# Congruence subgroups of braid groups 

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#### Abstract

In this paper we give a description of the generators of the prime level congruence subgroups of braid groups. Also, we give a new presentation of the symplectic group over a finite field, and we calculate symmetric quotients of the prime level congruence subgroups of braid groups. Finally, we find a finite generating set for the level-3 congruence subgroup of the braid group on 3 strands.


## 1 Introduction

Let $B_{n}$ be the braid group on $n$ strands. By evaluating the (unreduced) Burau representation $B_{n} \rightarrow \mathrm{GL}_{n-1}\left(\mathbb{Z}\left[t^{ \pm 1}\right]\right)$ at $t=-1$ we obtain a symplectic representation

$$
\rho: B_{n} \rightarrow \begin{cases}\mathrm{Sp}_{n-1}(\mathbb{Z}) & \text { if } n \text { is odd } \\ \left(\operatorname{Sp}_{n}(\mathbb{Z})\right)_{u} & \text { if } n \text { is even }\end{cases}
$$

where $\left(\operatorname{Sp}_{n}(\mathbb{Z})\right)_{u}$ is the subgroup of $\operatorname{Sp}_{n}(\mathbb{Z})$ fixing one vector $u \in \mathbb{Z}^{n}[17$, Proposition 2.1] (see also [9] and [1]).

For a positive integer $m$, the projection $\mathbb{Z} \rightarrow \mathbb{Z} / m$ induces a representation as follows:

$$
\rho_{m}: B_{n} \rightarrow \begin{cases}\operatorname{Sp}_{n-1}(\mathbb{Z} / m) & \text { if } n \text { is odd } \\ \left(\operatorname{Sp}_{n}(\mathbb{Z} / m)\right)_{u} & \text { if } n \text { is even }\end{cases}
$$

Note that if $m=1$, then $\rho_{1}=\rho$. For $i>1$ the kernel of $\rho_{m}$ is denoted by $B_{n}[m]$ and it is called the level-m congruence subgroup of $B_{n}$. The kernel of $\rho$ is called the braid Torelli group, and it is denoted by $\mathcal{B I}_{n}$. The group $\mathcal{B I}_{n}$ has been extensively studied by Hain [18], Brendle-Margalit [10, 12, and Brendle-Margalit-Putman 11 .

For $p$ prime, A'Campo proved that the homomorphism $\rho_{p}$ is surjective, by explicitly calculating the image of $\rho_{p}$ [1, Theorem $\left.1(1)\right]$. Wanjryb gave a presentation of $\operatorname{Sp}_{n-1}(\mathbb{Z} / p)$ and $\left(\operatorname{Sp}_{n}(\mathbb{Z} / p)\right)_{u}$ as quotients of $B_{n}$ [27, Theorem 1]. Let $P B_{n}$ be the pure braid group, that is, the kernel of the epimorphism $B_{n} \rightarrow S_{n}$, where $S_{n}$ is the symmetric group on $n$ letters. Our first result is an analogue of Wanjryb's theorem.

Theorem A For $p$ prime, the groups $\mathrm{Sp}_{n-1}(\mathbb{Z} / p)$ and $\left(\mathrm{Sp}_{n}(\mathbb{Z} / p)\right)_{u}$ admit a presentation as quotients of the pure braid group $P B_{n}$.

This result is given as Theorem 5.3 in the paper.
A result of Arnol'd shows that $B_{n}[2]=P B_{n}$, where $P B_{n}$ is the pure braid group [2]. Therefore, for every $k$ even, we have that $B_{n}[k] \unlhd P B_{n}$. Our second result extends A'Campo's theorem.

Theorem B For $m=2 p_{1} \ldots p_{k}$, where $p_{i} \geq 3$ are primes, we have that $P B_{n} / B_{n}[m]$ is isomorphic to $\bigoplus_{i=1}^{k} \operatorname{Sp}_{n-1}\left(\mathbb{Z} / p_{i}\right)$ if $n$ is odd, and $\bigoplus_{i=1}^{k}\left(\operatorname{Sp}_{n}\left(\mathbb{Z} / p_{i}\right)\right)_{u}$ if $n$ is even.

Theorem B is Theorem 5.1 (see also Theorem 5.2) in the paper.

We also characterize quotient groups of congruence subgroups of braid groups. The braid group $B_{n}$ surjects onto the symmetric group $S_{n}$. The kernel of this map is well known to be the pure braid group $P B_{n}$. Also, by a result established by A'rnold [2] the group $P B_{n}$ is isomorphic to $B_{n}[2]$. See also [9, Section 2] for further discussion. Therefore, we have $B_{n} / B_{n}[2] \cong S_{n}$. We generalize this result as stated in the following theorem.

Theorem C. For $p$ prime number, the group $B_{n}[p] / B_{n}[2 p]$ is isomorphic to $S_{n}$.
Theorem C is Theorem 6.1 in the paper.

Topological description of congruence subgroups. A key part of the paper is a topological interpretation of $B_{n}[p]$, for $p \geq 3$ prime, given in Section 4. The content of Section 4 was inspired by Powell, who based on Birman's work on the presentation of the symplectic group 8, Theorem 1], to show that the Torelli subgroup of the mapping class group is normally generated by bounding pair maps, and Dehn twists about separating simple closed curves [25, Theorem 2].

Theorems A and B are used to find normal generators for $B_{n}[m]$, where $m=2 p_{1} \ldots p_{k}$ and $p_{i}$ is an odd prime. Motivated by Section 4 it would be interesting to find a topological description of the generators of $B_{n}[m]$ in the future.

Related results. The mapping class group $\operatorname{Mod}(\Sigma)$ of an orientable surface $\Sigma$ is the group of isotopy classes of homeomorphisms that preserve the orientation of $\Sigma$, fix the boundary pointwise, and preserve the set of marked points setwise. We denote by $T_{c}$ a Dehn twist about a simple closed curve $c$. Let $\Sigma_{g}^{b}$ be a surface of genus $g \geq 1$ with $b$ boundary components, where $b \in\{1,2\}$. It is a special case of theorem of Birman-Hilden [7] that $B_{2 g+b}$ embeds into $\operatorname{Mod}\left(\Sigma_{g}^{b}\right)$ [15, Section 9.4]. We denote the image of this embedding by $\operatorname{SMod}\left(\Sigma_{g}^{b}\right)$. As mentioned in the previous page, the braid Torelli $\mathcal{B} \mathcal{I}_{2 g+b}$ is the kernel of the symplectic representation of $B_{2 g+b}$. Hain conjectured that $\mathcal{B} \mathcal{I}_{2 g+b}$ is isomorphic to the group generated by Dehn twists about separating simple closed curves inside $\operatorname{SMod}\left(\Sigma_{g}^{b}\right)$ [18]. This conjecture was proved by Brendle-Margalit-Putman [11, Theorem A], and also studied by Brendle-Margalit [10, 12]. By the definitions given in the beginning of the paper, the group $\mathcal{B} \mathcal{I}_{2 g+b}$ is a subgroup of $B_{2 g+b}[m]$, for any $m \in \mathbb{N}$.

For $m \geq 2$, consider $B_{2 g+b}[m]$ as a subgroup of $\operatorname{SMod}\left(\Sigma_{g}^{b}\right) \cong B_{2 g+b}$. A consequence of a work of Arnol'd shows that $B_{2 g+b}[2]$ is isomorphic to the pure braid group $P B_{2 g+b}$ [2] (see [9, Section 2] for explanation of this isomorphism). Combining the latter result with the work of Humphries [19, Theorem 1] we obtain that $B_{2 g+b}[2]$ is isomorphic to the normal closure of a square of a Dehn twist about nonseparating simple closed curve in $\operatorname{SMod}\left(\Sigma_{g}^{b}\right)$. Brendle-Margalit extended the latter result by proving that the normal closure of the $4^{t h}$ power of a Dehn twist about a nonseparating simple closed curve in $\operatorname{SMod}\left(\Sigma_{g}^{b}\right)$ is isomorphic to $B_{2 g+b}$ [4] [9, Main Theorem].

Let $\mathcal{T}_{2 g+b}(m)$ be the normal closure of the $m^{t h}$ power of a Dehn twist in $\operatorname{SMod}\left(\Sigma_{g}^{b}\right)$, where $g \geq 1$ and $b=1,2$. Coxeter proved that $\mathcal{T}_{2 g+b}(m)$ is a finite index subgroup of $\operatorname{SMod}\left(\Sigma_{g}^{b}\right)=B_{2 g+b}$ if and only if $(2 g+b-2)(m-2)<4$ [14, Section 10]. As mentioned above, $\mathcal{T}_{2 g+b}(2)=B_{2 g+b}[2]$. Furthermore, Humphries gave a complete description of when a group generated by $\left\{\mathcal{T}_{2 g+b}\left(m_{i}\right) \mid\right.$ $\left.m_{i} \in \mathbb{N}\right\}$, for finite number of $m_{i}$, is of finite index in $P B_{2 g+b}$ [20, Theorem 1]. In addition, Funar-Kohno proved that the intersection of all $\mathcal{T}_{2 g+b}(2 m)$, where $m \in \mathbb{N}$, is trivial [16, Theorem 1.1].

Finally, we note a more general definition of congruence subgroups of braid groups. Let $F_{n}$ be the free group of rank $n$. There is an inclusion $B_{n} \rightarrow \operatorname{Aut}\left(F_{n}\right)$ [5, Theorem 1.9]. Consider a characteristic subgroup $H$ of finite index in $F_{n}$. The kernel of $\operatorname{Aut}\left(F_{n}\right) \rightarrow \operatorname{Aut}\left(F_{n} / H\right)$ is called principal congruence subgroup, and any finite index subgroup of $\operatorname{Aut}\left(F_{n}\right)$ containing a principal congruence subgroup is called congruence subgroup. A group $G$ is said to have the congruence subgroup propery if every finite index subgroup of $G$ contains a principal congruence subgroup. Asada proved that $B_{n}$ satisfies the congruence subgroup property by using the notions of field extensions and profinite groups [3, Theorem 3A, Theorem 5]. In contrast with Asada's techniques, Thurston gave a more elementary proof to the congruence subgroup property of $B_{n}[22]$.

Outline of the paper. In Section 2 we give basic background on braid groups, hyperelliptic mapping class groups, the symplectic representation of braid groups, and the congruence subgroups of braid groups. In Section 3 we recall some key results about the congruence subgroups of symplectic groups. In Section 4 we give a topological interpretation of the generators of the prime level congruence subgroups of braid groups. In Section 5 we prove Theorems A and B. In Section 6 we prove Theorem C.

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## 2 Preliminaries

In this section we recall the definition of braid groups, hyperelliptic mapping class groups, and the symplectic representation of braid groups.

### 2.1 Definitions of braid groups



Figure 1: The action of $\sigma_{3}$ on a punctured disc.

Braid groups. For detailed description of the following definition, see Birman-Brendle's survey [6]. Let $\Sigma_{g, n}^{b}$ denote an orientable surface of genus $g$ with $n$ punctures and $b$ boundary components. If $n=0$ we will simply write $\Sigma_{g}^{b}$. If $g=0$ and $b=1$ then $\Sigma_{0, n}^{1}$ is homeomorphic to a punctured disc. We enumerate the punctures from left to right. The braid group $B_{n}$ on $n$ strands is defined to be the mapping class group $\operatorname{Mod}\left(\Sigma_{0, n}^{1}\right)$ of $\Sigma_{0, n}^{1}$. For $1 \leq i \leq n-1$ we denote by $\sigma_{i}$ the mapping classes that interchanges the punctures $i, i+1$ as depicted in Figure 1 for $i=3$. The mapping classes $\sigma_{i}$ are called half-twists. It turns out that $\sigma_{i}$ generate the braid group $B_{n}$. In fact we have the following presentation

$$
\left.\left\langle\sigma_{1}, \ldots, \sigma_{n-1}\right| \sigma_{i} \sigma_{i+1} \sigma_{i}=\sigma_{i+1} \sigma_{i} \sigma_{i+1}, \sigma_{i} \sigma_{j}=\sigma_{j} \sigma_{i} \text { when }|i-j|>1\right\rangle
$$

Consider the symmetric group $S_{n}$, and for $1 \geq i \geq n-1$ let $s_{i}$ denote the generators of $S_{n}$, that is the transpositions $(i, i+1)$. The map $B_{n} \rightarrow S_{n}$ defined by $\sigma_{i} \mapsto s_{i}$ is a well defined homomorphism with kernel the pure braid group $P B_{n}$. Let $1 \leq i<j \leq n-1$, we denote by $a_{i, j}$ the element $\sigma_{j-1} \ldots \sigma_{i}^{2} \ldots \sigma_{j-1}$. For $1 \leq i<j \leq n-1$ the group $P B_{n}$ admits a presentation with generators $a_{i, j}$ and relations
P1. $a_{r, s}^{-1} a_{i, j} a_{r, s}=a_{i, j}, 1 \leq r<s<i<j \leq n$ or $1 \leq i<r<s<j \leq n$,
P2. $a_{r, s}^{-1} a_{i, j} a_{r, s}=a_{r, j} a_{i, j} a_{r, j}^{-1}, 1 \leq r<s=i<j \leq n$,
P3. $a_{r, s}^{-1} a_{i, j} a_{r, s}=\left(a_{i, j} a_{s, j}\right) a_{i, j}\left(a_{i, j} a_{s, j}\right)^{-1}, 1 \leq r=i<s<j \leq n$,
P4. $a_{r, s}^{-1} a_{i, j} a_{r, s}=\left(a_{r, j} a_{s, j} a_{r, j}^{-1} a_{s, j}^{-1}\right) a_{i, j}\left(a_{r, j} a_{s, j} a_{r, j}^{-1} a_{s, j}^{-1}\right)^{-1}, 1 \leq r<i<s<j \leq n$.
For more details about definitions and presentations of $B_{n}$ and $P B_{n}$ see [6, Chapter 1].


Figure 2: Action of the hyperelliptic involution.

Hyperelliptic mapping class groups. Let $c$ be a nonseparating simple closed curve on a surface $\Sigma_{g, n}^{b}$. We denote by $T_{c}$ the Dehn twist about the curve $c$. Dehn twists about nonseparating simple closed curves generate $\operatorname{Mod}\left(\Sigma_{g}^{b}\right)$. Consider a hyperelliptic involution $\iota$ as depicted in Figure 2. For $b=1,2, \iota$ acts on $\Sigma_{g}^{b}$. Since $\iota$ does not fix the boundary components of $\Sigma_{g}^{b}$ pointwise, then $\iota \notin \operatorname{Mod}\left(\Sigma_{g}^{b}\right)$. We have a two fold branched cover $\Sigma_{g}^{b} \rightarrow \Sigma_{g}^{b} / \iota$. Topologically $\Sigma_{g}^{b} / \iota$ is homeomorphic to $\Sigma_{0,2 g+b}^{1}$ (see Figure 22). We note that if $q_{1}, q_{2}$ denote the boundary components of $\Sigma_{g}^{2}$, then $\iota\left(q_{1}\right)=q_{2}$.


Figure 3: Generators of the hyperelliptic mapping class group.

Consider the curves $c_{i}$ depicted in Figure 3, and let $\sigma_{i}$ be the generators of $B_{2 g+b}$. We define a map $\xi: B_{2 g+b} \rightarrow \operatorname{Mod}\left(\Sigma_{g}^{b}\right)$ by $\xi\left(\sigma_{i}\right)=T_{c_{i}}$. Since the braid, and the disjointness relations are satisfied by $\sigma_{i}$ and $T_{c_{i}}$, then $\xi$ is a homomorphism. The image of $\xi$ is called hyperelliptic mapping class group, and it is denoted by $\operatorname{SMod}\left(\Sigma_{g}^{b}\right)$. In fact we have $B_{2 g+b} \cong \operatorname{SMod}\left(\Sigma_{g}^{b}\right)$ 15. Theorem 9.2] (see also [24]).

### 2.2 Symplectic representation

In this section we will construct a representation for the braid group $B_{n}$. Firstly, we recall the definition of $\operatorname{Sp}_{2 n}(\mathbb{Z})$. Let $J$ be the $2 n \times 2 n$ matrix

$$
\left(\begin{array}{cc}
0 & I_{n} \\
-I_{n} & 0
\end{array}\right)
$$

The symplectic group with integer coefficients is defined to be

$$
\operatorname{Sp}_{2 n}(\mathbb{Z})=\left\{A \in \operatorname{GL}(2 n, \mathbb{Z}) \mid A^{T} J A=J\right\}
$$

We also define the symplectic group with coefficients in $\mathbb{Z} / m$ to be

$$
\mathrm{Sp}_{2 n}(\mathbb{Z} / m)=\left\{A \in \mathrm{GL}(2 n, \mathbb{Z}) \mid A^{T} J A \equiv J \bmod (m)\right\}
$$

where $m \in \mathbb{N}$. For a fixed $u \in \mathbb{Z}^{2 n}$, we also recall

$$
\left(\mathrm{Sp}_{2 n}(\mathbb{Z})\right)_{u}=\left\{t \in \operatorname{Sp}_{2 n}(\mathbb{Z}) \mid t(u)=u\right\}
$$

Consider $g \geq 1$ and $b=1,2$. Since $B_{2 g+b} \cong \operatorname{SMod}\left(\Sigma_{g}^{b}\right)$, we will use the action of $\operatorname{SMod}\left(\Sigma_{g}^{b}\right)$ on the first homology of $\Sigma_{g}^{b}$ to construct a representation for $B_{2 g+b}$.


Figure 4: Standard generators for $\mathrm{H}_{1}\left(\Sigma_{g}^{1}\right)$, and $\mathrm{H}_{1}^{P}\left(\Sigma_{g}^{2}, \mathbb{Z}\right)$.

Construction of the representation. We denote by $\iota_{a}$ the algebraic intersection number between curves of $\Sigma_{g}^{b}$ for $g \geq 1$ and $b=1,2$. The form $\iota_{a}$ is an alternating bilinear and nondegenerate. Every element of the mapping class group preserves $\iota_{a}$ [15, Section 6.3]. Consider $b=1$; the oriented curves $x_{i}, y_{i}$ of $\Sigma_{g}^{1}$ of Figure 4 form a symplectic basis for $\mathrm{H}_{1}\left(\Sigma_{g}^{1} ; \mathbb{Z}\right)$. The action of $\operatorname{SMod}\left(\Sigma_{g}^{1}\right)$ on $\mathrm{H}_{1}\left(\Sigma_{g}^{1} ; \mathbb{Z}\right)$ induces the following representation:

$$
\operatorname{SMod}\left(\Sigma_{g}^{1}\right) \rightarrow \operatorname{Sp}_{2 g}(\mathbb{Z})
$$

If $b=2$, the module $\mathrm{H}_{1}\left(\Sigma_{g}^{2} ; \mathbb{Z}\right)$ is not symplectic. Thus, we will consider a different module. Fix a point on each of the boundaries of $\Sigma_{g}^{2}$, and denote by $Q$ the set that contains those two points. Denote also by $P$ the set that contains the two boundary components. We set $\mathrm{H}_{1}^{P}\left(\Sigma_{g}^{2} ; \mathbb{Z}\right) \cong$ $\mathrm{H}_{1}\left(\Sigma_{g}^{2}, Q ; \mathbb{Z}\right) /\langle P\rangle$. The module $\mathrm{H}_{1}^{P}\left(\Sigma_{g}^{2} ; \mathbb{Z}\right)$ is symplectic [9, Section 2.1] (see also [26]). The basis of $\mathrm{H}_{1}^{P}\left(\Sigma_{g}^{2} ; \mathbb{Z}\right)$ is $x_{i}, y_{i}$ as indicated on the right hand side of Figure 4 The action of $\operatorname{SMod}\left(\Sigma_{g}^{2}\right)$ on $\mathrm{H}_{1}^{P}\left(\Sigma_{g}^{2} ; \mathbb{Z}\right)$ induces the following representation:

$$
\operatorname{SMod}\left(\Sigma_{g}^{2}\right) \rightarrow\left(\operatorname{Sp}_{2 g+2}(\mathbb{Z})\right)_{y_{g+1}}
$$

where $\left(\operatorname{Sp}_{2 g+2}(\mathbb{Z})\right)_{y_{g+1}}$ stands for the subgroup of $\operatorname{Sp}_{2 g+2}(\mathbb{Z})$ that fixes the vector $y_{g+1}$.
Since the map $\xi: B_{2 g+b} \rightarrow \operatorname{SMod}\left(\Sigma_{g}^{b}\right)$ is an isomorphism, we have a well defined representation

$$
\rho: B_{2 g+b} \rightarrow\left\{\begin{array}{cl}
\operatorname{Sp}_{2 g}(\mathbb{Z}) & \text { if } b=1 \\
\left(\operatorname{Sp}_{2 g+2}(\mathbb{Z})\right)_{y_{g+1}} & \text { if } b=2
\end{array}\right.
$$

Image of the representation. We denote also by $[c]$ the homology class of a curve $c$ in $\Sigma_{g}^{b}$. For $x, c$ nonseparating simple closed curves in $\Sigma_{g}^{b}$, the automorphism $T_{[c]}([x])=[x]+\iota_{a}(x, c)[c]$ is called a transvection [15, Section 6.6.3]. We remark that for every integer $m$, we have $T_{[c]}^{m}([x])=$ $[x]+m \iota_{a}(x, c)[c]$.

Let $T_{c_{i}}$ be a Dehn twist about a curve $c_{i}$ indicated in Figure 3. The image of $T_{c_{i}}$ under the symplectic representation is the transvection $T_{\left[c_{i}\right]}$. Also, since $\xi\left(\sigma_{i}\right)=T_{c_{i}}$ as explained in the previous section, we have $\rho\left(\sigma_{i}\right)=T_{\left[c_{i}\right]}$. We note also that $\rho\left(\sigma_{i}^{m}\right)=T_{\left[c_{i}\right]}^{m}$.

Kernel of the symplectic representation. Assume that $b=1,2, g \geq 0$, and recall that $B_{2 g+b}=\operatorname{Mod}\left(D_{2 g+b}\right)$. The kernel of the symplectic representation $\rho$ is denoted by $\mathcal{B} \mathcal{I}_{2 g+b}$, and it is called the braid Torelli. It is a result by Brendle-Margalit-Putman that $\mathcal{B I}_{2 g+b}$ is generated by Dehn twists about simple closed curves surrounding 3 or 5 number of puncture points [11, Theorem C].

Consider the isomorphism $\xi: B_{2 g+b} \rightarrow \operatorname{SMod}\left(\Sigma_{g}^{b}\right)$. The image of $\mathcal{B I}_{2 g+b}$ in $\operatorname{SMod}\left(\Sigma_{g}^{b}\right)$ under $\xi$ is denoted by $\mathcal{S I}\left(\Sigma_{g}^{b}\right)$. The latter group is well known as the hyperelliptic Torelli group. Furthermore, $\mathcal{S I}\left(\Sigma_{g}^{b}\right)$ is generated by Dehn twists about symmetric separating simple closed curves that bound a subsurface of genus 1 or 2 [11, Theorem A].

### 2.3 Congruence subgroups of braid groups

Let $m$ be a positive integer. The surjective homomorphisms $\mathrm{H}_{1}\left(\Sigma_{g}^{1} ; \mathbb{Z}\right) \rightarrow \mathrm{H}_{1}\left(\Sigma_{g}^{1} ; \mathbb{Z} / m\right)$ and $\mathrm{H}_{1}^{P}\left(\Sigma_{g}^{2} ; \mathbb{Z}\right) \rightarrow \mathrm{H}_{1}^{P}\left(\Sigma_{g}^{2} ; \mathbb{Z} / m\right)$ induce the following epimorphisms:

$$
\left\{\begin{array}{rl}
\mathrm{Sp}_{2 g}(\mathbb{Z}) & \rightarrow \mathrm{Sp}_{2 g}(\mathbb{Z} / m) \\
\left(\mathrm{Sp}_{2 g+2}(\mathbb{Z})\right)_{y_{g+1}} & \rightarrow
\end{array}\left(\mathrm{Sp}_{2 g+2}(\mathbb{Z} / m)\right)_{y_{g+1}} .\right.
$$

Thus we have a family of representations for the braid groups

$$
\rho_{m}: B_{2 g+b} \rightarrow\left\{\begin{array}{cl}
\mathrm{Sp}_{2 g}(\mathbb{Z} / m) & \text { if } b=1 \\
\left(\mathrm{Sp}_{2 g+2}(\mathbb{Z} / m)\right)_{y_{g+1}} & \text { if } b=2
\end{array}\right.
$$

where $g \geq 1$. The kernels of the representations $\rho_{m}$ are denoted by $B_{2 g+b}[m]$ and they are known as level-m congruence subgroups of braid groups.

## 3 Congruence subgroups of Symplectic groups

In this section we examine the structure of the congruence subgroups of symplectic groups.

Congruence subgroups and generators. The projection $\mathbb{Z} \rightarrow \mathbb{Z} / m$ induces a surjective homomorphism $\mathrm{Sp}_{2 n}(\mathbb{Z}) \rightarrow \mathrm{Sp}_{2 n}(\mathbb{Z} / m)$, whose kernel is the principal level $m$ congruence subgroup of $\mathrm{Sp}_{2 n}(\mathbb{Z})$ denoted by $\mathrm{Sp}_{2 n}(\mathbb{Z})[m]$. The group $\mathrm{Sp}_{2 n}(\mathbb{Z})[m]$ consists of all matrices of the form $I_{2 n}+m A$; where $A \in \operatorname{Sp}_{2 n}(\mathbb{Z})$. Furthermore, if $m$ is a multiple of $l$ then $\operatorname{Sp}_{2 n}(\mathbb{Z})[m] \triangleleft \operatorname{Sp}_{2 n}(\mathbb{Z})[l]$.

Next we give generators for $\operatorname{Sp}_{2 n}(\mathbb{Z})[p]$ when $p$ is any prime number. Let $r \in \mathbb{Z}$. We define $e_{i, j}(r)$ to be the $n \times n$ matrix with $(i, j)^{t h}$ entry equal to $r$ and 0 otherwise. Let $\beta_{i}(r)$ be the $n \times n$ matrix with $(i, i)^{t h}$ and $(i, i+1)^{t h}$ entries equal to $r,(i+1, i+1)^{t h}$ and $(i+1, i)^{t h}$ entries equal to $-r$ and 0 otherwise. Define also $s e_{i, j}(r)$ to be the $n \times n$ matrix with $(i, j)^{t h}$ and $(j, i)^{t h}$ entries equal to $r$ and 0 otherwise. For $1 \leq i \leq j \leq n$ we define:

$$
\mathcal{X}_{i, j}(r)=I_{2 n}+\left(\begin{array}{cc}
0 & 0 \\
s e_{i, j}(r) & 0
\end{array}\right), \quad \mathcal{Y}_{i, j}(r)=I_{2 n}+\left(\begin{array}{cc}
0 & s e_{i, j}(r) \\
0 & 0
\end{array}\right) .
$$

For $1 \leq i, j \leq n$ with $i \neq j$ we define:

$$
\mathcal{Z}_{i, j}(r)=I_{2 n}+\left(\begin{array}{cc}
e_{i, j}(r) & 0 \\
0 & -e_{i, j}(r)
\end{array}\right) .
$$

For $1 \leq i<n$

$$
\mathcal{W}_{i}(r)=I_{2 n}+\left(\begin{array}{cc}
\beta_{i}(r) & 0 \\
0 & -\beta_{i}(r)
\end{array}\right) .
$$

Finally,

$$
\mathcal{U}_{1}(r)=I_{2 n}+\left(\begin{array}{cc}
e_{1,1}(r) & e_{1,1}(r) \\
-e_{1,1}(r) & -e_{1,1}(r)
\end{array}\right) .
$$

The following theorem gives a nice description of $\operatorname{Sp}_{2 n}(\mathbb{Z})[p]$ as a group generated by the matrices above [13, Lemma 5.4].

Theorem 3.1 (Church-Putman). For $n \geq 2$ and for a prime number $p \geq 2$ the congruence subgroup $\mathrm{Sp}_{2 n}(\mathbb{Z})[p]$ is generated by the set

$$
\mathcal{S}=\left\{\mathcal{X}_{i, j}(p), \mathcal{Y}_{i, j}(p), \mathcal{Z}_{i, j}(p), \mathcal{W}_{i}(p), \mathcal{U}_{1}(p)\right\}
$$

where $i, j$ are indices defined as above.

We use Theorem 3.1 to prove the lemma below, since we do not know a concise proof in the literature. In particular, we use the generators of Theorem 3.1 to prove that $\mathrm{Sp}_{2 n}(\mathbb{Z} / b)$ can be expressed as a quotient of some congruence subgroup of $\operatorname{Sp}_{2 n}(\mathbb{Z})$ when $b$ is a prime number.

Lemma 3.2. Let $a$ and $b$ two distinct prime numbers. Then the following sequence is exact.

$$
1 \rightarrow \mathrm{Sp}_{2 n}(\mathbb{Z})[a b] \rightarrow \mathrm{Sp}_{2 n}(\mathbb{Z})[a] \rightarrow \mathrm{Sp}_{2 n}(\mathbb{Z} / b) \rightarrow 1
$$

Proof. The map $\operatorname{Sp}_{2 n}(\mathbb{Z})[a] \rightarrow \operatorname{Sp}_{2 n}(\mathbb{Z} / b)$ sends every matrix $A \in \operatorname{Sp}_{2 n}(\mathbb{Z})[a]$ into its $\bmod (b)$ reduction. First, we prove the surjectivity of the latter map. The generators of $\mathrm{Sp}_{2 n}(\mathbb{Z} / b)$ are $\mathcal{X}_{i, j}(1) \bmod (b)$ and $\mathcal{Y}_{i, j}(1) \bmod (b)$ where $1 \leq i<j \leq n$. Define $n$ to be the solution of the equation $a n \equiv 1 \bmod (b)$. Then, $\mathcal{X}_{i, j}(a)^{n} \equiv \mathcal{X}_{i, j}(1) \bmod (b)$ and $\mathcal{Y}_{i, j}(a)^{n} \equiv \mathcal{Y}_{i, j}(1) \bmod (b)$. This proves the surjectivity of the reduction map. The kernel of this reduction map contains matrices which satisfy $I_{2 n}+a A \equiv I_{2 n} \bmod (b)$. But since $a$ and $b$ are relatively primes, the latter equivalence holds if and only if $A=b B$ when $B$ is a symplectic matrix.

The following proposition gives a useful decomposition of $\mathrm{Sp}_{2 n}(\mathbb{Z} / m)$ [23, Theorem 5].
Proposition 3.3 (Newman-Smart). Let $m \in \mathbb{N}$ and write $m=p_{1}^{k_{1}} p_{2}^{k_{2}} \ldots p_{l}^{k_{l}}$, where $p_{i}^{k_{i}}$ are powers of prime numbers. Then

$$
\operatorname{Sp}_{2 n}(\mathbb{Z} / m)=\bigoplus_{i=1}^{l} \operatorname{Sp}_{2 n}\left(\mathbb{Z} / p_{i}^{k_{i}}\right)
$$

Newman-Smart also proved that the abelian group $\mathfrak{s p}_{2 n}(\mathbb{Z} / l)$ can be expressed as a quotient of congruence subgroups of $\mathrm{Sp}_{2 n}(\mathbb{Z})$, [23, Theorem 7].

Proposition 3.4 (Newman-Smart). Let $l, m \geq 2$ such that $l$ divides $m$. Then we have the following isomorphism.

$$
\operatorname{Sp}_{2 n}(\mathbb{Z})[m] / \operatorname{Sp}_{2 n}(\mathbb{Z})[m l] \cong \mathfrak{s p}_{2 n}(\mathbb{Z} / l)
$$

Lemma 3.2 and Propositions 3.3 and 3.4 play crucial role in Section 5, in which we explore the structure of congruence subgroups of braid groups.

## 4 Topological interpretation of prime level congruence subgroups

The purpose of this section is the characterization of the group $B_{2 g+b}[p]$ when $p$ is prime. Since $B_{2 g+b} \cong \operatorname{SMod}\left(\Sigma_{g}^{b}\right)$, it is convenient to study the kernel of the map

$$
\operatorname{SMod}\left(\Sigma_{g}^{b}\right) \rightarrow\left\{\begin{array}{cl}
\operatorname{Sp}_{2 g}(\mathbb{Z} / p) & \text { if } b=1 \\
\left(\operatorname{Sp}_{2 g+2}(\mathbb{Z} / p)\right)_{y_{g+1}} & \text { if } b=2
\end{array}\right.
$$

and we denote the map again by $\rho_{p}$. Also, we denote the kernel of $\rho_{p}$ by $B_{2 g+b}[p]$.
A'Campo proved that the homomorphism $\rho_{p}$ is surjective [1, Theorem 1 (1)]. Later Assion gave a presentation for $\mathrm{Sp}_{2 g}(\mathbb{Z} / 3)$ and $\left(\operatorname{Sp}_{2 g+2}(\mathbb{Z} / 3)\right)_{y_{g+1}}$ as quotients of braid groups 4]. Wajnryb improved the result of Assion and generalized it for any prime number greater than 2 [27, Theorem $1]$. We begin with the theorem of Wajnryb.

Theorem 4.1 (Wajnryb). Consider the curves $c_{i}$ depicted in Figure 3. Let $G_{2 g+b}$ be a group with generators $T_{c_{1}}, \ldots, T_{c_{2 g+b-1}}$ and relations $R 1$ to $R 6$ as follows.

$$
\begin{aligned}
& \text { R1. } T_{c_{i}} T_{c_{i+1}} T_{c_{i}}=T_{c_{i+1}} T_{c_{i}} T_{c_{i+1}} \\
& \text { R2. }\left[T_{c_{i}}, T_{c_{j}}\right]=1, \quad \text { for }|i-j|>1 ;
\end{aligned}
$$

R3. $T_{c_{1}}^{p}=1$;
R4. $\left(T_{c_{1}} T_{c_{2}}\right)^{6}=1, \quad$ for $p>3$;
R5. $T_{c_{1}}^{(p-1) / 2} T_{c_{2}}^{4} T_{c_{1}}^{-(p-1) / 2}=T_{c_{2}}^{2} T_{c_{1}} T_{c_{2}}^{-2}, \quad$ for $p>3$; and
R6. $\left(T_{c_{1}} T_{c_{2}} T_{c_{3}}\right)^{4}=A T_{c_{1}}^{2} A^{-1}$, for $n>4$, where $A=T_{c_{4}} T_{c_{3}}^{2} T_{c_{4}} T_{c_{2}}^{(p-1) / 2} T_{c_{3}}^{-1} T_{c_{2}}$.
Then $G_{2 g+1}$ is isomorphic to $\mathrm{Sp}_{2 g}(\mathbb{Z} / p)$, and $G_{2 g+2}$ is isomorphic to $\left(\operatorname{Sp}_{2 g+2}(\mathbb{Z} / p)\right)_{y_{n+1}}$.
As a consequence of Theorem 4.1 we obtain elements of $\operatorname{SMod}\left(\Sigma_{g}^{b}\right)$ which normally generate $B_{2 g+b}[p]$.

In the rest of the section we examine the elements of the relations of Theorem 4.1 in order to give a topological description for the generators of $B_{n}[p]$. We note that relations $R 1$ and $R 2$ are the defining relations in the presentation of the braid group.

We denote by $\left[c_{i}\right]$ the homology class of $c_{i}$, and by $T_{\left[c_{i}\right]}$ the transvection associated to the Dehn twist $T_{c_{i}}$ under the map

$$
\operatorname{SMod}\left(\Sigma_{g}^{b}\right) \rightarrow\left\{\begin{array}{cl}
\operatorname{Sp}_{2 g}(\mathbb{Z} / p) & \text { if } b=1 \\
\left(\operatorname{Sp}_{2 g+2}(\mathbb{Z} / p)\right)_{y_{g+1}} & \text { if } b=2
\end{array}\right.
$$

By definition, the action of a transvection $T_{[c]}^{m}$ on an element $u \in \mathrm{H}_{1}\left(\Sigma_{g}^{1}, \mathbb{Z}\right)$ (respectively $\mathrm{H}_{1}^{P}\left(\Sigma_{g}^{2}, \mathbb{Z}\right)$ ) is defined to be $T_{[c]}^{m}(u)=[u]+m \hat{i}(u,[c])[c]$, where $\hat{i}$ stands for the algebraic intersection number.

R3: Powers of Dehn twists. The $p^{t h}$ powers of Dehn twists about symmetric nonseparating simple closed curves are easy to check by looking at their image in the symplectic group. The symplectic representation sends $T_{c_{1}}^{p}$ into the following matrix:

$$
\left(\begin{array}{ll}
1 & p \\
0 & 1
\end{array}\right) \oplus I,
$$

where $I$ stands for the identity matrix of dimension depending on $g$ and $b$ (see Section 7.1.3). The $\bmod (p)$ reduction of the matrix above is the identity. Moreover, every Dehn twist about a non-separating curve is conjugate to $T_{c_{1}}$. As a consequence, every Dehn twist in $\operatorname{SMod}\left(\Sigma_{g}^{b}\right)$ raised to the power of $p$ lies in $B_{n}[p]$.

R4: Symmetric separating Dehn twists. By the chain relation the element $\left(T_{c_{1}} T_{c_{2}}\right)^{6}$ can be represented by a Dehn twist $T_{\gamma}$, where $\gamma$ is the symmetric separating curve bounding the genus 1 subsurface of $\Sigma_{g}^{b}$ as indicated in Figure 5 [15, Proposition 4.12]. We can generalize the relation R4 by considering a symmetric separating curve $\delta$ of a genus $k$ subsurface of $\Sigma_{g}^{b}$. By the chain relation there is a maximal chain of curves $a_{1}, \ldots, a_{2 k}$ in the subsurface of genus $k$ with boundary $\delta$ such that $\left(T_{a_{1}} \ldots T_{a_{2 k}}\right)^{4 k+2}=T_{\delta}$.

The fact that every symmetric separating simple closed curve $\delta$ is nullhomologous in $H_{1}\left(\Sigma_{g}^{1}\right)$ (respectively $H_{1}^{P}\left(\Sigma_{g}^{2}\right)$ ) implies that $T_{[\delta]}(x)=x+\iota_{a}(x,[\delta])=x+0=x$ for every $x \in H_{1}\left(\Sigma_{g}^{1}\right)$ (respectively $H_{1}^{P}\left(\Sigma_{g}^{2}\right)$ ), where $T_{[\delta]}$ is the corresponding transvection of $T_{\delta}$ as described in Section 2. Since for every symmetric separating curve $\delta$ in $\Sigma_{g}^{b}$ and $T_{\delta} \in B_{2 g+b}[p]$ we have that $\left(T_{a_{1}} \ldots T_{a_{2 k}}\right)^{4 k+2} \in \mathcal{S I}\left(\Sigma_{g}^{b}\right) \subset B_{2 g+b}[p]$.

R5: Mod-p involution maps. We begin by modifying the relation $R 5$ of Theorem 4.1
Lemma 4.2. The relation $R 5$ given above is equivalent to:

$$
\left(T_{c_{1}}^{(p+1) / 2} T_{c_{2}}^{4}\right)^{2}=\left(T_{c_{1}} T_{c_{2}}\right)^{3}
$$

in $\mathrm{Sp}_{2 g}(\mathbb{Z} / p)$ (respectively $\left.\left(\mathrm{Sp}_{2 g+2}(\mathbb{Z} / p)\right)_{y_{g+1}}\right)$.


Figure 5: The curve $\gamma$ that bound a surface of genus 1.

Proof. We have that $\left(T_{c_{1}} T_{c_{2}}\right)^{3}=T_{c_{1}} T_{c_{2}}^{2} T_{c_{1}} T_{c_{2}}^{2}$. Then

$$
T_{c_{1}}^{(p-1) / 2} T_{c_{2}}^{4} T_{c_{1}}^{-(p-1) / 2}=T_{c_{1}}^{-1}\left(T_{c_{1}}^{(p+1) / 2} T_{c_{2}}^{4}\right)^{2} T_{c_{2}}^{-4}=T_{c_{2}}^{2} T_{c_{1}} T_{c_{2}}^{-2}
$$

On the other hand

$$
\left(T_{c_{1}}^{(p+1) / 2} T_{c_{2}}^{4}\right)^{2}=T_{c_{1}} T_{c_{1}}^{(p-1) / 2} T_{c_{2}}^{4} T_{c_{1}}^{-(p-1) / 2} T_{c_{2}}^{4}=T_{c_{1}} T_{c_{2}}^{2} T_{c_{1}} T_{c_{2}}^{2}
$$

Now we examine the relation of Lemma 4.2 ,

RHS. For $i=1,2,\left(T_{c_{1}} T_{c_{2}}\right)^{3}\left(\left[c_{i}\right]\right)=-\left[c_{i}\right]$, where $\left[c_{i}\right]$ stands for the homology class of $c_{i}$. Thus, the homeomorphism $\left(T_{c_{1}} T_{c_{2}}\right)^{3}$ acts as the hyperelliptic involution on the subsurface bounded by the boundary of the chain $\operatorname{ch}\left(c_{1}, c_{2}\right)$ (see Figure 5).

LHS. We have

$$
\begin{array}{r}
\left(T_{c_{1}}^{(p+1) / 2} T_{c_{2}}^{4}\right)^{2}\left(\left[c_{1}\right]\right)=-8 p\left[c_{2}\right]+\left(4 p^{2}+2 p-1\right)\left[c_{1}\right] \equiv-\left[c_{1}\right] \bmod (p), \\
\quad\left(T_{c_{1}}^{(p+1) / 2} T_{c_{2}}^{4}\right)^{2}\left(\left[c_{2}\right]\right)=2 p \frac{p+1}{2}\left[c_{1}\right]-(2 p+1)\left[c_{2}\right] \equiv-\left[c_{2}\right] \bmod (p)
\end{array}
$$

Therefore, $\left(T_{c_{1}}^{(p+1) / 2} T_{c_{2}}^{4}\right)^{2}$ acts as the hyperelliptic involution $\bmod (p)$ in the subspace of $\mathrm{H}_{1}\left(\Sigma_{g}^{1}, \mathbb{Z} / p\right)$ $\left(\operatorname{resp} \mathrm{H}_{1}^{P}\left(\Sigma_{g}^{2}, \mathbb{Z} / p\right)\right)$ spanned by $\left[c_{1}\right],\left[c_{2}\right]$.

We can generalize Relation $R 5$ as follows. For $k$ even, consider any chain $\operatorname{ch}\left(a_{1}, a_{2}, \ldots, a_{k}\right)$ of symmetric simple closed curves such that $T_{a_{i}} \in \operatorname{SMod}\left(\Sigma_{g, b}\right)$ for all $i \leq k$. Choose an $f \in \operatorname{SMod}\left(\Sigma_{g}^{b}\right)$ such that $f\left(\left[a_{i}\right]\right)=-\left[a_{i}\right]$. Then $\left(T_{a_{1}} \ldots T_{a_{k}}\right)^{k+1} f^{-1} \in B_{2 g+b}[p]$. We call this type of element an mod-p involution map.

R6: Mod-p center maps. We describe a generalized version of $\left(T_{c_{1}} T_{c_{2}} T_{c_{3}}\right)^{4}\left(A T_{c_{1}}^{-2} A^{-1}\right)$. Let $A_{1}$ be the trivial homeomorphism in $\operatorname{SMod}\left(\Sigma_{g}^{b}\right)$. For $k$ odd, and $k \geq 3$, define

$$
A_{k}=T_{c_{k+1}} T_{c_{k}}^{2} T_{c_{k+1}} T_{c_{k-1}}^{(p-1) / 2} T_{c_{k}}^{-1} T_{c_{k-1}} A_{k-2}
$$

First, we deal with the case $b=1$. (For $b=2$ the process is exactly the same.) Consider the symplectic bases $\left\{y_{i}, x_{i}\right\}$ for $\mathrm{H}_{1}\left(\Sigma_{g}^{1}, \mathbb{Z}\right)$ depicted on Figure 4 .
Lemma 4.3. For $k$ odd, we have that $A_{k} T_{\left[c_{1}\right]} A_{k}^{-1}=T_{\left[y_{(k+1) / 2}\right]}$ in $\mathrm{Sp}_{2 g}(\mathbb{Z} / p)$.
Note that if $k=3$, then $T_{\left[y_{2}\right]}=T_{\left[d_{3}\right]}$.
Proof. We need to prove that $A_{k}\left(\left[c_{1}\right]\right) \equiv\left[c_{1}\right]+\left[c_{3}\right]+\ldots+\left[c_{k}\right] \in \operatorname{Sp}_{2 g}(\mathbb{Z} / p)$. A direct calculation shows that $A_{3}\left(\left[c_{1}\right]\right) \equiv\left[c_{1}\right]+\left[c_{3}\right] \bmod (p)$. Assume that the theorem is true for $k-2$, that is $A_{k-2}\left(\left[c_{1}\right]\right)=\left[c_{1}\right]+\left[c_{3}\right]+\ldots+\left[c_{k-2}\right]$. Then $T_{c_{k+1}} T_{c_{k}}^{2} T_{c_{k+1}} T_{c_{k-1}}^{(p-1) / 2} T_{c_{k}}^{-1} T_{c_{k-1}}\left(\left[c_{k-2}\right]\right) \equiv\left[c_{k-2}\right]+$ $\left[c_{k}\right] \bmod (p)$. The proof of the lemma follows.


Figure 6: The chain relation of $R 6$.

Let $k$ be an odd integer, and consider also the odd chain $\operatorname{ch}\left(c_{1}, c_{2}, \ldots, c_{k}\right)$. By the chain relation we have that $\left(T_{c_{1}} \ldots T_{c_{k}}\right)^{k+1}=T_{d_{k}} T_{d_{k}^{\prime}}$, where $d_{k}=y_{(k+1) / 2}$, and $\left[d_{k}\right]=\left[d_{k}^{\prime}\right]=\left[y_{(k+1) / 2}\right]$ (see, for example, Figure 6). Thus, $\left(T_{\left[c_{1}\right]} \ldots T_{\left[c_{k}\right]}\right)^{k+1}=T_{\left[y_{(k+1) / 2}\right]}^{2} \in \mathrm{Sp}_{2 g}(\mathbb{Z} / p)$. On the other hand, according to Lemma 4.3 we have that $A_{k} T_{\left[c_{1}\right]}^{2} A_{k}^{-1}=T_{\left[y_{(k+1) / 2}\right]}^{2} \in \mathrm{Sp}_{2 g}(\mathbb{Z} / p)$. Hence, $\left(T_{c_{1}} \ldots T_{c_{k}}\right)^{k+1} A_{k} T_{c_{1}}^{-2} A_{k}^{-1} \in \overline{B_{n}}[p]$. Note that if $k=3$, the element $\left(T_{c_{1}} \ldots T_{c_{k}}\right)^{k+1} A_{k} T_{c_{1}}^{-2} A_{k}^{-1}$ is the same one as in the relation 6 of Theorem 4.1.

We can describe a generalized version of $\left(T_{c_{1}} \ldots T_{c_{k}}\right)^{k+1} A_{k} T_{c_{1}}^{-2} A_{k}^{-1}$. Consider any odd chain $\operatorname{ch}\left(a_{1}, a_{2}, \ldots, a_{k}\right)$, such that $T_{a_{i}} \in \operatorname{SMod}\left(\Sigma_{g}^{1}\right)$ for all $i \leq k$. Choose a homeomorphism $h \in$ $\operatorname{SMod}\left(\Sigma_{g}^{1}\right)$ such that $h\left(\left[a_{1}\right]\right)=\left[a_{1}\right]+\left[a_{3}\right]+\ldots+\left[a_{k}\right] \in \operatorname{Sp}_{2 g}\left(\Sigma_{g}^{1}\right)$. Then $\left(T_{a_{1}} \ldots T_{a_{k}}\right)^{k+1} h T_{a_{1}}^{-2} h^{-1}$ lies on $B_{2 g+1}[p]$. If we consider $\left(T_{a_{1}} \ldots T_{a_{k}}\right)^{k+1}$ as the center of the subgroup $K$ of $\operatorname{SMod}\left(\Sigma_{g}^{b}\right)$ generated by $T_{a_{1}} \ldots T_{a_{k}}$, then $h T_{a_{1}}^{-2} h^{-1}$ is the center $\bmod (p)$ of the same group. Note that the choice of $h$ is not unique. We call this type of element an mod-p center map.

Generators for congruence subgroups. As a corollary of Theorem4.1 we obtain the following theorem.

Theorem 4.4. If $p=3$, then $B_{2 g+b}[3]$ is generated by Dehn twists raised to the power of 3 , and for $2 g+b>4$ by mod-p center maps. For $p>3$ the subgroup $B_{2 g+b}[p]$ of $\operatorname{SMod}\left(\Sigma_{g}^{b}\right)$ is generated by Dehn twists raised to the power of $p$, by Dehn twists about symmetric separating curves, by mod-p involution maps, and for $2 g+b>4$ by mod-p center maps.

Finite set of generators. It is well known that every finite index subgroup of a finitely generated group, is finitely generated [21, Corollary 2.7.1]. The generating set in Theorem 4.4 is infinite. When $p=3$ and $g=1$ we can find a finite set of generators.

Theorem 4.5. The group $B_{3}[3]$ is generated by four elements.
Proof. Set $S=\left\{T_{c_{1}}^{3}, T_{c_{2}}^{3}, T_{c_{2}} T_{c_{1}}^{3} T_{c_{2}}^{-1}, T_{c_{2}}^{2} T_{c_{1}}^{3} T_{c_{2}}^{-2}\right\}$. We denote by $\Gamma$ the subgroup of $B_{3}[3]$ generated by $S$. We prove that if we conjugate elements of $S$ by $T_{c_{1}}$ or $T_{c_{2}}$, then the resulting elements lie in $\Gamma$. Since $B_{3}[3]$ is normally generated by $S$ and since $S$ generates a normal subgroup of $B_{3}$, then $\Gamma=B_{3}[3]$.

In the braid group we have the relation

$$
T_{c_{j}} T_{c_{j-1}} \ldots T_{c_{i}}^{3} \ldots T_{c_{j-1}}^{-1} T_{c_{j}}^{-1}=T_{c_{i}}^{-1} T_{c_{i+1}}^{-1} \ldots T_{c_{j}}^{3} \ldots T_{c_{i+1}} T_{c_{i}}
$$

We prove the theorem in three steps.
Step 1: Conjugates of $T_{c_{1}}^{3}, T_{c_{2}}^{3}$ :

$$
\begin{array}{r}
T_{c_{2}}^{-1} T_{c_{1}}^{3} T_{c_{2}}=T_{c_{2}}^{-3} T_{c_{2}}^{2} T_{c_{1}}^{3} T_{c_{2}}^{-2} T_{c_{2}}^{3} \in \Gamma \\
T_{c_{1}}^{-1} T_{2}^{3} T_{c_{1}}=T_{2} T_{c_{1}}^{3} T_{2}^{-1} \in \Gamma \\
T_{c_{1}} T_{c_{2}}^{3} T_{c_{1}}^{-1}=T_{c_{2}}^{-1} T_{c_{1}}^{3} T_{c_{2}}=T_{c_{2}}^{-3} T_{c_{2}}^{2} T_{c_{1}}^{3} T_{c_{2}}^{-2} T_{c_{2}}^{3} \in \Gamma .
\end{array}
$$

Step 2: Conjugates of $T_{c_{2}} T_{c_{1}}^{3} T_{c_{2}}^{-1}$ :

$$
\begin{array}{r}
T_{c_{1}} T_{c_{2}} T_{c_{1}}^{3} T_{c_{2}}^{-1} T_{c_{1}}^{-1}=T_{c_{2}}^{3} \in \Gamma \\
T_{c_{1}}^{-1} T_{c_{2}} T_{c_{1}}^{3} T_{c_{2}}^{-1} T_{c_{1}}=T_{c_{1}}^{-2} T_{c_{2}}^{3} T_{c_{1}}^{2}=T_{c_{1}}^{-3}\left(T_{c_{1}} T_{c_{2}}^{3} T_{c_{1}}^{-1}\right) T_{c_{1}}^{3}
\end{array}
$$

The latter is in $\Gamma$ by step 1 .
Step 3: Conjugates of $T_{c_{2}}^{2} T_{c_{1}}^{3} T_{c_{2}}^{-2}$ :

$$
\begin{aligned}
& T_{c_{1}}^{-1} T_{c_{2}}^{2} T_{c_{1}}^{3} T_{c_{2}}^{-2} T_{c_{1}}=T_{c_{1}}^{-1} T_{c_{2}}^{3} T_{c_{2}}^{-1} T_{c_{1}}^{3} T_{c_{2}} T_{c_{2}}^{-3} T_{c_{1}}= \\
& \quad\left(T_{c_{1}}^{-1} T_{c_{2}}^{3} T_{c_{1}}\right)\left(T_{c_{1}}^{-1} T_{c_{2}}^{-1} T_{c_{1}}^{3} T_{c_{2}} T_{c_{1}}\right)\left(T_{1}^{-1} T_{c_{2}}^{-3} T_{c_{1}}\right)
\end{aligned}
$$

The elements $\left(T_{c_{1}}^{-1} T_{c_{2}}^{3} T_{c_{1}}\right),\left(T_{c_{1}}^{-1} T_{c_{2}}^{-3} T_{c_{1}}\right)$ are in $\Gamma$ by step 1.

$$
T_{c_{1}}^{-1} T_{c_{2}}^{-1} T_{c_{1}}^{3} T_{c_{2}} T_{c_{1}}=T_{c_{2}}^{3}
$$

Finally, since $T_{c_{2}}^{2} T_{c_{1}}^{3} T_{c_{2}}^{-2}=T_{c_{2}}^{3} T_{c_{2}}^{-1} T_{c_{1}}^{3} T_{c_{2}} T_{c_{2}}^{-3}$, it suffices to check that $T_{c_{1}} T_{c_{2}}^{-1} T_{c_{1}}^{3} T_{c_{2}} T_{c_{1}}^{-1}$ is in $\Gamma$.
But we have that

$$
T_{c_{1}} T_{c_{2}}^{-1} T_{c_{1}}^{3} T_{c_{2}} T_{c_{1}}^{-1}=T_{c_{1}}^{2} T_{c_{2}}^{3} T_{c_{1}}^{-2}=T_{c_{1}}^{3} T_{c_{1}}^{-1} T_{2}^{3} T_{c_{1}} T_{c_{1}}^{-3}=T_{c_{1}}^{3} T_{c_{2}} T_{c_{1}}^{3} T_{c_{2}}^{-1} T_{c_{1}}^{-3} \in \Gamma
$$

This proves the theorem.
Since $T_{c_{2}}^{2} T_{c_{1}}^{3} T_{c_{2}}^{-2}=T_{c_{2}}^{3} T_{c_{2}}^{-1} T_{c_{1}}^{3} T_{c_{2}} T_{c_{2}}^{-3}$ we deduce that $\left\{T_{c_{1}}^{3}, T_{c_{2}}^{3}, T_{c_{2}} T_{c_{1}}^{3} T_{c_{2}}^{-1}, T_{c_{2}}^{-1} T_{c_{1}}^{3} T_{c_{2}}\right\}$ is also a generating set for $B_{3}[3]$.

## 5 Symplectic groups and pure braid groups

For $i \in \mathbb{N}$, let $p_{i}$ denote a prime number greater than 2 . In this section we characterize $B_{2 g+b}[m]$, where $m=2 p_{1} p_{2} \ldots p_{k}$ and $m=4 p_{1} p_{2} \ldots p_{k}$. Our strategy is to find a presentation for $P B_{2 g+b} / B_{2 g+b}[m]$. We recall that $\mathrm{H}_{1}\left(P B_{2 g+b}, \mathbb{Z} / 2\right)$ is $\mathfrak{s p}_{2 g}(\mathbb{Z} / 2)$, if $b=1$ and $\operatorname{Ann}\left(y_{g+1}\right)$ if $b=2$, where $\operatorname{Ann}\left(y_{g+1}\right)=$ $\left\{h \in \mathfrak{s p}_{2 g+2}(\mathbb{Z} / 2) \mid h\left(y_{g+1}\right)=0\right\}[9]$. The generators of $B_{2 g+b}$ are denoted by $\sigma_{i}$ and the generators of $P B_{2 g+b}$ are denoted by $a_{i, j}$ as in Section 2.

Theorem 5.1. For $m=2 p_{1} p_{2} \ldots p_{k}$, where $p_{i} \geq 3$ are prime numbers, we have

$$
P B_{2 g+b} / B_{2 g+b}[m]=\left\{\begin{array}{cl}
\bigoplus_{i=1}^{k} \mathrm{Sp}_{2 g}\left(\mathbb{Z} / p_{i}\right) & \text { if } b=1 \\
\bigoplus_{i=1}^{k}\left(\mathrm{Sp}_{2 g+2}\left(\mathbb{Z} / p_{i}\right)\right)_{y_{g+1}} & \text { if } b=2
\end{array}\right.
$$

Proof. We set $m=2 p_{1} p_{2} \ldots p_{k}$. We have the map

$$
\rho_{m}: B_{2 g+b} \rightarrow\left\{\begin{aligned}
\mathrm{Sp}_{2 g}(\mathbb{Z}) & \rightarrow \mathrm{Sp}_{2 g}(\mathbb{Z} / m) & & \text { if } b=1, \\
\left(\mathrm{Sp}_{2 g+2}(\mathbb{Z})\right)_{y_{g+1}} & \rightarrow\left(\mathrm{Sp}_{2 g+2}(\mathbb{Z} / m)\right)_{y_{g+1}} & & \text { if } b=2
\end{aligned}\right.
$$

with kernel $B_{2 g+b}[m]$. By Lemma 3.3 we know that

$$
\operatorname{Sp}_{2 g}(\mathbb{Z} / m)=\operatorname{Sp}_{2 g}(\mathbb{Z} / 2) \bigoplus_{i=1}^{k} \operatorname{Sp}_{2 g}\left(\mathbb{Z} / p_{i}\right)
$$

If we restrict to the pure braid group, then the image of the map $P B_{2 g+1} \rightarrow \mathrm{Sp}_{2 g}(\mathbb{Z})$ is the group $\mathrm{Sp}_{2 g}(\mathbb{Z})[2]$, (see [9, Theorem 3.3]). Furthermore, by Lemma 3.2 we have that the map $\mathrm{Sp}_{2 g}(\mathbb{Z})[2] \rightarrow \mathrm{Sp}\left(\mathbb{Z} / p_{i}\right)$ is surjective. Thus, the image of the map

$$
\operatorname{Sp}_{2 g}(\mathbb{Z}) \rightarrow \operatorname{Sp}_{2 g}(\mathbb{Z} / m)=\operatorname{Sp}_{2 g}(\mathbb{Z} / 2) \bigoplus_{i=1}^{k} \operatorname{Sp}_{2 g}\left(\mathbb{Z} / p_{i}\right)
$$

after we restrict to $\mathrm{Sp}_{2 g}(\mathbb{Z})[2]$, is the group $\bigoplus_{i=1}^{k} \mathrm{Sp}_{2 g}\left(\mathbb{Z} / p_{i}\right)$. Hence, have a short exact sequence

$$
1 \rightarrow B_{2 g+1}[m] \rightarrow P B_{2 g+1} \rightarrow \bigoplus_{i=1}^{k} \operatorname{Sp}_{2 g}\left(\mathbb{Z} / p_{i}\right) \rightarrow 1
$$

Likewise, since the image of the map $P B_{2 g+2} \rightarrow\left(\operatorname{Sp}_{2 g+2}(\mathbb{Z})\right)_{y_{g+1}}$ is $\left(\operatorname{Sp}_{2 g+2}(\mathbb{Z})[2]\right)_{y_{g+1}}$ (see [9, Theorem 3.3]), and since $\left(\mathrm{Sp}_{2 g+2}(\mathbb{Z} / m)\right)_{y_{g+1}}<\operatorname{Sp}_{2 g+2}(\mathbb{Z} / m)$, we can apply Lemma 3.3 and end up with the following exact sequence.

$$
1 \rightarrow B_{2 g+2}[m] \rightarrow P B_{2 g+2} \rightarrow \bigoplus_{i=1}^{k}\left(\operatorname{Sp}_{2 g+2}\left(\mathbb{Z} / p_{i}\right)\right)_{y_{g+1}} \rightarrow 1
$$

This completes the proof.
In the following statement we slightly generalize Lemma 5.1. The symplectic Lie algebra $\mathfrak{s p}_{2 n}(\mathbb{Z})$ consists of those elements $A \in \mathfrak{g l}_{2 n}(\mathbb{Z})$ which satisfy the relation $A^{T} J+J A=0$. We define also

$$
\operatorname{Ann}(u)=\left\{m \in \mathfrak{s p}_{2 n}(\mathbb{Z}) \mid m(u)=0\right\}
$$

where $\operatorname{Ann}(u)$ stands for the annihilator of the vector $u$. We have the following theorem.
Theorem 5.2. For $m=4 p_{1} p_{2} \ldots p_{k}$, where $p_{i} \geq 3$ are prime numbers, we have

$$
P B_{2 g+b} / B_{2 g+b}[m]=\left\{\begin{array}{cl}
\mathfrak{s p}_{2 g}(\mathbb{Z} / 2) \bigoplus_{i=1}^{k} \mathrm{Sp}_{2 g}\left(\mathbb{Z} / p_{i}\right) & \text { if } b=1 \\
\operatorname{Ann}(e) \bigoplus_{i=1}^{k}\left(\mathrm{Sp}_{2 g+2}\left(\mathbb{Z} / p_{i}\right)\right)_{y_{g+1}} & \text { if } b=2
\end{array}\right.
$$

Proof. We set $m=4 p_{1} p_{2} \ldots p_{k}$. By Lemma 3.3 we have that

$$
\operatorname{Sp}_{2 g}(\mathbb{Z} / m)=\operatorname{Sp}_{2 g}(\mathbb{Z} / 4) \bigoplus_{i=1}^{k} \operatorname{Sp}_{2 g}\left(\mathbb{Z} / p_{i}\right)
$$

We want to characterize the image of the map

$$
B_{2 g+b} \rightarrow\left\{\begin{array}{cl}
\mathrm{Sp}_{2 g}(\mathbb{Z} / 4) \bigoplus_{i=1}^{k} \mathrm{Sp}_{2 g}\left(\mathbb{Z} / p_{i}\right) & \text { if } b=1 \\
\left(\mathrm{Sp}_{2 g+2}(\mathbb{Z} / 4)\right)_{y_{g+1}} \bigoplus_{i=1}^{k}\left(\mathrm{Sp}_{2 g+2}\left(\mathbb{Z} / p_{i}\right)\right)_{y_{g+1}} & \text { if } b=2
\end{array}\right.
$$

For $b=1$ we only need to characterize the image of the restriction of the map above to $P B_{2 g+b}$. In particular, we want to compute the image of the map $P B_{2 g+1} \rightarrow \operatorname{Sp}_{2 g}(\mathbb{Z} / 4)$. We know that the image of the map $P B_{2 g+1} \rightarrow \mathrm{Sp}_{2 g}(\mathbb{Z})$ is $\mathrm{Sp}_{2 g}(\mathbb{Z})[2]$. Consider the inclusion

$$
\mathrm{Sp}_{2 g}(\mathbb{Z})[2] \hookrightarrow \mathrm{Sp}_{2 g}(\mathbb{Z}) .
$$

We quotient the above inclusion by $\mathrm{Sp}_{2 g}(\mathbb{Z})[4]$, and we get the following inclusion:

$$
\mathfrak{s p}_{2 g}(\mathbb{Z} / 2) \hookrightarrow \mathrm{Sp}_{2 g}(\mathbb{Z} / 4) .
$$

We finally have

$$
P B_{2 g+1} \rightarrow \mathrm{Sp}_{2 g}(\mathbb{Z})[2] \rightarrow \mathfrak{s p}_{2 g}(\mathbb{Z} / 2)<\mathrm{Sp}_{2 g}(\mathbb{Z} / 4)
$$

Hence, the image of the map $P B_{2 g+1} \rightarrow \mathrm{Sp}_{2 g}(\mathbb{Z} / 4)$ is the abelian group $\mathfrak{s p}_{2 g}(\mathbb{Z} / 2)$. Thus, we have

$$
P B_{2 g+b} / B_{2 g+b}[m] \cong \mathfrak{s p}_{2 g}(\mathbb{Z} / 2) \bigoplus_{i=1}^{k} \operatorname{Sp}_{2 g}\left(\mathbb{Z} / p_{i}\right)
$$

For $b=2$, the maps

$$
P B_{2 g+2} \rightarrow\left(\operatorname{Sp}_{2 g+2}(\mathbb{Z})[2]\right)_{y_{g+1}} \rightarrow \operatorname{Ann}\left(y_{g+1}\right)
$$

are both surjective, [9, Lemma 3.5]. But $\operatorname{Ann}\left(y_{g+1}\right)<\left(\operatorname{Sp}_{2 g+2}(\mathbb{Z} / 4)\right)_{y_{g+1}}$, and thus, the image of the map

$$
P B_{2 g+2} \rightarrow\left(\mathrm{Sp}_{2 g+2}(\mathbb{Z} / 4)\right)_{y_{g+1}}
$$

is the group $\operatorname{Ann}\left(y_{g+1}\right)$. Thus, we get

$$
P B_{2 g+2} / B_{2 g+2}[m] \cong \operatorname{Ann}\left(y_{g+1}\right) \bigoplus_{i=1}^{k}\left(\mathrm{Sp}_{2 g+2}\left(\mathbb{Z} / p_{i}\right)\right)_{y_{g+1}} .
$$

This completes the proof.
In order to find generators for $B_{2 g+1}[m]$, it suffices to find a presentation for $\mathrm{Sp}_{2 g}(\mathbb{Z} / p)$ in terms of pure braids. In the next proposition we prove that $\mathrm{Sp}_{2 g}(\mathbb{Z} / p)$ admits a presentation as a quotient of the pure braid group over some relations. These new relations are the generators for $B_{2 g+1}[2 p]$. Recall that the generators of $P B_{n}$ are defined to be $a_{i, j}=\sigma_{j-1} \ldots \sigma_{i+1} \sigma_{i}^{2} \sigma_{i+1}^{-1} \ldots \sigma_{j-1}^{-1}$, where $1 \leq i<j \leq n$.

Proposition 5.3. Fix a prime number $p$, and put $p=2 k+1$. Let $H_{n}$ be the group with generators $\left\{a_{i, j}\right\}$ with defining relations as follows:

PR1. $a_{i, i+1}^{k} a_{i+1, i+2}^{k} a_{i, i+1}^{k}=a_{i+1, i+2}^{k} a_{i, i+1}^{k} a_{i+1, i+2}^{k}$,
PR2. $a_{i, j}^{p}=1$,
PR3. $\left(a_{1,2} a_{1,3} a_{2,3}\right)^{2}=1$ for $p>3$,
PR4. $a_{r, s}^{-1} a_{i, j} a_{r, s}=a_{i, j}, 1 \leq r<s<i<j \leq n$ or $1 \leq i<r<s<j \leq n$,
PR5. $a_{r, s}^{-1} a_{i, j} a_{r, s}=a_{r, j} a_{i, j} a_{r, j}^{-1}, 1 \leq r<s=i<j \leq n$,
PR6. $a_{r, s}^{-1} a_{i, j} a_{r, s}=\left(a_{i, j} a_{s, j}\right) a_{i, j}\left(a_{i, j} a_{s, j}\right)^{-1}, 1 \leq r=i<s<j \leq n$,
PR7. $a_{r, s}^{-1} a_{i, j} a_{r, s}=\left(a_{r, j} a_{s, j} a_{r, j}^{-1} a_{s, j}^{-1}\right) a_{i, j}\left(a_{r, j} a_{s, j} a_{r, j}^{-1} a_{s, j}^{-1}\right)^{-1}, 1 \leq r<i<s<j \leq n$,
PR8. $a_{i, j}=a_{j-1, j}^{k+1} a_{j-2, j-1}^{k+1} \ldots a_{i, i+1} a_{i+1, i+2}^{k} \ldots a_{j-1, j}^{k}, 1<|i-j| \leq n$,
PR9. $a_{1,2} a_{1,3} a_{2,3}=C$, where

$$
\begin{aligned}
& C=\left(a_{1,2}^{(p+1) / 4} a_{2,3}^{2}\right)^{2}, \text { if }(p+1) / 2 \text { is even, } \\
& C=a_{1,2}^{(p+3) / 4} a_{1,3}^{2} a_{1,2}^{(p-1) / 4} a_{2,3}^{2}, \text { if }(p+1) / 2 \text { is odd. }
\end{aligned}
$$

PR10. $a_{1,2} a_{1,3} a_{1,4} a_{2,3} a_{2,4} a_{3,4}=B a_{1,4} B^{-1}$, where

$$
\begin{aligned}
& B=a_{3,5} a_{4,5} a_{2,3}^{k / 2} a_{3,4}^{-1}, \text { if } k \text { is even, } \\
& B=a_{3,5} a_{4,5} a_{2,3}^{k+1} a_{3,4}, \text { if } k \text { is odd. }
\end{aligned}
$$

If $n=2 g+1$ then $H_{n}$ is isomorphic to $\mathrm{Sp}_{2 g}(\mathbb{Z} / p)$. On the other hand if $n=2 g+2$, then $H_{n}$ is isomorphic to $\mathrm{Sp}_{2 g+2}(\mathbb{Z} / p)_{y_{g+1}}$.

Note that relations $P R 4, P R 5, P R 6, P R 7$ are relations in the presentation of the pure braid group given in Chapter 4. We begin with the group $G_{n}$ defined in Theorem4.1, and using Tietze transformations, we obtain the presentation of $H_{n}$.

Proof. By Theorem 4.1 the group $G_{n}$ has the following presentation:

$$
G_{n}=\left\langle\sigma_{i} \mid R 1, R 2, R 3, R 4, R 5, R 6\right\rangle,
$$

where $1 \leq i<2 g+b$. Let $a_{i, j}=\sigma_{j-1} \ldots \sigma_{i+1} \sigma_{i}^{2} \sigma_{i+1}^{-1} \ldots \sigma_{j-1}^{-1}$ and denote this relation by $P R 11$. Then include $P R 11$ into the presentation of $G_{n}$ and add the generator $a_{i, j}$ to obtain

$$
\left\langle\sigma_{i}, a_{i, j} \mid R 1, R 2, R 3, R 4, R 5, R 6, P R 11\right\rangle .
$$

Since $P B_{n}$ is a subgroup of $B_{n}$, this means that $R 1$ and $R 2$ can be used to deduce the relations PR4, PR5, PR6, PR7.

$$
\left\langle\sigma_{i}, a_{i, j} \mid R 1, R 2, R 3, R 4, R 5, R 6, P R 4, P R 5, P R 6, P R 7, P R 11\right\rangle .
$$

The relation $R 2$ can be deduced by $P R 11$ and $R 3$ and $P R 4$

$$
\left\langle\sigma_{i}, a_{i, j} \mid R 1, R 3, R 4, R 5, R 6, P R 2, P R 4, P R 5, P R 6, P R 7, P R 11\right\rangle .
$$

We derive two more relations from $P R 11$ and $R 3$.

$$
\sigma_{i}=a_{i, i+1}^{k+1}, \quad \sigma_{i}^{-1}=a_{i, i+1}^{k} .
$$

Then $P R 1$ is equivalent to $R 1, P R 2$ is equivalent to $R 3, P R 3$ is equivalent to $R 4, P R 9$ is equivalent to $R 5, P R 10$ is equivalent to $R 6$, and $P R 11$ is equivalent to $P R 8$. In other words,

$$
\left\langle\sigma_{i}, a_{i, j} \mid P R 1, P R 2, P R 4, P R 5, P R 6, P R 7, P R 8, P R 9, P R 10, \sigma_{i}=a_{i, i+1}^{k+1}, \sigma_{i}^{-1}=a_{i, i+1}^{k}\right\rangle
$$

Finally, for $1 \leq i<j \geq 2 g+b$ we have that

$$
\left\langle a_{i, j} \mid P R 1, P R 2, P R 4, P R 5, P R 6, P R 7, P R 8, P R 9, P R 10\right\rangle,
$$

which is the presentation of $H_{n}$.
As an application of Proposition 5.3, we can obtain generators for $B_{2 g+b}[2 p]$.
Corollary 5.4. For $k=(p-1) / 2$, the group $B_{2 g+b}[2 p]$ is normally generated by six types of elements:

$$
\begin{array}{r}
a_{i, j}^{p} \\
\left(a_{1,2} a_{1,3} a_{2,3}\right)^{2} \\
a_{1,2} a_{1,3} a_{2,3} C^{-1} \\
a_{1,2} a_{1,3} a_{1,4} a_{2,3} a_{2,4} a_{3,4} B a_{1,4}^{-1} B^{-1} \\
a_{i, i+1}^{k} a_{i+1, i+2}^{k} a_{i, i+1}^{k} a_{i+1, i+2}^{-k} a_{i, i+1}^{-k} a_{i+1, i+2}^{-k} \\
a_{j-1, j}^{k+1} a_{j-2, j-1}^{k+1} \ldots a_{i, i+1} a_{i+1, i+2}^{k} \ldots a_{j-1, j}^{k} a_{i, j}^{-1}
\end{array}
$$

Actually we can use Proposition 5.3 to find normal generators for any $B_{n}[m]$, where $m$ is either $2 p_{1} \ldots p_{k}$ or $4 p_{1} \ldots p_{k}$ and $p_{i} \geq 3$ are prime numbers.

## 6 Symmetric quotients of congruence subgroups

In this section we explore factor groups of congruence subgroups of braid groups. From Section 3 we know that $B_{n}[2] \cong P B_{n}$ and $B_{n} / B_{n}[2] \cong S_{n}$. In the next theorem we generalize the latter isomorphism.

Theorem 6.1. The quotient $B_{n}[p] / B_{n}[2 p]$ is isomorphic to $S_{n}$.
Before we proceed to the proof of Theorem 6.1, we will prove the following lemma.
Lemma 6.2. The groups $B_{n}[2 p]$ and $B_{n}[2] \cap B_{n}[p]$ are isomorphic.
Proof. It is obvious that $B_{n}[2 p]<B_{n}[2] \cap B_{n}[p]$. By Proposition 3.3 we have the decomposition $\mathrm{Sp}_{2 g}(\mathbb{Z} / 2 p)=\mathrm{Sp}_{2 g}(\mathbb{Z} / 2) \oplus \mathrm{Sp}_{2 g}(\mathbb{Z} / p)$. By the homomorphism $\rho: B_{n} \rightarrow \mathrm{Sp}_{2 g}(\mathbb{Z} / 2 p)$ we deduce that $\rho\left(B_{n}[2] \cap B_{n}[p]\right)$ is trivial. Hence $B_{n}[2] \cap B_{n}[p]<B_{n}[2 p]$.

Now we can prove the main theorem of the section.

Proof of Theorem 6.1. Denote by $s_{i}$ the transposition $i, i+1$, that is, the generators of $S_{n}$. We have the following presentation.

$$
\left.S_{n}=\left\langle s_{1}, \ldots, s_{n-1}\right| s_{i}^{2}=1, s_{i} s_{i+1} s_{i}=s_{i+1} s_{i} s_{i+1}, s_{i} s_{j}=s_{j} s_{i} \text { when }|i-j|>1\right\rangle .
$$

Consider the natural epimorphism $\tau: B_{n} \rightarrow S_{n}$ defined by $\tau\left(\sigma_{i}\right)=s_{i}$. Fix a prime number $p>2$; then the restriction $\tau: B_{n}[p] \rightarrow S_{n}$ is a surjective homomorphism as well. Indeed, we have that $\tau\left(\sigma_{i}^{p}\right)=s_{i}^{p}=s_{i}$, and for any other generator $g \in B_{n}[p]$ we have $\tau(g)=1$. Finally, $\operatorname{ker}(\tau)=B_{n}[2] \cap B_{n}[p]=B_{n}[2 p]$ by Lemma 6.2

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