EMBEDDABILITY OF RIGHT-ANGLED ARTIN GROUPS ON COMPLEMENTS OF TREES

EON-KYUNG LEE AND SANG-JIN LEE

ABSTRACT. For a finite simplicial graph Γ , let $A(\Gamma)$ denote the right-angled Artin group on Γ . Recently Kim and Koberda introduced the extension graph Γ^e for Γ , and established the Extension Graph Theorem: for finite simplicial graphs Γ_1 and Γ_2 if Γ_1 embeds into Γ_2^e as an induced subgraph then $A(\Gamma_1)$ embeds into $A(\Gamma_2)$. In this article we show that the converse of this theorem does not hold for the case Γ_1 is the complement of a tree and for the case Γ_2 is the complement of a path graph.

Keywords: right-angled Artin groups, extension graphs, embeddability. 2010 Mathematics Subject Classification: Primary 20F65; Secondary 05C25

1. INTRODUCTION

Throughout this article all graphs are assumed to be simplicial and undirected. Let Γ be a finite graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$.

For a subset $A \subset V(\Gamma)$, the subgraph Γ_1 of Γ with $V(\Gamma_1) = A$ and $E(\Gamma_1) = \{\{a, b\} \in E(\Gamma) : a, b \in A\}$ is called the subgraph of Γ induced by A or the induced subgraph of Γ on A. If a graph Γ_1 embeds into Γ as an induced subgraph, we write $\Gamma_1 \leq \Gamma$. The complement graph of Γ , denoted by $\overline{\Gamma}$, is the graph such that $V(\overline{\Gamma}) = V(\Gamma)$ and two vertices are adjacent in $\overline{\Gamma}$ if and only if they are not adjacent in Γ . If a group H embeds into a group G, we write $H \leq G$. For elements g, h of a group, let g^h and [g, h] denote the conjugate $h^{-1}gh$ and the commutator $g^{-1}h^{-1}gh$, respectively.

The right-angled Artin group $A(\Gamma)$ on Γ is defined by the presentation

$$A(\Gamma) = \langle v \in V(\Gamma) \mid [a, b] = 1 \text{ if } \{a, b\} \in E(\Gamma) \rangle.$$

It is well-known that two right-angled Artin groups $A(\Gamma_1)$ and $A(\Gamma_2)$ are isomorphic as groups if and only if Γ_1 and Γ_2 are isomorphic as graphs [Dro87] and that $\Gamma_1 \leq \Gamma_2$ implies $A(\Gamma_1) \leq A(\Gamma_2)$.

1.1. Embeddability between right-angled Artin groups. The following is a fundamental question for right-angled Artin groups [CSS08, KK13].

Date: July 3, 2018.

Question 1.1 (Embeddability Problem). Is there an algorithm to decide whether or not there exists an embedding between two given right-angled Artin groups?

Kim and Koberda [KK13] introduced the notion of extension graph Γ^e which is obtained from Γ through a combinatorial procedure, and developed the Extension Graph Theorem.

Definition 1.2 (Extension graph). For a finite graph Γ , the *extension graph* of Γ is the graph Γ^e such that the vertices of Γ^e are in one-to-one correspondence with the conjugates of vertices of Γ in $A(\Gamma)$, that is,

$$V(\Gamma^e) = \{ a^g \in A(\Gamma) : a \in V(\Gamma), g \in A(\Gamma) \}$$

and two vertices of Γ^e are adjacent if and only if they commute in $A(\Gamma)$, that is,

$$E(\Gamma^{e}) = \{ \{a^{g}, b^{h}\} : a^{g}, b^{h} \in V(\Gamma^{e}), \ [a^{g}, b^{h}] = 1 \text{ in } A(\Gamma) \}.$$

Extension graphs are usually infinite and locally infinite.

Theorem 1.3 (Extension Graph Theorem [KK13]). For finite graphs Γ_1 and Γ_2 , if $\Gamma_1 \leq \Gamma_2^e$ then $A(\Gamma_1) \leq A(\Gamma_2)$.

This theorem was a significant progress toward solving the Embeddability Problem. Recently, Casals-Ruiz showed the following.

Theorem 1.4 (Theorem 3.5 in [Cas15]). There exists an algorithm that given two finite graphs Γ_1 and Γ_2 decides whether or not Γ_1 embeds into Γ_2^e .

Due to Theorems 1.3 and 1.4, the Embeddability Problem is solvable for the class of right-angled Artin groups for which the converse of the Extension Graph Theorem holds. It is natural to ask for which graphs this converse holds.

Question 1.5 (Question 1.5 in [KK13]). For which graphs Γ_1 and Γ_2 , do we have $A(\Gamma_1) \leq A(\Gamma_2)$ if and only if $\Gamma_1 \leq \Gamma_2^e$?

If a graph Λ does not embed into a graph Γ as an induced subgraph, we say that Γ is Λ -free. Let P_n and C_n denote the path graph and the cycle on n vertices, respectively.

It is known that the converse of the Extension Graph Theorem holds, hence $A(\Gamma_1) \leq A(\Gamma_2)$ if and only if $\Gamma_1 \leq \Gamma_2^e$, for some important classes of graphs:

(i) Γ_1 is a forest [KK13, Corollary 1.9];

- (ii) Γ_2 is C_3 -free [KK13, Theorem 1.11];
- (iii) Γ_2 is C_4 -free and P_4 -free [CDK13, Theorem 5.1].

The converse of the Extension Graph Theorem does not always hold. Casals-Ruiz, Duncun and Kazachkov first gave an example [CDK13].

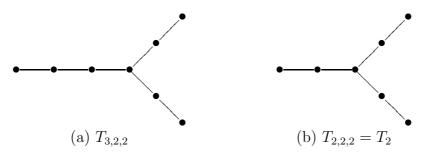


FIGURE 1. Tripods

1.2. Main result. The right-angled Artin groups on complements of trees form an important class of right-angled Artin groups because any right-angled Artin group embeds into a right-angled Artin group on the complement of a tree [KK15, LL16].

In this article we show that the converse of the Extension Graph Theorem does not hold for the case Γ_1 is the complement of a tree and for the case Γ_2 is the complement of a path graph.

Main Theorem (Corollary 4.5). There exist a finite tree T and a finite path graph P such that $A(\overline{T})$ embeds into $A(\overline{P})$ but \overline{T} does not embed into \overline{P}^e as an induced subgraph.

1.3. **Opposite convention.** From now on, we adopt the opposite of the usual convention for right-angled Artin groups as it is more appropriate for our approach:

$$G(\Gamma) = \langle v \in V(\Gamma) \mid [v_i, v_j] = 1 \text{ if } \{v_i, v_j\} \notin E(\Gamma) \rangle$$

Namely, $G(\Gamma) = A(\overline{\Gamma})$. For the extension graph we write $\Gamma^E = \overline{\overline{\Gamma}^e}$. The vertices of Γ^E coincide with the vertices of $\overline{\Gamma}^e$ and two vertices of Γ^E are adjacent if and only if they do not commute in $G(\Gamma)$, that is,

$$V(\Gamma^E) = \{ a^g \in G(\Gamma) : a \in V(\Gamma), g \in G(\Gamma) \},\$$

$$E(\Gamma^E) = \{ \{a^g, b^h\} : a^g, b^h \in V(\Gamma^E), [a^g, b^h] \neq 1 \text{ in } G(\Gamma) \}.$$

Note that for graphs Γ_1 and Γ_2 the following hold:

- (i) $\Gamma_1 \leqslant \Gamma_2$ if and only if $\overline{\Gamma}_1 \leqslant \overline{\Gamma}_2$;
- (ii) $\Gamma_1 \leqslant \Gamma_2^E$ if and only if $\overline{\Gamma}_1 \leqslant \overline{\Gamma}_2^e$.

Therefore the Extension Graph Theorem is equivalent to "For finite graphs Γ_1 and Γ_2 , if $\Gamma_1 \leq \Gamma_2^E$ then $G(\Gamma_1) \leq G(\Gamma_2)$."

1.4. Our approach toward proving Main Theorem. Let $T_{p,q,r}$ be the tripod whose three leaves contain p, q and r vertices, respectively. For instance, $T_{3,2,2}$ and $T_{2,2,2}$ are illustrated in Figure 1. Let T_2 denote the tripod $T_{2,2,2}$.

We obtain the Main Theorem by proving the following:

- (i) $G(T_2)$ embeds into $G(P_{22})$ (Theorem 3.3);
- (ii) T_2 does not embed into P_n^E as an induced subgraph for any *n* (Theorem 4.4).

The non-embeddability of T_2 into P_n^E gives rise to a question: Which trees T admit an embedding into P_n^E as an induced subgraph for some n? For this class of trees T, the right-angled Artin group $G(T) = A(\bar{T})$ embeds into $G(P_n) = A(\bar{P}_n)$ for some n. In Theorem 4.7, we obtain a characterization of such trees.

1.5. Universal family of graphs for right-angled Artin groups. Let us say that a collection \mathcal{G} of finite graphs is a *universal family of graphs for right-angled Artin groups* if for any right-angled Artin group $G(\Gamma_1)$ there exists $\Gamma_2 \in \mathcal{G}$ such that $G(\Gamma_1) \leq G(\Gamma_2)$. Kim and Koberda showed that the family of finite trees provides a universal family of graphs for right-angled Artin groups.

Theorem 1.6 ([KK15, LL16]). For any finite graph Γ , there exists a finite tree T such that $G(\Gamma) \leq G(T)$.

Using the fact that if Γ_1 is an edge-contraction of Γ_2 then $G(\Gamma_1) \leq G(\Gamma_2)$ [Kim08, KK13], we can see that the family of finite trees with degree ≤ 3 at each vertex is also universal [Kat16]. It would be interesting to find a universal family smaller than this. For instance, one can ask whether or not the family of path graphs is universal.

Question 1.7. Which right-angled Artin group embeds into $G(P_n)$ for some n?

Concerning the above question, Katayama proposed the following question [Kat16].

Question 1.8 (Question 5.2 in [Kat16]). Is it possible that $G(T_2) \leq G(P_n)$ for some n?

Theorem 3.3 gives an affirmative answer to the above question, and Theorem 4.7 gives a family of trees T with $G(T) \leq G(P_n)$.

We ask the same question as above for $T_{p,q,r}$: For which p,q,r, do we have $G(T_{p,q,r}) \leq G(P_n)$ for some n? It seems very hard to embed $G(T_{p,q,r})$ into $G(P_n)$ if p,q,r are large. We therefore propose the following:

Conjecture 1.9. If p, q, r are large enough, then $G(T_{p,q,r})$ does not embed into $G(P_n)$ for any n.

1.6. A remark on Theorem 3.14 in [Cas15]. Theorem 3.14 of [Cas15] claims the following: "For a forest F and a finite graph Γ , $G(F) \leq G(\Gamma)$ if and only if $F \leq \Gamma^{E*}$. In other words, it claims that the converse of the Extension Graph Theorem holds for complements of forests. This conflicts to our Main Theorem.

In the proof of Theorem 3.14 in [Cas15], the following argument is used. "For $1 \leq i \leq k$, let $g_i \in G(\Gamma)$ be a product of mutually commuting elements, i.e. $g_i = y_{i,1}y_{i,2}\cdots y_{i,r_i}$ such that $[y_{i,p}, y_{i,q}] = 1$ for all $1 \leq p < q \leq r_i$. Using commutator identities, the iterated commutator $[g_1, g_2, g_3, \ldots, g_k] = [\ldots [[g_1, g_2], g_3], \ldots, g_k]$ can be written as a product

$$\prod_{s=(j_1,\ldots,j_k),\ 1\leqslant j_i\leqslant r_i} [y_{1,j_1},y_{2,j_2},\ldots,y_{k,j_k}]^{g(s)}$$

for some elements $g(s) \in G(\Gamma)$." This is however not the case. For instance, let us denote the vertices of the path graph P_5 by x_1, \ldots, x_5 as follows.

$$x_1$$
 x_2 x_3 x_4 x_5

In $G(P_5)$, $[x_i, x_j] = 1$ if and only if $|i - j| \ge 2$. A direct computation shows

$$[x_2x_4, x_3, x_1, x_5] \neq 1.$$

Notice that $[x_2, x_3, x_1, x_5] = [x_4, x_3, x_1, x_5] = 1$. If the argument in [Cas15] were correct, then

$$x_2x_4, x_3, x_1, x_5] = [x_2, x_3, x_1, x_5]^g [x_4, x_3, x_1, x_5]^h$$

for some $g, h \in G(P_5)$, which results in $[x_2x_4, x_3, x_1, x_5] = 1$. It is a contradiction.

1.7. **Organization.** Section 2 reviews basic materials. Section 3 shows that $G(T_2)$ embeds into $G(P_{22})$. Section 4 shows that T_2 does not embed into P_n^E as an induced subgraph for any n.

2. Preliminaries

Let Γ be a finite graph. For a vertex $v \in V(\Gamma)$, the *link* of v in Γ is the set $\mathrm{Lk}_{\Gamma}(v) = \{ u \in V(\Gamma) \mid \{v, u\} \in E(\Gamma) \}$. We simply write $\mathrm{Lk}(v)$ for $\mathrm{Lk}_{\Gamma}(v)$ if Γ is clear from context. For $A \subset V(\Gamma)$, we denote by $\Gamma \setminus A$ the subgraph of Γ induced by $V(\Gamma) \setminus A$.

Each element in $V(\Gamma) \cup V(\Gamma)^{-1}$ is called a *letter*. An element in $G(\Gamma)$ can be expressed as a word, which is a finite product of letters. Abusing notation, we shall sometimes regard a word as the group element represented by that word. Let $w = a_1 \cdots a_k$ be a word in $G(\Gamma)$ where a_1, \ldots, a_k are letters. We say that w is *reduced* if any other word representing the same element in $G(\Gamma)$ as w has at least k letters.

For $g \in G(\Gamma)$, the support of g, denoted by $\operatorname{supp}(g)$, is the set of vertices v such that v or v^{-1} appears in a reduced word for g. It is known that $\operatorname{supp}(g)$ is well-defined.

Let w be a (possibly non-reduced) word in $G(\Gamma)$. A subword $v^{\pm 1}w_1v^{\mp 1}$ of w is called a *cancellation* of v in w if $\operatorname{supp}(w_1) \cap \operatorname{Lk}_{\Gamma}(v) = \emptyset$. If, furthermore, no letter in w_1 is equal to v or v^{-1} , it is called an *innermost cancellation* of v in w. It is known that w is reduced if and only if w has no innermost cancellation.

Let $\phi : \Gamma_2 \to \Gamma_1$ be a graph homomorphism between finite graphs, i.e. a function from $V(\Gamma_2)$ to $V(\Gamma_1)$ that maps adjacent vertices to adjacent vertices. Then ϕ induces a group homomorphism $\phi^* : G(\Gamma_1) \to G(\Gamma_2)$ defined by

$$\phi^*(v) = \prod_{v' \in \phi^{-1}(v)} v'$$

for $v \in V(\Gamma_1)$, where the product is defined to be the identity if $\phi^{-1}(v)$ is the empty set. Since ϕ is a graph homomorphism and since Γ_1 has no loops, no two vertices of $\phi^{-1}(v)$ are adjacent, hence the product is well-defined. Abusing notation, for a word w in $G(\Gamma_1)$, $\phi^*(w)$ denotes the word defined by the product. For this, we may fix a total order on $V(\Gamma_2)$ and write each product $\prod_{v' \in \phi^{-1}(v)} v'$ in the increasing order.

Definition 2.1. We say that $\phi : \Gamma_2 \to \Gamma_1$ is v'-surviving for $v' \in V(\Gamma_2)$ if for any reduced word w in $G(\Gamma_1)$, the word $\phi^*(w)$ has no innermost cancellation of v'.

The following lemma follows from well-known solutions to the word problem in right-angled Artin groups [Cha07].

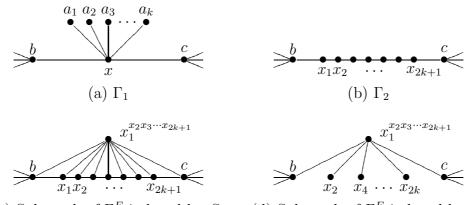
Lemma 2.2. Let $a \in V(\Gamma)$ and $w \in G(\Gamma)$. Then [a, w] = 1 if and only if $Lk(a) \cap supp(w) = \emptyset$, *i.e.* [a, c] = 1 for each $c \in supp(w)$.

Lemma 2.3. Let $w^{-1}bw$ be a reduced word for $b \in V(\Gamma)$ and $w \in G(\Gamma)$. Then

- (i) $\operatorname{supp}(w^{-1}bw) = \{b\} \cup \operatorname{supp}(w);$
- (ii) $\operatorname{supp}(w^{-1}bw)$ spans a connected subgraph of Γ .

Proof. (i) Notice that if w_1w_2 is a reduced word, then $\operatorname{supp}(w_1w_2) = \operatorname{supp}(w_1) \cup \operatorname{supp}(w_2)$ and that $\operatorname{supp}(w^{-1}) = \operatorname{supp}(w)$ for any word w. Since $w^{-1}bw$ is reduced, one has $\operatorname{supp}(w^{-1}bw) = \operatorname{supp}(w^{-1}) \cup \{b\} \cup \operatorname{supp}(w) = \{b\} \cup \operatorname{supp}(w)$.

(ii) Let Γ_0 be the subgraph of Γ induced by $\operatorname{supp}(w^{-1}bw)$. Assume that Γ_0 is not connected. Let Γ_1 be the component of Γ_0 containing b, and let $\Gamma_2 = \Gamma_0 \setminus \Gamma_1$. Then $w = w_2w_1$ for some nontrivial reduced words $w_1 \in G(\Gamma_1)$ and $w_2 \in G(\Gamma_2)$ because $[a_1, a_2] = 1$ for $a_1 \in V(\Gamma_1)$ and $a_2 \in V(\Gamma_2)$. Since each vertex of Γ_2 commutes with b, we have $w^{-1}bw = w_1^{-1}w_2^{-1}bw_2w_1 = w_1^{-1}bw_1$. This contradicts the hypothesis that $w^{-1}bw$ is reduced.



(c) Subgraph of Γ_2^E induced by S_1 (d) Subgraph of Γ_2^E induced by S_2

FIGURE 2. The graph Γ_1 embeds into Γ_2^E as an induced subgraph.

3. Two local moves on graphs

In this section, we propose two local moves on graphs that give rise to an embedding between right-angled Artin groups. Combining with a result in [LL16], we obtain $G(T_2) \leq G(P_{22})$.

Proposition 3.1. Let Γ_1 be a finite graph with a degree k + 2 vertex x for $k \ge 1$. Let $Lk(x) = \{a_1, \ldots, a_k\} \cup \{b, c\}$. Suppose each a_i has degree 1. See Figure 2(a). Let Γ_2 be the graph obtained from Γ_1 by deleting a_1, \ldots, a_k and then by replacing x with the path graph P_{2k+1} as in Figure 2(b). Then $\Gamma_1 \leq \Gamma_2^E$ and hence $G(\Gamma_1) \leq G(\Gamma_2)$.

Proof. By Lemmas 2.2 and 2.3, the element $x_1^{x_2x_3\cdots x_{2k+1}}$ in $G(\Gamma_2)$ commutes with $v \in V(\Gamma_2)$ if and only if $v \notin \{b, c, x_1, x_2, \dots, x_{2k+1}\}$. Hence the subgraph of Γ_2^E induced by $S_1 = V(\Gamma_2) \cup \{x_1^{x_2x_3\cdots x_{2k+1}}\}$ is as in Figure 2(c). The subgraph of Γ_2^E induced by $S_2 = S_1 \setminus \{x_1, x_3, \dots, x_{2k+1}\}$ is as in Figure 2(d), which is isomorphic to Γ_1 . Therefore $\Gamma_1 \leqslant \Gamma_2^E$ and hence $G(\Gamma_1) \leqslant G(\Gamma_2)$.

Proposition 3.2. Let Γ_1 be a finite graph containing a degree 3 vertex x with $Lk(x) = \{a, b, c\}$ as in Figure 3(a). Let Γ_2 be the graph obtained from Γ_1 by replacing the tripod centered at x with a 6-cycle as in Figure 3(b), where each vertex $v \in V(\Gamma_1) \setminus \{x\}$ is renamed as $v_1 \in V(\Gamma_2)$. Then $G(\Gamma_1) \leq G(\Gamma_2)$.

Proof. Let $\phi : \Gamma_2 \to \Gamma_1$ be the graph homomorphism defined by $\phi(v_1) = v$ for $v_1 \notin \{x_1, x_2, x_3\}$ and $\phi(x_i) = x$ for i = 1, 2, 3. Then $\phi^* : G(\Gamma_1) \to G(\Gamma_2)$ is the group homomorphism such that $\phi^*(x) = x_1 x_2 x_3$ and $\phi^*(v) = v_1$ for $v \neq x$. We will show that ϕ^* is injective.

Let
$$\Gamma'_1 = \Gamma_1 \setminus a$$
, $\Gamma'_2 = \Gamma_2 \setminus a_1$ and $\phi_1 = \phi|_{\Gamma'_2} : \Gamma'_2 \to \Gamma'_1$. See Figure 4.

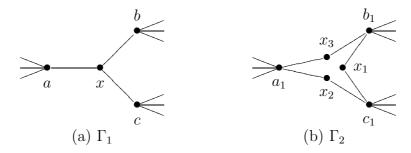


FIGURE 3. $G(\Gamma_1)$ embeds into $G(\Gamma_2)$.

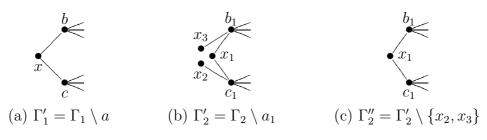


FIGURE 4. Graphs Γ'_1 , Γ'_2 and Γ''_2 .

Claim 1. ϕ_1 is v_1 -surviving for all $v_1 \in V(\Gamma'_2) \setminus \{x_2, x_3\}$. In particular, ϕ_1^* is injective.

Proof of Claim 1. Let $\Gamma_2'' = \Gamma_2' \setminus \{x_2, x_3\}$. Let $\iota : \Gamma_2'' \to \Gamma_2'$ be the inclusion. Then $\iota^* : G(\Gamma_2') \to G(\Gamma_2'')$ is an epimorphism such that $\iota^*(v_1) = v_1$ for all $v_1 \neq x_2, x_3$ and $\iota^*(x_2) = \iota^*(x_3) = 1$.

On the other hand, $\phi_1 \circ \iota : \Gamma_2'' \to \Gamma_1'$ is a graph isomorphism sending v_1 to vfor each $v_1 \in V(\Gamma_2'')$, hence $\iota^* \circ \phi_1^* : G(\Gamma_1') \to G(\Gamma_2'')$ is an isomorphism sending $w(x, b, c, \ldots)$ to $w(x_1, b_1, c_1, \ldots)$. In particular, if $w(x, b, c, \ldots)$ is a reduced word in $G(\Gamma_1')$, then $w(x_1, b_1, c_1, \ldots)$ is also a reduced word in $G(\Gamma_2'')$.

Assume that ϕ_1 is not v_1 -surviving for some $v_1 \in V(\Gamma'_2) \setminus \{x_2, x_3\}$. Then there exists a nontrivial reduced word w = w(x, b, c, ...) in $G(\Gamma'_1)$ such that the word

$$\phi_1^*(w) = w(x_1 x_2 x_3, b_1, c_1, \ldots)$$

has a cancellation of v_1 . This implies that

$$\iota^*(\phi_1^*(w)) = w(x_1, b_1, c_1, \ldots)$$

also has a cancellation of v_1 . This is a contradiction because $w(x_1, b_1, c_1, ...)$ is a reduced word in $G(\Gamma_2'')$. Therefore ϕ_1 is v_1 -surviving for all $v_1 \in V(\Gamma_2') \setminus \{x_2, x_3\}$.

If $w \in G(\Gamma'_1)$ is a nontrivial element, then $v \in \operatorname{supp}(w)$ for some $v \in V(\Gamma'_1)$, hence $v_1 \in \operatorname{supp}(\phi_1^*(w))$ because ϕ_1 is v_1 -surviving. This implies that ϕ_1^* is injective. \Box

Claim 2. For $w \in G(\Gamma'_1)$, if $x \in \operatorname{supp}(w)$ then either x_2 or x_3 belongs to $\operatorname{supp}(\phi_1^*(w))$.

Proof of Claim 2. We may assume that w = w(x, b, c, ...) is expressed as a reduced word as follows:

(1)
$$w = w_1 \cdot x^k \cdot t \cdot w_2 = w_1(b, c, \ldots) \cdot x^k \cdot t \cdot w_2(x, b, c, \ldots),$$

where $x \notin \text{supp}(w_1)$, $k \neq 0$ and $t \in \{b, c\}$. (The word w_1 is possibly empty.) This decomposition is obtained as follows.

Decompose the element w as $w = w_1u_1$ such that the word w_1u_1 is reduced and $x \notin \operatorname{supp}(w_1)$. We may assume that w_1 has the largest word length among all such decompositions. Then u_1 must start with x or x^{-1} . Let $u_1 = x^k u_2$ for $k \neq 0$. Take |k| as large as possible. Then u_2 cannot start with x or x^{-1} . Moreover, u_2 cannot start with a letter y such that [y, x] = 1, for otherwise we can make w_1 longer. Therefore either $u_2 = 1$ or u_2 starts with b or c. If $u_2 = 1$, then $w_1(b_1, c_1, \ldots) x_1^k x_2^k x_3^k$ is a reduced word representing $\phi^*(w) = \phi^*(w_1(b, c, \ldots)x^k)$, hence $x_2, x_3 \in \operatorname{supp}(\phi^*(w))$. If u_2 starts with b or c, then we have the desired decomposition of w as Eq. (1).

Without loss of generality, we may assume t = c. Then

$$w = w_1 \cdot x^k \cdot c \cdot w_2 = w_1(b, c, \ldots) \cdot x^k \cdot c \cdot w_2(x, b, c, \ldots),$$

$$\phi_1^*(w) = \phi_1^*(w_1) \cdot (x_1 x_2 x_3)^k \cdot c_1 \cdot \phi_1^*(w_2)$$

$$= w_1(b_1, c_1, \ldots) \cdot x_1^k x_2^k \cdot c_1 \cdot x_3^k \cdot \phi_1^*(w_2).$$

Let w_3 be a reduced word in $G(\Gamma'_2)$ representing $x_3^k \cdot \phi_1^*(w_2)$. Then

$$\phi_1^*(w) = w_1(b_1, c_1, \ldots) x_1^k x_2^k c_1 w_3.$$

Let w' be the word $w_1(b_1, c_1, \ldots) x_1^k x_2^k c_1 w_3$ in the above. Assume that w' has a cancellation of x_2 . Since w_3 is reduced, the cancellation must occur between $x_2^{\pm 1}$ in x_2^k and $x_2^{\pm 1}$ in w_3 , hence w' has a subword

$$x_2^{\pm 1} c_1 w_4 x_2^{\mp 1},$$

where $w_4 x_2^{\pm 1}$ is an initial subword of w_3 and $\operatorname{supp}(c_1 w_4) \cap \operatorname{Lk}(x_2) = \emptyset$. Since ϕ_1 is c_1 -surviving by Claim 1, we have $c_1 \in \operatorname{supp}(c_1 w_4)$. Since $c_1 \in \operatorname{Lk}(x_2)$, this contradicts $\operatorname{supp}(c_1 w_4) \cap \operatorname{Lk}(x_2) = \emptyset$. Therefore w' has no cancellation of x_2 , hence $x_2 \in \operatorname{supp}(w') = \operatorname{supp}(\phi_1^*(w))$.

Claim 3. ϕ is a_1 -surviving.

Proof of Claim 3. Assume that ϕ is not a_1 -surviving. Then there exists a reduced word w in $G(\Gamma_1)$ such that $\phi^*(w)$ has an innermost cancellation of a_1 . Hence the word w has a subword

$$a^{\pm 1}w_1a^{\mp 1}$$

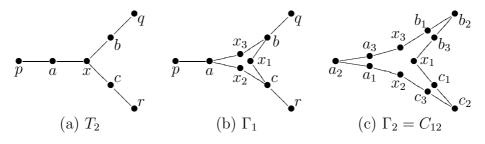


FIGURE 5. Graphs T_2 , Γ_1 and Γ_2 : $G(T_2) \leq G(\Gamma_1) \leq G(\Gamma_2)$.

such that w_1 is a nontrivial reduced word in $G(\Gamma_1 \setminus a) = G(\Gamma'_1)$ with $\operatorname{supp}(\phi^*(w_1)) \cap \operatorname{Lk}(a_1) = \emptyset$. Notice that $\phi^*(w_1) = \phi_1^*(w_1)$ and that $\operatorname{supp}(w_1) \cap \operatorname{Lk}(a) \neq \emptyset$ because w is a reduced word.

Assume that there exists $v \neq x$ in $\operatorname{supp}(w_1) \cap \operatorname{Lk}(a)$. Then $v_1 \in \operatorname{supp}(\phi^*(w_1)) = \operatorname{supp}(\phi_1^*(w_1))$ because ϕ_1 is v_1 -surviving by Claim 1 and w_1 is a word in $G(\Gamma'_1)$ with $v \in \operatorname{supp}(w_1)$. Since $v_1 \in \operatorname{supp}(\phi^*(w_1)) \cap \operatorname{Lk}(a_1)$, this contradicts $\operatorname{supp}(\phi^*(w_1)) \cap \operatorname{Lk}(a_1) = \emptyset$.

Therefore $\operatorname{supp}(w_1) \cap \operatorname{Lk}(a) = \{x\}$. Since $\operatorname{supp}(\phi^*(w_1)) \cap \operatorname{Lk}(a_1) = \emptyset$, the set $\operatorname{supp}(\phi^*(w_1))$ contains neither x_2 nor x_3 . This is impossible by Claim 2. Therefore ϕ is a_1 -surviving.

Let w be a nontrivial reduced word in $G(\Gamma_1)$. If $a \in \operatorname{supp}(w)$, then $\phi^*(w) \neq 1$ because ϕ is a_1 -surviving by Claim 3. If $a \notin \operatorname{supp}(w)$, then w is a nontrivial reduced word in $G(\Gamma'_1)$, hence $\phi^*(w) = \phi_1^*(w)$ is nontrivial by Claim 1. Therefore ϕ^* is injective.

Theorem 3.3. $G(T_2)$ embeds into $G(P_{22})$.

Proof. Let Γ_1 and Γ_2 be the graphs in Figure 5(b,c). Then $G(T_2)$ embeds into $G(\Gamma_1)$ by Proposition 3.2 and $G(\Gamma_1)$ embeds into $G(\Gamma_2)$ by Proposition 3.1. Notice that Γ_2 is the cycle C_{12} . By Theorem 3.16 in [LL16], $G(C_m)$ embeds into $G(P_{2m-2})$ for all $m \ge 3$. In particular, $G(C_{12})$ embeds into $G(P_{22})$. Consequently, $G(T_2)$ embeds into $G(P_{22})$.

4. Embeddability into extension graphs

In this section, we show that $T_2 \not\leq P_n^E$ for any n, and then give a characterization of trees that embeds into P_n^E for some n as an induced subgraph. Let Γ be a finite graph.

Lemma 4.1. Let $A \subset V(\Gamma)$ and $b^w \in V(\Gamma^E)$, where $b \in V(\Gamma)$ and $w \in G(\Gamma)$. Suppose that there is no edge between b^w and the vertices in A. Then there is an

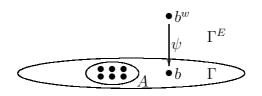


FIGURE 6. An inner automorphism ψ sends b^w into Γ fixing A pointwise.

inner automorphism ψ of $G(\Gamma)$ such that $\psi(a) = a$ for all $a \in A$ and $\psi(b^w) = b$. (See Figure 6.)

Proof. We may assume that $w^{-1}bw$ is reduced. Let $a \in A$. As $[b^w, a] = 1$, a commutes with each element of $\operatorname{supp}(b^w) = \{b\} \cup \operatorname{supp}(w)$, hence a commutes with w. Let ψ be the inner automorphism of $G(\Gamma)$ sending $g \in G(\Gamma)$ to wgw^{-1} , then $\psi(a) = a$ for all $a \in A$ and $\psi(b^w) = b$.

For a graph Λ , a subset A of $V(\Lambda)$ is called an *independent set* if there is no edge between any two vertices in A. Any independent subset A of $V(\Gamma^E)$ is a finite set because $G(\Gamma)$ contains a free abelian subgroup of rank |A| by the Extension Graph Theorem and because the maximum rank of a free abelian subgroup of $G(\Gamma)$ is the size of the largest independent subset of $V(\Gamma)$.

The following corollary shows that an independent subset of $V(\Gamma^E)$ is a conjugate of an independent subset of $V(\Gamma)$ by an element of $G(\Gamma)$. Similar arguments were used in [KK13] and [CDK13].

Corollary 4.2. Let $A \subset V(\Gamma^E)$ be an independent set. Then there is an inner automorphism ψ of $G(\Gamma)$ such that $\psi(A) \subset V(\Gamma)$.

Proof. Let $A = \{v_1, \ldots, v_m\} \subset V(\Gamma^E)$. By Lemma 4.1, if $v_1, \ldots, v_{k-1} \in V(\Gamma)$ for $1 \leq k \leq m$, then there exists an inner automorphism ψ of $G(\Gamma)$ such that $\psi(v_j) = v_j$ for $1 \leq j \leq k-1$ and $\psi(v_k) \in V(\Gamma)$. Since the composition of inner automorphisms is also an inner automorphism, we are done by using induction on |A|.

Lemma 4.3. Let $\{x, p, q\}$ be an independent subset of $V(P_n)$ for some $n \ge 5$ such that p lies between x and q. Let $b^w \in V(P_n^E)$ with $[b^w, p] = 1$ in $G(P_n)$, where $b \in V(P_n)$ and $w \in G(P_n)$. Then either $[b^w, x] = 1$ or $[b^w, q] = 1$.

Proof. The graph $P_n \setminus \text{Lk}(p)$ has three path components, say $P_n \setminus \text{Lk}(p) = \Gamma_1 \cup \{p\} \cup \Gamma_2$, where Γ_1 (resp. Γ_2) is the path component containing x (resp. q). Note that two vertices from distinct components commute with each other in $G(P_n)$.

As $[b^w, p] = 1$, one has $\operatorname{supp}(b^w) \cap \operatorname{Lk}(p) = \emptyset$ by Lemma 2.2. Since $\operatorname{supp}(b^w)$ spans a connected subgraph of P_n by Lemma 2.3, $\operatorname{supp}(b^w)$ is contained in one of Γ_1 , Γ_2 and $\{p\}$. Therefore b^w commutes with either x or q.

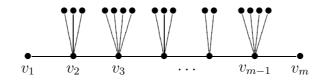


FIGURE 7. Hairy path graph

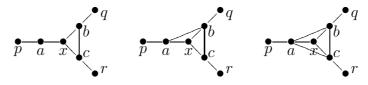
Theorem 4.4. The tripod T_2 does not embed into P_n^E as an induced subgraph for any n.

Proof. Assume $T_2 \leq P_n^E$ for some n. Let $\phi: T_2 \to P_n^E$ be the embedding. Label the vertices of T_2 as in Figure 5(a). Let v_x, v_p, v_q, v_r denote the images of x, p, q, r under ϕ , respectively. Since $\{x, p, q, r\}$ is an independent subset of $V(T_2)$, $\{v_x, v_p, v_q, v_r\}$ is also an independent subset of $V(P_n^E)$. By Corollary 4.2, we may assume that $\{v_x, v_p, v_q, v_r\} \subset V(P_n)$.

Since P_n is a path graph, at least two of v_p, v_q, v_r are in the same component of $P_n \setminus v_x$. Without loss of generality, we may assume that v_p and v_q are in the same component. Moreover, we may assume that v_x, v_p, v_q lie in P_n in this order.

For the vertex b of T_2 in Figure 5(a), let $\phi(b) = v_b^{w_b}$, where $v_b \in V(P_n)$ and $w_b \in G(P_n)$. Since [b, p] = 1, we have $[v_b^{w_b}, v_p] = 1$. By Lemma 4.3, either $[v_b^{w_b}, v_x] = 1$ or $[v_b^{w_b}, v_q] = 1$. This contradicts that $[b, x] \neq 1$ and $[b, q] \neq 1$. Therefore $T_2 \notin P_n^E$. \Box

The proof of Theorem 4.4 uses only the following properties of the tripod T_2 : in Figure 5(a), (i) $\{x, p, q, r\}$ is an independent subset of $V(T_2)$; (ii) the vertex *a* (resp. *b*, *c*) is adjacent to neither *q* nor *r* (resp. neither *p* nor *r*, neither *p* nor *q*). Therefore, by the same proof of Theorem 4.4, none of the following graphs embeds into P_n^E as an induced subgraph for any *n*.



By the above theorem together with Theorem 3.3, we obtain the following.

Corollary 4.5. There exist a finite tree T and a finite path graph P such that G(T) embeds into G(P) but T does not embed into P^E as an induced subgraph.

Definition 4.6. A finite tree T is called a *hairy path graph* if T contains a path graph P_m as an induced subgraph such that each vertex of $V(T) \setminus V(P_m)$ is adjacent to a vertex of P_m as in Figure 7.

Let T be the hairy path graph in Figure 8(a). Label the vertices of the path graph P_{10} as in Figure 8(b). By the same argument in the proof of Proposition 3.1, the

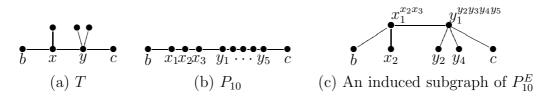


FIGURE 8. T embeds into P_{10}^E as an induced subgraph.

subgraph of P_{10}^E induced by $S = \{x_1^{x_2x_3}, y_1^{y_2\cdots y_5}, b, c, x_2, y_2, y_4\}$ is isomorphic to T as in Figure 8(c). Therefore T embeds into P_{10}^E as an induced subgraph.

Using Theorem 4.4, we obtain a characterization of trees that embeds into P_n^E as an induced subgraph.

Theorem 4.7. For a finite tree T, the following are equivalent.

- (i) $T \leq P_n^E$ for some n. (ii) $T_2 \leq T$.
- (iii) T is a hairy path graph.

Proof. (i) \Rightarrow (ii) It follows from Theorem 4.4.

(ii) \Rightarrow (iii) Let P_m be a longest path graph among induced subgraphs of T. Let $V(P_m) = \{v_1, \ldots, v_m\}$ such that v_i and v_{i+1} are adjacent for $i = 1, \ldots, m-1$. Let $v \in V(T) \setminus V(P_m)$. We will show that v is adjacent to v_i for some $2 \leq i \leq m-1$, hence T is a hairy path graph.

Since T is a tree, there exists a unique $v_i \in V(P_m)$ that is nearest to v. Since P_m is longest, $i \notin \{1, m\}$. If $i \in \{2, m-1\}$, then v must be adjacent to v_i because P_m is longest. If $3 \leq i \leq m-2$, then v must be adjacent to v_i because $T_2 \notin T$.

(iii) \Rightarrow (i) Let T be a hairy path graph containing P_m as a longest induced path subgraph as in Figure 7. Suppose each $v_i \in V(P_m)$ for $i = 2, \ldots, m-1$ is joined to k_i vertices in $V(T) \setminus V(P_m)$. Applying the argument in Proposition 3.1 (as in the discussion with the graphs in Figure 8 where m = 4, $k_2 = 1$ and $k_3 = 2$), we can see that $T \leq P_{m+2k}^E$, where $k = k_2 + \cdots + k_{m-1}$.

Acknowledgements

The first author was partially supported by NRF-2015R1C1A2A01051589. The second author was partially supported by NRF-2015R1D1A1A01056723. This paper was written as part of Konkuk University's research support program for its faculty on sabbatical leave in 2017.

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