# QUOTIENT AND BLOW-UP OF AUTOMORPHISMS OF GRAPHS OF GROUPS 

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#### Abstract

In this paper we study the quotient and "blow-up" of graph-of-groups $\mathcal{G}$ and of their automorphisms $H: \mathcal{G} \rightarrow \mathcal{G}$. We show that the existence of such a blow-up of any $\bar{H}: \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}$, relative to a given family of "local" graph-of-groups isomorphisms $H_{i}: \mathcal{G}_{i} \rightarrow \mathcal{G}_{i}$ depends crucially on the $H_{i}^{-1}$-conjugacy class of the correction term $\delta\left(E_{i}\right)$ for any edge $E_{i}$ of $\overline{\mathcal{G}}$, where $H$-conjugacy is a new but natural concept introduced here.

As an application we obtain a criterion as to whether a partial Dehn twist can be blown up relative to local Dehn twists, to give an actual Dehn twist. The results of this paper are also used crucially in the follow-up papers [12, 21, 22.


## 1. Introduction

Graphs-of-groups and Bass-Serre theory have played a central role in geometric group theory ever since this field came into existence in the 1980's. As a prime example we would like to mention its prominent role in the understanding of automorphisms of a hyperbolic group $G$, see [11, which is based on an essentially unique graph-of-groups decomposition of $G$, in case that $G$ is freely indecomposable.

If on the other hand the group $G$ is a free group $F_{n}$ of finite rank $n \geq 2$, then a special kind of graph-of-groups, called very small, plays an important role in the definition of the boundary of Culler-Vogtmann's Outer space $C V_{n}$, which is the analogue of Teichmüller space, for $\operatorname{Out}\left(F_{n}\right)$ in place of the mapping class group. The work presented here is mainly motivated by questions arising from this perspective, although we keep our set-up slightly more general.

Given a graph-of-groups $\mathcal{G}$ based on a finite connected graph $\Gamma=\Gamma(\mathcal{G})$, for any connected subgraph $\Gamma_{0} \subset \Gamma$ we denote by $\mathcal{G}_{0}$ the restriction of $\mathcal{G}$ to $\Gamma_{0}$. There is a natural way to define a quotient graph-of-groups $\overline{\mathcal{G}}=\mathcal{G} / \mathcal{G}_{0}$ which is obtained by "contracting" $\mathcal{G}_{0}$ into a single vertex $V_{0}$ with vertex group $G_{V_{0}} \cong \pi_{1}\left(\mathcal{G}_{0}\right)$, thus giving rise to a canonical isomorphism $\Theta: \pi_{1}(\mathcal{G}) \rightarrow$ $\pi_{1}(\overline{\mathcal{G}})$. By construction the quotient graph $\overline{\mathcal{G}}$ is compatible with the local graph-of-groups $\mathcal{G}_{0}$, in the sense that for any edge $E$ of $\overline{\mathcal{G}}$ with terminal vertex $V_{0}$ the canonical image of the edge group $G_{E}$ in the vertex group

[^0]$G_{V_{0}} \cong \pi_{1}\left(\mathcal{G}_{0}\right)$ is (up to conjugation) contained in one of the vertex groups of $\mathcal{G}_{0}$.

This quotient concept extends naturally to an isomorphism $H$ of $\mathcal{G}$ which acts as identity on the underlying graph $\Gamma$ : We can construct a quotient graph-of-groups isomorphism $\bar{H}: \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}$ which induces on the fundamental group an outer automorphism $\widehat{\bar{H}}$ that is conjugate via $\Theta$ to the outer automorphism $\widehat{H}$ induced by $H$, as shown in the diagram below.


The restriction $H_{0}$ of $H$ to $\mathcal{G}_{0}$ is called the local graph-of-groups isomorphism at $V_{0}$, and again certain natural compatibility conditions between the pairs $(\bar{H}, \overline{\mathcal{G}})$ and $\left(H_{0}, \mathcal{G}_{0}\right)$ are satisfied, which are stated precisely in Definition 6.1 below.

Of course, both, the quotient graph-of-groups $\overline{\mathcal{G}}$ and the quotient isomorphism $\bar{H}$, are also well-defined modulo more than one pairwise disjoint connected sub-graph-of-groups $\mathcal{G}_{i}$ of $\mathcal{G}$.

The main purpose of this paper is to study the converse of the above described quotient procedure, which we call the "blow-up" of a graph-ofgroups isomorphism. We prove (see Theorem 6.3):

Theorem 1.1. Let $\bar{H}: \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}$ be a graph-of-groups isomorphism which acts as identity on the graph $\bar{\Gamma}$ underlying $\overline{\mathcal{G}}$. Assume that for some vertices $V_{i}$ of $\bar{\Gamma}$ the group isomorphism $\bar{H}_{V_{i}}$ is induced by a local graph-of-groups automorphism $H_{i}: \mathcal{G}_{i} \rightarrow \mathcal{G}_{i}$ which also acts as identity on the underlying $\operatorname{graph} \Gamma\left(\mathcal{G}_{i}\right)$.

Then one can blow up $(\bar{H}, \overline{\mathcal{G}})$ via the family of $\left(H_{i}, \mathcal{G}_{i}\right)$ to obtain a blow-up graph-of-groups isomorphism $H: \mathcal{G} \rightarrow \mathcal{G}$, with induced outer automorphism $\widehat{H}=\widehat{\bar{H}}$, if and only if each $\left(H_{i}, \mathcal{G}_{i}\right)$ is compatible (in the sense of Definition 6.1) with $(\bar{H}, \overline{\mathcal{G}})$.

We are most interested in the special case where for any edge $E$ of $\overline{\mathcal{G}}$ the edge group $G_{E}$ is trivial. In this case the compatibility conditions from Definition 6.1 simplify to a property of the edge $E$ which we call "locally zero". Since this is a new concept, we will try to explain it here briefly:

Recall first that if $E$ terminates in the vertex $V_{i}$, then (as for any graph-of-groups isomorphism, see Definition (2.8) the correction term $\delta(E) \in G_{V_{i}}$ serves to make the edge and vertex isomorphisms $\bar{H}_{E}$ and $\bar{H}_{V}$ commute with the injective edge homomorphism $f_{E}: G_{E} \rightarrow G_{V_{i}}$.

Now, we say that $E$ is locally zero (see Definition 7.1) if the identification $G_{V_{i}} \cong \pi_{1} \mathcal{G}_{i}$ maps $\delta(E)$ to an element which is " $H_{i}^{-1}$-conjugate" to an element
that has $\mathcal{G}_{V_{i}}$-length equal to zero. If the local automorphism $H_{i}$ is equal to the identity map, then $H_{i}^{-1}$-conjugation will simply be the usual conjugation in $G_{V_{i}}$; in general though it is a more involved and quite delicate new notion, defined below in section 4 ,

In the last section of this paper we will apply Theorem 1.1 to the case of Dehn twist automorphisms of a free group $F_{n}$. Classically, a Dehn twist $D=\left(\mathcal{G},\left(z_{e}\right)_{e \in E(\mathcal{G})}\right)$ on a graph-of-groups $\mathcal{G}$ is defined by a family of twistors $\left(z_{e}\right)_{e \in E(\mathcal{G})}$, where each $z_{e}$ is in the center of the edge group $G_{e}$ of $\mathcal{G}$. It turns out (see Proposition 3.8 and Remark 3.12) that for free groups an alternative, slightly more restrictive definition of a Dehn twist is given by graph-of-groups isomorphisms $H: \mathcal{G} \rightarrow \mathcal{G}$ where all edge groups of $\mathcal{G}$ are trivial and $H$ acts as identity on the underlying graph and on every vertex group of $\mathcal{G}$.

Inspired by this alternative definition, we define (see Definition 3.13) a partial Dehn twist $D: \mathcal{G} \rightarrow \mathcal{G}$ with $\pi_{1}(\mathcal{G}) \cong F_{n}$, relative to some family of vertices $V_{1}, \ldots, V_{m}$ of the underlying graph $\Gamma(\mathcal{G})$, which differs from the above classical notion in that on these "exceptional vertices" $V_{i}$ the local automorphism $D_{V_{i}}: G_{V_{i}} \rightarrow G_{V_{i}}$ induced by $D$ may be non-trivial.

Of particular interest is the case where these non-trivial local automorphisms are all Dehn twist automorphisms themselves. This occurs naturally if one quotients a given Dehn twists modulo a family of pairwise disjoint subgraphs. The converse direction, however, is far less obvious, and the desired blow-up Dehn twist doesn't always exist. We prove here (see Corollary 7.2):

Corollary 1.2. (1) Let $\bar{D}: \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}$ be a partial Dehn twist, and assume that for some family of vertices $V_{i}$ of $\overline{\mathcal{G}}$ the vertex group automorphisms $D_{V_{i}}$ are induced by Dehn twists $D_{V_{i}}: \mathcal{G}_{V_{i}} \rightarrow \mathcal{G}_{V_{i}}$.

Then $\bar{D}$ can be blown up via the given family of local Dehn twists $D_{V_{i}}$ to give a graph-of-groups isomorphism $D: \mathcal{G} \rightarrow \mathcal{G}$ if and only if every edge $E_{i}$ of $\overline{\mathcal{G}}$ with terminal endpoint in one of the $V_{i}$ is locally zero.
(2) The blow-up automorphism $D: \mathcal{G} \rightarrow \mathcal{G}$ obtained in (1) is a Dehn twist, and hence $\widehat{D}=\widehat{\bar{D}}$ is a Dehn twist automorphism.

It turns out that the last conclusion of the above corollary is more subtle than it may appear at first sight. In order to explain this, we first consider the following two examples:

Example 1.3. (1) We consider a graph-of-groups isomorphism $\bar{H}: \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}$ defined as the follows:
(a) The graph $\Gamma(\overline{\mathcal{G}})$ underlying $\overline{\mathcal{G}}$ consists of a single edge $E$ and two distinct vertices $V=\tau(\bar{E}) \neq V_{1}=\tau(E)$. The graph-of-groups $\overline{\mathcal{G}}$ has trivial edge group $G_{E}$, hence trivial edge homomorphisms, and vertex groups $G_{V}=\langle a, b\rangle, G_{V_{1}}=\langle c\rangle$.
(b) The isomorphism $\bar{H}$ acts as the identiy on $\Gamma(\overline{\mathcal{G}})$ and induces trivial group automorphisms on $G_{E}$ and on $G_{V_{1}}$, while the local group automorphism $\bar{H}_{V}: G_{V} \rightarrow G_{V}$ is a Dehn twist automorphism which acts on the generators by the map $a \mapsto a$ and $b \mapsto b a$. The correction terms are $\delta(E)=1_{G_{V_{1}}}$ and $\delta(\bar{E})=a b a^{-1} b^{-1}$. In particular, $\bar{H}$ is a partial Dehn twist relative to the vertex $V$.
(2) We now consider a local graph-of-groups isomorphism $H_{V}: \mathcal{G}_{V} \rightarrow \mathcal{G}_{V}$ which induces the same Dehn twist automorphism as $\bar{H}_{V}$. The isomorphism $H_{V}$ is defined by the following data:
(a) The graph-of-groups $\mathcal{G}_{V}$ consists of a single vertex $v$ with $G_{v}=\langle x\rangle$ and a loop edge $e$ with trivial edge group. The isomorphism $H_{V}$ is a Dehn twist, in that it acts trivially on the underlying graph $\Gamma\left(\mathcal{G}_{V}\right)$, the edge group $G_{e}$ and vertex group $G_{v}$. We choose the correction terms to be $\delta(e)=x$ and $\delta(\bar{e})=1_{G_{v}}$.
(b) Then $H_{V}$ induces on its fundamental group $\pi_{1}\left(\mathcal{G}_{V}\right) \cong\left\langle x, t_{e}\right\rangle$ an automorphism which sends $x \mapsto x$ and $t_{e} \mapsto t_{e} x$. This is exactly the same automorphism as $\bar{H}_{V}$, modulo the identification map $\theta$ given by $a \mapsto x$ and $b \mapsto t_{e}$.
However, in this example $\bar{H}$ cannot be blown up via $H_{V}$, since the correction term $\delta(\bar{E})$ is mapped by $\theta$ to $x t_{e} x^{-1} t_{e}^{-1}$ which is not $H_{V}^{-1}$-conjugate to any element that has $G_{V}$-length equal to zero: the edge $\bar{E}$ is not locally-zero.

Example 1.4. Let $(\bar{H}, \overline{\mathcal{G}})$ be the partial Dehn twist defined in Example 1.3 , Instead of $\left(H_{V}, \mathcal{G}_{V}\right)$ we now consider a local Dehn twist $\left(H_{V}^{\prime}, \mathcal{G}_{V}^{\prime}\right)$ where $\mathcal{G}_{V}^{\prime}$ consists of a single vertex $v^{\prime}$ with $G_{v}^{\prime}=\langle x, y\rangle$, a loop edge $e^{\prime}$ with cyclic edge group $G_{e^{\prime}}=\langle z\rangle$, and edge homomorphisms that maps $z$ to $f_{\bar{e}^{\prime}}(z)=y$ and $f_{e^{\prime}}(z)=x$. The correction terms are $\delta\left(e^{\prime}\right)=x$ and $\delta\left(\bar{e}^{\prime}\right)=1_{G_{v^{\prime}}}$.

Then the fundamental group of $\mathcal{G}_{V}^{\prime}$ is $\pi_{1}\left(\mathcal{G}_{V}^{\prime}\right) \cong\left\langle x, y, t_{e^{\prime}} \mid y=t_{e^{\prime}} x t_{e^{\prime}}^{-1}\right\rangle \cong$ $\left\langle x, t_{e^{\prime}}\right\rangle$, and $H_{V}^{\prime}$ induces $\bar{H}_{V}$ via the identification $\theta^{\prime}: a \mapsto x ; b \mapsto t_{e^{\prime}}$.

Contrary to the previous example, one can indeed blow up ( $\bar{H}, \overline{\mathcal{G}}$ ) via $\left(H_{V}^{\prime}, \mathcal{G}_{V}^{\prime}\right)$ since the identification $\theta^{\prime} \operatorname{maps} \delta(\bar{E})$ to $x t_{e^{\prime}} x^{-1} t_{e^{\prime}}^{-1}=x y^{-1} \in G_{v^{\prime}}$ : in this example the edge $\bar{E}$ is locally-zero.

It is easy to see that both examples represent the same outer automorphisms of $F_{3}$. This shows that the last conclusion of Corollary 1.2, namely that the given partial Dehn twist $\bar{D}$ induces on $\pi_{1} \overline{\mathcal{G}}$ a Dehn twist automorphism $\widehat{\bar{D}}$, is not equivalent to the fact that the blow-up automorphism exists and is a Dehn twist. However, using the terminology of [9], it is shown in [21] that, when all the local Dehn twists $D_{V_{i}}$ are "efficient" (as is the case in Example (1.4), then the condition that all edges $E_{i}$ are locally zero is not just sufficient but also necessary for the last conclusion of Corollary 1.2 , This is used in [22] as an important ingredient of the proof that every linearly growing outer automorphism of a finitely generated free group $F_{n}$ is (up to taking powers) a Dehn twist automorphism.

Since this paper has been made public first, its results have already been used crucially in two applications:
(1) Corollary 1.2 is a vital ingredient in our algorithmic solution in [22] to the question which polynomially growing automorphisms of $F_{n}$ are (up to passing to a power) induced by a surface homeomorphism.
(2) In [12] a normal form (based on graph-of-groups and Dehn twists) for quadratically growing automorphisms is given, together with a method how to derive this normal form. One of the crucial steps in this procedure is to quotient subgraphs and to blow up vertices relative to local Dehn twists, as studied here.

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## 2. Basics of Graphs of groups and their Isomorphisms

In this section we recall some basic knowledge about graph-of-groups as well as their isomorphisms. Most of our notations are taken from (9]; we refer the readers to [19], [18] and [1] for more detailed information and discussions.

### 2.1. Basic Conventions.

Unless otherwise stated, a graph refers to a finite, non-empty, connected graph in the sense of Serre (cf. [19]).

We recall the notations here. For a graph $\Gamma$, we denote by $V(\Gamma), E(\Gamma)$ its vertex set and edge set respectively. An edge $e \in E(\Gamma)$ is oriented, and we denote by $\bar{e}$ the edge with inverse orientation, $\tau(e)$ its terminal vertex and $\tau(\bar{e})=\iota(e)$ its initial vertex.

Notice in particular that our graph $\Gamma$ is non-oriented. An orientation of $\Gamma$ refers to a subset $E^{+}(\Gamma) \subset E(\Gamma)$ such that $E^{+}(\Gamma) \cup \bar{E}^{+}(\Gamma)=E(\Gamma)$ and $E^{+}(\Gamma) \cap \bar{E}^{+}(\Gamma)=\emptyset$, where $\bar{E}^{+}(\Gamma)=\left\{\bar{e} \mid e \in E^{+}(\Gamma)\right\}$.

For an arbitrary group $G$, we denote by $a d_{x}: G \rightarrow G$ the inner automorphism defined by element $x \in G$, namely $a d_{x}(g)=x g x^{-1}$ for all $g \in G$.

### 2.2. Graphs of Groups.

Definition 2.1. A graph-of-groups $\mathcal{G}$ is defined by

$$
\mathcal{G}=\left(\Gamma,\left(G_{v}\right)_{v \in V(\Gamma)},\left(G_{e}\right)_{e \in E(\Gamma)},\left(f_{e}\right)_{e \in E(\Gamma)}\right)
$$

where:
(1) $\Gamma$ is a graph, called the underlying graph;
(2) each $G_{v}$ is a group, called the vertex group of $v$;
(3) each $G_{e}$ is a group, called the edge group of $e$, and we require $G_{e}=G_{\bar{e}}$ for every $e \in E(\Gamma)$;
(4) for each $e \in E(\Gamma)$, the map $f_{e}: G_{e} \rightarrow G_{\tau(e)}$ is an injective edge homomorphism.

For a graph-of-groups $\mathcal{G}$, we usually denote by $\Gamma(\mathcal{G})$ the graph underlying it. The vertex set of $\Gamma(\mathcal{G})$ is denoted by $V(\mathcal{G})$ while the edge set is denoted by $E(\mathcal{G})$.

Definition 2.2. For a graph-of-groups $\mathcal{G}$, its word group $W(\mathcal{G})$ is the free product of all vertex groups and of the free group generated by the stable letter $t_{e}$ for every $e \in E(\Gamma)$, i.e.

$$
W(\mathcal{G})=\underset{v \in V(\Gamma)}{*} G_{v} * F\left(\left\{t_{e} \mid e \in E(\Gamma)\right\}\right) .
$$

The path group (sometimes also called Bass group) of $\mathcal{G}$ is defined by

$$
\Pi(\mathcal{G})=W(\mathcal{G}) / R,
$$

where $R$ is the normal subgroup determined by the following relations:
$\diamond t_{e}=t_{\bar{e}}^{-1}$, for every $e \in E(\Gamma) ;$
$\diamond f_{\bar{e}}(g)=t_{e} f_{e}(g) t_{e}^{-1}$, for every $e \in E(\Gamma)$ and every $g \in G_{e}$.
Remark 2.3. A word $w \in W(\mathcal{G})$ can always be written in the form

$$
w=r_{0} t_{1} r_{1} \ldots r_{q-1} t_{q} r_{q} \quad(q \geq 0),
$$

where each $t_{i} \in\left\{t_{e} \mid e \in E(\Gamma)\right\}$ and each $r_{i} \in \underset{v \in V(\Gamma)}{*} G_{v}$.
The sequence $\left(t_{1}, t_{2}, \ldots, t_{q}\right)$ is called the path type of $w$, the number $q$ is called the path length, or sometimes the $\mathcal{G}$-length of $w$, denoted by $|w|_{\mathcal{G}}=q$. In this case, we say that $e_{1} e_{2} \ldots e_{q}$ is the path underlying $w$. Two path types $\left(t_{1}, t_{2}, \ldots, t_{q}\right)$ and $\left(t_{1}^{\prime}, t_{2}^{\prime}, \ldots, t_{s}^{\prime}\right)$ are said to be same if and only if $q=s$ and $t_{i}=t_{i}^{\prime}$ for each $1 \leq i \leq q$.
Definition 2.4. Let $w \in W(\mathcal{G})$ be a word of the form $w=r_{0} t_{1} r_{1} \ldots r_{q-1} t_{q} r_{q}$. The word $w$ is said to be connected if $r_{0} \in G_{\tau\left(\bar{e}_{1}\right)}, r_{q} \in G_{\tau\left(e_{q}\right)}$, and $\tau\left(e_{i}\right)=$ $\tau\left(\bar{e}_{i+1}\right)$ with $r_{i} \in G_{\tau\left(e_{i}\right)}$, for $i=1,2, \ldots, q-1$. We sometimes call such $w a$ connecting word from $\tau\left(\bar{e}_{1}\right)$ to $\tau\left(e_{q}\right)$, or a word from $\tau\left(\bar{e}_{1}\right)$ to $\tau\left(e_{q}\right)$.

Moreover, if $w$ is connected and $\tau\left(e_{q}\right)=\tau\left(\bar{e}_{1}\right)$, we say that $w$ is a closed connected word issued at the vertex $\tau\left(e_{q}\right)$.
Definition 2.5. Let $w=r_{0} t_{1} r_{1} \ldots r_{q-1} t_{q} r_{q} \in W(\mathcal{G}), w$ is said to be reduced if it satisfies:
$\diamond$ if $q=0$, then $w=r_{0}$ isn't equal to the unit element;
$\diamond$ if $q>0$, then whenever $t_{i}=t_{i+1}^{-1}$ for some $1 \leq i \leq q-1$ we have $r_{i} \notin f_{e_{i}}\left(G_{e_{i}}\right)$.
Moreover the word $w$ is said to be cyclically reduced if it is reduced and if $q>0$ and $t_{1}=t_{q}^{-1}$, then $r_{q} r_{0} \notin f_{e_{q}}\left(G_{e_{q}}\right)$.

We recall the following well known facts.
Proposition 2.6. For any graph-of-groups $\mathcal{G}$, the following holds:
(1) Every non-trivial element of $\Pi(\mathcal{G})$ can be represented as a reduced word.
(2) Every reduced word is a non-trivial element in $\Pi(\mathcal{G})$.
(3) If $w_{1}, w_{2} \in W(\mathcal{G})$ are two reduced words representing the same element in $\Pi(\mathcal{G})$, then $w_{1}$ and $w_{2}$ are of the same path type. In particular, $w_{2}$ is connected if and only if $w_{1}$ is connected.

In fact, suppose $w_{1}=r_{0} t_{1} r_{1} \ldots r_{q-1} t_{q} r_{q}$ and $w_{2}=r_{0}^{\prime} t_{1} r_{1}^{\prime} \ldots r_{q-1}^{\prime} t_{q} r_{q}^{\prime}$, then there exist elements $h_{i} \in G_{e_{i}}(i=1,2, \ldots, q)$ such that:

$$
r_{0}^{\prime}=r_{0} f_{\bar{e}_{1}}\left(h_{1}\right) ; r_{i}^{\prime}=f_{e_{i}}\left(h_{i}\right) r_{i} f_{\bar{e}_{i+1}}\left(h_{i+1}^{-1}\right) \text { for }(i=1,2, \ldots, q-1)
$$

$$
\text { and } r_{q}^{\prime}=f_{e_{q}}\left(h_{q}\right) r_{q}
$$

Definition 2.7. For any $v_{0} \in V(\Gamma)$ the fundamental group based at $v_{0}$, denoted by $\pi_{1}\left(\mathcal{G}, v_{0}\right)$, consists of the elements in $\Pi(\mathcal{G})$ that are represented by closed connected words issued at $v_{0}$.

For a vertex $w_{0} \in V(\Gamma)$ different from $v_{0}$, we have $\pi_{1}\left(\mathcal{G}, v_{0}\right) \cong \pi_{1}\left(\mathcal{G}, w_{0}\right)$. In fact, let $W \in \Pi(\mathcal{G})$ be represented by a a connected word with underlying path from $v_{0}$ to $w_{0}$. The restriction of $a d_{W}: \Pi(\mathcal{G}) \rightarrow \Pi(\mathcal{G})$ to $\pi_{1}\left(\mathcal{G}, w_{0}\right)$ induces an isomorphism from $\pi_{1}\left(\mathcal{G}, w_{0}\right)$ to $\pi_{1}\left(\mathcal{G}, v_{0}\right)$. Sometimes we write $\pi_{1}(\mathcal{G})$ when the choice of basepoint doesn't make a difference.

### 2.3. Graph-of-Groups Isomorphism.

Definition 2.8. Let $\mathcal{G}_{1}, \mathcal{G}_{2}$ be two graphs of groups. Denote $\Gamma_{1}=\Gamma\left(\mathcal{G}_{1}\right)$ and $\Gamma_{2}=\Gamma\left(\mathcal{G}_{2}\right)$. An isomorphism $H: \mathcal{G}_{1} \rightarrow \mathcal{G}_{2}$ is a tuple of the form

$$
H=\left(H_{\Gamma},\left(H_{v}\right)_{v \in V\left(\Gamma_{1}\right)},\left(H_{e}\right)_{e \in E\left(\Gamma_{1}\right)},(\delta(e))_{e \in E\left(\Gamma_{1}\right)}\right)
$$

where
(1) $H_{\Gamma}: \Gamma_{1} \rightarrow \Gamma_{2}$ is a graph isomorphism;
(2) $H_{v}: G_{v} \rightarrow G_{H_{\Gamma}(v)}$ is a group isomorphism, for any $v \in V\left(\Gamma_{1}\right)$;
(3) $H_{e}=H_{\bar{e}}: G_{e} \rightarrow G_{H_{\Gamma}(e)}$ is a group isomorphism, for any $e \in E\left(\Gamma_{1}\right)$;
(4) for every $e \in E\left(\Gamma_{1}\right)$, the correction $\operatorname{term} \delta(e) \in G_{\tau\left(H_{\Gamma}(e)\right)}$ is an element such that

$$
H_{\tau(e)} f_{e}=a d_{\delta(e)} f_{H_{\Gamma}(e)} H_{e}
$$

Remark 2.9. A graph-of-groups isomorphism $H: \mathcal{G}_{1} \rightarrow \mathcal{G}_{2}$ induces an isomorphism $H_{*}: \Pi\left(\mathcal{G}_{1}\right) \rightarrow \Pi\left(\mathcal{G}_{2}\right)$ defined on the generators by:

$$
\begin{aligned}
& H_{*}(g)=H_{v}(g), \text { for } g \in G_{v}, v \in V\left(\Gamma_{1}\right) \\
& H_{*}\left(t_{e}\right)=\delta(\bar{e}) t_{H_{\Gamma}(e)} \delta(e)^{-1}, \text { for } e \in E\left(\Gamma_{1}\right)
\end{aligned}
$$

It is easy to verify that $H_{*}$ preserves the relations $t_{e} t_{\bar{e}}=1$ for any $e \in E(\mathcal{G})$ and $f_{\bar{e}}(g)=t_{e} f_{e}(g) t_{e}^{-1}$, for any $e \in E(\mathcal{G})$ and $g \in G_{e}$.

Furthermore, the restriction of $H_{*}$ to $\pi_{1}\left(\mathcal{G}_{1}, v\right)$, where $v \in V\left(\Gamma_{1}\right)$, defines an isomorphism $H_{* v}: \pi_{1}\left(\mathcal{G}_{1}, v\right) \rightarrow \pi_{1}\left(\mathcal{G}_{2}, H_{\Gamma}(v)\right)$.

As in [9], we define the outer isomorphism induced by a group isomorphism $f: G_{1} \rightarrow G_{2}$ as the equivalence class

$$
\widehat{f}=\left\{a d_{g} f: G_{1} \rightarrow G_{2} \mid g \in G_{2}\right\}
$$

Hence $H_{* v}$ induces an outer isomorphism $\widehat{H}_{* v}: \pi_{1}\left(\mathcal{G}_{1}, v\right) \rightarrow \pi_{1}\left(\mathcal{G}_{2}, H_{\Gamma}(v)\right)$.

Observe that when choosing a different vertex $v_{1}$ as basepoint, we may choose a word $W \in \Pi\left(\mathcal{G}_{1}\right)$ with underlying path from $v_{1}$ to $v$ to obtain the following commutative diagram:

$$
\begin{array}{lll}
\pi_{1}\left(\mathcal{G}_{1}, v\right) \xrightarrow{H_{* v}} & \pi_{1}\left(\mathcal{G}_{2}, H_{\Gamma}(v)\right) \\
a d_{W} \downarrow \\
\pi_{1}\left(\mathcal{G}_{1}, v_{1}\right) \xrightarrow[H_{* v_{1}}]{ } & \pi_{1}\left(\mathcal{G}_{2}, H_{\Gamma}\left(v_{1}\right)\right)
\end{array}
$$

By Lemma 2.2 and Lemma 3.10 in [9, the map $\widehat{H}_{* v}$ determines an outer isomorphism $\widehat{H}_{* v_{1}}: \pi_{1}\left(\mathcal{G}_{1}, v_{1}\right) \rightarrow \pi_{1}\left(\mathcal{G}_{2}, H_{\Gamma}\left(v_{1}\right)\right)$ which is independent of the choice of $W$. Hence the isomorphism $H: \mathcal{G}_{1} \rightarrow \mathcal{G}_{2}$ induces a well-defined outer isomorphism $\widehat{H}: \pi_{1}\left(\mathcal{G}_{1}\right) \rightarrow \pi_{1}\left(\mathcal{G}_{2}\right)$ which doesn't depend on the choice of basepoint.

Remark 2.10 (Composition, Inverse). For two graph-of-groups isomorphisms $H^{\prime}: \mathcal{G}_{1} \rightarrow \mathcal{G}_{2}, H^{\prime \prime}: \mathcal{G}_{2} \rightarrow \mathcal{G}_{3}$, the composition of $H^{\prime}$ and $H^{\prime \prime}$ is an isomorphism $H^{\prime \prime} H^{\prime}=H: \mathcal{G}_{1} \rightarrow \mathcal{G}_{3}$ given (for any $v \in V\left(\Gamma_{1}\right)$, $e \in E\left(\Gamma_{1}\right)$ ) precisely by: $H_{\Gamma}=H_{\Gamma}^{\prime \prime} H_{\Gamma}^{\prime} ; H_{v}=H_{H_{\Gamma}^{\prime}(v)}^{\prime \prime} H_{v}^{\prime} ; H_{e}=H_{H_{\Gamma}^{\prime}(e)}^{\prime \prime} H_{e}^{\prime}$; $\delta(e)=H_{\tau\left(H_{\Gamma}^{\prime}(e)\right)}^{\prime \prime}\left(\delta^{\prime}(e)\right) \delta^{\prime \prime}\left(H_{\Gamma}^{\prime}(e)\right)$. Moreover, $H$ satisfies $H_{*}=H_{*}^{\prime \prime} H_{*}^{\prime}$ and $\widehat{H}=\widehat{H}^{\prime \prime} \widehat{H}^{\prime}$.

For any graph-of-groups isomorphism $H: \mathcal{G}_{1} \rightarrow \mathcal{G}_{2}$ the inverse isomorphism is $H^{-1}: \mathcal{G}_{2} \rightarrow \mathcal{G}_{1}$, which satisfies $H_{*}^{-1}=\left(H^{-1}\right)_{*}$ and $\widehat{H}^{-1}=\widehat{H^{-1}}$, is defined (for all $\left.v \in V\left(\Gamma_{2}\right), e \in E\left(\Gamma_{2}\right)\right)$ by: $\left(H^{-1}\right)_{\Gamma}=\left(H_{\Gamma}\right)^{-1} ;\left(H^{-1}\right)_{v}=$ $\left(H_{H_{\Gamma}^{-1}(v)}\right)^{-1} ;\left(H^{-1}\right)_{e}=\left(H_{H_{\Gamma}^{-1}(e)}\right)^{-1} ; \delta^{-1}(e)=H_{H^{-1}(\tau(e))}^{-1}\left(\delta\left(H_{\Gamma}^{-1}(e)\right)^{-1}\right)$.

### 2.4. A Natural Equivalence Between Graphs Of Groups.

Suppose that $\mathcal{G}, \mathcal{G}^{\prime}$ are two graphs-of-groups, and that $\mathcal{G}^{\prime}$ equals to $\mathcal{G}$ everywhere except that for some edge $e_{0} \in E(\mathcal{G})$ one has $f_{e_{0}}^{\prime}=a d_{w_{e_{0}}^{-1}} \circ f_{e_{0}}$, where $w_{e_{0}}$ is an element in $G_{\tau\left(e_{0}\right)}$. Then there is a natural isomorphism between $\mathcal{G}$ and $\mathcal{G}^{\prime}$.

More concretely, define $H_{0}: \mathcal{G} \rightarrow \mathcal{G}^{\prime}$ by the rules:
(1) $H_{0, \Gamma}=i d_{\Gamma(\mathcal{G})}$;
(2) $H_{0, v}=i d_{G_{v}}$ for any $v \in V(\mathcal{G}) ; H_{0, e}=i d_{G_{e}}$ for any $e \in E(\mathcal{G})$;
(3) $\delta_{0}\left(e_{0}\right)=w_{e_{0}}$, and $\delta_{0}(e)=1$ when $e \neq e_{0}$.

Then it's easy to verify that $H_{0}$ is a well-defined graph-of-groups isomorphism, since the additional compatibility requirement holds automatically for all edges $e_{0} \neq e \in \Gamma_{0}$, and for $e_{0}$ we have

$$
H_{0, \tau\left(e_{0}\right)} \circ f_{e_{0}}=f_{e_{0}}=a d_{w_{e_{0}}} \circ f_{e_{0}}^{\prime}=a d_{\delta_{0}\left(e_{0}\right)} \circ f_{e_{0}}^{\prime}=a d_{\delta_{0}\left(e_{0}\right)} \circ f_{e_{0}}^{\prime} \circ H_{0, e_{0}} .
$$

The above isomorphism gives rise to a natural notion of "equivalent" graphs-of-groups, where the equivalence relation is generated by isomorphisms of the above type $H_{0}$ as elementary equivalence. This notion of
"equivalent" graph-of-groups, although not really established in the literature, is natural, in that it preserves (up to canonical isomorphisms) the fundamental group. It also shows up in the prime feature of graph-of-groups, meaning Bass-Serre theory: Given a group $G$ that acts on a (simplicial) tree $T$, for the associated graph-of-groups decomposition $G \cong \pi_{1}\left(\mathcal{G}_{T}\right)$ the "quotient" graph-of-groups $\mathcal{G}_{T}$ of $T$ modulo $G$ is only well defined up to precisely this equivalence relation.

Lemma 2.11. Let $\mathcal{G}, \mathcal{G}^{\prime}, H_{0}, e_{0}$ and $w_{0}$ be as defined as above.
Let $H: \mathcal{G} \rightarrow \mathcal{G}$ be a graph-of-groups automorphism, and let $H^{\prime}=\left(H_{\Gamma}^{\prime}\right.$, $\left.\left(H_{v}^{\prime}\right)_{v \in V\left(\mathcal{G}^{\prime}\right)},\left(H_{e}^{\prime}\right)_{e \in E\left(\mathcal{G}^{\prime}\right)},\left(\delta^{\prime}(e)\right)_{e \in E\left(\mathcal{G}^{\prime}\right)}\right)$ be equal to $H$ everywhere except that $\delta^{\prime}\left(e_{0}\right)=H_{\tau\left(e_{0}\right)}\left(w_{e_{0}}\right)^{-1} \delta\left(e_{0}\right) w_{e_{0}}$.

Then $H^{\prime}$ determines a well-defined graph-of-groups automorphism which is conjugate to $H$ via $H_{0}$. More precisely, we have $H^{\prime}=H_{0} \circ H \circ H_{0}^{-1}$, and hence in particular $\widehat{H}^{\prime}=\widehat{H}_{0} \circ \widehat{H} \circ \widehat{H}_{0}^{-1}$.
Proof. In order to show that $H^{\prime}$ is a well-defined graph-of-groups isomorphism, it is sufficient to verify that $H_{\tau\left(e_{0}\right)} \circ f_{e_{0}}^{\prime}=a d_{\delta^{\prime}\left(e_{0}\right)} \circ f_{e_{0}}^{\prime} \circ H_{e_{0}}$.

For every $g \in G_{e_{0}}$ we have

$$
\begin{aligned}
a d_{\delta^{\prime}\left(e_{0}\right)} \circ f_{e_{0}}^{\prime} \circ H_{e_{0}}(g) & =H_{\tau\left(e_{0}\right)}\left(w_{e_{0}}\right)^{-1} \delta\left(e_{0}\right) w_{e_{0}} f_{e_{0}}^{\prime}\left(H_{e_{0}}(g)\right) w_{e_{0}}^{-1} \delta\left(e_{0}\right)^{-1} H_{\tau\left(e_{0}\right)}\left(w_{e_{0}}\right) \\
& =H_{\tau\left(e_{0}\right)}\left(w_{e_{0}}\right)^{-1} \delta\left(e_{0}\right) f_{e_{0}}\left(H_{e_{0}}(g)\right) \delta\left(e_{0}\right)^{-1} H_{\tau\left(e_{0}\right)}\left(w_{e_{0}}\right) \\
& =H_{\tau\left(e_{0}\right)}\left(w_{e_{0}}\right)^{-1} H_{\tau\left(e_{0}\right)}\left(f_{e_{0}}(g)\right) H_{\tau\left(e_{0}\right)}\left(w_{e_{0}}\right) \\
& =H_{\tau\left(e_{0}\right)}\left(w_{e_{0}}^{-1} f_{e_{0}}(g) w_{e_{0}}\right) \\
& =H_{\tau\left(e_{0}\right)} \circ f_{e_{0}}^{\prime}(g)
\end{aligned}
$$

Moreover, we have the following diagram commutes:


The only non-trivial part here is to verify that the following equation holds for all edges:

$$
H_{0, H_{\Gamma_{0}}(\tau(e))}(\delta(e)) \delta_{0}\left(H_{\Gamma}(e)\right)=H_{H_{\Gamma_{0}}(\tau(e))}^{\prime}\left(\delta_{0}(e)\right) \delta^{\prime}\left(H_{\Gamma_{0}}(e)\right)
$$

When $e \neq e_{0}$, this equation follows from $\delta(e)=\delta^{\prime}(e)$ which holds automatically by definition.

When $e=e_{0}$, we Compute both, the left and the right hand side:

$$
\begin{aligned}
\text { Left: } H_{0, H_{\Gamma_{0}}(\tau(e))}(\delta(e)) \delta_{0}\left(H_{\Gamma}(e)\right) & =\delta\left(e_{0}\right) \delta_{0}\left(e_{0}\right)=\delta\left(e_{0}\right) w_{e_{0}} \\
\text { Right: } H_{H_{\Gamma_{0}}(\tau(e))}^{\prime}\left(\delta_{0}(e)\right) \delta^{\prime}\left(H_{\Gamma_{0}}(e)\right) & =H_{\tau\left(e_{0}\right)}^{\prime}\left(\delta_{0}\left(e_{0}\right)\right) \delta^{\prime}\left(e_{0}\right) \\
& =H_{\tau\left(e_{0}\right)}^{\prime}\left(w_{e_{0}}\right) \delta^{\prime}\left(e_{0}\right) \\
& =H_{\tau\left(e_{0}\right)}\left(w_{e_{0}}\right) \delta^{\prime}\left(e_{0}\right)=\delta\left(e_{0}\right) w_{e_{0}}
\end{aligned}
$$

Hence we have shown that the equation holds for all edges, and that $H^{\prime}=H_{0} \circ H \circ H_{0}^{-1}$, which implies $\widehat{H}^{\prime}=\widehat{H}_{0} \circ \widehat{H} \circ \widehat{H}_{0}^{-1}$.

Remark 2.12. Similar to the equivalence between $\mathcal{G}$ and $\mathcal{G}^{\prime}$ discussed above, Lemma 2.11 gives rise to an equivalence between graph-of-groups automorphisms, which is natural in the following sense:

Assume that some group $G$ acts on a simplicial tree $T$, and let $\mathcal{G}_{T}$ be the "quotient graph-of-groups" of $T$ modulo $G$ mentioned above. Assume furthermore that for some outer automorphism $\varphi$ of $G$ the tree $T$ is " $\varphi$ invariant", by which one means that the translation length function $\|\cdot\|_{T}$, defined on the conjugacy classes of $G$ by setting all edge lengths equal to 1 , is preserved by $\varphi$ :

$$
\|\varphi[g]\|_{T}=\|[g]\|_{T} \quad \text { for all } \quad g \in G
$$

Now Bass-Serre theory is set up in such a way that this $\varphi$-invariance of $T$ is equivalent to the existence of a graph-of-groups automorphism $H: \mathcal{G}_{T} \rightarrow \mathcal{G}_{T}$ which induces the given automorphism $\varphi$.

However, in this situation the automorphism $H$ is determined only up to an equivalence which is generated by the elementary equivalence $H \sim H^{\prime}$, where $H$ and $H^{\prime}$ are precisely as given in Lemma 2.11 above.

A statement analogous to Lemma 2.11 with $e_{0}$ replaced by a family of edges can be derived by applying Lemma 2.11 iteratively.

## 3. Dehn Twists

### 3.1. Classical Dehn Twists.

We first recall the classical definition of a Dehn twist as given in [8].
Definition 3.1 (Classical Dehn twist). An automorphism $D: \mathcal{G} \rightarrow \mathcal{G}$ of a graph-of-groups $\mathcal{G}$ is called a (classical) Dehn twist if it satisfies:
(1) $D_{\Gamma}=i d_{\Gamma}$;
(2) $D_{v}=i d_{G_{v}}$, for all $v \in V(\Gamma)$;
(3) $D_{e}=i d_{G_{e}}$, for all $e \in E(\Gamma)$;
(4) for each $G_{e}$, there is an element $\gamma_{e} \in Z\left(G_{e}\right)$ such that $\delta(e)=f_{e}\left(\gamma_{e}\right)$, where $Z\left(G_{e}\right)$ denotes the center of $G_{e}$.
We denote a Dehn twist defined as above by $D=D\left(\mathcal{G},\left(\gamma_{e}\right)_{e \in E(\mathcal{G})}\right)$.
Definition 3.2 (Twistor). Given a Dehn twist $D=D\left(\mathcal{G},\left(\gamma_{e}\right)_{e \in E(\mathcal{G})}\right)$, we define the twistor of an edge $e \in E(\Gamma)$ by setting $z_{e}=\gamma_{\bar{e}} \gamma_{e}^{-1}$. Then for any edge $e$ we have $z_{e} \in Z\left(G_{e}\right)$ and $z_{\bar{e}}=\gamma_{e} \gamma_{\bar{e}}^{-1}=z_{e}^{-1}$.
Remark 3.3. The induced automorphism $D_{*}: \Pi(\mathcal{G}) \rightarrow \Pi(\mathcal{G})$ is defined on generators as follows:

$$
\begin{aligned}
& D_{*}(g)=g, \text { for } g \in G_{v}, v \in V(\Gamma) ; \\
& D_{*}\left(t_{e}\right)=t_{e} f_{e}\left(z_{e}\right), \text { for every } e \in E(\Gamma) .
\end{aligned}
$$

In particular, the induced automorphism on the fundamental group, $D_{* v}$ : $\pi_{1}(\mathcal{G}, v) \rightarrow \pi_{1}(\mathcal{G}, v)$, where $v \in V(\Gamma)$, is called a Dehn twist automorphism.

Remark 3.4. It follows from Proposition 5.4 in [9] that in many situations a Dehn twist on a given graph-of-groups is uniquely determined by its twistors. Thus sometimes we may define a Dehn twist by its twistors $\left(z_{e}\right)_{e \in E(\Gamma)}$ (With $z_{e} \in Z\left(G_{e}\right)$ and $z_{\bar{e}}=z_{e}^{-1}$, for each $\left.e \in E(\Gamma)\right)$. In this case, we may conversely define:

$$
\gamma_{e}= \begin{cases}z_{e}^{-1}, & e \in E^{+}(\Gamma) \\ 1, & e \in E^{-}(\Gamma) .\end{cases}
$$

Definition 3.5. A group automorphism $\varphi: G \rightarrow G$ is said to be a Dehn twist automorphism if it is represented by a graph-of-groups Dehn twist. In other words, there exists a graph-of-groups $\mathcal{G}$, a Dehn twist $D: \mathcal{G} \rightarrow \mathcal{G}$, and an isomorphism $\theta: G \rightarrow \pi_{1}(\mathcal{G}, v)$ such that $\varphi=\theta^{-1} \circ D_{* v} \circ \theta$.

In this case the induced outer automorphism $\widehat{\varphi} \in \operatorname{Out}(G)$ is called a Dehn twist outer automorphism.

Remark 3.6. Notice that, for a given Dehn twist automorphism $\varphi: G \rightarrow G$, its Dehn twist representative is in general not unique.

In the special case where $G$ is a free group, such "unique" Dehn twist representatives are given by efficient Dehn twists. For details see [8] or Section 3.3 in [21].
Remark 3.7. It is easy to see that every multiple Dehn twist homeomorphism $h$ on a compact surface $S$ (possibly with finitely many boundary components), as defined in [10], gives rise to a graph-of-groups Dehn twist $D: \mathcal{G} \rightarrow \mathcal{G}$. Here $\mathcal{G}$ is a graph-of-groups decomposition of the surface group $\pi_{1}(S)$ modeled the decomposition of $S$ by the twistor curves, and the Dehn twist $D$ on $\mathcal{G}$ defines the same outer automorphism on $\pi_{1}(\mathcal{G}) \cong \pi_{1}(S)$ as the given multiple Dehn twist homeomorphism $h$. See [23] and section 6 of [22] for a more detailed explanation.
3.2. General Dehn Twists. There are several places in the literature (e.g. see [7, 11, 15]) where the notion of a "Dehn twist" by means of graph-of-groups automorphisms have been defined; these definitions all agree in essence, but are slightly distinct in their technical specifications. Thus subsection is meant as contribution to a unification of these concepts.

Among the various alternatives to classical Dehn twists as presented in the previous subsection, the idea to simplify the concept of twists on nontrivial edge group elements through keeping the edge groups trivial and working with "interesting" correction terms has its strong merits, but also some defaults (see the opening paragraph of subsection 3.3). It leads to graph-of-groups automorphisms as considered in the next proposition; these are particularly important as they occur naturally in the context of BestvinaHandel's train track maps in 4].

Proposition 3.8. Let $\mathcal{G}$ be a graph-of-groups such that the edge groups $G_{e}$ are trivial for all $e \in E(\mathcal{G})$. Let $H: \mathcal{G} \rightarrow \mathcal{G}$ be an automorphism such that
$\diamond H_{\Gamma}$ acts on $\Gamma(\mathcal{G})$ as identity;
$\diamond H_{v}: G_{v} \rightarrow G_{v}$ is the identity map, for all $v \in V(\mathcal{G})$.
Then the induced automorphism $\widehat{H} \in \operatorname{Out}\left(\pi_{1} \mathcal{G}\right)$ is a Dehn twist automorphism.

The proof of this proposition is postponed (see Remark 3.11); we first want to enlarge our concept of a "Dehn wist automorphism" to include other notions mentioned above. Our Definition 3.9 below includes in particular also the concept set up in [11, which to our knowledge is the most general among the ones presently in the literature. Be aware, however, that in [11] Levitt's terminology for "Dehn twist" corresponds to what is called here "Dehn twist outer automorphism" (see Definition 3.5).

In any case, it will be shown in Proposition 3.10 below that all these generalizations differ from the original "classical" Dehn twist concept only in their presentation and not really in substance.

Definition 3.9 (General Dehn twist). Let $\mathcal{G}$ be a graph-of-groups. An automorphism $D: \mathcal{G} \rightarrow \mathcal{G}$ is called a general Dehn twist if

$$
\begin{aligned}
& \diamond D_{\Gamma}=i d_{\Gamma} ; \\
& \diamond D_{v}=i d_{G_{v}}, \text { for any vertex } v \in V(\mathcal{G}) ; \\
& \diamond D_{e}=i d_{G_{e}}, \text { for any edge } e \in E(\mathcal{G}) ; \\
& \diamond \delta(e) \in C\left(f_{e}\left(G_{e}\right)\right) \text {, where } C\left(f_{e}\left(G_{e}\right)\right) \text { denotes the centeralizer of } f_{e}\left(G_{e}\right) \\
& \quad \text { in } G_{\tau(e)}, \text { for any } e \in E(\Gamma) .
\end{aligned}
$$

Proposition 3.10. The notions of a "classical Dehn" twist and of a "general Dehn" twist are equivalent in the following sense:
(1) Every classical Dehn twist is a general Dehn twist.
(2) To every general Dehn twist $D: \mathcal{G} \rightarrow \mathcal{G}$ we can canonically associate a classical Dehn twist $D^{\prime}: \mathcal{G}^{\prime} \rightarrow \mathcal{G}^{\prime}$ and an isomorphism $\theta_{v}: \pi_{1}(\mathcal{G}, v) \rightarrow \pi_{1}\left(\mathcal{G}^{\prime}, v^{\prime}\right)$ (for any vertex $v$ of $\mathcal{G}$ and a corresponding vertex $v^{\prime}$ of $\mathcal{G}^{\prime}$ ) such that $D_{* v^{\prime}}^{\prime} \circ \theta_{v}=\theta_{v} \circ D_{* v}$.
In particular, the outer automorphism $\widehat{D}$ defined by a general Dehn twist is a Dehn twist automorphism.

Proof. (1) This follows immediately from $f_{e}\left(Z\left(G_{e}\right)\right) \subset C\left(f_{e}\left(G_{e}\right)\right)$.
(2) We first consider a special case:

Let $D: \mathcal{G} \rightarrow \mathcal{G}$ be a general Dehn twist, with all correction terms as defined in the classical case except for a single edge $e$ : we suppose $\delta(e) \in$ $C\left(f_{e}\left(G_{e}\right)\right) \subset G_{\tau(e)}$ but $\delta(e) \notin f_{e}\left(Z\left(G_{e}\right)\right)$.

Then we define a classical Dehn twist $D^{\prime}: \mathcal{G}^{\prime} \rightarrow \mathcal{G}^{\prime}$ which is obtained as follows:
$\diamond$ The graph-of-groups $\mathcal{G}^{\prime}$ is obtained from $\mathcal{G}$ by introducing a new vertex $v_{0}$ which subdivides the edge $e$ into $e^{\prime}$ and $e^{\prime \prime}$, with $\iota\left(e^{\prime}\right)=\iota(e)$, $\tau\left(e^{\prime}\right)=v_{0}=\iota\left(e^{\prime \prime}\right)$ and $\tau\left(e^{\prime \prime}\right)=\tau(e)$, by setting

$$
G_{e^{\prime}}=G_{e} \quad \text { and } \quad G_{e^{\prime \prime}}=G_{v_{0}}=\left\langle f_{e}\left(G_{e}\right), \delta(e)\right\rangle,
$$

and by defining the edge homomorphisms through

$$
\begin{aligned}
& f_{e^{\prime}}(x)=f_{e}(x), f_{f_{\bar{e}^{\prime}}}(x)=f_{\bar{e}}(x) \quad \text { for all } \quad x \in G_{e^{\prime}}=G_{e}, \\
& \quad \text { and } \quad f_{e^{\prime \prime}}=i d_{G_{e^{\prime \prime}}}, f_{\bar{e}^{\prime \prime}}=i d_{G_{e^{\prime \prime}}} .
\end{aligned}
$$

$\diamond$ The Dehn twist $D^{\prime}$ is defined by setting

$$
D_{e^{\prime}}^{\prime}=i d_{G_{e^{\prime}}}, \quad D_{e^{\prime \prime}}^{\prime}=i d_{G_{e^{\prime \prime}}}, \quad D_{v_{0}}^{\prime}=i d_{G_{v_{0}}},
$$

and by choosing

$$
\delta^{\prime}\left(\bar{e}^{\prime}\right)=\delta(\bar{e}), \delta^{\prime}\left(e^{\prime}\right)=1_{G_{v_{0}}}, \delta^{\prime}\left(\bar{e}^{\prime \prime}\right)=1_{G_{v_{0}}}, \delta^{\prime}\left(e^{\prime \prime}\right)=\delta(e) .
$$

The data for $D^{\prime}$ are equal to those for $D$ everywhere else.
Since $\delta(e)$ is assumed to lie in $C\left(f_{e}\left(G_{e}\right)\right)$, any element in $f_{e}\left(G_{e}\right)$ commutes with $\delta(e)$. As $\delta(e)$ also commutes with itself, we derive immediately that $\delta^{\prime}\left(e^{\prime \prime}\right)=\delta(e)$ commutes with all elements contained in $\left\langle f_{e}\left(G_{e}\right), \delta(e)\right\rangle=$ $f_{e^{\prime \prime}}\left(G_{e^{\prime \prime}}\right)$, i.e. $\delta^{\prime}\left(e^{\prime \prime}\right) \in Z\left(f_{e^{\prime \prime}}\left(G_{e^{\prime \prime}}\right)\right)=f_{e^{\prime \prime}}\left(Z\left(G_{e^{\prime \prime}}\right)\right)$. Thus $D^{\prime}$ is a classical Dehn twist as given in Definition 3.5,

The Dehn twist $D^{\prime}$ is "equal" to $D$ in the following sense: for every vertex $v \neq v_{0}$ from $\mathcal{G}^{\prime}$ there is a corresponding vertex for $\mathcal{G}$ which we also call $v$.

Consider the homomorphism $\theta: \Pi(\mathcal{G}) \rightarrow \Pi\left(\mathcal{G}^{\prime}\right)$ defined on generators by $t_{e} \mapsto t_{e^{\prime}} t_{e^{\prime \prime}}$ and by $g \mapsto g$ otherwise. Then we claim that, for every vertex $v \neq v_{0}$, the restriction of $\theta$ to $\pi_{1}(\mathcal{G}, v)$ defines an isomorphism $\theta_{v}: \pi_{1}(\mathcal{G}, v) \rightarrow$ $\pi_{1}\left(\mathcal{G}^{\prime}, v\right)$.

To see this, we first observe that because of $G_{v_{0}}=f_{\bar{e}^{\prime \prime}}\left(G_{e^{\prime \prime}}\right)$ a reduced word in $\Pi\left(\mathcal{G}^{\prime}\right)$ can not contain as subword any word of type $t_{\bar{e}^{\prime \prime}} g t_{e^{\prime \prime}}$ with $g \in G_{v_{0}}$. Since furthermore $e^{\prime \prime}$ and $\bar{e}^{\prime}$ are the only edges issuing from $v_{0}$, it follows for any reduced word $W \in \pi_{1}\left(\mathcal{G}^{\prime}, v\right)$ with $v \neq v_{0}$, after appropriately applying the relation $f_{e^{\prime}}(g)=t_{e^{\prime}}^{-1} f_{\bar{e}^{\prime}}(g) t_{e^{\prime}}$ from Definition [2.2, that any occurrence of $t_{e^{\prime \prime}}$ in $W$ is preceded by $t_{e^{\prime}}$ and any $t_{\bar{e}^{\prime \prime}}$ is succeeded by $t_{\bar{e}^{\prime}}$. This proves the surjectivity of the map $\theta_{v}$, since by Proposition 2.6 (1) it suffices to consider reduced words. The injectivity is a direct consequence of part (3) of the same proposition.

To conclude the proof in the special case we now observe that $\theta$ gives rise to the following diagram:


It follows directly from the definition of the maps involved that this diagram is commutative; the only non-trivial argument needed is given by:

$$
\begin{aligned}
\theta \circ D_{*}\left(t_{e}\right)=\theta\left(\delta(\bar{e}) t_{e} \delta(e)^{-1}\right) & =\delta(\bar{e}) t_{e^{\prime}} t_{e^{\prime \prime}} \delta(e)^{-1} \\
& =D_{*}^{\prime}\left(t_{e^{\prime}}\right) D_{*}^{\prime}\left(t_{e^{\prime \prime}}\right)=D_{*}^{\prime}\left(t_{e^{\prime}} t_{e^{\prime \prime}}\right) \\
& =D_{*}^{\prime} \circ \theta\left(t_{e}\right)
\end{aligned}
$$

In the general case, where $D$ is a general Dehn twist which may have more than one correction term defined in the "non-classical way", we may apply the above treated special case repeatedly to each of the "non-classical" correction terms, to eventually obtain a classical Dehn twist.

Remark 3.11. One obtains now the statement of Proposition 3.8 as direct consequence of Proposition 3.10 (2), since the graph-of-group automorphism $H$ from Proposition 3.8 is clearly a general Dehn twist.

### 3.3. Partial Dehn twists.

The type of Dehn twists as considered in Proposition 3.8 has a strong appeal, due to its simplicity. It is furthermore of natural interest because it is used in relative train track theory (see [4]). However, it should be noted that not every outer Dehn twist automorphism, even for a free group of finite rank, can be represented by such simple graph-of-groups automorphisms, as illustrated by the following example:
Remark 3.12. We consider $F_{3}=F(a, b, c)$ and the automorphism $\varphi \in$ $\operatorname{Out}\left(F_{3}\right)$ which acts on the generators by sending $a \mapsto a, b \mapsto b a$ and $c \mapsto$ $\left(a b a^{-1} b^{-1}\right) c\left(a b a^{-1} b^{-1}\right)^{-1}$. This is the Dehn twist automorphism induced by the partial Dehn twist given in Example 1.3.

In order to see that this Dehn twist automorphism can not be realized by a Dehn twist $D: \mathcal{G} \rightarrow \mathcal{G}$ as in Proposition 3.8, i.e. based on a graph-of-groups $\mathcal{G}$ with all edge groups trivial, we observe that for such a graph-of-groups every vertex group is a free factor of $\pi_{1} \mathcal{G} \cong F(a, b, c)$. From the algebraic prescriptions $a \mapsto a, b \mapsto b a, c \mapsto\left(a b a^{-1} b^{-1}\right) c\left(a b a^{-1} b^{-1}\right)^{-1}$ we derive that there must be two edges in $\mathcal{G}$, with precisely one of them a loop edge. As a consequence that there must also be precisely two vertices in $\mathcal{G}$, and none of the vertex groups can be trivial. It follows that both vertex groups must have rank 1. Since the conjugacy class of the two twisters are prescribed by the action on $\pi_{1} \mathcal{G}$, it follows that one of the edges at one of its endpoints must have as correction term an element conjugate to $a b a^{-1} b^{-1}$. Such a commutator, however, can not be contained in any of the vertex groups, if the latter is a free factor of $F(a, b, c)$ of rank 1 . Hence a graph-of-groups $\mathcal{G}$ as required for the Dehn twist in question does not exist.

In order to make up for this defect, we now introduce the following notion of partial Dehn twist, which is more thoroughly studied in [21], [22], as well as Section 7 of this article .

Definition 3.13 (Partial Dehn twist). Let $\mathcal{G}$ be a graph-of-groups, and let $\mathcal{V}_{0} \subset V(\mathcal{G})$ be a set of vertices which has the property that any edge $e$ with $\tau(e)=v \in \mathcal{V}_{0}$ has trivial edge group $G_{e}$.

A partial Dehn twist relative to $\mathcal{V}_{0}$ is a graph-of-groups isomorphism $H$ : $\mathcal{G} \rightarrow \mathcal{G}$ such that
$\diamond H_{\Gamma}: \Gamma(\mathcal{G}) \rightarrow \Gamma(\mathcal{G})$ is the identity;
$\diamond H_{e}=i d_{G_{e}}$ for all edges $e \in E(\mathcal{G})$;
$\diamond H_{v}=i d_{G_{v}}$ for all vertices $v \notin \mathcal{V}_{0}$ (while $H_{v}$ is any group isomorphism for all $\left.v \in \mathcal{V}_{0}\right)$;
$\diamond \delta(e) \in C\left(f_{e}\left(G_{e}\right)\right)$ for all edges $e \in E(\mathcal{G})$.
The case which is of most interest to us is that of a partial Dehn twist where all non-trivial vertex group automorphisms are Dehn twist automorphisms themselves. In order to simplify the notation, we include the identity map as trivial Dehn twist defined by a degenerate graph-of-groups that is based on the graph which consists of a single vertex only. We define:

Definition 3.14. A graph-of-groups automorphism $H: \mathcal{G} \rightarrow \mathcal{G}$ is a partial Dehn twist relative to a family of Dehn twist automorphisms if $H$ is a partial Dehn twist as in Definition 3.13, with the specification that for every vertex $v \in V(\mathcal{G})$ the map $H_{v}$ is a (possibly trivial) Dehn twist automorphism.

## 4. H-conjugation

Recall from Proposition 2.6 that for any element $W$ in the Bass group $\Pi(\mathcal{G})$, represented by some reduced word $w \in W(\mathcal{G})$, then any other reduced $w^{\prime} \in W(\mathcal{G})$ which also represents $W$ is connected if and only if $w$ is connected. We hence call $W$ in this case a connected element of $\Pi(\mathcal{G})$. Similarly, the initial and terminal vertices $\iota(W)$ and $\tau(W)$ are well defined.

Definition 4.1. Let $H: \mathcal{G} \rightarrow \mathcal{G}$ be an isomorphism of a graph-of-groups. Let $W_{1}, W_{2}$ be non-trivial connected elements in the Bass group $\Pi(\mathcal{G})$. Then $W_{1}$ is said to be $H$-conjugate to $W_{2}$ if there exists a connected element $W \in \Pi(\mathcal{G})$ such that $W_{1}=W W_{2} H_{*}(W)^{-1}$. This connected element $W$ is called $H$-conjugator.

Lemma 4.2. H-conjugacy is an equivalence relation on the set of non-trivial connected elements in $\Pi(\mathcal{G})$.

Proof. Reflexivity and symmetry are obvious. In order to show transitivity we observe that from $W_{1}=W W_{2} H_{*}(W)^{-1}$ and $W_{2}=W^{\prime} W_{3} H_{*}\left(W^{\prime}\right)^{-1}$, where $W$ and $W^{\prime}$ are connected elements, one deduces $W_{1}=W W^{\prime} W_{3} H_{*}\left(W W^{\prime}\right)^{-1}$. Here $W W^{\prime}$ is connected since $W_{1}$ and $W_{2}$ are non-trivial and connected, which implies that $W$ terminates at $\iota\left(W_{2}\right)$ while $W^{\prime}$ initiates at $\iota\left(W_{2}\right)$.

Remark 4.3. Let $H$ be an isomorphism of a graph-of-groups $\mathcal{G}$. Two nontrivial connected elements $W_{1}, W_{2} \in \Pi(\mathcal{G})$ are $H$-conjugate to each other if and only if $W_{1}^{-1}$ and $W_{2}^{-1}$ are $H^{-1}$-conjugate to each other.

Definition 4.4. Let $H: \mathcal{G} \rightarrow \mathcal{G}$ be an isomorphism of a graph-of-groups, and let $W \in \Pi(\mathcal{G})$ be a non-trivial connected element.

Then $W$ is said to be $H$-zero if there exists a connected element $W^{\prime} \in \Pi(\mathcal{G})$ such that the (possibly trivial) element $W^{\prime} W H_{*}\left(W^{\prime}\right)^{-1}$ is contained in a single vertex group of $\mathcal{G}$.

Remark 4.5. There is an important particular reason why the trivial element of $\Pi(\mathcal{G})$ is not contained in any of the $H$-conjugacy classes as they are defined in the above set-up. It is a rather tricky issue, which has been dealt with in detail in Section 3 of [12].

This leads, however, to the following "unexpected" situation, due to the fact (obtained directly from Definitions 4.1 and 4.4) that an element $W \in$ $\Pi(\mathcal{G})$ is $H$-zero if and only if any $H$-conjugate $W^{\prime} \in \Pi(\mathcal{G})$ is also $H$-zero: It could well be that all elements in the $H$-conjugacy class of $W$ are $H$-zero, but none of them is actually contained in some vertex group of $\mathcal{G}$, since the only such which is $H$-conjugate to $W$ is the trivial element.

Remark 4.6. It is also important noting that in the special case, where $\pi_{1} \mathcal{G}$ is a free group of finite rank and $H: \mathcal{G} \rightarrow \mathcal{G}$ is based on the identity map $H_{\Gamma}=i d_{\Gamma(\mathcal{G})}$, one can decide algorithmically whether a non-trivial connected element $W \in \Pi(\mathcal{G})$ is $H$-zero or not.

Indeed, by Definition 4.4, $W$ is $H$-zero if and only if one can write $W$ as product

$$
W=W_{1} g H_{*}\left(W_{1}\right)^{-1}
$$

where $W_{1} \in \Pi(\mathcal{G})$ is also connected, and $g$ is contained in some vertex group $G_{v}$ of $\mathcal{G}$. By properly chosing $W_{1}$ we can assume here that $W_{1}$ is written as reduced word, and that there is no cancellation (other than within $G_{v}$ ) in the above product, which is hence reduced and of even $\mathcal{G}$-length $2 r$, for $r=\left|W_{1}\right|_{\mathcal{G}}$.

Thus, if $W$ is $H$-zero, it can be written as product $W=W^{\prime} W^{\prime \prime}$, with $W^{\prime}$ and $W^{\prime \prime}$ of $\mathcal{G}$-length $r$. The element $W^{\prime}$ is not quite determined by $W$, but for all possible choices we always have $W^{\prime-1} W_{1} \in G_{v}$. We conclude that $W$ is $H$-zero if and only if for any $W^{\prime}$ as above we have that

$$
W^{\prime-1} W H_{*}\left(W^{\prime}\right)
$$

has $\mathcal{G}$-length 0 (which is equivalent to being contained in some vertex group).
For a formal proof of the last conclusion we observe that $g^{\prime}:=W^{\prime-1} W_{1} \in$ $G_{v}$ implies $W^{\prime \prime}=W^{\prime-1} W=W^{\prime-1} W_{1} g H_{*}\left(W_{1}\right)^{-1}=g^{\prime} g H_{*}\left(W_{1}\right)^{-1}$ and hence $W^{\prime \prime} H_{*}\left(W^{\prime}\right)=g^{\prime} g H_{*}\left(W_{1}\right)^{-1} H_{*}\left(W^{\prime}\right)=g^{\prime} g H_{*}\left(W_{1}^{-1} W^{\prime}\right)=g^{\prime} g H_{*}\left(W_{1}^{-1} W^{\prime}\right)=$ $g^{\prime} g H_{*}\left(g^{\prime-1}\right)$, where $H_{*}\left(g^{\prime-1}\right) \in G_{v}$ follows from our assumption $H_{\Gamma}=i d_{\Gamma(\mathcal{G})}$.

## 5. QuOTIENT GRAPH-OF-GROUPS ISOMORPHISM

### 5.1. Quotient graph-of-groups.

Let $\mathcal{G}$ be a graph-of-groups and $\mathcal{G}_{0}$ be a sub-graph-of-groups of $\mathcal{G}$. By sub-graph-of-groups we simply mean the restriction of $\mathcal{G}$ to a connected
subgraph Denote $\Gamma=\Gamma(\mathcal{G})$ and $\Gamma_{0}=\Gamma\left(\mathcal{G}_{0}\right)$; by definition $\Gamma_{0}$ is a connected subgraph of $\Gamma$.

We will denote by $\overline{\mathcal{G}}=\mathcal{G} / \mathcal{G}_{0}$ the quotient graph-of-groups of $\mathcal{G}$ by $\mathcal{G}_{0}$, which we define now in detail:

The graph underlying $\overline{\mathcal{G}}$, denoted by $\bar{\Gamma}=\Gamma / \Gamma_{0}$ with $V(\bar{\Gamma})=(V(\Gamma) \backslash$ $\left.V\left(\Gamma_{0}\right)\right) \cup\left\{V_{0}\right\}$ and $E(\bar{\Gamma})=E(\Gamma) \backslash E\left(\Gamma_{0}\right)$, is obtained precisely by contracting $\Gamma_{0}$ into a vertex $V_{0}$ through the map:

$$
\begin{aligned}
& q: \Gamma \rightarrow \bar{\Gamma} \\
& x \mapsto V_{0} \quad \text { if } \mathrm{x} \in E\left(\Gamma_{0}\right) \text { or } V\left(\Gamma_{0}\right) \\
& x \mapsto X \quad \text { otherwise }
\end{aligned}
$$

Thus on $\Gamma \backslash \Gamma_{0}$ the map $q$ sends $x$, by which we mean either an edge or a vertex, to its natural correspondence in $\bar{\Gamma}$. We denote $q(x)$ by the same but capitalized letter $X \in \bar{\Gamma}$, to avoid confusion.

We choose a vertex

$$
\begin{equation*}
p_{0} \in V\left(\Gamma_{0}\right) \tag{5.1}
\end{equation*}
$$

as basepoint and set $G_{V_{0}}=\pi_{1}\left(\mathcal{G}_{0}, p_{0}\right)$.
For every $v \in V(\Gamma)$ and $V=q(v) \neq V_{0}$, we set $G_{V}=G_{v}$. For every $e \in E(\Gamma)$ and $E=q(e) \in E(\bar{\Gamma})$, let $G_{E}=G_{e}$, and $f_{E}=f_{e}$ if $\tau(E) \neq V_{0}$. If $\tau(E)=V_{0}$, we choose a word

$$
\begin{equation*}
\gamma_{E} \in \Pi\left(\mathcal{G}_{0}\right) \subset \Pi(\mathcal{G}) \tag{5.2}
\end{equation*}
$$

from $p_{0}$ to $\tau(e) \in V\left(\Gamma_{0}\right)$ and define the edge homomorphism of $E$ to be $f_{E}:=a d_{\gamma_{E}} \circ f_{e}$.

We define a group homomorphism $\theta$ from $\Pi(\overline{\mathcal{G}})$ to $\Pi(\mathcal{G})$ via:

$$
\begin{aligned}
\theta: \Pi(\overline{\mathcal{G}}) & \rightarrow \Pi(\mathcal{G}) \\
t_{E} & \mapsto t_{e} \gamma_{E}^{-1} \quad \text { if } \tau(E)=V_{0}, \tau(\bar{E}) \neq V_{0} \\
t_{E} & \mapsto \gamma_{\bar{E}} t_{e} \gamma_{E}^{-1} \quad \text { if } \tau(E)=\tau(\bar{E})=V_{0} \\
\theta & \text { acts as identity elsewhere }
\end{aligned}
$$

Through the identification $G_{V_{0}}=\pi_{1}\left(\mathcal{G}_{0}, p_{0}\right)$ we see immediately that the restriction of $\theta$ to $\pi_{1}\left(\overline{\mathcal{G}}, V_{0}\right)$ defines an isomorphism from $\pi_{1}\left(\overline{\mathcal{G}}, V_{0}\right)$ to $\pi_{1}\left(\mathcal{G}, p_{0}\right)$ which we denote by $\theta_{0}$ :

The inverse map $\theta_{0}^{-1}$ is given through introducing a cancelling pair $\gamma_{E}^{-1} \gamma_{E}$ after the stable letter $t_{e}$ for any edge $e$ with $\tau(e) \in V\left(\Gamma_{0}\right)$, and $\gamma_{\bar{E}}^{-1} \gamma_{\bar{E}}$ before the stable letter $t_{e}$ for any edge $e$ with $\tau(\bar{e}) \in V\left(\Gamma_{0}\right)$. One then maps $t_{e}$ to $\gamma_{\bar{E}}^{-1} t_{E} \gamma_{E}$ if both $\tau(e), \tau(\bar{e}) \in V\left(\Gamma_{0}\right)$, and one maps $t_{e}$ to $t_{E} \gamma_{E}$ if only $\tau(e) \in V\left(\Gamma_{0}\right)$.

[^1]Remark 5.1. For later purposes the reader should note here that for any edge $E=q(e)$ of $\Gamma(\overline{\mathcal{G}})$ with $\tau(E)=V_{0}$ there exist a vertex $v(E):=\tau(e) \in$ $V\left(\mathcal{G}_{0}\right)$ such that for the "connecting word" $\gamma_{E} \in \Pi\left(\mathcal{G}_{0}\right)$ from $p_{0}$ to $v(E)$ one has $\theta \circ f_{E}\left(G_{E}\right) \subset \gamma_{E} G_{v(E)} \gamma_{E}^{-1}$.

### 5.2. Quotient graph-of-groups isomorphism.

In this subsection we define the notion of a quotient graph-of-groups isomorphism.

The graphs-of-groups $\mathcal{G}, \mathcal{G}_{0}, \overline{\mathcal{G}}$ and the group homomorphisms $\theta$ and $\theta_{0}$ are defined as in the previous subsection. In particular, let $V_{0}, \gamma_{E}$ and $p_{0}$ be as given there.

Let $H: \mathcal{G} \rightarrow \mathcal{G}$ be a graph-of-groups isomorphism which acts as identity on the graph $\Gamma$. The map $H_{0}: \mathcal{G}_{0} \rightarrow \mathcal{G}_{0}$, obtained by restricting $H$ to $\mathcal{G}_{0}$, is called the local graph-of-groups isomorphism.

In order to define $\bar{H}: \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}$ we set:
(1) $\bar{H}_{V_{0}}=H_{0, * p_{0}}: \pi_{1}\left(\mathcal{G}_{0}, p_{0}\right) \rightarrow \pi_{1}\left(\mathcal{G}_{0}, p_{0}\right)$;
(2) $\delta(E)=H_{*}\left(\gamma_{E}\right) \delta(e) \gamma_{E}^{-1}$, for all $E$ such that $\tau(E)=V_{0}$;
(3) $\bar{H}$ "equals" $H$ on the rest of $\overline{\mathcal{G}}$ (modulo replacing $x$ by $X$ as explained in the previous subsection). In particular, $\bar{H}_{\bar{\Gamma}}$ is the identity on the quotient graph $\bar{\Gamma}$.

Proposition 5.2. The above conditions (1) - (3) give a well defined graph-of-groups isomorphism $\bar{H}: \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}$. It induces an outer automorphism $\widehat{\bar{H}}$ which is conjugate to $\widehat{H}$ via the isomorphism $\theta_{0}: \pi_{1}\left(\overline{\mathcal{G}}, V_{0}\right) \rightarrow \pi_{1}\left(\mathcal{G}, p_{0}\right)$.

Proof. In order to show that the above data (1) - (3) give a well defined graph-of-groups isomorphism, the only non-trivial step to verify is the condition that $\bar{H}_{V_{0}} \circ f_{E}=a d_{\delta(E)} \circ f_{E} \circ \bar{H}_{E}$ for all $E=q(e)$ with $\tau(E)=V_{0}$. Denote $v=\tau(e) \in V\left(\Gamma_{0}\right)$.

Observe first that for any $h \in G_{v}$, since $H_{0}$ induces an isomorphism on $\Pi\left(\mathcal{G}_{0}\right)$, we have:

$$
\bar{H}_{V_{0}}\left(\gamma_{E} h \gamma_{E}^{-1}\right)=H_{0 *}\left(\gamma_{E}\right) H_{v}(h) H_{0 *}\left(\gamma_{E}^{-1}\right)=H_{*}\left(\gamma_{E}\right) H_{v}(h) H_{*}\left(\gamma_{E}^{-1}\right)
$$

For any $g \in G_{E}$ we compute, where the fourth equality uses the previous observation, for $h=f_{e}(g)$ :

$$
\begin{aligned}
a d_{\delta(E)} \circ f_{E} \circ \bar{H}_{E}(g) & =H_{*}\left(\gamma_{E}\right) \delta(e) \gamma_{E}^{-1} f_{E}\left(\bar{H}_{E}(g)\right) \gamma_{E} \delta(e)^{-1} H_{*}\left(\gamma_{E}^{-1}\right) \\
& =H_{*}\left(\gamma_{E}\right) \delta(e) f_{e}\left(H_{e}(g)\right) \delta(e)^{-1} H_{*}\left(\gamma_{E}^{-1}\right) \\
& =H_{*}\left(\gamma_{E}\right) H_{v}\left(f_{e}(g)\right) H_{*}\left(\gamma_{E}^{-1}\right) \\
& =\bar{H}_{V_{0}}\left(\gamma_{E} f_{e}(g) \gamma_{E}^{-1}\right) \\
& =\bar{H}_{V_{0}} \circ f_{E}(g)
\end{aligned}
$$

In order to illustrate the previous proof we propose the diagram below: every face of the "cube" pictured there commutes, up to inner automorphisms
on the front and back faces, defined by the correction terms $\delta(E)$ and $\delta(e)$ respectively.


Here the map $G_{e} \rightarrow G_{E}$ is the identity, while the map $G_{v} \rightarrow G_{V_{0}}$ is $a d_{\gamma_{E}}$.
After having thus proved the first sentence of the proposition, we now turn to the second: we want to show that the following diagram commutes.


Notice that the group homomorphism $\theta$ acts on all vertex groups other than $G_{V_{0}}$ as identity. On the other hand, for $E \in E(\overline{\mathcal{G}})$ with $\tau(E)=V_{0}$ but $\tau(\bar{E}) \neq V_{0}$ we have:

$$
\begin{aligned}
\theta \circ \bar{H}_{*}\left(t_{E}\right)=\theta\left(\delta(\bar{E}) t_{E} \delta(E)^{-1}\right) & =\delta(\bar{E}) t_{e} \gamma_{E}^{-1} \delta(E)^{-1} \\
& =\delta(\bar{e}) t_{e} \delta(e)^{-1} H_{*}\left(\gamma_{E}\right)^{-1} \\
& =H_{*}\left(t_{e} \gamma_{E}^{-1}\right) \\
& =H_{*} \circ \theta\left(t_{E}\right)
\end{aligned}
$$

A similar computation applies to $E \in E(\overline{\mathcal{G}})$ with $\tau(E)=\tau(\bar{E})=V_{0}$.
Therefore we obtain $\theta_{0} \circ \bar{H}_{* V_{0}}=H_{* p_{0}} \circ \theta_{0}$, and hence $\widehat{\bar{H}}$ and $\widehat{H}$ are conjugate to each other through the outer isomorphism $\widehat{\theta}_{0}$.

Remark 5.3. Note in particular, for all $E$ with $\tau(E)=V_{0}$, we have by definition that $\delta(E)$ is $H_{0}^{-1}$-conjugate to an element with $\mathcal{G}_{0}$-length equal to zero, and hence is $H_{0}^{-1}$-zero: The word $\gamma_{E}$ satisfies

$$
H_{0 *}\left(\gamma_{E}^{-1}\right) \theta(\delta(E)) \gamma_{E} \in G_{v(E)} .
$$

Remark 5.4. Our formal construction of quotient graph-of-group isomorphism $\bar{H}$, constructed as above, depends on the choice of base point $p_{0}$ in $\Gamma_{0}$ and of "connecting words" $\gamma_{E}$ for all edges $E$ with $\tau(E)=V_{0}$, as set up in (5.2) in order to define $\theta$ and $\theta_{0}$.

However, the outer automorphism $\widehat{\bar{H}}$ induced by $\bar{H}: \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}$ depends neither on the choice of the $\gamma_{E}$ nor on the choice of $p_{0}$, up to conjugation by the natural isomorphism $\theta_{0}^{-1} \theta_{0}^{\prime}$, where $\theta_{0}^{\prime}$ is the map analogous to $\theta_{0}$ defined through an alternative choice of $p_{0}$ and and the $\gamma_{E}$.

This is a direct consequence of the statement in Proposition 5.2 that $\widehat{\bar{H}}$ is conjugate to $\widehat{H}$ via $\theta_{0}$.

Alternatively, a direct proof, without passing through $\widehat{H}$, can be given by applying Lemma 2.11 this yields the slightly stronger result that a second quotient automorphism $\bar{H}^{\prime}: \overline{\mathcal{G}}^{\prime} \rightarrow \overline{\mathcal{G}}^{\prime}$ is conjugated to $\bar{H}$ by a graph-ofgroups isomorphism $F: \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}^{\prime}$.
Remark 5.5. We may apply this quotient procedure above on several disjoint connected subgraphs-of-groups of $\mathcal{G}$ and obtain the analoguous conclusion that the quotient graph-of-groups isomorphism $\bar{H}$ is well defined and induces an outer automorphism conjugate to $\widehat{H}$.
Remark 5.6. For simplicity of notations, we sometimes represent the simultaneous quotienting of the graph-of-groups $\mathcal{G}$ and of the isomorphism $H$ by referring to the quotient pair $(\bar{H}, \overline{\mathcal{G}})$, obtained from $(H, \mathcal{G})$ modulo the pair $\left(H_{0}, \mathcal{G}_{0}\right)$.

## 6. BLOWING UP GRAPH-OF-GROUPS AUTOMORPHISM

In this section, we will reverse the quotient construction in the previous section; we emphasize this reversal by the choice of our notation.

For simplicity of the presentation, we only give the blow-up construction at a single vertex. However (for example through iterating this procedure), one can generalize the technique described in this section directly to a blowup construction at several vertices simultaneously to obtain Theorem 1.1 in the Introduction.

### 6.1. Assumptions.

Let $\bar{H}: \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}$ and $H_{0}: \mathcal{G}_{0} \rightarrow \mathcal{G}_{0}$ be graph-of-groups automorphisms which act as the identity on their underlying graphs $\bar{\Gamma}$ and $\Gamma_{0}$ respectively. Let $V_{0}$ be a vertex of $\bar{\Gamma}$. We consider the following assumptions:
(A1) There exist a vertex $p_{0} \in V\left(\mathcal{G}_{0}\right)$ and a group isomorphism $\theta_{0}: G_{V_{0}} \rightarrow$ $\pi_{1}\left(\mathcal{G}_{0}, p_{0}\right)$ such that

$$
\theta_{0} \circ H_{V_{0}}=H_{0 *, p_{0}} \circ \theta_{0} ;
$$

(A2) Compatibility requirement for graph-of-groups:
For any edge $E$ of $\Gamma(\overline{\mathcal{G}})$ with $\tau(E)=V_{0}$ there exist some vertex $v(E) \in V\left(\mathcal{G}_{0}\right)$ and a "connecting word" $\gamma_{E} \in \Pi\left(\mathcal{G}_{0}\right)$ from $p_{0}$ to $v(E)$ such that $\theta_{0} \circ f_{E}\left(G_{E}\right) \subset \gamma_{E} G_{v(E)} \gamma_{E}^{-1}$;
(A3) Compatibility requirement for isomorphism:
The word $\gamma_{E}$ also satisfies $H_{0 *}\left(\gamma_{E}^{-1}\right) \theta_{0}(\delta(E)) \gamma_{E} \in G_{v(E)}$.
Definition 6.1. Assume that condition (A1) is satisfied. Then the pair ( $H_{0}, \mathcal{G}_{0}$ ), which is called the local graph-of-groups isomorphism associated to $V_{0}$, is said to be compatible with $(\bar{H}, \overline{\mathcal{G}})$ if both compatibility requirements (A2) and (A3) are satisfied.

Remark 6.2. We'd like to note:
(1) The compatibility requirement for isomorphism implies, for any $\delta(E)$ with $\tau(E)=V_{0}$, that the element $\theta_{0}(\delta(E))$ is $H_{0}^{-1}$-zero.

Conversely, in order to derive the compatibility requirement for isomorphism for any correction term $\delta(E)$ such that $\theta_{0}(\delta(E))$ is $H_{0}^{-1}$ zero, one needs the additional hypothesis that $\delta(E)$ is $G_{E}$-compatible: by this we mean that there is a connected word $\gamma_{E} \in \Pi(\mathcal{G})$ which satisfies both assumptions (A2) and (A3) above.
(2) For each $E$ with $\tau(E)=V_{0}$, there may exist more then one pair of $\left(v(E), \gamma_{E}\right)$ such that the above conditions hold.

### 6.2. Existence of the blow-up.

Theorem 6.3. Let $\bar{H}: \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}$ and $H_{0}: \mathcal{G}_{0} \rightarrow \mathcal{G}_{0}$ be graph-of-groups isomorphisms which act as the identity on their underlying graphs, and let $V_{0} \in V(\overline{\mathcal{G}})$ be a vertex for which condition (A1) is satisfied.

Then there exists a "blow-up" graph-of-groups $\mathcal{G}$ and a "blow-up" graph-ofgroups isomorphism $H: \mathcal{G} \rightarrow \mathcal{G}$, which contains $H_{0}$ as local graph-of-groups isomorphism and yields modulo $\left(H_{0}, \mathcal{G}_{0}\right)$ the quotient pair $(\bar{H}, \overline{\mathcal{G}})$, if and only if the conditions (A2) and (A3) are satisfied.

In particular, $H$ and $\bar{H}$ induce outer automorphisms which are conjugate.
Proof. For the convenience of the reader we divide this proof in 6 steps; in the first two we present the data which will serve to define $\mathcal{G}$ and $H$ respectively.
(1) The graph $\Gamma(\mathcal{G})$, with vertex set $V(\mathcal{G})=V(\overline{\mathcal{G}}) \backslash\left\{V_{0}\right\} \cup V\left(\mathcal{G}_{0}\right)$, is obtained from $\Gamma(\mathcal{G})$ and $\Gamma\left(\mathcal{G}_{0}\right)$ by replacing every edge $E$ with terminal vertex $\tau(E)=$ $V_{0}$ by an edge $e$ with terminal vertex $v(E)$. The analogous replacement is done for $\bar{E}$. If $E$ has both endpoints distinct from $V_{0}$ we leave them as they are, but rename $E$ by the corresponding small letter $e$.

We set $G_{e}=G_{E}$ and define, if $\tau(e)=v(E)$, the edge injection by $f_{e}(g)=$ $\gamma_{E}^{-1} \theta_{0}\left(f_{E}(g)\right) \gamma_{E}$, for every $g \in G_{e}$. For $\tau(e)=\tau(E)$ we define $f_{e}=f_{E}$.
(2) The isomorphism $H: \mathcal{G} \rightarrow \mathcal{G}$ is equal to $H_{0}$ or to $\bar{H}$ when restricted to $\mathcal{G}_{0}$ or to $\overline{\mathcal{G}} \backslash\left\{V_{0}\right\}$ respectively, except that in the case $\tau(e)=v(E)$ we modify the correction term to $\delta(e)=H_{0 *}\left(\gamma_{E}^{-1}\right) \theta_{0}(\delta(E)) \gamma_{E}$.
(3) In order to show that the data defined above in (1) give a well defined graph-of-groups one only needs to verify that for any edge $e$ of $\mathcal{G}$ the edge injection $f_{e}$ has its image in the vertex group $G_{\tau(e)}$. If $\tau(e)$ is not contained
in $V\left(\mathcal{G}_{0}\right)$, then this is immediate from the definition. If $\tau(e) \in V\left(\mathcal{G}_{0}\right)$, then we use the compatibility requirement for graph-of-groups, which gives (A2), $f_{e}\left(G_{e}\right)=\gamma_{E}^{-1} \theta_{0}\left(f_{E}\left(G_{E}\right)\right) \gamma_{E} \subset G_{v(E)}$.
(4) We now want to show that the data defined above in (2) give a welldefined graph-of-groups isomorphism. This is equivalent to showing for every edge $e$ of $\mathcal{G}$ the equality stated in condition (4) of Definition 2.8, Again, for $\tau(e) \notin V\left(\mathcal{G}_{0}\right)$ this is a direct consequence of our set-up. For $\tau(e) \in V\left(\mathcal{G}_{0}\right)$ we compute (where the third equality uses the definition of $f_{e}$ from (2) above, and the fifth equality uses condition (A1)):

$$
\begin{aligned}
a_{\delta(e)} \circ f_{e} \circ H_{e}(g) & =\delta(e) f_{e}\left(H_{e}(g)\right) \delta(e)^{-1} \\
& =H_{0 *}\left(\gamma_{E}^{-1}\right) \theta_{0}(\delta(E)) \gamma_{E} f_{e}\left(H_{e}(g)\right) \gamma_{E}^{-1} \theta_{0}\left(\delta(E)^{-1}\right) H_{0 *}\left(\gamma_{E}\right) \\
& =H_{0 *}\left(\gamma_{E}^{-1}\right) \theta_{0}\left(\delta(E) f_{E}\left(H_{E}(g)\right) \delta(E)^{-1}\right) H_{0 *}\left(\gamma_{E}\right) \\
& =H_{0 *}\left(\gamma_{E}^{-1}\right) \theta_{0}\left(H_{V_{0}}\left(f_{E}(g)\right)\right) H_{0 *}\left(\gamma_{E}\right) \\
& =H_{0 *}\left(\gamma_{E}^{-1}\right) H_{0 *, P_{0}}\left(\theta_{0}\left(f_{E}(g)\right)\right) H_{0 *}\left(\gamma_{E}\right) \\
& =H_{0 *}\left(\gamma_{E}^{-1} \theta_{0}\left(f_{E}(g)\right) \gamma_{E}\right)=H_{0 *}\left(f_{e}(g)\right)=H_{v(E)} \circ f_{e}(g)
\end{aligned}
$$

(5) We now observe from our construction above that the blow-up graph-ofgroups isomorphism $H$ contains $H_{0}$ as local graph-of-groups isomorphism. We furthermore have already a base point $p_{0}$ as well as connecting words $\gamma_{E}$ from $p_{0}$ to $\tau(e)$ for each edge $e$ with terminal vertex $\tau(e) \in V\left(\mathcal{G}_{0}\right)$ specified, so that one can readily apply Proposition 5.2 to obtain a quotient graph-ofgroups isomorphism, which, since $p_{0}$ and all $\gamma_{E}$ are as chosen before, must agree with the isomorphism $\bar{H}: \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}$.

In particular it follows that the induced outer automorphisms $\widehat{H}$ and $\widehat{\bar{H}}$ are conjugate.
(6) Finally, it follows from Remarks 5.1 and 5.3 that any blow-up pair $(H, \mathcal{G})$ which quotients modulo $\left(H_{0}, \mathcal{G}_{0}\right)$ to the given pair $(\bar{H}, \overline{\mathcal{G}})$ must necessarily satisfy the conditions (A2) and (A3).

## 7. Partial Dehn Twist case

In this section we will apply Theorem 6.3 to the special case where all edges $E$ with terminal vertex $V_{0}$ have trivial edge group:

$$
G_{E}=\{1\} \quad \text { if } \quad \tau(E)=V_{0}
$$

In this case the compatibility conditions from Definition 6.1 simplify considerably, as condition (A2) is trivially satisfied. Regarding the other compatibility condition, we observe that a connecting word $\gamma_{E}$ which satisfies condition (A3) exists, if and only if the $\theta_{0}$-image of the correction term $\delta(E)$ is $H_{0}^{-1}$-zero, in the terminology of section 4. Hence we define:

Definition 7.1. Let $\bar{H}: \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}$ and $H_{0}: \mathcal{G}_{0} \rightarrow \mathcal{G}_{0}$ be graph-of-groups isomorphisms which act as the identity on their underlying graphs, and let
$V_{0} \in V(\overline{\mathcal{G}})$ be a vertex with an isomorphism $\theta_{0}: G_{V_{0}} \rightarrow \pi_{1}\left(\mathcal{G}_{0}, p_{0}\right)$ as in condition (A1).

Then an edge $E$ of $\overline{\mathcal{G}}$ is said to be locally zero if the correction term of $E$ has image $\theta_{0}(\delta(E))$ which is $H_{0}^{-1}$-zero. In other words, there exists a connected word $\gamma_{E} \in \Pi\left(\mathcal{G}_{0}\right)$ such that $H_{0 *}\left(\gamma_{E}^{-1}\right) \theta_{0}(\delta(E)) \gamma_{E} \in G_{v}$ for some vertex $v$ of $\mathcal{G}_{0}$.

We will now specialize further to the case of partial Dehn twists $\bar{H}: \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}$ as defined in Definition 3.13, which have trivial vertex group isomorphisms $\bar{H}_{V}$ except at vertices $V_{i}$ that belong to a subset $\mathcal{V}_{0} \subset V(\overline{\mathcal{G}})$. At those "special" vertices we want to assume furthermore that $\bar{H}_{V_{i}}$ is a Dehn twist automorphism, given via an isomorphism $\theta_{i}: G_{V_{i}} \rightarrow \pi_{1}\left(\mathcal{G}_{i}, p_{i}\right)$ as in condition (A1) by some local Dehn twist $D_{i}: \mathcal{G}_{i} \rightarrow \mathcal{G}_{i}$ (see Definition 3.9), so that $\bar{H}$ is a partial Dehn twist relative to a family of local Dehn twists.

Corollary 7.2. Let $\bar{H}: \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}$ be a partial Dehn twist relative to $\mathcal{V}_{0} \subset V(\overline{\mathcal{G}})$, and assume that for each $V_{i} \in \mathcal{V}_{0}$ the vertex group isomorphism $H_{V_{i}}$ is a Dehn twist automorphism represented by a "local" Dehn twist $D_{i}: \mathcal{G}_{i} \rightarrow \mathcal{G}_{i}$.

Then one can blow up $(\bar{H}, \overline{\mathcal{G}})$ via $\left(D_{i}, \mathcal{G}_{i}\right)$ to obtain a Dehn twist $D: \mathcal{G} \rightarrow \mathcal{G}$ if and only if every edge $E$ of $\overline{\mathcal{G}}$ with $\tau(E) \in \mathcal{V}_{0}$ is locally zero.

In this case the induced outer automorphism $\widehat{\bar{H}}: \pi_{1}(\overline{\mathcal{G}}) \rightarrow \pi_{1}(\overline{\mathcal{G}})$ is a Dehn twist automorphism.

Proof. Let $V_{i}$ be a vertex contained in $\mathcal{V}_{0}$. Since any edge $E$ terminating at $V_{i}$ has trivial edge group, the compatibility requirement for graph-ofgroups (A2) holds automatically. On the other hand, the assumption that any edge $E$ with $\tau(E) \in \mathcal{V}_{0}$ is locally zero is equivalent to the compatibility requirement for isomorphism (A3). We can hence apply Theorem 6.3 to directly obtain the desired "if and only if" statement.

Since $\bar{H}$ is a partial Dehn twist and the $D_{i}$ are Dehn twists, it follows directly from the construction of the blow-up isomorphism in the proof of Theorem 6.3 that $D$ satisfies the first three properties of Definition 3.9 ,

In order to see that the fourth condition of Definition 3.9 is also satisfied, we consider three cases: If an edge $E$ from $\overline{\mathcal{G}}$ has endpoint outside of $\mathcal{V}_{0}$, then all relevant data for $E$ and the corresponding edge $e$ in $\mathcal{G}$ coincide, so that we can simply use the last condition of Definition 3.13. If $\tau(E) \in \mathcal{V}_{0}$, then $G_{E}=\{1\}$ follows from Definition [3.13, so that the fourth condition of Definition 3.9 is automatically satisfied. Finally, if the edge $e$ from $\mathcal{G}$ in question is an edge of some of the local graph-of-groups $\mathcal{G}_{i}$, then we use that $D_{i}: \mathcal{G}_{i} \rightarrow \mathcal{G}_{i}$ itself is assumed to satisfy Definition [3.9, so that in particular its fourth condition of this definition holds for $e$.

As a consequence we obtain from Proposition 3.10 that $\widehat{D}$ is a Dehn twist automorphism. But Theorem 6.3 also states that the outer automorphisms induced by $\bar{H}$ and by $D$ are conjugate, which shows that $\widehat{\bar{H}}$ is also a Dehn twist automorphism.

Corollary 7.2 gives the possibility to decide the existence of a blow-up Dehn twist relative to particular given local Dehn twist representatives. A harder but more interesting question is whether one can blow up the given partial Dehn twist $\bar{H}: \overline{\mathcal{G}} \rightarrow \overline{\mathcal{G}}$ to a global Dehn twist with respect to some family of local Dehn twists that induce the vertex automorphisms of $\bar{H}$. In other words (using the terminology of Definition 3.14):
"When does a partial Dehn twist with Dehn twist automorphism on the vertices induce a Dehn twist automorphism ?"

The subtlety of this question is illustrated by the Examples 1.3 and 1.4 in the Introduction. A complete answer is given in [21].

Remark 7.3. Assume that in the situation of Corollary 7.2 the following data are given:
(a) The graph-of-groups $\overline{\mathcal{G}}$ and $\mathcal{G}_{i}$ have free groups of finite rank as vertex and edge groups, with chosen bases for each of them, and that the edge maps are given in the usual fashion by specifying the images of the edge groups basis elements as words in the basis of the adjacent vertex group.
(b) The graph-of-groups automorphisms $\bar{H}$ and $H_{i}$ are similarly given by specifying the images of the given basis elements as word in those bases, and by specifying for every edge $E$ of $\overline{\mathcal{G}}$ with terminal vertex $V_{i} \in \mathcal{V}_{0}$ the $\theta_{i}$-image of the correction term $\delta(E)$ as word $W(\delta(E)) \in$ $\Pi\left(\mathcal{G}_{i}\right)$.
(c) For every edge $E$ with $\tau(E)=V_{i} \in \mathcal{V}_{0}$, a vertex $v_{E}$ of $\mathcal{G}_{i}$ and connecting words $\gamma_{E} \in \Pi\left(\mathcal{G}_{i}\right)$ are specified which satisfy

$$
H_{i *}\left(\gamma_{E}^{-1}\right) W(\delta(E)) \gamma_{E} \in G_{v_{E}}
$$

Then from these data one derives directly (in an algorithmic way) the analogous data needed to define the blow-up graph $\mathcal{G}$ and the blow-up isomorphism $H$. Indeed, the precise instructions for this procedure are given in the parts (1) and (2) of proof of Theorem 6.3.

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[^1]:    ${ }^{1}$ Be aware that this definition of sub-graph-of-groups is different from the one in 1 .

