A TWO-DIMENSIONAL SLICE THROUGH THE PARAMETER SPACE OF TWO-GENERATOR KLEINIAN GROUPS

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Abstract. We describe all real points of the parameter space of two-generator Kleinian groups with a parabolic generator, that is, we describe a certain two-dimensional slice through this space. In order to do this we gather together known discreteness criteria for two-generator groups and present them in the form of conditions on parameters. We complete the description by giving discreteness criteria for groups generated by a parabolic and a π-loxodromic elements whose commutator has real trace and present all orbifolds uniformized by such groups.

1. Introduction

A two-generator subgroup $\Gamma = \langle f, g\rangle$ of $\mathrm{PSL}(2,\mathbb{C})$ is determined up to conjugacy by its parameters $\beta = \beta(f) = \mathrm{tr}^2 f - 4$, $\beta' = \beta(g) = \mathrm{tr}^2 g - 4$, and $\gamma = \gamma(f, g) = \mathrm{tr}[f, g] - 2$ whenever $\gamma \neq 0$ [6]. So the conjugacy class of an ordered pair $\{f, g\}$ can be identified with a point in the parameter space $\mathbb{C}^3 = \{(\beta, \beta', \gamma)\}$ whenever $\gamma \neq 0$. The subspace $\mathcal{K}$ of $\mathbb{C}^3$ that corresponds to the discrete non-elementary groups $\Gamma = \langle f, g\rangle$ is called the *parameter space of two-generator Kleinian groups*. Note that a two-generator Kleinian group Γ can be represented by several points in $\mathcal{K}$, since the same group can have different generating pairs.

Among all two-generator subgroups of $\mathrm{PSL}(2,\mathbb{C})$, we distinguish the class of $\mathcal{RP}$ *groups* (two-generator groups with real parameters):

$$\mathcal{RP} = \{\Gamma : \Gamma = \langle f, g\rangle \text{ for some } f, g \in \mathrm{PSL}(2,\mathbb{C}) \text{ with } (\beta, \beta', \gamma) \in \mathbb{R}^3\}.$$

The aim of this paper is to completely determine all points in $\mathbb{C}^3$ that are parameters for the discrete non-elementary $\mathcal{RP}$ groups with one generator parabolic:

$$S_\infty = \{(\gamma, \beta) : (\beta, 0, \gamma) \text{ are parameters for some } \langle f, g\rangle \in \mathcal{DRP}\},$$

where $\mathcal{DRP}$ denotes the class of all discrete non-elementary $\mathcal{RP}$ groups. Geometrically, S_∞ is a two-dimensional slice through the six-dimensional parameter space $\mathcal{K}$.

The slice S_∞ intersects the well-known Riley slice $(0, 0, \gamma)$, $\gamma \in \mathbb{C}$, which consists of all Kleinian groups generated by two parabolics.

Consider the sequence of slices $\{S_n\}_{n=2}^{\infty}$, where

$$S_n = \{(\gamma, \beta) : (\beta, -4\sin^2(\pi/n), \gamma) \text{ are parameters for some } \langle f, g\rangle \in \mathcal{DRP}\}.$$

Date: July 28, 2018.

1991 *Mathematics Subject Classification.* Primary: 30F40; Secondary: 20H10, 22E40, 57M60.

Key words and phrases. Kleinian group, discrete group, hyperbolic orbifold.

The first author was supported by Gettysburg College Research and Professional Development Grant, 2005–2006. The research of the second author was supported by FP6 Marie Curie IIF Fellowship and carried out at LATP (UMR CNRS 6632).

The first slice S_2 of this sequence is of great interest in the theory of discrete groups. This slice consists of all parameters for discrete $\mathcal{RP}$ groups with an elliptic generator of order 2 and was investigated in [5]. It was shown that if $\langle f, g\rangle$ has parameters (β, β', γ), then there exists a group $\langle f, h\rangle$ with parameters $(\beta, -4, \gamma)$ such that if $\gamma \neq 0, \beta$, then $\langle f, h\rangle$ is discrete whenever $\langle f, g\rangle$ is. Hence, the slice S_2 gives necessary discreteness conditions for a group with parameters (β, β', γ), where β and γ are real. It follows that every S_n with $n > 2$, including S_∞, is a subset of S_2.

Since a parabolic element can be viewed as the limit of a sequence of primitive elliptic elements of order n as $n \to \infty$, the following two questions for $\{S_n\}$ and S_∞ naturally arise.

(1) Is it true that for every point $x \in S_\infty$ there exists a sequence $\{x_k\}_{k=2}^{\infty}$ with $x_k \in S_k$ that converges to x?
(2) Is it true that for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that the ε-neighbourhood of S_∞ contains S_n for all $n > N$?

Note that the structure of S_n for $n > 2$ is unknown.

We work out S_∞ by splitting the plane (γ, β) into several parts. It turns out that $\Gamma = \langle f, g\rangle$ has an invariant plane in one of the following cases: (1) $\gamma < 0$ and $\beta \leq -4$; (2) $\gamma > 0$ and $\beta \geq -4$. Such discrete groups were investigated, for example, in [13] and [8, 14, 15], respectively. If $\gamma < 0$ and $\beta > -4$, then Γ is truly spatial (non-elementary and without invariant plane) and this case is treated in [11]. We get these dicreteness criteria together and transform them into conditions on β and γ if it was not done before.

So the last case to consider is when $\gamma > 0$ and $\beta < -4$. In this case Γ is truly spatial with f π-loxodromic. We complete the study of the slice S_∞ by giving discreteness criteria for such groups.

The paper is organised as follows. In Section 2, discreteness criteria are given for truly spatial $\mathcal{RP}$ groups Γ generated by a π-loxodromic and a parabolic elements (Theorems 2.1 and 2.6). In Section 3, for each such discrete Γ we obtain a presentation and the Kleinian orbifold $Q(\Gamma)$ (Theorem 3.1). Section 4 is devoted to the analysis of the parameter space. We completely describe the slice S_∞ by giving explicit formulas for the parameters β and γ. We also program the obtained formulas in the package Maple 7.0 and plot a part of S_∞ on the (γ, β)-plane to give an idea of how it looks like.

2. Discreteness criteria

Recall that an element $f \in \mathrm{PSL}(2, \mathbb{C})$ with real $\beta(f)$ is *elliptic*, *parabolic*, *hyperbolic*, or *π-loxodromic* according to whether $\beta(f) \in [-4, 0)$, $\beta(f) = 0$, $\beta(f) \in (0, +\infty)$, or $\beta(f) \in (-\infty, -4)$. If $\beta(f) \notin [-4, +\infty)$, then f is called *strictly loxodromic*.

An elliptic element f of order n is said to be *non-primitive* if it is a rotation through $2\pi q/n$, where q and n are coprime $(1 < q < n/2)$. If f is a rotation through $2\pi/n$, then it is called *primitive*.

Theorem 2.1. *Let $f \in \mathrm{PSL}(2, \mathbb{C})$ be a π-loxodromic element, $g \in \mathrm{PSL}(2, \mathbb{C})$ be a parabolic element, and let $\Gamma = \langle f, g\rangle$ be a non-elementary $\mathcal{RP}$ group without invariant plane. Then*

(1) *there exist unique elements* $h_1, h_2 \in \mathrm{PSL}(2,\mathbb{C})$ *such that* $h_1^2 = fg^{-1}f^{-1}g^{-1}$ *and* $(h_1g)^2 = 1$, $h_2^2 = f^{-1}g^{-1}f^2gf^{-1}$ *and* $(h_2fg^{-1}f^{-1})^2 = 1$.
(2) *the group* Γ *is discrete if and only if one of the following conditions holds:*
   (i) h_1 *is either a hyperbolic, or parabolic, or primitive elliptic element of even order* $m \geq 4$, *and* h_2 *is either a hyperbolic, or parabolic, or primitive elliptic element of order* $p \geq 3$;
   (ii) h_1 *is a primitive elliptic element of odd order* $m \geq 3$, *and* h_2h_1 *is either a hyperbolic, or parabolic, or primitive elliptic element of order* $k \geq 3$.

Basic geometric construction. We will construct a group Γ^* that contains $\Gamma = \langle f, g\rangle$ as a subgroup of finite index. The idea is to find Γ^* so that a fundamental polyhedron for a discrete Γ^* can be easily constructed. It will be clear from the construction that Γ is commensurable with a reflection group which either coincides with Γ^* or is an index 2 subgroup of Γ^*. The construction presented below will be used throughout Sections 2 and 3 and we shall use the notation introduced here.

Let f and g be as in the statement of Theorem 2.1. Since Γ is a non-elementary $\mathcal{RP}$ group without invariant plane, there exists an invariant plane of g, say η, which is orthogonal to the axis of f [9, Theorem 2].

Denote by M the fixed point of g and by ω the plane that passes through M and f (we denote elements and their axes by the same letters when it does not lead to any confusion). Note that f keeps ω invariant. Since f is orthogonal to η, ω is also orthogonal to η. Let e be the half-turn with the axis $\omega \cap \eta$. Then e passes through M and is orthogonal to f.

Let e_f and e_g be half-turns such that

$$f = e_f e \quad \text{and} \quad g = e_g e. \tag{2.1}$$

Then e_f is orthogonal to ω and e_g lies in η.

Let τ be the plane passing through e_g orthogonally to η and let $\sigma = e_f(\tau)$. The planes τ and ω are parallel and M is their common point on the boundary $\partial\mathbb{H}^3$. Since e_f is orthogonal to ω, the planes σ and ω are also parallel with the common point $e_f(M)$ on $\partial\mathbb{H}^3$. Since $e_f(M) \neq M$, the planes ω, σ, and τ do not have a common point in $\overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \partial\mathbb{H}^3$. Therefore, there exists a unique plane δ orthogonal to all ω, σ, and τ. It is clear that $e_f \subset \delta$.

Consider two extensions of Γ: $\widetilde{\Gamma} = \langle f, g, e\rangle$ and $\Gamma^* = \langle f, g, e, R_\omega\rangle$. (We denote the reflection in a plane κ by R_κ.) One can show that $\widetilde{\Gamma} = \langle e_f, e_g, e\rangle$ and $\Gamma^* = \langle e_f, R_\eta, R_\omega, R_\tau\rangle$. From (2.1), it follows that $\widetilde{\Gamma}$ contains Γ as a subgroup of index at most 2. Moreover, $\widetilde{\Gamma}$ is the orientation preserving subgroup of Γ^* and, hence, Γ^* contains Γ as a subgroup of finite index. Therefore, Γ, $\widetilde{\Gamma}$, and Γ^* are either all discrete, or all non-discrete. We then concentrate on the group Γ^*.

Let $\mathcal{P}^*$ be the infinite volume polyhedron bounded by η, ω, τ, σ, and δ. $\mathcal{P}^*$ has five right dihedral angles (between faces lying in η and ω, η and τ, δ and ω, δ and τ, and δ and σ). The plane σ may either intersect with, or be parallel to, or be disjoint from each of τ and η.

If σ and τ intersect, then we denote the dihedral angle of $\mathcal{P}^*$ between them by $2\pi/m$, where $m > 2$, m is not necessary an integer. We keep the notation $2\pi/m$ taking $m = \infty$ and $m = \overline{\infty}$ for parallel or disjoint σ and τ, respectively. Similarly, we denote the "dihedral angle" between η and σ by π/p, where $p > 2$ is real, ∞,

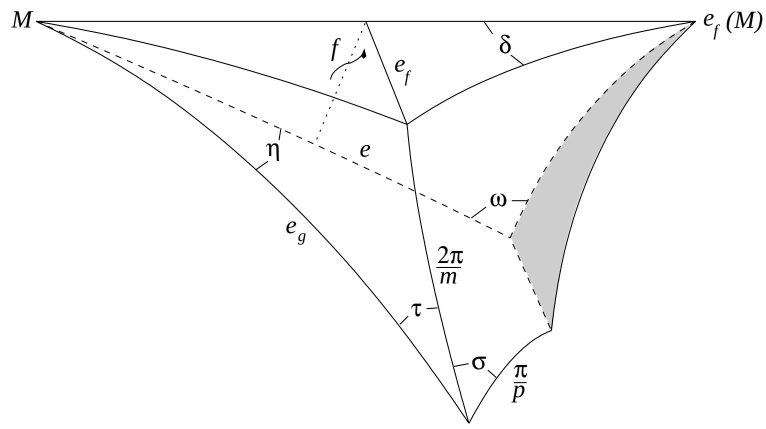


FIGURE 1. Polyhedron $\mathcal{P}^*$

or $\overline{\infty}$. (We regard $\overline{\infty} > \infty > x$, $x/\infty = x/\overline{\infty} = 0$, $\infty/x = \infty$, $\overline{\infty}/x = \overline{\infty}$ for any positive real x.) $\mathcal{P}^*$ exists in $\mathbb{H}^3$ for all $m > 2$ and $p > 2$ by [16].

In Figure 1, $\mathcal{P}^*$ is drawn under assumption that $m < \infty$, $p < \infty$, and $1/2 + 1/p + 2/m > 1$. The shaded triangle shows the hyperbolic plane orthogonal to η, σ, and ω. Note that this plane is not a face of $\mathcal{P}^*$ and is shown only to underline the combinatorial structure of $\mathcal{P}^*$. In figures, we do not label dihedral angles of $\pi/2$ in order to not overload the picture.

Suppose now that $m < \infty$, that is σ and τ intersect. Let ξ be the plane passing through e_f orthogonally to δ. Then ξ is orthogonal to ω. One can see that $\sigma = R_\xi(\tau)$ and ξ is the bisector of the dihedral angle of $\mathcal{P}^*$ made by τ and σ.

Let $\mathcal{Q}^*$ be the polyhedron bounded by η, τ, ω, δ, and ξ. $\mathcal{Q}^*$ has six dihedral angles of $\pi/2$; the dihedral angle between τ and ξ is equal to π/m with $2 < m < \infty$. Denote the "dihedral angle" between η and ξ by π/k, where $k > 2$ is real, $k = \infty$, or $k = \overline{\infty}$. $\mathcal{Q}^*$ exists in $\mathbb{H}^3$ for all $m > 2$ and $k > 2$ by [16]. Note that R_ξ is not necessary in Γ^*, but if it is and if Γ^* is discrete, then we will see that $\mathcal{Q}^*$ is a fundamental polyhedron for Γ^*. In Figure 2, $\mathcal{Q}^*$ is drawn under assumption that $1/2 + 1/k + 1/m > 1$.

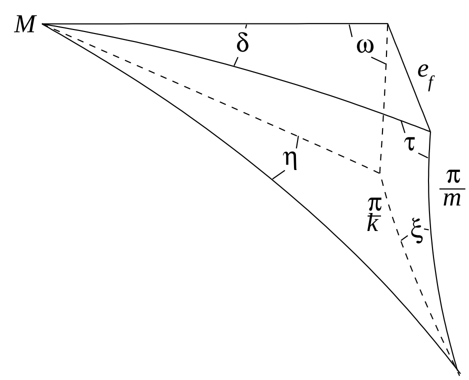


FIGURE 2. Polyhedron $\mathcal{Q}^*$

Lemma 2.2. *Let $f \in \mathrm{PSL}(2,\mathbb{C})$ be a π-loxodromic element, $g \in \mathrm{PSL}(2,\mathbb{C})$ be a parabolic element, and let $\Gamma = \langle f, g\rangle$ be a non-elementary $\mathcal{RP}$ group without invariant plane. Then there exist unique elements $h_1, h_2 \in \mathrm{PSL}(2,\mathbb{C})$ such that*

(1) $h_1^2 = fg^{-1}f^{-1}g^{-1}$ *and* $(h_1g)^2 = 1$,
(2) $h_2^2 = f^{-1}g^{-1}f^2gf^{-1}$ *and* $(h_2fg^{-1}f^{-1})^2 = 1$.

Moreover, the elements h_1 and h_2 are not strictly loxodromic.

Proof. First, note that $R_\sigma = e_f R_\tau e_f$ and $g = R_\tau R_\omega$. Therefore,

$$R_\sigma R_\omega = e_f R_\tau e_f R_\omega = e_f R_\tau R_\omega e_f = e_f g e_f = fg^{-1}f^{-1}. \tag{2.2}$$

Let us show that if we take $h_1 = R_\xi R_\tau = R_\sigma R_\xi$, then the assertion (1) of the lemma hold. Indeed,

$$h_1^2 = R_\sigma R_\tau = (R_\sigma R_\omega)(R_\omega R_\tau) = fg^{-1}f^{-1}g^{-1}.$$

Moreover, $h_1 g = (R_\xi R_\tau)(R_\tau R_\omega) = R_\xi R_\omega$. Since ξ and ω are orthogonal, $(R_\xi R_\omega)^2 = 1$. Hence, $(h_1g)^2 = 1$. Note also that since h_1 is a product of two reflections, h_1 is not strictly loxodromic.

Now let us show that h_1 is unique. The element $fg^{-1}f^{-1}g^{-1}$ is uniquely determined as an element of $\mathrm{PSL}(2,\mathbb{C})$.

If $fg^{-1}f^{-1}g^{-1}$ is parabolic, it has only one square root h_1. Suppose that $fg^{-1}f^{-1}g^{-1}$ is hyperbolic. Then it has exactly two square roots, one of which is h_1 defined above and the other, denoted $\overline{h}_1$, is a π-loxodromic element with the same axis and translation length as h_1. Clearly, $(\overline{h}_1g)^2 \neq 1$.

If $fg^{-1}f^{-1}g^{-1}$ is elliptic, then it also has two square roots h_1 and $\overline{h}_1$, both are elliptic elements. The element $\overline{h}_1$ is elliptic with the same axis as h_1 and with rotation angle $(\pi - 2\pi/m)$, while h_1 is a rotation through $2\pi/m$ in the opposite direction. Again, $(\overline{h}_1g)^2 \neq 1$.

Now we take

$$h_2 = R_\eta R_\sigma = (R_\eta R_\tau)(R_\tau R_\sigma) = e_g h_1^{-2} = efgf^{-1}.$$

Then

$$h_2^2 = f^{-1}g^{-1}f^2gf^{-1} \quad \text{and} \quad (fg^{-1}f^{-1}h_2)^2 = 1.$$

These two conditions determine h_2 uniquely. □

Note that the elements h_1, h_2 defined in Lemma 2.2 determine combinatorial and metric structures of $\mathcal{P}^*$. For example, if h_1 is elliptic, then its rotation angle is equal to the dihedral angle of $\mathcal{P}^*$ between σ and τ. If h_2 is elliptic, then its rotation angle is equal to the doubled dihedral angle of $\mathcal{P}^*$ between η and σ. Vice versa, if the metric structure of $\mathcal{P}^*$ is fixed, then the types of elements h_1 and h_2 can be determined.

The same can be said about $\mathcal{Q}^*$ and the elements h_1 and h_2h_1. The element h_2h_1 is responsible for the mutual position of the planes η and ξ (see the proof of Lemma 2.5).

Lemmas 2.3–2.5 below give some necessary conditions for discreteness of Γ via conditions on elements h_1 and h_2. One needs to keep in mind the connection between these elements and the polyhedra $\mathcal{P}^*$ and $\mathcal{Q}^*$.

Lemma 2.3. *If Γ is discrete, then h_1 is either a hyperbolic, or parabolic, or primitive elliptic element of order $m \geq 3$.*

Proof. The subgroup $H = \langle g, fgf^{-1}\rangle$ of Γ keeps δ invariant and is conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Since Γ is discrete, H must be discrete. By [15] or [2], the group H is discrete if and only if either

(1) $fg^{-1}f^{-1}g^{-1} = h_1^2$ is a hyperbolic, or a parabolic, or a primitive elliptic element, or

(2) h_1 is a primitive elliptic element of odd order m, where $m \geq 3$.

If h_1^2 is parabolic of hyperbolic, then h_1 is parabolic or hyperbolic, respectively. If h_1^2 is a primitive elliptic element, then h_1 is a primitive elliptic of even order $m \geq 4$. □

Lemma 2.4. *If Γ is discrete, then h_2 is either a hyperbolic, or parabolic, or primitive elliptic element of order $p \geq 3$.*

Proof. Let κ be the plane orthogonal to η, σ, and ω. The subgroup $H = \langle e, fgf^{-1}\rangle$ of $\widetilde{\Gamma}$ keeps the plane κ invariant and is conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. By [15], H is discrete if and only if $h_2 = efgf^{-1}$ is either a hyperbolic, or parabolic, or primitive elliptic element of order $p \geq 3$. □

Lemma 2.5. *If Γ is discrete and h_1 is a primitive elliptic element of odd order, then h_2h_1 is either a hyperbolic, or parabolic, or primitive elliptic element of order $k \geq 3$.*

Proof. Recall that $\Gamma^* = \langle e_f, R_\eta, R_\tau, R_\omega\rangle$. Since h_1 has odd order and $h_1^2 \in \Gamma^*$, $h_1 \in \Gamma^*$. Since, moreover, $h_1 = R_\xi R_\tau$, $e_f = R_\delta R_\xi$, and $R_\tau \in \Gamma^*$, both R_ξ and R_δ are also in Γ^*. Further, since the plane ξ is orthogonal to ω, the group $\langle R_\eta R_\delta, e_f\rangle$ keeps ω invariant and is conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. It is clear that $\langle R_\eta R_\delta, e_f\rangle$ is discrete if and only if $R_\eta R_\xi = h_2h_1$ is a hyperbolic, parabolic, or primitive elliptic element of order $k \geq 3$ [15]. □

Proof of Theorem 2.1. Lemma 2.2 proves existence and uniqueness of elements h_1 and h_2. Now we prove part (2) of the theorem.

If Γ is discrete then h_1 is either a hyperbolic, or parabolic, or primitive elliptic element of order $m \geq 3$ by Lemma 2.3. We split the discrete groups Γ into two families. The first family consists of those groups for which h_1 is hyperbolic, parabolic, or primitive elliptic of even order. By Lemma 2.4, for these groups h_2 is a hyperbolic, parabolic, or primitive elliptic element.

The second family consists of the discrete groups with h_1 elliptic of odd order. Then by Lemma 2.5, h_2h_1 is a hyperbolic, or parabolic, or primitive elliptic element of order $k \geq 3$. (Note that in this case h_2 is necessarily hyperbolic or primitive elliptic.)

So if Γ is discrete, then either (2)(i) or (2)(ii) of Theorem 2.1 can occur. Clearly, if neither (2)(i) nor (2)(ii) holds, then Γ is not discrete by Lemmas 2.3–2.5.

Now prove that each of (2)(i) and (2)(ii) is a sufficient condition for Γ to be discrete. In each of the two cases we will give a fundamental polyhedron for Γ^* to show, by using the Poincaré polyhedron theorem [3], that Γ^* is discrete.

Suppose that (2)(i) holds. Then since m is even, the group G_1 generated by the side pairing transformations R_η, R_ω, R_σ, R_τ, and e_f and the polyhedron $\mathcal{P}^*$ satisfy the Poincaré polyhedron theorem, G_1 is discrete and $\mathcal{P}^*$ is its fundamental polyhedron. Obviously, $G_1 = \Gamma^*$.

Suppose that (2)(ii) holds. Then the group G_2 generated by the side pairing transformations R_η, R_ω, R_ξ, R_τ, and R_δ and the polyhedron $\mathcal{Q}^*$ satisfy the Poincaré theorem, G_2 is discrete, and $\mathcal{Q}^*$ is its fundamental polyhedron.

In the proof of Lemma 2.5 it was shown that, for m odd, $R_\xi \in \Gamma^*$ and $R_\delta \in \Gamma^*$. Moreover, $e_f = R_\xi R_\delta$. Hence, $G_2 = \Gamma^*$, so Γ^* is discrete.

Theorem 2.1 is proved. $\square$

Our next goal is to compute parameters $(\beta(f), \beta(g), \gamma(f,g))$ for both series of discrete groups listed in Theorem 2.1.

If $f \in \mathrm{PSL}(2,\mathbb{C})$ is a loxodromic element with translation length d_f and rotation angle θ_f, then

$$\mathrm{tr}^2 f = 4\cosh^2 \frac{d_f + i\theta_f}{2}$$

and $\lambda_f = d_f + i\theta_f$ is called the *complex translation length* of f.

Note that if f is hyperbolic then $\theta_f = 0$ and $\mathrm{tr}^2 f = 4\cosh^2(d_f/2)$. If f is elliptic then $d_f = 0$ and $\mathrm{tr}^2 f = 4\cos^2(\theta_f/2)$. If f is parabolic then $\mathrm{tr}^2 f = 4$; by convention we set $d_f = \theta_f = 0$.

We define the set

$$\mathcal{U} = \{u : u = i\pi/p \text{ for some } p \in \mathbb{Z}, p \geq 2\} \cup [0, +\infty).$$

In other words, the set $\mathcal{U}$ consists of all complex translation half-lengths $u = \lambda_f/2$ for hyperbolic, parabolic, and primitive elliptic elements f. Furthermore, we define a function $t : \mathcal{U} \to \{2,3,4,\dots\} \cup \{\infty, \overline{\infty}\}$ as follows:

$$t(u) = \begin{cases} p & \text{if} \quad u = i\pi/p, \\ \infty & \text{if} \quad u = 0, \\ \overline{\infty} & \text{if} \quad u \in (0, +\infty). \end{cases}$$

Given $u \in \mathcal{U}$ and f with $\mathrm{tr}^2 f = 4\cosh^2 u$, $t(u)$ determines the type of f and, moreover, its order if f is elliptic. Note also that since we regard $\infty/n = \infty$ and $\overline{\infty}/n = \overline{\infty}$, an expression of the form $(t(u), n) = 1$ with $n > 1$ means, in particular, that $t(u)$ is finite.

Theorem 2.6. *Let $f, g \in \mathrm{PSL}(2,\mathbb{C})$ with $\beta(f) < -4$, $\beta(g) = 0$, and $\gamma(f,g) > 0$. Then $\Gamma = \langle f, g\rangle$ is discrete if and only if one of the following holds:*

(1) *$\gamma(f,g) = 4\cosh^2 u$ and $\beta(f) = -4\cosh^2 v/\gamma(f,g) - 4$, where $u, v \in \mathcal{U}$ with $t(u) \geq 4$, $(t(u), 2) = 2$, and $t(v) \geq 3$;*

(2) *$\gamma(f,g) = 4\cosh^2 u$ and $\beta(f) = -4\cosh^2 v - 4$, where $u, v \in \mathcal{U}$ with $t(u) \geq 3$, $(t(u), 2) = 1$, and $t(v) \geq 3$.*

Proof. Obviously, $\beta(f) < -4$ and $\beta(g) = 0$ if and only if f is π-loxodromic and g is parabolic. With this choice of $\beta(f)$ and $\beta(g)$, $\gamma(f,g) > 0$ if and only if the group $\Gamma = \langle f, g\rangle$ is a non-elementary $\mathcal{RP}$ group without invariant plane [9]. This means that the hypotheses of Theorem 2.6 are equivalent to the hypotheses of Theorem 2.1. Therefore, in order to prove Theorem 2.6 it is sufficient to calculate the parameters $\beta(f)$ and $\gamma(f,g)$ for both families of the discrete groups listed in Theorem 2.1.

Let σ' be the image of σ under R_ω, that is $R_{\sigma'} = R_\omega R_\sigma R_\omega$. Using the identity (2.2) and the fact that $g = R_\tau R_\omega$, we have

$$[f,g] = fgf^{-1}g^{-1} = (R_\omega R_\sigma)(R_\omega R_\tau) = (R_{\sigma'} R_\omega)(R_\omega R_\tau) = R_{\sigma'} R_\tau.$$

Note that σ' and τ are disjoint and δ is orthogonal to both of them. Therefore, $[f, g]$ is a hyperbolic element with the axis lying in δ and the translation length $2d$, where d is the distance between σ' and τ. Hence, since $\gamma(f, g) > 0$,

$$\gamma(f, g) = \mathrm{tr}[f, g] - 2 = +2\cosh d - 2.$$

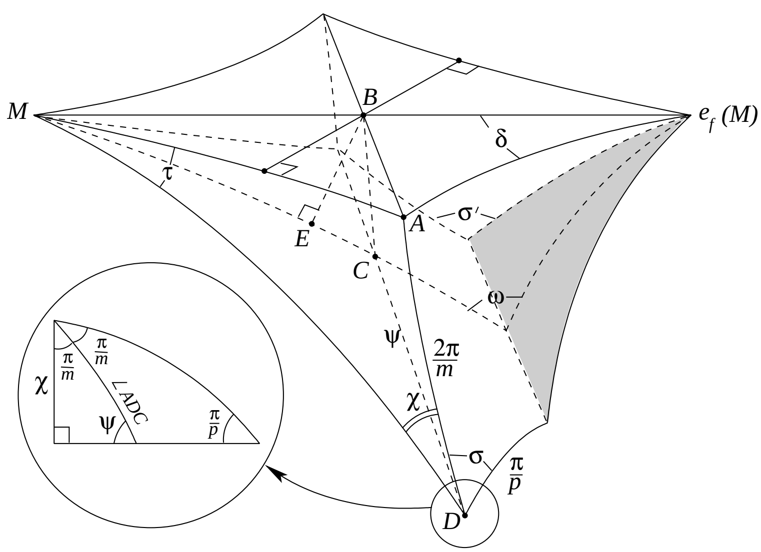


FIGURE 3.

Now, using generalised triangles in the plane δ, it is not difficult to calculate that

$$\gamma(f, g) = \begin{cases} 4\cos^2(\pi/m) & \text{if} \quad 3 \le m < \infty, \\ 4 & \text{if} \quad m = \infty, \\ 4\cosh^2(d(\sigma, \tau)/2) & \text{if} \quad m = \overline{\infty}, \end{cases}$$

where $d(\sigma, \tau)$ is the distance between σ and τ if they are disjoint. Hence,

$$\gamma(f, g) = 4\cosh^2 u,$$

where $u \in \mathcal{U}$, $t(u) = m \ge 3$.

Let us calculate $\beta(f)$. The element f is π-loxodromic if and only if $\mathrm{tr}^2 f = 4\cosh^2(T + i\pi/2) = -4\sinh^2 T$, where $2T$ is the translation length of f. That is,

$$\beta(f) = -4\sinh^2 T - 4.$$

Note that T is the distance between e and e_f. It is measured in ω and equals BE (see Figure 3).

Suppose that we are in case (2)(i) of Theorem 2.1, that is $(t(u), 2) = 2$, and that σ and τ intersect. Recall that ξ is the bisector of the dihedral angle of $\mathcal{P}^*$ made by σ and τ. Let ψ be the angle that ξ makes with η. Note that $\psi = \angle BCE$. From the link of D, we have that

$$\cos\chi = \frac{\cos(\pi/p)}{\sin(2\pi/m)} = \frac{\cos\psi}{\sin(\pi/m)}$$

and, therefore,

$$\cos\psi = \frac{\cos(\pi/p)}{2\cos(\pi/m)}. \tag{2.3}$$

Further, from the link of D,

$$\cos\angle ADC = \frac{\cos\psi\cdot\cos(\pi/m)}{\sin\psi\cdot\sin(\pi/m)}. \tag{2.4}$$

From the $\triangle ABM$, $\cosh^2 AB = 1/\sin(\pi/m)$ and, from the quadrilateral $ABCD$,

$$\sinh BC = \frac{\cos\angle ADC}{\sinh AB} \tag{2.5}$$

Finally, from $\triangle BCE$,

$$\sinh T = \sinh BE = \sin\psi\cdot\sinh BC. \tag{2.6}$$

Combining (2.3)–(2.6), we have that

$$\sinh^2 T = \frac{\cos^2(\pi/p)}{4\cos^2(\pi/m)} = \frac{\cos^2(\pi/p)}{\gamma(f,g)}.$$

Similar calculations can be done for parallel or disjoint σ and τ. Hence, $\beta(f) = -\sinh^2 T - 4 = -\cosh^2 v/\gamma(f,g) - 4$, where $v \in \mathcal{U}$, $t(v) \geq 3$.

Now note that in case (2)(ii) of Theorem 2.1, the angle $\psi = \angle BCE$ must be of the form π/k, $k \geq 3$ is an integer, ∞, or $\overline{\infty}$. Then we need to recompute the formulas (2.4)–(2.6) with $\psi = \pi/k$:

$$\cos\angle ADC = \frac{\cos(\pi/k)\cdot\cos(\pi/m)}{\sin(\pi/k)\cdot\sin(\pi/m)}, \quad \sinh BC = \frac{\cos\phi}{\sinh a} = \frac{\cos(\pi/k)}{\sin(\pi/k)}.$$

Then

$$\sinh T = \sin\psi\cdot\sinh BC = \cos(\pi/k).$$

Hence, $\beta(f) = -4\cosh^2 v - 4$, where $v \in \mathcal{U}$, $t(v) \geq 3$. □

3. Orbifolds

Denote by $\Omega(\Gamma)$ the discontinuity set of a Kleinian group Γ. The *Kleinian orbifold* $Q(\Gamma) = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ is said to be an orientable 3-orbifold with a complete hyperbolic structure on its interior $\mathbb{H}^3/\Gamma$ and a conformal structure on its boundary $\Omega(\Gamma)/\Gamma$.

We need the following (Kleinian) group presentations:

- $PH[\infty, m; q] = \langle x, y, s \mid x^\infty = s^2 = (xs)^2 = (ys)^2 = (xyxy^{-1})^m = (y^{-1}xys)^q = 1\rangle$,
- $P[\infty, m; q] = \langle w, x, y, z \mid w^\infty = x^2 = y^2 = z^2 = (wx)^2 = (wy)^2 = (yz)^2 = (zx)^q = (zw)^m = 1\rangle$,
- $\mathcal{S}_2[\infty, m; q] = \langle x, L \mid x^\infty = (xLxL^{-1})^m = (xL^2x^{-1}L^{-2})^q = 1\rangle$,
- $GTet_1[\infty, m; q] = \langle x, y, z \mid x^\infty = y^2 = z^\infty = (xy)^m = (yzy^{-1}z^{-1})^q = [x, z] = 1\rangle$.

Here m and q are integers greater than 1, or ∞ or $\overline{\infty}$ with the following convention. If we have a relation of the form $w^n = 1$ with $n = \overline{\infty}$, then we simply remove the relation $w^n = 1$ from the presentation (in fact, this means that the element w is hyperbolic). Further, if $n = \infty$ and we keep the relation $w^n = 1 \sim w^\infty = 1$, we get a Kleinian group presentation where parabolics are indicated. To get an abstract group presentation, we need to remove all relations of the form $w^\infty = 1$.

Theorem 3.1. *Let $\Gamma = \langle f, g\rangle$ be a non-elementary discrete $\mathcal{RP}$ group without invariant plane. Let $\beta(f) \in (-\infty, -4)$ and let $\beta(g) = 0$. Then $\gamma(f, g) = 4\cosh^2 u$, where $u \in \mathcal{U}$, $t(u) \geq 3$, and one of the following holds:*

1. *If $(t(u), 2) = 2$ and $\beta(f) = -4\cosh^2 v/\gamma(f, g) - 4$, where $v \in \mathcal{U}$, $t(v) \geq 3$, $(t(v), 2) = 1$, then Γ is isomorphic to $PH[\infty, t(u)/2; t(v)]$.*
2. *If $(t(u), 2) = 2$ and $\beta(f) = -4\cosh^2 v/\gamma(f, g) - 4$, where $v \in \mathcal{U}$, $t(v) \geq 4$, $(t(v), 2) = 2$, then Γ is isomorphic to $\mathcal{S}_2[\infty, t(u)/2; t(v)/2]$.*
3. *If $(t(u), 2) = 1$ and $\beta(f) = -4\cosh^2 v - 4$, where $v \in \mathcal{U}$, $t(v) \geq 3$, $(t(v), 2) = 1$, then Γ is isomorphic to $P[\infty, t(u); t(v)]$.*
4. *If $(t(u), 2) = 1$ and $\beta(f) = -4\cosh^2 v - 4$, where $v \in \mathcal{U}$, $t(v) \geq 4$, $(t(v), 2) = 2$, then Γ is isomorphic to $GTet_1[\infty, t(u); t(v)/2]$.*

Proof. Suppose $(t(u), 2) = 2$, that is the dihedral angle of $\mathcal{P}^*$ between σ and τ is $2\pi/m$ with m even, ∞, or $\overline{\infty}$. Consider a polyhedron $\widetilde{\mathcal{P}}$ bounded by σ, τ, $\sigma' = R_\omega(\sigma)$, $\tau' = R_\omega(\tau)$, η, and δ. Applying the Poincaré theorem to $\widetilde{\mathcal{P}}$ and the side pairing transformations g, $g' = R_\sigma R_\omega$, e, and e_f, one can see that $\langle g, g', e_f, e\rangle$ is isomorphic to $\widetilde{\Gamma}$ and has the presentation

$$\langle f, g, e \,|\, g^\infty = e^2 = (ef)^2 = (eg)^2 = (gfgf^{-1})^{m/2} = (f^{-1}gfe)^p = 1\rangle.$$

If p is odd, then $e \in \langle f, g\rangle$ and $\widetilde{\Gamma} = \Gamma \cong PH[\infty, m/2; p]$.

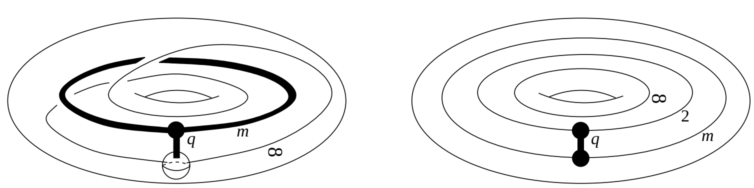
If p is even, ∞, or $\overline{\infty}$, then $\widetilde{\Gamma}$ contains Γ as a subgroup of index 2 and has presentation $\mathcal{S}_2[\infty, m/2; p/2]$. In order to see this, one can apply the Poincaré theorem to a polyhedron $\mathcal{P}$ bounded by τ, σ, τ', σ', η, and $e_f(\eta)$, and side-pairing transformations f, g, and $g' = fg^{-1}f^{-1}$.

The proof for $(t(u), 2) = 1$ is analogous. In this case we need to use the polyhedron $\mathcal{Q}^*$ as the starting point. □

(a) $\pi_1^{orb}(Q) \cong PH[\infty, m; q]$
$m \geq 2$, $q \geq 3$

(b) $\pi_1(Q)^{orb} \cong P[\infty, m; q]$
$m \geq 3$, $q \geq 3$

FIGURE 4. Orbifolds embedded in $\mathbb{S}^3$



(a) $\pi_1^{orb}(Q) \cong \mathcal{S}_2[\infty, m; q]$
$m \geq 2$, $q \geq 2$

(b) $\pi_1^{orb}(Q) \cong GTet_1[\infty, m; q]$
$m \geq 3$, $q \geq 2$

FIGURE 5. Orbifolds embedded in Seifert fibred spaces

The orbifolds $Q(\Gamma)$ for the groups described in Theorem 3.1 can be obtained from corresponding fundamental polyhedra. In Figures 4 and 5, we schematically draw singular sets, cusps, and boundary components of $Q(\Gamma)$ by using fat vertices and fat edges. Roughly speaking, a fat vertex is either an interior point, or is removed, or removed together with its regular neighbourhood depending on the indices. A fat edge can be labelled by ∞ or $\overline{\infty}$. If the index at a fat edge is ∞, then the egde corresponds to a cusp, and if the index is $\overline{\infty}$, the edge is removed together with its regular neighbourhood. For details, see [12].

In Figure 4, orbifolds are embedded in $\mathbb{S}^3$ so that ∞ is a non-singular interior point of $Q(\Gamma)$. Note that the volume of $Q(PH[\infty, m; q])$ is always infinite and $Q(P[\infty, m; q])$ is always non-compact.

Let $T(n)$ be a Seifert fibred solid torus obtained from a trivial fibred solid torus $D^2 \times \mathbb{S}^1$ by cutting it along $D^2 \times \{x\}$ for some $x \in \mathbb{S}^1$, rotating one of the discs through $2\pi/n$ and glueing back together.

Denote by $\mathcal{S}(n)$ a space obtained by glueing two copies of $T(n)$ along their boundaries fibre to fibre. Clearly, $\mathcal{S}(n)$ is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ and is n-fold covered by trivially fibred $\mathbb{S}^2 \times \mathbb{S}^1$. There are two critical fibres whose length is n times shorter than the length of a regular fibre.

In Figure 5(a), orbifolds are embedded in Seifert fibre spaces $\mathcal{S}(2) = T(2) \cup T(2)$. We draw only the solid torus that contains singular points (or boundary components). The other fibred torus is meant to be attached and is not shown. If $m < \infty$, the orbifold $Q(\mathcal{S}_2[\infty, m; q])$ is embedded in $\mathcal{S}(2)$ in such a manner that the axis of order m lies on a critical fibre of $\mathcal{S}(2)$. The removed regular fibre gives rise to a cusp.

In Figure 5(b), orbifolds are embedded in trivially fibred space $\mathbb{S}^2 \times \mathbb{S}^1$. The rank 2 cusp corresponds to the subgroup of $GTet_1[\infty, m; q]$ generated by x and z.

4. Structure of the slice S_∞

Recall that

$$S_\infty = \{(\gamma, \beta) : (\beta, 0, \gamma) \text{ are parameters for some } \langle f, g\rangle \in \mathcal{DRP}\},$$

where $\mathcal{DRP}$ denotes the class of all non-elementary discrete $\mathcal{RP}$ groups.

To investigate the slice S_∞, we split the plane (γ, β) as follows.

1. If $\beta = -4$ then by [9, Theorem 2], the group $\langle f, g\rangle$ has an invariant plane. We use [5] to find all discrete groups on the line $\beta = -4$.
2. If $\beta > -4$ and $\gamma > 0$ then the group $\langle f, g\rangle$ is conjugate to a subgroup of $\mathrm{PSL}(2, \mathbb{R})$. More precisely, if $-4 < \beta < 0$ then f is elliptic and the axis of f is orthogonal to an invariant plane of g and if $\beta = 0$ then the fixed points of f and g lie in their common invariant plane. Discreteness criteria in terms of traces of f, g, and fg were given in [14]. For $\beta > 0$, an algorithm to decide whether f and g generate a discrete group was given in [8].
3. If $\beta > -4$ and $\gamma < 0$ then f is elliptic, parabolic, or hyperbolic and the group $\langle f, g\rangle$ is known to be truly spatial. Discrete such groups are described in [11], where β and γ are found explicitly.
4. If $\beta < -4$ and $\gamma < 0$ then f is π-loxodromic whose axes lies in an invariant plane of g. Then this plane is invariant under action of $\langle f, g\rangle$ and f acts as

a glide-reflection on it. A geometrical description of such discrete groups was given in [13].

5. The case of $\beta < -4$ and $\gamma > 0$ was treated in Section 2 of the present paper.

We will obtain explicit formulas for β and γ in the cases 2 and 4 above and completely describe the structure of the slice S_∞. We will pay special attention to the subsets of S_∞ corresponding to free groups.

First, we need the following elementary facts.

Lemma 4.1. *If $f, g \in \mathrm{PSL}(2,\mathbb{C})$ and g is parabolic, then*

$$\gamma(f,g) = (\mathrm{tr}(fg) - \mathrm{sign}(\mathrm{tr}g)\cdot \mathrm{tr}f)^2.$$

Proof. By the Fricke identity, we have

$$\begin{aligned}\gamma(f,g) &= \mathrm{tr}[f,g] - 2\\ &= \mathrm{tr}^2 f + \mathrm{tr}^2 g + \mathrm{tr}^2(fg) - \mathrm{tr}f\cdot \mathrm{tr}g\cdot \mathrm{tr}(fg) - 4\\ &= (\mathrm{tr}(fg) - \mathrm{sign}(\mathrm{tr}g)\cdot \mathrm{tr}f)^2,\end{aligned}$$

since $\mathrm{tr}^2 g = 4$. □

Lemma 4.2. *If $f, g \in \mathrm{PSL}(2,\mathbb{C})$ and $\mathrm{tr}g = 2$, then*

$$\mathrm{tr}(fg^k) = k(\mathrm{tr}(fg) - \mathrm{tr}f) + \mathrm{tr}f.$$

Proof. By substituting $\mathrm{tr}g = 2$ into the recurrent formula

$$\mathrm{tr}(fg^k) = \mathrm{tr}(fg^{k-1})\mathrm{tr}g - \mathrm{tr}(fg^{k-2}),$$

we immediately get the result. □

Remark 4.3. *Suppose that f is non-primitive elliptic of finite order n, i.e., $\beta(f) = -4\sin^2(q\pi/n)$, where $(q,n) = 1$, $1 < q < n/2$. Then there exists an integer r so that f^r is primitive of the same order. Obviously, $\langle f, g\rangle = \langle f^r, g\rangle$ and $\beta(f^r) = -4\sin^2(\pi/n)$. By [7], $\gamma(f^r, g) = (\beta(f^r)/\beta(f))\gamma(f,g)$.*

It is natural to introduce the constant

$$C(q,n) = \frac{\sin^2(q\pi/n)}{\sin^2(\pi/n)} = \frac{\beta(f)}{\beta(f^r)} \geq 1$$

that plays an important role in parameters calculation concerning groups with elliptic elements. It is also convenient to consider a parabolic element f as a limit rotation of order $n = \infty$ and write $0 = \beta(f) = -4\sin^2(\pi/n)$ with $C(q,n) = C(1,n) = 1$.

4.1. $-4 \leq \beta \leq 0$**.** This means that f is either elliptic or parabolic. Obviously, if f is elliptic of infinite order, then $\langle f, g\rangle$ is not discrete. So we assume that $\beta = -4\sin^2(q\pi/n)$, where $(q,n) = 1$ and $1 \leq q < n/2$, including $\beta = 0$.

Theorem 4.4. *Let $\Gamma = \langle f, g\rangle \subset \mathrm{PSL}(2,\mathbb{C})$ have parameters $(\beta, 0, \gamma)$ with $\gamma \in \mathbb{R}\backslash\{0\}$. Let $\beta = -4\sin^2(q\pi/n)$, where $(q,n) = 1$ and $1 \leq q < n/2$, including $\beta = 0$. Then Γ is discrete if and only if one of the following holds:*

(1) *$\gamma = -4C(q,n)\cosh^2 u$, where $u \in \mathcal{U}$ and $t(u) \geq 3$;*
(2) *$\gamma = 4C(q,n)(\cos(\pi/n) + \cosh u)^2$, where $u \in \mathcal{U}$;*
(3) *$\beta = 0$ and $\gamma = 4(1 + \cos(2\pi/k))^2$, where $k \geq 3$ is odd.*

Proof. Let us prove the theorem for $q = 1$; in order to get the result for $q > 1$, we only need to apply Remark 4.3.

If $n = 2$ then $\beta = -4$ and, by [5, Theorem 4.15], Γ is discrete if and only if $\gamma = \pm 4\cosh^2 u$, where $u \in \mathcal{U}$ with $t(u) \geq 3$.

If $2 < n \leq \infty$ and $\gamma < 0$, then, by [11, Corollary 2.5], Γ is discrete if and only if $\gamma = -4\cosh^2 u$, where $u \in \mathcal{U}$ and $t(u) \geq 3$.

Assume that $2 < n < \infty$ and $\gamma > 0$. In this case Γ is conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ and we can apply Knapp's results [14] to compute γ. Conjugate Γ so that ∞ is the fixed point of g. By replacing, if necessary, f with f^{-1} and g with g^{-1}, we may assume that

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} -1 & \tau \\ 0 & -1 \end{pmatrix},$$

where $ad - bc = 1$, $a + d = -2\cos(\pi/n)$ with $n \in \mathbb{Z}$, $b > 0$, and $\tau > 0$.

One can show that $\mathrm{tr}(fg) < 2$. By [14, Proposition 4.1], Γ is discrete if and only if $\mathrm{tr}(fg) \leq -2$ or $\mathrm{tr}(fg) = -2\cos(\pi/k)$, where $k \geq 2$ is an integer, that is $\mathrm{tr}(fg) = -2\cosh u$, where $u \in \mathcal{U}$. Hence, by Lemma 4.1, $\gamma = (\mathrm{tr}(fg) + \mathrm{tr} f)^2 = (2\cosh u + 2\cos(\pi/n))^2$.

So it remains to consider the case when $n = \infty$ (i.e., $\beta = 0$) and $\gamma > 0$. Again, we normalize Γ so that g is as above and $f = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$. By [14, Proposition 4.2], such a group is discrete if and only if $\tau \geq 4$ or $\tau = 2 + 2\cos(2\pi/k)$ for an integer $k \geq 3$. Since in this case $\gamma = \tau^2$, we have that $\gamma \geq 16$ or $\gamma = (2 + 2\cos(2\pi/k))^2$, which can be written as $\gamma = 4(1 + \cosh u)^2$, where $u \in \mathcal{U}$, or $\gamma = 4(1 + \cos(2\pi/k))^2$ for odd $k \geq 3$. □

Remark 4.5. *If $-4 \leq \beta \leq 0$ then Γ is discrete and free if and only if $\beta = 0$ and $\gamma \in (-\infty, -4] \cup [16, +\infty)$.*

The parameters from the infinite strip $-4 \leq \beta \leq 0$ are displayed in Figure 6. If $\beta = -4\sin^2(q\pi/n)$ is fixed, then there exist values $\gamma_1(\beta) < 0$ and $\gamma_2(\beta) > 0$ so that Γ is discrete in the union of two rays $(-\infty, \gamma_1(\beta)] \cup [\gamma_2(\beta), +\infty)$. There are only countably many discrete groups in $(\gamma_1(\beta), \gamma_2(\beta))$ with accumulation points $\gamma_1(\beta)$ and $\gamma_2(\beta)$.

Moreover, if we denote $\beta_n^q = -4\sin^2(q\pi/n)$, then

$$\gamma_1(\beta_n^q) < \gamma_1(\beta_n^1) < \gamma_2(\beta_n^1) < \gamma_2(\beta_n^q) \quad \text{for all } 1 < q < n/2.$$

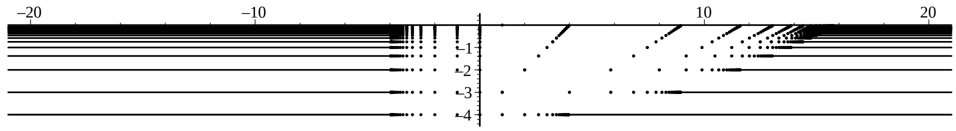
4.2. $\beta > 0$**.** In this case f is hyperbolic.

Theorem 4.6 ([11, Corollary 2.5])**.** *Let $\Gamma = \langle f, g \rangle \subset \mathrm{PSL}(2,\mathbb{C})$ have parameters $(\beta, 0, \gamma)$ with $\beta > 0$ and $\gamma < 0$. Then Γ is discrete if and only if $\gamma = -4\cosh^2 u$, where $u \in \mathcal{U}$, $t(u) \geq 3$.*

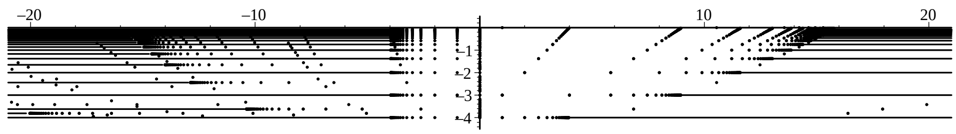
Remark 4.7. *From [11], Γ with parameters $(\beta, 0, \gamma)$, where $\beta \geq 0$ and $\gamma < 0$ is free if and only if (γ, β) lies in the region*

$$A = \{(\gamma, \beta) : \gamma \leq -4, \beta \geq 0\}.$$

Theorem 4.8. *Let $\Gamma = \langle f, g \rangle \subset \mathrm{PSL}(2,\mathbb{C})$ have parameters $(\beta, 0, \gamma)$ with $\beta > 0$ and $\gamma > 0$. Let $k = \left\lceil \dfrac{\sqrt{\beta + 4} - 2}{\sqrt{\gamma}} \right\rceil$. The group Γ is discrete if and only if one of the following holds:*



$\beta = -4\sin^2(\pi/n)$ with $n \in \mathbb{Z}$



$\beta = -4\sin^2(q\pi/n)$ with $(q,n) = 1$ and $1 \le q < n/2$

FIGURE 6. Structure of the strip $-4 \le \beta \le 0$

(1) $\beta = (k\sqrt{\gamma}+2)^2 - 4$ *and* $\gamma = 16\cosh^4 u$, *where* $u \in \mathcal{U}$ *and* $t(u) \ge 3$*;*
(2) $\beta = (k\sqrt{\gamma} \pm 2\cos(q\pi/n))^2 - 4$ *and* $\gamma = 4C(q,n)(\cos(\pi/n)+\cosh u)^2$, *where* $(q,n) = 1$, $1 \le q < n/2$, *and* $u \in \mathcal{U}$*;*
(3) $\beta = (k\sqrt{\gamma} - 2\cosh u)^2 - 4$ *and* $\gamma > 4(1+\cosh u)^2$, *where* $u \ge 0$.

Proof. Since $\gamma > 0$, the axis of f lies in an invariant plane of g, so $\Gamma = \langle f, g\rangle$ is conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. In [8], an algorithm for determining whether such a group is discrete was given. We will apply this algorithm and calculate parameters for each discrete group.

Normalize Γ so that ∞ is the fixed point of g and ± 1 are the fixed points of f. Then we can write

$$f = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}, \text{ where } a^2 - b^2 = 1,\ a > 1,\ b, \tau \in \mathbb{R}.$$

By replacing f with f^{-1} and g with g^{-1}, we may assume that $b < 0$ and $\tau > 0$.

Let k be a positive integer such that $\mathrm{tr}(fg^k) \le 2$ and $\mathrm{tr}(fg^\ell) > 2$ for all ℓ with $0 \le \ell < k$.

By Lemmas 4.1 and 4.2, we have that $k^2\gamma = k^2(\mathrm{tr}(fg) - \mathrm{tr} f)^2 = (\mathrm{tr}(fg^k) - \mathrm{tr} f)^2$. Since $\mathrm{tr}(fg^k) \le 2$ and $\mathrm{tr} f > 2$,

$$\mathrm{tr} f = k\sqrt{\gamma} + \mathrm{tr}(fg^k). \tag{4.7}$$

We distinguish three cases:

1. $\mathrm{tr}(fg^k) = 2$, that is fg^k is parabolic. From (4.7),

$$\beta = (k\sqrt{\gamma}+2)^2 - 4.$$

By Theorem 4.4, $\langle fg^k, g\rangle$ and, hence, $\langle f, g\rangle$ is discrete if and only if

$\gamma = \gamma(fg^k, g) = 4(1+\cosh v)^2$, where $v \in \mathcal{U}$, or
$\gamma = 4(1+\cos(2\pi/k))^2$, where $k \ge 3$ is odd.

These expressions can be rearranged and combined as $\gamma = 16\cosh^4 u$, where $u \in \mathcal{U}$ and $t(u) \ge 3$.

2. $-2 < \mathrm{tr}(fg^k) < 2$, that is fg^k is elliptic and $\mathrm{tr}(fg^k) = \pm 2\cos(q\pi/n)$, where $(q,n) = 1$ and $1 \le q < n/2$. Hence, from (4.7),

$$\beta = (k\sqrt{\gamma} \pm 2\cos(q\pi/n))^2 - 4.$$

By Theorem 4.4, $\langle fg^k, g\rangle$ and, hence, $\langle f, g\rangle$ is discrete if and only if

$$\gamma = 4C(q,n)(\cos(\pi/n)+\cosh u)^2, \quad \text{where } u \in \mathcal{U}.$$

3. $\mathrm{tr}(fg^k) \le -2$, that is fg^k is hyperbolic or parabolic so we can write $\mathrm{tr}(fg^k) = -2\cosh u$, where $u \ge 0$. Then

$$\beta = (k\sqrt{\gamma} - 2\cosh u)^2 - 4.$$

Consider the group $\langle g^{k-1}f, g\rangle$. The element $g^{k-1}f$ is hyperbolic with $\mathrm{tr}(g^{k-1}f) > 2$. Therefore, one can normalize $\langle g^{k-1}f, g\rangle$ so that the attracting and repelling fixed points of $g^{k-1}f$ are x_a and x_r, respectively, and $x_a < x_r$. Since $\mathrm{tr}(g^k f) \le -2$, such a group is dicrete and free by [8, Case II]. So by Lemma 4.1, we have that

$$\begin{aligned} \gamma = \gamma(fg^{k-1}, g) &= (\mathrm{tr}(fg^k) - \mathrm{tr}(fg^{k-1}))^2 \\ &= (2\cosh u + 2\cosh v)^2, \end{aligned}$$

where v is any positive real number.

It remains to compute k. Since $\mathrm{tr}(fg^k) = 2a + b\tau k \le 2$, we have that $k \ge (-2a+2)/(b\tau)$. Computing $\gamma = b^2\tau^2$, we get $b\tau = -\sqrt{\gamma}$. So $k = \left\lceil \dfrac{\sqrt{\beta+4}-2}{\sqrt{\gamma}} \right\rceil$. □

It follows from [8] that Γ is free if and only if (γ, β) lies in one of the regions

$$C_k = \{(\gamma,\beta) : \gamma \ge 16, ((k-1)\sqrt{\gamma}+2)^2 \le \beta + 4 \le (k\sqrt{\gamma}-2)^2\},\ k = 1,2,3\ldots$$

4.3. $\beta < -4$**.** First, consider $\gamma < 0$. In this case the axis of the π-loxodromic generator f lies in an invariant plane of g [9], so $\langle f, g\rangle$ keeps this plane invariant.

Theorem 4.9. *Let $\Gamma = \langle f, g\rangle \subset \mathrm{PSL}(2,\mathbb{C})$ have parameters $(\beta, 0, \gamma)$ with $\beta < -4$ and $\gamma < 0$. Let $k = \left\lceil \dfrac{\sqrt{-\beta-4}}{\sqrt{-\gamma}} \right\rceil$. Then the group $\langle f, g\rangle$ is discrete if and only if one of the following holds:*

1. *$-4(\beta+4) = \left((2k-1)\sqrt{-\gamma} \pm \sqrt{-\gamma - 8(1+\cosh u)}\right)^2$, where $u \in \mathcal{U}$;*
2. *$4(\beta+4) = (2k-1)^2\gamma$ and $\gamma = -16\cos^2(\pi/p)$, where $p \ge 3$ is odd;*
3. *$\beta = k^2\gamma - 4$ and $\gamma = -4\cosh^2 u$, where $u \in \mathcal{U}$ and $t(u) \ge 3$.*

Proof. Let $\delta = \{(z,t) : \mathrm{Im}\ z = 0\}$ be the invariant plane of Γ. Since the axis of f lies in δ, we can normalize Γ so that the fixed point of g is ∞, the fixed points of f are ± 1, and

$$f = \begin{pmatrix} ai & bi \\ bi & ai \end{pmatrix}, \quad g = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}, \quad \text{where } b^2 - a^2 = 1,\ a > 1,\ b, \tau \in \mathbb{R}.$$

Further, replacing f with f^{-1} and g with g^{-1}, we can assume that $b < 0$ and $\tau > 0$. Since b is negative, $+1$ is the repelling fixed point of f and -1 is attracting.

Let e be the half-turn whose axis passes through the fixed point of g orthogonally to the axis of f. That is e fixes 0 and ∞. Let e_f and e_1 be half-turns such that $f = ee_f$ and $g = e_1 e$. Since f is π-loxodromic, the axis of e_f intersects the axis of f (and the plane δ) orthogonally; denote the intersection point by A. Further, since g is parabolic and keeps δ invariant, the axis of e_1 fixes ∞ and lies in the plane δ. It is easy to calculate that

$$e = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_f = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix}, \quad e_1 = \begin{pmatrix} i & \tau \\ 0 & -i \end{pmatrix}.$$

Consider half-turns $e_{k-1} = g^{k-1}e$ and $e_k = g^k e$ such that A lies in the region bounded by the axes of e_{k-1} and e_k in the plane δ, see Figure 7. It is easy to calculate that $A = -a/b - j/b$. Since e_k fixes ∞ and $\tau k/2$, we have that

$$A \in \left\{(z,t) : \frac{\tau(k-1)}{2} < \mathrm{Re} z \leq \frac{\tau k}{2},\ \mathrm{Im}\, z = 0,\ t > 0\right\}.$$

Hence, we can immediately determine k.

$$\frac{\tau(k-1)}{2} < -\frac{a}{b} \leq \frac{\tau k}{2}. \tag{4.8}$$

Therefore, since $2a = -i\mathrm{tr} f = \sqrt{-\beta - 4}$ and $b\tau = -\sqrt{-\gamma}$,

$$k = \left\lceil -\frac{2a}{b\tau} \right\rceil = \left\lceil \frac{\sqrt{-\beta-4}}{\sqrt{-\gamma}} \right\rceil.$$

It is easy to see that Γ is discrete if and only if $\widetilde{\Gamma} = \langle e_f, e_{k-1}, e_k \rangle$ is. Following [13], we give geometric conditions for $\widetilde{\Gamma}$ to be discrete.

Suppose that $A \notin axis(e_k)$; see Figure 7(a). By [13], $\widetilde{\Gamma}$ is discrete if either

(a) the angle ϕ between e_{k-1} and $e_f(e_k)$ is of the form π/p, where $p \geq 2$ is an integer, ∞, or $\overline{\infty}$; or

(b) $\phi = 2\pi/p$, where $p \geq 3$ is odd and the bisector of ϕ passes through A.

Suppose that $A \in axis(e_k)$; see Figure 7(b). By [13], $\widetilde{\Gamma}$ is discrete if

(c) the angle ψ made by $axis(e_{k-1})$ and $axis(\tilde{e}_f)$ is of the form π/p, $p \geq 3$ is an integer, ∞, or $\overline{\infty}$, where $\tilde{e}_f = e_k e_f$ is the half-turn whose axis passes through A orthogonally to $axis(e_k)$ in the plane δ.

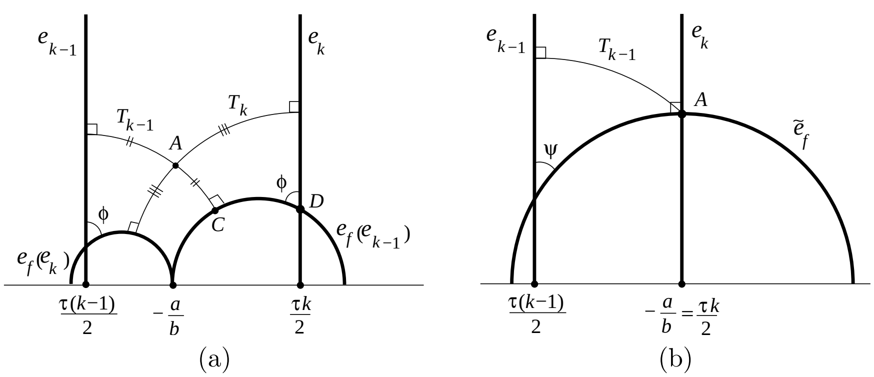


FIGURE 7. The invariant plane δ

There are no other discrete groups. So, we need to calculate the parameters β and γ in each of the cases (a), (b), and (c).

Assume that we are in case (a) or (b). Then each $g^\ell f = e_\ell e_f$, $\ell \in \mathbb{Z}$, is a π-loxodromic element with translation length $2T_\ell$ and $\mathrm{tr}(g^\ell f) = \pm 2i \sinh T_\ell$, where T_ℓ is the distance between e_ℓ and A. Moreover, from the matrix representation, $\mathrm{tr}(g^\ell f) = 2ai + b\tau\ell i$. The inequalities (4.8) enable us to determine the signs of $\mathrm{tr}(fg^{k-1})$ and $\mathrm{tr}(fg^k)$:

$$\mathrm{tr}(fg^k) = -2i\sinh T_k \quad \text{and} \quad \mathrm{tr}(fg^{k-1}) = +2i\sinh T_{k-1}.$$

Suppose that $p < \infty$. Simple calculations in the plane δ show that

$$\sinh CD = \frac{1 + \cos\phi \cosh(2T_{k-1})}{\sin\phi \sinh(2T_{k-1})}$$

and, on the other hand,

$$\sinh CD = \frac{\sinh T_k + \cos\phi \sinh T_{k-1}}{\sin\phi \cosh T_{k-1}}.$$

So, we obtain

$$2(1+\cos\phi) = 4\sinh T_{k-1}\sinh T_k = \mathrm{tr}(fg^{k-1})\mathrm{tr}(fg^k).$$

Applying Lemmas 4.1 and 4.2 and the facts that $\mathrm{tr}f = i\sqrt{-\beta-4}$ and $\mathrm{tr}(fg) - \mathrm{tr}f = b\tau i = -i\sqrt{-\gamma}$, we get

$$\begin{aligned} 2(1+\cos\phi) &= [(k-1)(\mathrm{tr}(fg) - \mathrm{tr}f) + \mathrm{tr}f] \cdot [k(\mathrm{tr}(fg) - \mathrm{tr}f) + \mathrm{tr}f] \\ &= k(k-1)(\mathrm{tr}(fg) - \mathrm{tr}f)^2 + (2k-1)\cdot \mathrm{tr}f \cdot (\mathrm{tr}(fg) - \mathrm{tr}f) + \mathrm{tr}^2 f \\ &= k(k-1)\gamma + (2k-1)\sqrt{-\beta-4}\sqrt{-\gamma} + \beta + 4. \end{aligned}$$

Hence, $-4(\beta+4) = ((2k-1)\sqrt{-\gamma} \pm \sqrt{-8(1+\cos\phi) - \gamma})^2$, where $\phi = \pi/p$, $p \geq 2$ is an integer. Analogous calculation can be done for $p = \infty$ and $p = \overline{\infty}$, and we obtain item (1) of the theorem.

In case (b), in addition, $T_{k-1} = T_k$. Then $\mathrm{tr}(fg^k) = -\mathrm{tr}(fg^{k-1})$ and by Lemmas 4.1 and 4.2 we have

$$2\sqrt{-\beta-4} = (2k-1)\sqrt{-\gamma}.$$

Therefore, $2(1+\cos\phi) = -\mathrm{tr}^2(fg^k) = (-k\sqrt{-\gamma} + \sqrt{-\beta-4})^2 = -\gamma/4$. Hence, since $\phi = 2\pi/p$, $\gamma = -16\cos^2(\pi/p)$.

Now assume that we are in case (c) and $p < \infty$. Since in this case $e_k e_f = \tilde{e}_f$ is an ellitic element of order 2, $\mathrm{tr}(g^k f) = 0$. Therefore, since $\mathrm{tr}(g^k f) = -ki\sqrt{-\gamma} + i\sqrt{-\beta-4}$, we have that $\beta = k^2\gamma - 4$.

Further, since $\mathrm{tr}(fg^{k-1}) = 2i\sinh T_{k-1}$ and, from the plane δ, $\sinh T_{k-1} = \cos\psi$, we have that

$$\begin{aligned} 4\cos^2\psi = 4\sinh^2 T_{k-1} &= -((k-1)(\mathrm{tr}(fg) - \mathrm{tr}f) + \mathrm{tr}f)^2 \\ &= (-(k-1)\sqrt{-\gamma} + \sqrt{-\beta-4})^2 \\ &= (-(k-1)\sqrt{-\gamma} + k\sqrt{-\gamma})^2 \\ &= -\gamma. \end{aligned}$$

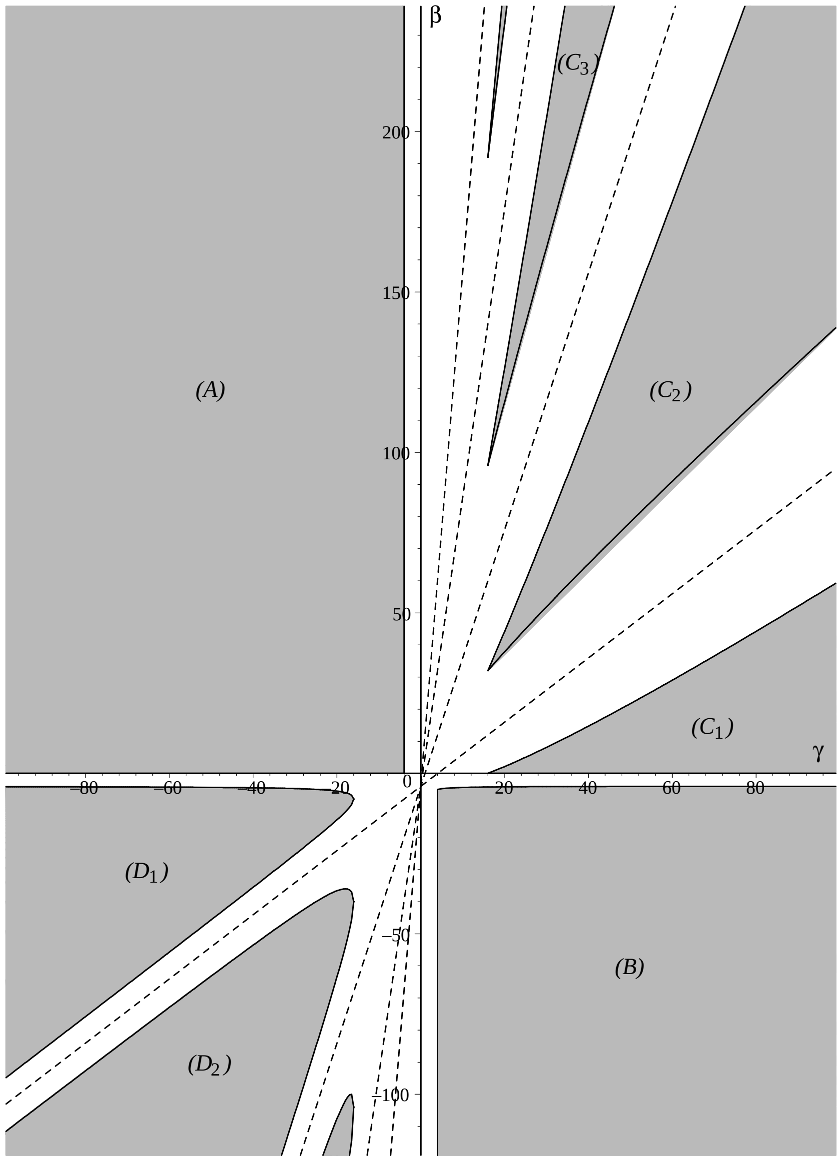
Thus, $\gamma = -4\cos^2(\pi/p)$, where $p \geq 3$ is an integer. Analogous calculations can be done for $p = \infty$ and $p = \overline{\infty}$ and we obtain item (3) of the theorem. □

Remark 4.10. *If $\beta < -4$ and $\gamma < 0$, then $\langle f, g\rangle$ is free if and only if (γ, β) lies in one of the regions D_k, $k = 1, 2, 3, \dots$, given by*

$$D_k = \{(\gamma,\beta) : \gamma \leq -16,\ \frac{((2k-1)\sqrt{-\gamma} - \sqrt{-\gamma-16})^2}{-4} \geq \beta + 4 \geq \frac{((2k-1)\sqrt{-\gamma} + \sqrt{-\gamma-16})^2}{-4}\}.$$

When $\gamma > 0$, the parameters were described in Theorem 2.6. Here we just note that for $\gamma > 0$ and $\beta < 0$, the group $\langle f, g\rangle$ is free if and only if (γ, β) lies in the region

$$B = \{(\gamma,\beta) : \gamma \geq 4,\ \beta + 4 \leq -4/\gamma\}.$$



$$A = \{(\gamma, \beta) : \gamma \le -4,\ \beta \ge 0\}$$
$$B = \{(\gamma, \beta) : \gamma \ge 4,\ \beta + 4 \le -4/\gamma\}$$
$$C_k = \{(\gamma, \beta) : \gamma \ge 16,\ ((k-1)\sqrt{\gamma} + 2)^2 \le \beta + 4 \le (k\sqrt{\gamma} - 2)^2\}$$
$$D_k = \{(\gamma, \beta) : \gamma \le -16,$$
$$\frac{((2k-1)\sqrt{-\gamma} + \sqrt{-\gamma - 16})^2}{-4} \le \beta + 4 \le \frac{((2k-1)\sqrt{-\gamma} - \sqrt{-\gamma - 16})^2}{-4}\}$$

Dashed lines $\beta = k^2\gamma - 4$, $k = 1, 2, 3, \ldots$

FIGURE 8. The discrete free groups

Finally, we are able to draw those subsets of S_∞ that correspond to discrete free groups. These subsets are shown in Figure 8. The dashed lines $\beta = k^2\gamma - 4$ are plotted to show a certain symmetry of S_∞.

The other discrete groups contain elliptic elements. Their parameters are represented by lines, parabolas, hyperbolas, and points accumulating, as orders of elliptic elements tend to ∞, to the regions of free groups.

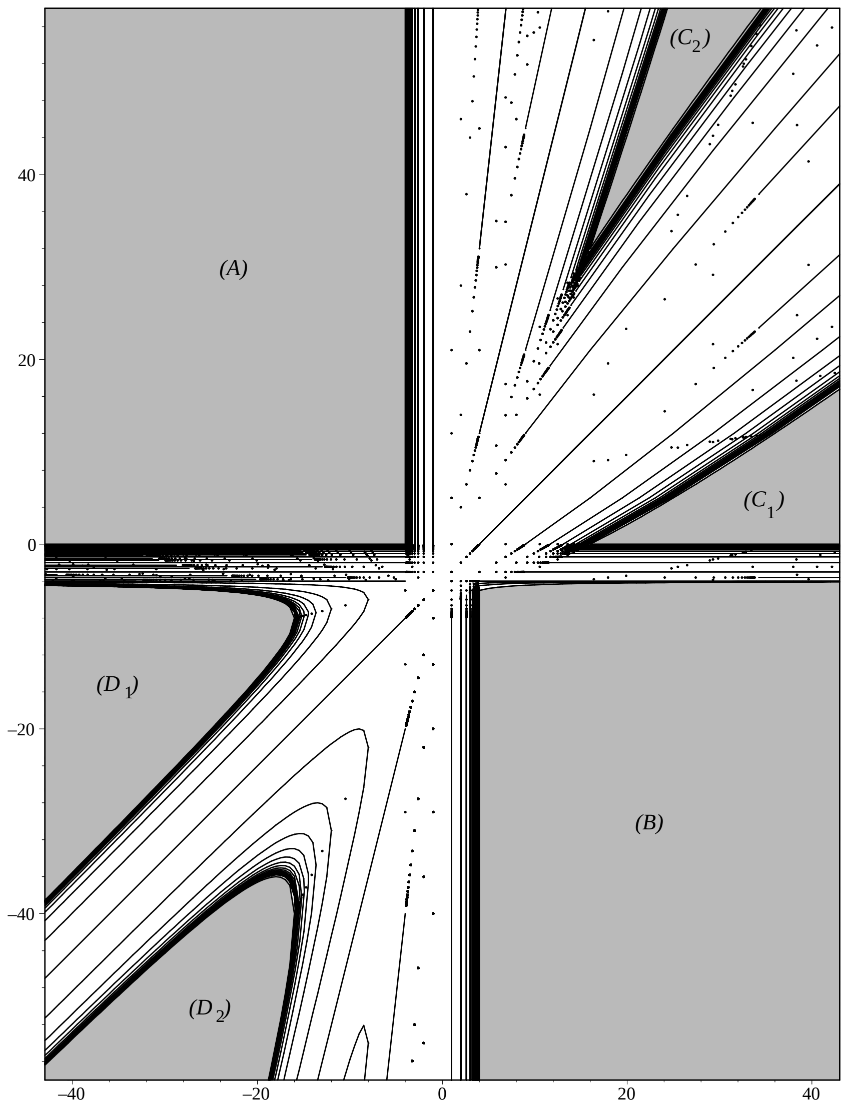


FIGURE 9. The structure of the slice S_∞

In Figure 9, the whole picture for the slice S_∞ is shown to give an idea of the structure of S_∞. The formulas for β and γ obtained in Theorems 2.6, 4.4, 4.6, 4.8, and 4.9, were programmed with the package Maple 7.0 for some (sufficiently large) values of independent variables like $n, q \in \mathbb{Z}$ and $u, v \in \mathcal{U}$ and plotted on the plane (γ, β).

The most interesting families of parameters appear when γ and β are of the same sign. For a fixed k, the hyperbolas

$$-4(\beta+4)=\Big((2k-1)\sqrt{-\gamma}\pm\sqrt{-\gamma-8(1+\cos(\pi/p))}\Big)^2,$$

where $p\geq 2$ is an integer, form a one-parameter family of curves converging to the boundary of D_k as $p\to\infty$. Each hyperbola has the asymptotes $\beta=(k-1)^2\gamma-4k(1+\cos(\pi/p))+4$ and $\beta=k^2\gamma+4k(1+\cos(\pi/p))-4$, which are obviously parallel to $\beta=(k-1)^2\gamma-4$ and $\beta=k^2\gamma-4$, respectively.

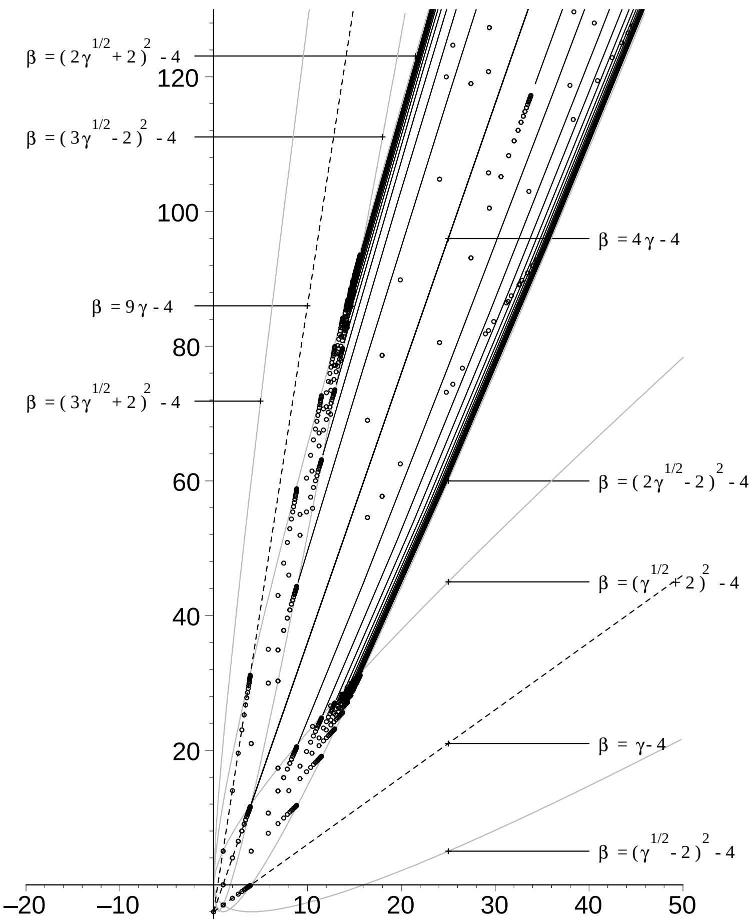


FIGURE 10. The structure of Σ_2

For $\gamma>0$ and $\beta>0$, consider a one-parameter family of parabolas $\beta_k=(k\sqrt{\gamma}\pm 2)^2-4$. Let Σ_k be the domain bounded by β_k:

$$\Sigma_k=\{(\gamma,\beta):(k\sqrt{\gamma}-2)^2\leq\beta+4\leq(k\sqrt{\gamma}+2)^2\}.$$

Within each Σ_k, the parameters for discrete groups are given by

$$\begin{cases}\beta=(k\sqrt{\gamma}\pm 2\cos(q\pi/n))^2-4,\\ \gamma=4C(q,n)(\cos(\pi/n)+\cosh u)^2,\end{cases}$$

where $(q,n)=1$, $1 \le q < n/2$, and $u \in \mathcal{U}$. Note that for $n=2$, we have $\beta = k^2\gamma - 4$ and $\gamma = 4\cosh^2 u$. As $n \to \infty$, the curves $\beta = (k\sqrt{\gamma} \pm 2\cos(q\pi/n))^2 - 4$ accumulate to the boundary of Σ_k, i.e., to the boundaries of C_{k-1} and C_k (see Figure 10 for an example of Σ_k for $k=2$).

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