On the intersection of tame subgroups in groups acting on

trees

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Abstract

Let G be a group acting on a tree T with finite edge stabilizers of bounded order. We provide, in some very interesting cases, upper bounds for the complexity of the intersection $H \cap K$ of two tame subgroups H and K of G in terms of the complexities of H and K. In particular, we obtain bounds for the Kurosh rank $Kr(H \cap K)$ of the intersection in terms of Kurosh ranks Kr(H) and Kr(K), in the case where H and K act freely on the edges of T.

1 Introduction

In 1954, Howson [9] showed that the intersection of two finitely generated subgroups Hand K of a free group F is also finitely generated and provided an upper bound for the rank $r(H \cap K)$ of $H \cap K$ in terms of r(H) and r(K). The Hanna Neumann conjecture, proved independently by Friedman [8] and Mineyev [12] in 2011, says that $\overline{r}(H \cap K) \leq \overline{r}(H)\overline{r}(K)$, where $\overline{r}(A) = \max\{0, r(A) - 1\}$ is the reduced rank of a free group A.

For free products the situation is analogous. Let Γ be a group. The Kurosh rank, denoted $Kr(\Gamma)$, of a free product decomposition $\Gamma = *_{i \in I}G_i$ of Γ is defined to be the number of (non-trivial) factors G_i . By the Kurosh subgroup theorem, any subgroup Hof Γ inherits a free product decomposition $H = *_{j \in J}H_j * F$, where each H_j is nontrivial and conjugate to a subgroup of a free factor of Γ and F is a free group. The (subgroup) Kurosh rank of H of Γ with respect to the above splitting of Γ , is the sum |J| + r(F), which we again denote by Kr(H). The reduced Kurosh rank of H is defined to be $\overline{Kr}(H) = \max\{0, Kr(H) - 1\}$.

Free products also have the Howson property, in the following sense: if H, K are subgroups of Γ of finite Kurosh rank, then $H \cap K$ also has finite rank (see [15, Theorem 2.13 (1)] for a proof). In [11], Ivanov proved that if Γ is torsion free, then $\overline{Kr}(H \cap K) \leq 2\overline{Kr}(H)\overline{Kr}(K)$. It is shown in [1], that if Γ is right-orderable, then the coefficient 2 can be replaced by 1. The problem of finding bounds for the "rank" of the intersection of subgroups in free products and more generally in groups satisfying the Howson property has also been considered in [14, 4, 10, 6, 7, 16, 17, 2].

In this paper, we obtain, under appropriate hypotheses, bounds for the complexity of the intersection of tame subgroups in groups acting on trees with finite edge stabilizers.

Let G be a group acting on a (simplicial) tree T without inversions. A vertex v of T is called (G-) degenerate if $G_v = G_e$ for some edge e incident to v. The corresponding vertex $[v]_G$ of the quotient graph T/G is also called degenerate. Let H be a subgroup of G. We denote by r(T/H) the rank of the fundamental group of T/H and by $V_{ndeg}(T/H)$ the set of H-non-degenerate vertices of T/H. The complexity $C_T(H)$ of H with respect to T is defined to be the sum $C_T(H) = r(T/H) + |V_{ndeg}(T/H)| \in [0, \infty]$, if H contains hyperbolic elements, and 1 otherwise. The reduced complexity of H with respect to T, is defined as $\overline{C}_T(H) = max\{C_T(H) - 1, 0\}$. The subgroup H of G is called tame if either H fixes a vertex, or H contains a hyperbolic element and the quotient graph T_H/H is finite, where T_H is the unique minimal H-invariant subtree of T. By [15, Theorem 2.13], if each edge stabilizer is finite, then the intersection of two tame subgroups H, K of G is again tame. In the case where $H \cap K$ fixes a vertex, we obviously have $\overline{C}_T(H \cap K) \leq \overline{C}_T(H) \cdot \overline{C}_T(K)$.

Finitely generated subgroups are examples of tame subgroups. In the case of free products, finite Kurosh rank implies tameness (see Lemma 2.3) and the complexity of a non-trivial subgroup is exactly its Kurosh rank (see section 2 for more details). Our first main result is the following.

Theorem 3.3. Let G be a group acting on a tree T with finite quotient and finite stabilizers of edges and let H, K be tame subgroups of G such that $H \cap K$ does not fix a vertex of T.

1. If T_H/H and T_K/K do not contain degenerate vertices of valence two, then

$$\overline{C}_T(H \cap K) \le \left(6NM + 12(M-1)N\right) \cdot \overline{C}_T(H) \cdot \overline{C}_T(K),$$

where $N = \max\{|G_x \cap HK| : x \in ET\}$ and $M = \max\{M_H, M_K\} \le \max\{|G_x| : x \in ET\}.$

2. Suppose H and K satisfy the following property: for each H-degenerate (resp. Kdegenerate) vertex v of T, the stabilizer H_v (resp. K_v) stabilizes each edge in the star of v. Then

$$\overline{C}_T(H \cap K) \le 6N \cdot \overline{C}_T(H) \cdot \overline{C}_T(K).$$

In particular, if H, K act freely on the edges of T, then

$$\overline{K}_T(H \cap K) \le 6N \cdot \overline{K}_T(H) \cdot \overline{K}_T(K).$$

In the special case where both H and K act freely on T, the above inequality was proved by Zakharov in [16].

Now let $G = *_A G_i * F$ be the free product of the amalgamated free product of G_i 's with a finite amalgamated subgroup A and F, such that A is normal in each G_i . Following Dicks and Ivanov [6], we define $a_3(G_i/A) = \min \{ |\Gamma| : \Gamma \text{ is a subgroup of } G_i/A \text{ with } |\Gamma| \ge 3 \}$ and $\theta(G_i/A) = \left\{ \frac{a_3(G_i/A)}{a_3(G_i/A)-2} \right\} \in [1,3]$, where $\frac{\infty}{\infty-2} := 1$.

We represent G as the fundamental group of a graph of groups (\mathcal{G}, Ψ) , where Ψ is the wedge of copies of [0, 1] (one copy for each factor G_i) and a bouquet of circles (one for each free generator of F). To each copy of [0, 1] and to the wedge point we associate the group A, and to each circle we associate the trivial group. To each of the remaining vertices we associate a factor G_i . Let T be the corresponding universal tree.

Theorem 3.6. Let $G = *_A G_i * F$ be the free product of the amalgamated free product of G_i 's with a finite amalgamated subgroup A and F, such that A is normal in each G_i . We consider the natural action of G on T defined above. Suppose that H and K are tame subgroups (with respect to T) of G which act freely on the edges of T. Then $H \cap K$ is tame and

$$\overline{K}_T(H \cap K) \le 2 \cdot \theta \cdot N \cdot \overline{K}_T(H) \cdot \overline{K}_T(K) \le 2 \cdot \theta \cdot |A| \cdot \overline{K}_T(H) \cdot \overline{K}_T(K) + \frac{1}{2} \cdot \frac{1}$$

where $\theta = \max\{\theta(G_i/A) : i \in I\}$ and $N = \max\{|gAg^{-1} \cap HK| : g \in G\}.$

As an immediate corollary we obtain the main result of [11] mentioned above.

It should be noted that the arguments in the proof of Theorem 3.6, work in a slightly more general setting as well. Thus, with essentially the same proof, we obtain Theorem 3.9 (see also Remark 3.7): If H, K are tame subgroups of a free product $*_A G_i$ with a finite and normal amalgamated subgroup A, then $\overline{C}_T(H \cap K) \leq 2 \cdot \theta \cdot |A \cap HK| \cdot \overline{C}_T(H) \cdot \overline{C}_T(K)$, where $\theta = \max\{\theta(G_i/A) : i \in I\}$ and T is defined as above for F = 1.

After posting the first version of this paper on the arXiv, the authors learned from A. Zakharov that he, in collaboration with S. Ivanov, had also recently obtained (unpublished) upper bounds for the Kurosh rank of the intersection of free product subgroups in groups acting on trees with finite edge stabilizers.

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2 Preliminaries

To fix our notation, we first recall the definition of a graph.

Definition 2.1. A graph X consists of a (nonempty) set of vertices VX, a set of edges EX, a fixed-point free involution $^{-1}: EX \to EX$ $(e \mapsto e^{-1})$ and a map $i: EX \to VX$. The vertex i(e) is called the *initial* vertex of the edge e. The terminal vertex t(e) of e is defined by $t(e) = i(e^{-1})$.

Throughout, let G be a group acting on a (simplicial) tree T (without inversions, i.e. $ge \neq e^{-1}$ for any $g \in G$ and $e \in EX$). By Bass-Serre theory, for which we refer to [5, 13], this is equivalent to saying that G is the fundamental group of the corresponding graph of groups $(\mathcal{G}, T/G)$. If $x \in T$, we denote by $[x]_G$ the G-orbit of x and by G_x its stabilizer. An element $g \in G$ is *elliptic* if it fixes a vertex of T and hyperbolic otherwise. If H is a subgroup of G containing a hyperbolic element, then there is a unique minimal H-invariant subtree T_H which is the union of the axes of the hyperbolic elements of H.

We recall that a subgroup H of G is called *tame* if either H fixes a vertex, or H contains a hyperbolic element and the quotient graph T_H/H is finite. By [15, Prop. 2.2], the subtree T_H is a "core" for the action of H on T in the sense that $r(T/H) + |V_{ndeg}(T/H)| =$ $r(T_H/H) + |V_{ndeg}(T_H/H)|$, i.e. $C_T(H) = C_{T_H}(H)$. From this it follows that the complexity of a tame subgroup is finite.

Finitely generated subgroups of G are examples of tame subgroups, since a finitely generated group Γ acting by isometries on T, either fixes a point of T or else contains a hyperbolic isometry and the quotient graph T_{Γ}/Γ is finite.

Remark 2.2. We note that if the G-stabilizer of each edge is finite and there is a bound on their orders, then any subgroup of G consisting of elliptic elements fixes a vertex of T([15, Lem. 2.5]).

If we restrict attention to subgroups H of G that act edge-freely on T, then the Kurosh rank $K_T(H)$ of H (with respect T) is defined to be the complexity $C_T(H)$ of H.

Let $\Gamma = *_{i \in I} G_i$ be a free product and H a subgroup of Γ . By the Kurosh subgroup theorem, $H = *_{i \in I, g_i} (H \cap g_i G_i g_i^{-1}) * F$, where for each i, g_i ranges over a set of double coset representatives in $G_i \setminus \Gamma / H$ and F is a free group intersecting each conjugate $gG_i g^{-1}$ trivially. The (subgroup) Kurosh rank of H with respect the above free product decomposition of Γ , denoted by Kr(H), is the sum $|\Lambda| + rank(F)$, where $|\Lambda|$ is the number of all non-trivial factors $H \cap g_i G_i g_i^{-1}$. Note that the Kurosh rank of Γ is the number of non-trivial factors G_i .

It is not difficult to verify that the numbers $|\Lambda|$, rank(F) depend only on H and the given free product decomposition of Γ . In fact, if T is any Γ -tree corresponding to the given decomposition of Γ , then the Kurosh rank of H with respect to $\Gamma = *_{i \in I} G_i$ is equal to the Kurosh rank $K_T(H)$ of the associated free product decomposition of H coming from the action of H on T. Thus, if H is non-trivial, then $Kr(H) = K_T(H) = C_T(H)$. **Lemma 2.3.** Let G be a group acting on a tree T and H a subgroup of G that act edgefreely on T. If $K_T(H) < \infty$, then H is tame.

Proof. It suffices to consider the case when H contains a hyperbolic element. Let π : $T \to T/H$ be the natural projection given by $\pi(x) = [x]_H$. Since $K_T(H) < \infty$, there are finitely many vertices v_1, \ldots, v_n of T/H with non-trivial group and finitely many edges e_1, \ldots, e_m of T/H such that $X = T/H \setminus \{e_1, \ldots, e_m\}$ is a maximal tree of T/H. Let Y be the finite subgraph of T/H consisting of $\{e_1, \ldots, e_m\}$ and all geodesics in X between endpoints of the $e'_i s$ and v_1, \ldots, v_n . We claim that $\pi^{-1}(Y)$ is connected. To see this, let $p = x_1 \cdots x_k$ be a reduced path connecting vertices of $\pi^{-1}(Y)$ such that no edge of p lies in $\pi^{-1}(Y)$. Then $\pi(p)$ is contained in the complement $T/H \setminus Y$ of Y. Since Y contains the edges e_1, \ldots, e_m , each component C of $T/H \setminus Y$ is a tree, and it is not difficult to see that C intersects $T/H \setminus Y$ in only one vertex. It follows that there is an index i such that $\pi(x_i) = \pi(x_{i+1})^{-1}$. This means that $hx_i = x_{i+1}^{-1}$ for some $h \in H$ and hence h fixes the initial vertex v of x_i . From the construction of Y, v is degenerate and therefore h = 1, which contradicts the choice of p.

Thus, $\pi^{-1}(Y)$ is a connected *H*-invariant subgraph of *T*. It follows that $T_H \subseteq \pi^{-1}(Y)$. We conclude that T_H/H is finite, being a subgraph of *Y*.

3 Proofs of the main results

Let Y be a graph and v a vertex of Y. The star of v, denoted $Star_Y(v)$, is the set of edges of Y with initial vertex v, i.e. $Star_Y(v) = \{e \in EY | i(e) = v\}$. The valence or degree of v in Y, denoted $\deg_Y(v)$, is the number of edges in the star of v.

Lemma 3.1. Let G be a group acting on a tree T, let H be a tame subgroup of G containing hyperbolic elements and let \widetilde{X} be the graph obtained from $X = T_H/H$ by attaching a loop at each H-non-degenerate vertex. Then

$$\overline{C}_T(H) = \overline{r}(\widetilde{X}) = \frac{1}{2} \sum \left(\deg_{\widetilde{X}}([v]_H) - 2 \right),$$

where the sum is taken over all vertices $[v]_H$ of \widetilde{X} .

Proof. The reduced rank of a graph is equal to the number of its (geometric-oriented) edges minus the number of its vertices. The minimality of T_H implies that each vertex of X of valence one is H-non-degenerate. Therefore, every vertex of \widetilde{X} has valence at least two. Now an easy calculation shows that the sum $\sum (\deg_{\widetilde{X}}([v]_H) - 2)$, over all vertices $[v]_H$ of \widetilde{X} , is equal to $2\overline{r}(\widetilde{X})$. By construction of \widetilde{X} , we have $\overline{r}(\widetilde{X}) = \overline{C}_T(H)$ which completes the proof.

Lemma 3.2. Let G be a group acting on a tree T and let A and B be subgroups of G such that $A \subseteq B$. Suppose that A and B contain hyperbolic elements and that v is a vertex of T_B . We consider the graph map $\pi_B : T_A/A \longrightarrow T_B/B$ given by $\pi([x]_A) = [x]_B$.

- 1. $|Star([v]_A)| \leq |G_v \cap B| \cdot |Star([v]_B)|$ (provided that they are finite).
- 2. If, moreover, B_v is B-degenerate and stabilizes each edge in $Star_{T_B}(v)$, then the restriction $\pi_B : Star([v]_A) \longrightarrow Star([v]_B)$ is an embedding.

Proof. Suppose that $[e_1]_A$ and $[e_2]_A$ are two edges in the star of $[v]_A$ with $\pi([e_1]_A) = \pi([e_2]_A)$. Then there are $a_1, a_2 \in A$ and $b \in B$ such that $i(e_1) = a_1v$, $i(e_2) = a_2v$ and $e_1 = be_2$. It follows that $i(e_1) = bi(e_2)$ and thus $a_1v = ba_2v$. Hence $a_1^{-1}ba_2 \in G_v \cap B = B_v$. Now, if $[x]_A$ is an edge in the star of $[v]_A$ with $\pi([x]_A) = \pi([e_1]_A) = \pi([e_2]_A)$, then as before $i(x) = a_xv$ and $x = b_xe_2$ for some $a_x \in A$ and $b_x \in B$. If we assume further that $a_1^{-1}ba_2 = a_x^{-1}b_xa_2$, then $a_1^{-1}b = a_x^{-1}b_x$ and so $[e_1]_A = [be_2]_A = [a_1a_x^{-1}b_xe_2]_A = [b_xe_2]_A = [x]_A$. This means that each fiber of the restriction (on stars) has at most $|G_v \cap B|$ elements, and the first assertion follows.

Now, if B_v stabilizes each edge in $Star_{T_B}(v)$, then $a_1^{-1}ba_2$ stabilizes $a_2^{-1}e_2$ and therefore $[e_1]_A = [be_2]_A = [a_1a_2^{-1}e_2]_A = [e_2]_A$.

In view of this lemma, we define $M_B := \max\{|G_v \cap B| : v \text{ is a B-degenerate vertex of T }\}$. The following is our first main result.

Theorem 3.3. Let G be a group acting on a tree T with finite quotient and finite stabilizers of edges and let H, K be tame subgroups of G such that $H \cap K$ does not fix a vertex of T.

1. If T_H/H and T_K/K do not contain degenerate vertices of valence two, then

$$\overline{C}_T(H \cap K) \le \left(6NM + 12(M-1)N\right) \cdot \overline{C}_T(H) \cdot \overline{C}_T(K),$$

where $N = \max\{|G_x \cap HK| : x \in ET\}$ and $M = \max\{M_H, M_K\} \le \max\{|G_x| : x \in ET\}.$

2. Suppose H and K satisfy the following property: for each H-degenerate (resp. Kdegenerate) vertex v of T, the stabilizer H_v (resp. K_v) stabilizes each edge in the star of v. Then

$$\overline{C}_T(H \cap K) \le 6N \cdot \overline{C}_T(H) \cdot \overline{C}_T(K).$$

In particular, if H, K act freely on the edges of T, then

$$\overline{K}_T(H \cap K) \le 6N \cdot \overline{K}_T(H) \cdot \overline{K}_T(K).$$

Proof. Since $H \cap K$ does not fix a vertex, it follows from Remark 2.2 that $H \cap K$, H and K contain hyperbolic elements. Let $T_{H \cap K}$, T_H , T_K be the minimal subtrees of T

invariant under $H \cap K$, H, K, respectively. Let $\pi_H : T_{H \cap K}/H \cap K \longrightarrow T_H/H$ and $\pi_K : T_{H \cap K}/H \cap K \longrightarrow T_K/K$ be the natural projections (defined as in Lemma 3.2). We consider the map $\pi = (\pi_H, \pi_K) : T_{H \cap K}/H \cap K \longrightarrow T_H/H \times T_K/K$ given by $\pi([x]_{H \cap K}) =$ $([x]_H, [x]_K)$. By [3, Proposition 8.7], each fiber $\pi^{-1}([x]_H, [x]_K)$, where x is an edge or a vertex, has exactly $|H_x \setminus G_x \cap HK/K_x|$ elements. It follows that for each edge x the fiber $\pi^{-1}([x]_H, [x]_K)$ has at most N elements.

For convenience we simplify notation by setting $X = T_{H \cap K}/H \cap K$, $Y = T_H/H$ and $Z = T_K/K$. As in Lemma 3.1, we construct graphs \tilde{X} , \tilde{Y} and \tilde{Z} , by attaching a loop at each non-degenerate vertex of X, Y and Z, respectively.

1) By Lemma 3.1, it suffices to show that

$$\sum_{V\widetilde{X}} \left(\deg_{\widetilde{X}}([v]_{H\cap K}) - 2 \right) \le \left(3NM + 6N(M-1) \right) \sum_{V\widetilde{Y}} \left(\deg_{\widetilde{Y}}([v]_{H}) - 2 \right) \cdot \sum_{V\widetilde{Z}} \left(\deg_{\widetilde{Z}}([v]_{K}) - 2 \right)$$
(1)

For any pair of vertices $(a, b) \in Y \times Z$, we will show that

$$\sum_{v\in\pi^{-1}(a,b)} \left(\deg_{\widetilde{X}}(v)-2\right) \le \left(3NM+6N(M-1)\right) \cdot \left(\deg_{\widetilde{Y}}(a)-2\right) \cdot \left(\deg_{\widetilde{Z}}(b)-2\right)$$
(2)

from which (1) follows. The rest of the proof follows similar arguments to those given in [4], [6] and [11]. Let $\{v_1, \ldots, v_n\}$ be the vertices of $\pi^{-1}(a, b)$. Since the fiber of any edge of $Y \times Z$ contains at most N edges, we have

$$\sum_{i=1}^{n} \deg_X(v_i) \le N \cdot \deg_Y(a) \cdot \deg_Z(b).$$
(3)

We consider three cases depending on whether or not a and b are degenerate.

Case 1. Suppose that *a* is *H*-non-degenerate and *b* is *K*-non-degenerate. Then $\deg_Y(a) = \deg_{\tilde{Y}}(a) - 2$, $\deg_Z(b) = \deg_{\tilde{Z}}(b) - 2$ while $\deg_X(v_i)$ is equal to $\deg_{\tilde{X}}(v_i) - 2$ or $\deg_{\tilde{X}}(v_i)$. Hence

$$\sum_{i=1}^{n} \left(\deg_{\widetilde{X}}(v_i) - 2 \right) \le \sum_{i=1}^{n} \deg_X(v_i) \le N \cdot \deg_Y(a) \cdot \deg_Z(b) = N \cdot \left(\deg_{\widetilde{Y}}(a) - 2 \right) \cdot \left(\deg_{\widetilde{Z}}(b) - 2 \right) \cdot \left(\deg_{\widetilde{Z}}(b) - 2 \right) \cdot \left(\deg_{\widetilde{Y}}(a) - 2 \right) \cdot \left(\deg_{\widetilde{Z}}(b) - 2 \right) \cdot \left(\deg_{\widetilde{Z}}(b) - 2 \right) \cdot \left(\deg_{\widetilde{Y}}(a) - 2 \right) \cdot \left(\deg_{\widetilde{Z}}(b) - 2 \right) \cdot \left(\deg_{\widetilde{Y}}(a) - 2 \right) \cdot \left(\deg_{\widetilde{Y}}(b) - 2 \right) \cdot \left(\deg_{\widetilde{Y}(b)} - 2 \right) \cdot \left(\operatorname{S}(b) - 2 \right) \cdot \left(\operatorname{S}(b) - 2 \right)$$

Case 2. Exactly one of a, b, say b, is degenerate. Then each v_i is $(H \cap K)$ -degenerate as well, and thus $\deg_Y(a) = \deg_{\widetilde{Y}}(a) - 2$, $\deg_X(v_i) = \deg_{\widetilde{X}}(v_i)$ and $\deg_{\widetilde{Z}}(b) = \deg_Z(b) > 2$. Also, by Lemma 3.2, for each i we have $\deg_X(v_i) \le M \deg_Z(b)$.

If $n \leq N \cdot \deg_Y(a)$, then

$$\sum_{i=1}^{n} \left(\deg_{\widetilde{X}}(v_{i}) - 2 \right) = \sum_{i=1}^{n} \left(\deg_{X}(v_{i}) - 2 \right) \le n \cdot \left(M \deg_{Z}(b) - 2 \right) \le N \cdot \deg_{Y}(a) \left(M \deg_{Z}(b) - 2 \right) \\ \le N \cdot \deg_{Y}(a) \left(M \left(\deg_{Z}(b) - 2 \right) + 2(M - 1) \right) \\ = NM \left(\deg_{\widetilde{Y}}(a) - 2 \right) \left(\deg_{\widetilde{Z}}(b) - 2 \right) + 2N(M - 1) \left(\deg_{\widetilde{Y}}(a) - 2 \right) \quad (4) \\ \le \left(NM + 2N(M - 1) \right) \cdot \left(\deg_{\widetilde{Y}}(a) - 2 \right) \cdot \left(\deg_{\widetilde{Z}}(b) - 2 \right),$$

where the last inequality follows because $\deg_{\widetilde{Z}}(b) > 2$.

On the other hand, if $n \ge N \cdot \deg_Y(a)$, then

$$\sum_{i=1}^{n} \left(\deg_{\widetilde{X}}(v_i) - 2 \right) = \sum_{i=1}^{n} \deg_X(v_i) - 2n \le N \cdot \deg_Y(a) \cdot \deg_Z(b) - 2N \cdot \deg_Y(a)$$
$$= N \deg_Y(a) \cdot \left(\deg_Z(b) - 2 \right) = N \left(\deg_{\widetilde{Y}}(a) - 2 \right) \left(\deg_{\widetilde{Z}}(b) - 2 \right).$$
(5)

Case 3. Finally, suppose that a, b are degenerate in Y, Z, respectively. Then each vertex v_i is $(H \cap K)$ -degenerate as well and $\deg_{\widetilde{Y}}(a) = \deg_Y(a) > 2$, $\deg_{\widetilde{X}}(v_i) = \deg_X(v_i)$, $\deg_{\widetilde{Z}}(b) = \deg_Z(b) > 2$. Moreover, by Lemma 3.2, $\deg_X(v_i) \le \min\{M \deg_Y(a), M \deg_Z(b)\}$. Suppose that $\deg_Z(b) = \min\{\deg_Y(a), \deg_Z(b)\}$ and hence $\deg_Y(a) = \max\{\deg_Y(a), \deg_Z(b)\}$ (the other case is handled in the same way).

If $n \leq N \cdot \deg_Y(a)$, then

$$\sum_{i=1}^{n} \left(\deg_{\widetilde{X}}(v_i) - 2 \right) = \sum_{i=1}^{n} \left(\deg_X(v_i) - 2 \right) \le n \cdot \left(M \deg_Z(b) - 2 \right) \le N \cdot \deg_Y(a) \cdot \left(M \deg_Z(b) - 2 \right).$$

On the other hand, if $n \ge N \cdot \deg_Y(a)$, then

$$\sum_{i=1}^{n} \left(\deg_{\widetilde{X}}(v_i) - 2 \right) = \sum_{i=1}^{n} \deg_X(v_i) - 2n \le N \cdot \deg_Y(a) \cdot \deg_Z(b) - 2N \cdot \deg_Y(a) \le N \cdot \deg_Y(a) \cdot \left(\deg_Z(b) - 2 \right).$$

Thus, in each case we have

$$\sum_{i=1}^{n} \left(\deg_{\widetilde{X}}(v_i) - 2 \right) \le N \cdot \deg_Y(a) \cdot \left(M \deg_Z(b) - 2 \right).$$
(6)

Since $\deg_Y(a) \ge 3$, or equivalently, $\deg_Y(a) \le 3(\deg_Y(a) - 2)$, it follows that

$$\sum_{i=1}^{n} \left(\deg_{\widetilde{X}}(v_i) - 2 \right) \le 3N \cdot \left(\deg_{Y}(a) - 2 \right) \cdot \left(M \deg_{Z}(b) - 2 \right) \\ \le 3N \cdot \left(\deg_{Y}(a) - 2 \right) \cdot \left(M \left(\deg_{Z}(b) - 2 \right) + 2(M - 1) \right) \\ = 3NM \cdot \left(\deg_{Y}(a) - 2 \right) \cdot \left(\deg_{Z}(b) - 2 \right) + 6N(M - 1) \cdot \left(\deg_{Y}(a) - 2 \right) \\ \le \left(3NM + 6N(M - 1) \right) \cdot \left(\deg_{\widetilde{Y}}(a) - 2 \right) \cdot \left(\deg_{\widetilde{Z}}(b) - 2 \right).$$
(7)

This completes the proof of part 1) of the theorem.

2) To prove the second part, again by Lemma 3.1, it suffices to show that

$$\sum_{v \in \pi^{-1}(a,b)} \left(\deg_{\widetilde{X}}(v) - 2 \right) \le 3N \left(\deg_{\widetilde{Y}}(a) - 2 \right) \cdot \left(\deg_{\widetilde{Z}}(b) - 2 \right), \tag{8}$$

for each pair of vertices $(a, b) \in Y \times Z$. Proceeding exactly as before, we distinguish three cases. In Case 1, where both a and b are non-degenerate, we get the same inequality. In Cases 2 and 3, by Lemma 3.2 (2), we can now use 1 instead of M. Thus in Cases 2 and 3, we obtain respectively (from 4-5 and 7) the inequalities

$$\sum_{i=1}^{n} \left(\deg_{\widetilde{X}}(v_i) - 2 \right) \le N \left(\deg_{\widetilde{Y}}(a) - 2 \right) \cdot \left(\deg_{\widetilde{Z}}(b) - 2 \right)$$
(9)

and

$$\sum_{i=1}^{n} \left(\deg_{\widetilde{X}}(v_i) - 2 \right) \le 3N \left(\deg_{\widetilde{Y}}(a) - 2 \right) \cdot \left(\deg_{\widetilde{Z}}(b) - 2 \right).$$
(10)

It remains only to consider the case when both a and b are degenerate (in which case we are in Case 3) and $\deg_Y(a) = 2$, where a is the vertex of maximal degree. If $\deg_Y(a) = 2$, then $\deg_Z(b) = 2$ too, and inequality 8 follows since, by Lemma 3.2 (2), $\deg_{\widetilde{X}}(v_i) = \deg_X(v_i) \le$ $\min\{\deg_Y(a), \deg_Z(b)\}$ for each i.

Corollary 3.4. ([16, Theorem 1]) Let G be a group acting on a tree T with finite quotient and finite stabilizers of edges and let H, K be finitely generated subgroups of G which intersect trivially each vertex stabilizer (and hence they are free groups). Then $H \cap K$ is finitely generated and

$$\overline{r}(H \cap K) \le 6N \cdot \overline{r}(H) \cdot \overline{r}(K),$$

where $N = \max\{|G_x \cap HK| : x \in ET\}.$

Corollary 3.5. Let G be a group acting on a tree T with finite quotient, finite stabilizers of edges and infinite vertex stabilizers. If H and K are subgroups of finite index in G, then

$$\overline{C}_T(H \cap K) \le 2N \cdot \overline{C}_T(H) \cdot \overline{C}_T(K).$$

Proof. If the *G*-stabilizer of every vertex is infinite and both *H* and *K* are of finite index in *G*, then each vertex stabilizer is also infinite under the action of *H* or *K* (being of finite index in the corresponding *G*-stabilizer) and thus Cases 2 and 3 do not occur. \Box

Following [6], given a group G, we define $a_3(G) = \min \{ |\Gamma| : \Gamma \text{ is a subgroup of } G \text{ with } |\Gamma| \ge 3 \}$ and $\theta(G) = \{ \frac{a_3(G)}{a_3(G)-2} \} \in [1,3]$, where $\frac{\infty}{\infty-2} := 1$.

In the sequel, we prove that if H, K act freely on the edges, then the coefficient 6 in the above theorem can be replaced by a number 2θ , where $\theta \in [1,3]$, by imposing some extra hypotheses on the structure of G.

Let G_i , $i \in I$, be a family of groups together with a group A, let $\phi_i : A \longrightarrow G_i$ be a family of monomorphisms and let $*_AG_i$ be the amalgamated free product of G_i 's with amalgamated subgroup A (with respect to ϕ_i). We can think of each ϕ_i as an inclusion. Let F be a free group and let $G = *_AG_i * F$ be the free product of F and $*_AG_i$. We construct a graph of groups (\mathcal{G}, Ψ) with fundamental group G as follows. The graph Ψ consists of a wedge of open edges $e_i = [u_0, u_i], i \in I$ (i.e. one for each factor G_i and distinct endpoints u_0 and $u_i, i \in I$, $0 \notin I$), together with a wedge of loops l_j , one for each free generator of F, attached at a vertex u_0 with vertex group A. To each edge e_i we associate the group A, to each loop l_j we associate the trivial group and to each vertex u_i we associate the group G_i . We denote by T the corresponding universal tree. **Theorem 3.6.** Let $G = *_A G_i * F$ be the free product of the amalgamated free product of G_i 's with a finite amalgamated subgroup A and F, such that A is normal in each G_i . We consider the natural action of G on T defined above. Suppose that H and K are tame subgroups (with respect to T) of G which act freely on the edges of T. Then $H \cap K$ is tame and

$$\overline{K}_T(H \cap K) \le 2 \cdot \theta \cdot N \cdot \overline{K}_T(H) \cdot \overline{K}_T(K) \le 2 \cdot \theta \cdot |A| \cdot \overline{K}_T(H) \cdot \overline{K}_T(K),$$

where $\theta = \max\{\theta(G_i/A) : i \in I\}$ and $N = \max\{|gAg^{-1} \cap HK| : g \in G\}.$

Proof. We proceed as in the proof of Theorem 3.3. With the notation of that proof, we have to prove that

$$\sum_{i=1}^{n} \left(\deg_{\widetilde{X}}(v_i) - 2 \right) \le \theta \cdot N \cdot \left(\deg_{\widetilde{Y}}(a) - 2 \right) \cdot \left(\deg_{\widetilde{Z}}(b) - 2 \right)$$
(11)

for any pair of vertices $(a,b) \in Y \times Z$ (recall that $\{v_1, \ldots, v_n\}$ denotes the vertices of $\pi^{-1}(a,b)$). Since H, K act freely on the edges of T, it follows, by Lemma 3.2 (2), that we can use 1 instead of M. Suppose first that at least one of a and b is non-degenerate. Then the arguments of Cases 1, 2 of the proof of Theorem 3.3 apply to show that

$$\sum_{i=1}^{n} \left(\deg_{\widetilde{X}}(v_i) - 2 \right) \le N \cdot \left(\deg_{\widetilde{Y}}(a) - 2 \right) \cdot \left(\deg_{\widetilde{Z}}(b) - 2 \right)$$
(12)

and inequality 11 holds. Thus it suffices to consider the case where a is H-degenerate and b is K-degenerate (i.e. Case 3 in the proof of Theorem 3.3). In this case we have $\deg_Y(a) = \deg_{\widetilde{Y}}(a), \deg_Z(b) = \deg_{\widetilde{Z}}(b)$ and $\deg_X(v_i) = \deg_{\widetilde{X}}(v_i)$, while by Lemma 3.2 (2), $\deg_X(v_i) \leq \min\{\deg_Y(a), \deg_Z(b)\}$. For each $i \in \{1, \ldots, n\}$, choose a vertex, w_i , of T, so that $[w_i]_{H\cap K} = v_i \in \pi^{-1}(a, b)$. Note that all w_1, \ldots, w_n lie in the same G-orbit. There are two subcases to consider.

(i) w_i and u_0 are in the same *G*-orbit, i.e. $w_i = g_i u_0$ for some $g_i \in G$. Notice that $H_{w_i} = K_{w_i} = g_i A g_i^{-1} \cap H = g_i A g_i^{-1} \cap K = 1$. If one of the vertices *a* or *b* has valence 2, then (each) v_i has valence 2 as well and inequality 11 is obvious. If both *a* and *b* have valence at least 3, then

$$\sum_{i=1}^{n} \left(\deg_{\widetilde{X}}(v_i) - 2 \right) \le n \cdot \left(\deg_{\widetilde{Y}}(a) - 2 \right) \le |H_{w_i} \setminus G_{w_i} \cap HK/K_{w_i}| \left(\deg_{\widetilde{Y}}(a) - 2 \right) \cdot (3 - 2)$$
$$\le N \cdot \left(\deg_{\widetilde{Y}}(a) - 2 \right) \cdot \left(\deg_{\widetilde{Z}}(b) - 2 \right) \le \theta(G) \cdot N \cdot \left(\deg_{\widetilde{Y}}(a) - 2 \right) \cdot \left(\deg_{\widetilde{Z}}(b) - 2 \right)$$

(ii) w_i and u_j are in the same *G*-orbit for some $j \in I$, i.e. there exists $g_i \in G$ such that $w_i = g_i u_j$. As before, we may assume that both *a* and *b* have valence at least 3.

If $n \leq N$, then

$$\sum_{i=1}^{n} \left(\deg_{\widetilde{X}}(v_i) - 2 \right) \le N \cdot \left(\deg_{\widetilde{Y}}(a) - 2 \right) \cdot (3 - 2) \le N \cdot \left(\deg_{\widetilde{Y}}(a) - 2 \right) \cdot \left(\deg_{\widetilde{Z}}(b) - 2 \right).$$

Suppose now that n > N. Let $\mathcal{R} = \{g_{\lambda}\}_{\lambda \in \Lambda}$ be a set of representatives of left cosets of A in G_j . From the construction of X = T/G, the stars of different vertices of $\pi^{-1}(a, b)$ are disjoint, while each edge in the star of $[g_i u_j]_{\Gamma}$, where $\Gamma = H \cap K, H$ or K, is of the form $[xe_j]_{\Gamma}$ and its terminal vertex is $[g_i u_j]_{\Gamma}$. It follows that there is $\gamma \in \Gamma$ such that $\gamma x u_j = g_i u_j$ and thus $g_i^{-1} \gamma x \in G_j$. If we write $g_i^{-1} \gamma x$ as $g_{\lambda(x)} a$, where $g_{\lambda(x)} \in \mathcal{R}$ and $a \in A$, then $[xe_j]_{\Gamma} = [\gamma^{-1}g_i g_{\lambda(x)} ae_j]_{\Gamma} = [g_i g_{\lambda(x)} ae_j]_{\Gamma} = [g_i g_{\lambda(x)} e_j]_{\Gamma}$. It follows that there exists a subset \mathcal{R}_{Γ}^i of \mathcal{R} such that $Star([w_i]_{\Gamma}) = \{[g_i g_{\lambda} e_j]_{\Gamma} : g_{\lambda} \in \mathcal{R}_{\Gamma}^i\}$ and $|\mathcal{R}_{\Gamma}^i| = |Star([w_i]_{\Gamma})|$. In particular, $|\mathcal{R}_{H}^i| = |Star_Y(a)|$ and $|\mathcal{R}_{K}^i| = |Star_Z(b)|$.

Fix $i \in \{1, \ldots, n\}$. For any $k \in \{1, \ldots, n\}$, let C_k be the subset of $\mathcal{R}_H \times \mathcal{R}_K := \mathcal{R}^i_H \times \mathcal{R}^i_K$ consisting of all pairs (g_λ, g_μ) such that $([g_i g_\lambda e_j]_H, [g_i g_\mu e_j]_K)$ is the image under π of some edge $[xe_j]_{H\cap K}$ in the star of $[g_k u_j]_{H\cap K} = v_k$ in X. Let $\phi: G_j \to G_j/A$ denote the natural epimorphism. Note that the restriction of ϕ on \mathcal{R} is a bijection.

We will show that $\phi(C_k) = \{(\phi(g_\lambda), \phi(g_\mu)) : (g_\lambda, g_\mu) \in C_k\}$ is a single-quotient subset of $\phi(\mathcal{R}_H) \times \phi(\mathcal{R}_K)$, in the terminology of [6], i.e. that the product $\phi(g_\lambda) \cdot \phi(g_\mu)^{-1}$ is constant for all pairs $(g_\lambda, g_\mu) \in C_k$. Suppose that y_1 and y_2 are edges in the star of v_k and that $\pi(y_t) = ([g_i g_{\lambda(t)} e_j]_H, [g_i g_{\mu(t)} e_j]_K), t = 1, 2$. We want to show that $\phi(g_{\lambda(1)}) \cdot \phi(g_{\mu(1)})^{-1} =$ $\phi(g_{\lambda(2)}) \cdot \phi(g_{\mu(2)})^{-1}$. From the above analysis, we can write $y_t = [g_k g_{s(t)} e_j]_{H\cap K}$ for some $g_{s(t)} \in \mathcal{R}^k_{H\cap K}, t = 1, 2$, and thus $\pi(y_t) = ([g_k g_{s(t)} e_j]_H, [g_k g_{s(t)} e_j]_K)$. It follows that $([g_k g_{s(1)} e_j]_H, [g_k g_{s(1)} e_j]_K) = ([g_i g_{\lambda(1)} e_j]_H, [g_i g_{\mu(1)} e_j]_K)$, and that $([g_k g_{s(2)} e_j]_H, [g_k g_{s(2)} e_j]_K) =$ $([g_i g_{\lambda(2)} e_j]_H, [g_i g_{\mu(2)} e_j]_K)$. Hence there are $h_1, h_2 \in H, k_1, k_2 \in K$ and $a_1, a_2, a_3, a_4 \in A$ such that

$$g_k g_{s(1)} = h_1 g_i g_{\lambda(1)} a_1, \quad g_k g_{s(2)} = h_2 g_i g_{\lambda(2)} a_3$$
$$g_k g_{s(1)} = k_1 g_i g_{\mu(1)} a_2, \quad g_k g_{s(2)} = k_2 g_i g_{\mu(2)} a_4$$

By normality of A in G_j , the stabilizer of any edge in the star of w_i is equal to $g_i A g_i^{-1}$. Therefore our assumption that w_i is H, K degenerate implies that $H \cap G_{w_i} = H \cap g_i A g_i^{-1}$ and $K \cap G_{w_i} = K \cap g_i A g_i^{-1}$. Now, from the first two equalities above we deduce that

$$h_2^{-1}h_1 = g_i g_{\lambda(2)} a_3 g_{s(2)}^{-1} g_{s(1)} a_1^{-1} g_{\lambda(1)}^{-1} g_i^{-1} \in H \cap g_i G_j g_i^{-1} = H \cap G_{w_i} = H \cap g_i A g_i^{-1} = 1,$$
(13)

while from the last two

$$k_2^{-1}k_1 = g_i g_{\mu(2)} a_4 g_{s(2)}^{-1} g_{s(1)} a_2^{-1} g_{\mu(1)}^{-1} g_i^{-1} \in K \cap g_i G_j g_i^{-1} = K \cap G_{w_i} = K \cap g_i A g_i^{-1} = 1.$$
(14)

The above relations imply that $g_{\lambda(2)}a_3g_{s(2)}^{-1}g_{s(1)}a_1^{-1}g_{\lambda(1)}^{-1} = 1$ and $g_{\mu(2)}a_4g_{s(2)}^{-1}g_{s(1)}a_2^{-1}g_{\mu(1)}^{-1} = 1$. Thus, $g_{\lambda(1)}g_{\mu(1)}^{-1} = g_{\lambda(2)}a_3g_{s(2)}^{-1}g_{s(1)}a_1^{-1}a_2g_{s(1)}^{-1}g_{s(2)}a_4^{-1}g_{\mu(2)}^{-1}$, from which it follows that $\phi(g_{\lambda(1)}) \cdot \phi(g_{\mu(1)})^{-1} = \phi(g_{\lambda(2)}) \cdot \phi(g_{\mu(2)})^{-1}$.

Our aim is to apply [6, Corollary 3.5], which requires pairwise-disjoint, single-quotient subsets. Note that if the intersection $C_k \cap C_s$ is nonempty, then there are edges y_1 and y_2 in $Star_X(v_k)$ and $Star_X(v_s)$, respectively, such that $\pi(y_1) = \pi(y_2)$. Thus, for each $k = 1, \ldots, n$, we choose a subset F_k of $Star_X(v_k)$ with $|F_1| + \cdots + |F_n|$ maximum such that the restriction of π on the union $\bigcup_{k=1}^{n} F_k$ is an injection. In particular, they are pairwisedisjoint. Since the inverse image of any edge of $Y \times Z$ under π contains at most N elements, we have $|Star_X(v_1)| + \cdots + |Star_X(v_n)| \leq N(|F_1| + \cdots + |F_n|)$. If $C_{F(k)}$ denotes the subset of C_k corresponding to edges of F_k , then $C_{F(1)}, \ldots, C_{F(n)}$ are pairwise-disjoint. It follows that $\phi(C_{F(1)}), \ldots, \phi(C_{F(n)})$ are pairwise-disjoint, single-quotient subsets of $\phi(\mathcal{R}_H) \times \phi(\mathcal{R}_K)$ and [6, Corollary 3.5] applies to show that

$$\sum_{k=1}^{n} \left(\left| \phi(C_{F(k)}) \right| - 2 \right) \le \theta(G_j/A) \cdot \left(\left| \phi(\mathcal{R}_H) \right| - 2 \right) \cdot \left(\left| \phi(\mathcal{R}_K) \right| - 2 \right)$$

Finally

$$\begin{split} \sum_{k=1}^{n} \left(\deg_{\widetilde{X}}(v_k) - 2 \right) &= \sum_{k=1}^{n} \left(\deg_X(v_k) - 2 \right) = \sum_{k=1}^{n} |Star_X(v_k)| - 2n \le N \cdot \sum_{k=1}^{n} |F_k| - 2N \\ &= N \cdot \sum_{k=1}^{n} \left(|C_{F(k)}| - 2 \right) = N \cdot \sum_{k=1}^{n} \left(|\phi(C_{F(k)})| - 2 \right) \\ &\le N \cdot \theta(G_j/A) \cdot \left(|\phi(\mathcal{R}_H)| - 2 \right) \cdot \left(|\phi(\mathcal{R}_K)| - 2 \right) \\ &= N \cdot \theta(G_j/A) \cdot \left(|\mathcal{R}_H| - 2 \right) \cdot \left(|\mathcal{R}_K| - 2 \right) \\ &\le N \cdot \theta \cdot \left(\deg_{\widetilde{Y}}(a) - 2 \right) \cdot \left(\deg_{\widetilde{Z}}(b) - 2 \right). \end{split}$$

This completes the proof.

Remark 3.7. The analogous theorem with the same proof is valid for fundamental groups of graphs of groups (\mathcal{G}, Ψ) defined as follows. The subject graph Ψ is the same as the one defined previously (prior to Theorem 3.6). To the terminal vertex u_i of e_i we associate the group G_i , to the common initial vertex of e_i 's we associate the finite group A, and to each open edge e_i we associate a subgroup A_i of A normally embedded in G_i such that $A_{i_0} = A$ for some i_0 (this means that the "central" vertex is G-degenerate and thus $G_{w_i} = gAg^{-1}$ in Case (i) of the proof). To each loop we associate the trivial group. We need normality of A_i in G_i in order to make the natural map $G_i \to G_i/A_i$ a homomorphism (and thus the same arguments in Case (ii) work equally well to this more general setting).

As a corollary, we obtain the main result of Ivanov in [11] (in fact our proof can be slightly modified to generalize [6, Theorem 6.3] as well).

Corollary 3.8. Suppose that H_1 , H_2 are subgroups of a free product $G = *_{a \in I}G_a$ and H_1 , H_2 have finite Kurosh rank $K(H_1)$, $K(H_2)$. Then the intersection $H_1 \cap H_2$ also has finite Kurosh rank and

$$\overline{K}r(H_1 \cap H_2) \le 2 \cdot \theta(G) \cdot \overline{K}r(H_1) \cdot \overline{K}r(H_2).$$

In particular, if G is torsion-free (or more generally, every finite subgroup of G has order at most 2), then

$$\overline{K}r(H_1 \cap H_2) \le 2 \cdot \overline{K}r(H_1) \cdot \overline{K}r(H_2).$$

Proof. By Lemma 2.3 and the comments preceding it, the subgroup Kurosh rank is equal to the Kurosh rank with respect to T (i.e. $Kr(\cdot) = K_T(\cdot)$, where T is as above) and finite Kurosh rank implies tameness.

In the case of free products with a finite, normal subgroup amalgamated, we can use the same arguments to improve the bound for the complexity of the intersection of tame subgroups.

Let G_i , $i \in I$, be a family of groups together with a group A and let $G = *_A G_i$ be the amalgamated free product of G_i 's with amalgamated subgroup A (with respect to a family of monomorphisms, regarded as inclusions). We construct a tree of groups (\mathcal{G}, T_0) with fundamental group G as usual. The tree T_0 consists of a wedge of open edges $e_i = [u_0, u_i], i \in I$ (one for each factor G_i) attached at a vertex v_0 (where $0 \notin I$) with vertex group A. To each edge we associate the group A and to each vertex v_i we associate the group G_i . We denote by T the corresponding universal tree.

Theorem 3.9. Let $G = *_A G_i$ be the amalgamated free product of G_i 's with a finite and normal amalgamated subgroup A. We consider the action of G on T defined above. If H and K are tame subgroups (with respect to T) of G, then $H \cap K$ is tame and

$$\overline{C}_T(H \cap K) \le 2 \cdot \theta \cdot |A \cap HK| \cdot \overline{C}_T(H) \cdot \overline{C}_T(K),$$

where $\theta = \max\{\theta(G_i/A) : i \in I\}.$

Proof. The proof is exactly the same as the proof of Theorem 3.6. There are two things to note:

- (a) The normality of A in G and the fact that v₀ is a G-degenerate vertex imply that for each subgroup B of G the B-stabilizer of the star of any B-degenerate vertex v of T is equal to B_v and therefore Lemma 3.2 (2) applies (i.e we can again use 1 instead of M to obtain inequality 12).
- (b) Using the notation of the proof of Theorem 3.6, the relations 13 and 14 now give $g_{\lambda(2)}a_3g_{s(2)}^{-1}g_{s(1)}a_1^{-1}g_{\lambda(1)}^{-1} \in A$ and $g_{\mu(2)}a_4g_{s(2)}^{-1}g_{s(1)}a_2^{-1}g_{\mu(1)}^{-1} \in A$. Since A is the kernel of ϕ , we again conclude that $\phi(g_{\lambda(1)}) \cdot \phi(g_{\mu(1)})^{-1} = \phi(g_{\lambda(2)}) \cdot \phi(g_{\mu(2)})^{-1}$.

Remark 3.10. In general, there are examples (see [11, 16]) showing that the bounds obtained in the previous two theorems are sharp.

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