatorname{Aut}\left(K\left\langle z_{1}, \ldots, z_{n}\right\rangle\right) \rightarrow$ $\operatorname{Aut}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ (induced by the projection $\left.K\left\langle z_{1}, \ldots, z_{n}\right\rangle \rightarrow K\left[x_{1}, \ldots, x_{n}\right]\right)$ is the identity map. Then
(1) After a coordinate change $\Psi$ provides a correspondence between the standard torus actions $x_{i} \mapsto \lambda_{i} x_{i}$ and $z_{i} \mapsto \lambda_{i} z_{i}$.
(2) Images of elementary automorphisms

$$
x_{j} \mapsto x_{j}, j \neq i, \quad x_{i} \mapsto x_{i}+f\left(x_{1}, \ldots, \widehat{x}_{i}, \ldots, x_{n}\right)
$$

are elementary automorphisms of the form

$$
z_{j} \mapsto z_{j}, j \neq i, \quad z_{i} \mapsto z_{i}+f\left(z_{1}, \ldots, \widehat{z_{i}}, \ldots, z_{n}\right)
$$

(Hence image of tame automorphism is tame automorphism).
(3) $\psi\left(H_{n}\right)=G_{n}$. Hence $\psi$ induces a map between the completion of the groups of $\operatorname{Aut}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ and $\operatorname{Aut}\left(K\left\langle z_{1}, \ldots, z_{n}\right\rangle\right)$ with respect to the augmentation subgroup structure.

## Proof of Theorem 1.9

Any automorphism (including wild automorphisms such as the Nagata example) can be approximated by a product of elementary automorphisms with respect to augmentation topology. In the case of the Nagata automorphism corresponding to

$$
\operatorname{Aut}\left(K\left\langle x_{1}, \ldots, x_{n}\right\rangle\right),
$$

all such elementary automorphisms fix all coordinates except $x_{1}$ and $x_{2}$. Because of (2) and (3) of Proposition 3.13, the lifted automorphism would be an automorphism induced by an automorphism of $K\left\langle x_{1}, x_{2}, x_{3}\right\rangle$ fixing $x_{3}$. However, it is impossible to lift the Nagata
automorphism to such an automorphism due to the main result of 8 . Therefore, Theorem 1.9 is proved.

## 4. Automorphisms of the polynomial algebra and the approach of Bodnarchuk-Rips

Let $\Psi \in \operatorname{Aut}\left(\operatorname{Aut}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)\right)\left(\operatorname{resp} . \operatorname{Aut}\left(\operatorname{TAut}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)\right)\right.$, $\left.\operatorname{Aut}\left(\operatorname{TAut}_{0}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)\right), \operatorname{Aut}\left(\operatorname{Aut}_{0}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)\right)\right)$.
4.1. Reduction to the case when $\Psi$ is identical on $\mathrm{SL}_{n}$. We follow [23] and [16] using the classical theorem of Biatynicki-Birula [14, 15]:

Theorem 4.1 (Białynicki-Birula). Any effective action of torus $\mathbb{T}^{n}$ on $\mathbb{C}^{n}$ is linearizable (recall the definition 3.9).

Remark. An effective action of $\mathbb{T}^{n-1}$ on $\mathbb{C}^{n}$ is linearizable [15, 14. There is a conjecture whether any action of $\mathbb{T}^{n-2}$ on $\mathbb{C}^{n}$ is linearizable, established for $n=3$. For codimension $>2$, there are positive-characteristic counterexamples [2].

Remark. Kraft and Stampfli 23 proved (by considering periodic elements in $\mathbb{T}$ ) that an effective action $T$ has the following property: if $\Psi \in \operatorname{Aut}(A u t)$ is a group automorphism, then the image of $T$ (as a subgroup of Aut) under $\Psi$ is an algebraic group. In fact their proof is also applicable for the free associative algebra case. We are going to use this result.

Returning to the case of automorphisms $\varphi \in$ Aut $_{\text {Ind }}$ Aut preserving the Ind-group structure, consider now the standard action $x_{i} \mapsto \lambda_{i} x_{i}$ of the $n$-dimensional torus $\mathbb{T} \leftrightarrow T^{n} \subset$ $\operatorname{Aut}\left(\mathbb{C}\left[x_{1}, \ldots, x_{n}\right]\right)$ on the affine space $\mathbb{C}^{n}$. Let $H$ be the image of $T^{n}$ under $\varphi$. Then by Theorem $4.1 H$ is conjugate to the standard torus $T^{n}$ via some automorphism $\psi$. Composing $\varphi$ with this conjugation, we come to the case when $\varphi$ is the identity on the maximal torus. Then we have the following

Corollary 4.2. Without loss of generality, it is enough to prove Theorem 1.1 for the case when $\left.\varphi\right|_{\mathbb{T}}=\mathrm{Id}$.

Now we are in the situation when $\varphi$ preserves all linear mappings $x_{i} \mapsto \lambda_{i} x_{i}$. We have to prove that it is the identity.

Proposition 4.3 (E. Rips, private communication). Let $n>2$ and suppose $\varphi$ preserves the standard torus action on the commutative polynomial algebra. Then $\varphi$ preserves all elementary transformations.

Corollary 4.4. Let $\varphi$ satisfy the conditions of Proposition 4.3. Then $\varphi$ preserves all tame automorphisms.

Proof of Proposition 4.3. We state a few elementary lemmas.

Lemma 4.5. Consider the diagonal action $T^{1} \subset T^{n}$ given by automorphisms: $\alpha: x_{i} \mapsto$ $\alpha_{i} x_{i}, \beta: x_{i} \mapsto \beta_{i} x_{i}$. Let $\psi: x_{i} \mapsto \sum_{i, J} a_{i J} x^{J}, i=1, \ldots, n$, where $J=\left(j_{1}, \ldots, j_{n}\right)$ is the multi-index, $x^{J}=x^{j_{1}} \cdots x^{j_{n}}$. Then

$$
\alpha \circ \psi \circ \beta: x_{i} \mapsto \sum_{i, J} \alpha_{i} a_{i J} x^{J} \beta^{J},
$$

In particular,

$$
\alpha \circ \psi \circ \alpha^{-1}: x_{i} \mapsto \sum_{i, J} \alpha_{i} a_{i J} x^{J} \alpha^{-J} .
$$

Applying Lemma 4.5 and comparing the coefficients we get the following
Lemma 4.6. Consider the diagonal $T^{1}$ action: $x_{i} \mapsto \lambda x_{i}$. Then the set of automorphisms commuting with this action is exactly the set of linear automorphisms.

Similarly (using Lemma 4.5) we obtain Lemmas 4.7, 4.9, 4.10,
Lemma 4.7. a) Consider the following $T^{2}$ action:

$$
x_{1} \mapsto \lambda \delta x_{1}, x_{2} \mapsto \lambda x_{2}, x_{3} \mapsto \delta x_{3}, x_{i} \mapsto \lambda x_{i}, i>3 .
$$

Then the set $S$ of automorphisms commuting with this action is generated by the following automorphisms:

$$
x_{1} \mapsto x_{1}+\beta x_{2} x_{3}, x_{i} \mapsto \varepsilon_{i} x_{i}, i>1,\left(\beta, \varepsilon_{i} \in K\right)
$$

b) Consider the following $T^{n-1}$ action:

$$
x_{1} \mapsto \lambda^{I} x_{1}, x_{j} \mapsto \lambda_{j} x_{j}, j>1\left(\lambda^{I}=\lambda_{2}^{i_{2}} \cdots \lambda_{n}^{i_{n}}\right) .
$$

Then the set $S$ of automorphisms commuting with this action is generated by the following automorphisms:

$$
x_{1} \mapsto x_{1}+\beta \prod_{j=2}^{n} x_{j}^{i_{j}},(\beta \in K) .
$$

Remark. A similar statement for the free associative case is true, but one has to consider the set $\hat{S}$ of automorphisms $x_{1} \mapsto x_{1}+h, x_{i} \mapsto \varepsilon_{i} x_{i}, i>1,(\varepsilon \in K$, and the polynomial $h \in K\left\langle x_{2}, \ldots, x_{n}\right\rangle$ has total degree $J$ - in the free associative case it is not just monomial anymore).

Corollary 4.8. Let $\varphi \in \operatorname{Aut}\left(\operatorname{TAut}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)\right)$ stabilizing all elements from $\mathbb{T}$. Then $\varphi(S)=S$.

Lemma 4.9. Consider the following $T^{1}$ action:

$$
x_{1} \mapsto \lambda^{2} x_{1}, x_{i} \mapsto \lambda x_{i}, i>1 .
$$

Then the set $S$ of automorphisms commuting with this action is generated by the following automorphisms:

$$
x_{1} \mapsto x_{1}+\beta x_{2}^{2}, x_{i} \mapsto \lambda_{i} x_{i}, i>2,\left(\beta, \lambda_{i} \in K\right)
$$

Lemma 4.10. Consider the set $S$ defined in the previous lemma. Then $[S, S]=\left\{u v u^{-1} v^{-1}\right\}$ consists of the following automorphisms

$$
x_{1} \mapsto x_{1}+\beta x_{2} x_{3}, x_{2} \mapsto x_{2}, x_{3} \mapsto x_{3},(\beta \in K) .
$$

Lemma 4.11. Let $n \geq 3$. Consider the following set of automorphisms

$$
\psi_{i}: x_{i} \mapsto x_{i}+\beta_{i} x_{i+1} x_{i+2}, \beta_{i} \neq 0, x_{k} \mapsto x_{k}, k \neq i
$$

for $i=1, \ldots, n-1$. (Numeration is cyclic, so for example $x_{n+1}=x_{1}$ ). Let $\beta_{i} \neq 0$ for all i. Then all of $\psi_{i}$ can be simultaneously conjugated by a torus action to

$$
\psi_{i}^{\prime}: x_{i} \mapsto x_{i}+x_{i+1} x_{i+2}, x_{k} \mapsto x_{k}, k \neq i
$$

for $i=1, \ldots, n$ in a unique way.

Proof. Let $\alpha: x_{i} \mapsto \alpha_{i} x_{i}$. Then by Lemma 4.5 we obtain

$$
\alpha \circ \psi_{i} \circ \alpha^{-1}: x_{i} \mapsto x_{i}+\beta_{i} x_{i+1} x_{i+2} \alpha_{i+1}^{-1} \alpha_{i+2}^{-1} \alpha_{i}
$$

and

$$
\alpha \circ \psi_{i} \circ \alpha^{-1}: x_{k} \mapsto x_{k}
$$

for $k \neq i$.
Comparing the coefficients of the quadratic terms, we see that it is sufficient to solve the system:

$$
\beta_{i} \alpha_{i+1}^{-1} \alpha_{i+2}^{-1} \alpha_{i}=1, i=1, \ldots, n-1 .
$$

As $\beta_{i} \neq 0$ for all $i$, this system has a unique solution.
Remark. In the free associative algebra case, instead of $\beta x_{2} x_{3}$ one has to consider $\beta x_{2} x_{3}+\gamma x_{3} x_{2}$.

### 4.2. The lemma of Rips.

Lemma 4.12 (E. Rips). Let $\operatorname{Char}(K) \neq 2,|K|=\infty$. Linear transformations and $\psi_{i}^{\prime}$ defined in Lemma 4.11 generate the whole tame automorphism group of $K\left[x_{1}, \ldots, x_{n}\right]$.

Proposition 4.3 follows from Lemmas 4.6, 4.7, 4.9, 4.10, 4.11, 4.12, Note that we have proved an analogue of Theorem 1.1 for tame automorphisms.

Proof of Lemma 4.12, Let $G$ be the group generated by elementary transformations as in Lemma 4.11. We have to prove that is isomorphic to the tame automorphism subgroup fixing the augmentation ideal. We are going to need some preliminaries.

Lemma 4.13. Linear transformations of $K^{3}$ and

$$
\psi: x \mapsto x, y \mapsto y, z \mapsto z+x y
$$

generate all mappings of the form

$$
\phi_{m}^{b}(x, y, z): x \mapsto x, y \mapsto y, z \mapsto z+b x^{m}, \quad b \in K .
$$

Proof of Lemma 4.13. We proceed by induction. Suppose we have an automorphism

$$
\phi_{m-1}^{b}(x, y, z): x \mapsto x, y \mapsto y, z \mapsto z+b x^{m-1} .
$$

Conjugating by the linear transformation $(z \mapsto y, y \mapsto z, x \mapsto x)$, we obtain the automorphism

$$
\phi_{m-1}^{b}(x, z, y): x \mapsto x, y \mapsto y+b x^{m-1}, z \mapsto z .
$$

Composing this on the right by $\psi$, we get the automorphism

$$
\varphi(x, y, z): x \mapsto x, y \mapsto y+b x^{m-1}, z \mapsto z+y x+x^{m} .
$$

Note that

$$
\phi_{m-1}(x, y, z)^{-1} \circ \varphi(x, y, z): x \mapsto x, y \mapsto y, z \mapsto z+x y+b x^{m} .
$$

Now we see that

$$
\psi^{-1} \phi_{m-1}(x, y, z)^{-1} \circ \varphi(x, y, z)=\phi_{m}^{b}
$$

and the lemma is proved.

Corollary 4.14. Let $\operatorname{Char}(K) \nmid n($ in particular, $\operatorname{Char}(K) \neq 0)$ and $|K|=\infty$. Then $G$ contains all the transformations

$$
z \mapsto z+b x^{k} y^{l}, y \mapsto y, x \mapsto x
$$

such that $k+l=n$.

Proof. For any invertible linear transformation

$$
\varphi: x \mapsto a_{11} x+a_{12} y, y \mapsto a_{21} x+a_{22} y, z \mapsto z ; a_{i j} \in K
$$

we have

$$
\varphi^{-1} \phi_{m}^{b} \varphi: x \mapsto x, y \mapsto y, z \mapsto z+b\left(a_{11} x+a_{12} y\right)^{m} .
$$

Note that sums of such expressions contain all the terms of the form $b x^{k} y^{l}$. The corollary is proved.

### 4.3. Generators of the tame automorphism group.

Theorem 4.15. If $\operatorname{Char}(K) \neq 2$ and $|K|=\infty$, then linear transformations and

$$
\psi: x \mapsto x, y \mapsto y, z \mapsto z+x y
$$

generate all mappings of the form

$$
\alpha_{m}^{b}(x, y, z): x \mapsto x, y \mapsto y, z \mapsto z+b y x^{m}, \quad b \in K .
$$

Proof of theorem 4.15. Observe that

$$
\alpha=\beta \circ \phi_{m}^{b}(x, z, y): x \mapsto x+b y^{m}, y \mapsto y+x+b y^{m}, z \mapsto z,
$$

where $\beta: x \mapsto x, y \mapsto x+y, z \mapsto z$. Then

$$
\gamma=\alpha^{-1} \psi \alpha: x \mapsto x, y \mapsto y, z \mapsto z+x y+2 b x y^{m}+b y^{2 m} .
$$

Composing with $\psi^{-1}$ and $\phi_{2 m}^{2 b}$ we get the desired

$$
\alpha_{m}^{2 b}(x, y, z): x \mapsto x, y \mapsto y, z \mapsto z+2 b y x^{m}, \quad b \in K .
$$

Corollary 4.16. Let $\operatorname{Char}(K) \nmid n$ and $|K|=\infty$. Then $G$ contains all transformations of the form

$$
z \mapsto z+b x^{k} y^{l}, y \mapsto y, x \mapsto x
$$

such that $k=n+1$.
The proof is similar to the proof of Corollary 4.14. Note that either $n$ or $n+1$ is not a multiple of $\operatorname{Char}(K)$ so we have

Lemma 4.17. If $\operatorname{Char}(K) \neq 2$ then linear transformations and

$$
\psi: x \mapsto x, y \mapsto y, z \mapsto z+x y
$$

generate all mappings of the form

$$
\alpha_{P}: x \mapsto x, y \mapsto y, z \mapsto z+P(x, y), \quad P(x, y) \in K[x, y] .
$$

We have proved Lemma 4.12 for the three variable case. In order to treat the case $n \geq 4$ we need one more lemma.

Lemma 4.18. Let $M(\vec{x})=a \prod x_{i}^{k_{i}}, \quad a \in K,|K|=\infty$, $\operatorname{Char}(K) \nmid k_{i}$ for at least one of $k_{i}$ 's. Consider the linear transformations denoted by

$$
f: x_{i} \mapsto y_{i}=\sum a_{i j} x_{j}, \operatorname{det}\left(a_{i j}\right) \neq 0
$$

and monomials $M_{f}=M(\vec{y})$. Then the linear span of $M_{f}$ for different $f$ 's contains all homogenous polynomials of degree $k=\sum k_{i}$ in $K\left[x_{1}, \ldots, x_{n}\right]$.

Proof. It is a direct consequence of the following fact. Let $S$ be a homogenous subspace of $K\left[x_{1}, \ldots, x_{n}\right]$ invariant with respect to $G L_{n}$ of degree $m$. Then $S=S_{m / p^{k}}^{p^{k}}, p=$ Char $(K), S_{l}$ is the space of all polynomials of degree $l$.

Lemma 4.12 follows from Lemma 4.18 in a similar way as in the proofs of Corollaries 4.14 and 4.16 .
4.4. Aut(TAut) for general case. Now we consider the case when Char $(K)$ is arbitrary, i.e. the remaining case $\operatorname{Char}(K)=2$. Still $|K|=\infty$. Although we are unable to prove the analogue of Proposition 4.3, we can still play on the relations.

Let

$$
M=a \prod_{i=1}^{n-1} x_{i}^{k_{i}}
$$

be a monomial, $a \in K$. For polynomial $P(x, y) \in K[x, y]$ we define the elementary automorphism

$$
\psi_{P}: x_{i} \mapsto x_{i}, i=1, \ldots, n-1, x_{n} \mapsto x_{n}+P\left(x_{1}, \ldots, x_{n-1}\right)
$$

We have $P=\sum M_{j}$ and $\psi_{P}$ naturally decomposes as a product of commuting $\psi_{M_{j}}$. Let $\Psi \in \operatorname{Aut}(\operatorname{TAut}(K[x, y, z]))$ stabilizing linear mappings and $\phi$ (Automorphism $\phi$ defined in Lemma 4.13). Then according to the corollary 4.8 $\Psi\left(\psi_{P}\right)=\prod \Psi\left(\psi_{M_{j}}\right)$. If $M=a x^{n}$ then due to Lemma 4.13

$$
\Psi\left(\psi_{M}\right)=\psi_{M}
$$

We have to prove the same for other type of monomials:

Lemma 4.19. Let $M$ be a monomial. Then

$$
\Psi\left(\psi_{M}\right)=\psi_{M}
$$

Proof. Let $M=a \prod_{i=1}^{n-1} x_{i}^{k_{i}}$. Consider the automorphism

$$
\alpha: x_{i} \mapsto x_{i}+x_{1}, i=2, \ldots, n-1 ; x_{1} \mapsto x_{1}, x_{n} \mapsto x_{n} .
$$

Then

$$
\alpha^{-1} \psi_{M} \alpha=\psi_{x_{1}^{k_{1}}} \prod_{i=2}^{n-1}\left(x_{i}+x_{1}\right)^{k_{i}}=\psi_{Q} \psi_{a x_{1}^{\sum_{i=2}^{n-1} k_{i}}} .
$$

Here the polynomial

$$
Q=x_{1}^{k_{1}}\left(\prod_{i=2}^{n-1}\left(x_{i}+x_{1}\right)^{k_{i}}-a x_{1}^{\sum k_{i}}\right) .
$$

It has the following form

$$
Q=\sum_{i=2}^{n-1} N_{i},
$$

where $N_{i}$ are monomials such that none of them is proportional to a power of $x_{1}$.
According to Corollary 4.8, $\Psi\left(\psi_{M}\right)=\psi_{b M}$ for some $b \in K$. We need only to prove that $b=1$. Suppose the contrary, $b \neq 1$. Then

$$
\begin{array}{r}
\Psi\left(\alpha^{-1} \psi_{M} \alpha\right)=\left(\prod_{\left[N_{i}, x_{1}\right] \neq 0} \Psi\left(\psi_{N_{i}}\right)\right) \circ \Psi\left(\psi_{a x_{1}^{x_{i=2}^{n-1} k_{i}}}\right)= \\
\left(\prod_{\left[N_{i}, x_{1}\right] \neq 0} \psi_{b_{i} N_{i}}\right) \circ \psi_{a x_{1} \sum_{i=2}^{n-1} k_{i}}
\end{array}
$$

for some $b_{i} \in K$.
On the other hand

$$
\Psi\left(\alpha^{-1} \psi_{M} \alpha\right)=\alpha^{-1} \Psi\left(\psi_{M}\right) \alpha=\alpha^{-1} \psi_{b M} \alpha=\left(\prod_{\left[N_{i}, x_{1}\right] \neq 0} \psi_{b N_{i}}\right) \circ \psi_{a x_{1}^{\sum_{i=2}^{n-1} k_{i}}}
$$

Comparing the factors $\psi_{a x_{1}^{\sum} \sum_{i=2}^{n-1} k_{i}}$ and $\psi_{a x_{1}^{\sum_{i=2}^{n-1} k_{i}}}$ in the last two products we get $b=1$. Lemma 4.19 and hence Proposition 4.3 are proved.

## 5. The approach of Bodnarchuk-Rips to automorphisms of

$$
\operatorname{TAut}\left(K\left\langle x_{1}, \ldots, x_{n}\right\rangle\right)(n>2)
$$

Now consider the free associative case. We treat the case $n>3$ on group-theoretic level and the case $n=3$ on Ind-scheme level. Note that if $n=2$ then $\operatorname{Aut}_{0}(K[x, y])=$ $\operatorname{TAut}_{0}(K[x, y]) \simeq \operatorname{TAut}_{0}(K\langle x, y\rangle)=\operatorname{Aut}_{0}(K\langle x, y\rangle)$ and description of automorphism group of such objects is known due to J. Déserti.

### 5.1. The automorphisms of the tame automorphism

group of $K\left\langle x_{1}, \ldots, x_{n}\right\rangle, n \geq 4$.

Proposition 5.1 (E. Rips, private communication). Let $n>3$ and let $\varphi$ preserve the standard torus action on the free associative algebra $K\left\langle x_{1}, \ldots, x_{n}\right\rangle$. Then $\varphi$ preserves all elementary transformations.

Corollary 5.2. Let $\varphi$ satisfy the conditions of the proposition 5.1. Then $\varphi$ preserves all tame automorphisms.

For free associative algebras, we note that any automorphism preserving the torus action preserves also the symmetric

$$
x_{1} \mapsto x_{1}+\beta\left(x_{2} x_{3}+x_{3} x_{2}\right), x_{i} \mapsto x_{i}, i>1
$$

and the skew symmetric

$$
x_{1} \mapsto x_{1}+\beta\left(x_{2} x_{3}-x_{3} x_{2}\right), x_{i} \mapsto x_{i}, i>1
$$

elementary automorphisms. The first property follows from Lemma 4.9. The second one follows from the fact that skew symmetric automorphisms commute with automorphisms of the following type

$$
x_{2} \mapsto x_{2}+x_{3}^{2}, x_{i} \mapsto x_{i}, i \neq 2
$$

and this property distinguishes them from elementary automorphisms of the form

$$
x_{1} \mapsto x_{1}+\beta x_{2} x_{3}+\gamma x_{3} x_{2}, x_{i} \mapsto x_{i}, i>1 .
$$

Theorem 1.2 follows from the fact that the forms $\beta x_{2} x_{3}+\gamma x_{3} x_{2}$ corresponding to general bilinear multiplication

$$
*_{\beta, \gamma}:\left(x_{2}, x_{3}\right) \mapsto \beta x_{2} x_{3}+\gamma x_{3} x_{2}
$$

lead to associative multiplication if and only if $\beta=0$ or $\gamma=0$; the approximation also applies (see section (3.3).

Suppose at first that $n=4$ and we are dealing with $K\langle x, y, z, t\rangle$.

Proposition 5.3. The group $G$ containing all linear transformations and mappings

$$
x \mapsto x, y \mapsto y, z \mapsto z+x y, t \mapsto t
$$

contains also all transformations of the form

$$
x \mapsto x, y \mapsto y, z \mapsto z+P(x, y), t \mapsto t .
$$

Proof. It is enough to prove that $G$ contains all transformations of the following form

$$
x \mapsto x, y \mapsto y, z \mapsto z+a M, t \mapsto t, \quad a \in K,
$$

where $M$ is a monomial.
Step 1. Let
or

$$
M=a \prod_{i=1}^{m} x^{k_{i}} y^{l_{i}} \quad \text { or } \quad M=a \prod_{i=1}^{m} y^{l_{0}} x^{k_{i}} y^{l_{i}}
$$

$$
M=a \prod_{i=1}^{m} x^{k_{i}} y^{l_{i}} \quad \text { or } \quad M=a \prod_{i=1}^{m} x^{k_{i}} y^{l_{i}} x^{k_{m+1}} .
$$

Define the height of $M, H(M)$, to be the number of segments comprised of a specific generator - such as $x^{k}$ - in the word $M$. (For instance, $H\left(a \prod_{i=1}^{m} x^{k_{i}} y^{l_{i}} x^{k_{m+1}}\right)=2 m+1$.) Using induction on $H(M)$, one can reduce to the case when $M=y x^{k}$. Let $M=M^{\prime} x^{k}$ such that $H\left(M^{\prime}\right)<H(M)$. (Case when $M=M^{\prime} y^{l}$ is obviously similar.) Let

$$
\begin{aligned}
& \phi: x \mapsto x, y \mapsto y, z \mapsto z+M^{\prime}, t \mapsto t . \\
& \alpha: x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto t+z x^{k} .
\end{aligned}
$$

Then

$$
\phi^{-1} \circ \alpha \circ \phi: x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto t-M+z x^{k} .
$$

The automorphism $\phi^{-1} \circ \alpha \circ \phi$ is the composition of automorphisms

$$
\beta: x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto t-M
$$

and

$$
\gamma: x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto t+z x^{k} .
$$

Observe that $\beta$ is conjugate to the automorphism

$$
\beta^{\prime}: x \mapsto x, y \mapsto y, z \mapsto z-M, t \mapsto t
$$

by a linear automorphism

$$
x \mapsto x, y \mapsto y, z \mapsto t, t \mapsto z .
$$

Similarly, $\gamma$ is conjugate to the automorphism

$$
\gamma^{\prime}: x \mapsto x, y \mapsto y, z \mapsto z+y x^{k}, t \mapsto t .
$$

We have thus reduced to the case when $M=x^{k}$ or $M=y x^{k}$.
Step 2. Consider automorphisms

$$
\alpha: x \mapsto x, y \mapsto y+x^{k}, z \mapsto z, t \mapsto t
$$

and

$$
\beta: x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto t+a z y
$$

Then

$$
\alpha^{-1} \circ \beta \circ \alpha: x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto t+a z x^{k}+a z y
$$

It is a composition of the automorphism

$$
\gamma: x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto t+a z x^{k}
$$

which is conjugate to the needed automorphism

$$
\gamma^{\prime}: x \mapsto x, y \mapsto y, z \mapsto z+y x^{k}, t \mapsto t
$$

and an automorphism

$$
\delta: x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto t+a z y
$$

which is conjugate to the automorphism

$$
\delta^{\prime}: x \mapsto x, y \mapsto y, z \mapsto z+a x y, t \mapsto t
$$

and then to the automorphism

$$
\delta^{\prime \prime}: x \mapsto x, y \mapsto y, z \mapsto z+x y, t \mapsto t
$$

(using similarities). We have reduced the problem to proving the statement

$$
G \ni \psi_{M}, \quad M=x^{k}
$$

for all $k$.
Step 3. Obtain the automorphism

$$
x \mapsto x, y \mapsto y+x^{n}, z \mapsto z, t \mapsto t
$$

This problem is similar to the commutative case of $K\left[x_{1}, \ldots, x_{n}\right]$ (cf. Section (4).
Proposition 5.3 is proved.
Returning to the general case $n \geq 4$, let us formulate the remark made after Lemma 4.7 as follows:

Lemma 5.4. Consider the following $T^{n-1}$ action:

$$
x_{1} \mapsto \lambda^{I} x_{1}, x_{j} \mapsto \lambda_{j} x_{j}, \quad j>1 ; \quad \lambda^{I}=\lambda_{2}^{i_{2}} \cdots \lambda_{n}^{i_{n}}
$$

Then the set $S$ of automorphisms commuting with this action is generated by the following automorphisms:

$$
x_{1} \mapsto x_{1}+H, x_{i} \mapsto x_{i} ; \quad i>1
$$

where $H$ is any homogenous polynomial of total degree $i_{2}+\cdots+i_{n}$.

Proposition 5.3 and Lemma 5.4 imply

Corollary 5.5. Let $\Psi \in \operatorname{Aut}\left(\operatorname{TAut}_{0}\left(K\left\langle x_{1}, \ldots, x_{n}\right\rangle\right)\right)$ stabilize all elements of torus and linear automorphisms,

$$
\phi_{P}: x_{n} \mapsto x_{n}+P\left(x_{1}, \ldots, x_{n-1}\right), x_{i} \mapsto x_{i}, i=1, \ldots, n-1 .
$$

Let $P=\sum_{I} P_{I}$, where $P_{I}$ is the homogenous component of $P$ of multi-degree $I$. Then
a) $\Psi\left(\phi_{P}\right): x_{n} \mapsto x_{n}+P^{\Psi}\left(x_{1}, \ldots, x_{n-1}\right), x_{i} \mapsto x_{i}, i=1, \ldots, n-1$.
b) $P^{\Psi}=\sum_{I} P_{I}^{\Psi}$; here $P_{I}^{\Psi}$ is homogenous of multi-degree $I$.
c) If I has positive degree with respect to one or two variables, then $P_{I}^{\Psi}=P_{I}$.

Let $\Psi \in \operatorname{Aut}\left(\operatorname{TAut}_{0}\left(K\left\langle x_{1}, \ldots, x_{n}\right\rangle\right)\right)$ stabilize all elements of torus and linear automorphisms,

$$
\phi: x_{n} \mapsto x_{n}+P\left(x_{1}, \ldots, x_{n-1}\right), x_{i} \mapsto x_{i}, i=1, \ldots, n-1 .
$$

Let $\varphi_{Q}: x_{1} \mapsto x_{1}, x_{2} \mapsto x_{2}, x_{i} \mapsto x_{i}+Q_{i}\left(x_{1}, x_{2}\right), i=3, \ldots, n-1, x_{n} \mapsto x_{n}$; $Q=\left(Q_{3}, \ldots, Q_{n-1}\right)$. Then $\Psi\left(\varphi_{Q}\right)=\varphi_{Q}$ by Proposition 5.3.

Lemma 5.6. a) $\varphi_{Q}^{-1} \circ \phi_{P} \circ \varphi_{Q}=\phi_{P_{Q}}$, where

$$
P_{Q}\left(x_{1}, \ldots, x_{n-1}\right)=P\left(x_{1}, x_{2}, x_{3}+Q_{3}\left(x_{1}, x_{2}\right), \ldots, x_{n-1}+Q_{n-1}\left(x_{1}, x_{2}\right)\right) .
$$

b) Let $P_{Q}=P_{Q}^{(1)}+P_{Q}^{(2)}, P_{Q}^{(1)}$ consist of all terms containing one of the variables $x_{3}, \ldots, x_{n-1}$, and let $P_{Q}^{(1)}$ consist of all terms containing just $x_{1}$ and $x_{2}$. Then

$$
P_{Q}^{\Psi}=P_{Q}^{\Psi}=P_{Q}^{(1) \Psi}+P_{Q}^{(2) \Psi}=P_{Q}^{(1) \Psi}+P_{Q}^{(2)}
$$

Lemma 5.7. If $P_{Q}^{(2)}=R_{Q}^{(2)}$ for all $Q$ then $P=R$.
Proof. It is enough to prove that if $P \neq 0$ then $P_{Q}^{(2)} \neq 0$ for appropriate $Q=$ $\left(Q_{3}, \ldots, Q_{n-1}\right)$. Let $m=\operatorname{deg}(P), Q_{i}=x_{1}^{2^{i+1} m} x_{2}^{2^{i+1} m}$. Let $\hat{P}$ be the highest-degree component of $P$, then $\hat{P}\left(x_{1}, x_{2}, Q_{3}, \ldots, Q_{n-1}\right)$ is the highest-degree component of $P_{Q}^{(2)}$. It is enough to prove that

$$
\hat{P}\left(x_{1}, x_{2}, Q_{3}, \ldots, Q_{n-1}\right) \neq 0
$$

Let $x_{1} \prec x_{2} \prec x_{2} \prec \cdots \prec x_{n-1}$ be the standard lexicographic order. Consider the lexicographically minimal term $M$ of $\hat{P}$. It is easy to see that the term

$$
\left.M\right|_{Q_{i} \mapsto x_{i}}, \quad i=3, n-1
$$

cannot cancel with any other term

$$
\left.N\right|_{Q_{i} \mapsto x_{i}}, \quad i=3, n-1
$$

of $\hat{P}\left(x_{1}, x_{2}, Q_{3}, \ldots, Q_{n-1}\right)$. Therefore $\hat{P}\left(x_{1}, x_{2}, Q_{3}, \ldots, Q_{n-1}\right) \neq 0$.
Lemmas 5.6 and 5.7 imply

Corollary 5.8. Let $\Psi \in \operatorname{Aut}\left(\operatorname{TAut}_{0}\left(K\left\langle x_{1}, \ldots, x_{n}\right\rangle\right)\right)$ stabilize all elements of torus and linear automorphisms. Then $P^{\Psi}=P$, and $\Psi$ stabilizes all elementary automorphisms and therefore the entire group $\operatorname{TAut}_{0}\left(K\left\langle x_{1}, \ldots, x_{n}\right\rangle\right)$.

We obtain the following

Proposition 5.9. Let $n \geq 4$ and let $\Psi \in \operatorname{Aut}^{\text {PAut }}\left(\operatorname{TAl}_{0}\left(K\left\langle x_{1}, \ldots, x_{n}\right\rangle\right)\right)$ stabilize all elements of torus and linear automorphisms. Then either $\Psi=\mathrm{Id}$ or $\Psi$ acts as conjugation by the mirror anti-automorphism.

Let $n \geq 4$. Let $\Psi \in \operatorname{Aut}\left(\operatorname{TAut}_{0}\left(K\left\langle x_{1}, \ldots, x_{n}\right\rangle\right)\right)$ stabilize all elements of torus and linear automorphisms. Denote by $E L$ an elementary automorphism

$$
E L: x_{1} \mapsto x_{1}, \ldots, x_{n-1} \mapsto x_{n-1}, x_{n} \mapsto x_{n}+x_{1} x_{2}
$$

(all other elementary automorphisms of this form, i.e. $x_{k} \mapsto x_{k}+x_{i} x_{j}, x_{l} \mapsto x_{l}$ for $l \neq k$ and $k \neq i, k \neq j, i \neq j$, are conjugate to one another by permutations of generators).

We have to prove that $\Psi(E L)=E L$ or $\Psi(E L): x_{i} \mapsto x_{i} ; i=1, \ldots, x_{n-1}, x_{n} \mapsto$ $x_{n}+x_{2} x_{1}$. The latter corresponds to $\Psi$ being the conjugation with the mirror antiautomorphism of $K\left\langle x_{1}, \ldots, x_{n}\right\rangle$.

Define for some $a, b \in K$

$$
x *_{a, b} y=a x y+b y x
$$

Then, in any of the above two cases,

$$
\Psi(E L): x_{i} \mapsto x_{i} ; i=1, \ldots, x_{n-1}, x_{n} \mapsto x_{n}+x_{1} *_{a, b} x_{2}
$$

for some $a, b$.
The following lemma is elementary:

Lemma 5.10. The operation $*=*_{a, b}$ is associative if and only if $a b=0$.

The associator of $x, y$, and $z$ is given by

$$
\begin{array}{r}
\{x, y, z\}_{*} \equiv(x * y) * z-x *(y * z)= \\
a b(z x-x z) y+a b y(x z-z x)=a b[y,[x, z]]
\end{array}
$$

Now we are ready to prove Proposition 5.9. For simplicity we treat only the case $n=4$ - the general case is dealt with analogously. Consider the automorphisms

$$
\begin{aligned}
& \alpha: x \mapsto x, y \mapsto y, z \mapsto z+x y, t \mapsto t, \\
& \beta: x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto t+x z, \\
& h: x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto t-x z .
\end{aligned}
$$

(Manifestly $h=\beta^{-1}$.) Then

$$
\gamma=h \alpha^{-1} \beta \alpha=[\beta, \alpha]: x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto t-x^{2} y .
$$

Note that $\alpha$ is conjugate to $\beta$ via a generator permutation

$$
\kappa: x \mapsto x, y \mapsto z, z \mapsto t, t \mapsto y, \kappa \circ \alpha \circ \kappa^{-1}=\beta
$$

and

$$
\Psi(\gamma): x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto t-x *(x * y) .
$$

Let

$$
\begin{aligned}
& \delta: x \mapsto x, y \mapsto y, z \mapsto z+x^{2}, t \mapsto t, \\
& \epsilon: x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto t+z y .
\end{aligned}
$$

Let $\gamma^{\prime}=\epsilon^{-1} \delta^{-1} \epsilon \delta$. Then

$$
\gamma^{\prime}: x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto t-x^{2} y .
$$

On the other hand we have

$$
\varepsilon=\Psi\left(\epsilon^{-1} \delta^{-1} \epsilon \delta\right): x \mapsto x, y \mapsto y, z \mapsto z, t \mapsto t-\left(x^{2}\right) * y .
$$

We also have $\gamma=\gamma^{\prime}$. Equality $\Psi(\gamma)=\Psi\left(\gamma^{\prime}\right)$ is equivalent to the equality $x *(x * y)=x^{2} * y$. This implies $x * y=x y$ and we are done.
5.2. The group $\operatorname{Aut}_{\operatorname{Ind}}(\operatorname{TAut}(K\langle x, y, z\rangle))$. This is the most technically loaded part of the present study. At the moment we are unable to accomplish the objective of describing the entire group $\operatorname{Aut} \operatorname{TAut}(K\langle x, y, z\rangle)$. In this section we will determine only its subgroup $\operatorname{Aut}_{\text {Ind }} \operatorname{TAut}_{0}(K\langle x, y, z\rangle)$, i.e. the group of Ind-scheme automorphisms, and prove Theorem 1.8. We use the approximation results of Section 3.3. In what follows we suppose that $\operatorname{Char}(K) \neq 2$. As in the preceding chapter, $\{x, y, z\}_{*}$ denotes the associator of $x, y, z$ with respect to a fixed binary linear operation $*$, i.e.

$$
\{x, y, z\}_{*}=(x * y) * z-x *(y * z) .
$$

Proposition 5.11. Let $\Psi \in \operatorname{Aut}_{\text {Ind }}\left(\operatorname{TAut}_{0}(K\langle x, y, z\rangle)\right)$ stabilize all linear automorphisms.

## Let

$$
\phi: x \mapsto x, y \mapsto y, z \mapsto z+x y .
$$

Then either

$$
\Psi(\phi): x \mapsto x, y \mapsto y, z \mapsto z+a x y
$$

or

$$
\Psi(\phi): x \mapsto x, y \mapsto y, z \mapsto z+b y x
$$

for some $a, b \in K$.
Proof. Consider the automorphism

$$
\phi: x \mapsto x, y \mapsto y, z \mapsto z+x y .
$$

Then

$$
\Psi(\phi): x \mapsto x, y \mapsto y, z \mapsto z+x * y,
$$

where $x * y=a x y+b y x$. Let $a \neq 0$. We can make the star product $*=*_{a, b}$ into $x * y=x y+\lambda y x$ by conjugation with the mirror anti-automorphism and appropriate linear substitution. We therefore need to prove that $\lambda=0$, which implies $\Psi(\phi)=\phi$.

The following two lemmas are proved by straightforward computation.

Lemma 5.12. Let $A=K\langle x, y, z\rangle$. Let $f * g=f g+\lambda f g$. Then $\{f, g, h\}_{*}=\lambda[g,[f, h]]$.
In particular $\{f, g, f\}_{*}=0, f *(f * g)-(f * f) * g=-\{f, f, g\}_{*}=\lambda[f,[f, g]]$, $(g * f) * f-g *(f * f)=\{g, f, f\}_{*}=\lambda[f,[f, g]]$.

Lemma 5.13. Let $\varphi_{1}: x \mapsto x+y z, y \mapsto y, z \mapsto z ; \varphi_{2}: x \mapsto x, y \mapsto y, z \mapsto z+y x$; $\varphi=\varphi_{2}^{-1} \varphi_{1}^{-1} \varphi_{2} \varphi_{1}$. Then modulo terms of order $\geq 4$ we have:

$$
\varphi: x \mapsto x+y^{2} x, y \mapsto y, z \mapsto z-y^{2} z
$$

and

$$
\Psi(\varphi): x \mapsto x+y *(y * x), y \mapsto y, z \mapsto z-y *(y * z) .
$$

Lemma 5.14. a) Let $\phi_{l}: x \mapsto x, y \mapsto y, z \mapsto z+y^{2} x$. Then

$$
\Psi\left(\phi_{l}\right): x \mapsto x, y \mapsto y, z \mapsto z+y *(y * x) .
$$

b) Let $\phi_{r}: x \mapsto x, y \mapsto y, z \mapsto z+x y^{2}$. Then

$$
\Psi\left(\phi_{r}\right): x \mapsto x, y \mapsto y, z \mapsto z+(x * y) * y .
$$

Proof. According to the results of the previous section we have

$$
\Psi\left(\phi_{l}\right): x \mapsto x, y \mapsto y, z \mapsto z+P(y, x)
$$

where $P(y, x)$ is homogenous of degree 2 with respect to $y$ and degree 1 with respect to $x$. We have to prove that $H(y, x)=P(y, x)-y *(y * x)=0$.

Let $\tau: x \mapsto z, y \mapsto y, z \mapsto x ; \tau=\tau^{-1}, \quad \phi^{\prime}=\tau \phi_{l} \tau^{-1}: x \mapsto x+y^{2} z, y \mapsto y, z \mapsto z$. Then $\Psi\left(\phi_{l}^{\prime}\right): x \mapsto x+P(y, z), y \mapsto y, z \mapsto z$.

Let $\phi_{l}^{\prime \prime}=\phi_{l} \phi_{l}^{\prime}: x \mapsto x+P(y, z), y \mapsto y, z \mapsto z+P(y, x)$ modulo terms of degree $\geq 4$.
Let $\tau: x \mapsto x-z, y \mapsto y, z \mapsto z$ and let $\varphi_{2}, \varphi$ be the automorphisms described in Lemma 5.13

Then

$$
T=\tau^{-1} \phi_{l}^{-1} \tau \phi_{l}^{\prime \prime}: x \mapsto x, y \mapsto y, z \mapsto z
$$

modulo terms of order $\geq 4$.
On the other hand

$$
\Psi(T): x \mapsto x+H(y, z)-H(y, x), y \mapsto y, z \mapsto z+P
$$

modulo terms of order $\geq 4$. Because $\operatorname{deg}_{y}\left(H(y, x)=2, \operatorname{deg}_{x}(H(y, x))=1\right.$ we get $H=0$. Proof of $b$ ) is similar.

Lemma 5.15. a) Let

$$
\psi_{1}: x \mapsto x+y^{2}, y \mapsto y, z \mapsto z ; \quad \psi_{2}: x \mapsto x, y \mapsto y, z \mapsto z+x^{2} .
$$

Then

$$
\begin{aligned}
& {\left[\psi_{1}, \psi_{2}\right]=\psi_{2}^{-1} \psi_{1}^{-1} \psi_{2} \psi_{1}: x \mapsto x, y \mapsto y, z \mapsto z+y^{2} x+x y^{2},} \\
& \Psi\left(\left[\psi_{1}, \psi_{2}\right]\right): x \mapsto x, y \mapsto y, z \mapsto z+(y * y) * x+x *(y * y) .
\end{aligned}
$$

b)

$$
\phi_{l}^{-1} \phi_{r}^{-1}\left[\psi_{1}, \psi_{2}\right]: x \mapsto x, y \mapsto y, z \mapsto z
$$

modulo terms of order $\geq 4$ but

$$
\begin{gathered}
\Psi\left(\phi_{l}^{-1} \phi_{r}^{-1}\left[\psi_{1}, \psi_{2}\right]\right): x \mapsto x, y \mapsto y \\
z \mapsto z+(y * y) * x+x *(y * y)-(x * y) * y-y *(y * x)= \\
=z+4 \lambda[x[x, y]]
\end{gathered}
$$

modulo terms of order $\geq 4$.

Proof. a) can be obtained by direct computation. b) follows from a) and the lemma 5.12

Proposition 5.11 follows from Lemma 5.15.
We need a few auxiliary lemmas. The first one is an analogue of the hiking procedure from [21, 3].

Lemma 5.16. Let $K$ be algebraically closed, and let $n_{1}, \ldots, n_{m}$ be positive integers. Then there exist $k_{1}, \ldots, k_{s} \in \mathbb{Z}$ and $\lambda_{1}, \ldots, \lambda_{s} \in K$ such that

- $\sum k_{i}=1$ modulo $\operatorname{Char}(K)$ (if $\operatorname{Char}(K)=0$ then $\sum k_{i}=1$ ).
- $\sum_{i} k_{i}^{n_{j}} \lambda_{i}=0$ for all $j=1, \ldots, m$.

For $\lambda \in K$ we define an automorphism $\psi_{\lambda}: x \mapsto x, y \mapsto y, z \mapsto \lambda z$.
The next lemma provides for some translation between the language of polynomials and the group action language. It is similar to the hiking process [3, 21].

Lemma 5.17. Let $\varphi \in K\langle x, y, z\rangle$. Let $\varphi(x)=x, \varphi(y)=y+\sum_{i} R_{i}+R^{\prime}, \varphi(z)=z+Q$. Let $\operatorname{deg}\left(R_{i}\right)=N$, let also the degree of all monomials in $R^{\prime}$ be greater than $N$, and let the degree of all monomials in $Q$ be greater than or equal to $N$. Finally, assume $\operatorname{deg}_{z}\left(R_{i}\right)=i$ and the $z$-degree of all monomials of $R_{1}$ greater than 0 .

Then
a) $\psi_{\lambda}^{-1} \varphi \psi_{\lambda}: x \mapsto x, y \mapsto y+\sum_{i} \lambda^{i} R_{i}+R^{\prime \prime}, z \mapsto z+Q^{\prime}$. Also the total degree of all monomials comprising $R^{\prime}$ is greater than $N$, and the degree of all monomials of $Q$ is greater than or equal to $N$.
b) Let $\phi=\Pi\left(\psi_{\lambda_{i}^{-1}} \varphi \psi_{\lambda_{i}}\right)^{k_{i}}$. Then

$$
\phi: x \mapsto x, y \mapsto y+\sum_{i} R_{i} \lambda_{i}^{k_{i}}+S, z \mapsto z+T
$$

where the degree of all monomials of $S$ is greater than $N$ and the degree of all monomials of $T$ is greater than or equal to $N$.

Proof. a) By direct computation. b) is a consequence of a).
Remark. In the case of characteristic zero, the condition of $K$ being algebraically closed can be dropped. After hiking for several steps, we need to prove just

Lemma 5.18. Let $\operatorname{Char}(K)=0$, let $n$ be a positive integer. Then there exist $k_{1}, \ldots, k_{s} \in \mathbb{Z}$ and $\lambda_{1}, \ldots, \lambda_{s} \in K$ such that

- $\sum k_{i}=1$.
- $\sum_{i} k_{i}^{n} \lambda_{i}=0$.

Using this lemma we can cancel out all the terms in the product in the Lemma 5.17 except for the constant one. The proof of Lemma 5.18 for any field of zero characteristic can be obtained through the following observation:

## Lemma 5.19.

$$
\begin{gathered}
\left(\sum_{i=1}^{n} \lambda_{i}\right)^{n}-\sum_{j}\left(\lambda_{1}+\cdots+\widehat{\lambda_{j}}+\cdots+\lambda_{n}\right)^{n}+\cdots+ \\
+(-1)^{n-k} \sum_{i_{1}<\cdots<i_{k}}\left(x_{i_{1}}+\cdots+x_{i_{k}}\right)^{n}+\cdots+(-1)^{n-1}\left(x_{1}^{n}+\cdots+x_{n}^{n}\right)=n!\prod_{i=1}^{n} x_{i}
\end{gathered}
$$

and if $m<n$ then

$$
\begin{gathered}
\left(\sum_{i=1}^{n} \lambda_{i}\right)^{m}-\sum_{j}\left(\lambda_{1}+\cdots+\widehat{\lambda_{j}}+\cdots+\lambda_{n}\right)^{m}+\cdots+ \\
+(-1)^{n-k} \sum_{i_{1}<\cdots<i_{k}}\left(x_{i_{1}}+\cdots+x_{i_{k}}\right)^{m}+\cdots+(-1)^{n-1}\left(x_{1}^{m}+\cdots+x_{n}^{m}\right)=0 .
\end{gathered}
$$

The lemma 5.19 allows us to replace the $n$-th powers by product of constants, after that the statement of Lemma 5.18 becomes transparent.

Lemma 5.20. Let $\varphi: x \mapsto x+R_{1}, y \mapsto y+R_{2}, z \mapsto z^{\prime}$, such that the total degree of all monomials in $R_{1}, R_{2}$ is greater than or equal to $N$. Then for $\Psi(\varphi): x \mapsto x+R_{1}^{\prime}, y \mapsto$ $y+R_{2}^{\prime}, z \mapsto z^{\prime \prime}$ with the total degree of all monomials in $R_{1}^{\prime}, R_{2}^{\prime}$ also greater than or equal to $N$.

Proof. Similar to the proof of Theorem 3.2,
Lemmas 5.20, 5.17, 5.16 imply the following statement.
Lemma 5.21. Let $\varphi_{j} \in \operatorname{Aut}_{0}(K\langle x, y, z\rangle), j=1,2$, such that

$$
\varphi_{j}(x)=x, \varphi_{j}(y)=y+\sum_{i} R_{i}^{j}+R_{j}^{\prime}, \varphi_{j}(z)=z+Q_{j} .
$$

Let $\operatorname{deg}\left(R_{i}^{j}\right)=N$, and suppose that the degree of all monomials in $R_{j}^{\prime}$ is greater than $N$, while the degree of all monomials in $Q$ is greater than or equal to $N$; $\operatorname{deg}_{z}\left(R_{i}\right)=i$, and the $z$-degree of all monomials in $R_{1}$ is positive. Let $R_{0}^{1}=0, R_{0}^{2} \neq 0$.

Then $\Psi\left(\varphi_{1}\right) \neq \varphi_{2}$.
Consider the automorphism

$$
\phi: x \mapsto x, y \mapsto y, z \mapsto z+P(x, y) .
$$

Let $\Psi \in$ Aut $_{\text {Ind }} \operatorname{TAut}_{0}(k\langle x, y, z\rangle)$ stabilize the standard torus action pointwise. Then

$$
\Psi(\phi): x \mapsto x, y \mapsto y, z \mapsto z+Q(x, y)
$$

We denote

$$
\bar{\Psi}(P)=Q
$$

Our goal is to prove that $\bar{\Psi}(P)=P$ for all $P$ if $\Psi$ stabilizes all linear automorphisms and $\bar{\Psi}(x y)=x y$. We proceed by strong induction on total degree. The base case corresponds to $k=1$ and $l=1$ and is assumed. We then heave

## Lemma 5.22.

$$
\bar{\Psi}\left(x^{k} y^{l}\right)=x^{k} y^{l}
$$

provided that $\bar{\Psi}(P)=P$ for all monomials $P(x, y)$ of total degree $<k+l$.

## Proof.

Let

$$
\begin{aligned}
& \phi: x \mapsto x, y \mapsto y, z \mapsto z+x^{k} y \\
& \varphi_{1}: x \mapsto x+y^{l}, y \mapsto y, z \mapsto z \\
& \varphi_{2}: x \mapsto x, y \mapsto y+x^{k}, z \mapsto z \\
& \varphi_{3}: x \mapsto x, y \mapsto y, z \mapsto z+x y \\
& h: x \mapsto x, y \mapsto y, z \mapsto z-x^{k+1}
\end{aligned}
$$

Then, for $k>1$ and $l>1$

$$
\begin{gathered}
g=h \varphi_{3}^{-1} \varphi_{1}^{-1} \varphi_{2}^{-1} \varphi_{3} \varphi_{1} \varphi_{2}: \\
x \mapsto x-y^{l}+\left(y-\left(x-y^{l}\right)^{k}\right)^{l} \\
y \mapsto y-\left(x-y^{l}\right)^{k}+\left(x-y^{l}+\left(y-\left(x-y^{l}\right)^{k}\right)^{l}\right)^{k} \\
z \mapsto z-x y-x^{k+1}+\left(x-y^{l}\right)\left(y-\left(x-y^{l}\right)^{k}\right)
\end{gathered}
$$

Observe that the height of $g(x)-x, g(y)-y$ and $g(z)-z$ is at least $k+l-1$, when $k>1$ or $l>1$. We then use Theorem 3.2 and the induction step. Applying $\Psi$ yields the result because $\Psi\left(\varphi_{i}\right)=\varphi_{i}, i=1,2,3$ and $\varphi\left(H_{N}\right) \subseteq H_{N}$ for all $N$. The lemma is proved.

Let

$$
M_{k_{1}, \ldots, k_{s}}=x^{k_{1}} y^{k_{2}} \cdots y^{k_{s}}
$$

for even $s$ and

$$
M_{k_{1}, \ldots, k_{s}}=x^{k_{1}} y^{k_{2}} \cdots x^{k_{s}}
$$

for odd $s, k=\sum_{i=1}^{n} k_{i}$. Then

$$
M_{k_{1}, \ldots, k_{s}}=M_{k_{1}, \ldots, k_{s-1}} y^{k_{s}}
$$

for even $s$ and

$$
M_{k_{1}, \ldots, k_{s}}=M_{k_{1}, \ldots, k_{s-1}} x^{k_{s}}
$$

for odd $s$.
We have to prove that $\bar{\Psi}\left(M_{k_{1}, \ldots, k_{s}}\right)=M_{k_{1}, \ldots, k_{s}}$. By induction we may assume that $\bar{\Psi}\left(M_{k_{1}, \ldots, k_{s-1}}\right)=M_{k_{1}, \ldots, k_{s-1}}$.

For any monomial $M=M(x, y)$ we define an automorphism

$$
\varphi_{M}: x \mapsto x, y \mapsto y, z \mapsto z+M .
$$

We also define the automorphisms

$$
\phi_{k}^{e}: x \mapsto x, y \mapsto y+z x^{k}, z \mapsto z
$$

and

$$
\phi_{k}^{o}: x \mapsto x+z y^{k}, y \mapsto y, z \mapsto z .
$$

We will present the case of even $s$ - the odd $s$ case is similar.
Let $D_{z x^{k}}^{e}$ be a derivation of $K\langle x, y, z\rangle$ such that $D_{z x^{k}}^{e}(x)=0, D_{z x^{k}}^{e}(y)=z x^{k}, D_{z x^{k}}^{e}(z)=$ 0 . Similarly, let $D_{z y^{k}}^{o}$ be a derivation of $K\langle x, y, z\rangle$ such that $D_{z y^{k}}^{o}(y)=0, D_{z x^{k}}^{o}(x)=z y^{k}$, $D_{z y^{k}}(z)^{o}=0$.

The following lemma is proved by direct computation:

Lemma 5.23. Let

$$
u=\phi_{k_{s}}^{e}{ }^{-1} \varphi\left(M_{k_{1}, \ldots, k_{s-1}}\right)^{-1} \phi_{k_{s}}^{e} \varphi\left(M_{k_{1}, \ldots, k_{s-1}}\right)
$$

for even $s$ and

$$
u=\phi_{k_{s}}^{o-1} \varphi\left(M_{k_{1}, \ldots, k_{s-1}}\right)^{-1} \phi_{k_{s}}^{o} \varphi\left(M_{k_{1}, \ldots, k_{s-1}}\right)
$$

for odd s. Then

$$
u: x \mapsto x, y \mapsto y+M_{k_{1}, \ldots, k_{s}}+N^{\prime}, z \mapsto z+D_{z x^{k}}^{e}\left(M_{k_{1}, \ldots, k_{s-1}}\right)+N
$$

for even $s$ and

$$
u: x \mapsto x+M_{k_{1}, \ldots, k_{s}}+N^{\prime}, y \mapsto y, z \mapsto z+D_{z x^{k}}^{o}\left(M_{k_{1}, \ldots, k_{s-1}}\right)+N
$$

for odd $s$, where $N, N^{\prime}$ are sums of terms of degree $>k=\sum_{i=1}^{s} k_{i}$.

Let $\psi\left(M_{k_{1}, \ldots, k_{s}}\right): x \mapsto x, y \mapsto y, z \mapsto z+M_{k_{1}, \ldots, k_{s}}$,

$$
\alpha_{e}: x \mapsto x, y \mapsto y-z, z \mapsto z, \alpha_{o}: x \mapsto x-z, y \mapsto y, z \mapsto z,
$$

Let $P_{M}=\Psi(M)-M$. Our goal is to prove that $P_{M}=0$.
Let

$$
v=\psi\left(M_{k_{1}, \ldots, k_{s}}\right)^{-1} \alpha_{e} \psi\left(M_{k_{1}, \ldots, k_{s}}\right) u \alpha_{e}^{-1}
$$

for even $s$ and

$$
v=\psi\left(M_{k_{1}, \ldots, k_{s}}\right)^{-1} \alpha_{o} \psi\left(M_{k_{1}, \ldots, k_{s}}\right) u \alpha_{o}^{-1}
$$

for odd $s$.
The next lemma is also proved by direct computation:
Lemma 5.24. a)

$$
v: x \mapsto x, y \mapsto y+H, z \mapsto z+H_{1}+H_{2}
$$

for even $s$ and

$$
v: x \mapsto x+H, y \mapsto y, z \mapsto z+H_{1}+H_{2}
$$

for odd $s$
b)

$$
\Psi(v): x \mapsto x, y \mapsto y+P_{M_{k_{1}}, \ldots, k_{s}}+\widetilde{H}, z \mapsto z+\widetilde{H_{1}}+\widetilde{H_{2}}
$$

for even $s$ and

$$
\Psi(v): x \mapsto x+P_{M_{k_{1}}, \ldots, k_{s}}+\widetilde{H}, y \mapsto y, z \mapsto z+\widetilde{H_{1}}+\widetilde{H_{2}}
$$

for odd $s$, where $H_{2}, \widetilde{H_{2}}$ are sums of terms of degree greater than $k=\sum_{i=1}^{s} k_{i}, H, \widetilde{H}$ are sums of terms of degree $\geq k$ and positive $z$-degree, $H_{1}, \widetilde{H_{1}}$ are sums of terms of degree $k$ and positive $z$-degree.

Proof of Theorem 1.8. Part b) follows from part a). In order to prove a) we are going to show that $\bar{\Psi}(M)=M$ for any monomial $M(x, y)$ and for any $\Psi \in \operatorname{Aut}_{\operatorname{Ind}}(\operatorname{TAut}(\langle x, y, z\rangle))$ stabilizing the standard torus action $T^{3}$ and $\phi$. The automorphism $\Psi\left(\Phi_{M}\right)$ has the form described in Lemma 5.24. But in this case Lemma 5.21 implies $\bar{\Psi}(M)-M=0$.

## 6. Some open questions concerning the tame automorphism group

As the conclusion of the paper, we would like to raise the following questions.
(1) Is it true that any automorphism $\varphi$ of $\operatorname{Aut}\left(K\left\langle x_{1}, \ldots, x_{n}\right\rangle\right)$ (in the group-theoretic sense - that is, not necessarily an automorphism preserving the Ind-scheme structure) for $n=3$ is semi-inner, i.e. is a conjugation by some automorphism or mirror anti-automorphism?
(2) Is it true that $\operatorname{Aut}\left(K\left\langle x_{1}, \ldots, x_{n}\right\rangle\right)$ is generated by affine automorphisms and automorphism $x_{n} \mapsto x_{n}+x_{1} x_{2}, x_{i} \mapsto x_{i}, i \neq n$ ? For $n \geq 5$ it seems to be easier and the answer is probably positive, however for $n=3$ the answer is known to be negative, cf. Umirbaev 33 and Drensky and Yu 18. For $n \geq 4$ we believe the answer is positive.
(3) Is it true that $\operatorname{Aut}\left(K\left[x_{1}, \ldots, x_{n}\right]\right)$ is generated by linear automorphisms and automorphism $x_{n} \mapsto x_{n}+x_{1} x_{2}, x_{i} \mapsto x_{i}, i \neq n$ ? For $n=3$ the answer is negative: see the proof of the Nagata conjecture [30, 31, 34]. For $n \geq 4$ it is plausible that the answer is positive.
(4) Is any automorphism $\varphi$ of $\operatorname{Aut}(K\langle x, y, z\rangle)$ (in the group-theoretic sense) semiinner?
(5) Is it true that the conjugation in Theorems 1.3 and 1.7 can be done by some tame automorphism? Suppose $\psi^{-1} \varphi \psi$ is tame for any tame $\varphi$. Does it follow that $\psi$ is tame?
(6) Prove Theorem 1.8 for $\operatorname{Char}(K)=2$. Does it hold on the set-theoretic level, i.e. $\operatorname{Aut}(\operatorname{TAut}(K\langle x, y, z\rangle))$ are generated by conjugations by antomorphism or the mirror anti-automorphism?

Similar questions can be formulated for nice automorphisms.

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[^1]:    ${ }^{1}$ In fact, the conjecture seeks to establish an isomorphism $\operatorname{Sympl}\left(K^{2 n}\right) \simeq \operatorname{Aut}\left(W_{n}(K)\right)$ for any field $K$ of characteristic zero in a functorial manner.

[^2]:    ${ }^{2}$ Without loss of generality we may assume that the coordinate functions $x_{i}$ correspond to the principal axes of $\hat{A}(t)$.

