

APPLICATIONS OF L SYSTEMS TO GROUP THEORY

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ABSTRACT. L systems generalise context-free grammars by incorporating parallel rewriting, and generate languages such as EDT0L and ET0L that are strictly contained in the class of indexed languages. In this paper we show that many of the languages naturally appearing in group theory, and that were known to be indexed or context-sensitive, are in fact ET0L and in many cases EDT0L. For instance, the language of primitives and bases in the free group on two generators, the Bridson-Gilman normal forms for the fundamental groups of 3-manifolds or orbifolds, and the co-word problem of Grigorchuk's group can be generated by L systems. To complement the result on primitives in rank 2 free groups, we show that the language of primitives, and primitive sets, in free groups of rank higher than two is context-sensitive. We also show the existence of EDT0L and ET0L languages of intermediate growth.

1. INTRODUCTION

In this paper we show that many of the context-sensitive or indexed languages arising in problems in group theory and combinatorics are in fact ET0L or even EDT0L. The merit of giving this new formal language characterisation is that ET0L and EDT0L languages are a strict subclass of the indexed ones, and also, the descriptions of these languages are simpler and more algebraic than those based on indexed grammars or nested-stack automata that were given in the literature for the sets considered here.

Both EDT0L and ET0L belong to the languages generated by L systems, which were introduced by Lindenmayer in the late 1960s in order to model the growth of various organisms. The acronym ET0L (respectively EDT0L) refers to *Extended, Table, 0 interaction, and Lindenmayer* (respectively *Deterministic*). ET0L and EDT0L languages have only recently featured in group theory; their first prominent appearance was in the work of the first two authors with Volker Diekert, in the context of equations in groups. The language of solutions of equations in free groups as tuples of reduced words is EDT0L [5], as are solutions in appropriate normal forms in virtually free groups [9] and partially commutative groups [10]. The result in [5] implies that the pattern languages studied by Jain, Miasnikov and Stephan are EDT0L as well [19].

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In Section 2 we give the necessary background on formal languages. In Section 3 we examine the gap between ETOL and indexed languages, and in Section 4 we revisit an example of Grigorchuk and Machì of a language of intermediate growth and show that it is EDTOL. In the remaining sections we show why Lindenmeyer languages are relevant for group theory by presenting several instances where L systems appear naturally. In Section 5 we describe the set of primitives in the free group on two generators as EDTOL, improving on the context-sensitive characterisation by Silva and Weil [27]. Furthermore, we show that the set of primitives, and primitive sets, in free groups of rank higher than 2 is context-sensitive. In Section 6 we construct an explicit grammar to prove that the co-word problem in Grigorchuk's group is ETOL, and thus take a different approach to that of Holt and Röver [18], who used nested-stack automata to describe the same set. In Section 7 we show that the Bridson-Gilman normal forms for the fundamental groups of 3-manifolds or orbifolds, proved in [2] to be indexed, are in fact ETOL. We conclude the paper with a list of open problems.

2. ETOL AND EDTOL LANGUAGES

L systems were introduced by Lindenmayer in order to model the growth of various organisms and capture the fact that growth happens in *parallel* everywhere in the organism. Therefore the rewriting system had to incorporate parallelism, as opposed to the sequential behavior of context-free grammars. The difference between sequential and parallel grammars is well illustrated by the following example (page 2 in [26]).

Example 1. Suppose we have an alphabet $A = \{a\}$ and a single rewriting rule $a \rightarrow a^2$, which is to be applied to a^3 . If we apply this rule to one a inside a^3 at a time, we get the set $\{a^i \mid i \geq 3\}$. If we apply the rule simultaneously to each a in a^3 we obtain a^6 after one rewrite, and the set of words obtained via parallel rewriting is $\{a^{3 \cdot 2^i} \mid i \geq 0\}$.

There is a vast literature on Lindenmayer systems, see for example [26, 25], with various acronyms such as DOL, DTOL, ETOL, HDTOL and so forth. The following inclusions hold: $\text{EDTOL} \subsetneq \text{ETOL} \subsetneq \text{indexed}$, and $\text{context-free} \subsetneq \text{ETOL}$. Furthermore, the classes of EDTOL and context-free languages are incomparable.

Let V be a finite alphabet. A *table* for V is a finite subset of $V \times V^*$, that can be represented as in Figure 1.

V	V^*
a	$\rightarrow a, b, aba$
b	$\rightarrow bbabb$
$\#$	$\rightarrow \#, b$

FIGURE 1. A table for $V = \{a, b, \#\}$.

If (c, v) is in some table t , we say that (c, v) is a *rule* for c and use the convention that if for some $c \in V$ no rule for c is specified in t , then t contains the rule (c, c) . We express the rewriting corresponding to the rule (c, v) in t as $c \xrightarrow{t} v$.

Definition 1 (ET0L). An *ET0L-system* is a tuple $H = (V, A, T, I)$, where

- (1) V is a finite alphabet,
- (2) $A \subseteq V$ is the subset of *terminal symbols*,
- (3) T is a finite set of *tables* for V , that is, each $t \in T$ is a finite subset of $V \times V^*$, and
- (4) $I \subseteq V^*$ is a finite set of words called *axioms*.

Let $t \in T$. We will write $u \xrightarrow{t} v$ to denote that a word $v \in V^*$ can be produced from $u \in V^*$ using the rules in t ; that is, if $u = c_1 \cdots c_m$ and $v = v_1 \cdots v_m$, where $c_i \in V$ and $v_i \in V^*$, we write $u \xrightarrow{t} v$ to mean v was obtained via rules $c_j \xrightarrow{t} v_j$ from table t , applied to each c_j appearing in u . More generally, $u \rightarrow v$ signifies $u \xrightarrow{t} v$ for some $t \in T$. If there exist $u_0, \dots, u_k \in V^*$ with $u_i \rightarrow u_{i+1}$ for $0 \leq i \leq k-1$, then we write $u_0 \xrightarrow{*} u_k$. The language *generated by* H is defined as

$$L(H) = \{v \in A^* \mid w \xrightarrow{*} v \text{ for some } w \in I\}.$$

A language is ET0L if it is equal to $L(H)$ for some ET0L system H .

Definition 2 (EDT0L). An *EDT0L-system* is an ET0L system where in each table there is exactly one rule for each letter in V . A language is EDT0L if it is equal to $L(H)$ for some EDT0L system H .

ET0L languages form a full *AFL* (abstract family of languages), that is, they are closed under homomorphisms, inverse homomorphisms, intersection with regular languages, union, concatenation and Kleene closure, while EDT0L are closed under all of the above except inverse homomorphism so do not form a full AFL [7, 1].

If for some words u_1, u_2, u_3 and tables t_1, t_2 we have $u_1 \xrightarrow{t_1} u_2$ and $u_2 \xrightarrow{t_2} u_3$ we will write $u_1 \xrightarrow{t_1 t_2} u_3$ to denote the composition of the rewriting. This can be naturally extended to any finite sequence of rewrites. Given a regular expression R over a finite set $T = \{t_1, \dots, t_n\}$, where t_1, \dots, t_n are tables, we will sometimes abuse the notation and use $u_1 \xrightarrow{R} u_2$ to denote that there is a word r in the language generated by the regular expression R such that $u_1 \xrightarrow{r} u_2$. Furthermore, if the system is deterministic, then every table is in fact a homomorphism on the free monoid V^* , and using this more algebraic notation we can give Asveld's equivalent definition for E(D)T0L languages as follows [1].

Definition 3. Let A be an alphabet and $L \subseteq A^*$. We say that L is an *ET0L* language if there is an alphabet C with $A \subseteq C$, a set H of tables (i.e. finite subsets of $C \times C^*$), a regular language $\mathcal{R} \subseteq H^*$ and a letter $c \in C$ such that

$$L = \{w \in A^* \mid c \xrightarrow{r} w \text{ for some } r \in \mathcal{R}\}.$$

In the case when every table $h \in H$ is deterministic, i.e. each $h \in H$ is in fact a homomorphism, we write $r(c) = w$ and say that L is EDTOL.

The set \mathcal{R} is called the *regular (or rational) control*, the symbol c the *start symbol* and C the *extended alphabet*.

Convention. In any description of rational control in this paper, the maps are always applied left to right, but in algebraic settings where f and g are morphisms $fg(a) := f(g(a))$.

3. NON-ETOL LANGUAGES

For completeness we present in this section a short survey of examples of languages which are not ETOL. The first examples turn out not to be indexed either.

Let $\varphi: \mathbb{N}^+ \rightarrow \mathbb{N}^+$ be such that $\lim_{n \rightarrow \infty} \varphi(n) = \infty$ and let U be an arbitrary infinite subset of \mathbb{N} . Set $K(\varphi, U) = \{(ba^{\varphi(k)})^k \mid k \in U\}$. This construction gives us an infinite family of languages. It was proved in [11, Theorem 2] that $K(\varphi, U)$ is not ETOL regardless of the choice of φ and U . We show in Lemma 5 that these languages are not indexed.

Lemma 4. [15, Theorem A] (*Shrinking lemma*) *Let L be an indexed language over a finite alphabet Σ and let $m > 0$ be a given integer. There is a constant $k > 0$ such that each word $w \in L$ with $|w| \geq k$ can be factorised as a product $w = w_1 \dots w_r$ such that the following conditions hold:*

- (i) $m < r \leq k$,
- (ii) $w_i \neq \epsilon$ for every $i \in \{1, \dots, r\}$,
- (iii) each choice of m factors is included in a proper subproduct which lies in L .

Lemma 5. *The language $K(\varphi, U)$ is not indexed for any choice of φ and U .*

Proof. The proof follows that of [15, Corollary 4]. Assume that $K(\varphi, U)$ is indexed and fix $m = 1$. Let $k \in \mathbb{N}$ be given by Lemma 4.

Pick $k' \in U$ such that $k' > k$ and $k' = \min(\varphi^{-1}(\varphi(k')))$. Note that such k' exists as $\lim_{n \rightarrow \infty} \varphi(n) = +\infty$. Then $w = (ba^{\varphi(k')})^{k'}$ can be factorised as $w = w_1 \dots w_r$, where $1 < r \leq k < k'$. As $r < |w|_b = k'$, by the pigeonhole principle we see that there is some $1 \leq i \leq r$ such that $|w_i|_b \geq 2$, i.e. w_i contains $ba^{\varphi(k')}b$ as a subword. By Lemma 4, w_i can be included in a proper subproduct v which lies in $K(\varphi, U)$. However, v contains $ba^{\varphi(k')}b$, so $v = (ba^{\varphi(k'')})^{k''}$ for some $k'' \in \mathbb{N}$ such that $\varphi(k') = \varphi(k'')$. As v is a proper subproduct of w we see that $k'' < k'$. However, this is a contradiction with $k' = \min(\varphi^{-1}(\varphi(k')))$, thus $K(\varphi, U)$ is not indexed. \square

Note that when φ is the identity function and $U = \mathbb{N}$ this was already established in [17, Theorem 5.3] and later in [15, Corollary 4].

An explicit example of an indexed language which is not ETOL was given in [14] as the language of tree-cuts. Informally speaking, a tree-cut is a

sequence of binary strings encoding the list of leaves of a proper rooted binary tree, i.e. a tree in which every vertex has exactly zero or two children. Formally, tree cuts can be defined in a recursive manner:

- (i) the sequence containing only the empty string, i.e. (ϵ) , is a tree-cut;
- (ii) suppose that $u_1, \dots, u_k, v_1, \dots, v_l \in \{0, 1\}^*$ such that the sequences (u_1, \dots, u_k) and (v_1, \dots, v_l) are tree-cuts, then the sequence

$$(0u_1, \dots, 0u_k, 1v_1, \dots, 1v_l)$$

is a tree-cut.

A sequence (v_1, \dots, v_k) , where $v_1, \dots, v_k \in \{0, 1\}$ is a tree-cut if and only if it can be obtained by repeatedly applying to the above mentioned rules. For example, the sequence $(000, 001, 01, 10, 11)$ is a tree-cut, but the sequence $(000, 111)$ is not. For more detail on tree-cuts see for example [14, Section 3].

Let a, b denote symbols distinct from 0 and 1. The *language of cuts* is then defined as

$$L_0 = \{av_10bv_11 \dots av_k0bv_k1 \mid (v_1, \dots, v_k) \text{ is a cut}\} \subseteq \{0, 1, a, b\}^*$$

Carefully checking the proof of [14, Lemma 3.3] one can verify that the language L_0 is accepted by a nested stack automaton and hence L_0 is indexed. It is then proved in [14, Lemma 3.4] that L_0 is not ETOL.

Another example of an indexed language that is not ETOL is given in [13, Corollary 2]. In fact, the paper suggests an infinite family of such languages. However, the statement of [13, Theorem 3] contains a misprint, so we give the correct statement here. For a word $w \in \Sigma^*$ let \bar{w} denote its mirror image, i.e. if $w = x_1 \dots x_n$, where $x_1, \dots, x_n \in \Sigma$, then $\bar{w} = x_n \dots x_1$.

Theorem 6. (see [13, Theorem 3]) *Let Σ be a finite alphabet and let Σ' be a copy of Σ distinct from Σ . Let $h: \Sigma^* \rightarrow \Sigma'^*$ be a homomorphism defined by $h(x) = x'$ for every $x \in \Sigma$.*

Let K be a context-free language over Σ such that K is not EDTOL. Then the language

$$M_K = \{k\overline{h(k)} \mid k \in K\} \subseteq (\Sigma \cup \Sigma')^*$$

is indexed but not ETOL.

It is a well known fact (see [12, Theorem 9]) that if K is the Dyck language on at least 8 letters, then K is context-free but not EDTOL, hence M_K is indexed but not ETOL. In fact we observe below in Proposition 26 that the word problem for free groups of rank at least 2 is not EDTOL.

4. GROWTH OF LANGUAGES

The growth of a language $L \subseteq \Sigma^*$ is the function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that $f(n)$ is the number of words in L of length n . If Σ is finite then the growth function is at most exponential. A language has *intermediate growth* if for any $\alpha, \beta > 1$ there is an integer $N \in \mathbb{N}$ such that for all $n > N$ we have

$n^\alpha < f(n) < \beta^n$. Bridson and Gilman showed that there are no context-free languages of intermediate growth [3], whereas Grigorchuk and Machì [16, Theorem 1] give the following example of a language of intermediate growth which is recognisable by a one-way deterministic non-erasing stack automaton (1DNESE), so is indexed. They define the set

$$A := \{ab^{i_1}ab^{i_2} \dots ab^{i_k} \mid 0 \leq i_1 \leq \dots \leq i_k, k \in \mathbb{N}\}$$

over the alphabet $\{a, b\}$.

Proposition 7. *The language A is EDTOL.*

Proof. We show that A is generated by an EDTOL system as in Definition 3, with the following data: the extended alphabet is $\{a, b, q, q'\}$, the start symbol q , and the maps are h_a, h_b, h_\S with $h_a(q) = qa q'$, $h_b(q) = qb$, $h_b(q') = q'b$, $h_\S(q) = h_\S(q') = \epsilon$.

The rational control is given by $\mathcal{R} = \{h_a, h_b\}^* h_a h_\S$.

Let $g \in \{h_a, h_b\}^*$. We can easily prove by induction on the length of g as a word over $\{h_a, h_b\}$ that $g(q) = qb^{i_1}aq'b^{i_2}a \dots aq'b^{i_k}$ where $0 \leq i_1 \leq \dots \leq i_k$, i.e the word starts with q , and then only q' appears. Then applying $h_a h_\S$ to $g(q)$ produces a word which starts with a and contains no more q, q' . \square

Corollary 8. *There exist EDTOL (and ETOL) languages of intermediate growth.*

5. PRIMITIVES AND BASES IN FREE GROUPS

Given an alphabet Σ , we will always assume that a tuple (w_1, \dots, w_t) , where $w_1, \dots, w_n \in \Sigma^*$, is encoded as a string $w_1\# \dots \#w_t$, where the symbol $\#$ is distinct from the ones contained in Σ .

A *free basis* of a free group is a tuple of elements that freely generate the group, and a *primitive (element)* is an element that belongs to some free basis of the group. In this section we show that the set of bases and the set of primitives, written as reduced words over the standard free basis, is EDTOL for free groups of rank 2, and context-sensitive for higher rank. It is an open question whether the higher rank context-sensitive characterisation can be lowered to indexed or E(D)TOL.

5.1. Bases and primitives in the free group F_2 are EDTOL. In [27] Silva and Weil showed that the set of primitives in the free group F_2 on generators $\{a, b\}$, written as reduced words over $\{a, b, a^{-1}, b^{-1}\}$, is context-sensitive and not context-free. In this section we show that both the set of bases and the set of primitives are EDTOL.

Proposition 9. *The set of free bases and the set of primitives in F_2 , as reduced words, are EDTOL.*

Proof. It is a classical result due to Nielsen that two elements $g, h \in F_2$ form a basis of F_2 if and only if the commutator $[g, h]$ is conjugate either to

$[a, b]$ or $[b, a]$. Hence the set of bases in F_2 can be seen as the values X, Y satisfying the equations

$$Z[X, Y]Z^{-1} = [a, b]$$

or

$$Z[X, Y]Z^{-1} = [b, a]$$

where Z can take any value.

By Corollary 2.2 in [5], for any equation in a free group the set of solutions, or any projections thereof, written as tuples in reduced words, is EDT0L. In this case the values of (X, Y) will provide the set of bases, and the values of X the set of primitives. \square

5.2. Primitives in free groups of higher rank. While in F_2 the set of free bases can be expressed in terms of solutions to equations, this does not hold in higher rank. It was shown in [21] that the set of primitives in free groups of higher rank is not definable in the first order theory of the group, and thus we cannot use our previous approach to give an EDT0L characterisation in this case. Another approach is to produce the set of primitives by applying all the maps in $\text{Aut}(F_k)$ to a free basis element. Since $\text{Aut}(F_k)$ is finitely generated, we would apply all the maps in $\text{Aut}(F_k)$, written as words over the generators of $\text{Aut}(F_k)$ - which can be seen as rational control, to a basis element which would play the role of the axiom. The set thus obtained would be EDT0L and contain all the primitives, but neither as reduced nor unique words.

Since we are interested in establishing a formal language characterisation for primitives as reduced words, we use a different approach, based on Stallings' folding of labeled graphs, and an algorithm for testing primitivity in a free group given in [6], to show here that they are context-sensitive. For the sake of keeping this note succinct, we do not define here the *Stallings graph* or what is meant by folding, but refer the reader to references such as [28] and [20]. If Γ is the Stallings graph of a finitely generated subgroup of a free group F , a *pinch* will signify the identification of two distinct vertices of Γ .

A *primitive set* (as opposed to the set of primitives) in a free group F is a set of elements that can be extended to a basis of F .

Lemma 10. [6, Theorem 4.4] *Let $n \leq k$ and let $W = \{w_1, \dots, w_n\}$ be a set of reduced words in $F_k = F(X)$, where $X = \{x_1, \dots, x_k\}$, such that for every $x \in X$ either x or x^{-1} appears in some $w \in W$. Let Γ be a graph representing the subgroup of F_k generated by W . Then W is a primitive set if and only if there is a sequence of pinches and folds, containing exactly $k - n$ pinches, that transforms Γ into the elementary wedge on X .*

In the case that not every generator or inverse appears in the list of w_i then the Lemma 10 can still decide primitivity by taking a smaller value for k .

A standard way to represent a Stallings graph $\Gamma = (V\Gamma, E\Gamma)$ over an alphabet X would be as a collection of labeled oriented edges, where an edge $\gamma \in E\Gamma$ is given as a triple $(i(\gamma), t(\gamma), l(\gamma))$, with $i(\gamma) \in V\Gamma$ the initial vertex of γ , $t(\gamma) \in V\Gamma$ the terminal vertex and $l(\gamma) \in X \cup X^{-1}$ the label of γ . Obviously, a word $w \in F(X)$ can be represented by a graph Γ_w such that $|V\Gamma_w|, |E\Gamma_w| \leq n$, where $n = |w|$. However, as there might be up to n vertices, and one needs $\log(n)$ space to encode the names of the vertices, the naïve implementation of Γ would require $n \log(n)$ space.

We propose a different representation of the Stallings graph, along with two additional types of moves, and show that Lemma 10 can be realised by a linear space algorithm. Instead of labelling an edge by a single symbol, we will label edges by subwords, that is, an edge $\gamma \in E\Gamma$ will be represented by a triple $(i(\gamma), t(\gamma), s(\gamma))$, where $i(\gamma)$ and $t(\gamma)$ are as before and $s(\gamma) \in F(X)$ is a reduced word. In this case we say that Γ is a *segment graph*. Adapting the terminology of Stallings graphs, we say that a segment graph Γ is *folded* if it does not contain a pair of incident edges whose labels share a common prefix or suffix. If a vertex v is incident to two edges with a common prefix or suffix, then we say that v is *unfolded*, and otherwise call it *folded*.

Folding at a vertex v applies to segment graphs in three possible ways, depending on whether two segments incident to v have exactly the same label, or only common proper prefixes. More precisely, let $\gamma_1, \gamma_2 \in E\Gamma$ satisfy $i(\gamma_1) = v = i(\gamma_2)$, $s(\gamma_1) = uw_1$ and $s(\gamma_2) = uw_2$ for some $u, w_1, w_2 \in F(X)$.

- I. If $w_1 = w_2 = \epsilon$, then this is a usual folding of graphs.
- II. If $w_1 = \epsilon$ and $w_2 \neq \epsilon$, then we split γ_2 into γ'_2 and γ''_2 by introducing a new vertex v' on γ_2 such that $i(\gamma'_2) = v$, $t(\gamma'_2) = v'$ and $s(\gamma'_2) = u$, and fold γ_1 with γ'_2 as in case I.
- III. If $w_1, w_2 \neq \epsilon$, then we introduce a new vertex v' and a new edge γ , such that $i(\gamma) = v$, $t(\gamma) = v'$, $s(\gamma) = u$, and replace edges γ_1, γ_2 by new edges γ'_1, γ'_2 such that $i(\gamma'_1) = v' = i(\gamma'_2)$, $t(\gamma'_1) = t(\gamma_1)$, $t(\gamma'_2) = t(\gamma_2)$, $s(\gamma'_1) = w_1$ and $s(\gamma'_2) = w_2$.

Pinching a segment graph is defined as follows. Let $\gamma, \gamma' \in E\Gamma$ satisfy $s(\gamma) = w_1w_2$ and $s(\gamma') = w'_1w'_2$ for some $w_1, w_2, w'_1, w'_2 \in F(X)$. We remove the edges γ, γ' , introduce new vertex v and up to four new edges $\gamma_1, \gamma_2, \gamma'_1, \gamma'_2$ given by

$$\begin{aligned} \gamma_1: & i(\gamma_1) = i(\gamma), t(\gamma_1) = v, s(\gamma_1) = w_1; \\ \gamma_2: & i(\gamma_2) = v, t(\gamma_2) = t(\gamma), s(\gamma_2) = w_2; \\ \gamma'_1: & i(\gamma'_1) = i(\gamma'), t(\gamma'_1) = v, s(\gamma'_1) = w'_1; \\ \gamma'_2: & i(\gamma'_2) = v, t(\gamma'_2) = t(\gamma'), s(\gamma'_2) = w'_2. \end{aligned}$$

If one of w_1, w_2, w'_1, w'_2 is empty, i.e. when we are pinching a segment with a vertex, we don't need to introduce the vertex v .

Both foldings of type III and pinches can cause the numbers of edges and vertices to increase; the resulting graph could potentially not fit on a linear size tape, but the discussion below shows that this is not the case.

Recall that if Γ is a Stallings graph corresponding to some finite set W then the *core* of Γ (or the *core graph* for $\langle W \rangle$), is the smallest subgraph of Γ in which we can read all the elements of $\langle W \rangle$ as reduced words along loops based at the base vertex of Γ . In particular, if Γ is finite then the core of Γ does not contain any vertices of degree one, with the exception of the base vertex of Γ . The core of a segment graph is defined in an analogous manner.

Remark 1. (1) A set $W = \{w_1, \dots, w_n\} \subseteq F_k$ is primitive if and only if the set $W^g = \{gw_1g^{-1}, \dots, gw_n g^{-1}\}$ is primitive for some (and thus for all) $g \in F_k$. As choosing a different base vertex in the core graph of $\langle W \rangle$ is the same as taking a conjugate of $\langle W \rangle$, we will assume that the base vertex in any segment graph has degree at least 2. In fact, unless the graph is homeomorphic to a loop, we may assume that the base vertex is of degree at least three.

(2) Furthermore, as the core graph will never contain any leaves other than the base vertex and as was already mentioned, we can always assume that the base vertex is of degree at least two, hence without loss of generality we may assume that the segment graph does not contain any leaves; this can be achieved by *pruning*, i.e. repeatedly removing vertices of degree 1 (along with corresponding edges). Note that for finite graphs this procedure always terminates.

We say that a segment graph Γ is *topological* if all its vertices have degree ≥ 3 , unless it consists of a single vertex, or a loop attached at a degree two vertex. Following Remark 1, we see that every segment graph can be transformed into a topological one (representing a subgroup conjugate to the original) by picking a different base vertex, pruning and *merging* edges: suppose $\gamma_1, \gamma_2 \in E\Gamma$ such that $t(\gamma_1) = v = i(\gamma_2)$ and $\deg(v) = 2$; then we can replace the edges γ_1, γ_2 by a new edge γ such that $i(\gamma) = i(\gamma_1)$, $t(\gamma) = t(\gamma_2)$ and $s(\gamma) = s(\gamma_1)s(\gamma_2)$, and remove vertex v .

The following lemma implies that pruning will need to be done at most once in the entire process corresponding to Lemma 10.

Lemma 11. *Let Γ be a folded graph that does not contain any vertex of degree 1 and let Γ' be a graph obtained from Γ by performing a pinch and a folding. Then Γ' does not contain vertices of degree 1.*

Proof. Suppose that there is $v' \in V\Gamma'$ such that $\deg(v') = 1$. Without loss of generality we may assume that v' is the initial vertex of its unique adjacent edge $\gamma \in E\Gamma'$. Note that the pinch and the subsequent folding uniquely define a surjective morphism of graphs $f: V\Gamma_0 \rightarrow V\Gamma'$, where Γ can be obtained from Γ_0 by merging. Note that Γ_0 is folded and it does not contain vertices of degree 1. Let $v \in f^{-1}(v')$ be arbitrary. Again, without loss of generality we may assume that v is the initial vertex of all of its adjacent edges. It follows that all the edges adjacent to v must share a common prefix with γ , meaning that v is not folded unless $\deg(v) = 1$. As Γ_0 is folded, $\deg(v) = 1$ for every $v \in f^{-1}(v')$, which contradicts the fact that

$\deg(u) > 1$ for every $u \in V\Gamma_0$. Thus Γ' does not contain vertices of degree 1. \square

The *topological rank* of a graph is the rank of its fundamental group.

Lemma 12. *Let Γ be a finite topological graph of topological rank r with $r > 1$. Then $|E\Gamma| \leq 3r - 3$ and $|V\Gamma| \leq 2r - 2$.*

Proof. As $r > 1$ we see that $\deg(v) \geq 3$ for every $v \in V\Gamma$. From the Euler characteristic formula $r = |E\Gamma| - |V\Gamma| + 1$, and since in any graph $2|E\Gamma| = \sum_{v \in V\Gamma} \deg(v)$ by assumption $2|E\Gamma| \geq 3|V\Gamma|$, which gives the inequalities above. \square

Remark 2. Folding does not cause the topological rank of a graph to increase, but pinching can increase it by 1.

Corollary 13. *Let Γ be a finite topological segment graph of topological rank r with a single unfolded vertex, and let Γ' be the folded topological graph obtained from Γ by folding, merging and pruning.*

If $r = 1$ then $|V\Gamma'| = |E\Gamma'| = 1$, otherwise $|V\Gamma'| \leq 2r - 2$ and $|E\Gamma'| \leq 3r - 3$.

Proof. Let r' denote the topological rank of the graph Γ' ; note that $r' \leq r$ by Remark 2.

If $r' = 1$ then, following the definition of topological graphs, we see that Γ' is a single vertex with an attached loop and $|V\Gamma'| = |E\Gamma'| = 1$.

If $r' > 1$ then the result follows by Lemma 12. \square

Proposition 14. *Let F_k be a free group of rank $k \geq 2$. Then the primitive sets in F_k can be recognised in linear space.*

Proof. We represent a graph with t edges on the tape of a Turing machine by the string

$$\#v_{i_1}|v_{i_2}|w_1\#\#\dots\#\#v_{i_{2t-1}}|v_{i_{2t}}|w_t\#,$$

where v_{i_j} are binary numbers and the factor $\#v|v'|w\#$ represents the segment γ with $i(\gamma) = v$, $t(\gamma) = v'$ and $s(\gamma) = w$.

On input a list of words w_1, w_2, \dots, w_n , if $n \geq k$ we return NO, else we write on the tape

$$\#v_0|v_0|w_1\#\#v_0|v_0|w_2\#\#\dots\#\#v_0|v_0|w_n\#$$

which describes the bouquet of loops labeled w_i at a vertex v_0 . Let k' be the rank of the free group generated by the letters appearing in the w_i , so $k' \leq k$.

We perform folds of types I, II, III by modifying the tape as follows:

- I. Scan the tape to find factors $\#v_i|v_j|u\#$ and $\#v_i|v_k|u\#$; erase the second factor and replace v_k by v_j everywhere on the tape.
- II. Scan the tape to find $\#v_i|v_j|u\#$ and $\#v_i|v_k|up\#$; overwrite the second factor with $\#v_j|v_k|p\#$.

- III. Scan the tape to find $\#v_i|v_j|uap\#$ and $\#v_i|v_k|ubq\#$ with $a \neq b, a, b \in X^{\pm 1}$; erase both factors and write $\#v_i|v|u\#\#v|v_j|ap\#\#v|v_k|bq\#$ where the binary number v is some value not already in use.

For pruning we scan the tape to find a factor $\#v|v'|u\#$ such that the v (or v') does not appear in any other factor. If such a factor is found, we delete it.

Similarly, for merging we scan the tape for a pair of factors $\#v|v'|u_1\#$ and $\#v'|v''|u_2\#$ (or $\#v''|v'|u_2\#$) such that v'' does not occur in any other factor. If such a pair is found, we replace them by $\#v|v''|u_1u_2\#$ (or $\#v|v''|u_1u_2^{-1}\#$, respectively).

We perform a pinch by choosing either:

- (1) two distinct vertices v_i, v_j and replacing v_j by v_i everywhere on the tape;
- (2) a vertex v and a segment (v_i, v_j, w_1w_2) with $|w_i| > 0$ and replacing $\#v_i|v_j|w_1w_2\#$ by $\#v_i|v|w_1\#\#v|v_j|w_2\#$;
- (3) two segments $(v_i, v_j, w_1w_2), (v_p, v_q, w_3w_4)$ with $|w_i| > 0$, and replacing their encodings by $\#v_i|v|w_1\#\#v|v_j|w_2\#\#v_p|v|w_3\#\#v|v_q|w_4\#$ where v is some value not already in use.

Note that folding and pruning moves decrease the number of letters appearing as labels of segments, and pinching and merging moves preserve this number.

The procedure starts by performing folding moves I, II, III in any order exhaustively, and each folding is followed by all possible pruning and merging. Then a pinch is applied, and the previous two steps repeat $k' - n$ times.

Following Lemma 11 we see that we will only need to perform pruning moves before the first pinch. Note that during this process, the number of letters present in the labels might decrease. In this case we need to decrease k' accordingly.

Return YES if the tape contains

$$\#v|v|a_{i_1}\#\#\dots\#\#v|v|a_{i_{k'-n}}\#$$

with all i_j distinct and $a_{i_j} \in X^{\pm 1}$, else return NO.

Termination of the algorithm is guaranteed since each fold strictly decreases the number of letters from $X^{\pm 1}$ appearing on the tape as labels of segments. The algorithm accepts precisely the primitive sets in F_k by Remark 1 and Lemma 10.

Since the rank of all topological segment graphs considered is smaller than the rank of the ambient free group, by Corollary 13 the number of vertices in use at any time is at most $2k - 2$, so in binary notation each vertex requires at most $\log(2k - 2)$ space, the total number of segments is at most $3k - 3$, and the number of letters from $X^{\pm 1}$ appearing on the tape is at most $\sum_{i=1}^n |w_i| = N$. Thus the amount of space required at any time

in the process is at most

$$(3k - 3)(2 \log(2k - 2) + 4) + \sum_{i=1}^n |w_i|,$$

which is linear in the input size. \square

From the proposition it follows that for every k there is a linearly bounded Turing machine T_k which recognises primitive sets in F_k , hence we can state the following corollary.

Corollary 15. *Let $F_k = F(X)$ be a free group over X , where $|X| = k$ and $n \leq k$. Then*

$$\begin{aligned} \mathcal{P}_{k,n} &= \{w_1 \# \dots \# w_n \mid \{w_1, \dots, w_n\} \text{ is a primitive set in } F_k\} \\ &\subseteq (X \cup X^{-1} \cup \{\#\})^* \end{aligned}$$

is a context-sensitive language.

In particular, the set $P_k (= \mathcal{P}_{k,1})$ of primitive elements is context-sensitive.

6. CO-WORD PROBLEM FOR GRIGORCHUK GROUP IS ETOL

In this section we show that the co-word problem for the Grigorchuk group is ETOL, improving on Holt and Röver's result in [18], where they showed it is indexed. It is still an open question whether the co-word problem for the Grigorchuk's group is context-free.

6.1. Generators and the word problem. We refer the reader to [8] for more details, here we give the essentials for our proof. Let \mathbf{T} denote the infinite rooted binary tree and let $\mathbf{A} = \text{Aut}(\mathbf{T})$. Note that $\mathbf{A} \simeq \mathbf{A} \wr C_2 \simeq (\mathbf{A} \times \mathbf{A}) \rtimes C_2$, where $C_2 = \langle \alpha \mid \alpha^2 = 1 \rangle$, so every $g \in \mathbf{A}$ can be uniquely expressed as $g = (g_L, g_R)\alpha_g$ for some $g_L, g_R \in \mathbf{A}$ and $\alpha_g \in C_2$. The Grigorchuk group is $G = \langle a, b, c, d \rangle \leq \mathbf{A}$, where the generators $a, b, c, d \in \mathbf{A}$ are given by

$$a = (1, 1)\alpha, \quad b = (a, c)1, \quad c = (a, d)1, \quad d = (1, b)1.$$

One can easily verify that the following identities hold in G :

$$(1) \quad \begin{aligned} a^2 = b^2 = c^2 = d^2 = 1, \\ bc = cb = d, \quad cd = dc = b, \quad db = bd = c. \end{aligned}$$

Starting with an arbitrary word $w \in \{a, b, c, d\}^*$, one can rewrite w via the identities in (1) to a word w' which represents the same element in G and does not contain any of

$$aa, bb, cc, dd, bc, cb, bd, db, cd, dc$$

as a subword. Thus w' has the form $w' = x_0 a x_1 \dots x_{n-1} a x_n$, where $x_i \in \{1, b, c, d\}$ for $0 \leq i \leq n$ and $x_i \neq 1$ for $1 < i < n - 1$, so we say that w' is *alternating* or *reduced*. Every non-trivial element $g \in G$ can be represented by a reduced word. The following remark has an analogous proof to that of the uniqueness of reduced words in free groups (see [23, Section I.1]).

Remark 3. Every word $w \in \{a, b, c, d\}^*$ can be rewritten via the identities in (1) to a unique reduced word.

However, not every reduced word represents a nontrivial element: the word $dadadada$ is reduced, yet it represents the identity in G .

Let G_1 be the subgroup of G consisting of all elements that fix the first level of \mathbf{T} . Then $|G : G_1| = 2$, $G_1 = \langle b, c, d, aba, aca, ada \rangle$ and

$$aba = (c, a)1, \quad aca = (d, a)1, \quad ada = (d, 1)1.$$

Every element $g \in G_1$ can be expressed as an alternating sequence $g = x_0 x_1^a \dots x_{n-1} x_n^a$, where $x_i \in \{1, b, c, d\}$ and $x_i^a \in \{1, aba, aca, ada\}$ for $0 \leq i \leq n$ such that $x_i \neq 1$ if $i > 0$ and $x_i^a \neq 1$ if $i < n$.

Let $\phi_L, \phi_R : G_1 \rightarrow G$ be the group homomorphisms defined as

$$\phi_L : \begin{cases} b & \rightarrow a, \\ c & \rightarrow a, \\ d & \rightarrow 1, \\ aba & \rightarrow c, \\ aca & \rightarrow d, \\ ada & \rightarrow b, \end{cases} \quad \phi_R : \begin{cases} b & \rightarrow c, \\ c & \rightarrow d, \\ d & \rightarrow b, \\ aba & \rightarrow a, \\ aca & \rightarrow a, \\ ada & \rightarrow 1, \end{cases}$$

and define $\phi : G_1 \rightarrow G \times G$ as $\phi(g) = (\phi_L(g), \phi_R(g))$. Then ϕ is injective, and $|\phi_i(w)| < \frac{1}{2}|w| + 1 < |w|$ for every reduced $w \in \{a, b, c, d\}^*$ representing some element in G_1 such that $|w| > 1$. Let L_S be the set of words with an odd number of a 's, that is,

$$L_S = \{w \in \{a, b, c, d\}^* \mid |w|_a \equiv 1 \pmod{2}\}.$$

Notice that the parity of $|w|_a$ is invariant under the rewrite rules (1). Indeed, if $w \in L_S$ then $w \neq_G 1$.

The following is the outline of the word problem algorithm in Grigorchuk's group (see [8, Section VIII.E] for more details and Figure 2 for an example). We use ϵ to denote the empty string.

- (1) reduce w
- (2) if $w = \epsilon$ answer YES (meaning $w =_G 1$),
- (3) if $w \in L_S$ answer NO (meaning $w \neq_G 1$),
- (4) otherwise answer $(\phi_L(w) =_G 1 \text{ and } \phi_R(w) =_G 1)$.

Obviously, $w \neq_G 1$ if and only if there are sequences $w_0, \dots, w_n \in \{a, b, c, d\}$ and $\phi_1, \dots, \phi_n \in \{\phi_L, \phi_R\}$ such that $w_n =_G w$, $w_{i-1} =_G \phi_i(w_i)$ for $i = 1, \dots, n$ and $w_0 \in L_S$. The main idea behind our grammar is to invert this process, i.e. start with a word in $w_0 \in L_S$ and generate a sequence of words w_1, \dots, w_n such that $w_{i-1} =_G \phi_i(w_i)$.

6.2. ETOL grammar. In this subsection we introduce the ETOL grammar used to generate the co-word problem for Grigorchuk's group and give an informal explanation of the roles of the corresponding tables.

Our grammar works over the extended alphabet

$$\Sigma = \{S_0, S_1, a, b, c, d, \delta, \#\},$$

where S_0 is the start symbol, along with tables s, p, h_L, h_R, u, t and rational control

$$\mathcal{R} = s^* \{p^* \{h_L, h_R\} u^* t\}^*.$$

The process of generating words can be split in three phases. It is important to note that the second and third phase can occur multiple times, as can be seen in Example 2.

Phase 1: Generate L_S . First we generate the language L_S of words with an odd number of occurrences of a . This is done by table s , which can be seen as the grammar version of a two state finite automaton producing the regular language L_S . We call the words in L_S *seeds*.

$$(s) \quad \begin{array}{c} \text{Generate seeds} \\ \hline S_0 \rightarrow aS_1, bS_0, cS_0, dS_0 \\ S_1 \rightarrow aS_0, bS_1, cS_1, dS_1, \epsilon \end{array}$$

Lemma 16. *Let $w \in \{a, b, c, d\}^*$. Then $w \in L_S$ if and only if $S_0 \xrightarrow{s^*} w$. In particular, if $S_0 \xrightarrow{s^*} w$ then $w \neq_G 1$.*

Phase 2: Invert ϕ_L and ϕ_R . Once we have produced a seed w_0 , we generate a sequence of words w_1, \dots, w_n such that $w_{i-1} =_G \phi_i(w_i)$, and we create w_i from w_{i-1} by ‘inverting’ the maps ϕ_L and ϕ_R ; this is achieved via tables h_L, h_R :

$$(h_L, h_R) \quad \begin{array}{cc} \text{Invert } \phi_L & \text{Invert } \phi_R \\ \hline a \rightarrow b, c & a \rightarrow aba, aca \\ b \rightarrow ada & b \rightarrow d \\ c \rightarrow aba & c \rightarrow b \\ d \rightarrow aca & d \rightarrow c \\ \delta \rightarrow d & \delta \rightarrow ada \end{array}$$

Table p introduces a new symbol δ , which serves as a placeholder for the empty word that resulted from applying ϕ_L and ϕ_R (recall that $\phi_L(d) = 1$, $\phi_R(ada) = 1$).

$$(p) \quad \begin{array}{c} \text{Insert } \delta \\ \hline a \rightarrow a, \delta a, a\delta \\ b \rightarrow b, \delta b, b\delta \\ c \rightarrow c, \delta c, c\delta \\ d \rightarrow d, \delta d, d\delta \end{array}$$

Lemma 17. *Let $w, w' \in \{a, b, c, d\}^*$ and let $i \in \{L, R\}$ be given. Then $w =_G \phi_i(w')$ if and only if $w \xrightarrow{p^* h_i} w'$. In particular, if $w \xrightarrow{p^* h_i} w'$ and $w \neq_G 1$ then $w' \neq_G 1$.*

The proof follows immediately from an analysis of the tables.

Phase 3: Use group relations and insert trivial subwords. The following table inverts the reduction process induced by the identities in (1). We introduce the symbol # as a placeholder for the reduction process. This means that # signifies the position of a subword that reduces to the trivial word.

	Unreduce
(u)	$a \rightarrow a, \#a, a\#$
	$b \rightarrow b, \#b, b\#, c\#d, d\#c$
	$c \rightarrow c, \#c, c\#, b\#d, d\#b$
	$d \rightarrow d, \#d, d\#, b\#c, c\#b$
	$\# \rightarrow \#, a\#a, b\#b, c\#c, d\#d$

At this stage we remove all the occurrences of the placeholder #. This is achieved by table *t*:

	Tidy #
(t)	$\# \rightarrow \epsilon$

Lemma 18. *Let $w, w' \in \{a, b, c, d\}^*$ be arbitrary. Then w can be obtained from w' by reducing rules (induced by the identities in (1)) if and only if $w \rightarrow^{u^*t} w'$. In particular, if $w \rightarrow^{u^*t} w'$ and $w \neq_G 1$ then $w' \neq_G 1$.*

Example 2. Figure 2 demonstrates the word problem algorithm on input *bcddacbabcaa*, showing it to be nontrivial.

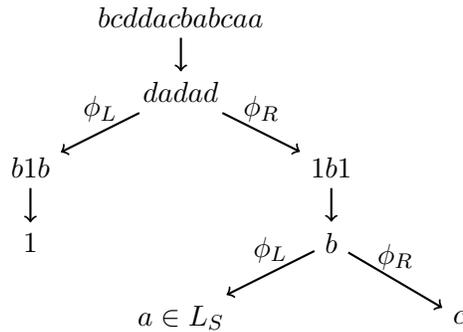


FIGURE 2. Word problem algorithm applied to *bcddacbabcaa*.

This word can be obtained from our grammar as follows.

Phase 1: $S_0 \xrightarrow{s} aS_1 \xrightarrow{s} a$

Phase 2: $a \xrightarrow{h_L} b$

Phase 3: $b \xrightarrow{t} b$

Phase 2: $b \xrightarrow{p} \delta b \xrightarrow{p} \delta b \delta \xrightarrow{h_R} dadad$

Phase 3: $dadad \xrightarrow{u} b\#c\#ac\#bab\#c \xrightarrow{u} b\#cd\#dac\#bab\#c\#$
 $\xrightarrow{u} b\#cd\#dac\#bab\#ca\#a \xrightarrow{t} bcddacbabcaa.$

6.3. Proof that the ETOL grammar generates exactly the co-word problem. Let L' be the language generated by the ETOL grammar with alphabet Σ , tables s, p, h_L, h_R, u, c given above and rational control

$$\mathcal{R} = s^* \{p^* \{h_L, h_R\} u^* t\}^*.$$

Note that the rational control \mathcal{R} is equivalent to $s^* \{p^* h_L u^* t, p^* h_R u^* t\}^*$. Following Definition 3, L' is ETOL. Now consider $L = L' \cap \{a, b, c, d\}^*$. As the class of ETOL languages is closed under intersection with regular languages, L is an ETOL language. Intersecting L' with $\{a, b, c, d\}^*$ effectively discards all words containing symbols S_0, S_1 . With this in mind, we will assume that we are only working with words that do not contain symbols S_0 and S_1 .

Combining Lemmas 16, 17 and 18 we can immediately show that our grammar produces words that represent non-trivial elements in the group.

Lemma 19. *Let $w \in \{a, b, c, d\}^*$ be arbitrary. If $w \in L$ then $w \neq_G 1$.*

Proof. If $w \in L$ then, by definition, there is a sequence of words

$$w_0, w_1, w'_1, \dots, w_n, w'_n \in \{a, b, c, d\}^*$$

and a sequence of tables $h_1, \dots, h_n \in \{h_L, h_R\}$ such that

- (1) $s \xrightarrow{s^*} w_0$,
- (2) $w_{i-1} \xrightarrow{p^* h_i} w'_i$ for $i = 1, \dots, n$
- (3) $w'_i \xrightarrow{u^* c} w_i$,
- (4) $w_n = w$.

We use induction on n . If $n = 0$ then by Lemma 16 we see that $w_0 \in L_S$, hence $w_0 \neq_G 1$.

Now suppose that the result has been established for all words that can be obtained via sequences of length $\leq n - 1$. Then $w_{n-1} \neq_G 1$ by the induction hypothesis. Using Lemma 17 we see that $w_{n-1} = \phi_i(w'_n)$, where $\phi_i = \phi_L$ if $h_i = h_L$ and $\phi_i = \phi_R$ if $h_i = h_R$, thus $w'_n \neq_G 1$. Using Lemma 18 we see that $w'_n =_G w_n$ and hence $w_n \neq_G 1$, so w indeed represents a nontrivial element of Grigorchuk's group. \square

The next lemma establishes the completeness of our grammar.

Lemma 20. *Let $\{a, b, c, d\}^*$ be arbitrary. If $w \neq_G 1$ then $w \in L$.*

Proof. Suppose that $w \neq_G 1$. Following the algorithm described in Subsection 6.1 there is a sequence of words $w_0, w'_0, \dots, w_n, w'_n$ and a sequence of maps $\phi_1, \dots, \phi_n \in \{\phi_L, \phi_R\}$ such that

- (1) w_i is the reduced word obtained by reducing w'_i for $i = 0, \dots, n$,
- (2) $|w_i|_a$ is even for $i \geq 1$ and odd for $i = 0$,
- (3) w'_{i-1} is obtained from w_i by applying ϕ_i ,
- (4) $w'_n = w$.

Again, we proceed by induction on n .

Suppose that $n = 0$, i.e. $w'_0 = w$. As the parity of the number of occurrences of the symbol a is invariant with respect to the reduction rules, we see that $|w|_a$ is odd. It follows by Lemma 16 that $S_0 \xrightarrow{s^*} w$ and thus $w \in L$.

Now suppose that the statement holds for all $w' \in \{a, b, c, d\}^*$ for which the word problem algorithm uses up to $n-1$ levels of recursion. In particular,

$$S_0 \xrightarrow{s^* \{p^* h_L u^* t, p^* h_R u^* t\}^*} w'_{n-1}.$$

Using Lemma 17 we see that $w'_{n-1} \xrightarrow{p^* h_n} w_n$, where $h_n = h_L$ if $\phi_n = \phi_L$ and $h_n = h_R$ if $\phi_n = \phi_R$. Similarly, using Lemma 18 we see that $w_n \xrightarrow{u^* t} w'_n$. Altogether we see that $w'_{n-1} \xrightarrow{p^* h_n u^* t} w$ and therefore $w \in L$. \square

Combining Lemma 19 and Lemma 20 we see that a word $w \in \{a, b, c, d\}^*$ represents a nontrivial element of Grigorchuk's group if and only if $w \in L$, which implies the main result of this section.

Theorem 21. *The co-word problem in Grigorchuk's group is an ETOL language.*

7. ETOL LANGUAGES AND 3-MANIFOLD GROUPS

The goal of this section is to prove Theorem 22, which is a strengthening of Theorem B in [2], proved there for indexed instead of ETOL languages. In 1996 Bridson and Gilman stated the theorem for all manifolds satisfying the geometrisation conjecture, but since then Perelman [24] proved that all compact 3-manifolds do, so we can state the result in full generality.

Theorem 22. *Let M be a compact 3-manifold or orbifold, and let $\mu : \Sigma^* \rightarrow \pi_1 M$ be a choice of generators. Then there exists a set of normal forms $L \subseteq \Sigma^*$ which satisfies the asynchronous fellow-traveler property and is an ETOL language.*

We follow in the footsteps of Bridson and Gilman, whose proof relies on (1) closure properties of AFL languages, and (2) showing that an appropriate set of normal forms for \mathbb{Z}^2 is ETOL. Note that standard regular normal forms for \mathbb{Z}^2 will not produce an appropriate language of normal forms for the extension $\mathbb{Z}^2 \rtimes \mathbb{Z}$, which needs to satisfy the properties detailed in [2, page 541].

For the sake of completeness we recall the relevant results on AFL languages. As in the paper of Bridson and Gilman (and much of the literature), we will call a set of normal forms satisfying the asynchronous fellow-traveler property a *combing*.

Proposition 23 ([2]). *Let \mathcal{A} be a full AFL class of languages (such as regular, context-free, indexed, or EDTOL).*

- (1) (Prop. 2.9) *If G_1 and G_2 both have an asynchronous \mathcal{A} -combing, then so does the free product $G_1 * G_2$.*
- (2) (Theorem 2.16) *Let G be a finitely generated group, and H a subgroup of finite index. Then G admits an asynchronous \mathcal{A} -combing if and only if H admits an asynchronous \mathcal{A} -combing.*

We first recall the crossing sequence $\kappa(m, n)$ of Bridson and Gilman, which gives the EDTOL normal form for \mathbb{Z}^2 . Let $(m, n) \in \mathbb{Z}^2$ have $m > 0, n \geq 0$, and consider the line $l(m, n^+)$ in the plane from $(0, 0)$ to (m, n^+) , where n^+ is chosen slightly larger than n , but small enough to ensure that (1) $l(m, n^+)$ does not contain any lattice points except $(0, 0)$, and (2) there are no lattice points in the interior of the triangle with vertices $(0, 0)$, (m, n) and (m, n^+) . For the line $l(m, n^+)$, the sequence formed by recording an h each time a horizontal line in the plane is crossed and a v each time a vertical line is crossed is called the crossing sequence $\kappa(m, n)$. For example, $\kappa(2, 3) = hvhhv$.

Theorem 24. *The set $L = \{\kappa(m, n) \mid m > 0, n \geq 0\}$ is an EDTOL language. That is, the indexed combing for \mathbb{Z}^2 in [2] is in fact EDTOL.*

Proof. The proof only focuses on the first quadrant in \mathbb{Z}^2 , and it can be easily extended to all of \mathbb{Z}^2 . As is described in [2], L can be generated by starting with an arbitrary v^k , $k > 0$, and alternately replacing all v 's by $h^i v$ and all h 's by $v^j h$. The EDTOL grammar thus first has to produce v^k , and then apply maps which mimic the morphisms described. Let $\{q, v, h\}$ be the extended alphabet, with q the start symbol. Let ϕ_q, ϕ_v, ϕ_h and ϕ_s be the maps defined by $\phi_q(q) = qv$, $\phi_v(v) = hv$, $\phi_h(h) = vh$ and $\phi_s(q) = v$.

Then $\phi_s \phi_q^{k-1}(q) = v^k$ generates the starting point of the crossing sequence, and then we apply any map in $\{\phi_v, \phi_h\}^*$ to v^k and obtain the set L . Thus by Definition 3 the set L is an EDTOL language. \square

We remark that the EDTOL characterisation for the \mathbb{Z}^2 combing cannot be lifted to $\mathbb{Z}^2 \rtimes \mathbb{Z}$ and other more general groups, because EDTOL languages do not form a full AFL, which is essential in several proofs in [2].

Proposition 25. *Every semidirect product of the form $\mathbb{Z}^2 \rtimes \mathbb{Z}$ admits an asynchronous EDTOL combing.*

Proof. The proof is exactly as that of Corollary 3.5 in [2]. More precisely, let t be a generator of \mathbb{Z} and L be the language of normal forms for \mathbb{Z}^2 from Theorem 24. By [2, Theorem 3.1], the language $L_0 = \{t^* \cup (t^{-1})^*\}L$ is an

asynchronous combing of $\mathbb{Z}^2 \rtimes \mathbb{Z}$, and since ETOL languages are closed under concatenation with a regular language and finite unions, we get that L_0 is ETOL. \square

Proof. (of Theorem 22) The work of Thurston, Epstein and Perelman implies that any $\pi_1 M$ as in the hypothesis is commensurable to the free product of an automatic group and (possibly) finite extensions of groups of the form $\mathbb{Z}^2 \rtimes \mathbb{Z}$. Thus the proof follows immediately from Propositions 23 and 25. \square

8. OPEN PROBLEMS

Among the formal languages naturally appearing in group theory none is more prominent than the word problem, that is, the set of words representing the trivial element in a finitely generated group. Since EDTOL languages are not closed under inverse homomorphism, *a priori* a group may have EDTOL word problem for one finite generating set but not for another. However, we do not know of any infinite group which has EDTOL word or co-word problem for some finite generating set. Since EDTOL languages are relatively close in complexity to context-free languages, one might wonder whether the groups with context-free word problem have EDTOL word problem. A first negative answer is given below.

Proposition 26. *Let F be the free group of rank at least two. Then the word problem in F is not EDTOL.*

Proof. It was proved in [22] that if a language L is a context-free generator, i.e. for every context-free language K there is a regular language R_K and a homomorphism h_K such that $K = h_K(L \cap R_K)$, then L is not EDTOL. It follows by the Chomsky-Schützenberger representation theorem [4] that every Dyck language on at least two letters is a context-free generator. It can be easily seen that the word problem in F_n , the free group on n generators, is isomorphic to D_n^* , the symmetric Dyck Language on n letters. It follows that if $n > 1$ then the word problem in F_n is not EDTOL. \square

Question 8.1. Is the word problem for \mathbb{Z} (for some or every finite generating set) EDTOL?

A related open problem is to determine the class of groups having ETOL word problem; a well known problem is to find a non-virtually free group with indexed word problem, so a reasonable conjecture here is that the only groups with ETOL word problem are virtually free.

Conjecture 8.2. *A group has EDTOL word problem if and only if it is finite. A group has ETOL word problem if and only if it is virtually free.*

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