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$\mathbb{Z}_2$-GRADED CODIMENSIONS OF UNITAL ALGEBRAS

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Abstract. We study polynomial identities of nonassociative algebras constructed by using infinite binary words and their combinatorial properties. Infinite periodic and Sturmian words were first applied for constructing examples of algebras with arbitrary real PI-exponent greater than one. Later we used these algebras for confirmation of the conjecture that PI-exponent increases precisely by one after adjoining an external unit to a given algebra. Here we prove the same result for these algebras for graded identities and graded PI-exponent, provided that the grading group is cyclic of order two.

1. Introduction

We study numerical invariants of polynomial identities of algebras over a field of characteristic zero. One of the most important characteristics of identities of an algebra A is its codimension sequence $\{c_n(A)\}$. In many cases this sequence is exponentially bounded and one can ask whether the limit

$$exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

exists. The answer is in general negative [22]. Nevertheless, there is a wide class of algebras where $exp(A)$ exists, for example, associative PI-algebras [11], [12], finite dimensional Lie algebras [9], [8], [20], finite dimensional Jordan and alternative algebras [10], and many others. Given an algebra A, one can consider an extension $A^\#$ of A, obtained from A by adjoining the external unit element. Then some natural questions arise: is the codimension sequence $c_n(A^\#)$ exponentially bounded, does $exp(A^\#)$ exist, does there exist a relationship between $exp(A)$ and $exp(A^\#)$?

It was first mentioned in [14] that $exp(A^\#)$ exists and is equal to either $exp(A)$ or $exp(A)+1$ for any associative PI-algebra A. One of the first examples of a non-associative algebra A with $exp(A^\#) = exp(A)+1$ was found in [21]. In the same paper it was conjectured that for any algebra A either $exp(A^\#) = exp(A)$ or $exp(A^\#) = exp(A)+1$. In [19] this conjecture was confirmed for a wide class of algebras associated with infinite Sturmian words. Similar results for certain kinds of Poisson algebras were found in [18]. Note also that $exp(A^\#) = exp(A)$, whenever A is a unital algebra [2].

If A is equipped with a group grading then one can also consider its graded identities and graded codimensions $\{c_n^{gr}(A)\}$. In the present paper we begin to study connections between asymptotics of $\{c_n(A)\}$ and $\{c_n^{gr}(A)\}$. We prove that for the class of algebras introduced in [6] and associated with Sturmian words, graded PI-exponents exist, $exp^{gr}(A) = exp(A)$, and $exp^{gr}(A^\#) = exp^{gr}(A)+1$ for

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the most natural $\mathbb{Z}_2$-grading. For all details concerning the polynomial identities and their numerical invariants we refer to [3], [13].

2. Preliminaries and main constructions

Let A be an algebra over a field F of characteristic zero and let $F\{X\}$ be the absolutely free algebra over F with an infinite set of generators X. A polynomial $f = f(x_1,\ldots,x_n)$, $x_1,\ldots,x_n \in X$, is called an identity of A if $f(a_1,\ldots,a_m) = 0$, whenever $a_1,\ldots,a_n \in A$. The set $Id(A)$ of all identities of A forms an ideal of $F\{X\}$. Denote by P_n the subspace of all multilinear polynomials on $x_1,\ldots,x_n$. Then $P_n \cap Id(A)$ is the set of all multilinear identities of A of degree n. It is well known that all identities of A are completely defined by the family of subspaces $\{P_n \cap Id(A)\}, n = 1,2,\ldots$. An important numerical invariant of identical relations of the algebra A is the sequence of codimensions

$$c_n(A) = \dim \frac{P_n}{P_n \cap Id(A)}.$$

In the case of exponentially bounded growth of $\{c_n(A)\}$, one can define the lower and the upper PI-exponents by setting

$$\underline{exp}(A) = \liminf_{n\to\infty} \sqrt[n]{c_n(A)}, \quad \overline{exp}(A) = \limsup_{n\to\infty} \sqrt[n]{c_n(A)}$$

and the ordinary PI-exponent

$$exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)},$$

provided that $\underline{exp}(A) = \overline{exp}(A)$. A powerful tool for studying asymptotics of codimensions is the representation theory of the symmetric group S_n. The group S_n acts naturally on the space P_n of multilinear polynomials

$$\sigma f(x_1,\ldots,x_n) = f(x_{\sigma(1)},\ldots,x_{\sigma(n)}).$$

Under this action, the subspaces $P_n, P_n \cap Id(A)$ and the quotient

$$P_n(A) = \frac{P_n}{P_n \cap Id(A)}$$

become S_n-modules. Consider the nth cocharacter of A, that is the character of $P_n(A)$, $\chi_n(A) = \chi(P_n(A))$, and its decomposition into irreducible components

$$\chi_n(A) = \sum_{\lambda\vdash n} m_\lambda \chi_\lambda, \tag{1}$$

where χ_λ denotes the irreducible S_n-character, corresponding to the partition λ of n, and the integer m_λ denotes its multiplicity in $\chi_n(A)$.

Denote by $d_\lambda = \deg \chi_\lambda$ the dimension of the irreducible S_n-module with the character χ_λ. It follows from (1) that

$$c_n(A) = \sum_{\lambda\vdash n} m_\lambda d_\lambda. \tag{2}$$

Another important numerical characteristic of $Id(A)$ is its nth colength

$$l_n(A) = \sum_{\lambda\vdash n} m_\lambda.$$

In many cases the sequence $l_n(A)$ is polynomially bounded while d_λ's in (2) are exponentially large. This means that the asymptotics of $c_n(A)$ is actually defined by the maximal value of d_λ with $m_\lambda \neq 0$.

For group graded algebras, identical relations and corresponding numerical invariants can also be considered. We restrict ourselves to the case of $\mathbb{Z}_2$-gradings. Consider the free algebra $F\{X,Y\}$ with two independent sets of generators X and Y. We can endow $F\{X,Y\}$ with a $\mathbb{Z}_2$-grading, by setting $\deg x = 0, \deg y = 1$, for all $x \in X, y \in Y$, and extending this grading to all monomials on $X \cup Y$. If $A = A_0 \oplus A_1$ is a $\mathbb{Z}_2$-graded algebra over F then a polynomial $f(x_1,\ldots,x_k,y_1,\ldots,y_m) \in F\{X,Y,\}$ is a graded identity of A if $f(a_1,\ldots,a_k,b_1,\ldots,b_m) = 0$, for all $a_1,\ldots,a_k \in A_0, b_1,\ldots,b_m \in A_1$.

The set of all graded identities of A forms a homogeneous in $\mathbb{Z}_2$-grading ideal Id^{gr} of $F\{X,Y\}$. The intersection $P_{k,m} \cap Id^{gr}(A)$ consists of all multilinear graded identities of degree k on even variables and of degree m on odd variables, where $P_{k,m}$ is the subspace of all polynomials multilinear on $x_1,\ldots,x_k,y_1,\ldots,y_m$. As before, the symmetric groups S_k, S_m act independently on even and odd variables and both $P_{k,m}$ and $P_{k,m} \cap Id^{gr}(A)$, and also

$$P_{k,m}(A) = \frac{P_{k,m}}{P_{k,m} \cap Id^{gr}(A)}$$

are $S_k \times S_m$-modules. One can decompose $P_{k,m}(A)$ into irreducible components and write

$$\chi(P_{k,m}(A)) = \sum_{\substack{\lambda \vdash k \\ \mu \vdash m}} m_{\lambda,\mu}\chi_{\lambda,\mu},$$

where $\chi_{\lambda,\mu}$ is the irreducible $S_k \times S_m$-character and $m_{\lambda,\mu}$ is its multiplicity. It is well known that $\chi_{\lambda,\mu} = \chi_\lambda \otimes \chi\mu$ and that

$$\deg \chi_{\lambda,\mu} = \deg \chi_\lambda \deg \chi_\mu = d_\lambda d_\mu.$$

Partial codimensions and colengths are defined as follows:

$$c_{k,m}(A) = \deg \chi(P_{k,m}(A)) = \dim P_{k,m}(A),$$

$$l_{k,m}(A) = \sum_{\substack{\lambda \vdash k \\ \mu \vdash m}} m_{\lambda,\mu}.$$

Finally, the graded nth codimension and the colength of A are equal to

$$c_n^{gr}(A) = \sum_{k=0}^{n} \binom{n}{k} c_{k,n-k}(A)$$

and

$$l_n^{gr}(A) = \sum_{k=0}^{n} l_{k,n-k}(A),$$

respectively.

Graded PI-exponents are defined similarly,

$$\underline{exp}^{gr}(A) = \liminf_{n\to\infty} \sqrt[n]{c_n^{gr}(A)},$$

$$\overline{exp}^{gr}(A) = \limsup_{n\to\infty} \sqrt[n]{c_n^{gr}(A)},$$

$$exp(A)^{gr} = \lim_{n\to\infty} \sqrt[n]{c_n^{gr}(A)}.$$

Generalizing (2), we get

$$c_{k,n-k}(A) = \sum_{\substack{\lambda\vdash k\\ \mu\vdash n-k}} m_{\lambda,\mu} d_\lambda d_\mu. \tag{3}$$

Graded and ordinary codimensions satisfy the relation

$$c_n(A) \le c_n^{gr}(A) \tag{4}$$

(see [7] or [1]).

We will use the following auxiliary function for computing codimensions. Let $x_1, \dots, x_d$ be non-negative real numbers such that $x_1 + \dots + x_d = 1$, $d \le 2$. Then

$$\Phi(x_1, \dots, x_d) = \frac{1}{x_1^{x_1} \cdots x_d^{x_d}}.$$

If $d = 2$ then we write

$$\Phi(x_1, x_2) = \Phi(x) = \frac{1}{x^x (1-x)^{1-x}}$$

instead of $\Phi(x_1, x_2)$, where $0 \le x \le 1$.

3. Sturmian words and Sturmian algebras

In this section we recall the construction of algebras based on infinite binary words and their combinatorial properties. First, let $K = k_1 k_2 \dots$ be an infinite word with integers $k_i \ge 2, i = 1, 2, \dots$. Denote by $A(K)$ a non-associative algebra with the basis

$$\{a, b, z_j^{(i)} \mid 1 \le j \le k_i,\ i \ge 1\}$$

and with the multiplication table given by

$$z_1^{(i)} a = z_2^{(i)}, \dots, z_{k_i-1}^{(i)} a = z_{k_i}^{(i)},\ z_{k_i}^{(i)} b = z_1^{(i+1)},\ i = 1, 2, \dots.$$

All other products are zero. Note that $A(K)$ is 2-step left nilpotent, that is $x_1(x_2x_3) \equiv 0$ is an identity of $A(K)$. It allows us to omit brackets in all products and write $x_1x_2x_3 \cdots x_n$ instead of $(\cdots((x_1x_2)x_3)\cdots)x_n$, keeping in mind that all non-left normed products are zero. Algebras of this type are used intensively in the study of numerical invariants of polynomial identities. For instance, in [6] the first examples of algebras with an arbitrary exponential growth $\alpha^n, 1 \le \alpha \in \mathbb{R}$, were presented. Examples of algebras with an intermediate growth $n^{n^\beta}, 0 < \beta < 1$, were constructed in [5]. Recently, examples of commutative algebras with polynomial codimension growth $n^\alpha, 3 < \alpha < 4$, were presented in [4]. Other important examples of abnormal codimension growth were constructed in [16], [22].

In the present paper we study identities on algebras $A(K)$ of special kind. Let $m \ge 2$ be an integer and let $w = w_1 w_2 \dots$ be an infinite word in the alphabet $\{0; 1\}$. We denote by $A(m, w)$ the algebra $A(K)$, where K is constructed as follows:

$$k_i = m + w_i,\ i = 1, 2, \dots\ .$$

Earlier, the algebras $A(m, w)$ have already been used for constructing a continuous family of unitary algebras with non-integer PI-exponents and for confirmation of the conjecture that $exp(A^\#) = exp(A) + 1$ (see [19]).

We recall some well known facts from the combinatorial theory of infinite words (see, for example, [17]). Given a binary word $w = w_1 w_2 \dots$, the complexity $Comp_w$ of w is the function $Comp_w \colon \mathbb{N} \to \mathbb{N}$, where $Comp_w(n)$ is the number of distinct subwords of w of length n. It is easy to see that for a periodic word w with period

T, the complexity function is bounded, $Comp_w(n) \leq T$. Moreover, it is well known that $Comp_w(n) \geq n+1$ for any aperiodic w. An infinite word w is called *Sturmian* if $Comp_w(n) = n+1$ for all $n \geq 1$.

For a finite word $x = x_1 \dots x_n$ in the alphabet $\{0; 1\}$, the height $h(x)$ and the length $|x|$ are defined as $h(x) = x_1 + \cdots + x_n$ and $|x| = n$, respectively. Then the *slope* $\pi(x)$ is defined by

$$\pi(x) = \frac{h(x)}{|x|}.$$

One can extend this notion to certain infinite binary words. Namely, if the limit

$$\pi(w) = \lim_{n \to \infty} \frac{h(w_1 \dots w_n)}{n}$$

exists then $\pi(w)$ is called the slope of w. Clearly, the limit does not exist in general. Nevertheless, for periodic and Sturmian words, the slope is well defined. In the next proposition we recall the basic properties which we will need in the sequel.

Proposition 1. *(*[17, Section 2.2]*). Let w be a Sturmian or periodic word. Then there exists a constant C such that*

1. *$|h(x) - h(y)| \leq C$, for any finite subwords x, y of w with $|x| = |y|$;*
2. *the slope $\pi(w)$ of w exists;*
3. *for any non-empty finite subword u of w,*
$$|\pi(u) - \pi(w)| \leq \frac{C}{|u|}; \text{ and}$$
4. *for any real $\alpha \in (0; 1)$, there exists a word w with $\pi(w) = \alpha$ and w is Sturmian or periodic, according to whether α is irrational or rational, respectively.*

We will use the following results.

Theorem 1. *(*[6, Theorem 5.1] *Let w be a Sturmian or periodic word with the slope $0 < \alpha < 1$. If $m \leq 2$ then for the algebra $A = A(m, w)$ the PI-exponent exists and $exp(A) = \Phi(\beta)$, where $\beta = \frac{1}{m+\alpha}$.*

$\square$

Theorem 2. *(*[24, Theorem 1]*) Let $A = A(m, w)$, where w is an infinite Sturmian or periodic word, and $m \geq 2$. Let $A^{\#}$ be the algebra obtained from A by adjoining an external unit. Then PI-exponent of $A^{\#}$ exists and $exp(A^{\#}) = exp(A) + 1$.*

$\square$

4. Gradings on Sturmian algebras

The algebra $A = A(m, w)$ can be equipped by a $\mathbb{Z}_2$-grading in different ways. We begin our study with the most natural case when generators of A are homogeneous. The algebra A is generated by the three elements $z_1^{(1)}, a, b$. Each generator can be even or odd, so we have eight options. Clearly, if $\deg z_1^{(1)} = \deg a = \deg b = 0$ then the grading is trivial and all identities and codimensions are the same as in the non-graded case. In the present paper we consider one of non-trivial cases when $z_1^{(1)}$ and a are even, whereas b is odd. At the end we will discuss the difference between distinct gradings.

Throughout this section, let $A = A(m, w)$ be the algebra defined in the previous section, where $m \geq 2$ is an integer and w is an infinite periodic or Sturmian word. Then a $\mathbb{Z}_2$-grading $A = A_0 \oplus A_1$ on A is uniquely defined by setting $\deg z_1^{(1)} = \deg a = 0, \deg b = 1$. First, we will give an upper bound for the graded codimension $c_n^{gr}(A)$.

Lemma 1. *Let $c_{k,n-k}(A)$ be the partial graded codimension of A. Then $c_{k,n-k}(A) \leq 2n^2$ for all large enough n.*

Proof. Consider a left-normed monomial $M = M(x_1, \ldots, x_k, y_1, \ldots, y_{n-k})$ on even $x_1, \ldots, x_k$ and odd $y_1, \ldots, y_{n-k}$. Then $M = x_i u_1 \cdots u_{n-1}$ or $M = y_i u_1 \cdots u_{n-1}$, where $u_1, \ldots, u_{n-1}$ are some x_j's, y_j's. Let for example,

$$M = x_k \cdots x_{i_1} \cdots x_{i_{k-1}} \cdots \ , \ \{i_1, \ldots, i_{k-1}\} = \{1, \ldots, k-1\}.$$

Then $M \equiv M_0$ modulo the graded ideal $Id^{gr}(A)$, where $M_0 = x_k \cdots x_1 \cdots x_{k-1} \cdots$, since any non-zero evaluation φ of M and M_0 in A can be obtained only if $\varphi(x_k) = z_j^{(i)}, \varphi(x_1) = \cdots = \varphi(x_{k-1}) = a$. Moreover, $\varphi(M) \neq 0$ if and only if the positions of $x_{i_1}, \ldots, x_{i_{k-1}}$ in M are in 1-1 correspondence with the positions of symbol 0 in the subword $\bar{w} = w_{t+1} \cdots w_{t+n-1}$ of length $n-1$, where the integer t can be computed from the condition $z_j^{(i)} = z_1^{(1)} u_1 \cdots u_t$ for proper $u_1, \ldots, u_t \in \{a, b\}$. Similarly, $y_1, \ldots, y_{n-k}$ in M can be ordered naturally. Since $Comp_w(n-1) = n$ for Sturmian word and $Comp_w$ is bounded in the periodic case, we conclude that the number of subwords $\bar{w}$ corresponding to monomials that do not vanish on A does not exceed $kn \leq n^2$ for sufficiently large n. The same upper bound takes place for monomials of the type $y_i u_1 \cdots u_{n-1}$, and we have completed the proof.

□

Lemma 2. *For any real number $\varepsilon > 0$ there exists an integer n_0 such that conditions $n \geq n_0$, $P_{k,n-k}(A) \neq 0$ imply the inequalities*

$$\beta - \varepsilon \leq \frac{n-k}{n} \leq \beta + \varepsilon, \tag{5}$$

where $\beta = \frac{1}{m+\alpha}$ and $\alpha = \pi(w)$ is the slope of the infinite word w defining $A = A(m, w)$.

Proof. Any non-zero product of n basis elements of A has the form

$$z_j^{(i)} \underbrace{a \cdots a}_{s_0} b \underbrace{a \cdots a}_{s_1} b \cdots b \underbrace{a \cdots a}_{s_r} b \underbrace{a \cdots a}_{s_{r+1}} = z_{1+s_{r+1}}^{(i+r+1)}, \tag{6}$$

where $0 \leq s_0, s_{r+1} \leq m$,

$$s_1 = m + w_{i+1} - 1, \ldots, s_r = m + w_{i+r} - 1, \tag{7}$$

$n = s_0 + s_{r+1} + 2 + mr + w_{i+1} + \cdots + w_{i+r}$. The number of factors b in this product is equal to $r+1$. Moreover, (6) is the value of monomials from $P_{k,n-k}$ with $n - k = r + 1$. Hence

$$\frac{n-k}{n} = \frac{r+1}{s_0 + s_{r+1} + 2 + mr + w_{i+1} + \cdots + w_{i+r}} \tag{8}$$

$$= \frac{1 + \frac{1}{r}}{m + \frac{s_0 + s_{r+2} + 2}{r} + \frac{w_{i+1} + \cdots + w_{i+r}}{r}}.$$

Since $s_0 + s_{r+1} + 2 \le 2m + 2$ and $w_{i+1} + \cdots + w_{i+r} \le r$, it follows that

$$r \ge \frac{n}{m+1} - 2.$$

In particular, $r \to \infty$ if $n \to \infty$. Moreover, the limit of $\frac{1}{r}(w_{i+1} + \cdots + w_{i+r})$, as $r \to \infty$, is equal to α, by Proposition 1. It follows that the right hand side of (8) goes to $\beta = \frac{1}{m+\alpha}$ as $n \to \infty$ and hence (5) holds.

$\square$

Lemmas 1 and 2 give an upper bound for graded codimensions of A.

Lemma 3. *For any $0 < \varepsilon \le \frac{1}{2} - \frac{1}{m+\alpha}$ there exists n_0 such that*

$$c_n^{gr}(A) \le 2n^3\Phi(\beta+\varepsilon)^n$$

for all $n \ge n_0$, where $\beta = \frac{1}{m+\alpha}$. In particular, $\overline{exp}^{gr}(A) \le \Phi(\beta)$.

Proof. By Lemmas 1 and 2 we have

$$c_n^{gr}(A) \le \sum_{\beta-\varepsilon \le \frac{n-k}{n} \le \beta+\varepsilon} \binom{n}{k} c_{k,n-k}(A) \le 2n^2 \sum_{\beta-\varepsilon \le \frac{n-k}{n} \le \beta+\varepsilon} \binom{n}{k}.$$

By Stirling's formula for factorials we have

$$\binom{n}{k} \le n\frac{n^n}{k^k(n-k)^{n-k}} = n\Phi(\frac{n-k}{n})^n.$$

Since $m \ge 2$ and $0 < \alpha < 1$, we have $\beta = \frac{1}{m+\alpha} < \frac{1}{2}$ and

$$\max_{\beta-\varepsilon \le \frac{n-k}{n}} \Phi(\frac{n-k}{n}) \le \Phi(\beta+\varepsilon)$$

as soon as $\beta + \varepsilon < \frac{1}{2}$ and n is sufficiently large. Hence

$$c_n^{gr}(A) \le 2n^3\Phi(\beta+\varepsilon)^n,$$

and we are done. $\square$

Now we are ready to prove main result of this section.

Theorem 3. *Let $A = A(m, w)$ be the algebra defined by an integer $m \ge 2$ and by an infinite periodic or Sturmian word w with the slope $\pi(w) = \alpha$. Suppose that the decomposition $A = A_0 \oplus A_1$ is a $\mathbb{Z}_2$-grading of A such that $a, z_1^{(1)} \in A_0, b \in A_1$. Then the graded PI-exponent $exp^{gr}(A)$ exists and*

$$exp^{gr}(A) = exp(A) = \Phi(\frac{1}{m+\alpha}).$$

Proof. According to Lemma 3, it is enough to show that $\underline{exp}^{gr}(A) \ge \Phi(\beta)$, where $\beta = \frac{1}{m+\alpha}$. Since A is not nilpotent, there exists for any n, a non-zero product of the type (6). In particular, given n, there exists $0 \le k \le n$ such that $P_{k,n-k} \ne 0$. Then $\frac{n-k}{n} \ge \beta - \varepsilon$ asymptotically for any fixed $\varepsilon > 0$ by Lemma 2, and by Stirling's formula we have

$$c_n^{gr}(A) \ge \binom{n}{k} c_{k,n-k}(A) \ge \binom{n}{k} \ge \frac{1}{n^2}\frac{n^n}{k^k(n-k)^{n-k}} \ge \frac{1}{n^2}\Phi(\beta-\varepsilon)^n.$$

It follows that $\underline{exp}^{gr}(A) \ge \Phi(\beta)$, and thus the proof has been completed. $\square$

5. Algebras with adjoint unit

In this section we study codimensions of algebras with an external unit. Given an algebra B, we denote by $B^{\#}$ the algebra obtained by adjoining the external unit to B. Note that if $C = \oplus_{g\in G} C_g$ is a G-graded algebra with unit 1 then 1 is a homogeneous element and $1 \in C_e$, where $e \in G$ is the identity element of the group G. Therefore in the case of a $\mathbb{Z}_2$-graded algebra B, its extension $B^{\#} = B \oplus 1$ has a unique $\mathbb{Z}_2$-grading $B^{\#} = B_0^{\#} \oplus B_1^{\#}$, where B is a homogeneous subalgebra of $B^{\#}$, namely, $B_0^{\#} = B_0 \oplus 1, B_1^{\#} = B_1$.

First, let $A_0 \oplus A_1$ be an arbitrary $\mathbb{Z}_2$-graded algebra. Denote by $R\{X,Y\}$ the relatively free $\mathbb{Z}_2$-graded algebra of the variety $var^{gr}(A)$ of graded algebras generated by A with two infinite sets X and Y of even and odd generators, respectively. That is, $R\{X,Y\} = F\{X,Y\}/Id^{gr}(A)$. Consider a partial $(k, n-k)$-cocharacter of A,

$$\chi_{k,n-k}(A) = \chi(P_{k,n-k}(A)) = \sum_{\substack{\lambda\vdash k\\ \mu\vdash n-k}} m_{\lambda,\mu}\chi_{\lambda,\mu}. \tag{9}$$

In order to bound the multiplicities $m_{\lambda,\mu}$ in (9) we denote by $R^{k,n-k}_{d_0,d_1}(A)$ the subspace of polynomials on $X_{d_0} = \{x_1,\dots,x_{d_0}\}$, $Y_{d_1} = \{y_1,\dots,y_{d_1}\}$ in $R\{X,Y\}$ of total degree k on X_{d_0} and total degree $n-k$ on Y_{d_1}. The same argument as in [23] gives us the next lemma.

Recall that the height $h(\lambda)$ of a partition $\lambda = (\lambda_1,\dots,\lambda_t)$ is the number t of its parts.

Lemma 4. *Let $m_{\lambda,\mu}, \lambda \vdash k, \mu \vdash n-k$, be the multiplicity from (9) with $h(\lambda) \le d_0, h(\mu) \le d_1$. Then*

$$m_{\lambda,\mu} \le \dim R^{k,n-k}_{d_0,d_1}(A).$$

$\square$

Now, let $A = A(m,w)$ be the algebra defined by an infinite binary word $w = w_1w_2\dots$. The following lemma holds for any w, not necessarily Sturmian or periodic.

Lemma 5. *Suppose that A is $\mathbb{Z}_2$-graded and that $\deg z_1^{(1)} = \deg a = 0, \deg b = 1$. Then*

$$\dim R^{k,n-k}_{d_0,d_1}(A) \le d_0 m^2 k Comp_w(n-k).$$

Proof. Denote by W the span of all monomials

$$x_0u_1\cdots u_{n-1} \tag{10}$$

in $R\{X,Y\}$, where $u_1,\dots,u_{n-1} \in X_{d_0}\cup Y_{d_1}$, $u_{i_1},\dots,u_{i_{k-1}} \in X_{d_0}$ for some

$$i_1,\dots,i_{k-1} \in \{1,\dots,n-1\},$$

while $u_j \in Y_{d_1}$, provided that $j \ne i_1,\dots,i_{k-1}$. Clearly, $\dim R^{k,n-k}_{d_0,d_1}(A) \le d_0 \dim W$.

Let $f = f(x_0,\dots,x_{d_0},y_1,\dots,y_{d_1}) \in F\{X,Y\}$ be a linear combination of monomials of the same type as (10). Then $f \equiv 0$ is an identity of A if and only if $\sigma(f) = 0$ for any homomorphism $\sigma\colon F\{X,Y\} \to A$ such that

$$\sigma(x_0) = z_j^{(i)}, \sigma(x_s) = a, \sigma(y_s) = b.$$

Hence $\dim W$ does not exceed the codimension of the intersection of all Ker σ in $F^{k,n-k}_{d_0,d_1}$, where $F^{k,n-k}_{d_0,d_1}$ is a subspace of $F\{X,Y\}$ defined similarly as $R^{k,n-k}_{d_0,d_1}(A)$. Consider the family of graded homomorphisms $\varphi_{ij}\colon F\{X,Y\} \to A$ such that

$$\varphi_{ij}(x_0) = z^{(i)}_j, \varphi_{ij}(x_s) = a, \varphi_{ij}(y_s) = b,$$

for all $x_s \in X, y_s \in Y$. Then either $\varphi_{ij}(x_0u_1\cdots u_{n-1}) = 0$ or $\varphi_{ij}(x_0u_1\cdots u_{n-1}) = z^{(i+r+1)}_{1+s_{r+1}}$, the element from (6). The latter equality takes place if and only if $s_0 = m-1-j+w_i, 0 \le s_{r+1} \le m-1+w_{i+r+1}, n = s_0+s_{r+1}+2+mr+w_{i+1}+\cdots+w_{i+r}$, relations (7) hold and all $x_1,\ldots,x_{d_0}$ stay on "correct" positions among $u_1,\ldots,u_{n-1}$, according to the word w. In particular, codim Ker φ_{ij} in $F^{k,n-k}_{d_0,d_1}$ is less than or equal to one. Moreover, Ker φ_{ij} = Ker $\varphi_{i'j}$ if the subwords $w_{i+1}\cdots w_{i+r+1}$, $w_{i'+1}\cdots w_{i'+r+1}$ coincide. It follows that the codimension of $\bigcap$Ker φ_{ij} in $F^{k,n-k}_{d_0,d_1}$ is at most $m^2Comp_w(r+1) = m^2Comp_w(n-k)$. Since $\dim W$ is equal to codim $\bigcap$Ker φ_{ij}, we have completed the proof of the lemma. $\square$

Next, we will find an upper bound for $\dim R^{k,n-k}_{d_0,d_1}(B^{\#})$ in terms of $\dim R^{k,n-k}_{d_0,d_1}(B)$ if B is a $\mathbb{Z}_2$-graded algebra.

Lemma 6. *Given a $\mathbb{Z}_2$-graded algebra B, suppose that* $\dim R^{k,n-k}_{d_0,d_1}(B) \le \theta k(n-k)^T$ *for all $0 \le k \le n$ and for some constant θ. Then*

$$\dim R^{k,n-k}_{d_0,d_1}(B^{\#}) \le \theta(k+1)^{d_0+2}(n-k+1)^{T+d_1}.$$

Proof. Note that a multihomogeneous polynomial $f(x_1,\ldots,x_{d_0},y_1,\ldots,y_{d_1})$ is a graded identity of $B^{\#}$ if and only if all multihomogeneous on $x_1,\ldots,x_{d_0},y_1,\ldots,y_{d_1}$ components of $f(1+x_1,\ldots,1+x_{d_0},y_1,\ldots,y_{d_1}) \in F\{X,Y\}^{\#}$ are identities of B. The total number of such components does not exceed $(k+1)^{d_0}(n-k+1)^{d_1}$, provided that the degree on $\{x_1,\ldots,x_{d_0}\}$ is at most k and the degree on $\{y_1,\ldots,y_{d_1}\}$ is equal to $n-k$.

Let $f_1,\ldots,f_N \in F^{k,n-k}_{d_0,d_1}$. Consider the linear combination $f = \lambda_1f_1+\cdots+\lambda_Nf_N$ with unknown coefficients $\lambda_1,\ldots,\lambda_N$. Any multihomogeneous component $g = g(x_1,\ldots,x_{d_0},y_1,\ldots,y_{d_1})$ of $f(1+x_1,\ldots,1+x_{d_0},y_1,\ldots,y_{d_1})$ gives us at most

$$\dim R^{j,n-k}_{d_0,d_1}(B) \le \theta j(n-k)^T$$

linear equations on $\lambda_1,\ldots,\lambda_N$, provided that $g \equiv 0$ is an identity of B and the degree of g on $x_1,\ldots,x_{d_0}$ is equal to j. Hence $f \equiv 0$ is an identity of $B^{\#}$ if $\lambda_1,\ldots,\lambda_N$ satisfy no more than $\widetilde{N}$ linear equations, where

$$\widetilde{N} = (k+1)^{d_0}(n-k)^{d_1}\theta(n-k)^T\sum_{j=0}^{k} j.$$

Note that

$$\widetilde{N} \le \theta(k+1)^{d_0+2}(n-k+1)^{T+d_1}. \tag{11}$$

Therefore if N is greater than the right hand side of (11) then $f_1,\ldots,f_N$ are linearly dependent modulo $Id^{gr}(B^{\#})$ and we have completed the proof. $\square$

Now we are ready to get an upper bound for graded colength of $A^{\#}$.

Lemma 7. *Let $A = A(m,w)$, where $m \ge 2$ is an integer and w is an infinite periodic or Sturmian word. Then*

$$l_{k,n-k}(A^{\#}) \le 3m^2(k+1)^8(n-k+1)^6$$

and

$$l_n^{gr}(A^\#) \le 3m^2(n+1)^{15}.$$

Proof. Consider a partial cocharacter of $A^\#$

$$\chi_{k,n-k}(A^\#) = \sum_{\substack{\lambda\vdash k\\ \mu\vdash n-k}} m_{\lambda,\mu}\chi_{\lambda,\mu}. \tag{12}$$

The linear subspace $I = Span\{z_j^{(i)} \mid i,j \ge 1\}$ forms a homogeneous ideal of $A^\#$ with zero multiplication and

$$\dim\left(A^\#/I\right)_0 = 2, \quad \dim\left(A^\#/I\right)_1 = 1.$$

Hence any multilinear polynomial alternating on 4 even variables or on 3 odd variables is an identity of $A^\#$. Standard argument implies that $m_{\lambda,\mu} \ne 0$ in (12) only if $h(\lambda) \le 3, h(\mu) \le 2$. By Lemma 5 we have

$$\dim R_{3,2}^{k,n-k}(A) \le 3m^2k(n-k+1) \le 3m^2k(n-k)^2.$$

Then by Lemmas 4 and 6,

$$m_{\lambda,\mu} \le \dim R_{3,2}^{k,n-k}(A^\#) \le 3m^2k(k+1)^5(n-k+1)^4.$$

The number of summands on the right hand side of (12) is not greater than $k^3(n-k)^2$, hence

$$l_{k,n-k}(A^\#) \le 3m^2k(k+1)^8(n-k+1)^6$$

and $l_n^{gr}(A^\#) \le 3m^2(n+1)^{15}$. $\square$

Now we specify necessary conditions for inequality $m_{\lambda,\mu} \ne 0$ in (12).

Lemma 8. *Let $A = A(m,w)$ and suppose that w is a Sturmian or periodic word with the slope α. Suppose that $m_{\lambda,\mu} \ne 0$ in (12), where $\lambda \vdash k, \mu \vdash n-k$. Then λ and μ satisfy the following conditions:*

1. *$\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_3 \le 1$;*
2. *$\mu = (\mu_1, \mu_2)$ with $\mu_2 \le 1$;*
3. *$\lambda_1 + \lambda_2 + \mu_1 = n$ or $n-1$; and*
4. *for any $0 < \varepsilon < \frac{1}{2} - \beta$ there exists an integer n_0 such that*

$$\mu_1 \le \frac{\beta+\varepsilon}{1-\beta-\varepsilon}\lambda_1$$

for all $n \ge n_0$, where $\beta = \frac{1}{m+\alpha}$.

Proof. Any multilinear polynomial f containing an alternating set of order 4 on even variables vanishes on $A^\#$. If $A^\#$ contains two alternating sets of order 3 on even variables then also $f \in Id^{gr}(A^\#)$. From the structure of essential idempotents of group ring FS_k it follows that $\lambda_4 = 0, \lambda_3 \le 1$. This proves 1. Similar argument gives us 2.

Let $\lambda = (\lambda_1, \lambda_2, 1), \mu = (\mu_1, 1)$. If M is an irreducible $S_k \times S_{n-k}$-submodule of $P_{k,n-k} \subset F\{X,Y\}$ with the character $\chi(M) = \chi_{\lambda,\mu}$ then M is generated by a polynomial $f(x_1,\dots,x_k,y_1,\dots,y_{n-k})$ alternating on x_1,x_2,x_3 and on y_1,y_2. If we evaluate x_1,x_2,x_3 on $\{1,a\}$ and y_1,y_2 on $\{b\}$ then we get zero. Otherwise, x_1,x_2,x_3 should be equal to $1,a,z_j^{(1)}$, and y_1,y_2 should be equal to $b,z_s^{(r)}$. In this case the value is also zero. Hence $\lambda_3+\mu_2 \le 1$, and we obtain 3.

Let us prove 4. If $m_{\lambda,\mu} \ne 0$ then there exists a polynomial

$$f = f(x_1,\dots,x_k,y_1,\dots y_{n-k})$$

which generates an irreducible $S_k \times S_{n-k}$-module with the character $\chi_{\lambda,\mu}$ and an evaluation $\varphi\colon X \cup Y \to A^{\#}$ such that $\varphi(f) \neq 0$. Moreover, f contains λ_2 disjoint alternating sets of $x's$ of order 2. The set of values $\{\varphi(x_1), \dots, \varphi(x_k)\}$ contains $\lambda_1 \geq p \geq \lambda_2 - 1$ elements a, at most one element $z_j^{(i)}$ and $k-p$ or $k-p-1$ units. The set $\{\varphi(y_1), \dots, \varphi(y_{n-k})\}$ contains at most one odd $z_j^{(i)}$ and $q = \mu_1 - 1$ or μ_1 elements b.

Furthermore, there exists a non-zero product

$$g = z_j^{(i)} a \cdots aba \cdots ab \cdots ba \cdots a$$

which is equal (up to a scalar factor) to $\varphi(f)$. Denote by $p = \deg_a g, q = \deg_b g$ the numbers of entries of a and b in g, respectively. If the total degree $N = \deg g = 1 + p + q$ increases then there exists a corellation between the growth of p and q (provided that $g \neq 0$). Namely,

$$\lim_{n\to\infty} \frac{q}{q+p} = \beta = \frac{1}{m+\alpha},$$

hence

$$\lim_{n\to\infty} \frac{q}{p} = \frac{\beta}{1-\beta}.$$

It follows that there exists r such that for any $q \geq r+1$ (and for corresponding p) we have

$$\frac{q}{p} \leq \frac{\beta + \varepsilon/2}{1 - (\beta + \varepsilon/2)} \qquad \text{and} \qquad \frac{1}{p} + \frac{\beta + \varepsilon/2}{1 - (\beta + \varepsilon/2)} \leq \frac{\beta + \varepsilon}{1 - (\beta + \varepsilon)}.$$

Since $\mu_1 - 1 \leq q$ and $p \leq \lambda_1$, we get

$$\frac{\mu_1 - 1}{\mu_1} \leq \frac{q}{p} \leq \frac{\beta + \varepsilon/2}{1 - (\beta + \varepsilon/2)}$$

and

$$\frac{\mu_1}{\lambda_1} \leq \frac{q}{p} + \frac{1}{\lambda_1} \leq \frac{q}{p} + \frac{1}{p} \leq \frac{\beta + \varepsilon}{1 - (\beta + \varepsilon)},$$

provided that $\mu_1 \geq r$.

On the other hand, if $\mu_1 < r$ then

$$\frac{n}{\mu_1} > \frac{n}{r} \qquad \text{and} \qquad \frac{\lambda_1}{\mu_1} > \frac{n}{2r} - 1$$

since $2\lambda_1 + \mu_1 \geq \lambda_1 + \lambda_2 + \mu_1 \geq n - 1$. Denote for short $\gamma = \frac{\beta+\varepsilon}{1-(\beta+\varepsilon)}$. Then for all $n \geq \frac{2(\gamma+1)}{\gamma} r$ we have

$$\frac{n}{2r} \geq \frac{1}{\gamma} + 1, \quad \frac{\lambda_1}{\mu_1} > \frac{1}{\gamma}.$$

This proves 4. $\square$

In order to get an upper bound for graded codimensions we need some properties of the function $\Phi(x_1, \dots, x_d)$ introduced in Section 2. Recall that $\Phi(x_1, x_2, x_3) = x_1^{-x_1} x_2^{-x_2} x_3^{-x_3}$, where $0 \leq x_1, x_2, x_3 \in \mathbb{R}$, $x_1 + x_2 + x_3 = 1$.

Lemma 9. *Let $x_3 = \gamma x_2$ for a fixed coefficient γ. Then*

$$\max \Phi(x_1, x_2, x_3) = \frac{1+\gamma}{\gamma^{\gamma/(\gamma+1)}} + 1.$$

Proof. Denote $x = x_1$. Then the relations $x_1 + x_2 + x_3 = 1$, $x_3 = \gamma x_2$ imply

$$x_2 = \frac{1-x}{1+\gamma}, \quad x_3 = \frac{\gamma}{1+\gamma}(1-x).$$

Denote also $\Phi(x_1, x_2, x_3) = f(x)$. Then

$$f^{-1}(x) = x^x \left(\frac{1-x}{1+\gamma}\right)^{\frac{1-x}{1+\gamma}} \left(\frac{\gamma(1-x)}{1+\gamma}\right)^{\frac{\gamma(1-x)}{1+\gamma}}$$

and

$$g(x) = \ln f^{-1}(x) = x\ln x + \frac{1-x}{1+\gamma}\ln\frac{1-x}{1+\gamma} + \frac{\gamma(1-x)}{1+\gamma}\ln\frac{\gamma(1-x)}{1+\gamma}.$$

Hence

$$g'(x) = \ln\frac{x}{(\frac{1-x}{1+\gamma})^{\frac{1}{1+\gamma}}(\frac{\gamma(1-x)}{1+\gamma})^{\frac{\gamma}{1+\gamma}}}$$

and $g'(\widetilde{x}) = 0$ only if

$$\widetilde{x} = \left(\frac{1-\widetilde{x}}{1+\gamma}\right)^{\frac{1}{1+\gamma}} \left(\frac{\gamma(1-\widetilde{x})}{1+\gamma}\right)^{\frac{\gamma}{1+\gamma}} \gamma^{\frac{\gamma}{1+\gamma}} = (1-\widetilde{x})\rho,$$

where

$$\rho = \frac{\gamma^{\frac{\gamma}{1+\gamma}}}{1+\gamma}.$$

That is,

$$\widetilde{x} = \frac{\rho}{1+\rho}.$$

Since

$$g'(x) = \ln\frac{x}{1-x} + const$$

on the interval $(0;1)$, we see that $f^{-1}(x)$ has a local minimum in $\widetilde{x}$. Direct computations show that $f^{-1}(\widetilde{x}) = \widetilde{x}$ and

$$\max\Phi = f(\widetilde{x}) = \frac{1}{\widetilde{x}} = 1 + \frac{1}{\theta} = \frac{1+\gamma}{\gamma^{\frac{\gamma}{1+\gamma}}} + 1.$$

$\square$

Now we are ready to compute the required upper bound for the upper PI-exponent.

Remark 1. *If we denote $\frac{\gamma}{\gamma+1}$ by θ then $\gamma = \frac{\theta}{1-\theta}$. In this case direct computations give us*

$$\frac{1+\gamma}{\gamma^{\frac{\gamma}{1+\gamma}}} = \frac{1}{\theta^\theta(1-\theta)^{1-\theta}} = \Phi(\theta).$$

Moreover, if $\gamma_1 < \gamma_2 \le 1$ then $\theta_1 < \theta_2$ and $\Phi(\theta_1) < \Phi(\theta_2)$.

Lemma 10.

$$\overline{exp}^{gr}(A^\#) \le exp^{gr}(A) + 1.$$

Proof. By (3)

$$c_n^{gr}(A^\#) \le l_n^{gr}(A^\#)\sum_k \sum_{\substack{\lambda\vdash k,\mu\vdash n-k\\ m_{\lambda,\mu}\neq 0}} \binom{n}{k} d_\lambda d_\mu. \tag{13}$$

First estimate a fixed summand $\binom{n}{k} d_\lambda d_\mu$ provided that $m_{\lambda,\mu} \neq 0$. By Lemma 8, we have $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $\lambda_3 \leq 1$, $\mu = (\mu_1, \mu_2)$, $\mu_2 \leq 1$. By the Hook formula for degree of an irreducible representation,

$$d_\lambda \leq \frac{k!}{\lambda_1! \lambda_2!}, \quad d_\mu \leq n - k - 1 \leq n.$$

Since $n - k = \mu_1$ or $\mu_1 - 1$, we have $n - k + 1 \geq \mu_1$ and $n(n-k)! \geq \mu_1!$. Also, $n - 2 \leq \lambda_1 + \lambda_2 + \mu_1 \leq n$, that is $n! \leq (\lambda_1 + \lambda_2 + \mu_1)!(n+2)^2$. Therefore

$$\binom{n}{k} d_\lambda d_\mu \leq \frac{n!}{k!(n-k)!} \cdot \frac{k!}{\lambda_1! \lambda_2!} \cdot n \leq (n+2)^4 \frac{(\lambda_1 + \lambda_2 + \mu_1)!}{\lambda_1! \lambda_2! \mu_1!}. \tag{14}$$

By the Stirling's formula

$$\frac{(\lambda_1 + \lambda_2 + \mu_1)!}{\lambda_1! \lambda_2! \mu_1!} \leq n \frac{(\lambda_1 + \lambda_2 + \mu_1)^{\lambda_1 + \lambda_2 + \mu_1}}{\lambda_1^{\lambda_1} \lambda_2^{\lambda_2} \mu_1^{\mu_1}} \leq n \Phi(x_1, x_2, x_3)^n, \tag{15}$$

where

$$x_1 = \frac{\lambda_2}{\lambda_1 + \lambda_2 + \mu_1}, \quad x_2 = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \mu_1}, \quad x_3 = \frac{\mu_1}{\lambda_1 + \lambda_2 + \mu_1}.$$

Denote $\mu_1 / \lambda_1 = \gamma$. Then $x_3 = \gamma x_2$, and by Lemma 9 and Remark 1,

$$\Phi(x_1, x_2, x_3) \leq \Phi(\theta) + 1,$$

where $\theta = \frac{\gamma}{\gamma+1}$. Fix an arbitrary small $\varepsilon > 0$. We can assume that

$$\frac{\beta + \varepsilon}{1 - \beta - \varepsilon} < 1.$$

Then by Lemma 8 we get that $\gamma < 1$, θ is an increasing function of γ on interval $(0;1)$ and $\theta \leq \frac{1}{2}$. Hence $\Phi(\theta)$ is also an increasing function of γ and

$$\Phi(x_1, x_2, x_3) \leq 1 + \Phi(\beta + \varepsilon)$$

for all sufficiently large n. Applying (13), (14), (15) and Lemma 7 and taking into account that the number of partitions $\lambda \vdash k$ with $h(\lambda) \leq 3, \lambda_3 \leq 1$ is not greater than n, we obtain

$$c_n^{gr}(A^\#) \leq 6m^2(n+2)^{22}(1 + \Phi(\beta + \varepsilon))^n,$$

from which it follows that

$$\overline{exp}^{gr}(A^\#) \leq 1 + \Phi(\beta) = 1 + exp^{gr}(A).$$

□

Theorem 4. *Let $A = A(m,w) = A_0 \oplus A_1$ be the algebra defined by an integer $m \geq 2$ and by Sturmian or periodic word w equipped with a $\mathbb{Z}_2$-grading, where generators $z_1^{(1)}$ and a are even whereas b is odd. Let $A^\#$ be obtained from A by adjoining the external unit. Then its graded PI-exponent exists and*

$$exp^{gr}(A^\#) = 1 + exp^{gr}(A).$$

Proof. By [19, Theorem 1], $exp(A^\#) = exp(A) + 1$. Hence $\underline{exp}^{gr}(A^\#) \geq exp(A^\#) = exp(A) + 1 = exp^{gr}(A) + 1$ by (4) and Theorem 3. Now our statement follows from Lemma 10. □

In conclusion, we discuss other $\mathbb{Z}_2$-gradings on $A = A(m,w)$. In the proof of Theorems 3 and 4 we have never used the fact that $\deg z_1^{(1)} = 0$. Hence the same results hold for graded codimensions if $\deg z_1^{(1)} = \deg b = 1, \deg a = 0$.

By slightly modifying arguments, one can prove Theorems 3 and 4, provided that $\deg a = 1, \deg b = 0$. Finally, if $\deg a = \deg b = 1$ then the argument is similar to that of [6], [19]. Therefore we can generalize Theorems 3 and 4 as follows.

Theorem 5. *Let $A = A(m, w)$ be the algebra defined by an integer $m \geq 2$ and by an infinite periodic or Sturmian word w with the slope $\pi(w) = \alpha$. Suppose that the decomposition $A = A_0 \oplus A_1$ is a $\mathbb{Z}_2$-grading such that the generators $a, b, z_1^{(1)}$ are homogeneous. Then the graded PI-exponent $exp^{gr}(A)$ exists and*

$$exp^{gr}(A) = exp(A) = \Phi(\frac{1}{m+\alpha}),$$

where

$$\Phi(x) = \frac{1}{x^x(1-x)^{(1-x)}}.$$

Moreover, if $A^\#$ is obtained from A by adjoining an external unit with the induced $\mathbb{Z}_2$-grading then $exp^{gr}(A^\#)$ also exists and

$$exp^{gr}(A^\#) = exp^{gr}(A) + 1.$$

$\square$

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