# ON RINGS OF DIFFERENTIAL ROTA-BAXTER OPERATORS 

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#### Abstract

Using the language of operated algebras, we construct and investigate a class of operator rings and enriched modules induced by a derivation or Rota-Baxter operator. In applying the general framework to univariate polynomials, one is led to the integrodifferential analogs of the classical Weyl algebra. These are analyzed in terms of skew polynomial rings and noncommutative Gröbner bases.


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## 1. Introduction

The ring of differential operators $\mathcal{F}[\partial]$ over a given differential ring $(\mathcal{F}, \partial)$ is a fundamental algebraic structure ${ }^{\text {P }}$ in the area of differential algebra [25, 22, 8], especially in differential Galois theory [36] and $\mathcal{D}$-module theory [1]. Building on this framework and specializing to the case of linear ordinary differential equations (LODEs), the larger ring of integro-differential operators $\mathcal{F}\left[\partial, \int\right] \supset \mathcal{F}[\partial]$ over an integro-differential ring $\left(\mathcal{F}, \partial, \int\right)$ was introduced in [26, 27] for describing, computing and factoring the Green's operators of regular boundary problems for LODEs. As one knows from the classical theory, such Green's operators will be integrals with the Green's function as its nucleus ${ }^{( }$. Algebraically speaking, the Green's operators are contained in the ring of integral operators $\mathcal{F}\left[\int\right] \subset \mathcal{F}\left[\partial, \int\right]$ associated to the Rota-Baxter algebra $\left(\mathcal{F}, \int\right)$.

In the present paper we introduce the ring of differential Rota-Baxter operators $\mathcal{F}[\partial, \oint]$ over a given differential Rota-Baxter algebra $(\mathcal{F}, \partial, \oint)$. Although closely related to the integro-differential operator ring $\mathcal{F}\left[\partial, \int\right]$, this ring has a more delicate algebraic structure and a distinct range of applicability. In fact, we shall see that the ring of integro-differential

[^0]operators is a quotient of $\mathcal{F}[\partial, \oint]$ : Loosely speaking, we may view $(\mathcal{F}, \partial, \oint)$ as an integrodifferential algebra whose integral is initialized at a "generic point"; the passage to the quotient is then interpreted as "fixing the integration constant" (Propositions 3.4 and 5.5). For the particular case of polynomial coefficients $\mathcal{F}=\mathbf{k}[x]$ over a field $\mathbf{k} \supseteq \mathbb{Q}$, this has been studied in the context of the (integro-differential) Weyl algebra [24], including the aforementioned specialization isomorphism that fixes the integration constant. For this setting we now provide also a generalization isomorphism that goes the opposite route of embedding the finer structure of differential Rota-Baxter operators into an integro-differential operator ring containing a generic point (Theorem 5.6).

As to be expected from the quotient relation mentioned above, the ring of differential Rota-Baxter operators $\mathcal{F}[\partial, \oint]$ has a broader range of applicability. In particular, various classical distribution spaces from analysis can be construed as modules over $\mathcal{F}[\partial, \oint]$ but not over $\mathcal{F}\left[\partial, \int\right]$, taking $\mathcal{F}=C^{\infty}(\mathbb{R})$ or $\mathcal{F}=\mathbb{R}[x]$ as coefficients (Example 3.\&(c)). This is in stark contrast to $\mathcal{F}=C^{\infty}(\mathbb{R})$ or $\mathcal{F}=\mathbb{R}[x]$ itself, which is both an $\mathcal{F}\left[\partial, \int\right]$-module and an $\mathcal{F}[\partial, \oint]$-module, with $\int:=\oint:=\int_{0}^{x}$ the standard Rota-Baxter operator on $\mathcal{F}$. This is so because distributions can be differentiated arbitrarily but in general they cannot be evaluated (any point can be in the singular support of a distribution). In particular, the crucial identity $\int f^{\prime}=f-f(0)$ for smooth functions $f \in C^{\infty}(\mathbb{R})$ fails to hold for distributions $f \in \mathcal{D}^{\prime}(\mathbb{R})$. The upshot is that distributions have a sheaf structure (restrictions to open subsets) but no evaluations ("restrictions to points").

Besides the ring of differential Rota-Baxter operators $\mathcal{F}[\partial, \oint]$, which is the main object introduced in this paper, we have already mentioned the related operator rings $\mathcal{F}[\partial], \mathcal{F}[J]$ and $\mathcal{F}\left[\partial, \int\right]$. It should be recalled [27, Prop. 17] that the latter ring is more than the sum of the two others. In fact (Proposition 3.4), we have $\mathcal{F}\left[\partial, \int\right]=\mathcal{F}[\partial] \dot{+}\left[\int\right] \backslash \mathcal{F} \dot{+}(\mathrm{E})$ as $\mathbf{k}$-modules, where $\mathrm{E}:=1_{\mathcal{F}}-\int \circ \partial$ is the induced evaluation. Having four different operator rings, it will be expedient to describe a universal algebraic setting that allows to generate these four operator rings - and possibly others - in a uniform manner (Example 3.4).

In fact, we shall use a slightly more special setting that is better adapted to our needs: While universal algebra applies to all varieties (categories whose objects are sets $A$ endowed with any number of $n$-operations $A^{n} \rightarrow A$, subject to laws in equational form), we shall only need $\mathbf{k}$-algebras endowed with one or several unary operations $A \rightarrow A$, usually known as operated algebras. This leads to significant simplifications: While the algorithmic machinery of universal algebra is generally dependent on rewriting and the Knuth-Bendix algorithm, the situation of operated algebras is amenable to Gröbner(-Shirshov) bases [7], (6). Moreover, the latter are closely related to and compatible with the skew polynomial approach used in [24] for constructing the integro-differential Weyl algebra.

It should be emphasized that we allow arbitrary laws to be imposed on operated algebras, not just multilinear laws as one might be led to expect from the examples. For ground rings of characteristic zero, we show how to transform arbitrary laws to multilinear ones, using a suitable polarization process. This may also be the reason why the usual treatment is based on multilinear laws. For instance (Example 2.19(b)), Rota-Baxter algebras of weight zero are normally defined through the axiom $\left(\int f\right)\left(\int g\right)=\int f \int g+\int g \int f$, which may be viewed as the polarized version of $\left(\int f\right)^{2}=2 \int f \int f$; in characteristic zero these identities are equivalent. For showing that $\int$ is a Rota-Baxter operator, the latter identity may be better (e.g. using induction on some degree of $f$ rather than double induction on $f$ and $g$ ).

Since we have in mind various applications for function algebras, we restrict ourselves in this paper to operator rings over commutative algebras. However, the construction would work in essentially the same way for noncommutative algebras, writing the laws in terms of noncommutative rather than commutative decorated words (Definition 2.1). This could be employed for operator rings over matrix-valued functions; however, we shall not pursue this further in the scope of the present paper.

Terminology and Notation. We use $\mathbb{N}=\{0,1,2, \ldots\}$ for the natural numbers with zero. If $M$ is a (multiplicative) monoid $M$ with zero element $0 \in M$, the subset of nonzero elements is denoted by $M^{\times}:=M \backslash\{0\}$. If $Z$ is any set, we write respectively $M(Z)$ and $C(Z)$ for the free monoid and the free commutative monoid on $Z$; its identity is denoted by 1 .

Unless specified otherwise, all rings and algebras are assumed to be associative and unital (whereas nonunital rings will be called rungs). All modules and algebras are over a fixed commutative ring $\mathbf{k}$, which will be specialized to a field of characteristic zero in Section 5 . All modules are taken to be left modules. A (commutative or noncommutative) ring without zero divisor is called a domain. We write $\operatorname{Alg}_{R}$ and $\operatorname{Mod}_{R}$ for the category of $R$-modules and $R$-algebras, respectively (suppressing the subscript $R$ in the case $R=\mathbf{k}$ ). We denote the $\mathbf{k}$-span of a set $Z$ by $\mathbf{k} Z$; the ring of noncommutative polynomials is thus given by $\mathbf{k}\langle X\rangle:=$ $\mathbf{k} M(X)$ and the ring of commutative polynomials by $\mathbf{k}[X]:=\mathbf{k} C(X)$.

Let $A$ be a $\mathbf{k}$-module with $\mathbf{k}$-submodule $A^{\prime}$. Then $A \backslash A^{\prime}$ denotes a linear complement of $A$, rather than the set-theoretic one. (We will only use this notation when such linear complements exist and the specific choice is irrelevant.)

For $\partial$ and $\int$ we employ operator notation as in analysis; for example we write $(\partial f)(\partial g)$ rather than $\partial(f) \partial(g)$. Juxtaposition has precedence over the operators so that we have for instance $\partial f g:=\partial(f g)$ and $\int f \int g:=\int\left(f \int g\right)$. Moreover, we use also the customary notation $f^{\prime}$ for the derivative $\partial f$, and by analogy $f^{\prime}$ for the antiderivative $\int f$.

Structure of the Paper. In Section 2 we start by introducing the appropriate tools for describing varieties and their laws in the framework of $\Omega$-operated algebras. The main result in this section is the reduction of arbitrary laws to homogeneous and multilinear laws (Corollary 2.18). We end the section by introducing the four basic varieties coming from analysis (Example 2.19). In Section 3, the operator ring for a given variety is introduced (Definition 3.2). Modules over the operator rings are described equivalently as a special class of $\Omega$-operated modules (Proposition 3.6). The operator rings and modules are exemplified in the four basic varieties (Proposition 3.4 and Example 3.8). Section $\mathrm{Z}_{6}$ is devoted to one of the four operator rings that is introduced here for the first time: the ring of differential Rota-Baxter operators. Here the main results are a left adjoint to the forgetful functor from integro-differential to differential Rota-Baxter algebras (Theorem 4.6) and the embedding of the differential Rota-Baxter operator ring into a suitable integro-differential operator ring (Theorem 4.8). Finally, we turn to the important special case of polynomial coefficients in Section 5, thus considering integro-differential and differential Rota-Baxter analogs for the Weyl algebra. The most important result is that the so-called integro-differential algebra introduced in [24] is in fact the ring of differential Rota-Baxter operators with polynomial coefficients (Corollary 5.4), which also implies the embedding result by specializing the generic one (Theorem 5.6).

## 2. Varieties of Operated Algebras

Recall that an $\Omega$-operated algebra $\left(A ; P_{\omega} \mid \omega \in \Omega\right)$ is an algebra $A$ together with certain k-linear operators $P_{\omega}: A \rightarrow A$. Here no restrictions are imposed on the operators $P_{\omega}$. The category of $\Omega$-operated algebras is denoted by $\operatorname{Alg}(\Omega)$, the full subcategory of commutative $\Omega$-operated algebras by $\mathbf{C A l g}(\Omega)$. For $S \subseteq A$, we use the notation $(S)$ for the operated ideal generated by $S$.

We describe now the free object $\mathfrak{C}_{\Omega}(X)$ of $\mathbf{C A l g}(\Omega)$ over a countable set of generators $X$. The construction proceeds via stages $\mathfrak{C}_{\Omega, n}(X)$ that are defined recursively as follows. We start with $\mathfrak{C}_{\Omega, 0}(X):=C(X)$. Then for each $\omega \in \Omega$ we create $\lfloor C(X)\rfloor_{\omega}:=\left\{\lfloor u\rfloor_{\omega} \mid u \in C(X)\right\}$ as a disjoint copy of $C(X)$ and define

$$
\mathfrak{C}_{\Omega, 1}(X):=C\left(X \uplus \biguplus_{\omega \in \Omega}\lfloor C(X)\rfloor_{\omega}\right),
$$

where $\uplus$ means disjoint union. Note that elements in $\lfloor C(X)\rfloor_{\omega}$ are merely symbols indexed by $C(X)$; for example, $\lfloor 1\rfloor_{\omega}$ is not the identity. The inclusion $X \hookrightarrow X \uplus \biguplus\left\lfloor\mathfrak{C}_{\Omega, 0}(X)\right\rfloor_{\omega}$ induces a monomorphism

$$
i_{0,1}: \mathfrak{C}_{\Omega, 0}(X)=C(X) \hookrightarrow \mathfrak{C}_{\Omega, 1}(X)=C\left(X \uplus \biguplus_{\omega \in \Omega}\left\lfloor\mathfrak{C}_{\Omega, 0}(X)\right\rfloor_{\omega}\right)
$$

of free commutative monoids through which we identify $\mathfrak{C}_{\Omega, 0}(X)$ with its image in $\mathfrak{C}_{\Omega, 1}(X)$. For $n \geq 2$, inductively assume that $\mathfrak{C}_{\Omega, n-1}(X)$ has been defined and the embedding

$$
i_{n-2, n-1}: \mathfrak{C}_{\Omega, n-2}(X) \hookrightarrow \mathfrak{C}_{\Omega, n-1}(X)
$$

has been obtained. Then we define

$$
\mathfrak{C}_{\Omega, n}(X):=C\left(X \uplus \biguplus_{\omega \in \Omega}\left\lfloor\mathfrak{C}_{n-1}(X)\right\rfloor_{\omega}\right)
$$

$\left.\left.\begin{array}{l}\text { Since } \mathfrak{C}_{\Omega, n-1}(X)=C\left(X \uplus \biguplus_{\omega \in \Omega} \text { injections }\right.\end{array} \mathfrak{C}_{\Omega, n-2}(X)\right\rfloor_{\omega}\right)$ is a free commutative monoid, once again the

$$
\left\lfloor\mathfrak{C}_{\Omega, n-2}(X)\right\rfloor_{\omega} \hookrightarrow\left\lfloor\mathfrak{C}_{\Omega, n-1}(X)\right\rfloor_{\omega}
$$

induce a monoid embedding

$$
\mathfrak{C}_{\Omega, n-1}(X)=C\left(X \uplus \biguplus_{\omega \in \Omega}\left\lfloor\mathfrak{C}_{n-2}(X)\right\rfloor_{\omega}\right) \hookrightarrow \mathfrak{C}_{\Omega, n}(X)=C\left(X \uplus \biguplus_{\omega \in \Omega}\left\lfloor\mathfrak{C}_{n-1}(X)\right\rfloor_{\omega}\right) .
$$

Finally we define the monoid

$$
\mathfrak{C}_{\Omega}(X):=\bigcup_{n \geq 0} \mathfrak{C}_{\Omega, n}(X)
$$

whose elements are called (commutative) $\Omega$-decorated bracket words in $X$.
Definition 2.1. Let $X$ be a set, $\star$ a symbol not in $X$ and $X^{\star}:=X \cup\{\star\}$.
(a) By an $\Omega$-decorated $\star$-bracket word on $X$ we mean any expression in $\mathfrak{C}_{\Omega}\left(X^{\star}\right)$ with exactly one occurrence of $\star$. The set of all $\Omega$-decorated $\star$-bracket words on $X$ is denoted by $\mathfrak{C}_{\Omega}^{\star}(X)$.
(b) For $q \in \mathfrak{C}_{\Omega}^{\star}(X)$ and $u \in \mathfrak{C}_{\Omega}(X)$, we define $q[u]:=q[\star \mapsto u]$ to be the $\Omega$-decorated bracket word in $\mathfrak{C}(X)$ obtained by replacing the letter $\star$ in $q$ by $u$.
(c) For $s=\sum_{i} c_{i} u_{i} \in \mathbf{k} \mathfrak{C}_{\Omega}(X)$, where $c_{i} \in \mathbf{k}, u_{i} \in \mathfrak{C}_{\Omega}(X)$ and $q \in \mathfrak{C}_{\Omega}^{\star}(X)$, we define $q[s]:=\sum_{i} c_{i} q\left[u_{i}\right]$, which is in $\mathbf{k} \mathfrak{C}_{\Omega}(X)$.
More generally, with $\star_{1}, \cdots \star_{n}$ distinct symbols not in $X$, set $X^{\star n}:=X \cup\left\{\star_{1}, \ldots, \star_{n}\right\}$.
(d) We define an $\Omega$-decorated $\left(\star_{1}, \ldots, \star_{n}\right)$-bracket word on $X$ to be an expression in $\mathfrak{C}_{\Omega}\left(X^{\star n}\right)$ with exactly one occurrence of each of $\star_{j}, 1 \leq j \leq n$. The set of all $\Omega$-decorated $\left(\star_{1}, \ldots, \star_{n}\right)$-bracket words on $X$ is denoted by $\mathfrak{C}_{\Omega}^{\star n}(X)$.
(e) For $q \in \mathfrak{C}_{\Omega}^{\star n}(X)$ and $u_{1}, \ldots, u_{n} \in \mathbf{k} \mathfrak{C}_{\Omega}(X)$, we define

$$
q\left[u_{1}, \ldots, u_{n}\right]:=q\left[\star_{1} \mapsto u_{1}, \ldots, \star_{n} \mapsto u_{n}\right]
$$

to be obtained by replacing the letter $\star_{j}$ in $q$ by $u_{j}$ for $1 \leq j \leq n$.
The notation $q[\theta]$ used above for $\theta=\{\star \mapsto u\}$ and $\theta=\left\{\star_{1} \mapsto u_{1}, \ldots, \star_{n} \mapsto u_{n}\right\}$ can be extended to any substitution $\theta: X^{\star n} \xrightarrow{\sim} X^{\star n}$; see below after Proposition 2.3.

Now we describe the free object in the category $\mathbf{C A l g}(\Omega)$. For each $\omega \in \Omega$ we introduce an operator $\left\rfloor_{\omega}: \mathfrak{C}_{\Omega}(X) \rightarrow \mathfrak{C}_{\Omega}(X)\right.$ acting as $u \mapsto\lfloor u\rfloor_{\omega}$. Then $\left(\mathfrak{C}_{\Omega}(X) ;\lfloor \rfloor_{\omega} \mid \omega \in \Omega\right)$ is a commutative operated monoid; its linear $\operatorname{span}\left(\mathbf{k}_{\Omega}(X) ;\lfloor \rfloor_{\omega \in \Omega} \mid \omega \in \Omega\right)$ is a commutative operated algebra. It is moreover free in the sense of the following proposition [14, 17]. In the language of universal algebra, $\mathbf{k} \mathfrak{C}_{\Omega}(X)$ appears as the term algebra in the variety of $\Omega$-operated algebras (1).

Proposition 2.2. The triple $\left.\left(\mathbf{k} \mathfrak{C}_{\Omega}(X) ;\right\rfloor_{\omega} \mid \omega \in \Omega ; j_{X}\right)$, with $j_{X}: X \hookrightarrow \mathfrak{C}_{\Omega}(X)$ the natural embedding, is the free commutative $\Omega$-operated algebra on $X$. In other words, for any commutative $\Omega$-operated algebra $A$ and any set map $f: X \rightarrow A$, there is a unique extension of $f$ to a homomorphism $\bar{f}: \mathbf{k} \mathfrak{C}_{\Omega}(X) \rightarrow A$ of $\Omega$-operated algebras.

In the remainder of this section, we assume that $\mathbf{k}$ is a $\mathbb{Q}$-algebra. We first define polarization for the non-commutative case and then induce polarization for the commutative case via a natural homomorphism. The term polarization is adopted from Rota's early study [ 0 , p. 928] of this normalization process (second line in the proof of Prop. 2.1).

The construction of noncommutative $\Omega$-decorated bracket words $\mathfrak{M}_{\Omega}(X)$ is parallel to the commutative case $\mathfrak{C}_{\Omega}(X)$, using everywhere $M(X)$ in place of $C(X)$; the reader is referred to [17] for details. Clearly, $\mathbf{k}_{\Omega}(X)$ is the quotient of $\mathbf{k} \mathfrak{M}_{\Omega}(X)$ modulo the commutators.

Proposition 2.3. The triple $\left(\mathbf{k} \mathfrak{M}_{\Omega}(X) ;\lfloor \rfloor_{\omega} \mid \omega \in \Omega ; j_{X}\right)$, with $j_{X}: X \hookrightarrow \mathfrak{M}_{\Omega}(X)$ again the natural embedding, is the free $\Omega$-operated algebra on $X$. This means for any $\Omega$-operated algebra $A$ and for any set map $f: X \rightarrow A$, there exists a unique extension of $f$ to a homomorphism $\bar{f}: \mathbf{k M}_{\Omega}(X) \rightarrow A$ of $\Omega$-operated algebras.

Operated algebras usually satisfy additional relations, for example the aforementioned Rota-Baxter axiom $\left(\int f\right)\left(\int g\right)=\int f \int g+\int g \int f$ in the case of Rota-Baxter algebras. We model such relations by decorated bracket words $E \subseteq \mathbf{k}_{\Omega}(Y)$ or $E \subseteq \mathbf{k} \mathfrak{M}_{\Omega}(Y)$, depending on whether we intend the commutative or noncommutative case. Note that here we use a new set of variables $Y$ that should be distinct from the set $X$ of generators (see Proposition 2.7 below for an example combining the two sets of variables). Since relations are closed under linear combinations, we may take $E \subseteq \mathbf{k}_{\Omega}(Y)$ or $E \subseteq \mathbf{k} \mathfrak{M}_{\Omega}(Y)$ to be $\mathbf{k}$-submodules. Elements $l \in E$ will be called laws of the corresponding variety. In the following, we assume the noncommutative case but everything can be translated easily to the commutative case to which we shall return explicitly before Lemma 2.15.

For any operated algebra $A$ and $\theta: Y \rightarrow A$, by the universal property of $\mathbf{k} \mathfrak{M}_{\Omega}(Y)$ as the free $\Omega$-operated algebra on $Y$, there is a unique morphism of $\Omega$-operated algebras $\bar{\theta}: \mathbf{k} \mathfrak{M}_{\Omega}(Y) \rightarrow A$ that extends $\theta$. We use the notation $l[\theta]:=\bar{\theta}(l)$ for the corresponding instance of an $l \in \mathbf{k} \mathfrak{M}_{\Omega}(Y)$; formally this is the element of $A$ obtained from $l$ upon replacing
every $y \in Y$ by $\theta(y) \in A$, and $\bigsqcup_{\omega}$ by $P_{\omega}$ for $\omega \in \Omega$. For the special case $A=\mathbf{k} \mathfrak{M}_{\Omega}(Y)$ this covers the substitutions mentioned in Definition 2.1.

Definition 2.4. Let $E$ be a submodule of $\mathbf{k M}_{\Omega}(Y)$.
(a) An E-related algebra is defined to be an $\Omega$-operated algebra $A$ such that $l[\theta]=0$ for any law $l \in E$ and any assignment $\theta: Y \rightarrow A$.
(b) The substitution closure $\mathcal{S}(E) \subseteq \mathbf{k M}_{\Omega}(Y)$ of the laws $E$ is defined to be the submodule spanned by all instances $l[\theta]$ with $l \in E$ and $\theta: Y \rightarrow \mathbf{k} \mathfrak{M}_{\Omega}(Y)$.
If $E=\{l\}$, we speak of an $l$-algebra, and we write $\mathcal{S}(l)$ for $\mathcal{S}(E)$.
Since it is usually clear from the context that $E$ denotes a set of laws (rather than the ground ring), we will often say $E$-algebra instead of $E$-related algebra. In the terminology of universal algebra, the category of $E$-algebras (for a fixed set of laws $E$ ) forms a variety, which we write here as $\operatorname{Alg}(\Omega \mid E)$. As mentioned in the Introduction, the concept of variety is more general since its operators need not be unary or linear. For our purposes, this extended generality is not needed and would only complicate matters, for instance using congruence relations in place of operated ideals [10, §1.2].

Lemma 2.5. Let $E$ be a submodule of $\mathbf{k} \mathfrak{M}_{\Omega}(Y)$. Then every $E$-algebra is an $\mathcal{S}(E)$-algebra, and vice versa.

Proof. The sufficiency is clear since we have $E \subseteq \mathcal{S}(E)$. For showing the necessity, let $A$ be an $E$-algebra, and take $l[\theta] \in \mathcal{S}(E)$ with $l \in E$ and $\theta: Y \rightarrow \mathbf{k} \mathfrak{M}_{\Omega}(Y)$. For any $\eta: Y \rightarrow A$, define $\tilde{\eta}: Y \rightarrow A$ by setting $\tilde{\eta}(y):=\theta(y)[\eta]$ for any $y \in Y$, that is, $\tilde{\eta}(y)$ is obtained by replacing $y$ in $\theta(y)$ by $\eta(y)$. Then we have $l[\theta][\eta]=l[\tilde{\eta}]=0$, as $A$ is an $E$-algebra and $l \in E$.

Example 2.6. Let $\Omega$ be a singleton and $E=\mathbf{k}\left\{\left\lfloor y_{1} y_{2}\right\rfloor-\left\lfloor y_{1}\right\rfloor y_{2}-y_{1}\left\lfloor y_{2}\right\rfloor\right\}$. Then

$$
\mathcal{S}(E)=\mathbf{k}\left\{\lfloor u v\rfloor-\lfloor u\rfloor v-u\lfloor v\rfloor \mid u, v \in \mathfrak{M}_{\Omega}(Y)\right\}
$$

is the substitution closure. (This describes the variety of differential algebras.)
Using substitution closure, it is easy to characterize the free E-algebra. Again, this is a special case of a well-known result in universal algebra [10, Prop. 1.3.6].

Proposition 2.7. For any submodule $E \subseteq \mathbf{k M}_{\Omega}(Y)$ and any set $X$, let $S_{E}=S_{E}(X)$ denote the operated ideal of $\mathbf{k} \mathfrak{M}_{\Omega}(X)$ generated by all $l[\theta]$ with $l \in E$ and $\theta: Y \rightarrow \mathbf{k} \mathfrak{M}_{\Omega}(X)$. Then the free $E$-algebra on $X$ is the quotient $F_{E}(X):=\mathbf{k} \mathfrak{M}_{\Omega}(X) / S_{E}$.

Further exploiting the linear structure of $\Omega$-operated algebras, it turns out that we may actually assume that $E$ consists of linear combinations of multilinear monomials sharing the same variables. Let us make this precise. Given a monomial $u \in \mathfrak{M}_{\Omega}(Y)$, we define its degree in $y \in Y$, denoted by $\operatorname{deg}_{y} u$, as the number of times that $y$ appears in $u$. Its total degree is given by $\operatorname{deg} u:=\sum_{y \in Y} \operatorname{deg}_{y} u$. Note that if $\operatorname{deg}_{y} u=n$ there exists $q \in \mathfrak{M}_{\Omega}^{\star n}(Y \backslash\{y\})$ such that $u=q[y, \ldots, y]$. We call $l$ multilinear $\cap$ if $\operatorname{deg}_{y} l=1$ for each variable $y$ appearing in $l$.

[^1]For any $l \in E$ and $y \in Y$, let $l_{y, n}(n \geq 0)$ denote the linear combination of those monomials of $l$ that have degree $n$ in $y$, with the convention that $l_{y, n}=\delta_{0, n} l$ if $y$ does not appear in $l$. Then $l$ has the unique homogeneous decomposition

$$
l=\sum_{n \geq 0} l_{y, n}
$$

into its $y$-homogeneous parts.
Definition 2.8. Let $l \in \mathbf{k} \mathfrak{M}_{\Omega}(Y)$ be homogeneous in $y \in Y$ with $\operatorname{deg}_{y} l=n$ such that one has $l=\sum_{i \leq k} c_{i} q_{i}[y, \ldots, y]$ with coefficients $c_{i} \in \mathbf{k}^{\times}$and monomials $q_{i} \in \mathfrak{M}_{\Omega}^{\star n}(Y \backslash\{y\})$. Then the polarization of $l$ in $y$ is

$$
\mathcal{P}_{y}(l):=\sum_{\tau \in S_{n}} \sum_{i \leq k} c_{i} q_{i}\left[\tau y_{1}, \ldots, \tau y_{n}\right]
$$

where the substitution variables $y_{1}, \ldots, y_{n} \in Y$ are mutually distinct.
Note that $l$ can be recovered (up to a multiple) from $\mathcal{P}_{y}(l)$ through replacing $y_{1}, \ldots, y_{n}$ by $y$; this process is called centralization. For terms containing more variables, we can also apply polarization so long as the terms are homogeneous in all variables.

Definition 2.9. Let $l \in \mathbf{k M}_{\Omega}(Y)$ be homogeneous in all its variables. Then we define its polarization $\mathcal{P}(l)$ as the result of successively polarizing all variables in $l$.

A different order of the variables in $l$ in the polarization process and a different choice of the substitution variables in $l$ amounts to a bijection of the substitution variables. Thus the polarization of $l$ is unique (up to bijection of variables) and multilinear. Renaming variables if necessary, we may further assume that the number of variables not appearing in $E$ is countably infinite; hence polarization will not run out of substitution variables.

Example 2.10. In this example, $\Omega$ is a singleton so that we may abbreviate $\lfloor\ldots\rfloor_{\omega}$ by $\lfloor\ldots\rfloor$ and $\mathfrak{M}_{\Omega}(Y)$ by $\mathfrak{M}(Y)$.
(a) Consider $l=\lfloor y\lfloor y\rfloor\rfloor \in \mathfrak{M}(Y)$ with $y \in Y$. Its polarization is given by $\mathcal{P}(l)=\mathcal{P}_{y}(l)=$ $\left\lfloor y_{1}\left\lfloor y_{2}\right\rfloor\right\rfloor+\left\lfloor y_{2}\left\lfloor y_{1}\right\rfloor\right\rfloor$, and we recover $2\lfloor y\lfloor y\rfloor\rfloor$ by $y_{1}, y_{2} \mapsto y$.
(b) Let $l=x^{2} y^{2} \in \mathfrak{M}(Y)$ with $x, y \in Y$. Then $\mathcal{P}_{y}(l)=x^{2} y_{1} y_{2}+x^{2} y_{2} y_{1}$ and hence $\mathcal{P}(l)=\mathcal{P}_{x}\left(\mathcal{P}_{y}(l)\right)=y_{3} y_{4} y_{1} y_{2}+y_{4} y_{3} y_{1} y_{2}+y_{3} y_{4} y_{2} y_{1}+y_{4} y_{3} y_{2} y_{1}$ with $y_{1}, y_{2}, y_{3}, y_{4} \in Y$.
(c) For $l=\left\lfloor y^{2}\right\rfloor-2\lfloor y\rfloor y \in \mathbf{k M}(Y)$ we get $\mathcal{P}_{y}(l)=\left\lfloor y_{1} y_{2}\right\rfloor+\left\lfloor y_{2} y_{1}\right\rfloor-2\left\lfloor y_{1}\right\rfloor y_{2}-2\left\lfloor y_{2}\right\rfloor y_{1}$.

Lemma 2.11. Let $E \subseteq \mathbf{k} \mathfrak{M}_{\Omega}(Y)$ be a submodule and $l \in \mathcal{S}(E)$ arbitrary. If $l=\sum_{i} l_{y, i}$ is the homogeneous decomposition of $l$ in $y$, we have $l_{y, i} \in \mathcal{S}(E)$ for each $i$.

Proof. If $n$ is the maximal degree of $l$ in $y$, clearly $l_{y, i}=0 \in \mathcal{S}(E)$ for $i>n$. Replacing $y$ in $l$ by $j y$ for $1 \leq j \leq n+1$, we obtain

$$
l[j y]=\sum_{i=0}^{n} j^{i} l_{y, i} .
$$

Regard these equations as a linear system in unknowns $l_{y, i}$. Then the coefficient matrix is non-singular as a Vandermonde matrix. Thus one can solve for $l_{y, 0}, \ldots, l_{y, n}$ as $\mathbb{Q}$-linear combinations of $l[j y] \in \mathcal{S}(E)$, which shows that the $l_{y, i}$ are themselves in $\mathcal{S}(E)$.

We can use the preceding technique to conclude that the polarized form of a law is always contained in the substitution closure. To see why this is so, consider a typical example $l=\lfloor y\lfloor y\rfloor\rfloor \in \mathfrak{M}(Y)$ with $y \in Y$. Replacing $y$ by $y_{1}+y_{2}$ with $y_{1}, y_{2} \in Y$, we have

$$
\left\lfloor\left(y_{1}+y_{2}\right)\left\lfloor y_{1}+y_{2}\right\rfloor\right\rfloor=\left\lfloor y_{1}\left\lfloor y_{1}\right\rfloor\right\rfloor+\left\lfloor y_{2}\left\lfloor y_{2}\right\rfloor\right\rfloor+\left\lfloor y_{1}\left\lfloor y_{2}\right\rfloor\right\rfloor+\left\lfloor y_{2}\left\lfloor y_{1}\right\rfloor\right\rfloor \in \mathcal{S}(l)
$$

and so $\mathcal{P}_{y}(l):=\left\lfloor y_{1}\left\lfloor y_{2}\right\rfloor\right\rfloor+\left\lfloor y_{2}\left\lfloor y_{1}\right\rfloor\right\rfloor \in \mathcal{S}(l)$ by Lemma 2.11. Let us state the general result.
Lemma 2.12. Let $E \subseteq \mathbf{k} \mathfrak{M}_{\Omega}(Y)$ be a submodule, and assume $l \in E$ is homogeneous in $y$. Then we have $\mathcal{P}_{y}(l) \in \mathcal{S}(E)$.

Proof. Since $\mathcal{P}_{y}$ is a $\mathbf{k}$-linear operator, we only need to consider $l \in \mathfrak{M}_{\Omega}(Y)$. We prove the result by induction on $\operatorname{deg}_{y} l$. If $\operatorname{deg}_{y} l=1$, we have $\mathcal{P}_{y}(l)=l \in \mathcal{S}(E)$. Assuming the result for $\operatorname{deg}_{y} l \leq n-1$, we consider the case $\operatorname{deg}_{y} l=n$. By our assumption on $l$, we have $l=q[y, \ldots, y]$ with $q \in \mathfrak{M}_{\Omega}^{\star n}(Y \backslash\{y\})$. Replacing $y$ by $z_{1}+z_{2}$, we obtain

$$
\begin{equation*}
\mathcal{S}(E) \ni q\left[z_{1}+z_{2}, \cdots, z_{1}+z_{2}\right]=q\left[z_{1}, \ldots, z_{1}\right]+q\left[z_{2}, \ldots, z_{2}\right]+\tilde{l} . \tag{1}
\end{equation*}
$$

Since $q\left[z_{1}, \ldots, z_{1}\right]$ and $q\left[z_{2}, \cdots, z_{2}\right]$ are in the $\mathbf{k}$-module $\mathcal{S}(E)$ we have $\tilde{l} \in \mathcal{S}(E)$. We determine the homogeneous decomposition $\tilde{l}=\sum_{\tilde{\sim}=1}^{n-1} \tilde{l}_{z_{1}, j}$ with respect to $z_{1}$. From Lemma 2.11, we know $\tilde{l}_{z_{1}, j} \in \mathcal{S}(E)$. We note that $\operatorname{deg}_{z_{1}} \tilde{l}<n$ and likewise $\operatorname{deg}_{z_{2}} \tilde{l}<n$. By the induction hypothesis, we have $\mathcal{P}_{z_{1}}\left(\tilde{l}_{z_{1}, j}\right) \in \mathcal{S}(E)$ for $0<j<n$. Moreover, $\tilde{l}_{z_{1}, j}$ is homogeneous in $z_{1}$ of degree $j>0$, hence $\tilde{l}_{z_{1}, j}$ is homogeneous in $z_{2}$ of degree $n-j<n$. From the definition of $\mathcal{P}_{z_{1}}\left(\tilde{l}_{z_{1}, j}\right)$, we see that $\mathcal{P}_{z_{1}}\left(\tilde{l}_{z_{1}, j}\right)$ is also homogeneous in $z_{2}$ of degree $n-j<n$. By the induction hypothesis again, we obtain now $\mathcal{P}_{z_{2}}\left(\mathcal{P}_{z_{1}}\left(\tilde{l}_{z_{1}, j}\right)\right) \in \mathcal{S}(E)$. Thus it suffices to prove

$$
\begin{equation*}
\mathcal{P}_{y}(l)=\mathcal{P}_{z_{2}}\left(\mathcal{P}_{z_{1}}\left(\tilde{l}_{z_{1}, j}\right)\right) \tag{2}
\end{equation*}
$$

for $0<j<n$. By its definition, $\mathcal{P}_{y}(l)$ is a sum of $n!$ terms each of the form $q\left[\tau y_{1}, \ldots, \tau y_{n}\right]$ with $\tau \in S_{n}$, so we write it as

$$
\begin{equation*}
\mathcal{P}_{y}(l)=\sum_{\tau \in S_{n}} q\left[\tau y_{1}, \ldots, \tau y_{n}\right] \tag{3}
\end{equation*}
$$

On the other hand, we note that

$$
q\left[z_{1}+z_{2}, \ldots, z_{1}+z_{2}\right]=\sum_{I \subseteq[n]} q\left[I ; z_{1}, z_{2}\right],
$$

where for each subset $I \subseteq[n]$, the term $q\left[I ; z_{1}, z_{2}\right]$ is obtained from $q$ by replacing $\star_{i}$ by $z_{1}$ for $i \in I$ and by $z_{2}$ otherwise. For $I=[n]$ and $I=\emptyset$ we obtain the first two terms in (11), while for $0<j<n$ we get

$$
\tilde{l}_{z_{1}, j}=\sum_{|I|=j} q\left[I ; z_{1}, z_{2}\right]
$$

Then we have

$$
\begin{equation*}
\mathcal{P}_{z_{2}}\left(\mathcal{P}_{z_{1}}\left(\tilde{l}_{z_{1}, j}\right)\right)=\sum_{|I|=j, \tau_{1}, \tau_{2}} q\left[I ; \tau_{1}, \tau_{2}\right] \tag{4}
\end{equation*}
$$

where $\tau_{1}$ ranges over all bijections $I \xrightarrow{\sim}[j]$ and $\tau_{2}$ over all bijections $[n] \backslash I \xrightarrow{\sim}[n] \backslash[j]$, and where $q\left[I ; \tau_{1}, \tau_{2}\right]:=q\left[\tau y_{1}, \ldots, \tau y_{n}\right]$ is obtained from $q \in \mathfrak{C}_{\Omega}^{\star n}(X)$ via the permutation $\tau \in S_{n}$ defined by $\tau(i)=\tau_{1}(i)$ for $i \in I$ and $\tau(i)=\tau_{2}(i)$ for $i \in[n] \backslash I$. By this construction, distinct triples $\left(I ; \tau_{1}, \tau_{2}\right)$ corresponds to distinct permutations $\tau \in S_{n}$, so distinct monomials
in Eq. (4) also correspond to distinct monomials in Eq. (33). However, there are exactly $\binom{n}{j} j!(n-j)!=n!$ such triples, so the sums in the two equations must agree, and the proof of (2) is complete.

We introduce now polarization for a collection of laws $E$. It turns out that the resulting module, spanned by multilinear monomials, defines the same variety as the original $E$.

Definition 2.13. Let $E$ be a submodule of $\mathbf{k} \mathfrak{M}_{\Omega}(Y)$. Then we define its polarization $\mathcal{P}(E)$ as the submodule of $\mathbf{k} \mathfrak{M}_{\Omega}(Y)$ spanned by the polarizations of all homogeneous components of elements of $E$.

Theorem 2.14. For any submodule $E \subseteq \mathbf{k} \mathfrak{M}_{\Omega}(Y)$, an $\Omega$-operated algebra is an $E$-algebra if and only if it is a $\mathcal{P}(E)$-algebra.

Proof. By construction, we have $E \subseteq \mathcal{P}(E)$. By Lemma 2.5, it suffices to prove that $\mathcal{P}(E)$ is contained in $\mathcal{S}(E)$. Choose a law $l \in E$ and a variable $y \in Y$ appearing in $l$. Then we have $\mathcal{P}_{y}(l) \in \mathcal{S}(E)$ from Lemma 2.12; repeating the process for the other variables completes the proof.

Let us now go back to the commutative case. The concepts of degree and total degree can of course be defined in the same way. By a straightforward induction on the depth of bracket words, one obtains the following normalization result.

Lemma 2.15. Every element of $\mathfrak{C}_{\Omega}(Y)$ can be uniquely written as a bracket word in which all variables of $Y$ appear in increasing order.

The lemma gives an embedding $\varrho: \mathbf{k}_{\Omega}(Y) \hookrightarrow \mathbf{k} \mathfrak{M}_{\Omega}(Y)$ as modules. On the other hand, as algebras, we have $\mathbf{k}_{\Omega}(Y) \cong \mathbf{k} \mathfrak{M}_{\Omega}(Y) / \sim$, where $\sim$ is the operated ideal of $\mathbf{k} \mathfrak{M}_{\Omega}(Y)$ generated by the set $\left\{u v-v u \mid u, v \in \mathfrak{M}_{\Omega}(Y)\right\}$. Let $\pi: \mathbf{k} \mathfrak{M}_{\Omega}(Y) \rightarrow \mathbf{k} \mathfrak{C}_{\Omega}(Y)$ be the natural projection. We carry over the notion of polarization from the noncommutative case, in the following natural way (by abuse of notation we continue to use the same symbol $\mathcal{P}$ for the commutative polarization).

Definition 2.16. Let $E$ be a submodule of $\mathbf{k} \mathfrak{C}_{\Omega}(Y)$. If $l \in \mathbf{k}_{\Omega}(Y)$ is homogeneous in all its variables, we define its polarization as $\mathcal{P}(l):=\pi(\mathcal{P}(\varrho(l)))$. Similarly, the polarization of the module is defined as $\mathcal{P}(E):=\pi(\mathcal{P}(\varrho(E)))$.

Example 2.17. As in Example 2.10, we suppress the (unique) operator labels.
(a) Let $l=x^{2} y^{2} \in \mathbf{k} \mathfrak{C}_{\Omega}(Y)$ with $x, y \in Y$. Then its polarization is $\mathcal{P}(l)=4 y_{1} y_{2} y_{3} y_{4}$ with $y_{1}, y_{2}, y_{3}, y_{4} \in Y$.
(b) For $l=\left\lfloor y^{2}\right\rfloor-2\lfloor y\rfloor y \in \mathbf{k} \mathfrak{C}_{\Omega}(Y)$ we have $\mathcal{P}_{y}(l)=2\left\lfloor y_{1} y_{2}\right\rfloor-2\left\lfloor y_{1}\right\rfloor y_{2}-2 y_{1}\left\lfloor y_{2}\right\rfloor$.

As an immediate corollary to Theorem 2.14, we obtain that also in the commutative case one may polarize all laws and still describe the same variety.

Corollary 2.18. For any submodule $E \subseteq \mathbf{k}_{\Omega}(Y)$, an $\Omega$-operated algebra is an $E$-algebra if and only if it is a $\mathcal{P}(E)$-algebra.

In this sense, it is no loss of generality if one requires that $E$-algebras be described by multilinear laws (but see our remarks in the Introduction). The classical examples for varieties of operated algebras are indeed of this form. For avoiding cumbersome notation, we shall henceforth dispense with the brackets in the main examples, writing $\partial f$ for $\lfloor f\rfloor_{\partial}$
and $\int f$ for $\lfloor f\rfloor_{f}$. Likewise, we shall often identify the operations $P_{\omega}: A \rightarrow A$ of an operated algebra $\left(A ; P_{\omega} \mid \omega \in \Omega\right)$ with their labels $\omega$. Note also that an $E$-algebra is to be understood as the corresponding $\mathbf{k} E$-algebra if $E$ is not already a $\mathbf{k}$-submodule of $\mathbf{k} \mathfrak{C}_{\Omega}(Y)$. Of course, equations of the form $l=r$ are a shorthand for $l-r \in \mathbf{k} \mathfrak{C}_{\Omega}(Y)$.
Example 2.19. Take $Y=\{f, g\}$ for the variables. Then the four main varieties for doing analysis are the following collections of $E$-algebras with operators $\Omega$.
(a) The variety Diff $_{\lambda}$ of differential $\mathbf{k}$-algebras [25, 22, (8] of weight $\lambda \in \mathbf{k}$ :

Here $\Omega\left(\mathbf{D i f f}_{\lambda}\right)=\{\partial\}$, and $E\left(\mathbf{D i f f}_{\lambda}\right):=\{\partial f g=(\partial f) g+f(\partial g)+\lambda(\partial f)(\partial g)\}$ consists only of the Leibniz axiom.
(b) The variety $\mathbf{R B}_{\lambda}$ of Rota-Baxter $\mathbf{k}$-algebras [3, 31, 32, 18] of weight $\lambda \in \mathbf{k}$ :

Here $\Omega\left(\mathbf{R B}_{\lambda}\right)=\left\{\int\right\}$ and $E\left(\mathbf{R B}_{\lambda}\right):=\left\{\left(\int f\right)\left(\int g\right)=\int f \int g+\int g \int f+\lambda \int f g\right\}$ consists of the Rota-Baxter axiom.
(c) The variety $\mathbf{D R B}_{\lambda}$ of differential Rota-Baxter $\mathbf{k}$-algebras [19] of weight $\lambda \in \mathbf{k}$ :

Now $\Omega\left(\mathbf{D R B}_{\lambda}\right)=\Omega\left(\mathbf{D i f f}_{\lambda}\right) \cup \Omega\left(\mathbf{R B}_{\lambda}\right)=\left\{\partial, \int\right\}$ contains both operators, and the laws are given by $E\left(\mathbf{D R B}_{\lambda}\right)=E\left(\mathbf{D i f f}_{\lambda}\right) \cup E\left(\mathbf{R B}_{\lambda}\right) \cup\left\{\partial \int f=f\right\}$. The last law is the so-called section axiom, which specifies a "weak coupling" between $\partial$ and $\int$.
(d) The variety $\mathbf{I D}_{\lambda}$ of integro-differential $\mathbf{k}$-algebras [20] of weight $\lambda \in \mathbf{k}$ :

This has the same operators $\Omega\left(\mathbf{I D}_{\lambda}\right)=\Omega\left(\mathbf{D R B}_{\lambda}\right)$ but different laws-the weak coupling of $\mathbf{D R B}_{\lambda}$ is replaced by a stronger coupling [2Q, Thm. 2.5]: From various equivalent formulations, we choose

$$
E\left(\mathbf{I D}_{\lambda}\right)=E\left(\mathbf{D i f f}_{\lambda}\right) \cup\left\{f \int g=\int f^{\prime} \int g+\int f g+\lambda \int f^{\prime} g, \quad \partial \int f=f\right\}
$$

where the middle law describes integration by parts (which is strictly stronger than the Rota-Baxter axiom of $\mathbf{R B}_{\lambda}$ ).
As in [27, Def. 8] we call a differential (differential Rota-Baxter, integro-differential) algebra ordinary if $\operatorname{ker} \partial=\mathbf{k}$. For example, $(\mathbf{k}[x], d / d x)$ is ordinary but $(\mathbf{k}[x, y], \partial / \partial x)$ is not.

The above varieties provide the basic motivation for our study of the operator rings (to be defined in the next section), which are crucial for solving boundary problems in an algebraic setting. While this is not the focus of the present paper, the reader may refer to the end of the next section for some remarks on this application (and especially on the role played by differential Rota-Baxter algebras).

## 3. Operator Rings and Modules

We begin now with the description of the operator rings for a given variety of operated algebras. This proceeds in two steps - we introduce first a class of operator rings that does not take into account any law that might be imposed on a given operated algebra (Definition 3.1). In the second step we can then impose the given laws in a suitable form onto the free operators constructed in the first step (Definition 3.2). As mentioned in the Introduction, we work here only with commutative coefficient algebras, so from now on everything is commutative (except of course the operator rings).
Definition 3.1. Let $\left(A ; P_{\omega} \mid \omega \in \Omega\right)$ be a commutative $\Omega$-operated algebra. Then we define the induced ring of free operators as the free product $A[\Omega]:=A * \mathbf{k}\langle\Omega\rangle$.

The obvious abuse of notation $A[\Omega]$ is harmless since confusion with the commutative polynomial ring is unlikely. See also Remark 3.3 for further justification of this notation.

If $A$ is an $E$-algebra for a submodule $E \subseteq \mathbf{k}_{\Omega}(Y)$, we would like an operator ring that reflects the laws of $E$; we will construct it as a suitable quotient of the free operators $A[\Omega]$, using the following translation from laws to operators. The operator corresponding to a specific law shall be called the induced relator. For example, a differential ring $(A, \partial)$ is an $\Omega$-operated algebra with $\Omega=\{\partial\}$ satisfying the Leibniz law $(f g)^{\prime}=f^{\prime} g+f g^{\prime}$, which induces the relator $\partial f-f \partial-f^{\prime} \in A[\Omega]$; see Proposition 3.4 (a) for more details.

From Corollary 2.18, we may assume that $E \subseteq \mathbf{k}_{\Omega}(Y)$ is spanned by homogeneous and multilinear elements. We may also assume that none of these is of total degree 0 since such laws are either redundant (if $l=0$ ) or else describe a trivial variety. Since $Y$ is countable, we can write its elements as $y_{j}(j \in \mathbb{N})$. Then every basis element of $E$ having total degree $k+1$ can be written in the variables $y_{0}, \ldots, y_{k}$ by a change of variables; the resulting variety remains the same by Lemma 2.5. We call such basis elements the standard laws for the variety. For the translation process, we think of the lead variable $y_{0}$ as the argument of the induced relator with $y_{1}, \ldots, y_{k}$ constituting its parameters. The latter can be instantiated by assignments, which we view as arbitrary maps $\mathfrak{a}: Y^{\prime} \rightarrow A$ on the parameter set $Y^{\prime}:=Y \backslash\left\{y_{0}\right\}$. Since arguments are processed from right to left, we shall use the order $y_{k}, \ldots, y_{1}, y_{0}$ in the sequel.

The induced relator $[l]_{\mathfrak{a}} \in A[\Omega]$ for a standard law $l$ under an assignment $\mathfrak{a}$ is now defined by recursion on the depth of $l$. Taking $l \mapsto[l]_{\mathfrak{a}}$ to be $\mathbf{k}$-linear, it suffices to consider monomials $l$. For the base case take $l \in \mathfrak{C}_{\Omega, 0}\left(y_{k}, \ldots, y_{1}, y_{0}\right)=C\left(y_{k}, \ldots, y_{1}, y_{0}\right)$ with $l$ of total degree $k+1$. By multilinearity $l=y_{k} \cdots y_{1} y_{0}$, and we set $[l]_{\mathfrak{a}}:=\mathfrak{a}\left(y_{k}\right) \cdots \mathfrak{a}\left(y_{1}\right)$. Now assume $[\ldots]_{\mathfrak{a}}$ has been defined for monomial standard laws of depth at most $n$ and consider $l \in \mathfrak{C}_{\Omega, n+1}\left(y_{k}, \ldots, y_{1}, y_{0}\right)$. By multilinearity and the definition of $\mathfrak{C}_{\Omega, n+1}$, there exists $t \in \mathfrak{C}_{\Omega, n+1}\left(y_{k}, \ldots, y_{1}\right)$ such that either $l=t y_{0}$ or $l=t\left\lfloor l^{\prime}\right\rfloor_{\omega}$ for a certain operator label $\omega \in \Omega$ and $l^{\prime} \in \mathfrak{C}_{\Omega, n}\left(y_{k}, \ldots, y_{1}, y_{0}\right)$ being a monomial standard law of depth $n$. We set $[l]_{\mathfrak{a}}:=\overline{\mathfrak{a}}(t)$ in the former case and use the recursion $[l]_{\mathfrak{a}}:=\overline{\mathfrak{a}}(t) \omega\left[l^{\prime}\right]_{\mathfrak{a}}$ in the latter, where $\overline{\mathfrak{a}}: \mathfrak{C}_{\Omega, n+1}\left(y_{k}, \ldots, y_{1}\right) \rightarrow A$ is the monoid homomorphism induced by the (restricted) assignment map $\mathfrak{a}:\left\{y_{k}, \ldots, y_{1}\right\} \rightarrow A$ through the universal property of $\mathfrak{C}_{\Omega, n+1}\left(y_{k}, \ldots, y_{1}\right)$. This completes the definition of $[l]_{\mathfrak{a}}$. We can now introduce the ring of $E$-operators as the quotient of the free operators modulo the translated variety laws.

Definition 3.2. Let $A$ be an $E$-algebra for a submodule $E \subseteq \mathbf{k}_{\Omega}(Y)$. Then we define the ring of $E$-operators as $A[\Omega \mid E]:=A[\Omega] /[E]$, where $[E] \subset A[\Omega]$ is the ideal generated by $[l]_{\mathfrak{a}}$ for all standard laws $l \in E$ and assignments $\mathfrak{a}: Y^{\prime} \rightarrow A$.

Let us now look at the classical linear operator rings for the varieties of Example 2.19. Each of them comes with a noncommutative Gröbner basis and term order, providing transparent canonical forms and enabling a computational treatment via the well-known Diamond Lemma [日, Thm. 1.2]. As in the latter reference, we write the elements of the Gröbner basis in the form $m \rightarrow p$ instead of $m-p$ in order to emphasize the role of the leading monomial $m$ and the tail polynomial $p$, suggesting their use as rewrite rules.

Remark 3.3. In the sequel, we identify identities with the varieties they define; for example we write $\mathcal{F}\left[\partial \mid \operatorname{Diff}_{\lambda}\right]$ for $\mathcal{F}\left[\partial \mid E\left(\right.\right.$ Diff $\left.\left._{\lambda}\right)\right]$, with $E\left(\right.$ Diff $\left._{\lambda}\right)$ taken from Example 2.19 (a). In practice, this notation is of course contracted to $\mathcal{F}[\partial]$, further justifying the abuse of notation mentioned after Definition 3.1.

Proposition 3.4. Let $>$ be any graded lexicographic term order on $\mathcal{F}[\Omega]$ satisfying $\partial>f$ for all $f \in \mathcal{F}$ if $\partial \in \Omega$. Then the following four linear operator rings can be characterized by Gröbner bases as follows (primes and backprimes refer to the operations in $\mathcal{F}$ ):
(a) Given $(\mathcal{F}, \partial) \in \operatorname{Diff}_{\lambda}$, the ring of differential operators $\mathcal{F}[\partial]:=\mathcal{F}\left[\partial \mid \mathbf{D i f f}_{\lambda}\right]$ has the Gröbner basis $\mathrm{GB}\left(\mathbf{D i f f}_{\lambda}\right)=\left\{\partial f \rightarrow f \partial+\lambda f^{\prime} \partial+f^{\prime} \mid(f \in \mathcal{F})\right\}$.
(b) Given $\left(\mathcal{F}, \int\right) \in \mathbf{R B}_{\lambda}$, the ring of integral operators $\mathcal{F}\left[\int\right]:=\mathcal{F}\left[\int \mid \mathbf{R B}_{\lambda}\right]$ has the Gröbner basis $\operatorname{GB}\left(\mathbf{R B}_{\lambda}\right)=\left\{\int f \int \rightarrow f^{\prime} \int-\int f^{\prime}-\lambda \int f \mid f \in \mathcal{F}\right\}$.
(c) Given $(\mathcal{F}, \partial, \oint) \in \mathbf{D R B}_{\lambda}$, we consider next the ring of differential Rota-Baxter operators $\mathcal{F}[\partial, \oint]:=\mathcal{F}\left[\partial, \oint \mid \mathbf{D R B}_{\lambda}\right]$. Its Gröbner basis is given by the combined rewrite rules $\mathrm{GB}\left(\mathbf{D R B}_{\lambda}\right)=\mathrm{GB}\left(\mathbf{D i f f}_{\lambda}\right) \cup \mathrm{GB}\left(\mathbf{R B}_{\lambda}\right) \cup\left\{\partial \int \rightarrow 1\right\}$.
(d) $\operatorname{For}\left(\mathcal{F}, \partial, \int\right) \in \mathbf{I D}_{\lambda}$, the ring of integro-differential operators $\mathcal{F}\left[\partial, \int\right]:=\mathcal{F}\left[\partial, \int \mid \mathbf{I D}_{\lambda}\right]$ has Gröbner basis $\mathrm{GB}\left(\mathbf{I D}_{\lambda}\right):=\mathrm{GB}\left(\mathbf{D R B}_{\lambda}\right) \cup\left\{\int f \partial \rightarrow \underline{f}-\int \underline{f}^{\prime}-\mathrm{E}(\underline{f}) \mathrm{E} \mid f \in \mathcal{F}\right\}$, provided the shift $f \mapsto f+\lambda f^{\prime}$ has an inverse $f \mapsto \underline{f}$.
We have $\mathcal{F}\left[\partial, \int\right]=\mathcal{F}[\partial]+\mathcal{F}[\mathcal{S}] \backslash \mathcal{F} \dot{+}(\mathrm{E})$ as $\mathbf{k}$-modules, where $\mathcal{F}\left[\partial, \int\right]$ contains both $\mathcal{F}[\partial]$ and $\mathcal{F}[J]$ as subalgebras. Moreover, if $I$ is the ideal generated by $\{\mathrm{E} f-\mathrm{E}(f) \mathrm{E} \mid f \in \mathcal{F}\}$, we have $\mathcal{F}\left[\partial, \int\right] \cong \mathcal{F}[\partial, \oint] / I$, where $\oint:=\int$ is viewed as part of $(\mathcal{F}, \partial, \oint) \in \mathbf{D R B}_{\lambda}$.

Proof. Let us first prove the four items stated in the proposition (viewing all axioms in the main variable $g$ and using arbitrary assignments $\mathfrak{a}$ with $\mathfrak{a}(f) \in \mathcal{F}$ shortened to $f$ ):
(a) Clearly, the only relators are $[l]_{\mathfrak{a}}=\partial f-f^{\prime}-f \partial-\lambda f^{\prime} \partial$, corresponding to the Leibniz axiom $l:=\partial f g-(\partial f) g-f(\partial g)-\lambda(\partial f)(\partial g)=0$. From $\partial>f$ one sees that the leading monomial is $\partial f$. There is just one S-polynomial coming from the overlap ambiguity $\partial f g$ between the rule $\partial f \rightarrow f \partial+\lambda f^{\prime} \partial+f^{\prime}$ and the (tacit) rule $f g \rightarrow f *_{\mathcal{F}} g$. Using the Leibniz rule in $\mathcal{F}$, one checks immediately that the S-polynomial reduces to zero, so $\mathrm{GB}\left(\mathrm{Diff}_{\lambda}\right)$ is indeed a Gröbner basis.
(b) Here the Rota-Baxter axiom $l:=\left(\int f\right)\left(\int g\right)-\int f \int g-\int g \int f-\lambda \int f g=0$ yields the relators $[l]_{\mathfrak{a}}=f^{`} \int-\int f \int-\int f^{\prime}-\lambda \int f$ whose leading monomial is $\int f \int$ because the term order is graded. One obtains an S-polynomial from the self-overlap $\int f \int \bar{f} \int$ of the rule $\int f \int \rightarrow f^{\prime} \int-\int f^{\prime}-\lambda \int f$. Again one checks that this S-polynomial reduces to zero, and $\operatorname{GB}\left(\mathbf{R B}_{\lambda}\right)$ is thus a Gröbner basis.
(c) The relators are those of (a) and (b), and additionally $\partial \int-1$ whose corresponding rule is clearly $\partial \int \rightarrow 1$ because of the grading. Apart from the previous ones, we have the additional overlap ambiguity $\partial \int f \int$, and again its S-polynomial immediately reduces to zero so that $\mathrm{GB}\left(\mathbf{D R B}_{\lambda}\right)$ is a Gröbner basis.
(d) From the definition $\mathrm{E}:=1-\int \partial$ and the Leibniz rule we have the tautological relation

$$
f g-\mathbf{E}(f g)=\int f^{\prime} g+\int f g^{\prime}+\lambda \int f^{\prime} g^{\prime}
$$

Let us start by recalling [20, Thm. 2.5(b)] that the well-known integration-by-parts axiom $g \int f-\int g^{\prime} \int f-\int f g-\lambda \int g^{\prime} f=0$ characterizing $\mathbf{I D}_{\lambda}$ is equivalent to the multiplicativity condition $\mathrm{E}(f g)=\mathrm{E}(f) \mathrm{E}(g)$. Indeed, by the definition of E , the

[^2]condition is $\int(f g)^{\prime}+\int f^{\prime} \cdot \int g^{\prime}=f \int g^{\prime}+g \int f^{\prime}$. From this one obtains the axiom by expanding $(f g)^{\prime}$ according to the (weighted) Leibniz rule and substituting $f \rightsquigarrow \int f$. Conversely, the axiom of $\mathbf{I D}_{\lambda}$ implies that im $\int$ is an ideal of $\mathcal{F}$. From the definition of E one has the identity $f g=\mathrm{E} f \mathrm{E} g+\mathrm{E} f \int g^{\prime}+\mathrm{E} g \int f^{\prime}+\int f^{\prime} \cdot \int g^{\prime}$, which yields multiplicativity upon applying E since E projects onto $\operatorname{ker} \partial$ so that $\mathrm{E} f \mathrm{E} g \in \operatorname{ker} \partial$ is left invariant (note that ker $\partial$ is a subalgebra of $\mathcal{F}$ because of the Leibniz rule). The three remaining terms are all in the ideal $\int f$, which is the complement of ker $\partial$ under the projection, hence they vanish under $\mathbf{E}$.

We exploit the equivalent characterization of $\mathbf{I D}_{\lambda}$ in terms $\mathbf{E}(f g)=\mathbf{E}(f) \mathbf{E}(g)$ by substituting the tautological relation from above to obtain the equivalent law

$$
l:=\int f g^{\prime}-f g+\int f^{\prime} g+\lambda \int f^{\prime} g^{\prime}+\mathrm{E}(f) \mathrm{E}(g)=0
$$

This axiom gives rise to the new relator $[l]_{\mathfrak{a}}=\int\left(f+\lambda f^{\prime}\right) \partial-f+\int f^{\prime}+\mathbf{E}(f) \mathbf{E}$, which yields the rule $\int f \partial \rightarrow \underline{f}-\int \underline{f}^{\prime}-\mathrm{E}(\underline{f}) \mathrm{E}$ upon replacing $f$ by $\underline{f}$ and picking $\int f \partial$ as the leading monomial due to the grading. Since the plain Rota-Baxter axiom is implied by the integration-by-parts axiom [20, Lem. 2.3(b)], the relator constructed in Item [(C) is also contained in the current relator ideal, hence the corresponding rule is admissible in $\mathrm{GB}\left(\mathbf{I D}_{\lambda}\right)$. For seeing that this is again a Gröbner basis, we refer to the proof of [27, Prop. 13]. The latter assumes that $\mathbf{k}$ is a field and uses a k-basis of $\mathcal{F}$ but this is only a convenience tuned to the algorithmic treatment. As pointed out after [28, Prop. 26], choosing a basis is avoided by factoring out the linear ideal (this happens in the formation of the free operators $A[\Omega]$ in the current setup). Note also that here we take $\Phi=\{\mathrm{E}\}$, which means all rules of [27, Table 1] with characters $\varphi, \psi \in \Phi$ on the right-hand side are absent. With this understanding, the above definition of $\mathcal{F}\left[\partial, \int\right]$ coincides with the one in [27, which therefore establishes $\mathrm{GB}\left(\mathbf{I D}_{\lambda}\right)$ as a Gröbner basis.

We prove now the $\mathbf{k}$-module decomposition

$$
\begin{equation*}
\mathcal{F}\left[\partial, \int \mid \mathbf{I D}_{\lambda}\right]=\mathcal{F}\left[\partial \mid \mathrm{Diff}_{\lambda}\right] \dot{+}\left[\int \mid \mathbf{R B}_{\lambda}\right] \backslash \mathcal{F} \dot{+}(\mathrm{E}) \tag{5}
\end{equation*}
$$

with $(\mathrm{E}) \subset \mathcal{F}\left[\partial, \int \mid \mathbf{I D}_{\lambda}\right]$ being the two-sided ideal generated by E . Note that here and in the rest of this proof, we renounce the abbreviation of $\mathcal{F}[\Omega \mid E]$ by $\mathcal{F}[\Omega]$ used in the statement of the proposition. This is because we need to distinguish the free operator ring from various $E$-operator rings. Furthermore, we write $\mathcal{F}[\Omega]_{E}$ for the $\mathbf{k}$-submodule of normal forms in $\mathcal{F}[\Omega]$ with respect to the reduction system induced by $E$ and the given term order on $\mathcal{F}[\Omega]$. By the well-known Diamond Lemma [\#, Thm. 1.2], we have $\mathcal{F}\left[\partial, \int\right]=\mathcal{F}\left[\partial, \int\right]_{\mathbf{I D}_{\lambda}}+\left[\mathbf{I D}_{\lambda}\right]$. We claim that it suffices to prove

$$
\begin{equation*}
\mathcal{F}\left[\partial, \int\right]_{\mathbf{I D}_{\lambda}}=\mathcal{F}[\partial]_{\mathrm{Diff}_{\lambda}} \dot{+\mathcal{F}}\left[\int\right]_{\mathbf{R B}_{\lambda}} \backslash \mathcal{F} \dot{+}(\mathrm{E})_{\mathbf{I D}_{\lambda}} \tag{6}
\end{equation*}
$$

[^3]Indeed, substituting the decomposition (6) into the Diamond-Lemma decomposition and then taking the quotient by $\left[\mathbf{I D}_{\lambda}\right]$ yields展

$$
\begin{equation*}
\mathcal{F}\left[\partial, \int \mid \mathbf{I D}_{\lambda}\right]=\frac{\mathcal{F}[\partial]_{\mathbf{D i f f}_{\lambda}}+\left[\mathbf{I D}_{\lambda}\right]}{\left[\mathbf{I D}_{\lambda}\right]}+\frac{\mathcal{F}\left[\int\right]_{\mathbf{R B}_{\lambda}} \backslash \mathcal{F}+\left[\mathbf{I D}_{\lambda}\right]}{\left[\mathbf{I D}_{\lambda}\right]}+\frac{(\mathrm{E})_{\mathbf{I D}_{\lambda}}+\left[\mathbf{I D}_{\lambda}\right]}{\left[\mathbf{I D}_{\lambda}\right]} \tag{7}
\end{equation*}
$$

Since $\mathrm{Diff}_{\lambda} \subset \mathbf{I D}_{\lambda}$, we may replace the first denominator on the right-hand side of (7) by $\left(\mathcal{F}[\partial]_{\text {Diff }_{\lambda}}+\left[\mathbf{D i f f}_{\lambda}\right]\right)+\left[\mathbf{I D}_{\lambda}\right]=\mathcal{F}[\partial]+\left[\mathbf{I D}_{\lambda}\right]$, using now the Diamond Lemma for Diff $_{\lambda}$. In the same way, the second denominator is given by $\mathcal{F}[J] \backslash \mathcal{F}+\left[\mathbf{I D}_{\lambda}\right]$. For the third denominator we get ( E ) directly from the Diamond Lemma. Applying the second isomorphism theorem to the first and second summand yields (5) since $\left[\mathbf{I D}_{\lambda}\right] \cap \mathcal{F}[\partial]=\left[\mathbf{D i f f}_{\lambda}\right]$ and $\left[\mathbf{I D}_{\lambda}\right] \cap(\mathcal{F}[\mathcal{S}] \backslash \mathcal{F})=\left[\mathbf{R B}_{\lambda}\right]$, noting that $(\mathrm{E}) /\left[\mathbf{I D}_{\lambda}\right]$ is just $(\mathrm{E}) \subset \mathcal{F}\left[\partial, \int \mid \mathbf{I D}_{\lambda}\right]$ in (5) .

We give now a proof of (6), which follows closely the more general argument given in [28], specifically Lemma 23 as well as Propositions 25 and 26 therein. Let us start by analyzing the irreducible monomials. We claim that each monomial $w \in \mathcal{F}\left[\partial, \int\right]_{\mathbf{I D}_{\lambda}}$ is either of the form $w=f \partial^{i} \in \mathcal{F}[\partial]_{\text {Diff }_{\lambda}}(f \in \mathcal{F}, i \geq 0)$ or $w=f \int g \in \mathcal{F}\left[\int\right]_{\mathbf{R B}_{\lambda}}(f, g \in \mathcal{F})$ or $f \int \partial^{i+1}(f \in \mathcal{F}, i \geq 0)$. If $w$ contains any occurrences of $\partial$, they must be in the tail of $w$ since $\partial f(f \in \mathcal{F})$ is reducible relative to $\left[\mathbf{D i f f}_{\lambda}\right] \subset \mathbf{I D}_{\lambda}$ and also $\partial \int$ relative to $\left[\partial \int-1\right] \subset\left[\mathbf{I D}_{\lambda}\right]$. This means we have $w=v \partial^{i}$ with prefix monomial $v \in \mathcal{F}\left[\int\right]$ and $i \geq 0$. But then $v$ can have at most one occurrence of $\int$ since $\int f \int(f \in \mathcal{F})$ is reducible relative to $\left[\mathbf{R B}_{\lambda}\right] \subset\left[\mathbf{I D}_{\lambda}\right]$. Hence we have either $w=g \int f \partial^{i}$ or $w=f \partial^{i}$ for some $f, g \in \mathcal{F}$. In the latter case, we obtain $w \in \mathcal{F}[\partial]_{\text {Diff }_{\lambda}}$ and are done. In the former case, we can must have $i=0$ or $f=1$ since otherwise $w$ is reducible relative to relative to $\left[\int f \partial-\underline{f}+\int \underline{f^{\prime}}+\mathbf{E}(\underline{f}) \mathbf{E}\right] \subset\left[\mathbf{I D}_{\lambda}\right]$. Hence we have either the case $w=g \int f \in \mathcal{F}\left[\int\right]_{\mathbf{R B}_{\lambda}}$, where $f=1$ is possible. Or else we have the irreducible monomial $w=g \int \partial^{i}$ with $i>0$.

Next we analyze the irreducible elements of $(\mathrm{E})_{\mathbf{I D}_{\lambda}}$; unlike those of $\mathcal{F}[\partial]_{\text {Diff }_{\lambda}}$ and $\mathcal{F}\left[\int\right]_{\text {Diff }_{\lambda}}$, these are not monomials. Since any element of $\left(\mathrm{E}_{\mathbf{I D}_{\lambda}}\right.$ can be written as a k-linear combination of $w \mathrm{E} \tilde{w} \neq 0$ with monomials $w, \tilde{w}$, it suffices to analyze those. As we have seen above, if $w$ contains any occurrences of $\partial$, they must be at its tail. But since $\partial \mathrm{E}=0$, there can in fact be no $\partial$ in $w$. By the above analysis of normal forms for $w$, the only remaining possibilities are $w=f$ and $w=f \int g$ for some $f, g \in \mathcal{F}$. But the latter is also excluded since $\int g \mathrm{E}=\int g-\int g \int \partial$ is reducible relative to $\left[\mathbf{R B}_{\lambda}\right] \subset \mathbf{I D}_{\lambda}$. Hence we conclude $w=f$. Regarding the monomial $\tilde{m}$, we it cannot start with any $g \in \mathcal{F}$ since $\mathrm{E} g=g-\int \partial g$ is reducible relative to $\left[\mathbf{D i f f}_{\lambda}\right] \subset \mathbf{I D}_{\lambda}$. Furthermore, $\tilde{w}$ cannot start with $\int$ since $\mathrm{E} \int=0$. By our analysis of irreducible monomials, this leaves with the only remaining possibility $\tilde{w}=\partial^{i}$. Altogether this show that $w \mathrm{E} \tilde{w}=f \mathrm{E} \partial^{i}$. We may thus conclude that all three $\mathbf{k}$-modules on the right-hand side of (6) are in fact left $\mathcal{F}$-modules with the following generators: While $\mathcal{F}[\partial]_{\mathbf{D i f f}_{\lambda}}$ is generated by $\partial^{i}(i \geq 0)$, and $\mathcal{F}\left[\int\right]_{\mathbf{R B}_{\lambda}} \backslash \mathcal{F}$ by $\int f(f \in \mathcal{F})$, the normal forms in $(\mathrm{E})_{\mathbf{I D}_{\lambda}}$ are generated by $\mathrm{E} \partial^{i}$.

For establishing (6), it is sufficient to show that each $U \in \mathcal{F}\left[\partial, \int\right]_{\mathbf{I D}_{\lambda}}$ splits uniquely as $U=U_{\partial}+U_{\int}+U_{\mathbf{E}}$, containing a part $U_{\partial} \in \mathcal{F}[\partial]_{\text {Diff }_{\lambda}}$, a part $U_{\mathcal{S}} \in \mathcal{F}\left[\int\right]_{\mathbf{R B}_{\lambda}} \backslash \mathcal{F}$, and finally a part $U_{\mathrm{E}} \in(\mathrm{E})_{\mathbf{I D}_{\lambda}}$. Each irreducible monomial $f \partial^{i}$ of $U$ is put into $U_{\partial}$, and each irreducible monomial $f \int g$ into $U_{\int}$. For irreducible monomials of the form $f \int \partial^{i+1}=f \partial^{i}-f_{\mathrm{E}} \partial^{i}$,

[^4]we put $f \partial^{i}$ into $U_{\partial}$ and $-f_{\mathrm{E}} \partial^{i}$ into $U_{\mathrm{E}}$. Thus we have $U=U_{\partial}+U_{J}+U_{\mathrm{E}}$; let us now prove uniqueness. Hence assume $\sum_{i} a_{i} \partial^{i}+\sum_{i} b_{i} \int c_{i}+\sum_{i} d_{i} \mathrm{E} \partial^{i}=0$, each sum having finitely many nonzero coefficients $a_{i}, b_{i}, c_{j}, d_{i} \in \mathcal{F}$. By the definition of $\mathbf{E}$, this is the same as $\sum_{i}\left(a_{i}+d_{i}\right) \partial^{i}+\sum_{i} b_{i} \int c_{i}-\sum_{i} d_{i} \int \partial^{i+1}=0$. By the definition of the free operator ring $\mathcal{F}\left[\partial, \int\right]$, all monomials are linearly independent over $\mathbf{k}$, hence $a_{i}+d_{i}=b_{i}=d_{i}=0$ and then also $a_{i}=0$. This completes the uniqueness proof for splitting $U$. We have now established the k-module decomposition (6) and therefore also (国). Since $\mathcal{F}[\partial]$ and $\mathcal{F}[J]$ are both closed under multiplication, they are subalgebras of $\mathcal{F}\left[\partial, \int\right]$.

Finally, let us prove the quotient statement $\mathcal{F}\left[\partial, \oint \mid \mathbf{I D}_{\lambda}\right] \cong \mathcal{F}\left[\partial, \oint \mid \mathbf{D R B}_{\lambda}\right] / I$, where for once we use the same symbol for the Rota-Baxter operator in $\mathbf{I D}_{\lambda}$ and $\mathbf{D R B}_{\lambda}$. (Recall that the notational distinction between $\int$ and $\oint$ is purely a convenience that allows us to suppress the laws to be factored out.) Writing out the definitions, we must thus prove

$$
\frac{\mathcal{F}[\partial, \oint]}{\left[\mathbf{I D}_{\lambda}\right]} \cong \frac{\mathcal{F}[\partial, \oint]}{\left[\mathbf{D R B}_{\lambda}\right]} /[\mathrm{E} f-\mathrm{E}(f) \mathrm{E} \mid f \in \mathcal{F}]
$$

which reduces to showing $\left[\mathbf{I D}_{\lambda}\right] /\left[\mathbf{D R B}_{\lambda}\right]=[\mathrm{E} f-\mathbf{E}(f) \mathrm{E} \mid f \in \mathcal{F}]$ by the third isomorphism theorem. Hence it suffices to show $\left[\mathbf{I D}_{\lambda}\right]=\left[\mathbf{D R B}_{\lambda}\right]+[\mathrm{E} f-\mathbf{E}(f) \mathbf{E} \mid f \in \mathcal{F}]$ as $\mathbf{k}$-modules, where the directness of the sum is obvious. For the inclusion from left to right, we must show that every $\int f \partial-\underline{f}+\int \underline{f}^{\prime}+\mathrm{E}(\underline{f}) \mathrm{E}$ is in $\left[\mathbf{I D}_{\lambda}^{\prime}\right]:=\left[\mathbf{D R B}_{\lambda}\right]+[\mathrm{E} f-\mathrm{E}(f) \mathrm{E} \mid f \in \mathcal{F}]$. Substituting $f+\lambda f^{\prime}$ for $\bar{f}$, we may also show that every $r_{f}:=\int\left(f+\lambda f^{\prime}\right) \partial-f+\int f^{\prime}+\mathrm{E}(f) \mathrm{E}$ is in $\left[\mathbf{I D}_{\lambda}^{\prime}\right]$. But we have indeed

$$
r_{f}=\int\left(f \partial+\lambda f^{\prime} \partial+f^{\prime}-\partial f\right)-(\mathrm{E} f-\mathrm{E}(f) \mathrm{E}) \in\left[\mathbf{I D}_{\lambda}^{\prime}\right]
$$

since the first summand is in $\mathbf{D i f f}_{\lambda} \subset \mathbf{D R B}_{\lambda}$ and the second in $[\mathbf{E} f-\mathbf{E}(f) \mathbf{E} \mid f \in \mathcal{F}]$. For the inclusion from right to left, it suffices to show that every $\mathrm{E} f-\mathrm{E}(f) \mathrm{E}$ is in $\mathbf{I D}_{\lambda}$. But we have just proved that $r_{f}+\mathbf{E} f-\mathbf{E}(f) \mathbf{E} \in\left[\mathbf{D i f f}_{\lambda}\right] \subset\left[\mathbf{I D}_{\lambda}\right]$. Since we have also $r_{f} \in \mathbf{I D}_{\lambda}$, the proof is completed.

As the name suggests, there is another important aspect to $E$-operators that we should consider here - they operate on suitable domains. These domains are a special class of modules that we shall now introduce. Recall first that an $\Omega$-operated module $\left(M ; p_{\omega} \mid \omega \in \Omega\right)$ over a commutative ring $A$ is an $A$-module $M$ with $A$-linear operators $p_{\omega}: M \rightarrow M$. As in the case of $\Omega$-operated algebras, no restrictions are imposed on the operators $p_{\omega}$. An operated morphism $\varphi:\left(M ; p_{\omega} \mid \omega \in \Omega\right) \rightarrow\left(M^{\prime} ; p_{\omega}^{\prime} \mid \omega \in \Omega\right)$ is an $A$-linear homomorphism $\varphi: M \rightarrow M^{\prime}$ such that $\varphi \circ p_{\omega}=p_{\omega}^{\prime} \circ \varphi$ for all $\omega \in \Omega$; the resulting category of $\Omega$-operated modules over $A$ is denoted by $\operatorname{Mod}_{A}(\Omega)$.

Now assume that $A \in \operatorname{CAlg}(\Omega)$ is an operated algebra. Then the free operators $T \in$ $A[\Omega]$ act naturally on the $\Omega$-operated $A$-module $M$. Since $T$ is a k-linear combination of noncommutative monomials $t \in M(\Omega \uplus A)$, it suffices to define $t \cdot m$ for $m \in M$. By the universal property of $M(\Omega \uplus A)$, we obtain a unique monoid action by setting $\omega \cdot m:=p_{\omega}(m)$ for $\omega \in \Omega$ and $a \cdot m:=a m$ for $a \in A$. Thus $M$ becomes an $A[\Omega]$-module, and we can now introduce the module-theoretic analog of $E$-algebras.

Definition 3.5. Fix an operated algebra $A \in \mathbf{C A l g}(\Omega)$ and a submodule $E \subseteq \mathbf{k}_{\Omega}(Y)$ of standard laws. Then an $E$-related module over $A$ is an operated module $M \in \operatorname{Mod}_{A}(\Omega)$ with $L \cdot m=0$ for all relators $L \in[E]$ and $m \in M$.

Again we will briefly speak of $E$-modules (since the context will make it clear that $E$ is a set of laws). They form a full subcategory of $\operatorname{Mod}_{A}(\Omega)$ denoted by $\operatorname{Mod}_{A}(\Omega \mid E)$. The role of the $E$-operator ring becomes clear now: Operators correspond to the natural action defined above if $A$ is an $E$-algebra. This can be made precise by the following statement.

Proposition 3.6. Let $A$ be an E-algebra for a submodule $E \subseteq \mathbf{k}_{\Omega}(Y)$. Then we have the isomorphism of categories $\operatorname{Mod}_{A}(\Omega \mid E) \cong \operatorname{Mod}_{A[\Omega \mid E]}$.

Proof. As noted above, an $E$-module $M \in \operatorname{Mod}_{A}(\Omega \mid E) \subseteq \operatorname{Mod}_{A}(\Omega)$ can also be viewed as an $A[\Omega]$-module under the natural action, and as such it satisfies $[E] \cdot M=0$. But then the action of $A[\Omega \mid E]$ with $(T+[E]) \cdot m:=T \cdot m$ is well-defined and gives $M$ the structure of an $A[\Omega \mid E]$-module. Conversely, every such module restricts to an operated module $M \in \operatorname{Mod}_{A}(\Omega)$ with $[E] \cdot M=0$.

Of course, every morphism of $\operatorname{Mod}_{A[\Omega \mid E]}$ is also a morphism of $\operatorname{Mod}_{A}(\Omega \mid E)$. For the other direction, let $\varphi$ be a morphism of $E$-modules. For showing $\varphi((T+[E]) \cdot m)=(T+[E]) \cdot \varphi(m)$ for $T \in A[\Omega]$ and $m \in M$, it suffices to show $\varphi(T \cdot m)=T \cdot \varphi(m)$. Since $\varphi$ is $\mathbf{k}$-linear, we may assume a monomial $T \in M(\Omega \uplus A)$ and use induction on the degree of $T$. The base case $T=1$ is trivial, hence assume the claim for monomials of degree $n$ and let $T$ have degree $n+1$. Then there exists $T^{\prime} \in M(\Omega \uplus A)$ of degree $n$ such that either $T=a T^{\prime}$ for $a \in A$ or $T=\omega T^{\prime}$ with $\omega \in \Omega$. In the former case the claim follows because $\varphi$ is $A$ linear, in the latter case because it is a morphism of $\Omega$-operated modules. This completes the proof that $\varphi$ is also a morphism of $\operatorname{Mod}_{A[\Omega \mid E]}$.

Fact 3.7. Some standard constructions for creating new modules also work in the operated setting. Let us mention a few that are also relevant for the examples to be given afterwards.
(a) If $A$ is an $E$-algebra and $S$ an arbitrary set, the free module $A^{S}$ is an $E$-module. The action of $p_{\omega}(\omega \in \Omega)$ on a module element $f \in A^{S}$ is defined by $\left(p_{\omega} f\right)(s):=P_{\omega}(f s)$ for $s \in S$. It is easy to see that for any free operator $L \in A[\Omega]$ and $f \in A^{S}$ one has $(L \cdot f)(s)=L \cdot f(s)$ for all $s \in S$, where the left action takes place in $A^{S}$ and the right action in $A$. Hence one obtains $L \cdot f=0$ for all relators $L \in[E] \subset A[\Omega]$ and all $f \in A^{S}$, which confirms that $A^{S}$ is an $E$-module. Note that $A^{S}$ is free as an $A$-module but generally not as an $A[\Omega \mid E]$-module (see Example $3.8($ a $)$ below).
(b) If $M_{1}, \ldots, M_{k}$ are $E$-modules over $A$, their direct product $M_{1} \times \cdots \times M_{k}$ is an $E$ module with operators $p_{\omega}(\omega \in \Omega)$ acting component-wise. If $M_{1}=\cdots=M_{k}=M$, this gives the free module $M^{S}$ over the finite set $S=\{1, \ldots, k\}$.
(c) Whenever $M$ is an $E$-module over $A$, the dual module $M^{*}$ is naturally an $E^{*}$-module with operators $p_{\omega}^{*}(\omega \in \Omega)$; here $p_{\omega}^{*}: M^{*} \rightarrow M^{*}$ is defined as the dual map of the $A$-linear map $p_{\omega}: M \rightarrow M$. If $L \in A[\Omega]$ is any free operator and $f \in M^{*}$ one checks immediately that $\left(L^{*} \cdot f\right)(m)=f(L \cdot m)$ for all $m \in M$. In other words, the action on $M^{*}$ is the dual of the action on $M$. In particular, one sees that $L^{*} \cdot f=0$ for all relators $L \in[E] \subset A[\Omega]$ and all $f \in M^{*}$, confirming that $M^{*}$ is $E^{*}$-related (meaning it satisfies the transpose of all relators induced by $E$ ). Since any $E$-algebra $A$ is also an $E$-module over itself, $A^{*}$ is also an $E^{*}$-module.
(d) If $M$ is an $E$-module with a submodule $M^{\prime} \subseteq M$ that is closed under all all operators $p_{\omega}(\omega \in \Omega)$, their restrictions to the submodule $M^{\prime}$ make the latter into an $E$-module or, more precisely, an $E$-related submodule of $M$.

Example 3.8. Let us now exemplify the concept of $E$-module for the four standard varieties given in Example 2.19, corresponding to the four operator rings of Proposition 3.4. We make again use of the convention stated in Remark 3.3.
(a) The Diff $_{\lambda}$-modules are commonly known as differential modules [33, Def. 1.2.4(iii)], usually taken with weight $\lambda=0$ over a differential field $(\mathcal{F}, \partial)$. Their equivalent formulation as $\mathcal{F}[\partial]$-modules is often used as an alternative definition [36, Def. 2.5]. Differential modules are crucial for differential Galois theory as they provide an abstract way of formulating linear differential equations. In the important special case when the underlying differential ring is $\mathcal{F}=\mathbf{k}[x]$, the operator ring is the Weyl algebra $A_{1}(\mathbf{k})$, and the corresponding $A_{1}(\mathbf{k})$-modules are known as $\mathcal{D}$-modules 111 since $\mathcal{D}:=A_{1}(\mathbf{k})=\mathbf{k}[x][\partial]$ is the underlying differential operator ring. For example, $\mathbf{k}[x]^{n}$ is a differential module by Fact 3.7 (b). If $\mathbf{k}$ is a field, any $\mathbf{k}$-basis $e_{1}, \ldots, e_{n}$ of $\mathbf{k}^{n}$ is of course a $\mathbf{k}[x]$-basis for $\mathbf{k}[x]^{n}$ but since $\partial e_{1}, \ldots, \partial e_{n}=0$ it is not an $A_{1}(\mathbf{k})$-basis. In other words, $\mathbf{k}[x]^{n}$ is free as a $\mathbf{k}[x]$-module but not as a $\mathbf{k}[x][\partial]$-module.

Another important class of examples with $\mathbf{k}=\mathbb{R}$ is concerned with vector fields on a manifold $M$. In detail, each vector field $V \in \mathfrak{X}(M)$ induces a covariant derivative $\nabla_{V}: \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ with characteristic property $\nabla_{V}(f W)=f^{\prime} W+f \nabla_{V}(W)$ for $f \in C^{\infty}(M)$ and $W \in \mathfrak{X}(M)$. The vector fields $\mathfrak{X}(M)$ thus form a differential module over the differential algebra $C^{\infty}(M)$.
(b) The category of $\mathbf{R B}_{\lambda}$-modules has been introduced in [15, Def. 2.1(a)] under the name of Rota-Baxter module for a given Rota-Baxter algebra $\left(\mathcal{F}, \int\right)$. Their equivalent description in terms of $\mathcal{F}\left[\int\right]$-modules is elaborated in [15, §2.2].
(c) Similarly, we introduce now the category of $\mathbf{D R B}_{\lambda}$-modules, which we may also call differential Rota-Baxter modules. In $\mathcal{D}$-module theory $\|$, it is often pointed out that various spaces of (real or complex valued) distributions are differential modules (for weight $\lambda=0$ and ground field $\mathbf{k}=\mathbb{R}$ or $\mathbf{k}=\mathbb{C}$ ) and hence $\mathcal{D}$-modules since distributions can be multiplied by smooth functions so they are in particular modules over $\mathcal{F}:=\mathbf{k}[x]$. It is seldom appreciated that some of these distribution spaces are in fact differential Rota-Baxter modules over $\mathcal{F}$ and hence $\mathcal{F}[\partial, \oint]$-modules. For example, let $\mathcal{D}^{\prime}(\mathbb{R})_{+} \subset \mathcal{D}^{\prime}(\mathbb{R})$ be the space of all distributions $T$ with left-bounded support, meaning $\operatorname{supp}(T) \subseteq\left[a, \infty\left[\right.\right.$ for some $a \in \mathbb{R}$. Analogously, we write $\mathcal{D}(\mathbb{R})_{-}$for the space of test functions with right-bounded support; it is clear that this is a (nonunital!) differential Rota-Baxter algebra with standard derivation $\partial$ and Rota-Baxter operator $\oint:=-\int_{x}^{\infty}$. In fact, $\mathcal{D}(\mathbb{R})_{-}$is a degenerate (nonunital) integro-differential algebra since the induced evaluation $\mathrm{E}:=1_{\mathcal{D}(\mathbb{R})_{+}}-\oint \circ \partial=0$ is trivially multiplicative. In other words, $\partial$ is bijective with $\oint$ as its inverse, and the strong RotaBaxter axiom $f \oint g=\oint f^{\prime} \oint g+\oint f g$ immediately follows from $f^{\prime} \oint g=(f \oint g)^{\prime}-f g$. If $H \in \mathcal{D}^{\prime}(\mathbb{R})_{+}$is the Heaviside function, the operator $\oint: \mathcal{D}^{\prime}(\mathbb{R})_{+} \rightarrow \mathcal{D}^{\prime}(\mathbb{R})_{+}$defined by the convolution $\oint T:=H \star T$ is known to be a two-sided inverse [12, §13.1] of the distributional derivative $\partial$. One checks that $\oint: \mathcal{D}^{\prime}(\mathbb{R})_{+} \rightarrow \mathcal{D}^{\prime}(\mathbb{R})_{+}$is the transpose of $\oint: \mathcal{D}(\mathbb{R})_{-} \rightarrow \mathcal{D}(\mathbb{R})_{-}$, just as the distributional derivative $\partial: \mathcal{D}^{\prime}(\mathbb{R})_{+} \rightarrow \mathcal{D}^{\prime}(\mathbb{R})_{+}$is (by definition) the transpose of the standard derivation $\partial: \mathcal{D}(\mathbb{R})_{-} \rightarrow \mathcal{D}(\mathbb{R})_{-}$. Thus we

[^5]obtain a Rota-Baxter module $\left(\mathcal{D}^{\prime}(\mathbb{R})_{+}, \partial, \oint\right)$, which is actually a degenerate integrodifferential module over the nonunital Rota-Baxter algebra $\mathcal{D}(\mathbb{R})_{-}$. Of course, one may apply a similar construction to endow the space $\mathcal{D}^{\prime}(\mathbb{R})$ _ of right-bounded distributions with the structure of a differential Rota-Baxter module over the nonunital differential Rota-Baxter algebra of left-bounded test functions.
(d) Finally, let us consider the category of $\mathbf{I D}_{\lambda}$-modules, which we also call integrodifferential modules. Again we shall give an important example from distribution theory. Endowing $\mathcal{E}(\mathbb{R}):=C^{\infty}(\mathbb{R})$ with the usual derivation $\partial=d / d x$ and the RotaBaxter operator $\oint f:=\int_{0}^{x} f(\xi) d \xi$ yields a "dually integro-differential" module $\mathcal{E}^{*}(\mathbb{R})$ by Fact 3.7 (c), as the dual of the integro-differential algebra $(\mathcal{E}(\mathbb{R}), \partial, \oint)$. Just as the one-sided distribution spaces of Item [C), this is in fact a differential module as well as a Rota-Baxter module since the relators $\partial f \rightarrow f \partial+f^{\prime}$ and $\oint f \oint \rightarrow f^{\prime} \oint-\oint f^{\prime}$ are skew-symmetric under transposition. Hence we should take the negated transposes of $\partial, \int: \mathcal{E}(\mathbb{R}) \rightarrow \mathcal{E}(\mathbb{R})$; this is of course standard practice in defining the distributional derivative [12, §4]. As the two signs cancel, we obtain the transposed section law $\oint \circ \partial=1_{\mathcal{E}^{\prime}(\mathbb{R})}$.

One checks that both $\partial$ and $\oint$ restrict to the topological dual $\mathcal{E}^{\prime}(\mathbb{R}) \subset \mathcal{E}^{*}(\mathbb{R})$ consisting of all continuous functionals $\mathcal{E}(\mathbb{R}) \rightarrow \mathbf{k}$, relative to the well-known locally convex topology of $\mathcal{E}(\mathbb{R})$; see for example [35, (7.8)]. Therefore $\mathcal{E}^{\prime}(\mathbb{R})$ is a differential Rota-Baxter submodule of $\mathcal{E}^{*}(\mathbb{R})$ by Fact $3.7(\mathrm{~d})$, except that the section law is transposed. In analysis, $\mathcal{E}^{\prime}(\mathbb{R})$ is known as the space of compactly supported distributions. It may seem surprising that $\partial$ is injective and $\oint$ surjective on $\mathcal{E}^{\prime}(\mathbb{R})$. In fact, one checks that $\operatorname{ker} \oint=\mathbb{R} \delta_{0}$ and $\operatorname{im}(\partial)=\left\{T \in \mathcal{E}^{\prime}(\mathbb{R}) \mid T(1)=0\right\}$. It is known [355, (10.4)] that in $\mathcal{D}^{\prime}(\mathbb{R}) \supset \mathcal{E}^{\prime}(\mathbb{R})$, the kernel of $\partial$ is given by the constant distributions; but since their support is $\mathbb{R}$, they are not in $\mathcal{E}^{\prime}(\mathbb{R})$. Conversely, the image of $\partial$ on $\mathcal{D}^{\prime}(\mathbb{R})$ is full [35, Cor. §11.2] since the constant function 1 is not compactly supported so the condition $T(1)=0$ is void. As we have seen, $\partial$ is not surjective on $\mathcal{E}^{\prime}(\mathbb{R})$, which is of course well-known [35, Ex. 11.2].

Is $\left(\mathcal{E}^{\prime}(\mathbb{R}), \partial, \oint\right)$ an integro-differential module (with transposed section law)? One must check if the (transposed) induced evaluation $\mathrm{E}:=1_{\mathcal{E}^{\prime}(\mathbb{R})}-\partial \circ \oint: \mathcal{E}^{\prime}(\mathbb{R}) \rightarrow \mathcal{E}^{\prime}(\mathbb{R})$ is multiplicative in the sense that $\mathrm{E}(f T)=\mathbf{E}(f) \mathbf{E}(T)$ for all $T \in \mathcal{E}^{\prime}(\mathbb{R})$ and $f \in \mathcal{E}(\mathbb{R})$. Since $\mathrm{E}: \mathcal{E}^{\prime}(\mathbb{R}) \rightarrow \mathcal{E}^{\prime}(\mathbb{R})$ is the transpose of $\mathrm{E}: \mathcal{E}(\mathbb{R}) \rightarrow \mathcal{E}(\mathbb{R})$, one has $\mathrm{E}(T)=T(1) \delta_{0}$ for any $T \in \mathcal{E}^{\prime}(\mathbb{R})$. But then one sees that

$$
\mathrm{E}\left(e^{x} \delta_{1}\right)=\delta_{1}\left(e^{x}\right) \cdot \delta_{0}=e \cdot \delta_{0} \neq \mathrm{E}\left(e^{x}\right) \mathrm{E}\left(\delta_{1}\right)=1 \cdot \delta_{0},
$$

which shows that $\mathcal{E}^{\prime}(\mathbb{R})$ is in fact not an integro-differential module. The problem is that the corresponding relator $\mathrm{E} f \rightarrow \mathrm{E}(f) \mathrm{E}$ gets transposed to $f \mathrm{E} \rightarrow \mathrm{E}(f) \mathrm{E}$, which yields the true identity $f \mathrm{E}(T)=f(0) \mathrm{E}(T)$ or $f T(1) \delta_{0}=f(0) T(1) \delta_{0}$. If $T(1)=0$, this is trivially valid; otherwise division by $T(1)$ yields the familiar sifting property of the Dirac distribution [12, p. 38]. Of course we may replace $\delta_{0}$ by $\delta_{c}$ for any $c \in \mathbb{R}$ if we use the Rota-Baxter operator $\int_{a}^{x}$ on $\mathcal{E}(\mathbb{R})$ instead of $\int_{0}^{x}$.
For seeing an honest integro-differential module, we refer to 30], where the algebraic distribution module ( $\mathcal{D} \mathcal{F}, \partial, f)$ over a given ordinary shifted integro-differential algebra $\mathcal{F}$, such as the classical example $\mathcal{F}=C^{\infty}(\mathbb{R})$, is constructed and investigated. This provides a purely algebraic structure (involving no topology, in particular taking $\mathcal{F}$ only as an integro-differential algebra with shift maps such as $f(x) \mapsto f(x-c)$ for $c \in \mathbb{R}$ in the
classical example), providing just piecewise functions and Dirac distributions on top of $\mathcal{F}$. In the classical example, this gives rise to the Heaviside function $H_{a}=H(x-a)$ and their derivatives $\delta_{a}$. Compared to the analytic distribution spaces of Example 3.8 (c), (d), this is a very small module. However, it contains exactly what is needed for specifying and computing the Green's operator $G \in \mathcal{F}\left[\partial, \int\right]$ of a LODE boundary problem [34, $\left.\S \S 2,3\right]$. Acting as $G: \mathcal{F} \rightarrow \mathcal{F}$, it can be assigned a Green's function $g(x, \xi)$. This is a (bivariate) function involving Heavisides and-for ill-posed problems- also Diracs, characterized by a distributional differential equation. For a comprehensive presentation, we refer the reader to [30], specifically Theorems 26 and 29 therein. The actual computation of the Green's operator $G$ on the basis of a given fundamental system is detailed in [27, the extraction of the Green's function $g(x, \xi)$ from $G$ in (2g].

## 4. Differential Rota-Baxter Operators

As pointed out earlier, the operator rings in Proposition 3.4(a), (b), (d) are known and defined elsewhere, but the ring in (c) is introduced here for the first time. In the rest of this paper, we will therefore concentrate on the ring $\mathcal{F}[\partial, \oint]$. As a first step, let us analyze its canonical forms, in a way similar to [28, Prop. 25] and the above k-module decomposition for $\mathcal{F}\left[\partial, \int\right]$. In the following, recall that $\mathcal{F}[\oint] \backslash \mathcal{F}$ denotes a linear complement rather than the set-theoretic one.

Lemma 4.1. Let $(\mathcal{F}, \partial, \oint) \in \mathrm{DRB}_{\lambda}$. Then we have $\mathcal{F}[\partial, \oint]=\mathcal{F}[\partial] \dot{+} \mathcal{F}[\oint] \backslash \mathcal{F} \dot{+}[\mathrm{E}]$, where $[\mathrm{E}]:=\mathbf{k}\left\{f \oint g \partial^{k} \mid f, g \in \mathcal{F} ; k>0\right\}$ is a rung that we call the evaluation rung.

Proof. The proof of the direct sum is completely analogous to that of the corresponding statement in Proposition 3.4, with (66) being replaced by

$$
\begin{equation*}
\mathcal{F}[\partial, \oint]_{\mathbf{D R B}_{\lambda}}=\mathcal{F}[\partial]_{\mathbf{D i f f}_{\lambda}} \dot{+\mathcal{F}}[\oint]_{\mathbf{R B}_{\lambda}} \backslash \mathcal{F} \dot{+}[\mathrm{E}]_{\mathbf{D R B}_{\lambda}} \tag{8}
\end{equation*}
$$

The analysis of irreducible monomials $w \in \mathcal{F}[\partial, \oint]_{\mathbf{D R B}_{\lambda}}$ is also the same, except that the remaining case $w=f \oint g \partial^{k}$ with $f, g \in \mathcal{F}$ and $k \geq 0$ cannot be reduced any further. We have of course $w \in[\mathrm{E}]$ if $k>0$ and $w \in \mathcal{F}[\oint]_{\mathbf{R B}_{\lambda}}$ otherwise. The direct sum (8) now follows immediately since the evaluation rung $[\mathrm{E}]$, unlike the evaluation ideal ( E ), is generated by irreducible monomials.

It remains to prove that $[\mathrm{E}]$ is multiplicatively closed, meaning $\left(f \oint g \partial^{k}\right)\left(\tilde{f} \oint \tilde{g} \partial^{\tilde{k}}\right) \in[\mathrm{E}]$. It suffices to ensure $w_{k}:=\oint g \partial^{k} \tilde{f} \oint \tilde{g} \partial \in[\mathrm{E}]$, and for that we use induction over $k>0$. For $k=1$ we have $\partial \tilde{f}=\tilde{f}^{\prime}+\tilde{f} \partial+\lambda \tilde{f}^{\prime} \partial$ and $w_{k}=\oint \tilde{f}^{\prime} g \oint \tilde{g} \partial+\oint \tilde{f} g \tilde{g} \partial+\lambda \oint \tilde{f}^{\prime} g \tilde{g} \partial \in[\mathrm{E}]$ after applying the Rota-Baxter rule in the first summand. For the induction step we consider $w_{k+1}$, assuming the claim holds for $k$. We obtain

$$
w_{k+1}=\oint g \partial^{k} \tilde{f}^{\prime} \oint \tilde{g} \partial+\oint g \partial^{k} \tilde{f} \tilde{g} \partial+\lambda \oint g \partial^{k} \tilde{f}^{\prime} \tilde{g} \partial
$$

where the first summand is contained in $[\mathrm{E}]$ by the induction hypothesis and the second expands into a linear combination of terms having the shape $\tilde{w}_{k} \partial^{l}(1 \leq l \leq k+1)$, which are clearly contained in $[\mathrm{E}]$ as well.

Note that both $\mathcal{F}[\oint]_{+}:=\mathcal{F}[\oint] \backslash \mathcal{F}$ and $\mathcal{F}[\partial]_{+}:=\mathcal{F}[\partial] \backslash \mathcal{F}$ are rungs, which feature in the alternative $\mathbf{k}$-module decomposition

$$
\mathcal{F}[\partial, \oint]=\mathcal{F} \dot{+} \mathcal{F}[\partial]_{+} \dot{+} \mathcal{F}[\oint]_{+} \dot{+}[\mathrm{E}] .
$$

Moreover, one checks immediately that $[\mathrm{E}]$ is actually an $\left(\mathcal{F}[\oint]_{+}, \mathcal{F}[\partial]_{+}\right)$-bimodule. According to the subsequent lemma, the evaluation rung is also closely related to the evaluation (hence its name). Note that we continue to call the projector $\mathrm{E}:=1_{\mathcal{F}}-\oint \partial$ the evaluation of the differential Rota-Baxter algebra $(\mathcal{F}, \partial, \oint)$ although it is not multiplicative (unless $\mathcal{F}$ is in fact an integro-differential algebra). However, it is still a projector onto ker $\partial$ along im $\int$. By abuse of language, the corresponding $\mathrm{e} \in \mathcal{F}[\partial, \oint]$ will also be referred to as evaluation.

Lemma 4.2. The evaluation rung $[\mathrm{E}]$ is a bimodule over $\mathbf{k}[\mathrm{E}]$, with E as right annihilator.
Proof. Since $\mathrm{E}^{2}=\mathrm{E}$, the ring $\mathbf{k}[\mathrm{E}]$ is the $\mathbf{k}$-span of 1 and $\mathbf{E}$. Therefore it suffices to verify the inclusion $\mathrm{E}[\mathrm{E}] \subseteq[\mathrm{E}]$ and $[\mathrm{E}] \mathrm{E}=0$. The latter is immediate from the definition of $[\mathrm{E}]$, the former follows from $\mathrm{E} f \oint g \partial^{k}=[\mathrm{E}(f), \oint] g \partial^{k} \in[\mathrm{E}]$ via the Rota-Baxter axiom; here the bracket denotes the commutator in $\mathcal{F}[\partial, \oint]$.

Before we study further properties of differential Rota-Baxter algebras and their operator rings, let us give two simple examples (the weight is zero for both).

Example 4.3. Let $\mathbf{k}$ have characteristic zero. The most basic example of a differential Rota-Baxter algebra is clearly the polynomial ring $\mathbf{k}[x]$, with standard derivation $\partial=d / d x$ and Rota-Baxter operator $\int=\int_{0}^{x}$ or more generally $\int_{a}^{x}$ for any initialization point $a \in \mathbf{k}$. Here we think of $\int_{a}^{x}: \mathbf{k}[x] \rightarrow \mathbf{k}[x]$ in purely algebraic terms, as the $\mathbf{k}$-linear map defined by $x^{k} \mapsto\left(x^{k+1}-a^{k+1}\right) /(k+1)$. This example will play a great role in Section 5 although it is not a genuine example (in the sense that it is also an integro-differential algebra).

Example 4.4. For seeing a natural example of a differential Rota-Baxter algebra that is not an integro-differential algebra, we call on analysis. Of course, the primordial example of an integro-differential algebra consists of the (real or complex valued) smooth functions $C^{\infty}(\mathbb{R})$ or $C^{\infty}[a, b]$; see [27, Ex. 5]. Here $\partial$ and $\int=\int_{\xi}^{x}(\xi \in \mathbb{R}$ or $\xi \in[a, b])$ are defined analytically.

A slight variation of this example leads to a differential Rota-Baxter algebra, namely the (real or complex valued) piecewise smooth functions $P C^{\infty}(\mathbb{R})$ or $P C^{\infty}[a, b]$. For example, we take all functions that are smooth on the whole domain minus finitely many points. The operations are defined as before except that $\partial f$ and $\int f$ is undefined at the points where $f$ is so. (The ring operations,,$+- *$ have to be defined carefully since singularities may cancel; the result is always to be taken with all removable singularities actually removed. This process is also well-known in complex analysis where meromorphic functions can be defined in a similar way.)

The piecewise smooth functions are clearly a differential Rota-Baxter algebra. However, they are not an integro-differential algebra for if they were, the evaluation $1-\partial \int$ would be multiplicative - which it cannot be for functions undefined on the initialization point $\xi$. For a more explicit example, let us take $P C^{\infty}[0,1]$ with initialization point $\xi=0$. The Heaviside function $h(x):=H(x-1 / 2) \in P C^{\infty}[0,1]$ is the characteristic function of the subinterval $[1 / 2,1]$, and we have $\int h=\int_{1 / 2}^{x} d x=x-1 / 2$ but $h \cdot \int 1=H(x-1 / 2) x$. This means we have $\int(h \cdot 1) \neq h \cdot \int 1$ although $h \in \operatorname{ker} \partial$, and [20, Rem. 2.6(c)] shows that $\left(P C^{\infty}[0,1], \partial, \int\right)$ is not an integro-differential algebra.

In Proposition 3.4 the relation between the operator rings $\mathcal{F}[\partial, \oint]$ and $\mathcal{F}\left[\partial, \int\right]$ is illuminated in one direction only: It shows the differential Rota-Baxter operators $\mathcal{F}[\partial, \oint]$ to have a finer structure from which one obtains the integro-differential operator ring $\mathcal{F}\left[\partial, \int\right]$ as a quotient. However, we shall see below (Proposition 4.8) that the finer ring $\mathcal{F}[\partial, \oint]$ can
also be embedded into a suitably "generic" integro-differential operator ring. Applying this to the special case of polynomial coefficients will enable us to give an operator-theoretic interpretation to the integro-differential Weyl algebra (Section 5).

As a preparation to this construction, let us first determine the free integro-differential algebra $(\tilde{\mathcal{F}}, \partial, \oint)$ over a given differential Rota-Baxter algebra $\left(\mathcal{F}, \partial, \int\right)$. In other words, we want to "extend" $\mathcal{F}$ just enough to build an integro-differential structure. Categorically speaking, the association $\mathcal{F} \mapsto \tilde{\mathcal{F}}$ is the left adjoint of the forgetful functor $\mathbf{I D}_{\lambda} \rightarrow \mathbf{D R B}_{\lambda}$. However, note that $\int$ is not an extension of $\oint$.
Proposition 4.5. Given $(\mathcal{F}, \partial, \oint) \in \mathbf{D R B}_{\lambda}$, construct $\tilde{\mathcal{F}}=\mathcal{F} \otimes_{K} \mathcal{F}$ over $K:=\operatorname{ker} \partial$, extending the derivation to $\partial: \tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{F}}, f \otimes f_{\varepsilon} \mapsto(\partial f) \otimes f_{\varepsilon}$ and defining $\int: \tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{F}}$ via $\int f \otimes f_{\varepsilon}:=(\oint f) \otimes f_{\varepsilon}-1 \otimes\left(f_{\varepsilon} \oint f\right)$. Then one obtains $\left(\tilde{\mathcal{F}}, \partial, \int\right) \in \mathbf{I D}_{\lambda}$ with evaluation $\mathrm{E}\left(f \otimes f_{\varepsilon}\right)=1 \otimes f f_{\varepsilon}$, and an embedding $\iota: \mathcal{F} \rightarrow \tilde{\mathcal{F}}, f \mapsto f \otimes 1$ of differential algebras.
Proof. Let us first reassure ourselves that $\partial: \tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{F}}$ is well-defined. It suffices to prove that $\sum_{i} f(i) \otimes f_{\varepsilon}(i)=0$ implies $\sum_{i} f(i)^{\prime} \otimes f_{\varepsilon}(i)=0$ for finite families $f(i), f_{\varepsilon}(i) \in \mathcal{F}$. Hence assume $\sum_{i} f(i) \otimes f_{\varepsilon}(i)=0$. By [13, Lem. 6.4], there are $c(i, j) \in K$ and $g(i) \in \mathcal{F}$ satisfying the relations $\sum_{j} c(i, j) g(j)=f(i)$ for all $i$ and $\sum_{i} c(i, j) f_{\varepsilon}(i)=0$ for all $j$, which yields $\sum_{i} f(i)^{\prime} \otimes f_{\varepsilon}(i)=\sum_{i j} c(i, j) g(j)^{\prime} \otimes f_{\varepsilon}(i)=\sum_{j} g(j)^{\prime} \otimes\left(\sum_{i} c(i, j) f_{\varepsilon}(i)\right)=0$ since $c(i, j)^{\prime}=0$. Next we note that $\iota: \mathcal{F} \rightarrow \tilde{\mathcal{F}}$ is injective since its image is $\mathcal{F} \otimes_{K} K \cong \mathcal{F}$; it is a morphism of $\operatorname{Diff}_{\lambda}$ because $\partial(f \otimes 1)=f^{\prime} \otimes 1$.

The same argument can be used to demonstrate that $\int: \tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{F}}$ is well-defined. Hence assume $\sum_{i} f(i) \otimes f_{\varepsilon}(i)=0$ as before; we show $\sum_{i}(\oint f(i)) \otimes f_{\varepsilon}(i)=\sum_{i} 1 \otimes\left(f_{\varepsilon}(i) \oint f(i)\right)$. Since $\tilde{c}(i, j):=\oint c(i, j) g(j)-c(i, j) \oint g(j) \in K$, we get $\tilde{c}(i, j) \otimes f_{\varepsilon}(i)=1 \otimes \tilde{c}(i, j) f_{\varepsilon}(i)$ and therefore

$$
\sum_{i}(\oint f(i)) \otimes f_{\varepsilon}(i)=\sum_{i, j} \tilde{c}(i, j) \otimes f_{\varepsilon}(i)=\sum_{i, j} 1 \otimes \tilde{c}(i, j) f_{\varepsilon}(i)=\sum_{i} 1 \otimes\left(f_{\varepsilon}(i) \oint f(i)\right),
$$

where the first equality uses $\sum_{i, j}(c(i, j) \oint g(j)) \otimes f_{\varepsilon}(i)=\sum_{j}(\oint g(j)) \otimes\left(\sum_{i} c(i, j) f_{\varepsilon}(i)\right)=0$ and the last $\sum_{i, j} 1 \otimes(c(i, j) \oint g(j)) f_{\varepsilon}(i)=\sum_{j} 1 \otimes\left(\sum_{i} c(i, j) f_{\varepsilon}(i)\right) \oint g_{j}=0$.

Using now the fact that $\oint: \mathcal{F} \rightarrow \mathcal{F}$ is a Rota-Baxter operator, a short calculation reveals that $\int: \tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{F}}$ is as well. Moreover, it is immediate that $\partial \int=1_{\tilde{\mathcal{F}}}$, so $\left(\tilde{\mathcal{F}}, \partial, \int\right)$ is at least a differential Rota-Baxter algebra. Its evaluation is given by

$$
\begin{aligned}
\mathrm{E}\left(f \otimes f_{\varepsilon}\right) & =f \otimes f_{\varepsilon}-\int f^{\prime} \otimes f_{\varepsilon}=f \otimes f_{\varepsilon}-\left(\oint f^{\prime}\right) \otimes f_{\varepsilon}+1 \otimes\left(f_{\varepsilon} \oint f^{\prime}\right) \\
& =\mathrm{E}_{\mathcal{F}}(f) \otimes f_{\varepsilon}+1 \otimes\left(f_{\varepsilon} \oint f^{\prime}\right)=1 \otimes f_{\varepsilon}\left(\mathbf{E}_{\mathcal{F}}(f)+\oint f^{\prime}\right)=1 \otimes f_{\varepsilon} f,
\end{aligned}
$$

where in the third step we have used the definition of the evaluation on $\mathcal{F}$ and in the fourth the fact that all tensors are over $K=\operatorname{ker} \partial=$ ime. From this we see that the evaluation on $\tilde{\mathcal{F}}$ is multiplicative, which implies $\left(\tilde{\mathcal{F}}, \partial, \int\right) \in \mathbf{I D}_{\lambda}$ by [20, Thm. 2.5(b)].
Theorem 4.6. The integro-differential algebra $\tilde{\mathcal{F}}$ defined in Proposition 4.5 is free over $\mathcal{F}$. In other words, any $\mathbf{D R B}_{\lambda}$-morphism $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ to an integro-differential algebra $\mathcal{G}$ factors as $\varphi=\tilde{\varphi} \circ \iota$ for a unique $\mathbf{I D}_{\lambda}$-morphism $\tilde{\varphi}: \tilde{\mathcal{F}} \rightarrow \mathcal{G}$.

Proof. Let us first prove uniqueness of $\tilde{\varphi}$. Assuming $\varphi=\tilde{\varphi} \circ \iota$, we have $\tilde{\varphi}(f \otimes 1)=\varphi(f)$. Moreover, $1 \otimes f_{\varepsilon}=\mathrm{E}\left(f_{\varepsilon} \otimes 1\right)$ implies $\tilde{\varphi}\left(1 \otimes f_{\varepsilon}\right)=\mathrm{E}_{\mathcal{G}}\left(\tilde{\varphi}\left(f_{\varepsilon} \otimes 1\right)\right)=\mathrm{E}_{\mathcal{G}}\left(\varphi\left(f_{\varepsilon}\right)\right)$ since $\tilde{\varphi}$ is
an $\mathbf{I D}_{\lambda}$-morphism and thus commutes with the evaluation. As $\tilde{\varphi}$ is a morphism of $\mathbf{k}$-algebras we obtain $\tilde{\varphi}\left(f \otimes f_{\varepsilon}\right)=\varphi(f) \mathrm{E}_{\mathcal{G}}\left(\varphi\left(f_{\varepsilon}\right)\right)$, which determines $\tilde{\varphi}: \tilde{\mathcal{F}} \rightarrow \mathcal{G}$ uniquely.

For proving existence, it suffices to show that defining $\tilde{\varphi}\left(f \otimes f_{\varepsilon}\right):=\varphi(f) \mathrm{E}_{\mathcal{G}}\left(\varphi\left(f_{\varepsilon}\right)\right)$ yields an $\mathbf{I D}_{\lambda}$-morphism $\tilde{\varphi}$. Indeed, it is a $\mathbf{k}$-algebra homomorphism since $\varphi$ and $\mathbf{E}_{\mathcal{G}}$ are; one sees immediately that it respects the derivation. Let us now check that $\tilde{\varphi}$ also respects the Rota-Baxter structure, meaning $\tilde{\varphi}\left(\int\left(f \otimes f_{\varepsilon}\right)\right)=\int_{\mathcal{G}} \tilde{\varphi}\left(f \otimes f_{\varepsilon}\right)$. For the left-hand side, we apply $\tilde{\varphi}$ to $\int f \otimes f_{\varepsilon}=(\oint f) \otimes f_{\varepsilon}-1 \otimes\left(f_{\varepsilon} \oint f\right)$ to obtain

$$
\varphi(\oint f) \mathrm{E}_{\mathcal{G}}\left(\varphi\left(f_{\varepsilon}\right)\right)-\mathrm{E}_{\mathcal{G}}\left(\varphi\left(f_{\mathcal{E}} \oint f\right)\right)=\mathrm{E}_{\mathcal{G}}\left(\varphi\left(f_{\varepsilon}\right)\right)\left(\varphi(\oint f)-\mathrm{E}_{\mathcal{G}}(\varphi(\oint f))\right)
$$

using the multiplicativity of $\varphi$ and $\mathrm{E}_{\mathcal{G}}$ on the second term. Since by definition $1_{\mathcal{G}}-\mathrm{E}_{\mathcal{G}}=\int_{\mathcal{G}} \partial_{\mathcal{G}}$, the parenthesized expression above is $\int_{\mathcal{G}} \partial_{\mathcal{G}} \varphi(\oint f)=\int_{\mathcal{G}} \varphi(\partial \oint f)=\int_{\mathcal{G}} \varphi(f)$. For the righthand side, using $\int_{\mathcal{G}}$ on $\tilde{\varphi}\left(f \otimes f_{\varepsilon}\right)=\varphi(f) \mathrm{E}_{\mathcal{G}}\left(\varphi\left(f_{\varepsilon}\right)\right)$ yields $\mathrm{E}_{\mathcal{G}}\left(\varphi\left(f_{\varepsilon}\right)\right) \int_{\mathcal{G}} \varphi(f)$ since $\left(\mathcal{G}, \partial_{\mathcal{G}}, \int_{\mathcal{G}}\right)$ is an integro-differential algebra and $\int_{\mathcal{G}}$ is linear over ker $\partial_{\mathcal{G}}=\operatorname{im} \mathrm{E}_{\mathcal{G}}$ by [20, Rem. 2.6(d)].

The crucial point of the embedding of $\mathcal{F}[\partial, \oint]$ into a ring of integro-differential operators is that $\int f \partial^{k}$, though a normal form of $\mathcal{F}[\partial, \oint]$, splits when viewed as an integro-differential operator. Its reduction to normal forms can be computed as follows.
Lemma 4.7. Let $\left(\mathcal{F}, \partial, \int\right)$ be an integro-differential algebra. Then we have

$$
\begin{equation*}
\int f \partial^{k}=\sum_{i=0}^{k-1}(-1)^{i}\left(f^{(i)}-\mathrm{E}\left(f^{(i)}\right) \mathrm{E}\right) \partial^{k-i-1}+(-1)^{k} \int f^{(k)} \tag{9}
\end{equation*}
$$

for all $k>0$.
Proof. We use induction on $k>0$. The base case $k=1$ follows from the $\mathbf{I D}_{\lambda}$-relator of Proposition 3.4(d). Assume now the claim holds for some $k>0$. Then we have

$$
\int f \partial^{k+1}=\sum_{i=0}^{k-1}(-1)^{i}\left(f^{(i)}-\mathrm{E}\left(f^{(i)}\right) \mathrm{E}\right) \partial^{k-i}+(-1)^{k} \int f^{(k)} \partial,
$$

and the last term yields $(-1)^{k} f^{(k)}+(-1)^{k+1} \mathrm{E}\left(f^{(k)}\right) \mathrm{E}+(-1)^{k+1} \int f^{(k+1)}$ by the case $k=1$. Incorporating the first two summands into the summation, one obtains (9) with $k+1$ in place of $k$, which completes the induction.

We can now provide the embedding of differential Rota-Baxter operators into a ring of integro-differential operators with "generic" integral. The punch line is that one must pass to the free integro-differential algebra introduced in Proposition 4.6. Since its Rota-Baxter operator introduces new integration constants, one may view it as being initialized at a generic point; this will become clearer in Section 5 .
Theorem 4.8. Let $(\mathcal{F}, \partial, \oint)$ be an ordinary differential Rota-Baxter algebra, and $\left(\tilde{\mathcal{F}}, \partial, \int\right)$ the free integro-differential algebra defined in Proposition 4.0. Then the assignment

$$
\begin{equation*}
f \partial^{k} \mapsto f \partial^{k}, \quad f \oint \tilde{f} \mapsto f \int \tilde{f}, \quad f \oint \tilde{f} \partial^{k} \mapsto f \int \tilde{f} \partial^{k} \tag{10}
\end{equation*}
$$

defines an algebra monomorphism $\psi: \mathcal{F}[\partial, \oint] \rightarrow \tilde{\mathcal{F}}\left[\partial, \int\right]$.
Proof. From Proposition 3.4 we know that $\mathcal{F}[\partial, \oint]=\mathcal{F}[\partial] \dot{\mathcal{F}}[\oint] \backslash \mathcal{F}+[\mathrm{E}]$, where the three components have normal forms $f \partial^{k}, f \oint \tilde{f}$ and $f \oint \tilde{f} \partial^{k}$, respectively. Hence the map $\psi$ is well-defined, and it is clearly k-linear. We can also describe $\psi$ in a different but equivalent
way: Recall that $\mathcal{F} * \mathbf{k}\langle\partial, \oint\rangle$ is a coproduct in the category of (noncommutative) algebras, with canonical injections $i_{1}: \mathcal{F} \rightarrow \mathcal{F} * \mathbf{k}\langle\partial, \oint\rangle$ and $i_{2}: \mathbf{k}\langle\partial, \oint\rangle \rightarrow \mathcal{F} * \mathbf{k}\langle\partial, \oint\rangle$. Similarly, $\tilde{\mathcal{F}} * \mathbf{k}\left\langle\partial, \int\right\rangle$ is a coproduct with canonical injections $\tilde{1}_{1}$ and $\tilde{1}_{2}$. Then by the universal property for the coproduct $\mathcal{F} * \mathbf{k}\langle\partial, \oint\rangle$, there is an algebra morphism $j: \mathcal{F} * \mathbf{k}\langle\partial, \oint\rangle \rightarrow \tilde{\mathcal{F}} * \mathbf{k}\left\langle\partial, \int\right\rangle$ such that $j \circ i_{1}=\tilde{1}_{1} \circ \iota$ and $j \circ i_{2}=\tilde{1}_{2} \circ i$, where $i: \mathbf{k}\langle\partial, \oint\rangle \rightarrow \mathbf{k}\left\langle\partial, \int\right\rangle$ is the (trivial) isomorphism that renames $\oint$ into $\int$. Writing $\left[\mathbf{R B}_{\lambda}\right] \subset \mathcal{F} * \mathbf{k}\langle\partial, \oint\rangle$ and $\left[\mathbf{I D}_{\lambda}\right] \subset \tilde{\mathcal{F}} * \mathbf{k}\left\langle\partial, \int\right\rangle$ for the relator ideals of $\mathcal{F}[\partial, \oint]$ and $\tilde{\mathcal{F}}\left[\partial, \int\right]$, respectively, we have $j\left[\mathbf{D R B}_{\lambda}\right] \subset\left[\mathbf{I D}_{\lambda}\right]$ and hence $\tilde{p} j\left[\mathbf{D R B}_{\lambda}\right]=0$ for the canonical projection $\tilde{p}: \tilde{\mathcal{F}} * \mathbf{k}\left\langle\partial, \int\right\rangle \rightarrow \tilde{\mathcal{F}}\left[\partial, \int\right]=\left(\tilde{\mathcal{F}} * \mathbf{k}\left\langle\partial, \int\right\rangle\right) /\left[\mathbf{I D}_{\lambda}\right]$. Writing $p: \mathcal{F} * \mathbf{k}\langle\partial, \oint\rangle \rightarrow \mathcal{F}[\partial, \oint]=(\mathcal{F} * \mathbf{k}\langle\partial, \oint\rangle) /\left[\mathbf{D R B}_{\lambda}\right]$ for the other projection, we conclude that $\tilde{p} j: \mathcal{F} * \mathbf{k}\langle\partial, \oint\rangle \rightarrow \tilde{\mathcal{F}}\left[\partial, \int\right]$ descends to an algebra morphism $\mathcal{F}[\partial, \oint] \rightarrow \tilde{\mathcal{F}}\left[\partial, \int\right]$, which is easily recognized as $\psi$ so that $\tilde{p} j=\psi p$.

It remains to prove that $\psi$ is injective. Recall that although $f \oint \tilde{f} \partial^{k} \in \mathcal{F}[\partial, \oint]$ is a normal form, this is not the case for its image $f \int \tilde{f} \partial^{k} \in \tilde{\mathcal{F}}\left[\partial, \int\right]$. In fact, we will apply Lemma 4.7 for rewriting the latter as a k-linear combination of $\tilde{\mathcal{F}}\left[\partial, \int\right]$-normal forms. Now to show that $\psi$ is injective, assume $\psi\left(\sum_{j} w_{j}\right)=0$ with $w_{j} \neq 0$. Since $\tilde{\mathcal{F}}\left[\partial, \int\right]=\tilde{\mathcal{F}}[\partial] \dot{\mathcal{F}}\left[\int\right] \backslash \tilde{\mathcal{F}} \dot{+}(\mathrm{E})$, those $\psi\left(w_{j}\right) \in \tilde{\mathcal{F}}\left[\partial, \int\right]$ in the sum $\sum_{j} \psi\left(w_{j}\right)=0$ that belong to $\mathcal{F}[\partial]$ and $\mathcal{F}\left[\int\right] \backslash \mathcal{F}$ must cancel with corresponding contributions in the expansion (9) of the other $\psi\left(w_{j}\right) \in \tilde{\mathcal{F}}\left[\partial, \int\right]$. Hence we are left with a sum of the form $\sum_{k, l} w_{k, l}=0$ of evaluation terms coming from $\psi\left(f_{l} \oint g_{l} \partial^{k}\right)$, which are given by

$$
w_{k, l}=\sum_{i=0}^{k-1}(-1)^{i+1} f_{l} \mathrm{E}\left(g_{l}^{(i)}\right) \mathrm{E} \partial^{k-i-1}=\sum_{i=0}^{k-1}(-1)^{i+1}\left(f_{l} \otimes g_{l}^{(i)}\right) \mathrm{E} \partial^{k-i-1} \in(\mathrm{E}) \subset \tilde{\mathcal{F}}\left[\partial, \int\right]
$$

Let $\bar{k}$ be the highest exponent $k$ occurring among the $\psi\left(f_{l} \oint g_{l} \partial^{k}\right)$, and set $w_{l}:=w_{l, \bar{k}}$. Since the $\mathbf{E} \partial^{i}$ are $\mathbf{k}$-linearly independent, extracting the highest-order terms $\mathrm{E} \partial^{\bar{k}-1}$, corresponding to $i=0$ in the above sum, yields the relation $\sum_{l} f_{l} \otimes g_{l}=0$. Applying the criterion [13, Lem. 6.4] there exist $a_{l m} \in \mathbf{k}$ and $h_{m} \in \mathcal{F}$ such that $\sum_{m} a_{l m} h_{m}=f_{l}$ for each $l$, and $\sum_{l} a_{l m} g_{l}=0$ for each $m$. This implies $\sum_{l} f_{l} \oint g_{l} \partial^{k}=\sum_{m} h_{m} \oint\left(\sum_{l} a_{l m} g_{l}\right) \partial^{k}=0$, which means that there are no $k$-th order terms $w_{j} \in[\mathrm{E}] \subset \mathcal{F}[\partial, \oint]$ in the original $\operatorname{sum} \psi\left(\sum_{j} w_{j}\right)=0$. By induction on $k$, we conclude that $\sum_{j} w_{j}$ has in fact no term $w_{j} \in[\mathrm{E}]$. But then there are no terms $w_{j} \in \mathcal{F}[\partial]$ or $w_{j} \in \mathcal{F}[\oint] \backslash \mathcal{F}$ since their images in $\tilde{\mathcal{F}}\left[\partial, \int\right]$ would have nothing to cancel. Hence $\sum_{j} w_{j}=0$, completing the proof that $\psi$ is injective.

Since the ring of integro-differential operators is rather well-understood [27, 28], it is advantageous to have a description of the less familiar and somewhat more subtle differential Rota-Baxter operator ring $\mathcal{F}[\partial, \oint]$ as a subring of $\tilde{\mathcal{F}}\left[\partial, \int\right]$. However, there is a price to paythe expansion of integration constants from $\mathbf{k} \subset \mathcal{F}$ to $\mathbf{k} \otimes_{\mathbf{k}} \mathcal{F} \subset \tilde{\mathcal{F}}$. This becomes even more transparent in the case of polynomial coefficients, which we describe next.

## 5. The Integro-Differential Weyl Algebra

It is most efficient for our purposes to view the classical Weyl algebra $A_{1}(\mathbf{k})$ in the language of skew polynomial rings [10, §7.3] since this allows a smooth passage to the Rota-Baxter case and moreover provides a convenient framework for algorithmic tasks [9]. Let us recall the basic setup (without the twist endomorphism that we shall not need here). For this section we assume that $\mathbf{k}$ is a field of characteristic zero.

If $A$ is a k-algebral with derivation $\delta: A \rightarrow A$ and $\xi$ an indeterminate, the skew polynomial ring $A[\xi ; \delta]$ is the free left $A$-module $\bigoplus_{n \geq 0} A \xi^{n}$ with $\mathbf{k}$-basis $\xi^{n}$. The multiplication on $A[\xi ; \delta]$ extends the one on $A$ through the rule $\xi a=a \xi+\delta(a)$, subject to the obvious identifications $\xi^{0}=1, \xi^{n+1}=\xi \xi^{n}$. Extending also the homotheties in the obvious way, one obtains a k-algebra $A[\xi ; \delta]$ that contains $A$ as subalgebra.

Using $A=\mathbf{k}[x]$ with the standard derivation $\delta=d / d x$ and indeterminate $\xi=\partial$ yields the one-dimensional Weyl algebra $A_{1}(\mathbf{k})=\mathbf{k}[x][\partial ; \delta]$. One can also introduce the $n$-dimensional Weyl algebra in a similar way, starting with the algebra $A=\mathbf{k}\left[x_{1}, \ldots, x_{n}\right]$ and the derivations $\delta_{k}=\partial / \partial x_{k}$ and adjoining the indeterminates $\xi_{k}=\partial_{k}$ to obtain the skew polynomial ring $A_{n}(\mathbf{k})=\mathbf{k}\left[x_{1}, \ldots, x_{n}\right]\left[\partial_{1}, \delta_{1}\right] \cdots\left[\partial_{n}, \delta_{n}\right]$. Here we restrict ourselves to the onedimensional case, for which we shall henceforth use the alternative notation $\mathrm{A}_{1}(\partial):=A_{1}(\mathbf{k})$ as in [24]. In view of the upcoming Rota-Baxter analogs, we refer to $\mathrm{A}_{1}(\partial)$ as the differential Weyl algebra.

To be more precise, we actually employ the opposite route of defining $\mathrm{A}_{1}(\partial):=\mathbf{k}[\partial][x ; \delta]$ where $\delta: \mathbf{k}[\partial] \rightarrow \mathbf{k}[\partial]$ is now defined as the negative of the standard derivation, so that $\delta\left(\partial^{n}\right):=-n \partial^{n-1}$. One sees immediately that both definitions are equivalent since the Weyl algebra enjoys the well-known automorphism $x \leftrightarrow-\partial$. The reason for this unusual definition is that, for the Rota-Baxter counterpart of $\mathrm{A}_{1}(\partial)$, only the second definition will work. 月 $^{2}$ Indeed, we introduce [24] the integro Weyl algebra $\mathrm{A}_{1}(\ell):=\mathbf{k}[\ell][x ; \delta]$ with the derivation $\delta\left(\ell^{n}\right):=+n \ell^{n+1}$. Note that here as in [24] we use $\ell$ rather than $\int$ for the RotaBaxter operator; this improves the readability of iterated integrals and emphasizes the dual nature of $\partial$ and $\ell$.

Both derivations are fully determined by their action on the generators, namely $\delta(\partial)=-1$ and $\delta(\ell)=\ell^{2}$. The former encodes the Leibniz axiom in the commutator form $[x, \partial]=1$, the latter the analogous Rota-Baxter axiom $[x, \ell]=\ell^{2}$. Let us now make this precise by comparing those Weyl algebras with the corresponding linear operator rings of Section 3 . From now on, all weights are zero; we shall suppress the subscript $\lambda=0$ for the standard varieties in Diff $_{0}, \mathbf{R B}_{0}, \mathbf{D R B} \mathbf{D}_{0}$ and $\mathbf{I D}_{0}$.

Lemma 5.1. We have $\mathrm{A}_{1}(\partial) \cong \mathbf{k}[x][\partial \mid \mathbf{D i f f}]$ and $\mathrm{A}_{1}(\ell) \cong \mathbf{k}[x][\ell \mid \mathbf{R B}]$ as $\mathbf{k}$-algebras.
Proof. By Proposition 3.4 we know that $\mathbf{k}[x][\partial \mid$ Diff $]$ and $\mathbf{k}[x][\ell \mid \mathbf{R B}]$ are respectively defined by the Leibniz relation $\partial f=f \partial+f^{\prime}$ and the Rota-Baxter relation $\ell f \ell=f^{\prime} \ell-\ell f^{\prime}$; the latter employs the notation $\ell$ instead of $\int$ for uniformity. Clearly, it is enough to require the relations on the $\mathbf{k}$-basis $x^{n}$ of $\mathbf{k}[x]$. But the Leibniz relation for $f=x^{n}$ follow immediately by a simple induction argument from the special case $f=x$, which is just the commutator relation $[\partial, x]=-1$. For the Rota-Baxter relation, we show now that it suffices to take the special case $f=1$, embodied in the commutation $[x, \ell]=\ell^{2}$. We use induction on $n>0$ to prove $n \ell x^{n-1} \ell \equiv x^{n} \ell-\ell x^{n}$ modulo the two-sided ideal $\left(x \ell-\ell x-\ell^{2}\right)$.

[^6]The base case $n=1$ being trivial, assume the claim for $n \geq 1$. Then we have

$$
\begin{aligned}
\ell x^{n} \ell & \equiv n \ell^{2} x^{n-1} \ell+\ell^{2} x^{n} \equiv n(x \ell-\ell x) x^{n-1} \ell+(x \ell-\ell x) x^{n} \\
\text { or } \quad(n+1) \ell x^{n} \ell & \equiv n x \ell x^{n-1} \ell+x \ell x^{n}-\ell x^{n+1} \equiv x^{n+1} \ell-\ell x^{n+1}
\end{aligned}
$$

where the last step uses the induction hypothesis. Hence we obtain

$$
\begin{equation*}
\mathbf{k}[x][\partial \mid \mathbf{D i f f}] \cong \mathbf{k}\langle x, \partial\rangle /(\partial x-x \partial-1) \quad \text { and } \quad \mathbf{k}[x][\ell \mid \mathbf{R B}] \cong \mathbf{k}\langle x, \ell\rangle /\left(x \ell-\ell x-\ell^{2}\right) \tag{11}
\end{equation*}
$$

Using the reductions $x \partial \rightarrow \partial x-1$ and $x \ell \rightarrow \ell x+\ell^{2}$ for the ideals in (11), this corresponds to the multiplication in the skew polynomial rings $\mathrm{A}_{1}(\partial)=\mathbf{k}[\partial][x ; \delta]$ and $\mathrm{A}_{1}(\ell)=\mathbf{k}[\ell][x ; \delta]$.

The integro Weyl algebra shares certain common features with its differential counterpart but also exhibits some striking differences [24]. While both are Noetherian integral domains, the differential Weyl algebra is a simple ring but the integro Weyl algebra is not. On the other hand, the latter comes with a natural grading whereas the former only enjoys filtration. In this paper, we are not concerned with their further study. Let us just mention the following noteworthy commutations.
Lemma 5.2. We have the commutations $\left[x^{i}, \ell\right]=i \ell x^{i-1} \ell$ and $\left[x, \ell^{j}\right]=j \ell^{j+1}$ in $A_{1}(\ell)$.
Proof. The first commutation is [24, Lem. 11] and follows also from the proof of Lemma 5.1 above. The second commutation is the defining property of $A_{1}(\ell)=\mathbf{k}[\ell][x ; \delta]$ as a skew polynomial ring.

For introducing the integro-differential Weyl algebra $\mathrm{A}_{1}(\partial, \ell)$ one needs a coefficient domain $\mathbf{k}[\partial, \ell]$ that contains $\mathbf{k}[\partial]$ as well as $\mathbf{k}[\ell]$, subject to the natural requirement $\partial \ell=1$. In other words, we set $\mathbf{k}[\partial, \ell]=\mathbf{k}\langle D, L\rangle /(D L-1)$ where $\partial$ and $\ell$ are the residue classes of $D$ and $L$, respectively. This ring has been studied extensively; see for example [21, 16]. The derivation $\delta$ on $\mathbf{k}[\partial, \ell]$ is determined uniquely as an extension of the derivations on $\mathbf{k}[\partial]$ and $\mathbf{k}[\ell]$. Defining now the integro-differential Weyl algebra by $\mathrm{A}_{1}(\partial, \ell):=\mathbf{k}[\partial, \ell][x ; \delta]$, it is immediately clear that $\mathrm{A}_{1}(\partial, \ell)$ contains $\mathrm{A}_{1}(\partial)$ and $\mathrm{A}_{1}(\ell)$ as subalgebras.

We refer again to [24 for some basic algebraic properties of $\mathrm{A}_{1}(\partial, \ell)$; for deeper and more general results one may consult [2] and the references therein. Let us only mention that $\mathrm{A}_{1}(\partial, \ell)$ is neither Noetherian nor free of zero divisors. Writing $\mathrm{E}:=1-\ell \partial \in \mathrm{A}_{1}(\partial, \ell)$ for what we call again the evaluation, we have $\mathrm{A}_{1}(\partial, \ell)=\mathrm{A}_{1}(\partial) \dot{+} \mathrm{A}_{1}(\ell) \backslash \mathbf{k}[x] \dot{+}(\mathrm{E})$ as $\mathbf{k}$-modules [24, (18)]. The resemblance with the decomposition of Lemma 4.1 is no coincidence as can be seen in Corollary 5.4 below.
Lemma 5.3. For $\mathrm{A}_{1}(\partial, \ell)$ one may choose the $\mathbf{k}$-bases $\mathfrak{B}_{i}:=\mathfrak{D} \cup \mathfrak{R} \cup \mathfrak{E}_{i}(i=1,2,3)$ containing the subbases $\mathfrak{D}=\left\{x^{i} \partial^{k} \mid i, k \geq 0\right\}$ and $\mathfrak{R}=\left\{x^{i} \ell^{j} \mid i \geq 0 ; j>0\right\}$ together with
(a) $\mathfrak{E}_{1}:=\left\{x^{i} \ell^{j} \mathcal{E}^{k} \mid i, j, k \geq 0\right\}$,
(b) $\mathfrak{E}_{2}:=\left\{x^{i} \ell^{j} \partial^{k} \mid i \geq 0 ; j, k>0\right\}$,
(c) $\mathfrak{E}_{3}:=\left\{x^{i} \ell x^{j} \partial^{k} \mid i, j \geq 0 ; k>0\right\}$.

Hence one has $\mathfrak{B}_{2}=\left\{x^{i} \ell^{j} \partial^{k} \mid i, j, k \geq 0\right\}$ for the case (b). Moreover, one may also use the subbasis $\mathfrak{R}^{\prime}:=\left\{x^{i} \ell x^{j} \mid i, j \geq 0\right\}$ in place of $\mathfrak{R}$.
Proof. The basis $\mathfrak{B}_{1}$ has already been derived; see the observation before [24, Lem. 19]. Both $\mathfrak{R}$ and $\mathfrak{R}^{\prime}$ are known to be $\mathbf{k}$-bases of $\mathrm{A}_{1}(\ell) \backslash \mathbf{k}[x] \leq \mathrm{A}_{1}(\partial, \ell)$, called the right basis and the mid basis; see [24, Cor. 12] and the remark before [24, Lem. 11]. Hence the subbases $\mathfrak{R}$ and $\mathfrak{R}^{\prime}$ are interchangeable.

Let us next prove that $\mathfrak{B}_{2}$ is also a basis. We write $\mathfrak{D}(n):=\left\{x^{i} \partial^{k} \mid 0 \leq i, k \leq n\right\}$ and $\mathfrak{R}(n):=\left\{x^{i} \ell^{j} \mid 0 \leq i \leq n, 0<j \leq n\right\}$ for the truncations of $\mathfrak{D}$ and $\mathfrak{R}$. Likewise we have $\mathfrak{E}_{1}(n):=\left\{x^{i} \ell^{j} \partial^{k} \mid 0 \leq i \leq n ; 0 \leq j, k<n\right\}$ and $\mathfrak{E}_{2}(n):=\left\{x^{i} \ell^{j} \partial^{k} \mid 0 \leq i \leq n ; 0<j, k \leq n\right\}$ for the truncated complements. Now set $\mathfrak{B}_{i}(n):=\mathfrak{D} \cup \mathfrak{R} \cup \mathfrak{E}_{i}$ for $i=1$ and $i=2$. Clearly we have $\left|\mathfrak{B}_{1}(n)\right|=\left|\mathfrak{B}_{2}(n)\right|$ and

$$
\underset{\longrightarrow}{\lim } \mathfrak{B}_{i}(n)=\mathfrak{B}_{i}
$$

for $i=1$ and $i=2$. Since

$$
\ell^{j} \mathrm{E} \partial^{k}=\ell^{j}(1-\ell \partial) \partial^{k}=\ell^{j} \partial^{k}-\ell^{j+1} \partial^{k+1}
$$

we see that $\mathfrak{B}_{2}(n)$ generates $\mathbf{k} \mathfrak{B}_{1}(n)$. But the latter has $\mathfrak{B}_{1}(n)$ for a basis since it is a subset of the $\mathbf{k}$-basis $\mathfrak{B}_{1}$ of $\mathrm{A}_{1}(\partial, \ell)$. Since $\mathfrak{B}_{2}(n)$ is thus a generating set of the same cardinality, we conclude that $\mathfrak{B}_{2}(n)$ is also a $\mathbf{k}$-basis of $\mathbf{k} \mathfrak{B}_{1}(n)$ and hence linearly independent, and so are those of $\mathfrak{B}_{2}=\lim \mathfrak{B}_{2}(n)$. It follows that $\mathfrak{B}_{2}$ is a $\mathbf{k}$-basis of $\mathrm{A}_{1}(\partial, \ell)$.
 define $\psi: \mathbf{k} \mathfrak{B}_{2} \rightarrow \mathbf{k} \mathfrak{B}_{1}$ by fixing $\mathfrak{D}$ and $\mathfrak{R}$ while setting

$$
\psi\left(x^{i} \ell^{j} \partial^{k}\right)= \begin{cases}x^{i}\left(\ell^{j-k}-\sum_{m=1}^{k} \ell^{j-m} \mathrm{E} \partial^{k-m}\right) & \text { if } j \geq k \\ x^{i}\left(\partial^{k-j}-\sum_{m=1}^{j} \ell^{j-m} \mathrm{E} \partial^{k-m}\right) & \text { if } j<k\end{cases}
$$

Similarly, we define $\varphi: \mathbf{k} \mathfrak{B}_{1} \rightarrow \mathbf{k} \mathfrak{B}_{2}$ by fixing again $\mathfrak{D}$ and $\mathfrak{R}$, and sending $x^{i} \ell^{j}{ }_{\mathrm{E}} \partial^{k}$ to $x^{i}\left(\ell^{j} \partial^{k}-\ell^{j+1} \partial^{k+1}\right)$. Let us now show that $\psi \circ \varphi=1$ and $\varphi \circ \psi=1$. Obviously it suffices now to consider $\mathfrak{E}_{1}$ and $\mathfrak{E}_{2}$. For $j \geq k$ one has

$$
\begin{aligned}
& (\varphi \circ \psi)\left(x^{i} \ell^{j} \partial^{k}\right)=\varphi\left(x^{i} \ell^{j-k}-\sum_{m=1}^{k} x^{i} \ell^{j-m} \mathrm{E} \partial^{k-m}\right) \\
& \quad=x^{i} \ell^{j-k}-\sum_{m=1}^{k} x^{i}\left(\ell^{j-m} \partial^{k-m}-\ell^{j-m+1} \partial^{k-m+1}\right)=x^{i} \ell^{j} \partial^{k}
\end{aligned}
$$

and for $j<k$ again

$$
\begin{aligned}
& (\varphi \circ \psi)\left(x^{i} \ell^{j} \partial^{k}\right)=\varphi\left(x^{i} \partial^{k-j}-\sum_{m=1}^{j} x^{i} \ell^{j-m} \mathrm{E} \partial^{k-m}\right) \\
& \quad=x^{i} \partial^{k-j}-\sum_{m=1}^{j} x^{i}\left(\ell^{j-m} \partial^{k-m}-\ell^{j-m+1} \partial^{k-m+1}\right)=x^{i} \ell^{j} \partial^{k}
\end{aligned}
$$

For the other direction, in case $j \geq k$ one obtains

$$
\begin{aligned}
& (\psi \circ \varphi)\left(x^{i} \ell^{j} \mathrm{E} \partial^{k}\right)=\psi\left(x^{i} \ell^{j} \partial^{k}-x^{i} \ell^{j+1} \partial^{k+1}\right) \\
& \quad=x^{i}\left(\sum_{m=0}^{k} \ell^{j-m} \mathrm{E} \partial^{k-m}-\sum_{m=1}^{k} \ell^{j-m} \mathrm{E} \partial^{k-m}\right)=x^{i} \ell^{j} \mathrm{E} \partial^{k}
\end{aligned}
$$

and in case $j<k$ likewise

$$
\begin{aligned}
& (\psi \circ \varphi)\left(x^{i} \ell^{j} \mathrm{E} \partial^{k}\right)=\psi\left(x^{i} \ell^{j} \partial^{k}-x^{i} \ell^{j+1} \partial^{k+1}\right) \\
& \quad=x^{i}\left(\sum_{m=0}^{j} \ell^{j-m} \mathrm{E} \partial^{k-m}-\sum_{m=1}^{j} \ell^{j-m} \mathrm{E} \partial^{k-m}\right)=x^{i} \ell^{j} \mathrm{E} \partial^{k} .
\end{aligned}
$$

Since $\mathfrak{B}_{1}$ is a $\mathbf{k}$-basis of $A_{1}(\partial, \ell)$ we have the isomorphism $A_{1}(\partial, \ell) \cong \mathbf{k} \mathfrak{B}_{1}$, which together with the isomorphism $\varphi: \mathbf{k} \mathfrak{B}_{1} \cong \mathbf{k} \mathfrak{B}_{2}$ yields $A_{1}(\partial, \ell) \cong \mathbf{k} \mathfrak{B}_{2}$, and this implies that $\mathfrak{B}_{2}$ is also a k-basis of $A_{1}(\partial, \ell)$ as already proved above.

For proving that $\mathfrak{B}_{3}$ is a $\mathbf{k}$-basis of $A_{1}(\partial, \ell)$, one proceeds similarly using the transition maps $\varphi: \mathbf{k} \mathfrak{B}_{2} \rightarrow \mathbf{k} \mathfrak{B}_{3}$ and $\psi: \mathbf{k} \mathfrak{B}_{3} \rightarrow \mathbf{k} \mathfrak{B}_{2}$ defined by

$$
\begin{aligned}
\varphi\left(x^{i} \ell^{j} \partial^{k}\right) & =\sum_{m=0}^{j-1} \frac{(-1)^{m}}{m!(j-m-1)!} x^{i+j-m-1} \ell x^{m} \partial^{k} \\
\psi\left(x^{i} \ell x^{j} \partial^{k}\right) & =\sum_{m=0}^{j} \frac{(-1)^{j-m} j!}{m!} x^{i+m} \ell^{j-m+1} \partial^{k}
\end{aligned}
$$

in view of the identities [24, (17)/(16)]. Alternatively, one may use the two truncated bases $\mathfrak{B}_{2}^{\prime}(n):=\mathfrak{D}(n) \cup \mathfrak{R}(n) \cup \mathfrak{E}_{2}^{\prime}(n)$ and $\mathfrak{B}_{3}(n):=\mathfrak{D}(n) \cup \mathfrak{R}(n) \cup \mathfrak{E}_{3}(n)$ converging to $\mathfrak{B}_{2}=\underset{\longrightarrow}{\lim } \mathfrak{B}_{2}^{\prime}(n)$ and $\mathfrak{B}_{3}=\underset{\longrightarrow}{\lim } \mathfrak{B}_{3}(n)$, where one defines

$$
\begin{aligned}
& \mathfrak{E}_{2}^{\prime}(n):=\left\{x^{i} \ell{ }^{j} \partial^{k} \mid 0 \leq i<n ; 0<j, k \leq n ; i+j \leq n\right\}, \\
& \mathfrak{E}_{3}(n):=\left\{x^{i} \ell x^{j} \partial^{k} \mid 0 \leq i, j<n ; 0<k \leq n ; i+j<n\right\} .
\end{aligned}
$$

The rest of the argument is then as above, with $\mathfrak{B}_{2}^{\prime}$ in place of $\mathfrak{B}_{1}$, and $\mathfrak{B}_{3}$ in place of $\mathfrak{B}_{2}$.
The three bases correspond to direct decompositions $\mathrm{A}_{1}(\partial, \ell)=\mathbf{k} \mathfrak{D} \dot{\mathbf{k}} \mathfrak{R}+\mathbf{k} \mathfrak{E}_{i}(i=$ $1,2,3)$ with standard components $\mathbf{k} \mathfrak{D}=A_{1}(\partial)$ and $\mathbf{k} \mathfrak{R}=A_{1}(\ell) \backslash \mathbf{k}[x]$. The extra component is either the evaluation ideal $\mathbf{k} \mathfrak{E}_{1}=(\mathbf{E})$, the left $\mathbf{k}[x]$-submodule $\mathbf{k} \mathfrak{E}_{2}$, or the evaluation rung $[\mathrm{E}]=\mathbf{k E}_{3}$.
Corollary 5.4. We have $\mathrm{A}_{1}(\partial, \ell) \cong \mathbf{k}[x][\partial, \oint \mid \mathbf{D R B}]$ as $\mathbf{k}$-algebras.
Proof. In view of the decomposition in Lemma 4.1 and the isomorphisms of Lemma 5.1, this follows immediately from Lemma 5.3 since the evaluation rung $[\mathrm{E}] \leq \mathbf{k}[x][\partial, \oint]$ has the k-basis $\left\{x^{i} \oint x^{j} \partial^{k} \mid i, j \geq 0, k>0\right\}$, which corresponds to $\mathfrak{E}_{3}$.

It is now easy to derive the following specialization isomorphism [24, Thm. 20] from the general quotient result on the differential Rota-Baxter operator rings.
Proposition 5.5. We have $A_{1}(\partial, \ell) /(\mathbf{E} x) \cong \mathbf{k}[x]\left[\partial, \int \mid \mathbf{I D}\right]$ as $\mathbf{k}$-algebras.
Proof. Using the isomorphism of Corollary 5.4, this follows from Proposition 3.4.
Note that here we have used the standard Rota-Baxter operator $\int: x^{k} \mapsto x^{k+1} /(k+1)$ for the integro-differential Weyl algebra and the corresponding integro-differential operator ring $\mathbf{k}[x]\left[\partial, \int\right]$. As can be seen from [24, Thm. 20], one can also start from any other integro-differential structure $\left(\partial, \int\right)$ on $\mathbf{k}[x]$ for obtaining a similar isomorphism except that one factors out the ideal $(\mathbf{E} x-c \mathrm{E})$ where $c:=\mathrm{E}(x) \in \mathbf{k}$ is the integration constant associated with the integral operator $\int$.

The specialization isomorphism (Proposition 5.5) can be interpreted as "simulating" integro-differential operators by differential Rota-Baxter operators (in the important case of polynomial coefficients). Since the structure of the latter is finer, this is in principle not surprising. However, we can also derive a corresponding generalization isomorphism that identifies the finer ring of differential Rota-Baxter operators as a subalgebra in an overarching integro-differential operator ring. To this end, we take our earlier result of the general theory (Theorem 4.8) and interpret it in the more concrete polynomial setting.

Theorem 5.6. For $\varepsilon$ transcendental over $\mathbf{k}$, endow $\tilde{\mathbf{k}}[x]=\mathbf{k}[x, \varepsilon]$ with derivation $\partial=\partial / \partial x$ and integral $\int=\int_{\varepsilon}^{x}$. Then there is a unique $\mathbf{k}$-algebra monomorphism

$$
\iota: \mathrm{A}_{1}(\partial, \ell) \hookrightarrow \tilde{\mathbf{k}}[x]\left[\partial, \int\right]
$$

that sends $\ell$ to $\int$ while fixing $x$ and $\partial$.
Proof. Applying Theorem 4.8 to $\mathcal{F}:=\mathbf{k}[x]$ we observe that $\tilde{\mathcal{F}}=\mathbf{k}[x] \otimes_{\mathbf{k}} \mathbf{k}[x] \cong \mathbf{k}[x, \varepsilon]$, defined by Proposition 4.5, has the derivation and integral as described in the current theorem. Indeed, $\partial\left(x^{i} \otimes x^{j}\right)=\partial\left(x_{i}\right) \otimes x^{j}$ means $\partial\left(x^{i} \varepsilon^{j}\right)=(\partial / \partial x) x^{i} \varepsilon^{j}$ for the derivation while $\int\left(x^{i} \otimes x^{j}\right)=\left(\oint x^{i}\right) \otimes x^{j}-1 \otimes\left(x^{j} \oint x^{i}\right)$, where $\oint$ denotes the standard Rota-Baxter operator on $\mathbf{k}[x]$, translates to

$$
\int x^{i} \varepsilon^{j}=\frac{x^{i+1}}{i+1} \varepsilon^{j}-\varepsilon^{j} \frac{\varepsilon^{i+1}}{i+1}=\int_{\varepsilon}^{x} x^{i} \varepsilon^{j}
$$

for the integral. Hence $\tilde{\mathcal{F}}$ coincides with $\tilde{\mathbf{k}}[x]$ as an integro-differential algebra.
Let us now consider the map $\iota: \mathrm{A}_{1}(\partial, \ell) \rightarrow \tilde{\mathbf{k}}[x]\left[\partial, \int\right]$. Since $x, \partial$ and $\ell$ generate $\mathrm{A}_{1}(\partial, \ell)$, the uniqueness claim follows. But we know from Corollary 5.4 that $\mathrm{A}_{1}(\partial, \ell) \cong \mathcal{F}[\partial, \oint]$, and with this identification the map $\iota$ is clearly the same as the $\mathbf{k}$-algebra monomorphism given in Theorem 4.8.

The intuitive idea behind the generalization isomorphism is that one adjoins a generic initialization point $\varepsilon$ for the integral $\int$. The associated (multiplicative) evaluation $\mathrm{E}=1-\int \partial$ sends $f(x, \varepsilon) \in \tilde{\mathbf{k}}[x]$ to $f(\varepsilon, \varepsilon) \in \widetilde{\mathbf{k}}:=\mathbf{k}[\varepsilon]$. This yields an isomorphic copy $\iota\left(\mathrm{A}_{1}(\partial, \ell)\right)$ of the integro-differential Weyl algebra in $\tilde{\mathbf{k}}[x]\left[\partial, \int\right]$. However, one should observe that $\iota\left(\mathrm{A}_{1}(\partial, \ell)\right)$ by itself is only a differential Rota-Baxter operator ring and not an integro-differential operator ring: The evaluation $f(x, \varepsilon) \mapsto f(\varepsilon, \varepsilon)$ does not restrict to a map on its coefficient domain $\mathbf{k}[x]$.

One may also derive a $\mathbf{k}$-basis of $\iota\left(\mathrm{A}_{1}(\partial, \ell)\right) \leq \tilde{\mathbf{k}}[x]\left[\partial, \int\right]$. For any integro-differential operator ring one has the relation $\int f \mathrm{E}=\left(\int f\right) \mathrm{E}$; see [27, Table 1] and Footnote $\mathbb{⿴}$ in the proof of Proposition 3.4. Setting $f=1$ and iterating $j$ times the integral $\int=\int_{\varepsilon}^{x}$ one obtains the relation $\int \cdots \int \mathrm{E}=(x-\varepsilon)^{j} / j$ ! E . Hence $\iota$ maps the basis elements $x^{i} \ell^{j} \mathrm{E} \partial^{k} \in \mathfrak{E}_{1}$ of Lemma 5.3 to $(1 / j!) x^{i}(x-\varepsilon)^{j} \mathrm{E} \partial^{k}$ while "fixing" those of $\mathfrak{D}$ and $\mathfrak{R}^{\prime}$.

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[^0]:    Date: July 16, 2018.
    ${ }^{\text {a }}$ See the end of this section for conventions on notation and terminology.
    ${ }^{\mathrm{b}}$ This is usually called the kernel $k(x, y)$ of an integral operator $f(x) \mapsto \int k(x, y) f(y) d y$. In algebra, we prefer the less common term nucleus for avoiding confusion with the kernel of a homomorphism.

[^1]:    ${ }^{\mathrm{c}}$ In rewriting, this means one can turn the equation $l=\lambda_{1} u_{1}+\cdots+\lambda_{m} u_{m}=0$ into a rewrite rule $u_{k} \rightarrow\left(\lambda_{1} / \lambda_{k}\right) u_{1}+\cdots+\left(\lambda_{k-1} / \lambda_{k}\right) u_{k-1}+\left(\lambda_{k+1} / \lambda_{k}\right) u_{k+1}+\cdots+\left(\lambda_{m} / \lambda_{k}\right) u_{m}$ for any leading term $u_{k}$, and the resulting rule will be linear in the sense of [1], Def. 6.3.1]. In fact, the rewriting terminology allows variables to be absent in terms; this is not needed for our present purposes.

[^2]:    ${ }^{\mathrm{d}}$ Note that we rely on the context to disambiguate the notations $\mathcal{F}[\partial, \oint]$ and $\mathcal{F}\left[\partial, \int\right]$. In the frame of this paper, a Rota-Baxter operator will always be denoted by $\oint$ when it comes from a differential Rota-Baxter algebra, and by $\int$ when it comes from an integro-differential algebra.
    ${ }^{e}$ This is of course always satisfied in the zero weight case (with trivial shift). But it is also satisfied in the classical example with weight $\lambda= \pm 1$ : the sequence space $M^{\mathbb{Z}}$ over a k-module $M$ with forward/backward difference as derivation. This has increment/decrement as mutually inverse shifts.

[^3]:    ${ }^{\mathrm{f}}$ The character $\mathrm{E} \in \Phi$ is not part of the operator set $\Omega$, and its appearance on the right-hand side is to be understood merely as an abbreviation $\mathrm{E}:=1-\int \partial$. Moreover, the corresponding rules with E on the left-hand side are not required in $E$ since they follow from the other rules.

[^4]:    ${ }^{\mathrm{g}}$ Note that $M=A \dot{+} B \dot{+} C \dot{+} Z$ implies $M / Z=(A+Z) / Z \dot{+}(B+Z) / Z \dot{+}(C+Z) / Z$ for arbitrary submodules $A, B, C, Z$ of some module $M$.
    ${ }^{\mathrm{h}}$ The proofs in [28] use only ring-theoretic properties of $\mathbf{k}$; no field or zero characteristic is required. They are more general in that they allow character sets $\Phi \supsetneq\{\mathrm{E}\}$.

[^5]:    ${ }^{\text {i }}$ The two occurrences of $\mathcal{D}$ in " $\mathcal{D}$-modules" and in the distribution space $\mathcal{D}^{\prime}(\mathbb{R})$ are unrelated. In fact, $\mathcal{D}^{\prime}(\mathbb{R})$ is the dual of the differentiable class $\mathcal{D}(\mathbb{R}):=C_{0}^{\infty}(\mathbb{R})$ of smooth functions with compact support.

[^6]:    ${ }^{\mathrm{j}}$ In contrast to $\left.\sqrt{10}, \S 7.3\right]$, we do not require that $A$ be a domain though if it is then $A[\xi ; \delta]$ is as well. One sees easily [23, §1.1.2] that the construction works for any $\mathbf{k}$-algebra $A$, and this will indeed be crucial for our definition of the integro-differential Weyl algebra.
    ${ }^{\mathrm{k}}$ Following the standard definition of the differential case, one would need $\delta(x)=-\ell^{2}$, which does not yield a derivation $\delta: \mathbf{k}[x] \rightarrow \mathbf{k}[x]$. Algorithmically speaking, the problem is that while the degrees of $\partial$ are decreasing, the ones of $\ell$ are increasing; see the remark before [24, Def. 2].

