QUOTIENT AND BLOW-UP OF AUTOMORPHISMS OF GRAPHS OF GROUPS

KAIDI YE

ABSTRACT. In this paper we study the quotient and "blow-up" of graphof-groups \mathcal{G} and of their automorphisms $H: \mathcal{G} \to \mathcal{G}$. We show that the existence of such a blow-up of any $\overline{H}: \overline{\mathcal{G}} \to \overline{\mathcal{G}}$, relative to a given family of "local" graph-of-groups isomorphisms $H_i: \mathcal{G}_i \to \mathcal{G}_i$ depends crucially on the H_i^{-1} -conjugacy class of the correction term $\delta(E_i)$ for any edge E_i of $\overline{\mathcal{G}}$, where *H*-conjugacy is a new but natural concept introduced here.

As an application we obtain a criterion as to whether a partial Dehn twist can be blown up relative to local Dehn twists, to give an actual Dehn twist. The results of this paper are also used crucially in the follow-up papers [12, 21, 22].

1. INTRODUCTION

Graphs-of-groups and Bass-Serre theory have played a central role in geometric group theory ever since this field came into existence in the 1980's. As a prime example we would like to mention its prominent role in the understanding of automorphisms of a hyperbolic group G, see [11], which is based on an essentially unique graph-of-groups decomposition of G, in case that G is freely indecomposable.

If on the other hand the group G is a free group F_n of finite rank $n \ge 2$, then a special kind of graph-of-groups, called *very small*, plays an important role in the definition of the boundary of Culler-Vogtmann's Outer space CV_n , which is the analogue of Teichmüller space, for $Out(F_n)$ in place of the mapping class group. The work presented here is mainly motivated by questions arising from this perspective, although we keep our set-up slightly more general.

Given a graph-of-groups \mathcal{G} based on a finite connected graph $\Gamma = \Gamma(\mathcal{G})$, for any connected subgraph $\Gamma_0 \subset \Gamma$ we denote by \mathcal{G}_0 the restriction of \mathcal{G} to Γ_0 . There is a natural way to define a quotient graph-of-groups $\overline{\mathcal{G}} = \mathcal{G}/\mathcal{G}_0$ which is obtained by "contracting" \mathcal{G}_0 into a single vertex V_0 with vertex group $G_{V_0} \cong \pi_1(\mathcal{G}_0)$, thus giving rise to a canonical isomorphism $\Theta : \pi_1(\mathcal{G}) \to \pi_1(\overline{\mathcal{G}})$. By construction the quotient graph $\overline{\mathcal{G}}$ is *compatible* with the local graph-of-groups \mathcal{G}_0 , in the sense that for any edge E of $\overline{\mathcal{G}}$ with terminal vertex V_0 the canonical image of the edge group G_E in the vertex group

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 $G_{V_0} \cong \pi_1(\mathcal{G}_0)$ is (up to conjugation) contained in one of the vertex groups of \mathcal{G}_0 .

This quotient concept extends naturally to an isomorphism H of \mathcal{G} which acts as identity on the underlying graph Γ : We can construct a quotient graph-of-groups isomorphism $\overline{H}:\overline{\mathcal{G}}\to\overline{\mathcal{G}}$ which induces on the fundamental group an outer automorphism $\widehat{\overline{H}}$ that is conjugate via Θ to the outer automorphism \widehat{H} induced by H, as shown in the diagram below.

$$\begin{array}{ccc} \pi_1(\mathcal{G}) & \xrightarrow{\widehat{H}} & \pi_1(\mathcal{G}) \\ \Theta & & \Theta \\ \pi_1(\overline{\mathcal{G}}) & \xrightarrow{\widehat{H}} & \pi_1(\overline{\mathcal{G}}) \end{array}$$

The restriction H_0 of H to \mathcal{G}_0 is called the *local graph-of-groups isomorphism* at V_0 , and again certain natural *compatibility conditions* between the pairs $(\overline{H}, \overline{\mathcal{G}})$ and (H_0, \mathcal{G}_0) are satisfied, which are stated precisely in Definition 6.1 below.

Of course, both, the quotient graph-of-groups $\overline{\mathcal{G}}$ and the quotient isomorphism \overline{H} , are also well-defined modulo more than one pairwise disjoint connected sub-graph-of-groups \mathcal{G}_i of \mathcal{G} .

The main purpose of this paper is to study the converse of the above described quotient procedure, which we call the "blow-up" of a graph-of-groups isomorphism. We prove (see Theorem 6.3):

Theorem 1.1. Let $\overline{H} : \overline{\mathcal{G}} \to \overline{\mathcal{G}}$ be a graph-of-groups isomorphism which acts as identity on the graph $\overline{\Gamma}$ underlying $\overline{\mathcal{G}}$. Assume that for some vertices V_i of $\overline{\Gamma}$ the group isomorphism \overline{H}_{V_i} is induced by a local graph-of-groups automorphism $H_i : \mathcal{G}_i \to \mathcal{G}_i$ which also acts as identity on the underlying graph $\Gamma(\mathcal{G}_i)$.

Then one can blow up $(\overline{H}, \overline{\mathcal{G}})$ via the family of (H_i, \mathcal{G}_i) to obtain a blow-up graph-of-groups isomorphism $H : \mathcal{G} \to \mathcal{G}$, with induced outer automorphism $\widehat{H} = \widehat{\overline{H}}$, if and only if each (H_i, \mathcal{G}_i) is compatible (in the sense of Definition 6.1) with $(\overline{H}, \overline{\mathcal{G}})$.

We are most interested in the special case where for any edge E of $\overline{\mathcal{G}}$ the edge group G_E is trivial. In this case the compatibility conditions from Definition 6.1 simplify to a property of the edge E which we call "locally zero". Since this is a new concept, we will try to explain it here briefly:

Recall first that if E terminates in the vertex V_i , then (as for any graphof-groups isomorphism, see Definition 2.8) the correction term $\delta(E) \in G_{V_i}$ serves to make the edge and vertex isomorphisms \overline{H}_E and \overline{H}_V commute with the injective edge homomorphism $f_E: G_E \to G_{V_i}$.

Now, we say that E is *locally zero* (see Definition 7.1) if the identification $G_{V_i} \cong \pi_1 \mathcal{G}_i$ maps $\delta(E)$ to an element which is " H_i^{-1} -conjugate" to an element

that has \mathcal{G}_{V_i} -length equal to zero. If the local automorphism H_i is equal to the identity map, then H_i^{-1} -conjugation will simply be the usual conjugation in G_{V_i} ; in general though it is a more involved and quite delicate new notion, defined below in section 4.

In the last section of this paper we will apply Theorem 1.1 to the case of Dehn twist automorphisms of a free group F_n . Classically, a *Dehn twist* $D = (\mathcal{G}, (z_e)_{e \in E(\mathcal{G})})$ on a graph-of-groups \mathcal{G} is defined by a family of *twistors* $(z_e)_{e \in E(\mathcal{G})}$, where each z_e is in the center of the edge group G_e of \mathcal{G} . It turns out (see Proposition 3.8 and Remark 3.12) that for free groups an alternative, slightly more restrictive definition of a Dehn twist is given by graph-of-groups isomorphisms $H : \mathcal{G} \to \mathcal{G}$ where all edge groups of \mathcal{G} are trivial and H acts as identity on the underlying graph and on every vertex group of \mathcal{G} .

Inspired by this alternative definition, we define (see Definition 3.13) a partial Dehn twist $D : \mathcal{G} \to \mathcal{G}$ with $\pi_1(\mathcal{G}) \cong F_n$, relative to some family of vertices V_1, \ldots, V_m of the underlying graph $\Gamma(\mathcal{G})$, which differs from the above classical notion in that on these "exceptional vertices" V_i the local automorphism $D_{V_i} : G_{V_i} \to G_{V_i}$ induced by D may be non-trivial.

Of particular interest is the case where these non-trivial local automorphisms are all Dehn twist automorphisms themselves. This occurs naturally if one quotients a given Dehn twists modulo a family of pairwise disjoint subgraphs. The converse direction, however, is far less obvious, and the desired blow-up Dehn twist doesn't always exist. We prove here (see Corollary 7.2):

Corollary 1.2. (1) Let $\overline{D} : \overline{\mathcal{G}} \to \overline{\mathcal{G}}$ be a partial Dehn twist, and assume that for some family of vertices V_i of $\overline{\mathcal{G}}$ the vertex group automorphisms D_{V_i} are induced by Dehn twists $D_{V_i} : \mathcal{G}_{V_i} \to \mathcal{G}_{V_i}$.

Then \overline{D} can be blown up via the given family of local Dehn twists D_{V_i} to give a graph-of-groups isomorphism $D: \mathcal{G} \to \mathcal{G}$ if and only if every edge E_i of $\overline{\mathcal{G}}$ with terminal endpoint in one of the V_i is locally zero.

(2) The blow-up automorphism $D: \mathcal{G} \to \mathcal{G}$ obtained in (1) is a Dehn twist, and hence $\widehat{D} = \widehat{\overline{D}}$ is a Dehn twist automorphism.

It turns out that the last conclusion of the above corollary is more subtle than it may appear at first sight. In order to explain this, we first consider the following two examples:

Example 1.3. (1) We consider a graph-of-groups isomorphism $\overline{H} : \overline{\mathcal{G}} \to \overline{\mathcal{G}}$ defined as the follows:

(a) The graph $\Gamma(\overline{\mathcal{G}})$ underlying $\overline{\mathcal{G}}$ consists of a single edge E and two distinct vertices $V = \tau(\overline{E}) \neq V_1 = \tau(E)$. The graph-of-groups $\overline{\mathcal{G}}$ has trivial edge group G_E , hence trivial edge homomorphisms, and vertex groups $G_V = \langle a, b \rangle$, $G_{V_1} = \langle c \rangle$.

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(b) The isomorphism \overline{H} acts as the identity on $\Gamma(\overline{\mathcal{G}})$ and induces trivial group automorphisms on G_E and on G_{V_1} , while the local group automorphism $\overline{H}_V : G_V \to G_V$ is a Dehn twist automorphism which acts on the generators by the map $a \mapsto a$ and $b \mapsto ba$. The correction terms are $\delta(E) = 1_{G_{V_1}}$ and $\delta(\overline{E}) = aba^{-1}b^{-1}$. In particular, \overline{H} is a partial Dehn twist relative to the vertex V.

(2) We now consider a local graph-of-groups isomorphism $H_V : \mathcal{G}_V \to \mathcal{G}_V$ which induces the same Dehn twist automorphism as \overline{H}_V . The isomorphism H_V is defined by the following data:

- (a) The graph-of-groups \mathcal{G}_V consists of a single vertex v with $G_v = \langle x \rangle$ and a loop edge e with trivial edge group. The isomorphism H_V is a Dehn twist, in that it acts trivially on the underlying graph $\Gamma(\mathcal{G}_V)$, the edge group G_e and vertex group G_v . We choose the correction terms to be $\delta(e) = x$ and $\delta(\overline{e}) = 1_{G_v}$.
- (b) Then H_V induces on its fundamental group $\pi_1(\mathcal{G}_V) \cong \langle x, t_e \rangle$ an automorphism which sends $x \mapsto x$ and $t_e \mapsto t_e x$. This is exactly the same automorphism as \overline{H}_V , modulo the identification map θ given by $a \mapsto x$ and $b \mapsto t_e$.

However, in this example \overline{H} cannot be blown up via H_V , since the correction term $\delta(\overline{E})$ is mapped by θ to $xt_ex^{-1}t_e^{-1}$ which is not H_V^{-1} -conjugate to any element that has G_V -length equal to zero: the edge \overline{E} is not locally-zero.

Example 1.4. Let $(\overline{H}, \overline{\mathcal{G}})$ be the partial Dehn twist defined in Example 1.3. Instead of (H_V, \mathcal{G}_V) we now consider a local Dehn twist (H'_V, \mathcal{G}'_V) where \mathcal{G}'_V consists of a single vertex v' with $G'_v = \langle x, y \rangle$, a loop edge e' with cyclic edge group $G_{e'} = \langle z \rangle$, and edge homomorphisms that maps z to $f_{\overline{e'}}(z) = y$ and $f_{e'}(z) = x$. The correction terms are $\delta(e') = x$ and $\delta(\overline{e'}) = 1_{G_{v'}}$.

Then the fundamental group of \mathcal{G}'_V is $\pi_1(\mathcal{G}'_V) \cong \langle x, y, t_{e'} | y = t_{e'} x t_{e'}^{-1} \rangle \cong \langle x, t_{e'} \rangle$, and H'_V induces \overline{H}_V via the identification $\theta' : a \mapsto x; b \mapsto t_{e'}$.

Contrary to the previous example, one can indeed blow up $(\overline{H}, \overline{\mathcal{G}})$ via (H'_V, \mathcal{G}'_V) since the identification θ' maps $\delta(\overline{E})$ to $xt_{e'}x^{-1}t_{e'}^{-1} = xy^{-1} \in G_{v'}$: in this example the edge \overline{E} is locally-zero.

It is easy to see that both examples represent the same outer automorphisms of F_3 . This shows that the last conclusion of Corollary 1.2, namely that the given partial Dehn twist \overline{D} induces on $\pi_1 \overline{\mathcal{G}}$ a Dehn twist automorphism \overline{D} , is not equivalent to the fact that the blow-up automorphism exists and is a Dehn twist. However, using the terminology of [9], it is shown in [21] that, when all the local Dehn twists D_{V_i} are "efficient" (as is the case in Example 1.4), then the condition that all edges E_i are locally zero is not just sufficient but also necessary for the last conclusion of Corollary 1.2. This is used in [22] as an important ingredient of the proof that every linearly growing outer automorphism of a finitely generated free group F_n is (up to taking powers) a Dehn twist automorphism.

Since this paper has been made public first, its results have already been used crucially in two applications:

- (1) Corollary 1.2 is a vital ingredient in our algorithmic solution in [22] to the question which polynomially growing automorphisms of F_n are (up to passing to a power) induced by a surface homeomorphism.
- (2) In [12] a normal form (based on graph-of-groups and Dehn twists) for quadratically growing automorphisms is given, together with a method how to derive this normal form. One of the crucial steps in this procedure is to quotient subgraphs and to blow up vertices relative to local Dehn twists, as studied here.

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2. Basics of Graphs of groups and their Isomorphisms

In this section we recall some basic knowledge about graph-of-groups as well as their isomorphisms. Most of our notations are taken from [9]; we refer the readers to [19], [18] and [1] for more detailed information and discussions.

2.1. Basic Conventions.

Unless otherwise stated, a graph refers to a finite, non-empty, connected graph in the sense of Serre (cf. [19]).

We recall the notations here. For a graph Γ , we denote by $V(\Gamma)$, $E(\Gamma)$ its vertex set and edge set respectively. An edge $e \in E(\Gamma)$ is oriented, and we denote by \overline{e} the edge with inverse orientation, $\tau(e)$ its terminal vertex and $\tau(\overline{e}) = \iota(e)$ its initial vertex.

Notice in particular that our graph Γ is non-oriented. An *orientation* of Γ refers to a subset $E^+(\Gamma) \subset E(\Gamma)$ such that $E^+(\Gamma) \cup \overline{E}^+(\Gamma) = E(\Gamma)$ and $E^+(\Gamma) \cap \overline{E}^+(\Gamma) = \emptyset$, where $\overline{E}^+(\Gamma) = \{\overline{e} \mid e \in E^+(\Gamma)\}$.

For an arbitrary group G, we denote by $ad_x : G \to G$ the inner automorphism defined by element $x \in G$, namely $ad_x(g) = xgx^{-1}$ for all $g \in G$.

2.2. Graphs of Groups.

Definition 2.1. A graph-of-groups \mathcal{G} is defined by

$$\mathcal{G} = (\Gamma, (G_v)_{v \in V(\Gamma)}, (G_e)_{e \in E(\Gamma)}, (f_e)_{e \in E(\Gamma)})$$

where:

- (1) Γ is a graph, called the *underlying* graph;
- (2) each G_v is a group, called the *vertex group* of v;
- (3) each G_e is a group, called the *edge group* of e, and we require $G_e = G_{\overline{e}}$ for every $e \in E(\Gamma)$;
- (4) for each $e \in E(\Gamma)$, the map $f_e : G_e \to G_{\tau(e)}$ is an injective edge homomorphism.

For a graph-of-groups \mathcal{G} , we usually denote by $\Gamma(\mathcal{G})$ the graph underlying it. The vertex set of $\Gamma(\mathcal{G})$ is denoted by $V(\mathcal{G})$ while the edge set is denoted by $E(\mathcal{G})$.

Definition 2.2. For a graph-of-groups \mathcal{G} , its word group $W(\mathcal{G})$ is the free product of all vertex groups and of the free group generated by the *stable letter* t_e for every $e \in E(\Gamma)$, i.e.

$$W(\mathcal{G}) = \underset{v \in V(\Gamma)}{*} G_v * F(\{t_e \mid e \in E(\Gamma)\}).$$

The path group (sometimes also called Bass group) of \mathcal{G} is defined by

$$\Pi(\mathcal{G}) = W(\mathcal{G})/R$$

where R is the normal subgroup determined by the following relations:

 $\begin{array}{l} \diamond \ t_e = t_{\overline{e}}^{-1}, \, \text{for every } e \in E(\Gamma); \\ \diamond \ f_{\overline{e}}(g) = t_e f_e(g) t_e^{-1}, \, \text{for every } e \in E(\Gamma) \, \text{and every } g \in G_e. \end{array}$

Remark 2.3. A word $w \in W(\mathcal{G})$ can always be written in the form

$$w = r_0 t_1 r_1 \dots r_{q-1} t_q r_q \quad (q \ge 0),$$

where each $t_i \in \{t_e \mid e \in E(\Gamma)\}$ and each $r_i \in \underset{v \in V(\Gamma)}{*} G_v$.

The sequence $(t_1, t_2, ..., t_q)$ is called the *path type* of w, the number q is called the *path length*, or sometimes the \mathcal{G} -length of w, denoted by $|w|_{\mathcal{G}} = q$. In this case, we say that $e_1e_2...e_q$ is the path underlying w. Two path types $(t_1, t_2, ..., t_q)$ and $(t'_1, t'_2, ..., t'_s)$ are said to be same if and only if q = s and $t_i = t'_i$ for each $1 \leq i \leq q$.

Definition 2.4. Let $w \in W(\mathcal{G})$ be a word of the form $w = r_0 t_1 r_1 \dots r_{q-1} t_q r_q$. The word w is said to be *connected* if $r_0 \in G_{\tau(\overline{e}_1)}$, $r_q \in G_{\tau(e_q)}$, and $\tau(e_i) = \tau(\overline{e}_{i+1})$ with $r_i \in G_{\tau(e_i)}$, for $i = 1, 2, \dots, q-1$. We sometimes call such w a *connecting word from* $\tau(\overline{e}_1)$ to $\tau(e_q)$, or a word from $\tau(\overline{e}_1)$ to $\tau(e_q)$.

Moreover, if w is connected and $\tau(e_q) = \tau(\overline{e}_1)$, we say that w is a closed connected word issued at the vertex $\tau(e_q)$.

Definition 2.5. Let $w = r_0 t_1 r_1 \dots r_{q-1} t_q r_q \in W(\mathcal{G})$, w is said to be reduced if it satisfies:

- \diamond if q = 0, then $w = r_0$ isn't equal to the unit element;
- \diamond if q > 0, then whenever $t_i = t_{i+1}^{-1}$ for some $1 \le i \le q-1$ we have $r_i \notin f_{e_i}(G_{e_i})$.

Moreover the word w is said to be *cyclically reduced* if it is reduced and if q > 0 and $t_1 = t_q^{-1}$, then $r_q r_0 \notin f_{e_q}(G_{e_q})$.

We recall the following well known facts.

Proposition 2.6. For any graph-of-groups \mathcal{G} , the following holds:

- (1) Every non-trivial element of $\Pi(\mathcal{G})$ can be represented as a reduced word.
- (2) Every reduced word is a non-trivial element in $\Pi(\mathcal{G})$.

(3) If $w_1, w_2 \in W(\mathcal{G})$ are two reduced words representing the same element in $\Pi(\mathcal{G})$, then w_1 and w_2 are of the same path type. In particular, w_2 is connected if and only if w_1 is connected.

In fact, suppose $w_1 = r_0 t_1 r_1 \dots r_{q-1} t_q r_q$ and $w_2 = r'_0 t_1 r'_1 \dots r'_{q-1} t_q r'_q$, then there exist elements $h_i \in G_{e_i}$ (i = 1, 2, ..., q) such that:

$$r'_0 = r_0 f_{\overline{e}_1}(h_1); r'_i = f_{e_i}(h_i) r_i f_{\overline{e}_{i+1}}(h_{i+1}^{-1}) \text{ for } (i = 1, 2, ..., q - 1);$$

and $r'_q = f_{e_q}(h_q) r_q.$

Definition 2.7. For any $v_0 \in V(\Gamma)$ the fundamental group based at v_0 , denoted by $\pi_1(\mathcal{G}, v_0)$, consists of the elements in $\Pi(\mathcal{G})$ that are represented by closed connected words issued at v_0 .

For a vertex $w_0 \in V(\Gamma)$ different from v_0 , we have $\pi_1(\mathcal{G}, v_0) \cong \pi_1(\mathcal{G}, w_0)$. In fact, let $W \in \Pi(\mathcal{G})$ be represented by a connected word with underlying path from v_0 to w_0 . The restriction of $ad_W : \Pi(\mathcal{G}) \to \Pi(\mathcal{G})$ to $\pi_1(\mathcal{G}, w_0)$ induces an isomorphism from $\pi_1(\mathcal{G}, w_0)$ to $\pi_1(\mathcal{G}, v_0)$. Sometimes we write $\pi_1(\mathcal{G})$ when the choice of basepoint doesn't make a difference.

2.3. Graph-of-Groups Isomorphism.

Definition 2.8. Let \mathcal{G}_1 , \mathcal{G}_2 be two graphs of groups. Denote $\Gamma_1 = \Gamma(\mathcal{G}_1)$ and $\Gamma_2 = \Gamma(\mathcal{G}_2)$. An isomorphism $H : \mathcal{G}_1 \to \mathcal{G}_2$ is a tuple of the form

$$H = (H_{\Gamma}, (H_v)_{v \in V(\Gamma_1)}, (H_e)_{e \in E(\Gamma_1)}, (\delta(e))_{e \in E(\Gamma_1)}),$$

where

- (1) $H_{\Gamma}: \Gamma_1 \to \Gamma_2$ is a graph isomorphism;
- (2) $H_v: G_v \to G_{H_{\Gamma}(v)}$ is a group isomorphism, for any $v \in V(\Gamma_1)$;
- (3) $H_e = H_{\overline{e}} : G_e \to G_{H_{\Gamma}(e)}$ is a group isomorphism, for any $e \in E(\Gamma_1)$;
- (4) for every $e \in E(\Gamma_1)$, the correction term $\delta(e) \in G_{\tau(H_{\Gamma}(e))}$ is an element such that

$$H_{\tau(e)}f_e = ad_{\delta(e)}f_{H_{\Gamma}(e)}H_e$$

Remark 2.9. A graph-of-groups isomorphism $H : \mathcal{G}_1 \to \mathcal{G}_2$ induces an isomorphism $H_* : \Pi(\mathcal{G}_1) \to \Pi(\mathcal{G}_2)$ defined on the generators by:

$$\begin{aligned} H_*(g) &= H_v(g), \text{ for } g \in G_v, v \in V(\Gamma_1); \\ H_*(t_e) &= \delta(\overline{e}) t_{H_{\Gamma}(e)} \delta(e)^{-1}, \text{ for } e \in E(\Gamma_1) \end{aligned}$$

It is easy to verify that H_* preserves the relations $t_e t_{\overline{e}} = 1$ for any $e \in E(\mathcal{G})$ and $f_{\overline{e}}(g) = t_e f_e(g) t_e^{-1}$, for any $e \in E(\mathcal{G})$ and $g \in G_e$.

Furthermore, the restriction of H_* to $\pi_1(\mathcal{G}_1, v)$, where $v \in V(\Gamma_1)$, defines an isomorphism $H_{*v}: \pi_1(\mathcal{G}_1, v) \to \pi_1(\mathcal{G}_2, H_{\Gamma}(v))$.

As in [9], we define the *outer isomorphism* induced by a group isomorphism $f: G_1 \to G_2$ as the equivalence class

$$\widehat{f} = \{ ad_g f : G_1 \to G_2 \mid g \in G_2 \}.$$

Hence H_{*v} induces an outer isomorphism $\widehat{H}_{*v}: \pi_1(\mathcal{G}_1, v) \to \pi_1(\mathcal{G}_2, H_{\Gamma}(v)).$

Observe that when choosing a different vertex v_1 as basepoint, we may choose a word $W \in \Pi(\mathcal{G}_1)$ with underlying path from v_1 to v to obtain the following commutative diagram:

$$\begin{array}{cccc} \pi_1(\mathcal{G}_1, v) & \xrightarrow{H_{*v}} & \pi_1(\mathcal{G}_2, H_{\Gamma}(v)) \\ ad_W & & ad_{H_*(W)} \\ \pi_1(\mathcal{G}_1, v_1) & \xrightarrow{H_{*v_1}} & \pi_1(\mathcal{G}_2, H_{\Gamma}(v_1)) \end{array}$$

By Lemma 2.2 and Lemma 3.10 in [9], the map \hat{H}_{*v} determines an outer isomorphism $\hat{H}_{*v_1} : \pi_1(\mathcal{G}_1, v_1) \to \pi_1(\mathcal{G}_2, H_{\Gamma}(v_1))$ which is independent of the choice of W. Hence the isomorphism $H : \mathcal{G}_1 \to \mathcal{G}_2$ induces a well-defined outer isomorphism $\hat{H} : \pi_1(\mathcal{G}_1) \to \pi_1(\mathcal{G}_2)$ which doesn't depend on the choice of basepoint.

Remark 2.10 (Composition, Inverse). For two graph-of-groups isomorphisms $H' : \mathcal{G}_1 \to \mathcal{G}_2, H'' : \mathcal{G}_2 \to \mathcal{G}_3$, the composition of H' and H'' is an isomorphism $H''H' = H : \mathcal{G}_1 \to \mathcal{G}_3$ given (for any $v \in V(\Gamma_1)$, $e \in E(\Gamma_1)$) precisely by: $H_{\Gamma} = H''_{\Gamma}H'_{\Gamma}$; $H_v = H''_{H'_{\Gamma}(v)}H'_v$; $H_e = H''_{H'_{\Gamma}(e)}H'_e$; $\delta(e) = H''_{\tau(H'_{\Gamma}(e))}(\delta'(e))\delta''(H'_{\Gamma}(e))$. Moreover, H satisfies $H_* = H''_*H'_*$ and $\widehat{H} = \widehat{H}''\widehat{H}'$.

For any graph-of-groups isomorphism $H: \mathcal{G}_1 \to \mathcal{G}_2$ the *inverse* isomorphism is $H^{-1}: \mathcal{G}_2 \to \mathcal{G}_1$, which satisfies $H_*^{-1} = (H^{-1})_*$ and $\widehat{H}^{-1} = \widehat{H^{-1}}$, is defined (for all $v \in V(\Gamma_2)$, $e \in E(\Gamma_2)$) by: $(H^{-1})_{\Gamma} = (H_{\Gamma})^{-1}$; $(H^{-1})_v = (H_{H_{\Gamma}^{-1}(v)})^{-1}$; $(H^{-1})_e = (H_{H_{\Gamma}^{-1}(e)})^{-1}$; $\delta^{-1}(e) = H_{H^{-1}(\tau(e))}^{-1}(\delta(H_{\Gamma}^{-1}(e))^{-1})$.

2.4. A Natural Equivalence Between Graphs Of Groups.

Suppose that \mathcal{G} , \mathcal{G}' are two graphs-of-groups, and that \mathcal{G}' equals to \mathcal{G} everywhere except that for some edge $e_0 \in E(\mathcal{G})$ one has $f'_{e_0} = ad_{w_{e_0}^{-1}} \circ f_{e_0}$, where w_{e_0} is an element in $G_{\tau(e_0)}$. Then there is a natural isomorphism between \mathcal{G} and \mathcal{G}' .

More concretely, define $H_0: \mathcal{G} \to \mathcal{G}'$ by the rules:

- (1) $H_{0,\Gamma} = id_{\Gamma(\mathcal{G})};$
- (2) $H_{0,v} = id_{G_v}$ for any $v \in V(\mathcal{G})$; $H_{0,e} = id_{G_e}$ for any $e \in E(\mathcal{G})$;
- (3) $\delta_0(e_0) = w_{e_0}$, and $\delta_0(e) = 1$ when $e \neq e_0$.

Then it's easy to verify that H_0 is a well-defined graph-of-groups isomorphism, since the additional compatibility requirement holds automatically for all edges $e_0 \neq e \in \Gamma_0$, and for e_0 we have

$$H_{0,\tau(e_0)} \circ f_{e_0} = f_{e_0} = ad_{w_{e_0}} \circ f'_{e_0} = ad_{\delta_0(e_0)} \circ f'_{e_0} = ad_{\delta_0(e_0)} \circ f'_{e_0} \circ H_{0,e_0}.$$

The above isomorphism gives rise to a natural notion of "equivalent" graphs-of-groups, where the equivalence relation is generated by isomorphisms of the above type H_0 as elementary equivalence. This notion of

"equivalent" graph-of-groups, although not really established in the literature, is natural, in that it preserves (up to canonical isomorphisms) the fundamental group. It also shows up in the prime feature of graph-of-groups, meaning Bass-Serre theory: Given a group G that acts on a (simplicial) tree T, for the associated graph-of-groups decomposition $G \cong \pi_1(\mathcal{G}_T)$ the "quotient" graph-of-groups \mathcal{G}_T of T modulo G is only well defined up to precisely this equivalence relation.

Lemma 2.11. Let \mathcal{G} , \mathcal{G}' , H_0 , e_0 and w_0 be as defined as above.

Let $H : \mathcal{G} \to \mathcal{G}$ be a graph-of-groups automorphism, and let $H' = (H'_{\Gamma}, (H'_v)_{v \in V(\mathcal{G}')}, (H'_e)_{e \in E(\mathcal{G}')}, (\delta'(e))_{e \in E(\mathcal{G}')})$ be equal to H everywhere except that $\delta'(e_0) = H_{\tau(e_0)}(w_{e_0})^{-1}\delta(e_0)w_{e_0}$.

Then H' determines a well-defined graph-of-groups automorphism which is conjugate to H via H_0 . More precisely, we have $H' = H_0 \circ H \circ H_0^{-1}$, and hence in particular $\widehat{H}' = \widehat{H}_0 \circ \widehat{H} \circ \widehat{H}_0^{-1}$.

Proof. In order to show that H' is a well-defined graph-of-groups isomorphism, it is sufficient to verify that $H_{\tau(e_0)} \circ f'_{e_0} = ad_{\delta'(e_0)} \circ f'_{e_0} \circ H_{e_0}$.

For every $g \in G_{e_0}$ we have

$$\begin{aligned} ad_{\delta'(e_0)} \circ f'_{e_0} \circ H_{e_0}(g) &= H_{\tau(e_0)}(w_{e_0})^{-1} \delta(e_0) w_{e_0} f'_{e_0}(H_{e_0}(g)) w_{e_0}^{-1} \delta(e_0)^{-1} H_{\tau(e_0)}(w_{e_0}) \\ &= H_{\tau(e_0)}(w_{e_0})^{-1} \delta(e_0) f_{e_0}(H_{e_0}(g)) \delta(e_0)^{-1} H_{\tau(e_0)}(w_{e_0}) \\ &= H_{\tau(e_0)}(w_{e_0})^{-1} H_{\tau(e_0)}(f_{e_0}(g)) H_{\tau(e_0)}(w_{e_0}) \\ &= H_{\tau(e_0)}(w_{e_0}^{-1} f_{e_0}(g) w_{e_0}) \\ &= H_{\tau(e_0)} \circ f'_{e_0}(g) \end{aligned}$$

Moreover, we have the following diagram commutes:

$$egin{array}{cccc} \mathcal{G} & \stackrel{H}{\longrightarrow} & \mathcal{G} \ & & & \downarrow H_0 \ & & & \downarrow H_0 \ & \mathcal{G}' & \stackrel{H}{\longrightarrow} & \mathcal{G}' \end{array}$$

The only non-trivial part here is to verify that the following equation holds for all edges:

$$H_{0,H_{\Gamma_0}(\tau(e))}(\delta(e))\delta_0(H_{\Gamma}(e)) = H'_{H_{\Gamma_0}(\tau(e))}(\delta_0(e))\delta'(H_{\Gamma_0}(e))$$

When $e \neq e_0$, this equation follows from $\delta(e) = \delta'(e)$ which holds automatically by definition.

When $e = e_0$, we Compute both, the left and the right hand side:

Left:
$$H_{0,H_{\Gamma_0}(\tau(e))}(\delta(e))\delta_0(H_{\Gamma}(e)) = \delta(e_0)\delta_0(e_0) = \delta(e_0)w_{e_0}$$

Right: $H'_{H_{\Gamma_0}(\tau(e))}(\delta_0(e))\delta'(H_{\Gamma_0}(e)) = H'_{\tau(e_0)}(\delta_0(e_0))\delta'(e_0)$
 $= H'_{\tau(e_0)}(w_{e_0})\delta'(e_0) = \delta(e_0)w_{e_0}$

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Hence we have shown that the equation holds for all edges, and that $H' = H_0 \circ H \circ H_0^{-1}$, which implies $\hat{H}' = \hat{H}_0 \circ \hat{H} \circ \hat{H}_0^{-1}$.

Remark 2.12. Similar to the equivalence between \mathcal{G} and \mathcal{G}' discussed above, Lemma 2.11 gives rise to an equivalence between graph-of-groups automorphisms, which is natural in the following sense:

Assume that some group G acts on a simplicial tree T, and let \mathcal{G}_T be the "quotient graph-of-groups" of T modulo G mentioned above. Assume furthermore that for some outer automorphism φ of G the tree T is " φ invariant", by which one means that the translation length function $\|\cdot\|_T$, defined on the conjugacy classes of G by setting all edge lengths equal to 1, is preserved by φ :

$$\|\varphi[g]\|_T = \|[g]\|_T$$
 for all $g \in G$

Now Bass-Serre theory is set up in such a way that this φ -invariance of T is equivalent to the existence of a graph-of-groups automorphism $H : \mathcal{G}_T \to \mathcal{G}_T$ which induces the given automorphism φ .

However, in this situation the automorphism H is determined only up to an equivalence which is generated by the elementary equivalence $H \sim H'$, where H and H' are precisely as given in Lemma 2.11 above.

A statement analogous to Lemma 2.11 with e_0 replaced by a family of edges can be derived by applying Lemma 2.11 iteratively.

3. Dehn Twists

3.1. Classical Dehn Twists.

We first recall the classical definition of a Dehn twist as given in [8].

Definition 3.1 (Classical Dehn twist). An automorphism $D : \mathcal{G} \to \mathcal{G}$ of a graph-of-groups \mathcal{G} is called a *(classical) Dehn twist* if it satisfies:

(1)
$$D_{\Gamma} = id_{\Gamma};$$

- (2) $D_v = id_{G_v}$, for all $v \in V(\Gamma)$;
- (3) $D_e = id_{G_e}$, for all $e \in E(\Gamma)$;
- (4) for each G_e , there is an element $\gamma_e \in Z(G_e)$ such that $\delta(e) = f_e(\gamma_e)$, where $Z(G_e)$ denotes the center of G_e .

We denote a Dehn twist defined as above by $D = D(\mathcal{G}, (\gamma_e)_{e \in E(\mathcal{G})}).$

Definition 3.2 (Twistor). Given a Dehn twist $D = D(\mathcal{G}, (\gamma_e)_{e \in E(\mathcal{G})})$, we define the *twistor* of an edge $e \in E(\Gamma)$ by setting $z_e = \gamma_{\overline{e}} \gamma_e^{-1}$. Then for any edge e we have $z_e \in Z(G_e)$ and $z_{\overline{e}} = \gamma_e \gamma_{\overline{e}}^{-1} = z_e^{-1}$.

Remark 3.3. The induced automorphism $D_* : \Pi(\mathcal{G}) \to \Pi(\mathcal{G})$ is defined on generators as follows:

 $D_*(g) = g$, for $g \in G_v$, $v \in V(\Gamma)$; $D_*(t_e) = t_e f_e(z_e)$, for every $e \in E(\Gamma)$.

In particular, the induced automorphism on the fundamental group, D_{*v} : $\pi_1(\mathcal{G}, v) \to \pi_1(\mathcal{G}, v)$, where $v \in V(\Gamma)$, is called a *Dehn twist automorphism*.

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Remark 3.4. It follows from Proposition 5.4 in [9] that in many situations a Dehn twist on a given graph-of-groups is uniquely determined by its twistors. Thus sometimes we may define a Dehn twist by its twistors $(z_e)_{e \in E(\Gamma)}$ (With $z_e \in Z(G_e)$ and $z_{\overline{e}} = z_e^{-1}$, for each $e \in E(\Gamma)$). In this case, we may conversely define:

$$\gamma_e = \begin{cases} z_e^{-1}, & e \in E^+(\Gamma) \\ 1, & e \in E^-(\Gamma). \end{cases}$$

Definition 3.5. A group automorphism $\varphi : G \to G$ is said to be a *Dehn* twist automorphism if it is represented by a graph-of-groups Dehn twist. In other words, there exists a graph-of-groups \mathcal{G} , a Dehn twist $D : \mathcal{G} \to \mathcal{G}$, and an isomorphism $\theta : G \to \pi_1(\mathcal{G}, v)$ such that $\varphi = \theta^{-1} \circ D_{*v} \circ \theta$.

In this case the induced outer automorphism $\widehat{\varphi} \in Out(G)$ is called a *Dehn* twist outer automorphism.

Remark 3.6. Notice that, for a given Dehn twist automorphism $\varphi : G \to G$, its Dehn twist representative is in general not unique.

In the special case where G is a free group, such "unique" Dehn twist representatives are given by *efficient* Dehn twists. For details see [8] or Section 3.3 in [21].

Remark 3.7. It is easy to see that every multiple Dehn twist homeomorphism h on a compact surface S (possibly with finitely many boundary components), as defined in [10], gives rise to a graph-of-groups Dehn twist $D: \mathcal{G} \to \mathcal{G}$. Here \mathcal{G} is a graph-of-groups decomposition of the surface group $\pi_1(S)$ modeled the decomposition of S by the twistor curves, and the Dehn twist D on \mathcal{G} defines the same outer automorphism on $\pi_1(\mathcal{G}) \cong \pi_1(S)$ as the given multiple Dehn twist homeomorphism h. See [23] and section 6 of [22] for a more detailed explanation.

3.2. General Dehn Twists. There are several places in the literature (e.g. see [7, 11, 15]) where the notion of a "Dehn twist" by means of graphof-groups automorphisms have been defined; these definitions all agree in essence, but are slightly distinct in their technical specifications. Thus subsection is meant as contribution to a unification of these concepts.

Among the various alternatives to classical Dehn twists as presented in the previous subsection, the idea to simplify the concept of twists on nontrivial edge group elements through keeping the edge groups trivial and working with "interesting" correction terms has its strong merits, but also some defaults (see the opening paragraph of subsection 3.3). It leads to graph-of-groups automorphisms as considered in the next proposition; these are particularly important as they occur naturally in the context of Bestvina-Handel's train track maps in [4].

Proposition 3.8. Let \mathcal{G} be a graph-of-groups such that the edge groups G_e are trivial for all $e \in E(\mathcal{G})$. Let $H : \mathcal{G} \to \mathcal{G}$ be an automorphism such that

 \diamond H_{Γ} acts on $\Gamma(\mathcal{G})$ as identity;

 $\diamond H_v: G_v \to G_v$ is the identity map, for all $v \in V(\mathcal{G})$.

Then the induced automorphism $\hat{H} \in Out(\pi_1 \mathcal{G})$ is a Dehn twist automorphism.

The proof of this proposition is postponed (see Remark 3.11); we first want to enlarge our concept of a "Dehn wist automorphism" to include other notions mentioned above. Our Definition 3.9 below includes in particular also the concept set up in [11], which to our knowledge is the most general among the ones presently in the literature. Be aware, however, that in [11] Levitt's terminology for "Dehn twist" corresponds to what is called here "Dehn twist outer automorphism" (see Definition 3.5).

In any case, it will be shown in Proposition 3.10 below that all these generalizations differ from the original "classical" Dehn twist concept only in their presentation and not really in substance.

Definition 3.9 (General Dehn twist). Let \mathcal{G} be a graph-of-groups. An automorphism $D: \mathcal{G} \to \mathcal{G}$ is called a *general Dehn twist* if

- $\diamond D_{\Gamma} = id_{\Gamma};$
- $\diamond D_v = id_{G_v}$, for any vertex $v \in V(\mathcal{G})$;
- $\diamond D_e = id_{G_e}$, for any edge $e \in E(\mathcal{G})$;
- $\delta(e) \in C(f_e(G_e))$, where $C(f_e(G_e))$ denotes the centeralizer of $f_e(G_e)$ in $G_{\tau(e)}$, for any $e \in E(\Gamma)$.

Proposition 3.10. The notions of a "classical Dehn" twist and of a "general Dehn" twist are equivalent in the following sense:

- (1) Every classical Dehn twist is a general Dehn twist.
- (2) To every general Dehn twist $D : \mathcal{G} \to \mathcal{G}$ we can canonically associate a classical Dehn twist $D' : \mathcal{G}' \to \mathcal{G}'$ and an isomorphism $\theta_v : \pi_1(\mathcal{G}, v) \to \pi_1(\mathcal{G}', v')$ (for any vertex v of \mathcal{G} and a corresponding vertex v' of \mathcal{G}') such that $D'_{*v'} \circ \theta_v = \theta_v \circ D_{*v}$.

In particular, the outer automorphism \widehat{D} defined by a general Dehn twist is a Dehn twist automorphism.

Proof. (1) This follows immediately from $f_e(Z(G_e)) \subset C(f_e(G_e))$.

(2) We first consider a special case:

Let $D : \mathcal{G} \to \mathcal{G}$ be a general Dehn twist, with all correction terms as defined in the classical case except for a single edge e: we suppose $\delta(e) \in C(f_e(G_e)) \subset G_{\tau(e)}$ but $\delta(e) \notin f_e(Z(G_e))$.

Then we define a classical Dehn twist $D': \mathcal{G}' \to \mathcal{G}'$ which is obtained as follows:

♦ The graph-of-groups \mathcal{G}' is obtained from \mathcal{G} by introducing a new vertex v_0 which subdivides the edge e into e' and e'', with $\iota(e') = \iota(e)$, $\tau(e') = v_0 = \iota(e'')$ and $\tau(e'') = \tau(e)$, by setting

$$G_{e'} = G_e$$
 and $G_{e''} = G_{v_0} = \langle f_e(G_e), \delta(e) \rangle$,

and by defining the edge homomorphisms through

$$\begin{aligned} f_{e'}(x) &= f_e(x), \ f_{\overline{e}'}(x) = f_{\overline{e}}(x) \quad \text{for all} \quad x \in G_{e'} = G_e \,, \\ \text{and} \quad f_{e''} &= id_{G_{e''}}, \ f_{\overline{e}''} = id_{G_{e''}}. \end{aligned}$$

 \diamond The Dehn twist D' is defined by setting

$$D'_{e'} = id_{G_{e'}}, \ D'_{e''} = id_{G_{e''}}, \ D'_{v_0} = id_{G_{v_0}},$$

and by choosing

$$\delta'(\overline{e}') = \delta(\overline{e}), \ \delta'(e') = 1_{G_{v_0}}, \ \delta'(\overline{e}'') = 1_{G_{v_0}}, \ \delta'(e'') = \delta(e).$$

The data for D' are equal to those for D everywhere else.

Since $\delta(e)$ is assumed to lie in $C(f_e(G_e))$, any element in $f_e(G_e)$ commutes with $\delta(e)$. As $\delta(e)$ also commutes with itself, we derive immediately that $\delta'(e'') = \delta(e)$ commutes with all elements contained in $\langle f_e(G_e), \delta(e) \rangle =$ $f_{e''}(G_{e''})$, i.e. $\delta'(e'') \in Z(f_{e''}(G_{e''})) = f_{e''}(Z(G_{e''}))$. Thus D' is a classical Dehn twist as given in Definition 3.5.

The Dehn twist D' is "equal" to D in the following sense: for every vertex $v \neq v_0$ from \mathcal{G}' there is a corresponding vertex for \mathcal{G} which we also call v.

Consider the homomorphism $\theta : \Pi(\mathcal{G}) \to \Pi(\mathcal{G}')$ defined on generators by $t_e \mapsto t_{e'}t_{e''}$ and by $g \mapsto g$ otherwise. Then we claim that, for every vertex $v \neq v_0$, the restriction of θ to $\pi_1(\mathcal{G}, v)$ defines an isomorphism $\theta_v : \pi_1(\mathcal{G}, v) \to \pi_1(\mathcal{G}', v)$.

To see this, we first observe that because of $G_{v_0} = f_{\overline{e}''}(G_{e''})$ a reduced word in $\Pi(\mathcal{G}')$ can not contain as subword any word of type $t_{\overline{e}''}gt_{e''}$ with $g \in G_{v_0}$. Since furthermore e'' and \overline{e}' are the only edges issuing from v_0 , it follows for any reduced word $W \in \pi_1(\mathcal{G}', v)$ with $v \neq v_0$, after appropriately applying the relation $f_{e'}(g) = t_{e'}^{-1}f_{\overline{e}'}(g)t_{e'}$ from Definition 2.2, that any occurrence of $t_{e''}$ in W is preceded by $t_{e'}$ and any $t_{\overline{e}''}$ is succeeded by $t_{\overline{e}'}$. This proves the surjectivity of the map θ_v , since by Proposition 2.6 (1) it suffices to consider reduced words. The injectivity is a direct consequence of part (3) of the same proposition.

To conclude the proof in the special case we now observe that θ gives rise to the following diagram:

$$\begin{array}{ccc} \pi_1(\mathcal{G}, v) & \xrightarrow{D_{*v}} & \pi_1(\mathcal{G}, v) \\ \theta_v & & & \downarrow \theta_v \\ \pi_1(\mathcal{G}', v) & \xrightarrow{D'_{*v}} & \pi_1(\mathcal{G}', v) \end{array}$$

It follows directly from the definition of the maps involved that this diagram is commutative; the only non-trivial argument needed is given by: K. YE

$$\begin{aligned} \theta \circ D_*(t_e) &= \theta(\delta(\overline{e})t_e\delta(e)^{-1}) = \delta(\overline{e})t_{e'}t_{e''}\delta(e)^{-1} \\ &= D'_*(t_{e'})D'_*(t_{e''}) = D'_*(t_{e'}t_{e''}) \\ &= D'_* \circ \theta(t_e). \end{aligned}$$

In the general case, where D is a general Dehn twist which may have more than one correction term defined in the "non-classical way", we may apply the above treated special case repeatedly to each of the "non-classical" correction terms, to eventually obtain a classical Dehn twist.

Remark 3.11. One obtains now the statement of Proposition 3.8 as direct consequence of Proposition 3.10 (2), since the graph-of-group automorphism H from Proposition 3.8 is clearly a general Dehn twist.

3.3. Partial Dehn twists.

The type of Dehn twists as considered in Proposition 3.8 has a strong appeal, due to its simplicity. It is furthermore of natural interest because it is used in relative train track theory (see [4]). However, it should be noted that not every outer Dehn twist automorphism, even for a free group of finite rank, can be represented by such simple graph-of-groups automorphisms, as illustrated by the following example:

Remark 3.12. We consider $F_3 = F(a, b, c)$ and the automorphism $\varphi \in$ Out (F_3) which acts on the generators by sending $a \mapsto a, b \mapsto ba$ and $c \mapsto (aba^{-1}b^{-1})c(aba^{-1}b^{-1})^{-1}$. This is the Dehn twist automorphism induced by the partial Dehn twist given in Example 1.3.

In order to see that this Dehn twist automorphism can not be realized by a Dehn twist $D: \mathcal{G} \to \mathcal{G}$ as in Proposition 3.8, i.e. based on a graph-of-groups \mathcal{G} with all edge groups trivial, we observe that for such a graph-of-groups every vertex group is a free factor of $\pi_1 \mathcal{G} \cong F(a, b, c)$. From the algebraic prescriptions $a \mapsto a, b \mapsto ba, c \mapsto (aba^{-1}b^{-1})c(aba^{-1}b^{-1})^{-1}$ we derive that there must be two edges in \mathcal{G} , with precisely one of them a loop edge. As a consequence that there must also be precisely two vertices in \mathcal{G} , and none of the vertex groups can be trivial. It follows that both vertex groups must have rank 1. Since the conjugacy class of the two twisters are prescribed by the action on $\pi_1 \mathcal{G}$, it follows that one of the edges at one of its endpoints must have as correction term an element conjugate to $aba^{-1}b^{-1}$. Such a commutator, however, can not be contained in any of the vertex groups, if the latter is a free factor of F(a, b, c) of rank 1. Hence a graph-of-groups \mathcal{G} as required for the Dehn twist in question does not exist.

In order to make up for this defect, we now introduce the following notion of *partial Dehn twist*, which is more thoroughly studied in [21], [22], as well as Section 7 of this article.

Definition 3.13 (Partial Dehn twist). Let \mathcal{G} be a graph-of-groups, and let $\mathcal{V}_0 \subset V(\mathcal{G})$ be a set of vertices which has the property that any edge e with $\tau(e) = v \in \mathcal{V}_0$ has trivial edge group G_e .

A partial Dehn twist relative to \mathcal{V}_0 is a graph-of-groups isomorphism $H : \mathcal{G} \to \mathcal{G}$ such that

- $\diamond H_{\Gamma} : \Gamma(\mathcal{G}) \to \Gamma(\mathcal{G})$ is the identity;
- $\diamond H_e = id_{G_e} \text{ for all edges } e \in E(\mathcal{G});$
- ♦ $H_v = id_{G_v}$ for all vertices $v \notin \mathcal{V}_0$ (while H_v is any group isomorphism for all $v \in \mathcal{V}_0$);
- $\diamond \ \delta(e) \in C(f_e(G_e))$ for all edges $e \in E(\mathcal{G})$.

The case which is of most interest to us is that of a partial Dehn twist where all non-trivial vertex group automorphisms are Dehn twist automorphisms themselves. In order to simplify the notation, we include the identity map as *trivial Dehn twist* defined by a degenerate graph-of-groups that is based on the graph which consists of a single vertex only. We define:

Definition 3.14. A graph-of-groups automorphism $H : \mathcal{G} \to \mathcal{G}$ is a *partial Dehn twist relative to a family of Dehn twist automorphisms* if H is a partial Dehn twist as in Definition 3.13, with the specification that for every vertex $v \in V(\mathcal{G})$ the map H_v is a (possibly trivial) Dehn twist automorphism.

4. H-CONJUGATION

Recall from Proposition 2.6 that for any element W in the Bass group $\Pi(\mathcal{G})$, represented by some reduced word $w \in W(\mathcal{G})$, then any other reduced $w' \in W(\mathcal{G})$ which also represents W is connected if and only if w is connected. We hence call W in this case a *connected* element of $\Pi(\mathcal{G})$. Similarly, the initial and terminal vertices $\iota(W)$ and $\tau(W)$ are well defined.

Definition 4.1. Let $H : \mathcal{G} \to \mathcal{G}$ be an isomorphism of a graph-of-groups. Let W_1, W_2 be non-trivial connected elements in the Bass group $\Pi(\mathcal{G})$. Then W_1 is said to be *H*-conjugate to W_2 if there exists a connected element $W \in \Pi(\mathcal{G})$ such that $W_1 = WW_2H_*(W)^{-1}$. This connected element W is called *H*-conjugator.

Lemma 4.2. *H*-conjugacy is an equivalence relation on the set of non-trivial connected elements in $\Pi(\mathcal{G})$.

Proof. Reflexivity and symmetry are obvious. In order to show transitivity we observe that from $W_1 = WW_2H_*(W)^{-1}$ and $W_2 = W'W_3H_*(W')^{-1}$, where W and W' are connected elements, one deduces $W_1 = WW'W_3H_*(WW')^{-1}$. Here WW' is connected since W_1 and W_2 are non-trivial and connected, which implies that W terminates at $\iota(W_2)$ while W' initiates at $\iota(W_2)$. \Box

Remark 4.3. Let H be an isomorphism of a graph-of-groups \mathcal{G} . Two nontrivial connected elements $W_1, W_2 \in \Pi(\mathcal{G})$ are H-conjugate to each other if and only if W_1^{-1} and W_2^{-1} are H^{-1} -conjugate to each other.

Definition 4.4. Let $H : \mathcal{G} \to \mathcal{G}$ be an isomorphism of a graph-of-groups, and let $W \in \Pi(\mathcal{G})$ be a non-trivial connected element.

Then W is said to be *H*-zero if there exists a connected element $W' \in \Pi(\mathcal{G})$ such that the (possibly trivial) element $W'WH_*(W')^{-1}$ is contained in a single vertex group of \mathcal{G} .

Remark 4.5. There is an important particular reason why the trivial element of $\Pi(\mathcal{G})$ is not contained in any of the *H*-conjugacy classes as they are defined in the above set-up. It is a rather tricky issue, which has been dealt with in detail in Section 3 of [12].

This leads, however, to the following "unexpected" situation, due to the fact (obtained directly from Definitions 4.1 and 4.4) that an element $W \in \Pi(\mathcal{G})$ is *H*-zero if and only if any *H*-conjugate $W' \in \Pi(\mathcal{G})$ is also *H*-zero: It could well be that all elements in the *H*-conjugacy class of *W* are *H*-zero, but none of them is actually contained in some vertex group of \mathcal{G} , since the only such which is *H*-conjugate to *W* is the trivial element.

Remark 4.6. It is also important noting that in the special case, where $\pi_1 \mathcal{G}$ is a free group of finite rank and $H : \mathcal{G} \to \mathcal{G}$ is based on the identity map $H_{\Gamma} = id_{\Gamma(\mathcal{G})}$, one can decide algorithmically whether a non-trivial connected element $W \in \Pi(\mathcal{G})$ is *H*-zero or not.

Indeed, by Definition 4.4, W is H-zero if and only if one can write W as product

$$W = W_1 g H_* (W_1)^{-1}$$
,

where $W_1 \in \Pi(\mathcal{G})$ is also connected, and g is contained in some vertex group G_v of \mathcal{G} . By properly chosing W_1 we can assume here that W_1 is written as reduced word, and that there is no cancellation (other than within G_v) in the above product, which is hence reduced and of even \mathcal{G} -length 2r, for $r = |W_1|_{\mathcal{G}}$.

Thus, if W is H-zero, it can be written as product W = W'W'', with W'and W'' of \mathcal{G} -length r. The element W' is not quite determined by W, but for all possible choices we always have $W'^{-1}W_1 \in G_v$. We conclude that Wis H-zero if and only if for any W' as above we have that

$$W'^{-1}WH_*(W')$$

has \mathcal{G} -length 0 (which is equivalent to being contained in some vertex group).

For a formal proof of the last conclusion we observe that $g' := W'^{-1}W_1 \in G_v$ implies $W'' = W'^{-1}W = W'^{-1}W_1gH_*(W_1)^{-1} = g'gH_*(W_1)^{-1}$ and hence $W''H_*(W') = g'gH_*(W_1)^{-1}H_*(W') = g'gH_*(W_1^{-1}W') = g'gH_*(W_1^{-1}W') = g'gH_*(g'^{-1})$, where $H_*(g'^{-1}) \in G_v$ follows from our assumption $H_{\Gamma} = id_{\Gamma(G)}$.

5. QUOTIENT GRAPH-OF-GROUPS ISOMORPHISM

5.1. Quotient graph-of-groups.

Let \mathcal{G} be a graph-of-groups and \mathcal{G}_0 be a sub-graph-of-groups of \mathcal{G} . By sub-graph-of-groups we simply mean the restriction of \mathcal{G} to a connected subgraph¹. Denote $\Gamma = \Gamma(\mathcal{G})$ and $\Gamma_0 = \Gamma(\mathcal{G}_0)$; by definition Γ_0 is a connected subgraph of Γ .

We will denote by $\overline{\mathcal{G}} = \mathcal{G}/\mathcal{G}_0$ the quotient graph-of-groups of \mathcal{G} by \mathcal{G}_0 , which we define now in detail:

The graph underlying $\overline{\mathcal{G}}$, denoted by $\overline{\Gamma} = \Gamma/\Gamma_0$ with $V(\overline{\Gamma}) = (V(\Gamma) \setminus V(\Gamma_0)) \cup \{V_0\}$ and $E(\overline{\Gamma}) = E(\Gamma) \setminus E(\Gamma_0)$, is obtained precisely by contracting Γ_0 into a vertex V_0 through the map:

$$q: \Gamma \to \overline{\Gamma}$$

$$x \mapsto V_0 \quad \text{if } \mathbf{x} \in E(\Gamma_0) \text{ or } V(\Gamma_0)$$

$$x \mapsto X \quad \text{otherwise}$$

Thus on $\Gamma \setminus \Gamma_0$ the map q sends x, by which we mean either an edge or a vertex, to its natural correspondence in $\overline{\Gamma}$. We denote q(x) by the same but capitalized letter $X \in \overline{\Gamma}$, to avoid confusion.

We choose a vertex

$$(5.1) p_0 \in V(\Gamma_0)$$

as basepoint and set $G_{V_0} = \pi_1(\mathcal{G}_0, p_0)$.

For every $v \in V(\Gamma)$ and $V = q(v) \neq V_0$, we set $G_V = G_v$. For every $e \in E(\Gamma)$ and $E = q(e) \in E(\overline{\Gamma})$, let $G_E = G_e$, and $f_E = f_e$ if $\tau(E) \neq V_0$. If $\tau(E) = V_0$, we choose a word

(5.2)
$$\gamma_E \in \Pi(\mathcal{G}_0) \subset \Pi(\mathcal{G})$$

from p_0 to $\tau(e) \in V(\Gamma_0)$ and define the edge homomorphism of E to be $f_E := ad_{\gamma_E} \circ f_e$.

We define a group homomorphism θ from $\Pi(\overline{\mathcal{G}})$ to $\Pi(\mathcal{G})$ via:

$$\theta: \Pi(\overline{\mathcal{G}}) \to \Pi(\mathcal{G})$$

$$t_E \mapsto t_e \gamma_E^{-1} \quad \text{if } \tau(E) = V_0, \ \tau(\overline{E}) \neq V_0$$

$$t_E \mapsto \gamma_{\overline{E}} t_e \gamma_E^{-1} \quad \text{if } \tau(E) = \tau(\overline{E}) = V_0$$

$$\theta \text{ acts as identity elsewhere}$$

Through the identification $G_{V_0} = \pi_1(\mathcal{G}_0, p_0)$ we see immediately that the restriction of θ to $\pi_1(\overline{\mathcal{G}}, V_0)$ defines an isomorphism from $\pi_1(\overline{\mathcal{G}}, V_0)$ to $\pi_1(\mathcal{G}, p_0)$ which we denote by θ_0 :

The inverse map θ_0^{-1} is given through introducing a cancelling pair $\gamma_E^{-1}\gamma_E$ after the stable letter t_e for any edge e with $\tau(e) \in V(\Gamma_0)$, and $\gamma_{\overline{E}}^{-1}\gamma_{\overline{E}}$ before the stable letter t_e for any edge e with $\tau(\overline{e}) \in V(\Gamma_0)$. One then maps t_e to $\gamma_{\overline{E}}^{-1}t_E\gamma_E$ if both $\tau(e), \tau(\overline{e}) \in V(\Gamma_0)$, and one maps t_e to $t_E\gamma_E$ if only $\tau(e) \in V(\Gamma_0)$.

¹ Be aware that this definition of sub-graph-of-groups is different from the one in [1].

Remark 5.1. For later purposes the reader should note here that for any edge E = q(e) of $\Gamma(\overline{\mathcal{G}})$ with $\tau(E) = V_0$ there exist a vertex $v(E) := \tau(e) \in$ $V(\mathcal{G}_0)$ such that for the "connecting word" $\gamma_E \in \Pi(\mathcal{G}_0)$ from p_0 to v(E) one has $\theta \circ f_E(G_E) \subset \gamma_E G_{v(E)} \gamma_E^{-1}$.

5.2. Quotient graph-of-groups isomorphism.

In this subsection we define the notion of a quotient graph-of-groups isomorphism.

The graphs-of-groups $\mathcal{G}, \mathcal{G}_0, \overline{\mathcal{G}}$ and the group homomorphisms θ and θ_0 are defined as in the previous subsection. In particular, let V_0, γ_E and p_0 be as given there.

Let $H: \mathcal{G} \to \mathcal{G}$ be a graph-of-groups isomorphism which acts as identity on the graph Γ . The map $H_0: \mathcal{G}_0 \to \mathcal{G}_0$, obtained by restricting H to \mathcal{G}_0 , is called the local graph-of-groups isomorphism.

In order to define $\overline{H}:\overline{\mathcal{G}}\to\overline{\mathcal{G}}$ we set:

- (1) $\overline{H}_{V_0} = H_{0,*p_0} : \pi_1(\mathcal{G}_0, p_0) \to \pi_1(\mathcal{G}_0, p_0);$ (2) $\delta(E) = H_*(\gamma_E)\delta(e)\gamma_E^{-1}$, for all E such that $\tau(E) = V_0;$
- (3) \overline{H} "equals" H on the rest of $\overline{\mathcal{G}}$ (modulo replacing x by X as explained in the previous subsection). In particular, $\overline{H}_{\overline{\Gamma}}$ is the identity on the quotient graph Γ .

Proposition 5.2. The above conditions (1) - (3) give a well defined graphof-groups isomorphism $\overline{H}:\overline{\mathcal{G}}\to\overline{\mathcal{G}}$. It induces an outer automorphism \overline{H} which is conjugate to \widehat{H} via the isomorphism $\theta_0: \pi_1(\overline{\mathcal{G}}, V_0) \to \pi_1(\mathcal{G}, p_0).$

Proof. In order to show that the above data (1) - (3) give a well defined graph-of-groups isomorphism, the only non-trivial step to verify is the condition that $\overline{H}_{V_0} \circ f_E = ad_{\delta(E)} \circ f_E \circ \overline{H}_E$ for all E = q(e) with $\tau(E) = V_0$. Denote $v = \tau(e) \in V(\Gamma_0)$.

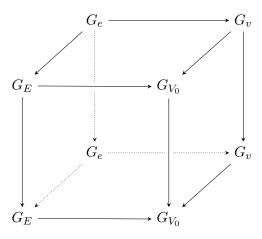
Observe first that for any $h \in G_v$, since H_0 induces an isomorphism on $\Pi(\mathcal{G}_0)$, we have:

$$\overline{H}_{V_0}(\gamma_E h \gamma_E^{-1}) = H_{0*}(\gamma_E) H_v(h) H_{0*}(\gamma_E^{-1}) = H_*(\gamma_E) H_v(h) H_*(\gamma_E^{-1})$$

For any $g \in G_E$ we compute, where the fourth equality uses the previous observation, for $h = f_e(g)$:

$$\begin{aligned} ad_{\delta(E)} \circ f_E \circ \overline{H}_E(g) &= H_*(\gamma_E)\delta(e)\gamma_E^{-1}f_E(\overline{H}_E(g))\gamma_E\delta(e)^{-1}H_*(\gamma_E^{-1}) \\ &= H_*(\gamma_E)\delta(e)f_e(H_e(g))\delta(e)^{-1}H_*(\gamma_E^{-1}) \\ &= H_*(\gamma_E)H_v(f_e(g))H_*(\gamma_E^{-1}) \\ &= \overline{H}_{V_0}(\gamma_E f_e(g)\gamma_E^{-1}) \\ &= \overline{H}_{V_0} \circ f_E(g) \end{aligned}$$

In order to illustrate the previous proof we propose the diagram below: every face of the "cube" pictured there commutes, up to inner automorphisms on the front and back faces, defined by the correction terms $\delta(E)$ and $\delta(e)$ respectively.



Here the map $G_e \to G_E$ is the identity, while the map $G_v \to G_{V_0}$ is ad_{γ_E} .

After having thus proved the first sentence of the proposition, we now turn to the second: we want to show that the following diagram commutes.

Notice that the group homomorphism θ acts on all vertex groups other than G_{V_0} as identity. On the other hand, for $E \in E(\overline{\mathcal{G}})$ with $\tau(E) = V_0$ but $\tau(\overline{E}) \neq V_0$ we have:

$$\theta \circ \overline{H}_*(t_E) = \theta(\delta(\overline{E})t_E\delta(E)^{-1}) = \delta(\overline{E})t_e\gamma_E^{-1}\delta(E)^{-1}$$
$$= \delta(\overline{e})t_e\delta(e)^{-1}H_*(\gamma_E)^{-1}$$
$$= H_*(t_e\gamma_E^{-1})$$
$$= H_* \circ \theta(t_E)$$

A similar computation applies to $E \in E(\overline{\mathcal{G}})$ with $\tau(E) = \tau(\overline{E}) = V_0$.

Therefore we obtain $\theta_0 \circ \overline{H}_{*V_0} = H_{*p_0} \circ \theta_0$, and hence $\widehat{\overline{H}}$ and \widehat{H} are conjugate to each other through the outer isomorphism $\widehat{\theta}_0$.

Remark 5.3. Note in particular, for all E with $\tau(E) = V_0$, we have by definition that $\delta(E)$ is H_0^{-1} -conjugate to an element with \mathcal{G}_0 -length equal to zero, and hence is H_0^{-1} -zero: The word γ_E satisfies

$$H_{0*}(\gamma_E^{-1})\theta(\delta(E))\gamma_E \in G_{v(E)}.$$

Remark 5.4. Our formal construction of quotient graph-of-group isomorphism \overline{H} , constructed as above, depends on the choice of base point p_0 in Γ_0 and of "connecting words" γ_E for all edges E with $\tau(E) = V_0$, as set up in (5.2) in order to define θ and θ_0 .

However, the outer automorphism \overline{H} induced by $\overline{H} : \overline{\mathcal{G}} \to \overline{\mathcal{G}}$ depends neither on the choice of the γ_E nor on the choice of p_0 , up to conjugation by the natural isomorphism $\theta_0^{-1}\theta'_0$, where θ'_0 is the map analogous to θ_0 defined through an alternative choice of p_0 and and the γ_E .

This is a direct consequence of the statement in Proposition 5.2 that \overline{H} is conjugate to \widehat{H} via θ_0 .

Alternatively, a direct proof, without passing through \widehat{H} , can be given by applying Lemma 2.11; this yields the slightly stronger result that a second quotient automorphism $\overline{H}': \overline{\mathcal{G}}' \to \overline{\mathcal{G}}'$ is conjugated to \overline{H} by a graph-of-groups isomorphism $F: \overline{\mathcal{G}} \to \overline{\mathcal{G}}'$.

Remark 5.5. We may apply this quotient procedure above on several disjoint connected subgraphs-of-groups of \mathcal{G} and obtain the analoguous conclusion that the quotient graph-of-groups isomorphism \overline{H} is well defined and induces an outer automorphism conjugate to \widehat{H} .

Remark 5.6. For simplicity of notations, we sometimes represent the simultaneous quotienting of the graph-of-groups \mathcal{G} and of the isomorphism H by referring to the quotient pair $(\overline{H}, \overline{\mathcal{G}})$, obtained from (H, \mathcal{G}) modulo the pair (H_0, \mathcal{G}_0) .

6. BLOWING UP GRAPH-OF-GROUPS AUTOMORPHISM

In this section, we will reverse the quotient construction in the previous section; we emphasize this reversal by the choice of our notation.

For simplicity of the presentation, we only give the blow-up construction at a single vertex. However (for example through iterating this procedure), one can generalize the technique described in this section directly to a blowup construction at several vertices simultaneously to obtain Theorem 1.1 in the Introduction.

6.1. Assumptions.

Let $\overline{H} : \overline{\mathcal{G}} \to \overline{\mathcal{G}}$ and $H_0 : \mathcal{G}_0 \to \mathcal{G}_0$ be graph-of-groups automorphisms which act as the identity on their underlying graphs $\overline{\Gamma}$ and Γ_0 respectively. Let V_0 be a vertex of $\overline{\Gamma}$. We consider the following assumptions:

(A1) There exist a vertex $p_0 \in V(\mathcal{G}_0)$ and a group isomorphism $\theta_0 : G_{V_0} \to \pi_1(\mathcal{G}_0, p_0)$ such that

$$\theta_0 \circ H_{V_0} = H_{0*,p_0} \circ \theta_0;$$

(A2) Compatibility requirement for graph-of-groups:

For any edge E of $\Gamma(\overline{\mathcal{G}})$ with $\tau(E) = V_0$ there exist some vertex $v(E) \in V(\mathcal{G}_0)$ and a "connecting word" $\gamma_E \in \Pi(\mathcal{G}_0)$ from p_0 to v(E) such that $\theta_0 \circ f_E(G_E) \subset \gamma_E G_{v(E)} \gamma_E^{-1}$;

(A3) Compatibility requirement for isomorphism: The word γ_E also satisfies $H_{0*}(\gamma_E^{-1})\theta_0(\delta(E))\gamma_E \in G_{v(E)}$.

Definition 6.1. Assume that condition (A1) is satisfied. Then the pair (H_0, \mathcal{G}_0) , which is called the *local graph-of-groups isomorphism associated to* V_0 , is said to be *compatible* with $(\overline{H}, \overline{\mathcal{G}})$ if both compatibility requirements (A2) and (A3) are satisfied.

Remark 6.2. We'd like to note:

- (1) The compatibility requirement for isomorphism implies, for any $\delta(E)$ with $\tau(E) = V_0$, that the element $\theta_0(\delta(E))$ is H_0^{-1} -zero. Conversely, in order to derive the compatibility requirement for isomorphism for any correction term $\delta(E)$ such that $\theta_0(\delta(E))$ is H_0^{-1} -zero, one needs the additional hypothesis that $\delta(E)$ is G_E -compatible: by this we mean that there is a connected word $\gamma_E \in \Pi(\mathcal{G})$ which satisfies both assumptions (A2) and (A3) above.
- (2) For each E with $\tau(E) = V_0$, there may exist more than one pair of $(v(E), \gamma_E)$ such that the above conditions hold.

6.2. Existence of the blow-up.

Theorem 6.3. Let $\overline{H} : \overline{\mathcal{G}} \to \overline{\mathcal{G}}$ and $H_0 : \mathcal{G}_0 \to \mathcal{G}_0$ be graph-of-groups isomorphisms which act as the identity on their underlying graphs, and let $V_0 \in V(\overline{\mathcal{G}})$ be a vertex for which condition (A1) is satisfied.

Then there exists a "blow-up" graph-of-groups \mathcal{G} and a "blow-up" graph-ofgroups isomorphism $H: \mathcal{G} \to \mathcal{G}$, which contains H_0 as local graph-of-groups isomorphism and yields modulo (H_0, \mathcal{G}_0) the quotient pair $(\overline{H}, \overline{\mathcal{G}})$, if and only if the conditions (A2) and (A3) are satisfied.

In particular, H and \overline{H} induce outer automorphisms which are conjugate.

Proof. For the convenience of the reader we divide this proof in 6 steps; in the first two we present the data which will serve to define \mathcal{G} and H respectively.

(1) The graph $\Gamma(\mathcal{G})$, with vertex set $V(\mathcal{G}) = V(\overline{\mathcal{G}}) \setminus \{V_0\} \cup V(\mathcal{G}_0)$, is obtained from $\Gamma(\mathcal{G})$ and $\Gamma(\mathcal{G}_0)$ by replacing every edge E with terminal vertex $\tau(E) = V_0$ by an edge e with terminal vertex v(E). The analogous replacement is done for \overline{E} . If E has both endpoints distinct from V_0 we leave them as they are, but rename E by the corresponding small letter e.

We set $G_e = G_E$ and define, if $\tau(e) = v(E)$, the edge injection by $f_e(g) = \gamma_E^{-1}\theta_0(f_E(g))\gamma_E$, for every $g \in G_e$. For $\tau(e) = \tau(E)$ we define $f_e = f_E$.

(2) The isomorphism $H: \mathcal{G} \to \mathcal{G}$ is equal to H_0 or to \overline{H} when restricted to \mathcal{G}_0 or to $\overline{\mathcal{G}} \setminus \{V_0\}$ respectively, except that in the case $\tau(e) = v(E)$ we modify the correction term to $\delta(e) = H_{0*}(\gamma_E^{-1})\theta_0(\delta(E))\gamma_E$.

(3) In order to show that the data defined above in (1) give a well defined graph-of-groups one only needs to verify that for any edge e of \mathcal{G} the edge injection f_e has its image in the vertex group $G_{\tau(e)}$. If $\tau(e)$ is not contained

in $V(\mathcal{G}_0)$, then this is immediate from the definition. If $\tau(e) \in V(\mathcal{G}_0)$, then we use the compatibility requirement for graph-of-groups, which gives (A2), $f_e(G_e) = \gamma_E^{-1} \theta_0(f_E(G_E)) \gamma_E \subset G_{v(E)}.$

(4) We now want to show that the data defined above in (2) give a welldefined graph-of-groups isomorphism. This is equivalent to showing for every edge e of \mathcal{G} the equality stated in condition (4) of Definition 2.8. Again, for $\tau(e) \notin V(\mathcal{G}_0)$ this is a direct consequence of our set-up. For $\tau(e) \in V(\mathcal{G}_0)$ we compute (where the third equality uses the definition of f_e from (2) above, and the fifth equality uses condition (A1)):

$$\begin{aligned} ad_{\delta(e)} \circ f_{e} \circ H_{e}(g) &= \delta(e)f_{e}(H_{e}(g))\delta(e)^{-1} \\ &= H_{0*}(\gamma_{E}^{-1})\theta_{0}(\delta(E))\gamma_{E}f_{e}(H_{e}(g))\gamma_{E}^{-1}\theta_{0}(\delta(E)^{-1})H_{0*}(\gamma_{E}) \\ &= H_{0*}(\gamma_{E}^{-1})\theta_{0}(\delta(E)f_{E}(H_{E}(g))\delta(E)^{-1})H_{0*}(\gamma_{E}) \\ &= H_{0*}(\gamma_{E}^{-1})\theta_{0}(H_{V_{0}}(f_{E}(g)))H_{0*}(\gamma_{E}) \\ &= H_{0*}(\gamma_{E}^{-1})H_{0*,P_{0}}(\theta_{0}(f_{E}(g)))H_{0*}(\gamma_{E}) \\ &= H_{0*}(\gamma_{E}^{-1}\theta_{0}(f_{E}(g))\gamma_{E}) = H_{0*}(f_{e}(g)) = H_{v(E)} \circ f_{e}(g) \end{aligned}$$

(5) We now observe from our construction above that the blow-up graph-ofgroups isomorphism H contains H_0 as local graph-of-groups isomorphism. We furthermore have already a base point p_0 as well as connecting words γ_E from p_0 to $\tau(e)$ for each edge e with terminal vertex $\tau(e) \in V(\mathcal{G}_0)$ specified, so that one can readily apply Proposition 5.2 to obtain a quotient graph-ofgroups isomorphism, which, since p_0 and all γ_E are as chosen before, must agree with the isomorphism $\overline{H}: \overline{\mathcal{G}} \to \overline{\mathcal{G}}$.

In particular it follows that the induced outer automorphisms \widehat{H} and $\overline{\overline{H}}$ are conjugate.

(6) Finally, it follows from Remarks 5.1 and 5.3 that any blow-up pair (H, \mathcal{G}) which quotients modulo (H_0, \mathcal{G}_0) to the given pair $(\overline{H}, \overline{\mathcal{G}})$ must necessarily satisfy the conditions (A2) and (A3).

7. PARTIAL DEHN TWIST CASE

In this section we will apply Theorem 6.3 to the special case where all edges E with terminal vertex V_0 have trivial edge group:

$$G_E = \{1\} \qquad \text{if} \qquad \tau(E) = V_0$$

In this case the compatibility conditions from Definition 6.1 simplify considerably, as condition (A2) is trivially satisfied. Regarding the other compatibility condition, we observe that a connecting word γ_E which satisfies condition (A3) exists, if and only if the θ_0 -image of the correction term $\delta(E)$ is H_0^{-1} -zero, in the terminology of section 4. Hence we define:

Definition 7.1. Let $\overline{H} : \overline{\mathcal{G}} \to \overline{\mathcal{G}}$ and $H_0 : \mathcal{G}_0 \to \mathcal{G}_0$ be graph-of-groups isomorphisms which act as the identity on their underlying graphs, and let

 $V_0 \in V(\overline{\mathcal{G}})$ be a vertex with an isomorphism $\theta_0 : G_{V_0} \to \pi_1(\mathcal{G}_0, p_0)$ as in condition (A1).

Then an edge E of $\overline{\mathcal{G}}$ is said to be *locally zero* if the correction term of E has image $\theta_0(\delta(E))$ which is H_0^{-1} -zero. In other words, there exists a connected word $\gamma_E \in \Pi(\mathcal{G}_0)$ such that $H_{0*}(\gamma_E^{-1})\theta_0(\delta(E))\gamma_E \in G_v$ for some vertex v of \mathcal{G}_0 .

We will now specialize further to the case of partial Dehn twists $\overline{H}: \overline{\mathcal{G}} \to \overline{\mathcal{G}}$ as defined in Definition 3.13, which have trivial vertex group isomorphisms \overline{H}_V except at vertices V_i that belong to a subset $\mathcal{V}_0 \subset V(\overline{\mathcal{G}})$. At those "special" vertices we want to assume furthermore that \overline{H}_{V_i} is a Dehn twist automorphism, given via an isomorphism $\theta_i: G_{V_i} \to \pi_1(\mathcal{G}_i, p_i)$ as in condition (A1) by some local Dehn twist $D_i: \mathcal{G}_i \to \mathcal{G}_i$ (see Definition 3.9), so that \overline{H} is a partial Dehn twist relative to a family of local Dehn twists.

Corollary 7.2. Let $\overline{H} : \overline{\mathcal{G}} \to \overline{\mathcal{G}}$ be a partial Dehn twist relative to $\mathcal{V}_0 \subset V(\overline{\mathcal{G}})$, and assume that for each $V_i \in \mathcal{V}_0$ the vertex group isomorphism H_{V_i} is a Dehn twist automorphism represented by a "local" Dehn twist $D_i : \mathcal{G}_i \to \mathcal{G}_i$.

Then one can blow up $(\overline{H}, \overline{\mathcal{G}})$ via (D_i, \mathcal{G}_i) to obtain a Dehn twist $D : \mathcal{G} \to \mathcal{G}$ if and only if every edge E of $\overline{\mathcal{G}}$ with $\tau(E) \in \mathcal{V}_0$ is locally zero.

In this case the induced outer automorphism $\widehat{\overline{H}} : \pi_1(\overline{\mathcal{G}}) \to \pi_1(\overline{\mathcal{G}})$ is a Dehn twist automorphism.

Proof. Let V_i be a vertex contained in \mathcal{V}_0 . Since any edge E terminating at V_i has trivial edge group, the compatibility requirement for graph-ofgroups (A2) holds automatically. On the other hand, the assumption that any edge E with $\tau(E) \in \mathcal{V}_0$ is locally zero is equivalent to the compatibility requirement for isomorphism (A3). We can hence apply Theorem 6.3 to directly obtain the desired "if and only if" statement.

Since H is a partial Dehn twist and the D_i are Dehn twists, it follows directly from the construction of the blow-up isomorphism in the proof of Theorem 6.3 that D satisfies the first three properties of Definition 3.9.

In order to see that the fourth condition of Definition 3.9 is also satisfied, we consider three cases: If an edge E from $\overline{\mathcal{G}}$ has endpoint outside of \mathcal{V}_0 , then all relevant data for E and the corresponding edge e in \mathcal{G} coincide, so that we can simply use the last condition of Definition 3.13. If $\tau(E) \in \mathcal{V}_0$, then $G_E = \{1\}$ follows from Definition 3.13, so that the fourth condition of Definition 3.9 is automatically satisfied. Finally, if the edge e from \mathcal{G} in question is an edge of some of the local graph-of-groups \mathcal{G}_i , then we use that $D_i: \mathcal{G}_i \to \mathcal{G}_i$ itself is assumed to satisfy Definition 3.9, so that in particular its fourth condition of this definition holds for e.

As a consequence we obtain from Proposition 3.10 that \widehat{D} is a Dehn twist automorphism. But Theorem 6.3 also states that the outer automorphisms induced by \overline{H} and by D are conjugate, which shows that $\widehat{\overline{H}}$ is also a Dehn twist automorphism. Corollary 7.2 gives the possibility to decide the existence of a blow-up Dehn twist relative to particular given local Dehn twist representatives. A harder but more interesting question is whether one can blow up the given partial Dehn twist $\overline{H}: \overline{\mathcal{G}} \to \overline{\mathcal{G}}$ to a global Dehn twist with respect to *some* family of local Dehn twists that induce the vertex automorphisms of \overline{H} . In other words (using the terminology of Definition 3.14):

"When does a partial Dehn twist with Dehn twist automorphism on the vertices induce a Dehn twist automorphism ?"

The subtlety of this question is illustrated by the Examples 1.3 and 1.4 in the Introduction. A complete answer is given in [21].

Remark 7.3. Assume that in the situation of Corollary 7.2 the following data are given:

- (a) The graph-of-groups $\overline{\mathcal{G}}$ and \mathcal{G}_i have free groups of finite rank as vertex and edge groups, with chosen bases for each of them, and that the edge maps are given in the usual fashion by specifying the images of the edge groups basis elements as words in the basis of the adjacent vertex group.
- (b) The graph-of-groups automorphisms \overline{H} and H_i are similarly given by specifying the images of the given basis elements as word in those bases, and by specifying for every edge E of $\overline{\mathcal{G}}$ with terminal vertex $V_i \in \mathcal{V}_0$ the θ_i -image of the correction term $\delta(E)$ as word $W(\delta(E)) \in$ $\Pi(\mathcal{G}_i)$.
- (c) For every edge E with $\tau(E) = V_i \in \mathcal{V}_0$, a vertex v_E of \mathcal{G}_i and connecting words $\gamma_E \in \Pi(\mathcal{G}_i)$ are specified which satisfy

$$H_{i*}(\gamma_E^{-1})W(\delta(E))\gamma_E \in G_{v_E}.$$

Then from these data one derives directly (in an algorithmic way) the analogous data needed to define the blow-up graph \mathcal{G} and the blow-up isomorphism H. Indeed, the precise instructions for this procedure are given in the parts (1) and (2) of proof of Theorem 6.3.

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